MATH 112B - HW2 (Raphael F. Levy - U30156477)

(1) Consider the i-b value problem: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, $0<x<\pi$, $t>0$; $u(x,0)=f(x)$; $u(0,t)=u(\pi,t)=0$

a) Now, by separation of variables, solve $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x}$.

$\Longrightarrow u(x,t) = X(x)T(t) \rightarrow \frac{\partial}{\partial t}(X.T) = D\frac{\partial^2}{\partial x^2}(X.T) + a\frac{\partial}{\partial x}(X.T)$

$\rightarrow XT' = DX''T + aX'T \xrightarrow[\text{by } DaXT]{\text{dividing}} \frac{T'}{DTa} = \frac{X''}{Xa} + \frac{X'}{XD} = -\lambda$

$\Longrightarrow$ The ODE's we obtain are: $T' = -\lambda DaT$ and $DX'' + aX' + \lambda X = 0$ //

b) Back to the original PDE, for the eigenvalue problem $X'' + \lambda X = 0$, $X(0)=X(\pi)=0$, we need to find all λ that give a non-trivial solution $X(x)$. Show that for $\lambda \leq 0$, the only solution is $X(x)=0$.

$\Longrightarrow$ Case $\lambda = 0$: $X''=0 \xrightarrow{\text{gen. sol.}} X(x) = ax+b$

$$\left.\begin{aligned} X(0) &= a.0+b=0 \Rightarrow b=0 \\ X(\pi) &= a.\pi = 0 \Rightarrow a=0 \end{aligned}\right\} X(x) = 0x+0=0$$

Case $\lambda<0$: let's write $\lambda = -\gamma^2$. In this case we have $\begin{cases} X''-\gamma^2 X = 0 \\ X(0)=X(\pi)=0 \end{cases}$

In this case, the general solution is given by $X(x) = a\cosh(\gamma x) + b\sinh(\gamma x)$

$X(0) = a\cosh(0) + b\sinh(0) = a\left(\frac{e^0+e^{-0}}{2}\right) + b\left(\frac{e^0-e^{-0}}{2}\right) = a.1 + \cancel{b.0} = a = 0$

$X(\pi) = b\sinh(\gamma\pi) = b\left(\frac{e^{\gamma\pi}-e^{-\gamma\pi}}{2}\right) = b\left(\frac{1-e^{-2\gamma\pi}}{2e^{-\gamma\pi}}\right) = 0 \Rightarrow b=0$

$\hookrightarrow X(x) = 0\cosh(\gamma x) + 0\sinh(\gamma x) = 0$

$\Longrightarrow$ We have shown that for $\lambda \leq 0$, the only solution is $X(x)=0$

---

(2) Consider the i-b value problem, defined on the interval $x\in[a,b]$: $\frac{\partial u}{\partial t} = \tilde{D}\frac{\partial^2 u}{\partial x^2}$, $a<x<b$, $t>0$

Reduce this problem to the original by a change of variables. $u(x,0)=\tilde{f}(x)$, $u(a,t)=u(b,t)=0$

$\Rightarrow$ Using the hint, let $\xi = \pi\frac{(x-a)}{(b-a)}$. When $x=a$, $\xi = \pi\frac{(a-a)}{(b-a)} = 0$. Therefore, for $a<x<b$,

When $x=b$, $\xi = \pi\frac{(b-a)}{(b-a)} = \pi$ $\qquad 0<\xi<\pi$

Now, to find $\frac{\partial^2 u}{\partial x^2}$ in terms of ξ: $\frac{\partial u}{\partial x} = \frac{du}{d\xi}\cdot\frac{d\xi}{dx}$; $\frac{\partial^2 u}{\partial x^2} = \frac{d}{d\xi}\left(\frac{du}{d\xi}\cdot\frac{d\xi}{dx}\right)\cdot\frac{d\xi}{dx}$, where $\frac{d\xi}{dx} =$

$= \frac{\partial}{\partial x}\left(\pi\frac{(x-a)}{(b-a)}\right) = \frac{\pi}{b-a}\frac{\partial}{\partial x}(x-a) = \frac{\pi}{b-a}\left(\frac{\partial x}{\partial x} - \cancel{\frac{\partial a}{\partial x}}\right) = \frac{\pi}{b-a}$

Now we have $\frac{\partial u}{\partial t} = \tilde{D}\left(\frac{d}{d\xi}\left(\frac{\partial u}{\partial \xi}\ \frac{\pi}{b-a}\right)\frac{\pi}{b-a}\right) \to \int \frac{d}{d\xi}\left(\frac{\partial u}{\partial \xi}\cdot\frac{\pi}{b-a}\right)d\xi = \frac{\partial u}{\partial \xi}\cdot\frac{\pi}{b-a} + C$

$u(\xi,0) = \tilde{f}(\xi)$, $u(0,t) = u(\pi,t) = 0$

$$\Longrightarrow \int \frac{\partial u}{\partial t}\,dt = \tilde{D}\left(\frac{\pi}{b-a}\right)^2 \int \frac{d^2}{d\xi^2}u\,d\xi \to u(\xi,t) = \underbrace{\left(\tilde{D}\left(\frac{\pi}{(b-a)}\right)^2\right)}_{\text{"}D\text{"}} \int \frac{d^2}{d\xi^2}u\,d\xi + C_1$$

Reducing to the original: $\frac{\partial u}{\partial t} = \left[\tilde{D}\left(\frac{\pi}{b-a}\right)^2\right]\frac{\partial^2 u}{\partial \xi^2}$, $0<\xi<\pi$; $u(\xi,0) = \tilde{f}(\xi) = f(\xi)$; $u(0,t) = u(\pi,t) = 0$.

---

(3) Consider the reaction-diffusion eq. on $x\in(-\infty,\infty)$: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru$, $t>0$. ($r \to$ const.)

a) Show that the following function satisfies the PDE: $u(x,t) = \frac{u_0}{2(\pi D t)^{1/2}}\exp\left\{rt - \frac{x^2}{4Dt}\right\}$

$\Rightarrow$ Starting with $r=0$ to verify, we can use Fourier Transform:

$u_t = Du_{xx} \to \widehat{u_t} = \widehat{Du_{xx}} \to \hat{u}_t = D(-is)^2\hat{u} = -Ds^2\hat{u} \to \hat{u}(s,t) = Ce^{-Ds^2t}$

OBS: $y' = ay \to y = ce^{at}$
OBS: $c = \hat{u}(s,0) = \widehat{u(x,0)} = \widehat{f(x)} = \hat{f}(s)$
OBS: $\int e^{-ax^2+bx}dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$

To get $u(x,t)$: $\check{\hat{u}}(s,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{C}_{\hat{f}(s)} e^{-Ds^2t}e^{-isx}ds = u(x,t)$

$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Ds^2t}e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Ds^2t-isx}ds \to a = Dt, b = -ix$

$\Rightarrow \frac{1}{2\pi}e^{(-ix)^2/4Dt}\sqrt{\frac{\pi}{Dt}} = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt} = u(x,t)$ //

Now, for $r>0$: $u_t = Du_{xx} + ru \to \hat{u}_t = \widehat{Du_{xx}+ru} = D(-is)^2\hat{u} + r\hat{u} = (-Ds^2+r)\hat{u}$

$\Rightarrow \hat{u}(s,t) = Ce^{(-Ds^2+r)t} \to \check{\hat{u}}(s,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ce^{(-Ds^2+r)t}e^{-isx}ds = u(x,t)$

$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-Ds^2+r)t}e^{-isx}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-Ds^2+r)t-isx}ds \to a = Dt, b = -ix \rightsquigarrow \frac{1}{2\pi}e^{rt+(-ix)^2/4Dt}\sqrt{\frac{\pi}{Dt}}$

$= \frac{1}{\sqrt{4\pi Dt}}e^{rt-\frac{x^2}{4Dt}}$ // $\rightsquigarrow$ what about the u_0?

MATH 112B - HW2 (Raphael F. Levy - U10156477)

③ b) Consider the level set $u(x,t) = U$ (const.). Show that on this contour, the ratio $\frac{x}{t}$ satisfies $\frac{x}{t} = \pm\left[4rD - \frac{2D}{t}\ln t - \frac{4D}{t}\ln\left(2\sqrt{\pi D}\,\frac{U}{u_0}\right)\right]^{\frac{1}{2}}$

$\Rightarrow$ From a), let's use that $u(x,t) = \frac{u_0}{2\sqrt{\pi D t}}\, e^{rt - \frac{x^2}{4Dt}} = U$. Therefore:

$$e^{rt - \frac{x^2}{4Dt}} = \frac{U}{u_0}\cdot 2\sqrt{\pi D t} \Rightarrow rt - \frac{x^2}{4Dt} = \ln \frac{U}{u_0}\cdot 2\sqrt{\pi D t} \Rightarrow \frac{x^2}{4Dt} = rt - \ln \frac{U}{u_0} 2\sqrt{\pi D t}$$

$$\Rightarrow x^2 = 4Dt\left(rt - \ln \frac{U}{u_0} 2\sqrt{\pi D t}\right) \Rightarrow x = \pm\sqrt{4Dt\left(rt - \ln \frac{U}{u_0} 2\sqrt{\pi D t}\right)}$$

$$\Rightarrow \frac{x}{t} = \pm\sqrt{\frac{4Dt\left(rt - \ln \frac{U}{u_0} 2\sqrt{\pi D t}\right)}{t^2}} = \pm\sqrt{\frac{4Drt^2 - 4Dt\ln\sqrt{t} - 4Dt\ln \frac{U}{u_0} 2\sqrt{\pi D}}{t^2}}$$

$$\Rightarrow \frac{x}{t} = \pm\sqrt{\frac{4Dr\cancel{t^2}}{\cancel{t^2}} - \frac{\overset{2}{\cancel{4}}D\cancel{t}\left(\frac{1}{2}\right)\ln t}{t^{\cancel{2}}} - \frac{4D\cancel{t}\,\frac{U}{u_0}2\sqrt{\pi D}}{t^{\cancel{2}}}} = \pm\sqrt{4Dr - \frac{2D}{t}\ln t - \frac{4D}{t}\left(2\sqrt{\pi D}\,\frac{U}{u_0}\right)}$$ //

c) Show that as $t \to \infty$, one can approximate the above by $\frac{x}{t} = \pm 2\sqrt{rD}$

$\Rightarrow$ As $t \to \infty$, $\lim_{t \to \infty} -\frac{2D}{t} \ln t \Rightarrow -2D \lim_{t \to \infty} \frac{\ln t}{t} \Rightarrow -2D \lim_{t \to \infty} \frac{\frac{1}{t}}{1} = \frac{1}{t} = 0$. As U, D and u_0 are all constants, the second ln also goes to 0 as $t \to \infty$, therefore we can approximate the above to $\frac{x}{t} = \pm \sqrt{4Dr} = 2\sqrt{rD}$.

③ a) (Cont.) We have verified that, from the PDE, $u(x,t)$ can be found to be $u(x,t) = \frac{1}{2(\pi D t)^{1/2}} \exp\{rt - \frac{x^2}{4Dt}\}$. Now, we must do the other way around, and confirm that the function satisfies the PDE.

$\Rightarrow \frac{u_0}{2(\pi D t)^{1/2}} \exp\{rt - \frac{x^2}{4Dt}\}$. Knowing $e^{b^2/4a}\sqrt{\frac{\pi}{a}} = \int e^{-ax^2+bx}$, we write $a = Dt$, $b = -ix$, and get

$$\Rightarrow = \frac{u_0}{2\pi} \exp\{rt + \frac{(-xi)^2}{4Dt}\}\sqrt{\frac{\pi}{Dt}} \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-Ds^2+r)t - isx} ds, \text{ where } \hat{f}(s) = \widehat{f(x)} = \widehat{u(x,0)}$$

$$\Rightarrow \frac{u_0}{2\pi}\int_{-\infty}^{\infty} e \rightsquigarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0 e^{(-Ds^2+r)t} e^{-isx} ds = \hat{u}(s,t) \Rightarrow \hat{u}(s,t) = u_0 e^{\underbrace{(-Ds^2+r)t}}$$

$$\Rightarrow (-Ds^2+r)\hat{u} = D(-is)^2\hat{u} + r\hat{u} = \widehat{Du_{xx}} + \widehat{ru} = \hat{u}_t$$

$$\Rightarrow Du_{xx} + ru = u_t$$

$$D\frac{\partial^2 u}{\partial x^2} + ru = \frac{\partial u}{\partial t} \quad //$$