MATH112B - HW5 (Raphael F. Levy - U10156477)

1) For the Final Project, I have chosen the project 7: Age-structured modeling in cell biology. The group is composed by: Raphael F. Levy, Deloris Franklin & ~~Xufei Zhang~~; Yige Gong.

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2) Suppose a system is characterized by a stochastic variable that can take values $\{0,1,2,3,4\}$. The prob. to go from i to $i+1$ is given by 0.1 for $i=1,2,3$. The prob. to go from i to $i-1$ is 0.2 for $i=1,2,3$. No other jumps can happen, with the exception of $i \to i$. a) Write down the transition matrix for this problem. b) Are there any absorbing states and transient states? Which ones? c) What is the probability that the system ends up in each of the absorbing states, starting from state $i=1$? Starting from $i=2$? $i=4$?

$\Rightarrow$ a)

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0.7 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.7 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

b) From the question, we know that it's impossible to leave states 0 and 4, as the probabilities of moving are $i \to i+1$ or $i \to i-1$ for $i \in \{1,2,3\}$. Therefore, $i=0$ and $i=4$ are absorbing states. $i=1,2,3$ are transient, as it is possible to leave and return to them as long as we don't get stuck in any absorbing state.

c) Notably, if we start at $i=4$, the probability of ~~being~~ ending up at 4 is 1, as it is an absorbing state, and the prob. to end up in 0 is 0. For $i=1$ and 2, let's define P_{ij} as the prob. of visiting j starting from i. In our case.

$$\begin{cases} P_{10} = 0.2 + 0.7P_{10} + 0.1P_{20} \\ P_{20} = 0 + 0.2P_{10} + 0.7P_{20} + 0.1P_{30} \\ P_{30} = 0 + 0.2P_{20} + 0.7P_{30} + 0.1\cancel{P_{40}}^{0} \end{cases}$$

$\to$ 0.2 of going straight to 0, 0.7 of going to 1 again, then go to 0, 0.1 of going to 2 then 1 then 0

$\to$ 0 of going to 0, 0.2 of going to 1, 0.7 of going to 2, 0.1 of going to 3

$$\longrightarrow \begin{cases} 0.3P_{10} = 0.2 + 0.1P_{20} \\ 0.3P_{20} = 0.2P_{10} + 0.1P_{30} \\ 0.3P_{30} = 0.2P_{20} \end{cases} \quad \Bigg| \quad P_{30} = \frac{2P_{20}}{3} \; (*) \qquad P_{20} = \frac{2P_{10} + \frac{2P_{20}}{3}}{3} \qquad P_{10} = \frac{2 + \frac{6}{7}P_{10}}{3}$$

$(*)$ $3P_{20} = 2P_{10} + \frac{2P_{20}}{3} \to \frac{7P_{20}}{3} = 2P_{10} \to P_{20} = \frac{6}{7}P_{10}$

$$3P_{10} = 2 + \frac{6}{7}P_{10} \to \frac{15}{7}P_{10} = 2 \to P_{10} = \frac{14}{15}, \quad P_{20} = \frac{6}{7}\cdot\frac{14}{15} = \frac{12}{15} = \frac{4}{5}, \quad P_{30} = \frac{2}{3}\cdot\frac{12}{15} = \frac{24}{45} = \frac{8}{15}$$

$\to$ Prob. of going to 0 starting at 1 is $\frac{14}{15}$, prob. of going to 0 starting at 2 is $\frac{4}{5}$, prob. of going to 0 starting at 4 is 0

$$\begin{cases} P_{14} = 0.2\cancel{P_{04}}^{0} + 0.7P_{14} + 0.1P_{24} \\ P_{24} = 0.2P_{14} + 0.7P_{24} + 0.1P_{34} \\ P_{34} = 0.2P_{24} + 0.7P_{34} + 0.1 \end{cases} \to \begin{cases} 0.3P_{14} = 0.1P_{24} \to P_{14} = \frac{1}{3}P_{24} \\ 0.3P_{24} = 0.2P_{14} + 0.1P_{34} \to P_{24} = \frac{\frac{2}{3}P_{24} + 1P_{34}}{3} = \frac{\frac{2P_{24}}{3} + \frac{1+2P_{24}}{3}}{3} = \frac{1+4P_{24}}{9} \to 5P_{24} = 1 \\ 0.3P_{34} = 0.2P_{24} + 0.1 \to P_{34} = \frac{1+2P_{24}}{3} \end{cases}$$

$$P_{24} = \frac{1}{5}$$

$$P_{14} = \frac{1}{3}\cdot\frac{1}{5} = \frac{1}{15}, \quad P_{34} = \frac{1+\frac{2}{5}}{3} = \frac{7}{15}$$

$\to$ Prob. of going to 4 starting from 1 is $\frac{1}{15}$, prob. of going to 4 starting from 2 is $\frac{1}{5}$, prob. of going to 4 starting at 4 is 1.

(3) Assume that in the previous system, the prob of $i \to i-1$ is 0.2 for states 1,2,3,4.

a)

	0	1	2	3	4
0	1	0	0	0	0
1	0.2	0.7	0.1	0	0
2	0	0.2	0.7	0.1	0
3	0	0	0.2	0.7	0.1
4	0	0	0	0.2	0.8

b) Now only 0 is an absorbing state, the other 4 are transient states

c)
$$\begin{cases} \rho_{10} = 0.2 + 0.7\rho_{10} + 0.1\rho_{20} \to \rho_{10} = 1 \\ \rho_{20} = 0.2\rho_{10} + 0.7\rho_{20} + 0.1\rho_{30} \to 2\rho_{20} = 2\rho_{10} \to \rho_{20} = \rho_{10} \\ \rho_{30} = 0.2\rho_{20} + 0.7\rho_{30} + 0.1\rho_{40} \to 2\rho_{30} = 2\rho_{20} \to \rho_{30} = \rho_{20} \\ \rho_{40} = 0.2\rho_{30} + 0.8\rho_{40} \to 2\rho_{40} = 2\rho_{30} \to \rho_{40} = \rho_{30} \end{cases}$$

$\hookrightarrow$ The probability to end up in the absorbing state is always 1, as there is a single absorbing state.

(4) Assume that in the previous system, the prob of $i \to i-1$ is 0.2 for states 1,2,3,4, and prob $i \to i+1$ is 0.1 for states 0, 1, 2, 3.

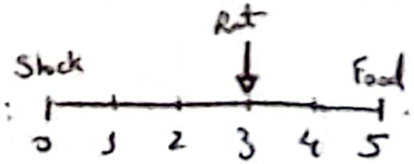
a)

	0	1	2	3	4
0	0.9	0.1	0	0	0
1	0.2	0.7	0.1	0	0
2	0	0.2	0.7	0.1	0
3	0	0	0.2	0.7	0.1
4	0	0	0	0.2	0.8

b) There are no absorbing states. However, as it is possible to get to every state independent from where you are, this would also mean that there are no transient states, and they are in fact all recurrent.

c) There are no absorbing states in this case.

(5) A rat is put into the linear maze:



a) Assume that the rat is equally likely to move right or left at each step. What is the probability that the rat finds food before getting shocked? b) As a result of learning, at each step the rat moves to the right with prob. $\frac{1}{2} + s$ and to the left with prob. $\frac{1}{2} - s$, where $0 < s < \frac{1}{2}$. What is the probability that the rat finds food before being shocked?

Correction: 5 a)

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0.5	0	0.5	0	0	0
2	0	0.5	0	0.5	0	0
3	0	0	0.5	0	0.5	0
4	0	0	0	0.5	0	0.5
5	0	0	0	0	0	1

$$\begin{cases} \rho_{15} = 0.5\rho_{25} + 0.5\cancel{\rho_{05}}^{0} \rightarrow \rho_{25} = 2\rho_{15} \\ \rho_{25} = 0.5\rho_{15} + 0.5\rho_{35} \rightarrow 1.5\rho_{15} = 0.5\rho_{35} \rightarrow 3\rho_{15} = \rho_{35} \\ \rho_{35} = 0.5\rho_{25} + 0.5\rho_{45} \rightarrow \rho_{35} = \rho_{15} + 0.5\rho_{45} \rightarrow 2\rho_{15} = 0.5\rho_{45} \rightarrow 4\rho_{15} = \rho_{45} \\ \rho_{45} = 0.5\rho_{35} + 0.5 \end{cases}$$

$$4\rho_{15} = 1.5\rho_{15} + 0.5$$

$$2.5\rho_{15} = 0.5 \rightarrow \rho_{15} = \frac{1}{5} \Rightarrow \boxed{\rho_{35} = \frac{3}{5}}$$

$$\begin{cases} \rho_{10} = 0.5 + 0.5\rho_{20} \rightarrow 4\rho_{40} = 0.5 + 1.5\rho_{40} \rightarrow \rho_{40} = \frac{1}{5} \\ \rho_{20} = 0.5\rho_{10} + 0.5\rho_{30} \rightarrow 2\rho_{40} = 0.5\rho_{10} \rightarrow \rho_{10} = 4\rho_{40} \\ \rho_{30} = 0.5\rho_{20} + 0.5\rho_{40} \rightarrow 1.5\rho_{40} = 0.5\rho_{20} \rightarrow \rho_{20} = 3\rho_{40} \\ \rho_{40} = 0.5\rho_{30} + 0.5\cancel{\rho_{50}}^{0} \rightarrow \rho_{30} = 2\rho_{40} \rightarrow \rho_{30} = \frac{2}{5} \end{cases}$$

$\Rightarrow$ The probability that the rat finds food before being shocked is 60%.

b)

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\frac{1}{2}-s$	0	$\frac{1}{2}+s$	0	0	0
2	0	$\frac{1}{2}-s$	0	$\frac{1}{2}+s$	0	0
3	0	0	$\frac{1}{2}-s$	0	$\frac{1}{2}+s$	0
4	0	0	0	$\frac{1}{2}-s$	0	$\frac{1}{2}+s$
5	0	0	0	0	0	1

$$\begin{cases} \rho_{15} = \left(\frac{1}{2}+s\right)\rho_{25} + \left(\frac{1}{2}-s\right)\cancel{\rho_{05}}^{0} \\ \rho_{25} = \left(\frac{1}{2}-s\right)\rho_{15} + \left(\frac{1}{2}+s\right)\rho_{35} \\ \rho_{35} = \left(\frac{1}{2}-s\right)\rho_{25} + \left(\frac{1}{2}+s\right)\rho_{45} \\ \rho_{45} = \left(\frac{1}{2}-s\right)\rho_{35} + \left(\frac{1}{2}+s\right) \end{cases}$$

$$\rho_{15} = \frac{1}{2}\rho_{25} + s\rho_{25} \rightarrow \rho_{25} = \frac{\rho_{15}}{\left(\frac{1}{2}+s\right)}$$

5 b) Cont.:

$$\begin{cases} p_{15} = \left(\frac{1}{2}+s\right)p_{25} + \cancel{\left(\frac{1}{2}-s\right)p_{05}}^{\,0} \\ p_{25} = \left(\frac{1}{2}-s\right)p_{15} + \left(\frac{1}{2}+s\right)p_{35} \\ p_{35} = \left(\frac{1}{2}-s\right)p_{25} + \left(\frac{1}{2}+s\right)p_{45} \\ p_{45} = \left(\frac{1}{2}-s\right)p_{35} + \left(\frac{1}{2}+s\right) \end{cases} \Rightarrow p_{25} = \left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)p_{25} + \left(\frac{1}{2}+s\right)p_{35}$$

$$p_{25}\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right) = \left(\frac{1}{2}+s\right)p_{35}$$

$$p_{25} = \frac{\left(\frac{1}{2}+s\right)p_{35}}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}$$

$$p_{35} = \frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)p_{35}}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)} + \left(\frac{1}{2}+s\right)p_{45} \Rightarrow p_{35}\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right) = \left(\frac{1}{2}+s\right)p_{45}$$

$$p_{35} = \frac{\left(\frac{1}{2}+s\right)p_{45}}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)} \Rightarrow p_{45} = \frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)p_{45}}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)} + \left(\frac{1}{2}+s\right)$$

$$p_{45}\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)}\right) = \frac{1}{2}+s \Rightarrow p_{45} = \frac{\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)}\right)}$$

$\Rightarrow$ The probability that the rat finds food before being shocked is given by

$$p_{35} = \frac{\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(\cancel{1}-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)} \cdot \frac{\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\frac{\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)}{\left(1-\left(\frac{1}{2}-s\right)\left(\frac{1}{2}+s\right)\right)}\right)}\right)} \quad //$$

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