



# BOOLEAN ALGEBRA: PART 1

- Principles
- Boolean functions



A Boolean algebra consists of

A set of elements  $B$

Binary operators  $\{+, \cdot\}$       Boolean sum and product

A unary operation  $\{ '\}$  (or  $\{ \bar{\phantom{x}} \}$ )      example:  $A'$  or  $\bar{A}$

...and the following axioms

1. The set  $B$  contains at least two elements  $\{a, b\}$  with  $a \neq b$
2. Closure:       $a+b$  is in  $B$        $a \cdot b$  is in  $B$        $a'$  is in  $B$
3. Commutative:       $a+b = b+a$        $a \cdot b = b \cdot a$
4. Associative:       $a+(b+c) = (a+b)+c$        $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
5. Two neutral elements denotes 0 and 1:       $a+0 = a$        $a \cdot 1 = a$
6. Distributive:       $a+(b \cdot c) = (a+b) \cdot (a+c)$        $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$
7. Complementarity: for every element  $a$ ,  $a'$  exists such that  
$$a+a' = 1 \qquad a \cdot a' = 0$$

# BOOLEAN ALGEBRA: EXHAUSTIVE DEFINITION



A SET OF TWO ELEMENTS denoted 0 and 1 and 3 operators defined below by Truth tables.

AND  $a \cdot b$  or  $ab$

a	b	ab
0	0	0
0	1	0
1	0	0
1	1	1

OR  $a+b$

a	b	a+b
0	0	0
0	1	1
1	0	1
1	1	1

NOT  $\overline{a}$  or  $a'$

a	a'
0	1
1	0



Substitute

$\{0, 1\}$  for  $B$

AND for  $\cdot$  Boolean Product. In logic this was  $\wedge$

OR for  $+$  Boolean Sum. In logic this was  $\vee$

NOT for  $'$  Complement. In logic this was  $\neg$

All the axioms hold for binary logic



Ambiguity:

$a + bc = a + (bc)$  or  $a+bc = (a+b) c$  ?

Priority rule: “and” has priority (must be executed before) over “or”.

So  $A + BC = A + (BC)$

Same rule as in standard arithmetic: multiplication has priority over addition.

Variables and their complements are sometimes called literals



Absorption:  $a + 1 = 1$

Dual:  $a \cdot 0 = 0$

Idempotent:  $a + a = a$

Dual:  $a \cdot a = a$

Idempotent:  $(a')' = a$

Uniting:  $a \cdot b + a \cdot b' = a$

Dual:  $(a+b) \cdot (a+b') = a$



Absorption:  $a + a \cdot b = a$

Dual:  $a \cdot (a + b) = a$

Absorption (#2):  $(a + b') \cdot b = a \cdot b$

Dual:  $(a \cdot b') + b = a + b$

de Morgan's:  $(a + b + \dots)' = a' \cdot b' \cdot \dots$

Dual:  $(a \cdot b \cdot \dots)' = a' + b' + \dots$

Multiplying & factoring:  $(a + b) \cdot (a' + c) = a \cdot c + a' \cdot b$

Dual:  $a \cdot b + a' \cdot c = (a + c) \cdot (a' + b)$

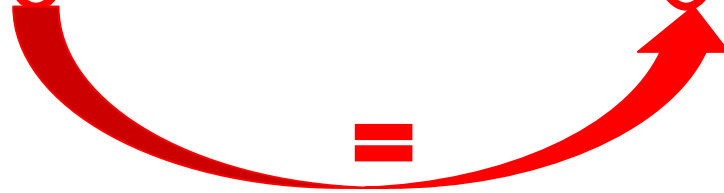
Consensus:  $(a \cdot b) + (b \cdot c) + (a' \cdot c) = a \cdot b + a' \cdot c$

Dual:  $(a + b) \cdot (b + c) \cdot (a' + c) = (a + b) \cdot (a' + c)$

# DEMONSTRATION USING TRUTH TABLE



a	b	(a+b)	(a+b')	(a+b)(a+b')
0	0	0	1	0
0	1	1	0	0
1	0	1	1	1
1	1	1	1	1

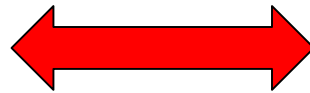


$$(a+b)(a+b') = a$$





abc	F(a,b,c)
000	0
001	0
010	1
011	1
100	0
101	1
110	0
111	1



Algebraic format  
 $F = A'B + AC$

Truth Table

REMARK: easy to derive Truth Table from algebraic expression  
 but going in the other direction is not straightforward



NAND  $(a \cdot b)'$  or  $(ab)'$

a	b	$(ab)'$
0	0	1
0	1	1
1	0	1
1	1	0

NOR  $(a+b)'$

a	b	$(a+b)'$
0	0	1
0	1	0
1	0	0
1	1	0



XOR (Ou Exclusif)  $a \oplus b$

<b>a</b>	<b>b</b>	<b><math>a \oplus b</math></b>
0	0	1
0	1	0
1	0	0
1	1	1



**DEFINITION:** For a boolean function of  $n$  variables  $x_{(n-1)}, x_{(n-2)}, \dots, x_2, x_1, x_0$ , a product term in which each of the  $n$  variables appears **at most once** (either in its complemented or uncomplemented form) is called a **general minterm**.

Thus, a **general minterm** is a logical expression of  $n$  variables that employs only the complement operator and the disjunction (AND) operator.

**DEFINITION (Wikipedia):** For a boolean function of  $n$  variables  $x_{(n-1)}, x_{(n-2)}, \dots, x_2, x_1, x_0$ , a product term in which each of the  $n$  variables appears **exactly once** (either in its complemented or uncomplemented form) is called a **(complete) minterm**.



Let us consider a  $n$  bit vector  $V = (v_{(n-1)}, v_{(n-2)}, \dots, v_2, v_1, v_0)$ , the complete minterm associated with  $V$  is defined by the formula

$$\text{Min}_V(x_{(n-1)}, x_{(n-2)}, \dots, x_2, x_1, x_0) = (1 \oplus x_{(n-1)} \oplus v_{(n-1)}) (1 \oplus x_{(n-2)} \oplus v_{(n-2)}) \dots (1 \oplus x_1 \oplus v_1) (1 \oplus x_0 \oplus v_0)$$



# MINTERM/MAXTERM: VECTOR /NUMBER



VECTOR	Minterm	Number	Name
(0,0,0,1,0)	$x'_4 \cdot x'_3 \cdot x'_2 \cdot x_1 \cdot x'_0$	2	$m_2$
(1,0,0,0,1)	$x_4 \cdot x'_3 \cdot x'_2 \cdot x'_1 \cdot x_0$	17	$m_{17}$
(1,0,0,1,1)	$x_4 \cdot x'_3 \cdot x'_2 \cdot x_1 \cdot x_0$	19	$m_{19}$
(0,1,1,1,0)	$x'_4 \cdot x_3 \cdot x_2 \cdot x_1 \cdot x'_0$	14	$m_{14}$

VECTOR	Maxterm	Number	Name
(0,0,0,1,0)	$x_4 + x_3 + x_2 + x'_1 + x_0$	2	$M_2$
(1,0,0,0,1)	$x'_4 + x_3 + x_2 + x_1 + x'_0$	17	$M_{17}$
(1,0,0,1,1)	$x'_4 + x_3 + x_2 + x'_1 + x'_0$	19	$M_{19}$
(0,1,1,1,0)	$x_4 + x'_3 + x'_2 + x'_1 + x_0$	14	$M_{14}$

# COMPLETE MINTERMS 3 VARIABLES



VECTOR	Minterm	Number	Name
(0,0,0)	$x'_2 \cdot x'_1 \cdot x'_0$	0	$m_0$
(0,0,1)	$x'_2 \cdot x'_1 \cdot x_0$	1	$m_1$
(0,1,0)	$x'_2 \cdot x_1 \cdot x'_0$	2	$m_2$
(0,1,1)	$x'_2 \cdot x_1 \cdot x_0$	3	$m_3$
(1,0,0)	$x_2 \cdot x'_1 \cdot x'_0$	4	$m_4$
(1,0,1)	$x_2 \cdot x'_1 \cdot x_0$	5	$m_5$
(1,1,0)	$x_2 \cdot x_1 \cdot x'_0$	6	$m_6$
(1,1,1)	$x_2 \cdot x_1 \cdot x_0$	7	$m_7$

# COMPLETE MAXTERMS 3 VARIABLES



VECTOR	Maxterm	Number	Name
(0,0,0)	$x_2+x_1+x_0$	0	$M_0$
(0,0,1)	$x_2+x_1+x'_0$	1	$M_1$
(0,1,0)	$x_2+x'_1+x_0$	2	$M_2$
(0,1,1)	$x_2+x'_1+x'_0$	3	$M_3$
(1,0,0)	$x'_2+x_1+x_0$	4	$M_4$
(1,0,1)	$x'_2+x_1+x'_0$	5	$M_5$
(1,1,0)	$x'_2+x'_1+x_0$	6	$M_6$
(1,1,1)	$x'_2+x'_1+x'_0$	7	$M_7$





Theorem: Let  $F$  be a Boolean function and  $\text{Sup}(F)$  (called support of  $F$ ) the set of Boolean vectors for which  $F$  value is 1, then

$$F = \sum_{V \text{ in } \text{Sup}(F)} \text{Min}_V$$



Theorem: Let  $F$  be a Boolean function and  $ASup(F)$  (called AntiSupport of  $F$ ) the set of Boolean vectors for which  $F$  value is 0, then

$$F = \prod_{V \text{ in } ASup(F)} \text{Max}_V$$

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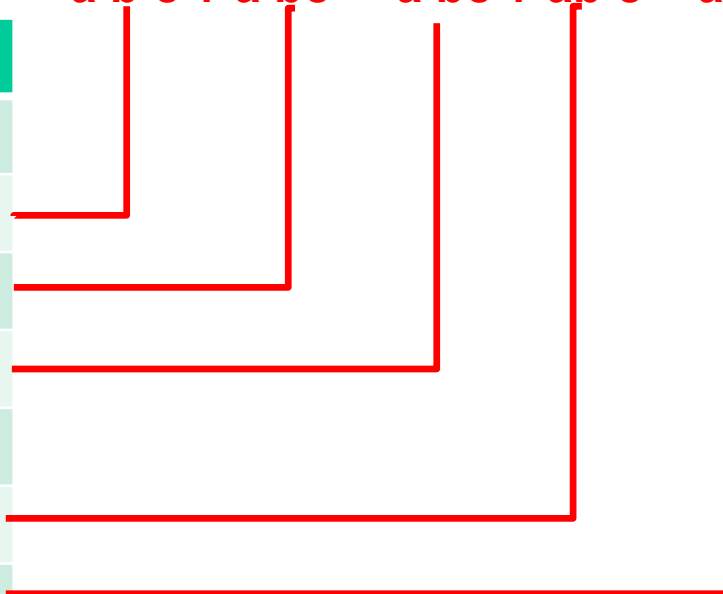
# MINTERM CANONICAL FORM (EXAMPLE)



$$m_1 + m_2 + m_3 + m_5 + m_6 = F(a,b,c) = \sum m(1,2,3,5,6)$$

$$a'b'c + a'bc' + a'bc + ab'c + abc'$$

	abc	F(a,b,c)
0	000	0
1	001	1
2	010	1
3	011	1
4	100	0
5	101	1
6	110	1
7	111	0



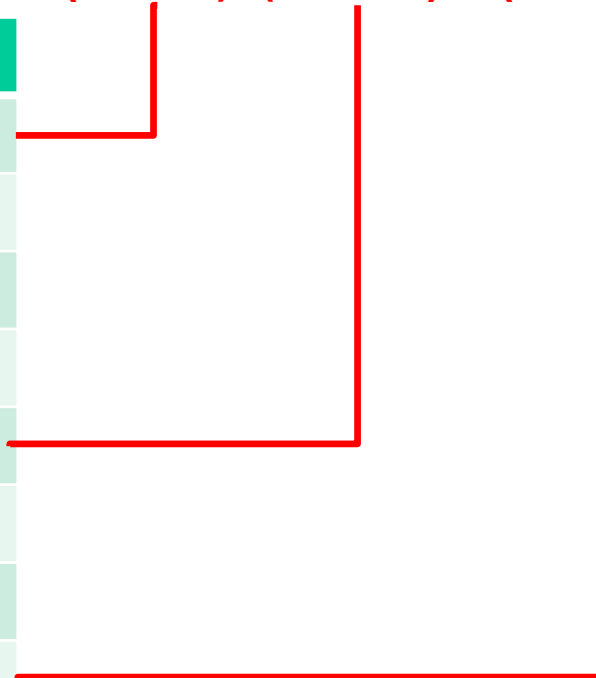
# MAXTERM CANONICAL FORM (EXAMPLE)



$$M_0 \cdot M_4 \cdot M_7 = F(a,b,c) = \prod M(0,4,7)$$

$(a+b+c) (a'+b+c) + (a'+b'+c')$

	abc	F(a,b,c)
0	000	0
1	001	1
2	010	1
3	011	1
4	100	0
5	101	1
6	110	1
7	111	0





A lot of material was found in Wikipedia.

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