Math 112B/212B, Introduction to Mathematical Biology, Homework 1

1. For the following functions, sketch the surface corresponding to $z = f(x, y)$, and the level curves in the xy plane: (a) $f(x, y) = exp\left(-\frac{x^2+y^2}{2}\right)$, (b) $f(x, y) = \sin x \cos y$.

2. For each of the functions in problem 1, find (1) $\frac{\partial f}{\partial x}$, (2) $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, (3) ∇f, (4) Δf.

3. For the following vectors, $\mathbf{v}$, find $\nabla \cdot \mathbf{v} = div\, \mathbf{v}$: (a) $\mathbf{v} = (y - z, z - x, x - y)$, (b) $\mathbf{v} = (\sin(xyz), \frac{1}{x}, z^2 + x^2)$.

4. For the following vector fields, determine if they are gradient flows. If so, find ϕ such that $\mathbf{F} = \nabla\phi$: (a) $\mathbf{F} = (x, y)$, (b) $\mathbf{F} = (\sin(xy), \cos(xy))$, (c) $\mathbf{F} = (x + y, x - y)$.

5. Consider the following reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{K}\right).$$

   Suppose that time is measured in days and distance in meters. (a) What are the units of D? (b) What are the units of r?

6. Suppose that $u(r, t)$ represents the density of particles/organisms, with r being the radial coordinate (distance from the center of coordinates), and θ the angular coordinate. The diffusion equation in 2D with radial symmetry is given by:

$$\frac{\partial u}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

   Suppose that we have a disk: $a < r < L$ with the following boundary conditions: $u(a, t) = 0$ (sink of radius a) and $u(L, t) = U_0$ (a source of radius L). (a) Solve the equation for u in steady-state (by setting $\frac{\partial u}{\partial t} = 0$). (b) Define

$$N = \int\int_{disk} u\, dA = \int_0^{2\pi}\int_0^L u(r, t) r\, dr\, d\theta.$$

   Compute this integral in steady state and interpret its meaning.