Math 112B/212B, Introduction to Mathematical Biology, Homework 7

Suppose a cell colony consists of two types of cells: type 0 and type 1. Cells of type 1 can divide at rate L and mutate into cells of type 0 with probability u. They can also die with rate D. Cells of type 0 can only transfom into cells of type 1, at a rate γ, by a process that is not coupled with division. Denote by i the number ofg cellsof type 0 and by j the number of cells of type 1. We have the following probabilities: during time Δt,

$$\begin{aligned} Prob(i,j \to i-1, j+1) &= \gamma i \Delta t, \\ Prob(i,j \to i, j+1) &= Lj(1-u)\Delta t, \\ Prob(i,j \to i+1, j) &= Lju\Delta t, \\ Prob(i,j \to i, j-1) &= Dj\Delta t, \\ Prob(i,j \to i, j) &= 1 - \Delta t(\gamma i + (L + D_j). \end{aligned}$$

Assum that initially, the nuimber of cells of type 0 is $M_0 > 0$ and the number of cells of type 1 is $M_1 > 0$.

1. Denote by $\varphi_{i,j}(t)$ the probability to have i cells of type 0 and j cells of type 1 at time t. Derive the Kolmogorov forward equation for $\varphi_{i,j}(t)$ and formulate the initial value problem. It should look like this:

$$\begin{aligned} \dot{\varphi}_{0,0} &= \ldots, \\ \dot{\varphi}_{0,j} &= \ldots, \quad j > 0, \\ \dot{\varphi}_{i,0} &= \ldots, \quad i > 0, \\ \dot{\varphi}_{i,j} &= \ldots, \quad i > 0, j > 0, \\ \varphi_{i,j}(0) &= \ldots. \end{aligned}$$

2. Define the expected number of cells of the two types as

$$X \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{i,j} i, \quad Y \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{i,j} j.$$

   Derive the initial value problem for these average populations. It should look like this:

$$\begin{aligned} \dot{X} &= \ldots, \\ \dot{Y} &= \ldots, \\ X(0) &= \ldots, \quad Y(0) = \ldots. \end{aligned}$$

   Optional: solve this linear system of ODEs.

3. Introdice the probability generating function:

$$\Psi(x,y,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{i,j} x^j y^x,$$

where x and y are NOT the same as the mean values in the pevious problem. Derive a PDE for $\Psi(x, y, t)$ together with the initial condition. It should look like this:

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \frac{\partial \Psi}{\partial x} \ldots + \frac{\partial \Psi}{\partial y} \ldots, \\ \Psi(x, y, 0) &= \ldots. \end{aligned}$$