

# Math 112B/212B, Introduction to Mathematical Biology, Homework 2

1. Consider the initial-boundary value problem,

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, t > 0, \quad (1)$$

$$u(x, 0) = f(x), \quad (2)$$

$$u(0, t) = u(\pi, t) = 0. \quad (3)$$

We solved this problem in class by using the separation of variable technique. This problem reviews some of the steps.

(a) The assumption we made was that  $u(x, t) = X(x)T(t)$ . When substituted into the PDE, this resulted in two ODEs:  $X'' + \lambda X = 0$  and  $T' = -\lambda DT$ . What ODEs would we obtain by the same method, if instead of equation (4), we had

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x}$$

(you do not have to carry out the solution of the ODEs).

(b) Back to the original PDE. For the eigenvalue problem,  $X'' + \lambda X = 0$ ,  $X(0) = X(\pi) = 0$ , we need to find all the values of  $\lambda$  that give a nontrivial solution,  $X(x)$ . Show that if  $\lambda \leq 0$ , the only solution is  $X(x) = 0$ . (Hint: do this separately for  $\lambda = 0$  and  $\lambda < 0$ , by writing the general solution and then solving for the constants using the boundary conditions).

2. Consider the initial-boundary value problem, defined on the interval  $x \in [a, b]$ :

$$\frac{\partial u}{\partial t} = \tilde{D} \frac{\partial^2 u}{\partial x^2}, \quad a < x < b, t > 0, \quad (4)$$

$$u(x, 0) = \tilde{f}(x), \quad (5)$$

$$u(a, t) = u(b, t) = 0. \quad (6)$$

Reduce this problem to problem (4-6) by a change of variables. (Hint: consider the new variable,  $\xi = \pi(x - a)/b$ ).

3. Consider the reaction-diffusion equation on  $x \in (-\infty, \infty)$ :

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + ru, \quad t > 0. \quad (7)$$

(a) Show that the following function satisfies the PDE:

$$u(x, t) = \frac{u_0}{2(\pi Dt)^{1/2}} \exp \left\{ rt - \frac{x^2}{4Dt} \right\}$$

(here  $u_0$  is an arbitrary constant).

(b) Consider the level set,  $u(x, t) = U$ , where  $U$  is a constant. Show that on this contour, the ratio  $x/t$  satisfies

$$\frac{x}{t} = \pm \left[ 4rD - \frac{2D}{t} \ln t - \frac{4D}{t} \ln \left( \sqrt{2\pi D} \frac{U}{u_0} \right) \right]^{1/2}.$$

(c) Show that as  $t \rightarrow \infty$ , one can approximate the above by

$$\frac{x}{t} = \pm 2\sqrt{rD}.$$