TD 9

→ Produits matrice-vecteur

1) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\times1+0\times2+3\times-1 \\ 1\times4+0\times5+6\times-1 \\ 7\times1+8\times0-1\times9 \\ 10\times1+11\times0-12\times-1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 10\times1 \\ 2\times-7 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -14 \\ 12 \\ 0 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\times1 \\ 1\times3+2\times-1+0\times-1 \\ 1\times3+-1\times2+3\times4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix}$

→ Produit Matrice-Matrice

1)

Pour que A×B existe → lignes B = colonnes A

$$\begin{array}{cc} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} & \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & 2 \\ -2 & -8 & 2 \end{pmatrix} \end{array}$$

$1\times1+3\times-1=-2$

$1\times4+-6\times1=-2$

$1\times7+-1\times9=-2$

$-1\times1+3\times1=2$

$-1\times4+6\times1=2$

$-1\times7+9\times1=2$

2)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 2 & 4 & 6 \\ 6 & 9 & 3 & 6 \\ 6 & 6 & 6 & 9 \end{pmatrix}$$

$1\times1+1\times3+1\times2=6$

$1\times2+1\times1+1\times3=6$

$1\times3+1\times2+1\times1=6$

$1\times4+1\times3+1\times2=9$

→ Forme échelonnée

1) $\begin{pmatrix} 3 & 1 & -4 & 0 & 5 \\ 2 & 0 & 0 & -1 & 9 \\ 0 & -5 & 3 & 2 & 0 \end{pmatrix}$

$L_1 \leftarrow 2L_1$ et $L_2 \leftarrow 3L_2$ $\begin{pmatrix} 6 & 2 & -8 & 0 & 10 \\ 6 & 0 & 0 & -3 & 27 \\ 0 & -5 & 3 & 2 & 0 \end{pmatrix}$

$L_2 \leftarrow L_2 - L_1$ $\begin{pmatrix} 6 & 2 & -8 & 0 & 10 \\ 0 & -2 & 8 & -3 & 17 \\ 0 & -5 & 3 & 2 & 0 \end{pmatrix}$

$L_2 \leftarrow 5L_2$ et $L_3 \leftarrow 2L_3$ $\begin{pmatrix} 6 & 2 & -8 & 0 & 10 \\ 0 & -10 & 40 & -15 & 85 \\ 0 & -10 & 6 & 4 & 0 \end{pmatrix}$

$L_3 \leftarrow L_3 - L_2$ $\begin{pmatrix} 6 & 2 & -8 & 0 & 10 \\ 0 & -10 & 40 & -15 & 85 \\ 0 & 0 & -34 & 19 & -85 \end{pmatrix}$

$L_1 \leftarrow 1/2\, L_1$ et $L_2 \leftarrow 1/5\, L_2$ $\begin{pmatrix} 3 & 1 & -4 & 0 & 5 \\ 0 & -2 & 8 & -3 & 17 \\ 0 & 0 & -34 & 19 & -85 \end{pmatrix}$

on recrit de bas à haut

2) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 8 & 4 & 2 & 1 \\ 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 8 & 4 & 2 & 1 \\ 8 & 4 & 2 & 0 \\ 8 & 4 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{pmatrix}$ $L_2 \leftarrow 2L_2$, $L_3 \leftarrow 4L_3$, $L_4 \leftarrow 8L_4$

$\begin{pmatrix} 8 & 4 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & -4 & -2 & -1 \end{pmatrix}$ $L_2 \leftarrow L_2 - L_1$, $L_3 \leftarrow L_3 - L_1$, $L_4 \leftarrow L_4 - L_1$ $\begin{pmatrix} 8 & 4 & 2 & 1 \\ 0 & -4 & -2 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $L_2 \leftrightarrow L_4$

$L_2 \leftarrow -L_2$, $L_3 \leftarrow -L_3$, $L_4 \leftarrow -L_4$ $\begin{pmatrix} 8 & 4 & 2 & 1 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $L_1 \leftarrow L_1 - L_2$, $L_2 \leftarrow L_2 - L_3$, $L_3 \leftarrow L_3 - L_4$ $\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $/8$, $/4$, $/2$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Systèmes linéaires

1) $\begin{cases} 3x + y - 4z = 12 \\ \quad 2y - z = -1 \\ \quad\quad 3z = 6 \end{cases} \iff \begin{pmatrix} 3 & 1 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \\ 6 \end{pmatrix}$

$\iff \left(\begin{array}{ccc|c} 3 & 1 & -4 & 12 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right) \underset{L_3 \leftarrow 1/3 L_3}{\iff} \left(\begin{array}{ccc|c} 3 & 1 & -4 & 12 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$

$L_1 \leftarrow +4L_3$

$L_2 \leftarrow +L_3 \iff \left(\begin{array}{ccc|c} 3 & 1 & 0 & 20 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \underset{L_1 \leftarrow 2L_1}{\iff} \left(\begin{array}{ccc|c} 6 & 2 & 0 & 40 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \underset{L_1 \leftarrow -L_2}{\iff} \left(\begin{array}{ccc|c} 6 & 0 & 0 & 39 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$

$\iff \left(\begin{array}{ccc|c} 1 & 0 & 0 & 13/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \end{array}\right)$ $\begin{matrix} L_1 \leftarrow /6 \\ L_2 \leftarrow /2 \\ L_3 \leftarrow L_3 \end{matrix}$ $\iff \begin{cases} x = 13/2 \\ y = 1/2 \\ z = 2 \end{cases}$

2) $\begin{cases} x + y + z = 3 \\ 2x + 7y - 2z = -1 \\ -x - y + z = 3 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 7 & -2 & -1 \\ -1 & -1 & 1 & 3 \end{array}\right) \underset{L_2 \leftarrow -2L_1}{\iff} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 5 & -4 & -7 \\ 0 & 0 & 2 & 6 \end{array}\right)$

$L_3 \leftarrow /2$

$\iff \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 5 & -4 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right) \underset{\substack{L_1 \leftarrow -L_3 \\ L_2 \leftarrow +4L_3}}{\iff} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right) \underset{L_2 \leftarrow /5}{\iff} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

$L_3 \leftarrow +L_1$

$L_1 \leftarrow -L_2$

$\iff \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

3) $\begin{cases} t+x+y+z=1 \\ 2t+x+3y+z=7 \\ 2t+2x+y+2z=7 \end{cases} \iff \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 7 \\ 2 & 2 & 1 & 2 & 7 \end{array}\right)$

$L_2 \leftarrow -2L_1 \quad L_3 \leftarrow -2L_1$

$\iff \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 5 \\ 0 & 0 & -1 & 0 & 5 \end{array}\right) \iff \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 6 \\ 0 & -1 & 0 & -1 & 10 \\ 0 & 0 & -1 & 0 & 5 \end{array}\right)$

$L_1 \leftarrow +L_3 \quad L_2 \leftarrow +L_3$

$\iff \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 16 \\ 0 & -1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & -5 \end{array}\right) \iff \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 16 \\ 0 & 1 & 0 & 1 & -10 \\ 0 & 0 & 1 & 0 & -5 \end{array}\right)$

$L_1 \leftarrow +L_2 \quad L_3 \leftarrow -L_3 \qquad L_2 \leftarrow -L_2$

$\iff x = 16 \quad y = -10 - t \quad z = -5$

4) $\begin{cases} x+y+2z=4 \\ y-z=-2 \\ -2x-2y-4z=-1 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ -2 & -2 & -4 & -1 \end{array}\right) \iff \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{array}\right)$

$L_3 \leftarrow +2L_1$

$\iff$ aucune solution car on a $0=7$ ce qui est impossible

→ Matrices à coefficients modulaires

1) $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$1+1+(-1)=1 \bmod 3 = 1$

$1+1+1=3 \bmod 3 = 0$

$-1 = -1 \times 3 + 2 \to -1 \bmod 3 = 2$

2) $= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$