

Math 112B/212B, Introduction to Mathematical Biology, Homework 2

1. Consider the initial-boundary value problem,

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, t > 0, \quad (1)$$

$$u(x, 0) = f(x), \quad (2)$$

$$u(0, t) = u(\pi, t) = 0. \quad (3)$$

We solved this problem in class by using the separation of variable technique. This problem reviews some of the steps.

(a) The assumption we made was that $u(x, t) = X(x)T(t)$. When substituted into the PDE, this resulted in two ODEs: $X'' + \lambda X = 0$ and $T' = -\lambda DT$. What ODEs would we obtain by the same method, if instead of equation (4), we had

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x}$$

(you do not have to carry out the solution of the ODEs).

(b) Back to the original PDE. For the eigenvalue problem, $X'' + \lambda X = 0$, $X(0) = X(\pi) = 0$, we need to find all the values of λ that give a nontrivial solution, $X(x)$. Show that if $\lambda \leq 0$, the only solution is $X(x) = 0$. (Hint: do this separately for $\lambda = 0$ and $\lambda < 0$, by writing the general solution and then solving for the constants using the boundary conditions).

2. Consider the initial-boundary value problem, defined on the interval $x \in [a, b]$:

$$\frac{\partial u}{\partial t} = \tilde{D} \frac{\partial^2 u}{\partial x^2}, \quad a < x < b, t > 0, \quad (4)$$

$$u(x, 0) = \tilde{f}(x), \quad (5)$$

$$u(a, t) = u(b, t) = 0. \quad (6)$$

Reduce this problem to problem (4-6) by a change of variables. (Hint: consider the new variable, $\xi = \pi(x - a)/b$).

3. Consider the reaction-diffusion equation on $x \in (-\infty, \infty)$:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + ru, \quad t > 0. \quad (7)$$

(a) Show that the following function satisfies the PDE:

$$u(x, t) = \frac{u_0}{2(\pi Dt)^{1/2}} \exp \left\{ rt - \frac{x^2}{4Dt} \right\}$$

(here u_0 is an arbitrary constant).

(b) Consider the level set, $u(x, t) = U$, where U is a constant. Show that on this contour, the ratio x/t satisfies

$$\frac{x}{t} = \pm \left[4rD - \frac{2D}{t} \ln t - \frac{4D}{t} \ln \left(\sqrt{2\pi D} \frac{U}{u_0} \right) \right]^{1/2}.$$

(c) Show that as $t \rightarrow \infty$, one can approximate the above by

$$\frac{x}{t} = \pm 2\sqrt{rD}.$$