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$$I = \int_{-\infty}^{+\infty} e^{-x^2}\, dx$$

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2}e^{-y^2}\, dx\, dy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\, r\, dr\, d\theta \\ &= \int_{0}^{2\pi} d\theta \cdot \int_{0}^{\infty} e^{-r^2}\, r\, dr \\ &= 2\pi \cdot \int_{0}^{\infty} re^{-r^2}\, dr \end{aligned}$$

$$\int_{0}^{\infty} re^{-r^2}\, dr$$

Let $u = r^2$ so that $du = 2r\, dr$

$$\begin{aligned} \int_{0}^{\infty} re^{-r^2}\, dr &= \int_{0}^{\infty} e^{-u}\, \frac{du}{2} \\ &= \frac{1}{2}\int_{0}^{\infty} e^{-u}\, du \\ &= \frac{1}{2} \end{aligned}$$

We substitute into the expression:

$$2\pi \cdot \frac{1}{2}$$
$$= \pi$$

Therefore:

$$I^2 = \pi$$
$$I = \sqrt{\pi}$$