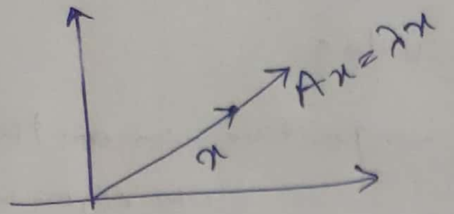
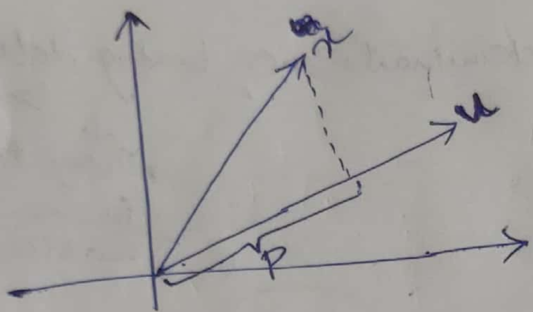
PCA



$$A = \begin{pmatrix} a_{11} & a \\ a & a_{12} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d(Ax)}{dx} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_2}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix} = A$$

$$\frac{d(x^T A x)}{dx} = 2Ax$$



~~$P = u \cdot v$ or $P = u^T v$ or $P = v^T u$~~

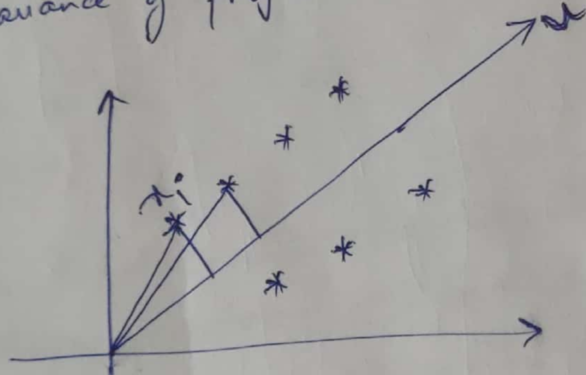
$$P = \vec{x} \cdot \vec{u}$$

$$\boxed{P = u^T x}$$

or

$$P = x^T u$$

Variance of projection.



$$\frac{1}{N}\sum_{n=1}^{N} \left(u^T x_n - u^T \bar{x}\right)^2$$

$$= u^T \left[\frac{\sum_{n=1}^{N} (x_n - \bar{x})(x_n - \bar{x})^T}{N} \right] u$$

variance of projection $= u^T S u$

You need to add a constraint inorder to optimize $u^T u = 1$

$\rightarrow$ variance of projection given $\|u\| = 1 \rightarrow$ $u^T u = 1$

$$\max(u^T S u)$$

$$\Rightarrow \max\left[(u^T S u) + \lambda(1 - u^T u)\right]$$

$$\Rightarrow \frac{d}{du}\left[u^T S u + \lambda(1 - u^T u)\right] = 0$$

$$\Rightarrow \boxed{Su = \lambda u}$$

$$\Rightarrow \boxed{\lambda = u^T S u}$$

So variance of projection = Eigen value given $u^T u = 1$

$$\boxed{\lambda = u^T S u}$$

So variance of projection $= u^T S u$

u should be eigen vectors.

S has a property called <u>Symetric positive semi-definate.</u>

(1) So here u is a orthonormal (orthogonal and Normalized) are $\perp r$ to each other.

i.e $\{u_1, u_2 \cdots u_k\}$

(2) $u^T = u^{-1}$

(3) $\lambda = u^{-1} S u$ → This is a diagnolizing operation.

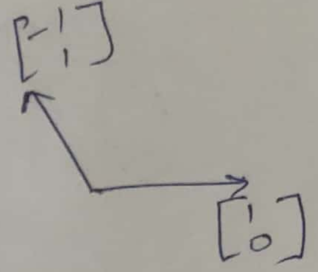
$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = u^{-1} S u$$

So a Symetric matrix is diagnolized by a matrix of its ~~orthogonal vectors~~. orthonormal eigen vectors.

$$S = u \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} u^{-1}$$

Diagnolize

A → chance the bases to it's Eigen vectors to diagnolize



$$\begin{bmatrix}3&0\\0&2\end{bmatrix} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}^{-1}\begin{bmatrix}3&1\\0&2\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$$

Change your coordinate system to use your eigen vectors as bases.

bases vector which are also eigen vectors → eigen bases

$$\begin{bmatrix}3&1\\0&2\end{bmatrix}^{100} \xrightarrow{\text{Convert to eigen bases}} \begin{bmatrix}3&0\\0&2\end{bmatrix}^{100} \longrightarrow$$ Convert this result back to standard system.

$$\underset{\text{her basis}}{\begin{bmatrix}2&-1\\1&1\end{bmatrix}}\underset{\text{her point}}{\begin{bmatrix}-1\\2\end{bmatrix}} = -1\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \underset{\text{our point}}{\begin{bmatrix}-4\\1\end{bmatrix}}$$

our grid → other grid

our grid ← other grid

$$\begin{bmatrix}2&-1\\1&1\end{bmatrix}^{-1}\underset{\text{our point}}{\begin{bmatrix}3\\2\end{bmatrix}} = \begin{bmatrix}5/3\\1/3\end{bmatrix}$$

$$A\begin{bmatrix}x_i\\y_i\end{bmatrix} = \begin{bmatrix}x_o\\y_o\end{bmatrix}$$

$$A^{-1}\cdot\begin{bmatrix}x_o\\y_o\end{bmatrix} = \begin{bmatrix}x_i\\y_i\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}^{-1}\underbrace{\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{\text{Transformation matrix in our language}}\underbrace{\overset{\text{changed basis}}{\begin{bmatrix}2&-1\\1&1\end{bmatrix}}\begin{bmatrix}-1\\2\end{bmatrix}}_{\text{Same vector in our language}}}_{\text{Transformed vector in our language}}}_{\text{Transformed vector in her language}}$$

Transformation matrix in her Language

$$[\underbrace{A^{-1}}_{\substack{\text{Shift in}\\\text{perspective}\\\text{(empathy)}}}\underbrace{MA}_{\substack{\text{transformation as}\\\text{we see it}}}]\vec{v}$$

Positive definite Matrix

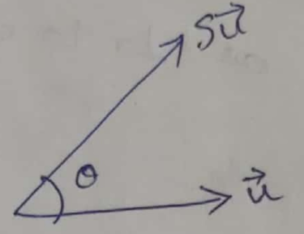
$\lambda = u^T S u$

Positive definite matrix is a Symmetric matrix.

Property

1) Every Eigen value is +ve.

i.e λ is always +ve.



$\theta < 90^\circ$

$u^T S u = |u| \cdot |Su| \cdot \cos\theta$

$Su = \lambda u$

$\Rightarrow u^T \cdot Su = u^T \lambda u$

$\Rightarrow \underbrace{u^T \cdot Su}_{\text{this is always } > 0} = \lambda \underbrace{\|u^2\|}_{\text{this is +ve}}$

So this should also be +ve.

Example

$S = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ det$|S| = -1$

Product of eigen values is -ve

So This is an indefinite matrix.

$S = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ det$|S| = 2$

so this is a definite matrix

$S = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$ det$|S| = 2$, but leading determinants are < 0.

$S = \begin{bmatrix} 3 & 4 \\ 4 & 16/3 \end{bmatrix}$

det $|S| = 0 \rightarrow$ it's singular,
So only one eigen value

The trace $3 + 16/3 \rightarrow$ positive, so the one eigen value is +ve.

$\lambda = 8\frac{1}{3}$

Positive $\boxed{\text{Semi}}$ definite

$\lambda_i \geq 0$, $u^T S u \geq 0$, deteminants ≥ 0, Leading deteminants > 0

Allway true in cov. matrices.

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SVD

$A_{m\times n}$

2 unit matrices which are orthogonal

1 rectangular diagnol matrix of singular values.

$$A_{m\times n} = U_{m\times m}\,\Sigma_{m\times n}\,V^T_{n\times n}, \quad UU^T = I, \quad VV^T = I$$

$$A^TA = V(\Sigma^T\Sigma)V^T$$

$$AA^T = U(\Sigma\Sigma^T)U^T$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$$AA^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}_{2\times 2}$$

Find eigen values

$$\begin{bmatrix} 11-\lambda & 1 \\ 1 & (11-\lambda) \end{bmatrix} = 0$$

$$(11-\lambda)^2 - 1^2 = 0$$

$$\Rightarrow \lambda_1 = 12, \lambda_2 = 10$$

Eigen vectors $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

↳ Need to do orthogonalization of this matrix

Using Gram-Schmidt Orthogonalization process.

Get λ's of U, V,

we get U, V^T

λ's of $U = 12, 10$

λ's of $V = 12, 10, 0$

$$\underset{2\times 3}{A} = \underset{2\times 2}{\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}} \begin{bmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}, \quad \text{IIrly } V = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -5 \end{bmatrix}, V^T \text{ and then orthogonalize}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

Application

People	
A B C	
	A
	T
	G
	C

Gene seq.

$m \times n$

trying to find a max. people-gene relation

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \frac{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}{\sqrt{5}} \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}{\sqrt{2}}$$

We want $u_1, \sigma_1, v_1^T \rightarrow$ this is the principal component.

$\lambda = u^T S u$

$\lambda = u^{-1} S u$ → This is how we diagnolize even a non-symmetric square matrix

$\Rightarrow \boxed{\underset{m \times m}{S} = u \lambda u^{-1}}$, u, u are orthogonal and same for a covariance matrix

$\underset{m \times n}{A} = U \Sigma V^T$, u, v are orthogonal and different.

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