Riemann-Roch

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September 9, 2024

For the definitions about divisors, let X be a noetherian integral scheme which is regular in codimension 1 meaning all local rings of codimension 1 are regular (where regular means minimal number of generators for maximal ideal is equal to Krull dimension).

Definition 1. A prime divisor on X is an irreducible subvariety of X with codimension 1.

Definition 2. A Weil Divisor on X is a formal $\mathbb{Z}$-linear combination of prime divisors of X. We denote the free abelian group consisting of all the Weil divisors by $\mathrm{Div}(X)$.

Definition 3 (Hartshorne p.140)**.** Let X be a scheme. For each open affine subset $U = SpecA$, let S be the set of elements of A which are not zero divisors, and let $K(U)$ be the localization of A by the multiplicative system S.

We call $K(U)$ the total quotient ring of A. For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings X we call the sheaf of total quotient rings of $\mathcal{O}$. On an arbitrary scheme, the sheaf $\mathcal{K}$ replaces the concept of function field of an integral scheme.

We denote by $\mathcal{K}^*$ the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings $\mathcal{K}$.

Similarly $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$.

Definition 4 (Hartshorne p.141)**.** A Cartier divisor is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. A Cartier divisor is principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}) \to T(X, \mathcal{K}^*/\mathcal{O}^*)$.

Two Cartier divisors are linearly equivalent if their difference is principal. (Although the group operation on $\mathcal{K}^*/\mathcal{O}^*$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.)

NEED LINE BUNDLE DIVISOR EQUIVALENCE - I.E. LINE BUNDLE ASSOCIATED WITH A DIVISOR SO WE CAN DO

NEED AN EQUIVALENCE OF DIVISOR DEFINITIONS BECAUSE LINE BUNDLE DIVISOR EQUIVALENCE REALLY NEEDS CARTIER BUT DOING INDUCTION BY ADDING POINTS TO THE DIVISOR IS REALLY USING A WEIL DIVISOR

NEED A COHOMOLOGY DEFINITION

NEED:- - COHOMOLOGY LONG EXACT SEQUENCE FOR EULER CHAR

- NEED TENSOR PRODUCT OF SHEAVES AND THAT TENSORING WITH LINE BUNDLES IS EXACT

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Lemma 5 (Skyscraper Sheaf has Vanishing Higher Cohomology)**.** *Given a skyscraper sheaf $\mathcal{F}$ on a topological space Y, for all $i \in \mathbb{N} \ \ \{0\}$, $H^i(\mathcal{F}, Y) = 0$.*

Proof. Hartshorne Chap 3 Prop 2.5 p.208 shows this by showing skscraper sheaves are flasque. I think I prefer showing that given any cover, can always refine it so only one set in the cover contains the point where the sheaf is supported. Hence the sheaf has no sections on intersections of sheaves of the cover and so neessarily has vanishing higher cohomology. □

Theorem 6 (Euler Characteristic Addititive)**.** *Given a short exact sequence of sheaves:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

the Euler characteristic is additive, that is:

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof sketch. This follows by first taking the corresponding long exact sequence of cohomology groups

$$0 \xrightarrow{\varphi_0'} H^0(X, \mathcal{F}') \xrightarrow{\varphi_0} H^0(X, \mathcal{F}) \xrightarrow{\varphi_0''} H^0(X, \mathcal{F}'')$$
$$\xrightarrow{\varphi_1'} H^1(X, \mathcal{F}') \xrightarrow{\varphi_1} H^1(X, \mathcal{F}) \xrightarrow{\varphi_1''} H^1(X, \mathcal{F}'') \xrightarrow{\varphi_2'} \ldots.$$

We then see that this splits as the following set of short exact sequences:

$$0 \to \mathrm{Coker}(\varphi_p') \xrightarrow{\varphi_p} H^p(X, \mathcal{F}) \xrightarrow{\varphi_p''} \mathrm{Im}(\varphi_p'') \to 0.$$

But since these are vector spaces, the dimension on these short exact sequences is additive, and so the alternating sum of the dimensions of our cohomology groups is 0. Rearranging this identity then gives the desired result. □

Theorem 7 (Riemann-Roch for curves)**.** *Given a smooth projective curve X over an algebraically closed field k and a divisor D on X, the following identity holds:*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$$

In this context, the word divisor here is implying that what definition we use doesn't make a difference (which is mainly where the smooth and projective assumptions are actually used).

For this, we probably want to cover some theory about divisors and line bundles in here.

Proof. We will prove this by an induction argument on D noting that any divisor D can be built up by adding or subtracting points P starting from the divisor 0. To that end, we will first note that $\chi(\mathcal{O}_X(0)) = \chi(\mathcal{O}_X) + 0$, hence our formula holds for the 0 divisor. Now, for our inductive case, suppose our formula holds for a divisor D, i.e. that $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$. We'll try to show it must then hold for $D + P$, where P is the divisor of a point. Then, we recall that we have the following exact sequence:

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0.$$

Now, we can twist this exact sequence by $\mathcal{O}_X(D+P)$, noting that since $\mathcal{O}_X(D+P)$ is a line bundle, $\mathcal{O}_X(D+P) \otimes_{\mathcal{O}_X} \mathcal{O}_P \simeq \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_P \simeq \mathcal{O}_P$. So, we end up with the following exact sequence:

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+P) \to \mathcal{O}_P \to 0$$

Hence, we have by Theorem 6 that $\chi(\mathcal{O}_X(D+P)) = \chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_{X,p})$. Now, we know from Theorem 5 that $\chi(\mathcal{O}_{X,p}) = 1$, hence:

$$\begin{aligned}\chi(\mathcal{O}_X(D+P)) &= \chi(\mathcal{O}_X(D)) + 1, \\ &= \chi(\mathcal{O}_X) + \deg(D) + 1, \text{ by inductive hypothesis,} \\ &= \chi(\mathcal{O}_X) + \deg(D+P).\end{aligned}$$

Now, we have another inductive case where we know our property holds for some divisor D, and we need to show it holds for $D - P$. The argument for this case is very similar, instead twisting our initial exact sequence by $\mathcal{O}_X(D)$ to get:

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to \mathcal{O}_P \to 0.$$

Then, again using the additivity of the Euler characteristic, we get:

$$\begin{aligned}\chi(\mathcal{O}_X(D-P)) &= \chi(\mathcal{O}_X(D)) - 1, \\ &= \chi(\mathcal{O}_X) + \deg(D) - 1, \ \text{by inductive hypothesis,} \\ &= \chi(\mathcal{O}_X) + \deg(D-P).\end{aligned}$$

Thus, by induction, our property is proven. $\square$