Study of the Series $\sum_{n=1}^{\infty} \frac{1}{n^2+\alpha}$

1. Convergence of the Series

We consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\alpha} \quad \text{with } \alpha \in \mathbb{R}$$

For large n, we have:

$$\frac{1}{n^2+\alpha} \sim \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges (a p-series with $p = 2 > 1$), the original series also converges for all real α, even for $\alpha < 0$ as long as no term diverges (i.e., $\alpha \neq -n^2$).

2. Closed-form Expression of the Series

We aim to show that for $\alpha > 0$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\alpha} = \frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) - \frac{1}{2\alpha}$$

Step 1: Use the Known Identity

For any $a > 0$, we use the classical identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a)$$

Step 2: Remove the $n = 0$ Term

$$\sum_{n\neq 0} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a) - \frac{1}{a^2}$$

Step 3: Keep Only Positive Terms

By symmetry, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2}\left(\frac{\pi}{a} \coth(\pi a) - \frac{1}{a^2}\right)$$

Step 4: Change Variable

Let $\alpha = a^2 \Rightarrow a = \sqrt{\alpha}$, then:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = \frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) - \frac{1}{2\alpha}$$

Final Result

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = \frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) - \frac{1}{2\alpha} \quad (\alpha > 0)}$$

3. Asymptotic Behavior

As $\alpha \to 0^+$

Let $x = \pi\sqrt{\alpha} \to 0$. Using the expansion:

$$\coth(x) = \frac{1}{x} + \frac{x}{3} + o(x)$$

Then:

$$\frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) = \frac{\pi}{2\sqrt{\alpha}} \left(\frac{1}{\pi\sqrt{\alpha}} + \frac{\pi\sqrt{\alpha}}{3} + o(\sqrt{\alpha}) \right) = \frac{1}{2\alpha} + \frac{\pi^2}{6} + o(1)$$

So the total sum is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = \left(\frac{1}{2\alpha} + \frac{\pi^2}{6} \right) - \frac{1}{2\alpha} + o(1) = \frac{\pi^2}{6} + o(1)$$

Limit as $\alpha \to 0^+$

$$\boxed{\lim_{\alpha \to 0^+} \sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = \frac{\pi^2}{6}}$$

As $\alpha \to +\infty$

We use:

$$\coth(x) = 1 + 2e^{-2x} + o(e^{-2x}) \quad \text{as } x \to \infty$$

Then:

$$\frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) = \frac{\pi}{2\sqrt{\alpha}} \left(1 + 2e^{-2\pi\sqrt{\alpha}} + o(e^{-2\pi\sqrt{\alpha}}) \right) = \frac{\pi}{2\sqrt{\alpha}} + o\left(\frac{1}{\sqrt{\alpha}} \right)$$

And:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = \frac{\pi}{2\sqrt{\alpha}} - \frac{1}{2\alpha} + o\left(\frac{1}{\sqrt{\alpha}} \right) \to 0$$

Limit as $\alpha \to +\infty$

$$\boxed{\lim_{\alpha \to +\infty} \sum_{n=1}^{\infty} \frac{1}{n^2 + \alpha} = 0}$$

4. Summary

- The series $\sum_{n=1}^{\infty} \frac{1}{n^2+\alpha}$ converges for all $\alpha \in \mathbb{R} \setminus \{-n^2\}$.
- For $\alpha > 0$, it admits the closed-form:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\alpha} = \frac{\pi}{2\sqrt{\alpha}} \coth(\pi\sqrt{\alpha}) - \frac{1}{2\alpha}$$

- Asymptotic behavior:

$$\lim_{\alpha \to 0^+} \sum_{n=1}^{\infty} \frac{1}{n^2+\alpha} = \frac{\pi^2}{6}, \quad \lim_{\alpha \to +\infty} \sum_{n=1}^{\infty} \frac{1}{n^2+\alpha} = 0$$