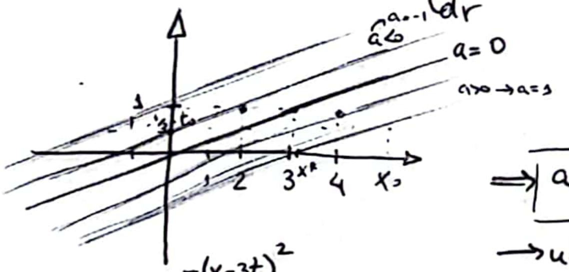
MATH112B-HW4 (Raphael F. Levy - U10156477)

1) Consider the transport eq. $\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0$, $c(x,t)=3$. a) Find the characteristics and draw them on the xt plane. b) Find the solution to the IVP with $u(x,0)=e^{-x^2}$. c) Draw this solution as a function of x for t=0,1,2.

$\Rightarrow$ a) To find $(X(r),T(r))$: $\begin{cases} \frac{dt}{dr} = 1 \Rightarrow t = r+k. \text{ By picking } k=0, \text{ we have } t=r \\ \frac{dx}{dr} = c(x,t) = 3 \Rightarrow \frac{dx}{dr} = \frac{dx}{dt} = 3 \Rightarrow \int dx = \int 3dt \Rightarrow x = 3t+a \Rightarrow \boxed{t = \frac{x-a}{3}} \end{cases}$

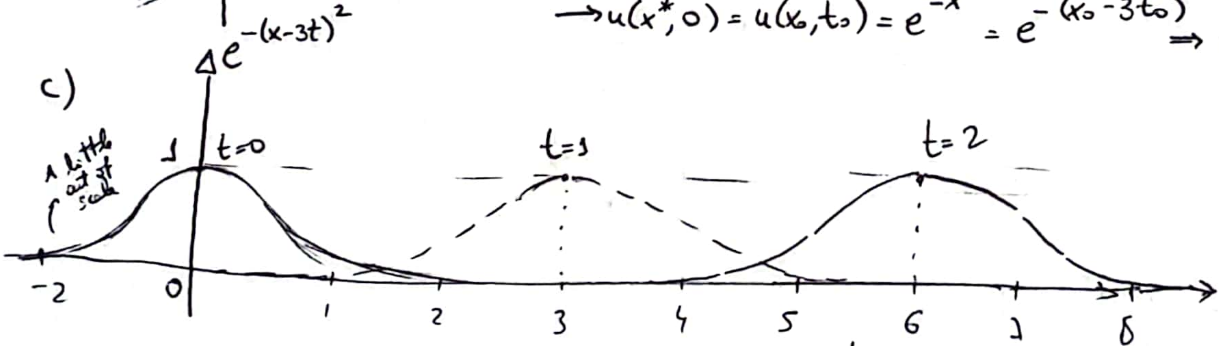


b) $u(x,0) = f_0(x) = e^{-x^2}$. At (x_0,t_0), we have $x_0 = 3t_0 + a$

$\Rightarrow \boxed{a = x_0 - 3t_0}$. At $(x^*,0)$, $x^* = 3.0 + a \to a = x^* = x_0 - 3t_0$

$\to u(x^*,0) = u(x_0,t_0) = e^{-x^{*2}} = e^{-(x_0-3t_0)^2} \Rightarrow u(x,t) = e^{-(x-3t)^2}$ //

c)



2) Consider the same transport equation with $c(x,t)=e^{-t}$. a) Find the characteristics and draw them on the xt plane. b) Find the solution to the IVP when $u(x,0)=e^{-x^2}$. Hint: to find the eq. of the char. lines, first divide the PDE by e^{-t}.

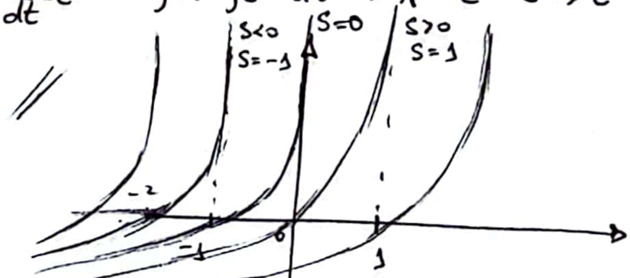
$\Rightarrow$ a) To find $(X(r),T(r))$: $\begin{cases} \frac{dt}{dr} = 1 \Rightarrow t = r+k. \text{ With } k=0, t=r \\ \frac{dx}{dr} = e^{-t} \Leftrightarrow \frac{dx}{dt} = e^{-t} \Rightarrow \int dx = \int e^{-t}dt \to x = -e^{-t} + s \Rightarrow e^{-t} = s - x \to -t = \ln(s-x) \end{cases}$

$t = -\ln(s-x)$

Or: $\frac{1}{e^{-t}}\frac{\partial u}{\partial t} + \frac{e^{-t}}{e^{-t}}\frac{\partial u}{\partial x} = 0$

$\rightsquigarrow e^{t}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \Rightarrow \begin{cases} \frac{dx}{dr} = 1 \to x = r \\ \frac{dt}{dr} = \frac{dt}{dx} = e^{t} \to \int dx = \int e^{-t}dt \end{cases}$

//



b) $u(x,0) = e^{-x^2}$. $x_0 = -e^{-t_0} + s \to \boxed{s = x_0 + e^{-t_0}} \Rightarrow x = -e^{-t} + x_0 + e^{-t_0}$

$x^* = -e^{0} + x_0 + e^{-t_0} = 1 + x_0 + e^{-t_0}$

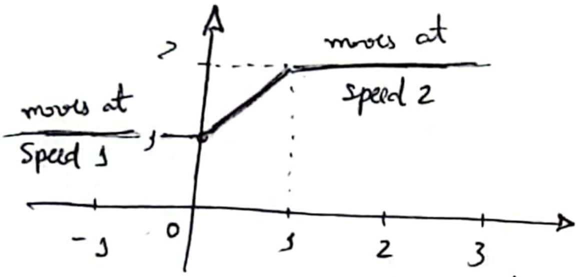
$u(x^*,0) = e^{-x^{*2}} = e^{-(1+x_0+e^{-t_0})^2} = u(x_0,t_0) \Rightarrow u(x,t) = e^{-(1+x+e^{-t})^2}$ //

3) Consider the inviscid Burger's eq with $u(x,0) = f_0(x) = \begin{cases} 1, x<0 \\ 1+x, 0 \le x \le 1 \\ 2, x>1 \end{cases}$. Find the eq. for the characteristics, and draw them on the xt plane. Will a shock form in this case?

$\Rightarrow$ The inviscid Burgers' equation is defined by: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$. So the char. lines have the eq.: $\frac{dx}{dt} = ? = u \longrightarrow$ Suppose $u = u(x,t) \overset{\text{along the char line}}{=} u(x(t),t)$

③ cont. By the chain rule, $\frac{du}{dt} = \frac{\partial u}{\partial x}\underbrace{\frac{dx}{dt}}_{= u} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial t} = 0$ as the original PDE. Therefore u, in fact, does not change along such lines. Since u does not change along the char. lines, it can be integrated: ~~$\int dx$~~ $\frac{dx}{dt} = u \Rightarrow \int dx = \int u\,dt \Rightarrow x = ut + K$. $\Downarrow$ $t = \frac{x-K}{u}$

OBS: $f(x) = \begin{cases} 1, & x<0 \\ 1+x, & 0 \le x \le 1 \\ 2, & x>1 \end{cases}$



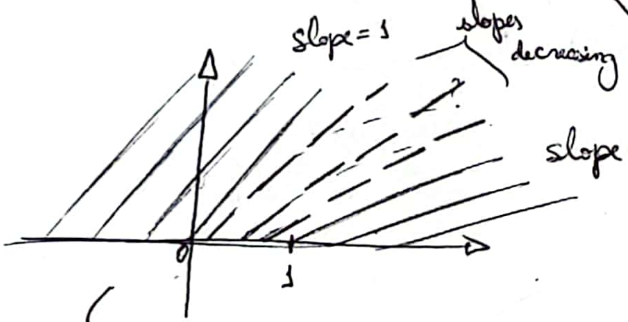
At $(x^*, 0)$:

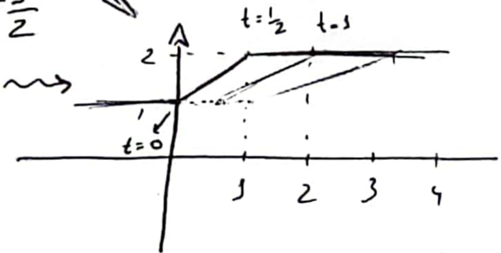
$x^* = u \cdot 0 + K \rightarrow K = x^*$

For any $x^* < 0$, $u = 1 \Rightarrow$ slope $= 1$

For any $x^* > 1$, $u = 2 \Rightarrow$ slope $= \frac{1}{2}$

For $0 \le x^* \le 1$, $u = 1 + x \Rightarrow$ slope $= \frac{1}{1+x}$?





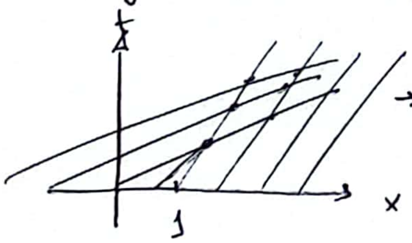
A shock will not be formed in this case, as the function "speeds up" faster in the upper part than in the lower part. This can be seen as the slope decreases, so the function will not "crash over".

④ Same as before, with $f(x) = \begin{cases} 2, & x<0 \\ 2-x, & 0 \le x \le 1 \\ 1, & x>1 \end{cases} \Rightarrow$ Given only $f(x)$ changed, our characteristic lines continue to be given by $x = ut + K \Leftrightarrow t = \frac{x-K}{u}$

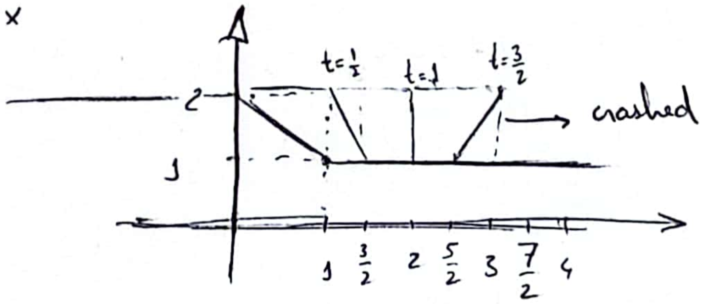
So, at $(x^*, 0)$: $x^* = u \cdot 0 + K \rightarrow K = x^*$

For any $x^* < 0$, $u = 2 \Rightarrow$ slope $= \frac{1}{2}$

For any $x^* > 1$, $u = 1 \Rightarrow$ slope $= 1$



→ The char. lines cross, this behavior is observed as the initial condition $f(x)$ has a region where it decreases. Therefore a shock will ~~happen~~ form in this case.



→ crashed over (not a function)