Math 112B/212B, Introduction to Mathematical Biology, Homework 6

1. Suppose a cell colony is described by the (constant population) Moran process. Suppose that if r_w is the division rate of the wild type cells and r_m is the division rate of the mutants. Then the probability of mutant fixation (that is, mutants taking over the whole population) starting from i mutant cells out of n cells, is given by

$$h_i = \frac{1 - \left(\frac{r_w}{r_m}\right)^i}{1 - \left(\frac{r_w}{r_m}\right)^N}.$$

(a) Assume that $r_m > r_w$. Calculate the probability of mutant fixation h_i for $0 < i < 10$ for very large population sizes (take the limit $\lim_{N\to\infty} h_i$). Plot this as a function of i. (b) The same with $r_m < r_w$.

2. Consider a Moran process with two types: the wild type with fitnes r_w and a mutant with fitness r_m. Suppose the mutant can experience a back-mutation, where, as a result of a mutant cell division, a wild-type daughter cells is created. We assume that this mutation happens with probability β every time a mutant divides, and no other mutations happen in the system. Consider the state space $j \in \{0, 1, \dots, N\}$, where j is the number of mutants. (a) Write down the probabilities $Prob(j \to j+1)$ and $Prob(j \to j-1)$ in a single step. (b) What are the absorbing states of this system?

3. Suppose a markov chain has states $0, 1, 2, 3$. Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is given by

$$\begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Suppose we have a continuous time birth process with immigration. That is, in time Δt, $Prob(j \to j+1) = (Lj + a)\Delta t$, $Prob(j \to j) = 1 - (Lj + a)\Delta t$, and all the other processes have probability zero. Denote by $\varphi_j(t)$ the propbability to have j individuals at time t. Assume that initially, there are zero individuals. (a) Write down the master equation for $\frac{d\varphi_j(t)}{dt}$, including the equation for $j = 0$ and the initial conditions for all $\varphi_j(t)$. (b) Find $\varphi_j(t)$ for $j = 0, 1, 2, \dots$. (c) Derive the ODE for the mean number of individuals,

$$\langle X \rangle = \sum_{j=0}^{\infty} \varphi_j(t) j,$$

and the initial condition. (d) Solve this ODE to find $\langle X \rangle$ as a function of time.