A paradox of diffusion market model related with existence of winning combinations of options*

Nikolai Dokuchaev†

The Institute of Mathematics and Mechanics, St.Petersburg State University, Russia

March, 1999

We consider strategies of investments into options and diffusion market model. It is shown that there exists a correct proportion between "put" and "call" in the portfolio such that the average gain is almost always positive for a generic Black and Scholes model. This gain is zero if and only if the market price of risk is zero. It is discussed a paradox related to the corresponding loss of option's seller.

Key words: Diffusion market model, investment in options, Black and Scholes, winning strategy

JEL classification: D52, D81, D84

1 Introduction

There are many works devoted investment strategies dealing with risky assets (stocks). We consider strategies of investments into options for a generic stochastic diffusion model of a financial market. It is assumed that there is a risky stock and a risk-free asset (bond), and that it is available European options "put" and "call" on that stock at the initial time We consider only strategies of selecting options portfolio at the initial time. The selection of this portfolio id the only action of the investor; after that, he/she waits until the expiration time to accept gain or loss.

We show that there exist a correct proportion between "put" and "call" options in the portfolio such that the average gain is almost always positive. This gain is zero if and only

*Supported by Russian Foundation for Basic Research grant 99-01-00886

†Correspondence address: The Institute of Mathematics and Mechanics, St.Petersburg State University, Bibliotechnaya pl.2, Petrodvoretz, St.Petersburg, 198904, Russia. E-mail: dokuchaev@pobox.spbu.ru; fax 7(812)4286998

if the market price of risk is zero, i.e. when the appreciation rate of the stock is equal to the interest rate of the risk free asset (i.e. $a \neq r$ in (2.1)–(2.2) below). It is discussed a paradox related to the corresponding loss of option's seller.

2 Definitions

Consider the generic model of financial market consisting of a risk-free asset (bond, or bank account) with price B_t, and a risky asset (stock) with price S_t. We are given a standard probability space with a probability measure $\mathbf{P}$ and a standard Brownian motion w_t. The bond and stock prices evolve as

$$B_t = e^{rt} B_0, \tag{2.1}$$

$$dS_t = aS_t dt + \sigma S_t dw_t. \tag{2.2}$$

Here $r \geq 0$ is the risk-free interest rate, , $\sigma > 0$ is the volatility, and $a \in \mathbf{R}$ is the appreciation rate. We assume that $t \in [0, T]$, where $T > 0$ is a given terminal time. The equation (2.2) is Itô's equation, and can be rewritten as

$$S_t = S_0 \exp\left(at - \frac{\sigma^2 t}{2} + \sigma w_t\right).$$

We assume that it is available the options "put" and "call" on that stock for that price defined by the Black-Scholes formula (see e.g. Strong (1994)).

Further, we assume that σ, r, $B_0 > 0$ and $S_0 > 0$ are given, but the constant a is unknown.

Let $p_{BS}(S_0, K, r, T, \sigma)$ denote Black-Scholes price for "put" option, and $c_{BS}(S_0, K, r, T, \sigma)$ denote Black-Scholes price for "call" option. Here S_0 is the initial stock price, K is the strike price, r is the risk-free interest rate, σ is the volatility, T is the expiration time.

We remind the Black-Scholes formula. Let

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

$$d \triangleq \frac{\ln(S_0/K) + T(r + \sigma^2/2)}{\sigma\sqrt{T}}, \quad d^- = d - \sigma\sqrt{T}. \tag{2.3}$$

Then

$$c_{BS}(S_0, K, r, T, \sigma) = S_0 \Phi(d) - K e^{-rT} \Phi(d^-), \tag{2.4}$$

$$p_{BS}(S_0, K, r, T, \sigma) = c_{BS}(S_0, K, r, T, \sigma) - S_0 + K e^{-rT} \tag{2.5}$$

(see e.g. Strong (1994)-Duffie (1988)).

We denote by X_0 the initial wealth of an investor (i.e. at the initial $t = 0$), and we denote by X_T the wealth of the investor at the terminal time $t = T$.

Consider a vector (K_p, μ_p, K_c, μ_c), such that $K_p > 0, \mu_p \geq 0, K_c > 0, \mu_c \geq 0$. We shall use this vector to describe the following strategy: buy a portfolio of options which consists of μ_p "put" options with the strike price K_p and μ_c options with the strike price K_c, with the same expiration time T; thus,

$$X_0 = \mu_p p_{BS}(S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(S_0, K_p, r, T, \sigma), \tag{2.6}$$

(We assume that the options are available for the Black-Scholes price). We have assumed that the investor does not take any other actions until the expiration time. In that case, the terminal wealth at time $t = T$ will be

$$X_T = \mu_p (K_p - S_T)^+ + \mu_c (S_T - K_c)^+. \tag{2.7}$$

Definition 2.1 *The vector (K_p, μ_p, K_c, μ_c) is said to be a strategy.*

For a case of risk-free "hold-only-bond" strategy, $X_T = e^{rT} X_0$. It is natural to compare the results of any investment with the risk-free investment.

Definition 2.2 *The difference $\mathbf{E}X_T - e^{rT} X_0$ is said to be the average gain.*

Note that the appreciation rate a in this definition is fixed but unknown. The average gain for a strategy depends on $a - r$. For example, for a "call" option holder, when $(\mu_p = 0)$, the average gain is positive if $a > r$.

3 The result

Let d_p and d_c be defined by (2.3), where d is defined after substituting $K = K_p$ or $K = K_c$ correspondingly.

Theorem 3.1 *Let $\mu_p > 0, \mu_c > 0$ and*

$$\frac{\mu_c}{\mu_p} = \frac{1 - \Phi(d_p)}{\Phi(d_c)}. \tag{3.1}$$

Then the average gain for the strategy (K_p, μ_p, K_c, μ_c) is positive for any $a \neq r$, i.e.

$$\mathbf{E}X_T > e^{rT}X_0 \quad \forall a \neq r. \tag{3.2}$$

Moreover,

$$\mathbf{E}X_T = e^{rT}X_0 \quad \text{if} \quad a = r. \tag{3.3}$$

For any (K_p, K_c), the proportion (3.3) is the only one which ensures (3.1): for any other proportion μ_c/μ_p there exists $\in \mathbf{R}$ such that the average gain is negative.

Corollary 3.1 *Let the variable a be random, independent of $w(\cdot)$ and such that $\mathbf{P}(a \neq r) > 0$. Then $\mathbf{E}X_T > e^{rT}X_0$ for the strategy from Theorem 3.1.*

Set

$$Q(x,t) \stackrel{\Delta}{=} \mu_p p_{BS}(x, K_p, r, T - t, \sigma) + \mu_c c_{BS}(x, K_c, r, T - t, \sigma),$$
$$\Delta(x,t) = \frac{dQ(x,t)}{dx}. \tag{3.4}$$

Corollary 3.2 *Let (K_p, μ_p, K_c, μ_c) be such as in Theorem 3.1, then*

$$\Delta(S_0, 0) = 0. \tag{3.5}$$

In terms of Strong (1994), the equality (3.5) means that the position is risk-neutral at time $t = 0$.

Example. Consider example with same parameters as in Strong (1994)[p. 109]. Let $K_p = K_c = \$25$, $S_0 = \$30$, $T = 0.25$ (i.e. the expiration time is 3 months=25 years); $r = 0.05$ (i.e 5% annual), $\sigma = 0.45$ (i.e. 45% annual). Then $d = d_p = d_c = 0.978$, $\Phi(d) = 0.836$, $\mu_c/\mu_p = 164/836$. Let S_0 be arbitrary, $K_p = K_c = S_0$, then $d = d_p = d_c = 0.168$, $\Phi(d) = 0.567$, $\mu_c/\mu_p = 433/567$.

4 A consequence for the seller and a paradox

Consider the result for a seller (writer) who has sold the options portfolio described in Theorem 3.1. The seller has received the premium X_0 and must pay X_T at the time $t = T$.

Let Y_t be the wealth, which was obtained by the seller from $Y_0 = X_0$ by some self-financing strategy. Let $\widehat{Y}_T = Y_T - X_T$ be the terminal wealth after paying obligations to options holder at the expiration time T.

Let us consider possible actions of the sellers after receiving the premium. The following strategy is most commonly presented in textbooks devoted to mathematical aspects of option pricing:

Strategy I: *To replicate the claim X_T using the replicating strategy.*
As is known, the Black-Scholes price is defined as a minimum initial wealth such that the option's random claim can be replicated. For the Strategy I, the number of shares is $\Delta(S_t, t)$ at any time $t \in [0, T]$, and $\widehat{Y}_T = 0$ a.s., i.e. *there is not neither risk nor any gain.* Thus, it is doubtful that the seller uses this strategy in practice.

Furthermore, it was mentioned in Strong (1994)[c. 53] that in practice the option writer rather just keeps premium as a really compensation for bearing added risk of for foregoing future price appreciation or depreciation. Thus, the second strategy is the following:

Strategy II: *To invest the premium X_0 into bonds, take no further actions and wait the outcome of the price movement similarly as the option's holder.*

The seller who sells only put (or only call) options and uses the Strategy II puts his/her stake on the random events $K_p \leq S_T$ (or $S_T \leq K_c$ correspondingly). A gain of the holder implies a loss for the writer. It looks as a fair game in a case of selling either put or call separately, because we know that the Black-Scholes price is fair and chances for gain should be equal for buyer and seller (otherwise, either ask or bid will prevail). But Theorem 3.1 implies that that the seller will receive non-positive in average gain which is negative in average for any $a \neq r$ if he/she sells *the combination described in Theorem 3.1.* In other words, we have a paradox:

A combination of two "fair" deals of selling puts and calls gives an "unfair" deal.

The second paradox can be formulated as following:

We know that the Black-Scholes price is fair for buyers as well as for sellers (otherwise, either ask or bid will prevail). However, we found an options combination such that buying is preferable, because the buyer has non-negative and almost always positive average gain, but the seller has either zero or negative average gain.

In practice, unlike as for our generic model, brokers use sophisticated measures such as insurance, long positions in stocks or other options, etc. to reduce the risk, but that does not affect the core of the paradoxes.

A possible explanation is that writer also use the premium X_0 to can receive a gain from $a \neq r$ *by using some other strategies* such as Merton's strategies which are possibly more effective then options portfolio. In other words, *a rational option's seller does not use neither risk-free replicating of claims nor "keep-only-bonds" strategy; he/she rather uses strategies which are able to explore $a \neq r$.*

5 Proofs

Proof of Theorem 3.1. Let $\mathbf{P}_a$ be the conditional probability measure given a. Let $\mathbf{E}_a$ be the corresponding expectation. We denote $\mathbf{E}_*$ the expectation which corresponds the risk neutral measure, when $a = r$. Set

$$h(a) \triangleq \mathbf{E}_a X_T.$$

By the definitions of X_T, it follows that

$$h(a) = \mu_p \mathbf{E}_* (K_p - e^{(a-r)T} S_T)^+ + \mu_c \mathbf{E}_* (e^{(a-r)T} S_T - K_c)^+. \tag{5.1}$$

As is known (see e.g. Strong (1994) and Duffie (1988)), the Black-Scholes price can be presented as

$$\begin{aligned} p_{BS}(S_0, K, r, T, \sigma) &= e^{-rT} \mathbf{E}_* (K - S_T)^+, \\ c_{BS}(S_0, K, r, T, \sigma) &= e^{-rT} \mathbf{E}_* (S_T - K)^+. \end{aligned} \tag{5.2}$$

Then

$$e^{-rT} h(a) = \mu_p p_{BS}(e^{(a-r)T} S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(e^{(a-r)T} S_0, K_c, r, T, \sigma).$$

By the put and call parity formula, it follows that

$$e^{-rT}h(a) = \mu_p\Big[c_{BS}(e^{(a-r)T}S_0, K_p, r, T, \sigma) - e^{(a-r)T}S_0 + K_p\Big] + \mu_c c_{BS}(e^{(a-r)T}S_0, K_c, r, T, \sigma).$$

The following proposition is well known (see e.g. Strong (1994)[p. 100]).

Proposition 5.1 *For any $T > 0, K > 0$, the following holds:*

$$\frac{\partial}{\partial x} c_{BS}(x, K, r, T, \sigma) = \Phi(d), \tag{5.3}$$

where d is defined by (2.3).

Let $d_c(a)$ and $d_p(a)$ be defined as d_p and d_c correspondingly with substituting $e^{(a-r)T}S_0$ as S_0. Set

$$y = y(a) \triangleq e^{(a-r)T}, \quad R(y) \triangleq h(T^{-1}\ln y).$$

We have that

$$R(y(a)) \equiv h(a),$$

then

$$\frac{dR(y)}{dy} = e^{rT}S_0\left[(\mu_p(\Phi(d_p(a)) - 1) + \mu_c\Phi(d_c(a))\right].$$

By (3.1) it follows that

$$\frac{dR}{dy}(y)|_{y=1} = 0. \tag{5.4}$$

(Note that $y = 1$ if and only if $a = 1$.) It is know that

$$\frac{\partial^2}{\partial x^2} p_{BS}(x, K_p, 0, T, \sigma) > 0, \quad \frac{\partial^2}{\partial x^2} c_{BS}(x, K_p, 0, T, \sigma) > 0$$

and the derivatives exist. Then $R''(y) > 0$ $(\forall y)$, i.e. $R(\cdot)$ is strongly convex. It follows that $a = r$ is the only one which minimize h. By (2.6), (5.1) it follows that

$$R(1) = e^{rT}X_0.$$

The uniqueness of the proportion (3.1) follows from the uniqueness of μ_c/μ_p which ensures (5.4). This completes the proof of Theorem 3.1.

Corrolary 3.1 follows from (3.2)–(3.3). Corrolary 3.2 follows from (5.3).

References

[1] Strong R. Speculative markets. 2nd ed. Harper Collins College Publishers, 1994.

[2] Duffie D. Security markets. Academic Press. Inc, 1988.