MATH342B - HW7 (Raphael F. Levy - U10156477)

(1) Denote by $\varphi_{i,j}(t)$ the prob. to have i cells of type 0 and j cells of type 1 at time t. Derive the Kolmogorov for. eq. for $\varphi_{i,j}(t)$ and formulate the IVP.

$\Rightarrow \varphi_{i,j}(t+\Delta t) = Lu j \Delta t\, \varphi_{i-1,j} + L(j-1)(1-u)\Delta t\, \varphi_{i,j-1} + D(j+1)\Delta t\, \varphi_{i,j+1} + \gamma(i+1)\Delta t\, \varphi_{i+1,j-1}$
$+ (1 - \Delta t(\gamma i + (L+D)j))\varphi_{i,j} \Rightarrow \dot{\varphi}_{i,j}(t) = \dfrac{\varphi_{i,j}(t+\Delta t) - \varphi_{i,j}(t)}{\Delta t}$

$-\varphi_{i,j}$ $\quad \Delta t$

$\dot{\varphi}_{i,j}^{(i,j>0)} = Lu j\, \varphi_{i-1,j} + L(j-1)(1-u)\varphi_{i,j-1} + D(j+1)\varphi_{i,j+1} + \gamma(i+1)\varphi_{i+1,j-1} - (\gamma i + (L+D)j)\varphi_{i,j}$ //

$\dot{\varphi}_{0,0} = 0$ (Def) $\rightarrow \varphi_{i,j}(0) = 0$ //

$\dot{\varphi}_{0,j}\ (j>0) = L(j-1)(1-u)\varphi_{0,j-1} + D(j+1)\varphi_{0,j+1} + \gamma\varphi_{1,j-1} - (L+D)j\varphi_{0,j}$ //

$\dot{\varphi}_{i,0}\ (i>0) = D\varphi_{i,1} - \gamma\varphi_{i,0}$ //

(2) $X \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{i,j}\cdot i;\ Y \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{i,j}\, j$

$\dot{X} = Lu\sum_{i,j}\varphi_{i-1,j}\, ij + L(1-u)\sum_{i,j}\varphi_{i,j-1}(j-1)i + D\sum_{i,j}\varphi_{i,j+1}(j+1)i + \gamma\sum_{i,j}\varphi_{i+1,j-1}(i+1)i - \gamma\sum_{i,j}\varphi_{i,j}\, i\, i$
$-(L+D)\sum_{i,j}\varphi_{i,j}\, j\, i \quad \Rightarrow k = i-1,\ i = k+1 \quad m = j-1,\ j = m+1$

$\hookrightarrow \dot{X} = Lu\sum\varphi_{k,j}(k+1)j + L(1-u)\sum\varphi_{i,m}\, mi + D\sum\varphi_{i,j+1}(j+1)i + \gamma\sum\varphi_{i+1,m}(i+1)i - \gamma\sum\varphi_{i,j}\, i^2$
$-(L+D)\sum\varphi_{i,j}\, ji$

$= Lu\cancel{\sum\varphi kj} + Lu\sum\varphi j + \cancel{L\sum\varphi mi} - \cancel{Lu\sum\varphi mi} + \cancel{D\sum\varphi ji} + D\sum\varphi i + \cancel{\gamma\sum\varphi i^2} + \gamma\sum\varphi i - \cancel{\gamma\sum\varphi i^2}$
$- \cancel{L\sum\varphi ji} - \cancel{D\sum\varphi ji} = Lu\sum\varphi j + D\sum\varphi i + \gamma\sum\varphi i = Lu\langle x\rangle + D\langle x\rangle + \gamma\langle x\rangle$

$\dot{Y} = Lu\sum\varphi_{ij}\, j\cdot j + L(1-u)\sum_{i,m} mj + D\sum\varphi(j+1)j + \gamma\sum\varphi(i+1)j - \gamma\sum\varphi_{ij} - (L+D)\sum\varphi_{jj}$

$= Lu\cancel{\sum\varphi j^2} + L\sum mj - Lu\sum mj + \cancel{D\sum\varphi j^2} + D\sum\varphi j + \cancel{\gamma\sum\varphi ji} + \gamma\sum\varphi j - \cancel{\gamma\sum\varphi ij} - \cancel{L\sum\varphi j^2}$
$- \cancel{D\sum\varphi j^2} \quad = L(1-u)\sum\varphi_{mj} + D\sum\varphi_j + \gamma\sum\varphi_j = L(1-u)\langle y\rangle + D\langle y\rangle + \gamma\langle y\rangle$

$\Rightarrow X(0) = M_0,\ \cancel{Y}(0) = M_1$

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③ $\Psi(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{i,j}\, x^j y^i \Rightarrow \Psi(x,y,0) = \sum\sum \underbrace{\varphi_{i,j}(0)}_{\substack{i=M_0 \\ j=M_1}} x^j y^i = X^{M_1}\cdot y^{M_0}$ //

$\hookrightarrow \dfrac{\partial\Psi}{\partial t} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \dfrac{\partial\varphi_{i,j}(t)}{\partial t} x^j y^i.$

$\hookrightarrow Lu\,\varphi_{i-1,j}: \sum_{i,j} Lu\,\varphi_{i-1,j}\, x^j y^i = Lu \sum_{i,j} \varphi_{i,j}\, x^j y^{i+1} = Luy\,\Psi$

$L(1-u)(j-1)\varphi_{i,j-1}: \sum_{i,j} L(1-u)(j-1)\varphi_{i,j-1}\, x^j y^i = L(1-u)\,x\,\dfrac{\partial\Psi}{\partial x}$

$D(j+1)\varphi_{i,j+1}: \sum_{i,j} D(j+1)\varphi_{i,j+1}\, x^j y^i = D\sum_{i,j} j\,\varphi_{i,j}\, x^{j-1} y^i x = D\,x\,\dfrac{\partial\Psi}{\partial x}$

$\gamma(i+1)\varphi_{i+1,j-1}: \sum_{i,j} \gamma(i+1)\varphi_{i+1,j-1}\, x^j y^i = \gamma x \sum_{i,j} i\,\varphi_{i,j}\, x^j y^{i-1} y = \gamma y\,\dfrac{\partial\Psi}{\partial y}$

$-(\gamma i + (L+D)j)\varphi_{i,j}: \sum_{i,j} -(\gamma i + (L+D)j)\varphi_{i,j}\, x^j y^i = -\gamma y\,\dfrac{\partial\Psi}{\partial y} - (L+D)\,x\,\dfrac{\partial\Psi}{\partial x}$

$$\Rightarrow \frac{\partial\Psi}{\partial t} = Luy\,\Psi + L(1-u)\,x\,\frac{\partial\Psi}{\partial x} + D\,x\,\frac{\partial\Psi}{\partial x} + \gamma y\,\frac{\partial\Psi}{\partial y} - \gamma y\,\frac{\partial\Psi}{\partial y} - (L+D)\,x\,\frac{\partial\Psi}{\partial x}$$

$$= Luy\,\Psi + \left(L(1-u)+D-(L+D)\right)x\,\frac{\partial\Psi}{\partial x} + \cancel{\gamma y\,\frac{\partial\Psi}{\partial y} - \gamma y\,\frac{\partial\Psi}{\partial y}} = Luy\,\Psi + Lu\,\frac{\partial\Psi}{\partial x}$$

$$\frac{\partial\Psi}{\partial t} = Lu\,\frac{\partial\Psi}{\partial x} + Luy\,\Psi$$