MATH 112B - HW1 (Raphael F. Levy - U10156477)

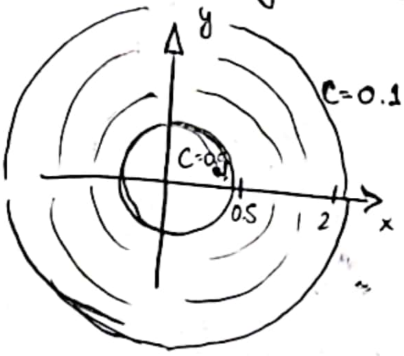
① Sketch the surface $z = f(x,y)$ and the level curves in the xy plane.

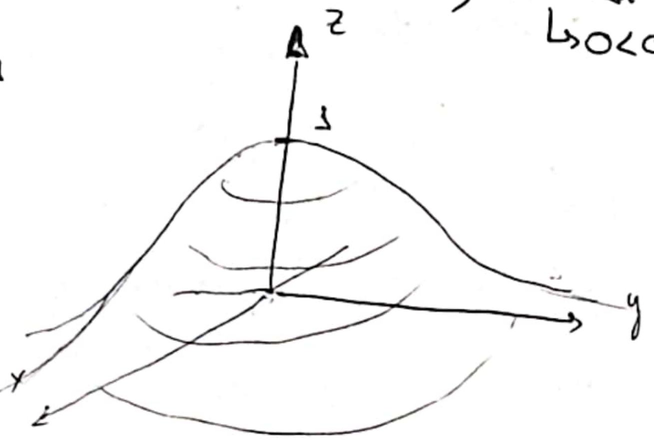
a) $f(x,y) = \exp\left(-\frac{(x^2+y^2)}{2}\right)$; b) $f(x,y) = \sin x \cos y$

$\Rightarrow$ a) $f(x,y) = C \Rightarrow e^{-\frac{(x^2+y^2)}{2}} = c \Rightarrow \ln e^{-\frac{(x^2+y^2)}{2}} = \ln c \Rightarrow -\frac{(x^2+y^2)}{2} = \ln c$

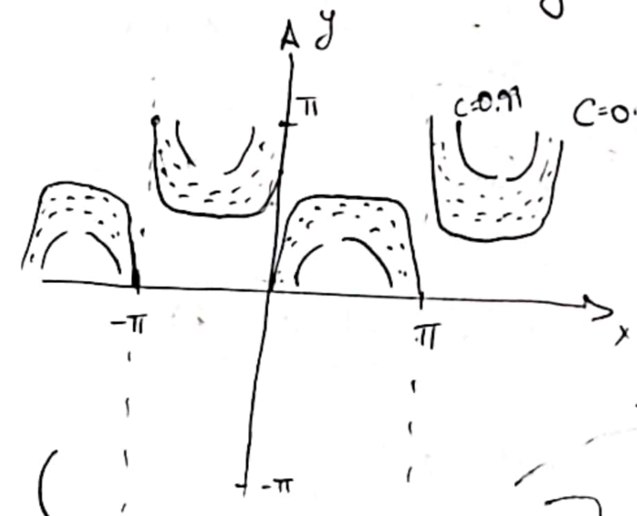
$\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{-2\ln c}$. So that $r > 0$, c must be smaller than 1

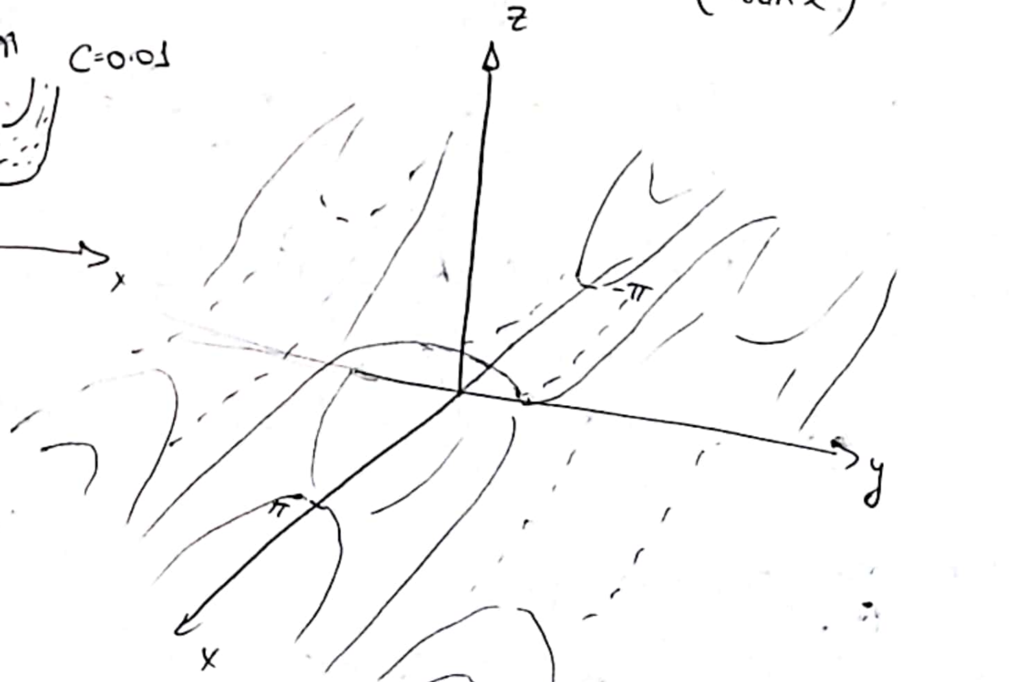
$\hookrightarrow 0 < c < 1$

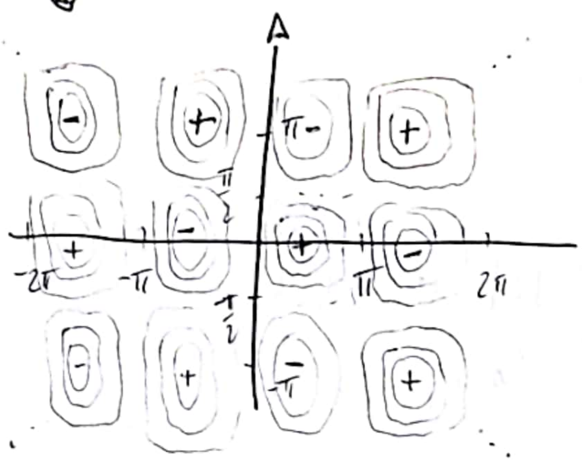




b) $f(x,y) = C \Rightarrow \sin x \cos y = C \Rightarrow \cos y = \frac{C}{\sin x} \Rightarrow y = \cos^{-1}\left(\frac{C}{\sin x}\right)$







$\hookrightarrow f(x,y) = 0$ in points $(K_1\pi, \frac{\pi}{2} + K_2\pi), (K_1, K_2) \in \mathbb{Z}^2$

$\rightarrow$ For $0 < x < \pi$, $\sin x$ is positive. For $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $\cos y$ is positive. So any adjacent quadrant must have a negative value of z ($\sin x$ or $\cos y$ will be negative). For every quadrant adjacent to that, z is positive, and so on.

② For a and b above, find (1) $\frac{\partial f}{\partial x}$, (2) $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, (3) ∇f, (4) Δf

a) (1) $\frac{\partial f}{\partial x} e^{\frac{-x^2-y^2}{2}} = x\left(-e^{\frac{-x^2-y^2}{2}}\right) \rightarrow \frac{\partial f}{\partial y} = y\left(-e^{\frac{-x^2-y^2}{2}}\right)$

(2) $\frac{\partial^2 f}{\partial x \partial y} e^{\frac{-x^2-y^2}{2}} = xye^{\frac{-x^2-y^2}{2}} = \frac{\partial^2 f}{\partial y \partial x}$

(3) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(-xe^{\frac{-x^2-y^2}{2}}, -ye^{\frac{-x^2-y^2}{2}}\right)$

(4) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (x^2-1)e^{\frac{-x^2-y^2}{2}} + (y^2-1)e^{\frac{-x^2-y^2}{2}} = e^{\frac{-x^2-y^2}{2}}(x^2+y^2-2)$

b) (1) $\frac{\partial f}{\partial x} \sin(x)\cos(y) = \cos(x)\cos(y) \rightarrow \frac{\partial f}{\partial y} = -\sin(x)\sin(y)$

(2) $\frac{\partial^2 f}{\partial x \partial y} \sin(x)\cos(y) = -\cancel{\cos}(x)\sin(y) = \frac{\partial^2 f}{\partial y \partial x}$

(3) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\cos x \cos y, -\sin x \sin y)$

(4) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (-\sin x \cos y) + (-\sin x \cos y) = -2(\sin x \cos y)$

(3) For the following vectors, find $\nabla V = \operatorname{div} V$: a) $V = (\overset{V_x}{y-z}, \overset{V_y}{z-x}, \overset{V_z}{x-y})$; b) $V = (\sin(xyz), \frac{1}{x}, x^2+z^2)$

Ans: a) $\operatorname{div} V(V_x, V_y, V_z) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \Rightarrow 0+0+0=0$

b) $\operatorname{div} V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \Rightarrow yz\cos(xyz) + 0 + 2z = z(2 + y\cos(xyz))$

(4) For the following vector fields, determine if they are gradient flows. If so, find ϕ such that $F = \nabla\phi$: a) $F = (x, y)$; b) $F = (\sin(xy), \cos(xy))$; c) $F = (x+y, x-y)$

Ans.: For a vector field to be a gradient field, there should be a function $f(x,y)$ s.t. $\vec{F} = (M, N) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. So $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$. If $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial N}{\partial x}$, we can find that function.

a) $F = (x, y) \rightarrow \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$. So it is a grad. field. Now, we find x and y such that $\frac{\partial f}{\partial x} = x$ and $\frac{\partial f}{\partial y} = y \Rightarrow f = \frac{x^2+y^2}{2} + C$.

$\int x\,dx = \frac{x^2}{2} + H$

$\int y\,dy = \frac{y^2}{2} + G$

In this case, $\phi = \frac{x^2+y^2}{2} + C$ //

b) $F = (\sin(xy), \cos(x,y)) \rightarrow \frac{\partial M}{\partial y} = x\cos(xy) \neq -y\sin(xy) = \frac{\partial N}{\partial x}$. This vector field is not a gradient flow.

c) $F = (x+y, x-y) \rightarrow \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. So it is a grad field. Now, we find x and y such that $\frac{\partial f}{\partial x} = x+y$ and $\frac{\partial f}{\partial y} = x-y \Rightarrow \int x+y\,dx = \frac{x^2}{2} + xy + \underset{\text{may depend on } y}{H}$; $\int x-y\,dy = xy - \frac{y^2}{2} + \underset{\text{may depend on } x}{G}$

For both to be equal, $\frac{x^2}{2} + xy + H = -\frac{y^2}{2} + xy + G \Rightarrow H = -\frac{y^2}{2} + C$, $G = \frac{x^2}{2} + C$

$\Rightarrow f(x,y) = \frac{x^2-y^2}{2} + xy + C = \phi$ //

(5) Consider the following reaction-diffusion equation: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left(1 - \frac{u}{K}\right)$. Suppose that time is measured in days and distance in meters. a) What are the units of D? b) What are the units of r?

Ans.: a) as the rate of diffusion is the movement of a substance through a unit of area per unit of time, in this case m.m and days, then the units of D are m^2/day. b) as the ~~[illegible]~~ growth rate, it is given in 1 per unit time, so in this case $\frac{1}{day}$.

$\Rightarrow [\text{unit of } u][T]^{-1} = [\text{unit of } D][\text{unit of } u][L]^{-2}$

$\Rightarrow [\text{unit of } D] = [L]^2[T]^{-1}$

⑥ Suppose that $u(r,t)$ represents the density of particles / organisms, with r being the radial coordinate (distance from the center of coordinates), and θ the angular coordinate. The dif. eq. in 2D with radial symmetry is given by $\frac{\partial u}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$. Suppose that we have a disk: $a<r<L$ with the boundary conditions: $u(a,t)=0$ (sink of radius a) and $u(L,t)=U_0$ (source of radius L). a) Solve the equation for u in steady state ($\frac{\partial u}{\partial t}=0$). b) Define $N=\iint_{Disk} u\, dA = \int_0^{2\pi}\int_0^L u(r,t)\, r\, dr\, d\theta$. Compute this integral in steady state and interpret it's meaning.

$\Rightarrow$ a) $\frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=0 \rightarrow \frac{D}{r}(ru''(r)+u'(r))=0$. Assuming $D>0$, let's define $u'(r)$ as $v(r)$: $\frac{D}{r}(rv'(r)+v(r))=0 \Rightarrow D\frac{dv}{dr} = -\frac{D}{r}v(r) \Rightarrow \int\frac{dv}{v} = \int -\frac{1}{r}dr + C_1$

$\Rightarrow \ln v = -\ln r + C_1 \Rightarrow e^{\ln v} = e^{(-\ln r + c_1)} = e^{(\ln(\frac{1}{r})+c_1)}$

$u'(r) = e^{\ln(\frac{1}{r})+c_1} = \frac{1}{r}e^{c_1} \Rightarrow u(r) = \int \frac{e^{c_1}}{r}dr = e^{c_1}\int\frac{1}{r}dr = e^{c_1}\ln|r| + C_2$

Bound. cond.: $u(a,t)=0 \rightarrow u(a) = e^{c_1}\ln|a| + C_2 = 0$
$u(L,t)=U_0 \rightarrow u(L) = e^{c_1}\ln|L| + C_2 = U_0$ (*)

b) $N = \iint_{Disk} u\,dA = \int_0^{2\pi}\int_0^L u(r,t)\,r\,dr\,d\theta$. In the steady state, $\frac{\partial u}{\partial t}=0$.

$\hookrightarrow \int_0^{2\pi}\int_0^L u(r)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^L (e^c\ln|r|+c)\cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\int_0^L cr + e^c r\ln r\,dr\right]d\theta$

$= \int_0^{2\pi}\left[c\int_0^L r\,dr + e^c\int_0^L r\ln(r)\,dr\right]d\theta = \int_0^{2\pi}\left[\left[\frac{cr^2}{2}\right]_0^L + \left[e^c\left(\frac{1}{2}r^2\ln(r) - \frac{r^2}{4}\right)\right]_0^L\right]d\theta$

$= \int_0^{2\pi}\left[\frac{cL^2}{2} + \frac{1}{4}e^cL^2(2\ln(L)-1)\right]d\theta = \int_0^{2\pi}\frac{L^2}{4}(2c+2e^c\ln(L)-e^c)\,d\theta$

$= \frac{L^2}{4}(2c+2e^c\ln(L)-e^c)\cdot\theta\Big]_0^{2\pi} = \frac{\pi}{2}L^2(2c+2e^c\ln(L)-e^c)$

$\rightarrow$ N is the total number of particles or organisms inside the disk.

(*) $e^{c_1}\ln|a| = -c_2 \rightarrow c_2 = \frac{U_0}{\ln(\frac{a}{L})}\ln|a|$

$e^{c_1}\ln|L| = U_0 - c_2 = U_0 + e^{c_1}\ln|a| \rightarrow U_0 = e^{c_1}\ln|L| - e^{c_1}\ln|a|$

$U_0 = -e^{c_1}\ln\left(\frac{a}{L}\right)$

$\rightarrow -e^{c_1} = \frac{U_0}{\ln(\frac{a}{L})} \Rightarrow +c_1 = -\ln\left(\frac{U_0}{\ln(\frac{a}{L})}\right)$