Midterm

Graded

Student

RAPHAEL FELBERG LEVY

Total Points

56 / 70 pts

Question 1

Q1

5 / 10 pts

+ 10 pts Correct

+ 1 pt Some work but of little relevance to the question

✔ **+ 5 pts** Find the ODE for stationary solution: $D\frac{d^2u}{dx^2} - ru = 0$

+ 5 pts Solved the ODE for the stationary solution. The general form is $u(x) = C_1 \exp(\sqrt{r/D}x) + C_2 \exp(-\sqrt{r/D}x))$.
(Notice that D>0 and r>0, so the characteristic equation has real roots)

− 1.5 pts Minor mistake/ incomplete in solving the ODE for the stationary solution.
Common mistake: didn't express it in terms of r and D.
(λ shouldn't be involved)

+ 1.5 pts Tried to solve the ODE for the stationary solution but incorrect

Question 2

Q2

20 / 25 pts

✔ **+ 5 pts** (a) $D\frac{\partial^2 u}{\partial x^2}$ term: increase of spatial flow (diffusion) of population. $-ru$ term: death/ removal of population

✔ **+ 5 pts** (b) Correct ODE for $T(t)$: we have $T' = -\lambda DT - rT$

✔ **+ 5 pts** (c) Solution: $u(x,t) = \sum_{n=0}^{\infty} c_n \cos(nx) \exp(-n^2Dt - rt)$

+ 5 pts (d) Solution of solution (9) in the long term will converge to $c_0 \cos(0x) \exp(-0Dt)$ since the $n = 0$ term doesn't converge to zero as t goes to infinity. But for problem (1-3) the solution goes to zero in the long-term behavior because of the $-rt$ term which exists even in $n = 0$ term.

+ 5 pts (e) biologically, the organisms die out because of the death effect due to the $-ru$ term

− 2 pts (a): didn't correctly explain the $-ru$ term on the right hand side or didn't correctly explain the first term

✔ **+ 5 pts** Partial credit for (d) (e)

+ 3 pts Partial credit for (b)

+ 3 pts Partial credit for (c)

+ 0 pts No solution or little relevant work

Question 3

Q3

10 / 10 pts

✔ **+ 10 pts** Correct: $\frac{1}{2+e^{-4}}$, from $u(x,t)=\phi(t-\ln|x|)$ and $\phi(-\ln|x|)=\frac{1}{2+x^2}$ we have $u(x,t)=\frac{1}{2+e^{2(\ln|x|-t)}}$

You can also solve this by identifying $u(1,2)$ is on the characteristic line of $u(e^{-2},0)$.

+ 7 pts Partial credit

+ 5 pts Partial credit

+ 3 pts Partial credit

− 1 pt minor mistake

+ 0 pts No solution / little relevant work

Question 4

Q4

8 / 10 pts

+ 10 pts Correct. The curve should be $\sin(x)=0$ and $y-y^2=0$. So it should be straight lines defined by $x=k\pi, k\in\mathbb{Z}$ and $y=0,1$

+ 4 pts Partial credit

+ 6 pts Partial credit.

✔ **+ 8 pts** Partial credit

Question 5

Q5

13 / 15 pts

✔ **+ 5 pts** (a) Correct: $u_t - au_{xx} = 0$

✔ **+ 5 pts** (b) Correct: $cm^2 \cdot s^{-1}$

+ 5 pts (c) Correct: cm

+ 3 pts (a) Correct with minor mistake

+ 3 pts (b) Correct with minor mistake

✔ **+ 3 pts** (c) Correct with minor mistake

+ 0 pts Incorrect

+ 3 pts Some partial results

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Spring 2024, Math 112B/212B Midterm (Komarova)

Do all problems, in any order. Show all your work, credit may not be given for an answer alone. You are not allowed to use books, notes or calculators.

1. *(10 points)* Consider the following reaction-diffusion equation, formulated as an initial-boundary value problem:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - ru,$$

where $D > 0, r > 0$ are some constants. Find a stationary solution for this equation that remains finite in $[0, \infty)$.

$\hookrightarrow u(x,t)=0$

$u = XT$

$\to XT' = DX''T - rXT$

Steady states: $u(x,t)=0 \quad \to X(x)T(t)=0$

$u(x,0)=f(x)$

$u(0,t)= u(\pi,t)=0$?

$\dots \frac{1}{2(\pi D t)^{\frac{1}{2}}} \exp\{r - \dots\}$

$\int \frac{\partial u}{\partial t}\, dt = \int D \frac{\partial^2 u}{\partial x^2}\, dx - \int r u \, dx$

$u = D\frac{\partial u}{\partial x} - r\int u \, du$?

2. *(25 points)* Suppose that the same equation as in problem 1 is formulated on a finite spatial domain, with the following initial and boundary conditions:

death rate

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - ru, \quad 0 < x < \pi, \quad t > 0, \quad r > 0 \tag{1}$$

$$u(x,0) = f(x), \tag{2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0. \tag{3}$$

D is the diffusion coefficient, r is death rate

(a) If $u(x,t)$ describes density of some microorganism, what does each term on the right hand side represent?

$D\frac{\partial^2 u}{\partial x^2}$ represents the diffusion of the microorganisms in space (x), and ru represents the growth rate of that microorganism, which in this case as it is negative, it is the death rate

(b) On page 6 you will find the solution for a similar problem for a diffusion equation, for your reference. Apply the separation of variables method, $u(x,t) = X(x)T(t)$, to problem (1-3), to show that the problem for $X(x)$ is the same as in equation (8), but the ODE for $T(t)$ is different. What ODE does $T(t)$ satisfy?

$XT' = DX''T - rXT$ → divide by DXT: $\frac{T'}{DT} = \frac{X''}{X} - \frac{r}{D} = \lambda$

For X to be equal, ru must go to the other side of the equation /D /X 1/XD

$T' = -\lambda DT$

$DX'' - rX = -XD\lambda \rightarrow DX'' - rX + XD\lambda = 0$

→ $XT' + rXT = DX''T$? $= -\lambda$

$DX'' + (\lambda D - r)X = 0$

$\frac{T'}{DT} + \frac{r}{D} = \frac{X''}{X} = -\lambda$

$\boxed{\Rightarrow X'' + \lambda X = 0;\ T' = (-\lambda - \frac{r}{D})DT = -\lambda DT - rT}$

(c) Write the solution $u(x,t)$ of problem (1-3) (that we obtain instead of equation (9)).

$\ldots \rightarrow T_n' = -Dn^2 T_n - rT_n \Rightarrow T_n(t) = Ce^{-Dn^2 t} - r t$? (*)

$\lambda = n^2$

$\int T'(t)\,dt = T(t)$ $\quad \int T(t)dt =$

$u(x,t) = \sum_{n=0}^{\infty} C_n \cos(nx) e^{-(Dn^2+r)t}$ ↪ $C_1 e^{-Dn^2 t} - C_2 e^{-rt}$?

(d) Compare the long-term behavior of solution (9) and the solution of problem (1-3).

Solution 9: $u(x,t) = \sum_{n=0}^{\infty} c_n \cos(nx) e^{-Dn^2 t}$

Solution above: $\sum_{n=0}^{\infty} C_n \cos(nx) e^{-(Dn^2+r)t}$ → As $t \to \infty$, both will go to 0... there must be a mistake in (*)

$u(x,t) = \frac{1}{\pi}\int_0^{\pi} f(x)dx = f_{avg}$

(e) Explain the difference in the long-term behavior from the biological perspective.

→ From the original equation, the solution tends to it's average, as the no-flux condition doesn't let the density to escape. However, with the death rate, in the long term the solution should tend to 0, as there is a negative-flux (death rate), and there is no entry (birth-rate)

3. *(10 points)* Supose we are given the following transport equation:

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0.$$

It is known that the solution $u(x,t)$ remains constant along the characteristics of the form

$$t = \ln|x| + c,$$

where c is a constant. It is also known that along the line $t = 0$, the solution is given by $u(x,0) = \frac{1}{2+x^2}$. Find the value $u(1,2)$.

$\to$ By char. lines: $u(X(r), T(r))$: $\frac{du}{dr} = \frac{\partial u}{\partial t}\frac{dt}{dr} + \frac{\partial u}{\partial x}\frac{dx}{dr} = 0$

$$\begin{cases} \frac{dt}{dr} = 1 \Rightarrow t = r + \cancel{c} \to t = r \\ \frac{dx}{dr} = c \Rightarrow \frac{dx}{dt} = c \to \int dx = \int c\,dt \to x = ct + a \end{cases}$$

$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0$

$\to t = \frac{x-a}{c}$

$t = \ln|x| + c \to \ln|x| = t - c$

$x = e^{t-c}$

$\hookrightarrow c = t - \ln|x|$

$\cancel{X_0 = e^{t_0 - (t - \ln|x|)}}$

$x^* = e$

$t_0 = \ln|x_0| + c$

$\boxed{c = t_0 - \ln|x_0|}$

$u(x_0, t_0) = u(x^*, 0) = \frac{1}{2+(x^*)^2}$

$x = e^{t-c} = e^{t + \ln|x_0| - t_0}$

At $(x^*, 0)$: $x^* = e^{0 + \ln|x_0| - t_0} = e^{\ln|x_0| - t_0}$

$u(x_0, t_0) = u(x^*, 0) = e^{\ln|x_0| - t_0}$

$\boxed{u(x,t) = e^{\ln|x| - t}}$

$u(1,2) = e^{\ln|1| - 2} = \frac{e^{\ln|1|}}{e^2} = \frac{1}{e^2} = e^{-2}$ //

$\cancel{u(1,0) = e^{\ln|1|} = 1 = \frac{1}{2+1}} ?$

$u(x_0, t_0) = u(x^*, 0) = \frac{1}{2 + (e^{\ln|x_0| - t_0})^2} \to u(x,t) = \frac{1}{2 + (e^{\ln|x| - t})^2} \to u(1,2) = \frac{1}{2 + (e^{\ln|1| - 2})^2}$

$= \frac{1}{2 + \left(\frac{1}{e^2}\right)^2} = \boxed{\frac{1}{2 + \frac{1}{e^4}}}$

$$\begin{array}{r} 0.5 \\ 0.5 \\ \hline 25 \\ 00 \\ \hline 0.25 \end{array}$$

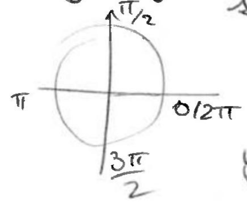
4. *(10 points)* For the function $f(x, y) = \sin(x)(y - y^2)$, sketch the level curves that correspond to the level $f = 0$.

$$f(x,y) = c \rightarrow \sin(x)(y-y^2) = c \rightarrow y - y^2 = \frac{c}{\sin(x)} \rightarrow y - y^2 = 0$$

$$y^2 = y$$

$$y(1-y) = 0$$

$$-y^2 + y - \frac{c}{\sin(x)} = 0$$



$$\Downarrow$$

$$y^2 - y + \frac{c}{\sin(x)} = 0$$

$$y = \frac{1 \pm \sqrt{1 - 4\frac{c}{\sin(x)}}}{2}$$

$$\sin(x)(y-y^2) = 0$$

$$\rightarrow \sin(x) = 0 \rightarrow x = k\pi$$

or

$$(y - y^2) = 0 \rightarrow y(1-y) = 0$$

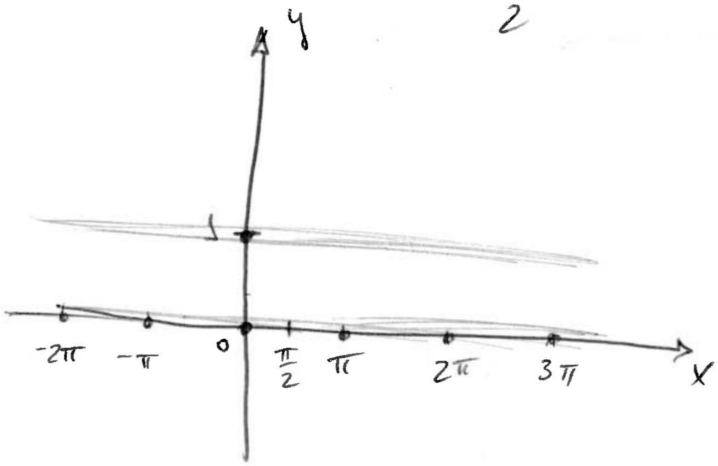
$$y = 0 \text{ or } y = 1$$

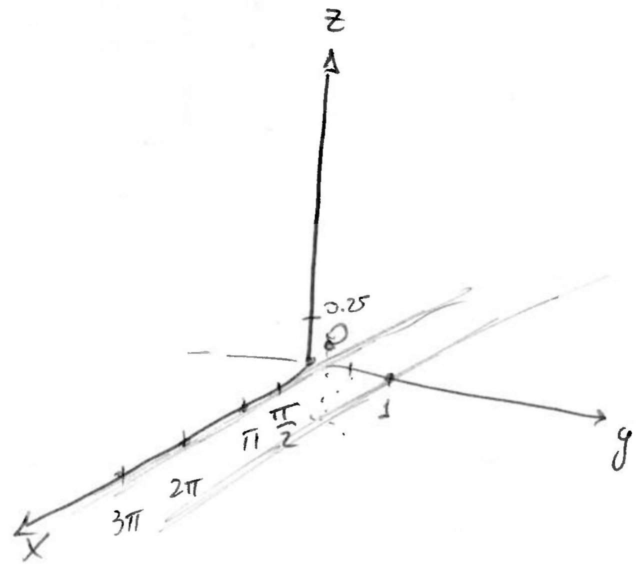
$$y^2 - y + 0 = 0$$

$$x_v = -\frac{b}{2a} = \frac{1}{2}$$

$$y_v = -\frac{\Delta}{4a} = \frac{1}{4}$$

$$\sin(x) = \frac{c}{y - y^2}$$





$$f\left(\frac{\pi}{2}, 0.5\right) = 1 \cdot (0.25) = 0.25$$

$$y = \frac{1 \pm \sqrt{1 - \frac{4c}{\sin(x)}}}{2} \rightarrow \text{At } c = 0: \; y = \frac{1 \pm \sqrt{1}}{2} = \frac{1 \pm 1}{2} = 1 \text{ or } 0$$

$\rightarrow$ At $z = 0$ there are only 2 points ??

5. *(15 points)* Write a short answer to each question.

(a) Suppose the density of bacteria satisfies the transport equation in 1D,

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

and the flux is proportional to the spatial gradient of the bacterial density (with the proportionality coefficient given by $(-a)$, $a > 0$). What is the PDE that $u(x,t)$ satisfies?

spatial density

$$\rightarrow q \equiv -a \frac{\partial u}{\partial x} \qquad \rightarrow q \equiv -a D \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}$$

(b) Consider the reaction–diffusion equation,

growth rate

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + bx\left(1 - \frac{x}{K}\right). \qquad (4)$$

consumption

Suppose time is measured in seconds, distance in cm, and the density $u(x,t)$ in cm^{-1}. What are the units of D in equation (4)?

As D is the diffusion coefficient, it will be measured in units of area over units of time, in this case cm^2/s.

(c) What are the units of K in the equation in equation (4)?

1/time

$$[\text{unit of } U][T]^{-1} = [\text{unit of } D][X]^{-2} + [\text{unit of } b][X]\left(1 - \frac{[X]}{[K]}\right)$$

$$[\text{unit of } U][T]^{-1} = [X]^2[T]^{-1}[\text{unit of } u][X]^{-2} + [T]^{-1}[X]\left(1 - [X][K]^{-1}\right)$$

$$[\text{unit of } U][T]^{-1} = [\text{unit of } U][T]^{-1} + [X][T^{-1}] - [X]^2[K]^{-1}[T]^{-1}$$

$$[X][T]^{-1} - [X]^2[K]^{-1}[T^{-1}] = 0 \quad \text{divide by } [X][T]^{-1}$$

$$1 - [X][K^{-1}] = 0$$

$$[X][K]^{-1} = 1 \rightarrow [K]^{-1} = \frac{1}{X} = [X]^{-1} \rightarrow K \text{ has units of } 1/cm.$$

Extra paper