Math 112B/212B, Introduction to Mathematical Biology, Homework 3

1. Suppose that $n(x,t)$ is the density of microorganisms (cells) and $g(x,t)$ is the concentration of glucose. Assuming that the motility of cells is much slower than the diffusion of the glucose, we can describe the co-dynamics of cells and the glucose by the following system:

$$\frac{\partial n}{\partial t} = kng, \tag{1}$$
$$\frac{\partial g}{\partial t} = D\frac{\partial^2 g}{\partial x^2} - ckng, \tag{2}$$

   where D is the diffusion coefficient of glucose, and k and c are positive constants. In this problem, we will work out a traveling wave solution for this system.

   (a) What processes are described by the terms kng and $-ckng$? (b) Define $z = x - vt$, and assume that the solutions depend on x and t only through this variable: $n(x,t) = N(z), g(x,t) = G(z)$. Show how we can derive the following ODEs for these functions:

$$-vN' = kNG, \tag{3}$$
$$-vG' = DG'' - ckNG. \tag{4}$$

   (c) To derive a system of two first-order ODEs, multiply equation (3) by c, add it to equation (4), and integrate once. (*Hint:* please do not forget the constant of integration).

   (d) Compare your result with the following system:

$$G' = -\frac{v}{D}G - \frac{cv}{D}N + a, \tag{5}$$
$$N' = -\frac{kNG}{v}. \tag{6}$$

   Set $a = \frac{cv}{D}N_0$ (this will help simplify the calculations). Find all the steady states and perform linear stability analysis.

   (e) On the phase-plane (N, G) show the steady states, and draw the heteroclinic trajectory (that is, the trajectory that connects a saddle point and a sink). Sketch the functions $N(z)$ and $G(z)$ that corresponds to that trajectory.

2. Consider the following general system of reaction-diffusion equations:

$$\frac{\partial u_1}{\partial t} = f_1(u_1, u_2) + D_1\frac{\partial^2 u_1}{\partial x^2}, \tag{7}$$
$$\frac{\partial u_2}{\partial t} = f_2(u_1, u_2) + D_2\frac{\partial^2 u_2}{\partial x^2}. \tag{8}$$

Determine whether or not a homogeneous steady state that is stable in the absence of diffusion can be obtained. If so, give explicit conditions for instability to arise, and determine which modes would be most destabilizing.

(a) Lotka-Volterra:

$$f_1 = au_1 - bu_1u_2, \quad f_2 = -qu_2 + du_1u_2.$$

(b) Glycolytic oscillator:

$$f_1 = \delta - ku_1 - u_1u_2^2, \quad f_2 = ku_1 + u_1u_2^2 - u_2.$$