MATH 112B - HW3 (Raphael F. Levy - U10156477)

① Suppose that $n(x,t)$ is the density of microorganisms and $g(x,t)$ is the concentration of glucose. Assuming that the motility of cells is much slower than the diffusion of the glucose, we can describe the co-dynamics of cells and glucose by the following system:

$$\begin{cases} \frac{\partial n}{\partial t} = Kng \\ \frac{\partial g}{\partial t} = D\frac{\partial^2 g}{\partial x^2} - cKng \end{cases}$$

, where D is the diffusion coef. of glucose, and K and c are const. > 0.

a) What processes are described by the terms Kng and -cKng?

⟹ The first term describes the growth/reproduction of cells influenced by the concentration of glucose, K representing the rate in which microorganisms grow in response to glucose. The second represents the consumption/depletion of glucose by the microorganisms, c representing the rate of consumption.

b) Define $z = x - vt$, and assume that the solutions depend on x and t only through this variable: $n(x,t) = N(z)$, $g(x,t) = G(z)$. Show how we can derive the following ODEs for these functions.

$$\begin{cases} -vN' = KNG \\ -vG' = DG'' - cKNG \end{cases}$$

⟹ Given that $z = x - vt$, we have

$$\begin{cases} \frac{\partial n}{\partial t} = \frac{dn}{dz}\frac{\partial z}{\partial t} = \frac{dn}{dz}(-v); \quad \frac{\partial g}{\partial t} = \frac{dg}{dz}\frac{\partial z}{\partial z} = \frac{dg}{dz}(-v) \\ \frac{\partial n}{\partial x} = \frac{dn}{dz}\frac{\partial z}{\partial x} = \frac{dn}{dz}; \quad \frac{\partial g}{\partial x} = \frac{dg}{dz}\frac{\partial z}{\partial x} = \frac{dg}{dz} \end{cases}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{d^2 g}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2 g}{dz^2}$$

⟹ $n(x,t) = N(z) = N(x-vt)$
$g(x,t) = G(z) = G(x-vt)$

$$\frac{\partial n}{\partial t} = (-v)\frac{dn}{dz} = -vN' = KNG$$

$$\frac{\partial g}{dt} = (-v)\frac{dg}{dz} = -vG' = DG'' - cKNG$$

① c) To derive a system of two first-order DE's, multiply $[-VN' = KNG]$ by add it to $[-VG' = DG'' - cKNG]$ and integrate once.

$\Rightarrow$ 1) $-VcN' = cKNG$ ⊛

2) $-VcN' - vG' = cKNG + DG'' - cK\cancel{NG} \Rightarrow -VcN' - vG' = DG''$

$$\int -VcN' dN + \int -vG' dG = \int DG'' dG \Rightarrow -Vc\int N' dN - v\int G' dG = D\int G'' dG$$

$$\rightarrow -VcN - vG + C = DG' \rightarrow \begin{cases} \dfrac{dG}{dz} = -\dfrac{VcN}{D} - \dfrac{V}{D}G + \dfrac{C}{D} \\ \dfrac{dN}{dz} = -\dfrac{\cancel{c}KNG}{V\cancel{c}} = \dfrac{-KNG}{V} \end{cases}$$

d) Compare your result with the following system:

$$\begin{cases} G' = -\dfrac{V}{D}G - \dfrac{cV}{D}N + a \\ N' = -\dfrac{KNG}{V} \end{cases}$$

. Set $a = \dfrac{cV}{D}N_0$. Find all steady states and perform linear stability analysis.

$\Rightarrow$ We can see that the equations are the same, the only difference between them is a better definition of the constant in d). Given this, we may rewrite $\dfrac{C}{D} = \dfrac{cVN_0}{D}$.

$\Rightarrow$ Steady states: $\begin{cases} G' = 0 \\ N' = 0 \end{cases} \Rightarrow \begin{cases} -\dfrac{V}{D}G - \dfrac{cV}{D}N + \dfrac{cV}{D}N_0 = 0 \Rightarrow -VG + cV(-N + N_0) = 0 \\ \qquad V(-G + c(N_0 - N)) = 0 \\ \dfrac{-KNG}{V} = 0 \quad \Rightarrow -KNG = 0 \rightarrow \text{As } K > 0 \text{, either } N = 0 \text{ or } G = 0 \end{cases}$

If $N = 0 \Rightarrow -\dfrac{V}{D}G + \dfrac{cV}{D}N_0 = 0 \rightarrow cN_0 = G$

If $G = 0 \Rightarrow -\dfrac{cV}{D}N + \dfrac{cV}{D}N_0 = 0 \rightarrow N_0 = N$

$\hookrightarrow$ Fixed points $(N, G) = (0, cN_0)$ and $(N_0, 0)$

$$\rightarrow J = \begin{pmatrix} -\dfrac{KG}{V} & -\dfrac{KN}{V} \\ -\dfrac{cV}{D} & -\dfrac{V}{D} \end{pmatrix}$$

$f_1 = N'$, $f_2 = G'$

$u_1 = N$, $u_2 = G$

$$J(0, cN_0) = \begin{pmatrix} -\dfrac{KcN_0}{V} & 0 \\ -\dfrac{cV}{D} & -\dfrac{V}{D} \end{pmatrix}; \quad J(N_0, 0) = \begin{pmatrix} 0 & -\dfrac{KN_0}{V} \\ -\dfrac{cV}{D} & -\dfrac{V}{D} \end{pmatrix}$$

⊛ This Jacobian was done incorrectly. The correct can be found in the next page

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1 d) Cont.: At $(\overset{N}{0}, \overset{G}{cN_0})$: $J = \cancel{\begin{pmatrix} -\frac{KcN_0}{V} & 0 \\ -\frac{V}{D}C & -\frac{V}{D} \end{pmatrix}}$. Now we need to find the values of $\lambda_1\lambda_2$ and $\lambda_1+\lambda_2$.

$$J = \begin{pmatrix} -V\frac{C}{D} & -\frac{V}{D} \\ -\frac{KcN_0}{V} & 0 \end{pmatrix}$$

↳ $\lambda^2 - (\text{Tr } A)\lambda + \text{Det } \lambda = 0$

OBS: $\begin{cases} \overset{f_1}{G'} = 0 \\ \underset{f_2}{N'} = 0 \end{cases} \Rightarrow \begin{cases} -\frac{V}{D}G - \frac{cV}{D}N + \frac{cV}{D}N_0 = 0 \\ -\frac{KNG}{V} = 0 \end{cases}$

$$J = \begin{pmatrix} -\frac{V}{D}C & -\frac{V}{D} \\ -\frac{KG}{V} & -\frac{K}{V}N \end{pmatrix}$$

$$\to \lambda^2 - \left(-\frac{VC}{D}\right)\lambda + \left[-\left(-\frac{V}{D}\right)\left(-\frac{KcN_0}{\cancel{V}}\right)\right] = 0$$

$$\lambda^2 + \frac{VC}{D}\lambda - \frac{KcN_0}{D} = 0$$

$$\Rightarrow \lambda_1\lambda_2 = -\frac{KcN_0}{D} < 0, \quad \lambda_1+\lambda_2 = -\frac{VC}{D}$$

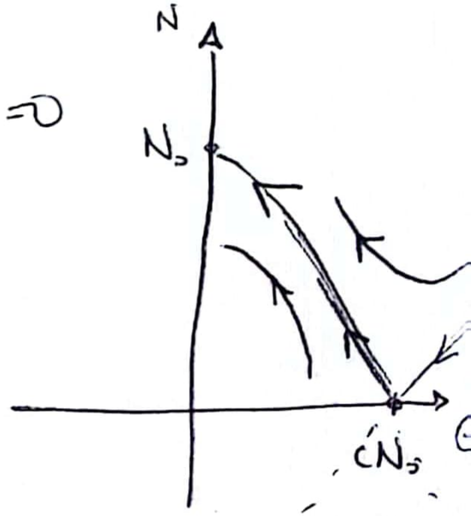
→ Since $\lambda_1\lambda_2 < 0$, $(0, cN_0)$ is a saddle point

At $(\overset{N}{N_0}, \overset{G}{0})$: $J = \begin{pmatrix} -\frac{V}{D}C & -\frac{V}{D} \\ 0 & -\frac{K}{V}N_0 \end{pmatrix} \longrightarrow \lambda^2 - \left(-\frac{VC}{D_{/V}} - \frac{K}{V_{/D}}N_0\right)\lambda + \left(-\frac{\cancel{V}}{D}C\right)\left(-\frac{KN_0}{\cancel{V}}\right) = 0$

$$\lambda^2 + \left(\frac{V^2C + KDN_0}{VD}\right)\lambda + \frac{KCN_0}{D} = 0$$

$$\lambda_1\lambda_2 = \frac{KcN_0}{D} > 0, \quad \lambda_1+\lambda_2 = -\left(\frac{V^2C + KDN_0}{VD}\right) < 0$$

↳ In this case, both eigenvalues are negative, so it's a sink



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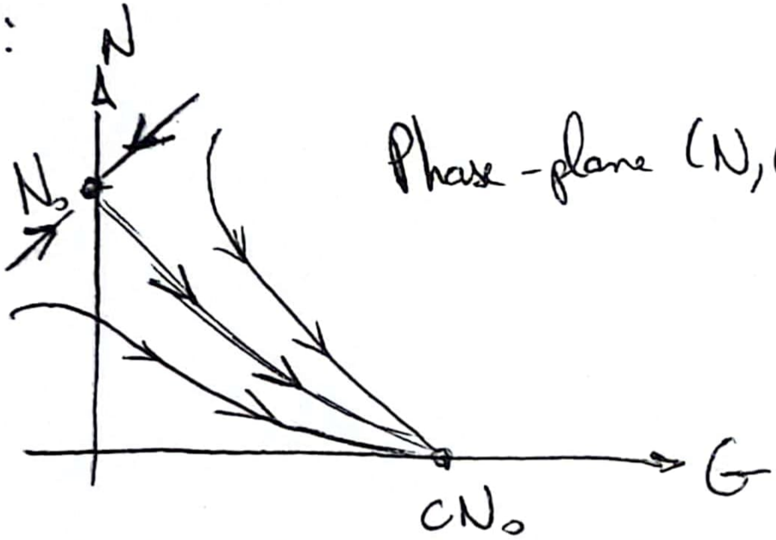
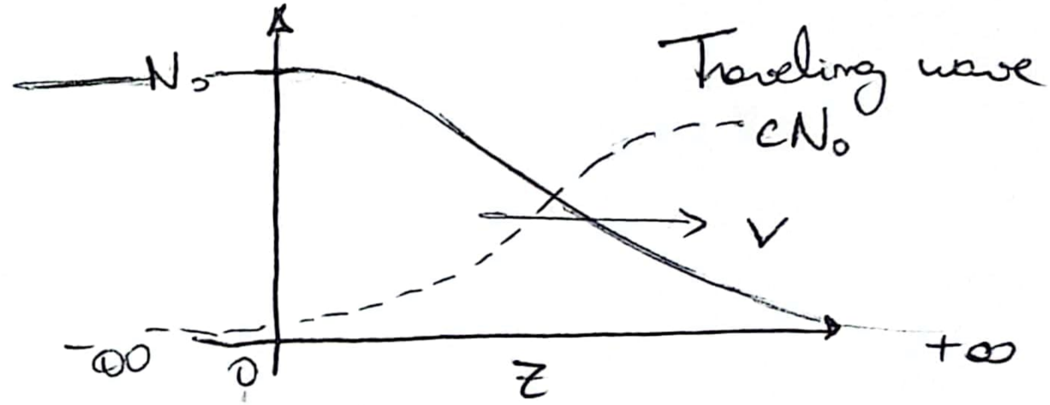
e)

Phase-plane (N,G)





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(2) Consider the following general system of reaction-diffusion equations:

$$\frac{\partial u_1}{\partial t} = f_1(u_1,u_2) + D_1 \frac{\partial^2 u_1}{\partial x^2}$$

$$\frac{\partial u_2}{\partial t} = f_2(u_1,u_2) + D_2 \frac{\partial^2 u_2}{\partial x^2}$$

Determine wether or not a homogeneous steady state that is stable in the absence of diffusion can be obtained. If so, give explicit conditions for instability to arise, and determine which modes would be most destabilizing.

a) Lotka-Volterra: $f_1 = au_1 - bu_1u_2$, $f_2 = -qu_2 + du_1 du_2$

b) glycolytic oscillator: $f_1 = \delta - ku_1 - u_1u_2^2$, $f_2 = ku_1 + u_1u_2^2 - u_2$

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② a) $f_1 = au_1 - bu_1u_2$. We must have that $a_{11} + a_{22} < 0$

$f_2 = -qu_2 + du_1u_2$

$a_{11}a_{22} - a_{12}a_{21} > 0$

$a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1D_2}(a_{11}a_{22} - a_{12}a_{21})^{1/2} > 0$

$$\begin{cases} a_{11} = \frac{\partial f_1}{\partial u_1} ; \; a_{12} = \frac{\partial f_1}{\partial u_2} \\ a_{21} = \frac{\partial f_2}{\partial u_1} ; \; a_{22} = \frac{\partial f_2}{\partial u_2} \end{cases} \Rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a - bu_2 & -bu_1 \\ du_2 & -q + du_1 \end{pmatrix}$$

$\overline{\frac{\partial u_1}{\partial t}} = \overline{\frac{\partial u_2}{\partial t}} = 0$; $\overline{\frac{\partial u_1}{\partial x}} = \overline{\frac{\partial u_2}{\partial x}} = 0$. We obtain our first homogeneous steady state by setting $f_1(u_1, u_2) = 0 \Rightarrow au_1 - bu_1u_2 = 0$

$f_2(u_1, u_2) = 0 \Rightarrow -qu_2 + du_1u_2 = 0$

$\rightarrow$ ~~$u_1 = \frac{bu_1u_2}{a}$~~ $a\bar{u}_1 = b\bar{u}_1\bar{u}_2 \Rightarrow a = b\bar{u}_2 \rightarrow \bar{u}_2 = \frac{a}{b}$

$\rightarrow -q\bar{u}_2 = -d\bar{u}_1\bar{u}_2 \Rightarrow q = d\bar{u}_1 \rightarrow \bar{u}_1 = \frac{q}{d}$

$\Rightarrow a_{11}' = a - b\bar{u}_2 = a - \frac{ba}{b} = a - a = 0$ | $a_{21}' = d\bar{u}_2 = \frac{da}{b}$

$a_{12}' = -b\bar{u}_1 = -\frac{bq}{d}$

$a_{22}' = -q + d\bar{u}_1 = -q + \frac{dq}{d} = -q + q = 0$

$$J = \begin{pmatrix} 0 & -\frac{bq}{d} \\ \frac{da}{b} & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}(J) = a_{11} + a_{22} \boxed{= 0}$$

$$\mathrm{Det}(J) = a_{11}a_{22} - a_{12}a_{21} = 0 - \left(-\frac{bq}{d}\right)\left(\frac{da}{b}\right) = aq > 0$$

$a_{11}D_2 + a_{22}D_1 = 0$ $\Rightarrow$ We cannot obtain a homogeneous steady state given these conditions.

② a) Cont.: we have previously verified that there is no homogeneous steady state as $Tr(J) = a_{11} + a_{22} = 0+0=0$. But we may still calculate the 3rd condition:

$a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2}\,(a_{11}a_{22} - a_{12}a_{21})^{1/2} > 0$, however this immediatly fails, given $a_{11} = a_{22} = 0$, so we cannot find any equilibrium for this problem.

b) $f_1 = \delta - Ku_1 - u_1 u_2^2$
$f_2 = Ku_1 + u_1 u_2^2 - u_2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -K-u_2^2 & -2u_1u_2 \\ K+u_2^2 & 2u_1u_2 - 1 \end{pmatrix} \rightarrow \begin{matrix} f_1(u_1,u_2)=0 \\ f_2(u_1,u_2)=0 \end{matrix} \Rightarrow f_1 + f_2 = \delta - u_2 = 0$$

$\delta = u_2$

$$\Rightarrow K\bar{u}_1 + \bar{u}_1\delta^2 - \delta = 0 \Rightarrow \boxed{\bar{u}_1 = \frac{\delta}{K+\delta^2}\ ;\ \bar{u}_2 = \delta}$$ steady state

$2\delta^2 > \delta^2 - K$

$K + 2\delta^2 > \delta^2$

$$a_{11} = \left.\frac{\partial f_1}{\partial u_1}\right|_{\bar{u}_1,\bar{u}_2} = -K - u_2^2\Big|_{\bar{u}_1,\bar{u}_2} = -K-\delta^2 \ldots \Rightarrow a_{12} = \frac{-2\delta^2}{K+\delta^2}\ ;\ a_{21} = K+\delta^2;\ a_{22} = \frac{2\delta^2}{K+\delta^2} - 1$$

$-K < \delta^2$

$$a_{11} + a_{22} = \frac{-(K+\delta^2)^2 + \delta^2 - K}{K+\delta^2} < 0 \Rightarrow \text{We need that } \delta^2 - K < (K+\delta^2)^2$$

$$\Rightarrow \begin{cases} K+\delta^2 > 0 \Rightarrow K > -\delta^2 \\ \delta^2 - K < (K+\delta^2)^2 \\ \quad \delta^2 - K < 2\delta^2 \end{cases}$$

>0

$a_{11}a_{22} - a_{12}a_{21} = -2\delta^2 + K + \delta^2 - (-2\delta^2) = K + \delta^2 > 0$

$\Rightarrow$ Now, we must find D_1 and D_2 that satisfy: $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1D_2}(a_{11}a_{22} - a_{12}a_{21}) > 0$

$$-(K+\delta^2)D_2 + \frac{2\delta^2 D_1}{K+\delta^2} - D_1 > 2\sqrt{D_1D_2}\sqrt{K+\delta^2}$$

② b) Cont.: $-(K+\delta^2)D_2 + \left(\frac{2\delta^2}{K+\delta^2} - 1\right) D_1 > 2\sqrt{D_1 D_2}\sqrt{K+\delta^2} > 0$

$$\boxed{2\sqrt{(D_1 D_2)(K+\delta^2)} > 0}$$

D₁ and D₂ must satisfy the inequality above

$\Rightarrow -(K+\delta^2)D_2 < 0$

$\frac{2\delta^2}{K+\delta^2} = \frac{\delta^2+\delta^2}{K+\delta^2}$, $2\delta^2 > 0$, $K+\delta^2 > 0$ $\rightarrow \frac{2\delta^2}{K+\delta^2} - 1 = \frac{2\delta^2 - K - \delta^2}{K+\delta^2} = \frac{\delta^2 - K}{\delta^2+K} < \frac{2\delta^2}{\delta^2+K}$

$\hookrightarrow$ If $\delta^2 < K$, $\frac{2\delta^2}{K+\delta^2} < 1$, so $\left(\frac{2\delta^2}{K+\delta^2} - 1\right)D_1 < 0$, which won't satisfy the condition. For it to be satisfied, we would need to have $\delta^2 > K \rightarrow \delta^2 - K > 0$

$$\Rightarrow \frac{-(K+\delta^2)D_2 + \left(\frac{\delta^2 - K}{\delta^2+K}\right)D_1}{\sqrt{D_1 D_2}} > 2\sqrt{K+\delta^2} > 0$$

. We need $D_1\left(\frac{\delta^2-K}{\delta^2+K}\right) > (\delta^2+K)D_2$

$\rightarrow D_1 > D_2(\delta^2+K)\left(\frac{\delta^2+K}{\delta^2-K}\right)$