

Master's Thesis

- Université Paris-Dauphine -

MSc 203 – Financial Markets

The valuation and hedging of multi-underlying autocallable structured products

By

Raphael LEDERMAN

Master's Thesis Tutor: Aymeric KALIFE

Master's Thesis Supervisor: Gaëlle LE FOL

April 27, 2017

With the help and support of Société Générale's exotic equity derivatives trading team

Abstract

This paper has been written in the context of an internship in exotic equity derivatives trading with the team specialized in single stocks at Société Générale.

The purpose of this Master's Thesis is to analyze the challenges linked to the valuation and hedging processes of multi-underlying Autocallables. We will provide a detailed analysis of the structure and components of such exotic products, and study the various expositions they generate from the point of view of the investor. We will then present the characteristics of pricing models involved in Autocallables valuation, with a particular focus on non-constant volatility and asynchronous correlation models. These theoretical explanations will allow us identifying the main issue linked to Autocallables risk management: the ubiquity of discontinuities throughout the entire payoff. Such discontinuities imply potentially explosive sensitivities, and we will show how the use of gaps can help smoothing the Greeks. Moreover, we will propose a thorough presentation of the Longstaff-Schwartz Monte Carlo numerical resolution method, which is the main algorithm used at Société Générale to value Autocallables and gaps. Finally, we will study different derivatives, from Vanilla options to exotic products (Variance Swaps, Calls vs. Call...), used to implement consistent hedging strategies in order to mitigate the expositions arising from the sale of multi-underlying autocallable structures.

Table of Contents

1. Introduction	4
2. Presentation of the product's structure and sensitivities	5
2. 1. The structure of a typical multi-underlying Autocallable on Worst-Of	5
2. 2. Reminder on the sensitivities: the Greeks	7
2. 3. The product's expositions from the investor's perspective	10
3. Theoretical models used to provide consistent quotes and price autocallable structures	15
3. 1. Volatility models: from the constant Black-Scholes assumption to stochastic diffusion.....	15
3. 1. 1. The limits of a constant volatility model	16
3. 1. 2. Local volatility models: skew dynamics and term structure.....	16
3. 1. 3. A few words on stochastic volatility models	19
3. 2. Correlation models: from historical to implied correlation	20
3. 2. 1. The case of correlation in asynchronous markets.....	21
3. 2. 2. The implied correlation quoting process	23
3. 3. Succinct presentation of interest rate, foreign exchange and dividend models	25
4. Analysis of the challenges linked to the hedging process	26
4. 1. The crucial smoothing of discontinuities: the impact of gaps	26
4. 1. 1. European digitals and triangular gaps.....	27
4. 1. 2. American digitals and rectangular gaps	28
4. 1. 3. Autocallables and intelligent gaps.....	29
4. 2. Longstaff-Schwartz Monte Carlo: the essential algorithm used to value autocallable structures and gaps.....	31
4. 2. 1. Presentation of the algorithm's purpose and method	31
4. 2. 2. Numerical illustration	32
4. 3. Products and strategies available to hedge volatility exposition	37

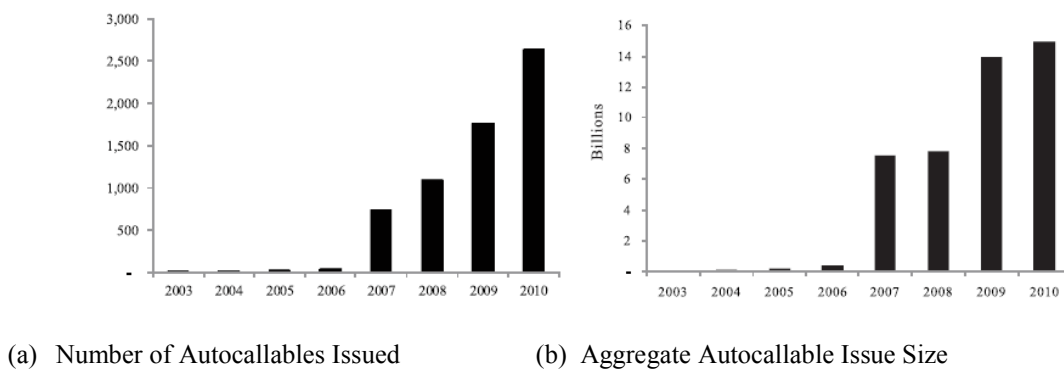
4. 3. 1. Main issues linked to the hedging of a regular Down-and-In Put option	37
4. 3. 2. Volatility hedging with Variance and Volatility Swaps	38
4. 3. 3. Volatility hedging with Vanilla options.....	40
4. 4. Correlation exposition hedging and dispersion strategies	45
4. 4. 1. A widely used correlation trade: the Calls vs. Call	45
4. 4. 2. The potential benefits of dispersion strategies.....	46
5. Conclusion.....	48
6. References	50
7. Appendix	51
7. 1. Some precisions regarding the pricing of PDI barrier options	51
7. 2. Empirical study: the implementation of a new implied correlation Bid quoting methodology.....	51

1. Introduction

“Toxic waste...it is a sad day when derivatives are described as toxic waste. Are these financial products really so, particularly those of exotic nature, or is it in fact people’s grasp and usage of them that is the source of toxicity?”. This open-ended question put by Mohamed Bouzoubaa in the introduction of his book *Exotic Options and Hybrids – A Guide to Structuring, Pricing and Trading* reflects well the commonly accepted fixed ideas linked to structured products.

Structured products first appeared on the 15th of August 2003 and were created to meet investors’ precise needs in terms of payoff and exposition as they allow a yield enhancing strategy. Unlike classic Vanilla options, they have a wide variety of features (averaging, barriers, triggers...) that makes them more complex to value and dynamically hedge all through their life. Indeed, such features can imply discontinuities in the payoff, leading to potentially explosive Greeks around certain thresholds: there comes the preponderant concept of smoothing, that is to say using additional derivatives in order to mitigate the variation of sensitivities. As an exotic derivatives trader, it is absolutely indispensable to be able to fully understand and monitor these additional features as well as different kinds of high order risk. The exotic derivatives market has been growing at a very fast pace for the past fifteen years, with investment banks constantly innovating in order to meet more and more specific requests: as always, competitiveness is the key, and tailor-made structured products are becoming a commonplace. But even if rivalry between banks or financial technology firms is getting tougher, and some structures are reaching high levels of complexity, it is possible to observe the progressive standardization of certain products.

Figure 1: Number and Total Issue Size of Autocallable Structured Products, January 2003 – June 2010, *Source Modeling Autocallable Structured Products*



There exist a large variety of structured products, and their characteristics depend on the profile of the investor. An important part of the business is generated by Private Banking counterparties: most of their clients are high net worth individuals or family offices mainly willing to take long positions on major industries through relatively standard structured products. Indeed, for compliance and regulatory purposes, structured products sold to retail clients mustn’t contain more than four “extra features” (activation barriers, conditional coupons, averaging of the performance, memory effect...). Moreover, as the financial crisis had an impact on investment mentalities and strategies, a major part of the deals now consist in “guaranteed capital” products. Such products can allow investors to get back their invested capital as long as certain conditions (like performance thresholds) are satisfied: this “risk-free” component (which stays effectively risk-free as long as the issuer does not default) makes it more attractive for investors in volatile and risky environment. While retail clients are often aimed at with these relatively basic structured products, a considerable part of the business is more oriented toward hedge funds. As they are willing to invest into more complex products, it leaves room for compulsive innovation: a higher number of extra features, and customized

expositions to a wide range of variables. But as this part of the activity is profitable, it also requires the implementation of powerful algorithms and pricing tools: the different models used to value some of these products require the generation and calibration of an increasing amount of parameters (with stochastic volatility models for instance). But with the financial crisis, and the implementation of numerous regulatory processes, relatively standard and transparent products have re-gained a lot of attractiveness. This is why in terms of nominal and exposition, they represent a consequent part of exotic derivatives books. TypiCally, Autocallables (also Called “auto-trigger” structures) have become very popular during the last decade: they now represent a vast majority of the exotic derivatives desks’ activity thanks to their relative simplicity and interesting yield levels (of course the level of simplicity can vary greatly depending on the additional features added to the structure). These products, characterized by a lower volatility than regular equity investments, are the perfect answer to the general post-crisis resentment and misunderstanding on exotic derivatives. Moreover, while being reasonably simple and accessible to all kind of investors, they are still a source of competitiveness and challenges for traders.

This is why we chose to lead an in-depth study on a typical multi-underlying autocallable structure and its implications in terms of pricing and risk management. Throughout this Master’s Thesis, we will therefore answer the following question: what are the various sensitivities linked to a typical multi-underlying autocallable structure, and how do they impact its valuation and hedging process?

We will first center our analysis on the different expositions and sensitivities implied by the Autocallable from the point of view of the investor. After having examined the different models used to price autocallable structures, with a particular focus on non constant volatility and asynchronous correlation models, we will then study the challenges linked to discontinuous payoffs risk management and sensitivities smoothing. Finally, we will bring answers to the crucial question of building consistent hedging strategies. Please note that we did our best to incorporate empirical studies all through this paper, including the analysis of an Autocallable’s sensitivities to parameters shifts, the impact of gaps on the smoothing of Greeks, the building of volatility hedging strategies and the design of a new methodology in order to generate implied correlation Bid quotes.

2. Presentation of the product’s structure and sensitivities

Let’s now present the structure and characteristics of a typical multi-underlying auto-Callable structured product from the point of view of the investor. Understanding the product’s dynamics and sensitivities “in the shoes” of the client is key in order to fully grasp the challenges linked to its pricing and hedging.

1. 1. The structure of a typical multi-underlying Autocallable on Worst-Of

It is composed of a short Worst-Of European Put Down-and-In that gets activated if the worst performing stock of the basket crosses a pre-determined barrier at maturity. If the PDI is activated, the investor is exposed to capital losses and his final redemption amount is indexed to the performance of the worst performing stock of the basket.

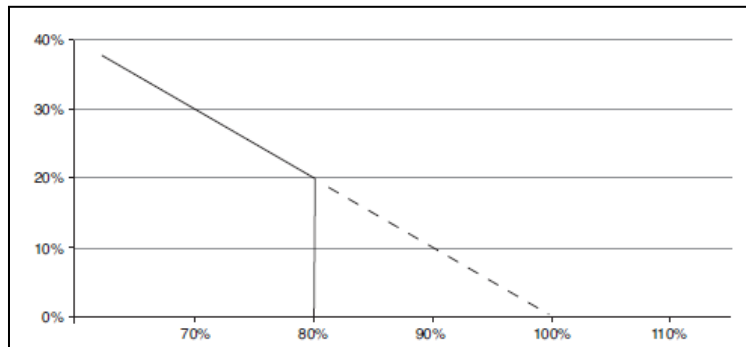
Here is the payoff of the Worst-Of Put Down-and-In:

$$(1) \quad WOF \text{ PDI} = \max[0; K - \min(\text{Perf}_i(T))] \times \mathbf{1}_{\{\min_{i=1,\dots,n} \min_{t \in [0,T]} \text{Perf}_i(t) \leq H\}}$$

where $\text{Perf}_i(t) = \frac{S_i(t)}{S_i(0)} - 1$ is the return at time t of the i th asset of the basket.

The WOF feature makes this option more expensive than the regular DI Put option with same characteristics: this is why it is very widely used in the context of yield enhancement or for creating more upside participation. Indeed, the investor accepts to bear the risk from selling the PDI in order to generate more funding that is used in the structured product to increase the participation.

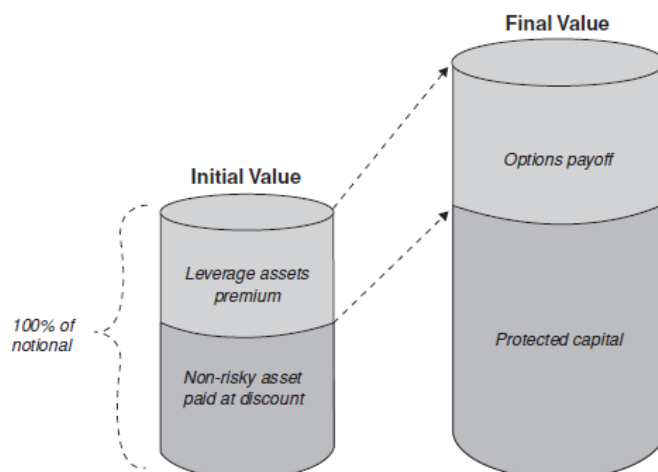
Figure 2: Payoff of a long Worst-Of Down-and-In At-the-Money Put option with a knock-in barrier at 80%, *Source Exotic Options and Hybrids – A Guide to Structuring, Pricing and Trading*



If the worst performing underlying crosses the automatic recall barrier at a valuation date, the product is automatically recalled, and pays a certain coupon. The autocallability feature can be whether daily, monthly, yearly, or based on any schedule determined in accordance with the client. This automatic recall component in the structure corresponds to a digital Call option, that is to say an option with a binary payoff (the concept of digital risk will be detailed more in the following parts). A typical automatic recall barrier is 100% of the initial strike level (that is to say equivalent to a null performance). If the WOF stays below the autocall barrier throughout the entire product's life, and is above the Down-and In Knock-In threshold at maturity, the client gets back his guaranteed capital at the expiration date, plus a Bonus coupon (this part of the structure can correspond to a zero coupon bond). Moreover, it is possible to add an Eagle coupon feature: if the worst performing stock stays between the Eagle limit and the autocall limit at pre-determined valuation dates (the Eagle limit being superior to the WOF PDI Knock-In threshold and inferior to the automatic Recall barrier), the investor can earn a certain coupon.

The protected capital feature is made possible by the particular “wrapper” of the product: it is issued as a European Medium Term Note (structured note). This wrapper has a pre-determined legal status and allows meeting the clients' needs in terms of regulatory issues. The structured note is composed of a non-risky asset providing a percentage of protected capital (like the zero coupon bond mentioned above for instance), and a risky asset offering leverage potential (the WOF PDI for instance). The level of the barriers and the coupon can vary in order to obtain the most competitive price and fit the client's needs and investing preferences: the riskier the optional part, the higher the potential coupon. The different possible barrier levels have a very important impact on the valuation as they impact the probabilities of automatic recall and therefore the autocallable digitals prices. Indeed, autocallable digitals (that is to say discontinuities in the payoff linked to potential knock-out events) can be valued using discounted conditional probabilities. For instance, the probability of autocalling on the second year is conditional on not autocalling at the end of the first year: we only have to compute and discount all the conditional probabilities and multiply them by the respective coupon levels to be potentially received in order to compute an estimate of the autocallable digitals price. In this way we understand why the product's thresholds and barriers must be chosen carefully in order to obtain a balanced structure that matches investors' requests.

Figure3: Decomposition of a standard note, *Source Exotic Options and Hybrids – A Guide to Structuring, Pricing and Trading*



1. 2. Reminder on the sensitivities: the Greeks

Delta

$$(2) \quad \Delta = \frac{\partial P}{\partial S}$$

It represents the first-order sensitivity of the derivative's price to a move in S . For a small change ε in S , the price of the option will move by $\Delta * \varepsilon$. To hedge against movements in the underlying asset, the buyer of a Call option must buy Delta units of the underlying: he would then obtain a Delta-neutral portfolio that has to be hedged dynamically in order to remain Delta-neutral throughout the option's life. In the case of a portfolio P made of several options O_1, O_2, \dots, O_n on the same underlying asset S , the sensitivity of the price of the portfolio P to moves in the S corresponds to the sum of individual Deltas (it is additive):

$$(3) \quad \Delta_P = \Delta_{O_1} + \Delta_{O_2} + \Delta_{O_n}$$

The Deltas under the Black-Scholes assumptions are given by:

$$(4) \quad \Delta_{Call} = e^{-qT} N(d_1)$$

$$(5) \quad \Delta_{Put} = e^{-qT} [N(d_1) - 1]$$

Gamma

$$(6) \quad \Gamma = \frac{\partial^2 P}{\partial S^2}$$

It represents the second-order sensitivity of the option to a movement in the underlying asset's price. It gives a second-order correction to Delta, as the option is a non-linear function of the underlying price. It also underlines the first-order sensitivity of Delta to a movement in the underlying.

The Gamma for Vanilla options (both Calls and Put s) under the Black-Scholes assumptions is given by:

$$(7) \quad \Gamma = \frac{N'(d_1)e^{-qt}}{S\sigma\sqrt{T}}$$

$$\text{Where } N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Vega

It represents the sensitivity of the option price to a movement in the volatility of the underlying asset. This sensitivity contradicts the essential Black-Scholes assumption on constant volatility, but is absolutely indispensable in order to manage a book of derivatives. The Vega is greatest around the ATM level (when moneyness is null) and falls exponentially on both sides.

Under Black-Scholes, the Vega (for both Call and Put options) is given by:

$$(8) \quad V = Se^{-qT}N'(d_1)\sqrt{T}$$

Theta

It represents the rate at which the option price varies over time, on a daily basis. Being long a Call or a Put option implies having a negative Theta: as time passes, the option has less time to expiry and therefore loses time value.

Under Black-Scholes, the Theta of a Call option is given by:

$$(9) \quad \Theta_{Call} = -\frac{S\sigma e^{-qT}N'(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2) + qSe^{-qT}N(d_1)$$

$$(10) \quad \Theta_{Put} = -\frac{S\sigma e^{-qT}N'(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2) - qSe^{-qT}N(-d_1)$$

Rho

It represents the sensitivity of an option price to a movement in interest rates. As interest rates only have a first-order impact on the option price, Call and Put options are almost linear in interest rates. This effect is linked to a dual influence of interest rates, firstly on the cost of Delta hedging, and secondly on price discounting.

Under Black-Scholes, the Rho is given by:

$$(11) \quad Rho_{Call} = KTe^{-rT}N(d_2)$$

$$(12) \quad Rho_{Put} = -KTe^{-rT}N(d_2)$$

Dividend sensitivity

It is the first-order sensitivity of the option price to a movement in the dividend payments. If we consider discrete dividend payments, it seems impossible to obtain an analytic expression from standard models. In this case, the sensitivity of a product can only be computed by analyzing the impact of shifting future dividend values on the option price. On the other hand, if we include a continuously compounded dividend yield q , we can derive the sensitivity with respect to a change in the dividend yield from the Black-Scholes model in the case of European options:

$$(13) \quad \delta_{Call} = \frac{\partial C}{\partial q} = -Ste^{-qT}N(d_1) = -ST\Delta_{Call}$$

$$(14) \quad \delta_{Put} = \frac{\partial P}{\partial q} = -STe^{-qT}(1 - N(d_1)) = -ST\Delta_{Put}$$

It is nevertheless important to note that while this formula shows the impact of a change in the dividend yield on the option value, it does not take into account that a company announcing a degradation of its dividend yield might see its stock value and volatility evolve. Moreover, as it is not possible to obtain an analytical expression for American options, the sensitivity formula written above is verified only for European options.

Vomma (or Volga)

It represents the second-order sensitivity of the option price to a movement in the implied volatility of the underlying asset: it corresponds to options that are convex in volatility, that is to say options that have Vega convexity.

$$(15) \quad \text{Vomma} = \frac{\partial^2 P}{\partial \sigma^2}$$

Vanna

It represents the sensitivity of the option price to a movement in both the underlying asset's price and its volatility. It can be seen as the sensitivity of an option's Delta to a movement in the volatility of the underlying. It is a good indicator of how much a Delta hedge is going to change if volatility moves.

$$(16) \quad \text{Vanna} = \frac{\partial^2 P}{\partial \sigma \partial S} = \frac{\partial P}{\partial S} \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \Delta}{\partial \sigma}$$

The following Greeks correspond to the particular case of multi-underlying derivatives.

Correlation Delta

It represents the first order sensitivity of the price of a multi-asset option to a move in the correlations between underlyings. This must be computed for every pair of underlying assets of the product, and can evolve greatly with movements in other parameters, especially underlyings' prices.

Cross Gamma

It represents the sensitivity of a multi-asset derivative to a movement in two of the underlying assets. More precisely, it can also be interpreted as the impact of a movement in S_i on the Delta of the option with respect to S_j when $i \neq j$.

It can be written as follows:

$$(17) \quad \Gamma_{S_i, S_j} = \frac{\partial^2 P}{\partial S_i \partial S_j}$$

Diagonal Gamma

It corresponds to the same concept as the crossed Gamma, but in the particular case where $i = j$. It therefore represents the impact of a movement in S_i on the Delta of the option with respect to this particular stock. It can be written:

$$(18) \quad \Gamma_{S_i, S_i} = \frac{\partial^2 P}{\partial S_i^2}$$

1. 3. Presentation of the product's expositions from the investor's perspective.

The autocallable structure described in the first section accumulates sensitivities to a wide range of variables: let's describe these expositions from the point of view of the investor. In order to analyze such sensitivities, we decided to base our reasoning on a product with specific parameters chosen in order provide the best illustration.

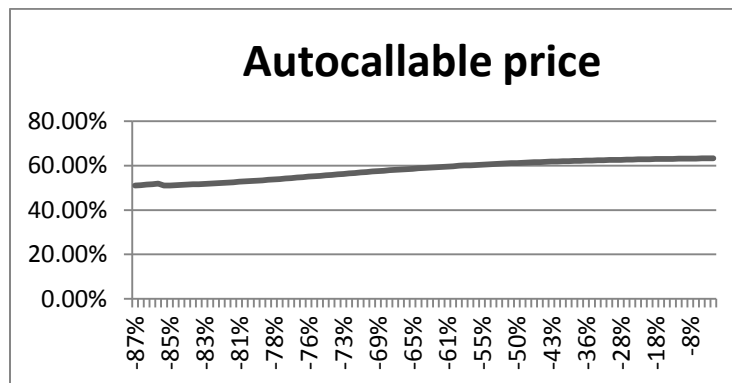
The following tab sums up the main characteristics of the Autocallable we will study throughout our analysis.

Type	Autocallable on WOF
Currency	USD
Underlying Stocks	Toyota-Ford-Renault
Guaranteed Capital	100%
Strike Date	06/04/2017
Maturity Date	06/04/2020
Automatic Recall Valuation Dates	06/04/2018 08/04/2018 06/04/2020
Automatic Recall Barrier	100%
Bonus Coupon if Recall	10%
WOF PDI Type	European at maturity
WOF PDI Knock-In Barrier	70%
WOF PDI Strike	100%
Eagle Limit	85%
Rebate Coupon if Eagle Limit < WOF < Recall Barrier on Recall Valuation Dates	10%

First-order exposition to the underlying stocks movements

This structure has a positive first order exposition to the underlyings as the product is globally “bullish”: the investor bets on the fact that all of the stocks of the basket will be above their initial level at the valuation dates. This positive Delta exposition is mainly concentrated on the worst performing underlying as the product's payoff depends positively on this precise stock (the “Worst-Of” of the basket). The Delta of the product with respect to the top performing stock is therefore almost null. But as the worst performing stock of the basket is not the same throughout the entire life of the product, the Delta of the product with respect to each stock can vary rapidly from a moment to another depending on which underlying becomes the worst performer. This instability creates discontinuities in the dynamic composition of the optimal replicating portfolio: these digitals on the first partial derivatives with respect to each stock make the daily Delta hedging more expensive.

Figure 4: Impact of the WOF performance on the Autocallable price



On this graph we can see that the more the worst performing underlying decreases, the more the price of the product drops: the structure has a positive Delta exposition that is concentrated on the WOF.

Second-order exposition to the underlying stocks movements

The second order sensitivity to the underlying stocks is globally negative. Indeed, the investor is selling an American Down-and-In Put on Worst-Of, he is therefore “short Gamma” on the worst performing underlying. This can be intuitively explained by the concave shape of the product’s payoff. The overall Delta of the structure indeed decreases when the underlying stocks increase: it roughly goes from approximately 1 when the short PDI is deep In-the-Money, to 0 when the worst performing stock crosses the autocall barrier. But we need to analyze more precisely the Gamma exposition in order to have a better understanding of our product’s second-order risk. Here, it is important to analyze the Cross and Diagonal Gamma expositions as our product is based on a basket of underlying stocks.

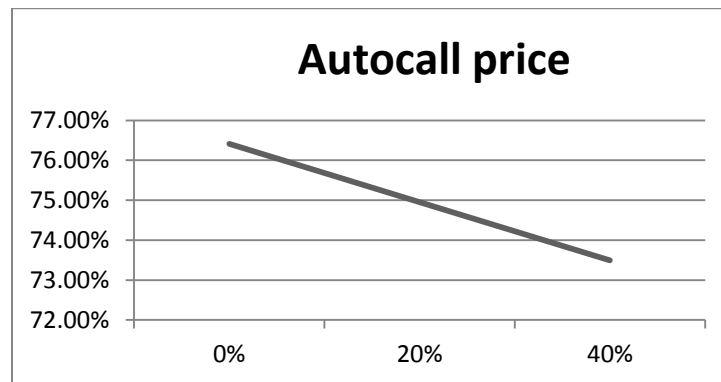
These Greeks allow having a more accurate view on the second order sensitivities of the product. First, our particular autocallable product on Worst-Of has a positive diagonal Gamma exposition on the worst performing stock: the Delta on this underlying roughly evolves from 1 to 0 from the case where the stock has an extremely negative performance to the one where it reaches the autocall barrier. On the other side, this product has a positive cross Gamma exposition. The intuition behind this statement can be explained as follows: when every stocks of the basket are at the exact same level, the Delta is almost equitably split between them. Then, if two stocks perform positively, the third one becomes the Worst-Of of the basket, and the Delta exposition automatically transfers to this stock. Therefore, the Delta on one stock depends positively on the performance of the other stocks. The same reasoning can be applied to the case where the worst performing stock grows and is replaced by another stock at the bottom of the basket. In this case, the Delta of the new Worst-Of goes from a very small to a higher level: the cross Gamma is, again, positive.

The question is therefore to evaluate the overall Gamma position of the product, knowing that it has a negative diagonal but a positive cross Gamma exposition. To answer that question, we have to measure the relative weights of the diagonal and the cross Gamma exposition in the overall Gamma exposition. In the case where all components of the basket are at the exact same level and all increase simultaneously by 1%, the Deltas on each stock stay equitably split (as no Worst-Of is identified) but their values decrease slightly (indeed, they tend to 0 in the scenario where all stocks cross the automatic recall barrier). To conclude, we can say that the product has a negative overall Gamma exposition.

Exposition to dividends

This product also is sensitive to dividends, as their values directly impact the underlying stocks future levels, and therefore the payoff of the product. This first order exposition is negative in the case of our bullish autocallable structure: the investor is “short dividend”.

Figure 5: Impact of a dividend shift on the Autocallable price



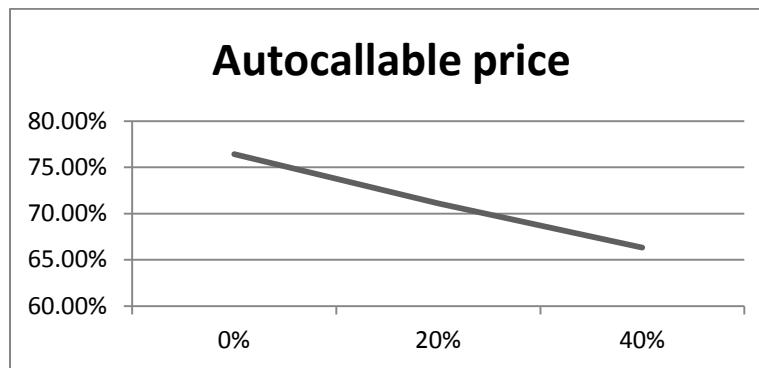
This graph shows the negative exposition to dividends: when shifting the dividend matrix, the price of the structure declines.

Exposition to volatility

At first sight, it would seem that an investor purchasing a standard autocallable structure would be “selling” volatility (“short Vega”). The reasoning would be the same as for the second order sensitivity: the client is selling the PDI, he is therefore selling volatility (which is linked to the intrinsic asymmetry that characterizes the option). Nevertheless, it is important to note that the Vega sensitivity can be a lot more complex depending on the specific features of each product. Indeed, the PDI is not the only element to bring volatility exposure to the investor: conditional coupons throughout the product’s life and autocallability at maturity also impact the Vega, as they imply several discontinuities (“digital” events) in the payoff.

Therefore, even if the structure’s “global” sensitivity to volatility is negative, it is more interesting and relevant to analyze the specific Vega sensitivities across time until maturity. In this way, we should rather evaluate the Vega for each time period by progressively shifting parts of the volatility term structure. This process, called “VegaT” at SG, makes it possible to clearly conclude on the local exposition to volatility across varying maturities: we look at “Vega buckets”, each corresponding to the volatility sensitivity of a particular maturity on the term structure of implied volatilities. Moreover, not only Vega exposure can differ across volatility maturities, but it can also be ambiguous for a single volatility maturity. For instance, on the one hand, the Vega induced by the PDI is negative (as the investor is selling the PDI) and concentrated at maturity (as most PDI are European and observed at maturity in these structures). On the other hand, the autocallability feature can have a positive impact on the sensitivity to volatility at maturity, depending on the size of the Bonus coupon (as a greater realized volatility would higher the automatic recall probability, and therefore the payment of the final coupon with early termination). We understand that the Vega analysis is not straight forward here, but depends on multiple factors: the trader’s quotes on stock volatilities must take into account all these subtleties. But despite these local variations across the term structure, the investor is globally selling volatility: the PDI has the biggest impact on the Vega.

Figure 6: Impact of an implied volatility surface shift on the Autocallable price



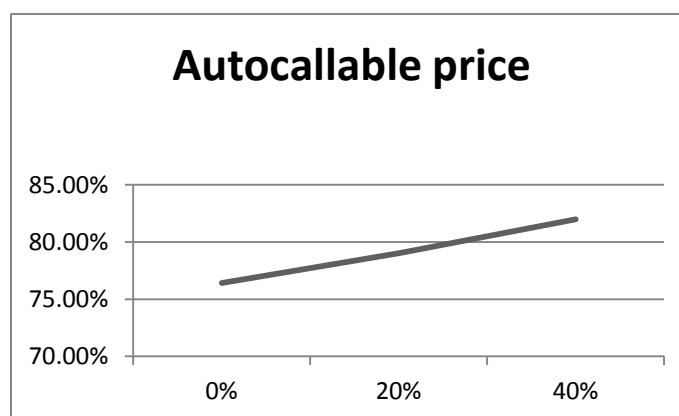
Here we can clearly see that the product is “vol ask”: when we shift the whole volatility surface (that is to say we shift volatilities all across varying strikes and maturities), the price of the product decreases.

It is important to note that the investor also is sensitive to the evolution of the implied volatility smile with this product (the concepts of “smile” and implied volatility surface will be discussed in the section dedicated to volatility models). Indeed, the investor is short a Put Down-and-In on Worst-Of: the value of this option depends on the level of implied volatilities, and therefore on the implied volatility smile. When selling a PDI, the investor takes a position on the skew, as the option price depends both on the At-the-Money implied volatility and on the implied volatility near the knock-in barrier level. If the smile grows during the life of the Autocallable, the implied volatility for low strike options becomes higher, and the investor sold the PDI at a level of implied volatility lower than the one then prevailing. The investor is therefore “short implied volatility smile” when purchasing this product.

Exposition to correlation

The product also has an exposition the correlation between the underlying stocks. Indeed, as the investor is selling a PDI on the worst performer of the basket, he is “long correlation”. A greater dispersion between the stock returns would likely result in a lower payoff, increasing the probability of crossing the PDI barrier, and therefore losing a part of the capital invested. Moreover, when correlation is low, there is a smaller chance that the worst performing stock of the basket ends up above the automatic recall barrier.

Figure 7: Impact of a stock correlation shift on the Autocallable price

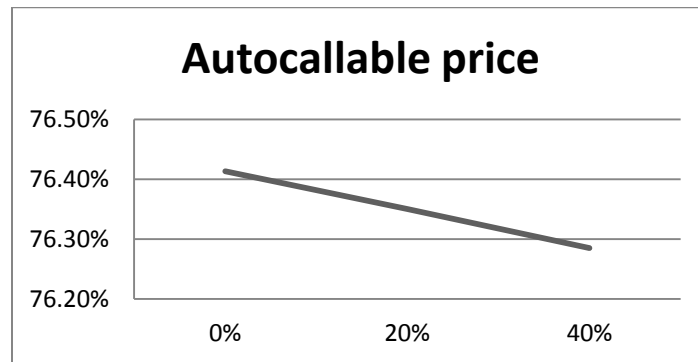


This previous graph illustrates well the fact that the structure is “correl bid”, that is to say that its price increases with implied correlation. Here, we shifted the correlations by pairs of stock, and the impact on the structure’s price was positive.

Exposition to interest rates

As the structure has a zero coupon bond embedded in it, it also has an important first order sensitivity to interest rates moves. More precisely, the structure has an overall negative exposition to interest rates: as interest rates grow, present value of future cash-flows is smaller, hence the negative relationship between the product and interest rates.

Figure 8: Impact of an interest rate shift on the Autocallable price



We obtained this graph by shifting the “repo” rates (rates at which central banks lend money to commercial banks): it shows well the structure’s negative exposition to interest rates moves.

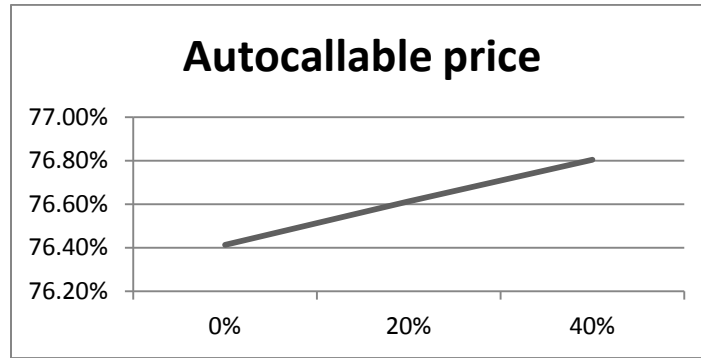
Exposition to the correlation between underlying stock returns and interest rates

Another central characteristic of this autocallable product is its un-deterministic lifetime. As the product can potentially be automatically recalled, it is impossible to compute its future maturity *a priori*. In this way, we have to calculate the fugit, which is an indicator of the products theoretical maturity. This number is based on the difference between the start and the expiration date, on which we apply certain conditional exit probabilities. This implies a particular exposition to the correlation between stock returns and interest rates. Indeed, if the product early recalls, the investor gets back his invested capital plus a Bonus coupon. He then has to re-invest his capital at current interest rates: he is therefore “long correlation” between stock returns and interest rates. He will take advantage of both a stock return and an interest rate increase as the first would imply a higher probability of early recall, and the second would allow the buyer to invest the proceeds of the product at better conditions.

Exposition to the correlation between underlying stock returns and foreign exchange rates

It is also important to take into account the exposition of this product to the correlation between forex and stock returns. Indeed, in the case of a product denominated in a currency that is different from the stocks’, there is an impact on the dynamic Delta hedging. In order to Delta hedge such a product, the investor has to short the underlying stocks in their respective currencies. If the exchange rate between the underlyings’ and the product’s currencies fluctuates, the investor has to adjust his Delta hedge. For instance, if the product’s currency depreciates, the investor has to short fewer stocks: he must buy back a certain amount of shares. But if at the same moment the stock prices go up, the investor is exposed to additional costs. On the contrary, if the stock prices fall, he will benefit from it and be able to buy back shares at a lower price. Therefore the product, if denominated in a currency that is different from the stocks’ currencies, has a positive exposition to the equity-foreign exchange rate correlation.

Figure 9: Impact of a stock return / FX correlation shift on the Autocallable price



This positive sensitivity is underlined by the graph above: we shifted up correlation between underlying stocks and foreign exchange rates and obtained a slightly increasing price.

3. Theoretical models used to provide consistent quotes and price autocallable structures

Different types of models are used in order to price structured products: each variable can be modeled according to different conventions, and a wrong choice in the parameters and their calibration can imply serious valuation and hedging consequences. We will now focus on the main volatility and correlation models that allow pricing exotic derivatives. We will then briefly present the most widely used interest rate, foreign exchange and dividend models at SG.

2. 1. Volatility models: from the constant Black-Scholes assumption to stochastic diffusion

In order to understand the different models linked to this parameter, it is first necessary to distinguish the concepts of realized and implied volatility. While realized (or historical) volatility is an observable statistical measure called standard deviation, implied volatility is the parameter we can infer from traded options prices.

Considering a set of N price observations $S(t_1), S(t_2), \dots, S(t_N)$, we can define the continuously compounded return r_i between time t_{i-1} and t_i as:

$$(19) \quad r_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right)$$

Then an unbiased estimate of the variance of the price returns on day t_N is:

$$(20) \quad \sigma_{t_N}^2 = \frac{1}{N-1} \sum_{i=1}^N (r_{N-i} - \bar{r})^2$$

where \bar{r} is the mean of the returns r_i given by:

$$(21) \quad \bar{r} = \frac{1}{N} \sum_{i=1}^N r_{N-i}$$

If we assume that the mean return \bar{r} is equal to zero, replace $N - 1$ by N and compute returns as percentage returns, we can simplify the variance formula to:

$$(22) \quad \sigma_{t_N}^2 = \frac{1}{N} \sum_{i=1}^N r_{N-i}^2$$

3. 1. 1. The limits of a constant volatility model

This historical volatility, while giving us information about the past fluctuations of an asset, does not necessary contains information about the current market sentiment. The implied volatility, used to quote Vanilla options, shows on the other hand the market's consensus on the forward looking volatility of the underlying asset. While the two parameters might be close, they are not equal as they don't represent the same concept: some may use realized volatility as a proxy for implied volatility when it cannot be implied from liquid tradable instruments, but it is only to get a very rough estimate. It is important to choose a volatility model that fits properly the available data: this model will determine the underlying's implied volatility surface dynamics and characteristics (which has a non-negligible impact on derivatives pricing). A constant single volatility model seems over-simplistic and unrealistic: it would imply a plane implied volatility across both strikes and maturities. This assumption would lead to a single volatility level and would fit few option prices available in the market: only the ones with no "skew" sensitivity *per se*. Using a constant volatility to price a single Call option seems fine, as for a Call Spread: even if this strategy has a dependency on the skew level, it can be modeled as a linear combination of two Vanilla options, using therefore two different constant implied volatilities levels depending on each strike. And how about barrier options? Such derivative products, while depending strongly on the future behavior of the skew, can't be broken down into Vanilla options, and a single constant implied volatility wouldn't allow pricing correctly the payoff. Here, there is a very high importance given to model risk (the risk that a derivative is modeled incorrectly).

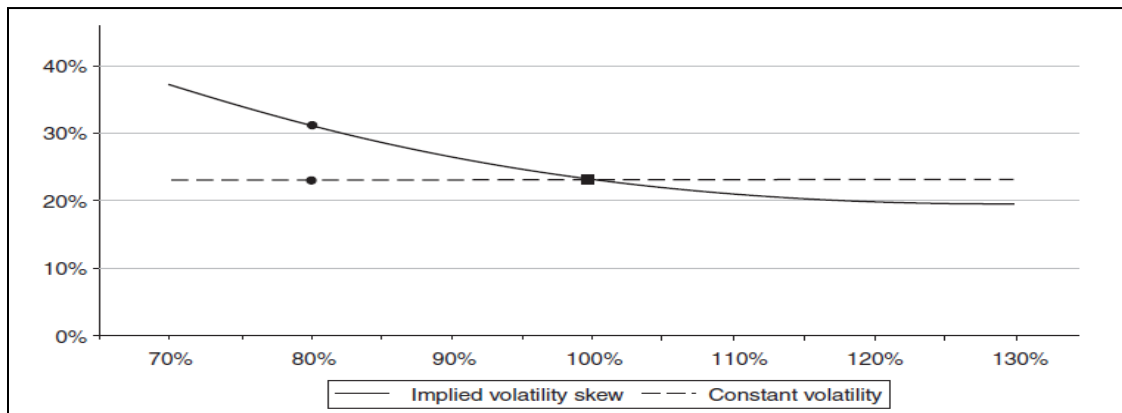
Using such a constant volatility model to price all derivatives would lead to serious misprices knowing that it would take into account neither the implied volatility smile, nor its term structure. Here, it is preponderant to understand well the concept of the smile, or "smirk". A simple model such as Black-Scholes uses the fundamental assumption that underlying assets' implied volatilities are constant. This assumption makes sense to the extent that a single underlying stock logically has a unique historical volatility, intrinsically linked to its variations through time. Nevertheless, it is easy for market practitioners to observe how implied volatility can vary across a range of options for a single underlying by deducing implied volatility levels from available options prices. It shows the limit of a constant volatility model such as Black-Scholes in an environment where derivatives prices are driven by infinite factors, including agents' anticipations.

3. 1. 2. Local volatility models: skew dynamics and term structure

This phenomenon of varying implied volatility levels across strikes can be explained historically by the emergence and the impact of market crash events since 1987. From then, we have been able to observe fat tails in stock returns' distribution, driven by temporary extreme moves in the market. In order to buy protection from such negatively

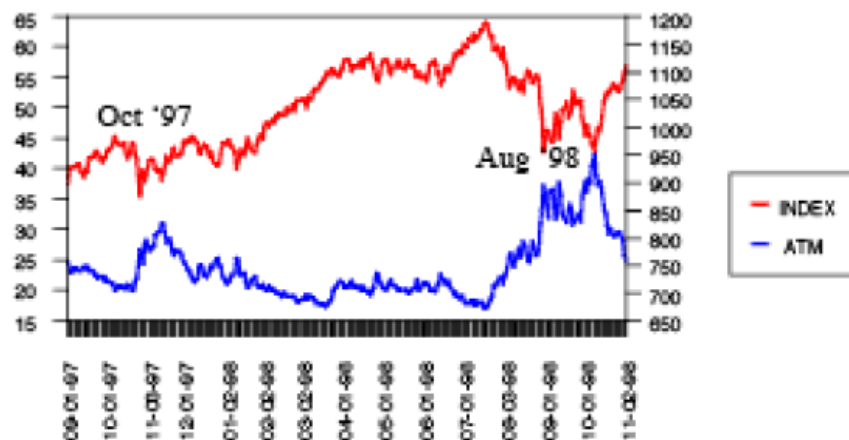
skewed stock returns, numerous investors (characterized by a long position on the market) have been buying Deep Out-of-the-Money Puts that would prevent them from a complete loss in case of a global downturn. This created a strong demand for low strike options, and had a direct positive impact on their price. Inflated option prices then mechanically impacted the levels of implied volatility. On the plot using the range of strikes as the X axis and the implied volatility level as the Y axis, a smile (or “smirk”) appeared.

Figure 10: Implied volatility skew vs constant volatility, *Source Exotic Options and Hybrids – A Guide to Structuring, Pricing and Trading*



Another way of understanding this skew shaped implied volatility plot is to consider the inverse relationship between stock returns and At-the-Money implied volatility. Indeed, in periods of bull market, the future tends to seem less uncertain as the economy is flourishing, and volatilities stay at relatively low levels. On the contrary, in periods of bear market, future prospects are darker and volatilities tend to go up (especially in our economies characterized by a high potential systematic risk). This relation between stock returns and volatility, while not always empirically verified, can help conceptualizing the implied volatility smile. The following graph underlines well the negative correlation between At-the-Money volatility and index level.

Figure 11: Three-months implied volatilities of SPX options, *Source Patterns of Volatility Change*



In order to take into account the smile, it is therefore important to use a volatility model that takes the strike level as an input. The goal is to include the idea of a relationship between stock returns and volatility in the generation of future instantaneous volatility. It is possible to measure this implied skew with liquid derivative instruments: while spread options (typically a 90%-100% Put Spread composed of a long ATM Put and a short 90% Put) allow trading the steepness of the skew, butterfly spreads (for instance a 90%-100%-110% Butterfly Put Spread composed of a long 90% Put, 2 short ATM Puts and a long 110% Put) allow capturing the curvature of the implied volatility skew.

Moreover, implied volatility does not only vary across the range of different strikes: it also varies throughout maturities. This creates a term structure of implied volatility with various possible shapes. For instance in relatively calm periods, the implied volatility term structure could have an increasing shape as returns tend to be more uncertain on the long term and short-term volatilities are relatively low. The term structure, while reflecting the market's expectations of the impact of near-term events on volatility levels, is also a good indicator of the mean-reverting effect of volatility. This shape can be very different from an underlying to another: for instance the S&P500 currently has a term structure slope a bit less than two times steeper than the Eurostoxx50. It is therefore important to consider the effect of varying maturities on the implied volatility levels. Indeed, the underlying's future instantaneous volatility tree should be determined using time as an explanatory variable: the more long-dated the option, the higher the uncertainty on future stock returns and therefore the higher the implied volatility. Moreover, the implied volatility term structure isn't linear; it is characterized by a curved shape as short-dated options' prices are more sensible to changes in maturity than long-dated ones. This relationship between implied volatility and maturity is evolving dynamically through time and a single rigid equation would fail to encompass the full range of possible term structures (different slopes, mean-reversion processes, curvatures...).

This whole three-dimensional implied volatility surface (implied volatility, strike, maturity) can be generated with available market data like option prices thanks to the use of econometric regression models and interpolating methods and can then be expressed in an implied volatility matrix. The complexity of implied volatility sensitivities and variations shows why a constant single volatility model seems unrealistic in order to properly price derivative products. By not taking into account the full behavior of implied volatility, one would disregard a preponderant element and therefore misprice and mishedge the products he trades.

Local volatility models represent a way of modeling the implied volatility surface without introducing additional sources of randomness: the only source of uncertainty remains the underlying asset's price. These one-factor models offer a consistent structure for pricing options: they can account for skew and term structure while allowing risk-neutral dynamics. In this way, Dupire (1994) and Derman and Kani (1994) proved that it was possible to find a particular log-return distribution (and not a log-normal one as in the Black-Scholes model) that corresponds to the full set of Vanilla options available in the market. In a local volatility model, we can therefore define the volatility of the underlying asset as a function of the asset price and time (in a non-random way).

Once we have analyzed the dynamics of the current underlying implied volatility surface, it is important to be able to analyze its variations with underlying movements in the future. A consistent volatility model makes it possible to compute a certain implied volatility skew as a linear function of the At-the-Money implied volatility, the slope of the skew (itself depending on time), and the moneyness. But in order to price and hedge properly a product, it is indispensable to have a view on the sensitivity of the skew to the underlying asset's level, the main question being the following: how does the smile evolve when the spot moves. Several "rules" exhibiting different views on the implied volatility skew behavior exist. They depend on which one of the variables impacting the implied volatility level is considered as "sticky" (or invariant).

The first and simplest rule is the "Sticky Strike" rule: it states that no matter what the level of the underlying asset is, the implied volatility embedded in the option price stays the same.

It can be written as follows:

$$(23) \quad \Sigma(S, K, t) = \Sigma_0 - b(t)(K - S_0)$$

In this simple “model”, implied volatility doesn’t have any dependence on the underlying level across time: each individual option is given its own future tree of instantaneous volatility and keeps it throughout time. This rule is the closest from the Black-Scholes assumption on volatility, and pricing an option with this model provides the same Delta as Black-Scholes. With the Sticky Strike rule, we consider that when the underlying asset rises, At-the-Money implied volatility falls. This consideration has its limits: constantly lowering implied volatility as the underlying asset rises would lastly lead to a null implied volatility at a point where the probability of crash is becoming higher. This model is the most commonly used one to price autocallable structures at Société Générale.

The second rule is the “Sticky Moneyness” rule. This is a more accurate view of the implied volatility: it states that the option’s volatility doesn’t depend on the strike, but on the moneyness K/S .

In mathematical terms, assuming S and K both close to S_0 :

$$(24) \quad \Sigma(S, K, t) = \Sigma_0 - b(t)(K - S)$$

With this rule, we consider that the At-the-Money volatility level stays constant when the underlying asset evolves. In this way, we assume that the market reverts to a constant and determined At-the-Money volatility level. Moreover, implied volatility is at the same level for all values of S and K for which S/K is the same. The sticky Delta rule is close to the sticky moneyness rule, as the Black-Scholes Delta itself depends closely on moneyness K/S . This rule states that implied volatility should not only be a function of moneyness K/S , but of the Black-Scholes Delta which depends on $\frac{\ln S/K}{\Sigma\sqrt{\tau}}$. With this view, we assume that the At-the-Money implied volatility for an option with fixed strike K increases with the underlying asset level. Therefore, we can conclude that under the sticky Delta rule, an option has a greater Delta than an option with the same Black-Scholes implied volatility.

We can now see how the choice of the volatility model can impact the pricing of the structure and how to draw conclusions on the influence of the implied volatility surface. Let’s price the product with a constant volatility model, and then price it with a local volatility model using the regular Sticky Strike convention.

	Constant volatility model	Local volatility model– Sticky strike
Price	76.5959930286 %	74.2637828128 %

We can see that by taking into account the implied volatility surface in our pricing models, we obtain a lower price: this is mainly explained by the fact that the product is short implied volatility skew as we explained before (because of the short WOF PDI). By doing a simple subtraction, we can obtain the impact of the implied volatility smile and term structure on the option price, and we can clearly see that this impact is negative (-2.3322102 %).

3. 1. 3. A few words on stochastic volatility models

Here, the relationships we have described between implied volatility, strikes and maturities are non-stochastic. But there exist another wide category of volatility models: stochastic volatility models. They are characterized by a non-deterministic volatility process. In such models, volatility, just as the underlying asset, is diffused according to

certain drifts and volatilities. It needs the generation and calibration of massive amounts of data as it takes into account several variables: short-term and long-term volatilities mean levels (for mean-reverting models), short-term and long-term volatilities of volatilities, correlations between short term and long-term volatilities... As it requires a lot of tools, these models are not used to price every single product: it is only used for certain particular derivatives characterized by an important sensitivity to Vega convexity (non-linear sensitivity to volatility) or to forward skew.

The random volatility assumption captures the Vega convexity just as the random underlying price assumption captures the convexity in the underlying's price. For Out-of-the-Money Vanilla options, the cost of Vega convexity is included in the implied volatility skew, but for more complex payoffs we have to use a stochastic volatility, with its own "vol-of-vol" (also Called Volga, or Vomma at SG). Moreover, forward-starting options need to be priced with a model that generates forward skews: local volatility models tend to give a forward skew that flattens out progressively with time, while it has no reason to do so. This is why the use of stochastic volatility models is also indispensable when trading cliquet structures (that is to say series of consecutive forward-starting options). Indeed, when pricing such structures with forward starting Variance Swaps for instance, it can be important to be able to control instantaneous forward variance dynamics and calibrate parameters to fit the term structure of vol-of-vol embedded in realized variance options prices. For the moment, exotic derivatives traders at SG don't use stochastic volatility models to price all structured products as it would be very time-consuming. These products are priced with local volatility model using the Sticky Strike convention, and external Vomma fees are computed (by shocking the volatility surface) and charged to the client as an additional cost.

2. 2. Correlation models: from historical to implied correlation

The correlation model also is very important in the valuation and hedging of multi-underlying derivatives. Indeed, the effect of correlation is absolutely preponderant when valuing multi-asset options: by adding different underlying stocks to a payoff, the concept of diversification appears, and exposure to an individual asset can be lowered. Historical correlations (which can also be called realized or statistical correlations) are calculated by pairs of stocks on a monthly basis, taking into account daily returns on a large period. There exist several different conventions to calculate historical correlations between pairs of stocks (depending on the frequency for instance), but they all underline the same concept: the idea of an existing linear relationship between two variables. One could calculate correlation using only stock returns during the last 6 months. Another approach could be to compute the daily average between 3 months correlations through the last 3 years. Thus, there is no "right" way to compute historical correlations between pairs of stock: each convention corresponds to a different analysis.

The realized correlation between two variables X and Y can be defined as:

$$(25) \quad \rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E^2(X)} \sqrt{E(Y^2) - E^2(Y)}}$$

where Cov is the covariance, Var is the variance, μ is the mean and σ is the standard deviation.

If we now consider a time series of daily log return observations of two assets, correlation can be computed as:

$$(26) \quad \rho_{X,Y} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{[\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2]^{\frac{1}{2}}}$$

3. 2. 1. The case of correlation in asynchronous markets

Moreover, in the case where the underlying stocks of a basket are listed in different time zones, an asynchronous correlation can be computed in order to take into account the fact that securities are not always simultaneously observable. Indeed, some areas don't have any overlap at all, like Japanese and European exchanges. In this situation, taking solely the correlation between stock levels at the respective close in each country wouldn't be the most relevant. As an important part of the structured products sold by SG are based on "world baskets", we will now provide further details on the correlation model used for asynchronous markets.

When computing the Delta at the market close of each security, "stale" values are used: they correspond to the most recent close level for the other underlying securities trading in market places around the globe. Products are Delta-hedged at the close of each underlying's market using these "stale" values: this can have strong implications in terms of volatility, correlation and covariance measurements.

As an illustration, let's write the Profit and Loss function of an option on a basket of two underlying stocks tradable on two separate market places like Japan and Europe. We will then be able to see how asynchronicity in the dynamic Delta-hedging process can impact correlations and volatilities estimators:

$$(27) \quad P \& L = -\left(f(S_{t+\Delta}^1, S_{t+\Delta-\delta}^2, t + \Delta) - f(S_t^1, S_{t-\delta}^2, t)\right) + \Delta_1(S_{t+\Delta}^1 - S_t^1) + \Delta_2(S_{t+\Delta-\delta}^2 - S_{t-\delta}^1)$$

$$\Delta_1 = \frac{df}{dS_1}(S_t^1, S_{t-\delta}^2, t)$$

$$\Delta_2 = \frac{df}{dS_2}(S_{t-\Delta}^1, S_{t-\delta}^2, t - \delta)$$

$$(28) \quad P \& L = -\left(f(S_{t+\Delta}^1, S_{t+\Delta-\delta}^2, t + \Delta)\right) + \frac{df}{dS_1}(S_{t+\Delta}^1 - S_t^1) + \left[\frac{df}{dS_2} + \frac{d^2f}{dS_1 dS_2}(S_{t-\Delta}^1 S_t^1)\right](S_{t+\Delta-\delta}^2 - S_{t-\delta}^1) \\ = -\left(f(S_t^1 + \delta S_1^+, S_{t-\delta}^2 + \delta S_2^+, t + \Delta) - f(S_t^1, S_{t-\delta}^2, t)\right) + \frac{df}{dS_1} \delta S_1^+ + \left[\frac{df}{dS_2} + \frac{d^2f}{dS_1 dS_2} \delta S_1^+\right] \delta S_2^+$$

Assuming f is a synchronous Black-Scholes PDE with volatility and correlation parameters σ_1 , σ_2 and ρ :

$$(29) \quad f(S_{t+\Delta}^1, S_{t+\Delta-\delta}^2, t + \Delta) - f(S_t^1, S_{t-\delta}^2, t) =$$

$$= \frac{df}{dS_1} \delta S_1^+ + \frac{df}{dS_2} \delta S_2^+ + \frac{1}{2} S_1^2 \frac{d^2f}{dS_1^2} \left[\left(\frac{\delta S_1^+}{S_1}\right)^2 - \hat{\sigma}_1^2 \Delta \right] + \frac{1}{2} S_2^2 \frac{d^2f}{dS_2^2} \left[\left(\frac{\delta S_2^+}{S_2}\right)^2 - \hat{\sigma}_2^2 \Delta \right] \\ + S_1 S_2 \frac{d^2f}{dS_1 dS_2} \left[\frac{\delta S_1^+ \delta S_2^+}{S_1 S_2} - \rho \hat{\sigma}_1 \hat{\sigma}_2 \Delta \right]$$

We are now able to plug this expression into the expression of the P&L:

$$\begin{aligned}
 (30) \quad P \& L &= \frac{df}{ds_1} \delta S_1^+ + \frac{df}{ds_2} \delta S_2^+ + \frac{1}{2} \left[S_1^2 \frac{d^2 f}{ds_1^2} \left(\left(\frac{\delta S_1^+}{S_1} \right)^2 - \hat{\sigma}_1^2 \Delta + S_2^2 \frac{d^2 f}{ds_2^2} \left(\left(\frac{\delta S_2^+}{S_2} \right)^2 - \hat{\sigma}_2^2 \Delta \right) + 2 S_1 S_2 \frac{d^2 f}{ds_1 ds_2} \left(\frac{\delta S_1^+ \delta S_2^+}{S_1 S_2} - \rho \hat{\sigma}_1 \hat{\sigma}_2 \Delta \right) \right] + \frac{df}{ds_1} \delta S_1^+ + \left[\frac{df}{ds_2} + \frac{d^2 f}{ds_1 ds_2} \delta S_1^- \right] \delta S_2^+ \\
 &= \frac{1}{2} S_1^2 \frac{d^2 f}{ds_1^2} \left[\left(\frac{\delta S_1^+}{S_1} \right)^2 - \hat{\sigma}_1^2 \Delta \right] + \frac{1}{2} S_2^2 \frac{d^2 f}{ds_2^2} \left[\left(\frac{\delta S_2^+}{S_2} \right)^2 - \hat{\sigma}_2^2 \Delta \right] + S_1 S_2 \frac{d^2 f}{ds_1 ds_2} \left[\frac{(\delta S_1^- + \delta S_1^+) \delta S_2^+}{S_1 S_2} - \rho \hat{\sigma}_1 \hat{\sigma}_2 \Delta \right]
 \end{aligned}$$

From this expression we can draw conclusions on how volatilities and correlations should be measured:

$$(31) \quad \hat{\sigma}_1^2 = \frac{1}{\Delta} < r_1^2 >$$

$$(32) \quad \hat{\sigma}_2^2 = \frac{1}{\Delta} < r_2^2 >$$

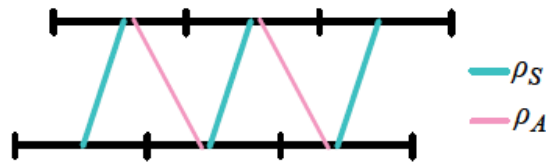
$$(33) \quad \hat{\rho} \hat{\sigma}_1 \hat{\sigma}_2 = \frac{1}{\Delta} < (r_1^- + r_1^+) r_2^2 >$$

- Volatilities and correlations estimators are only based on daily returns (which is consistent with the fact that Delta hedge is done at the close of each market place on a daily basis).
- Volatilities are estimated normally on each underlying stock using its own close values.
- Covariance is estimated by multiplying an underlying stock return (here the first one) by the sum of the second underlying stock returns that covers the same time period.

The main conclusion to keep in mind is that two different types of correlation exist, and the “proper” value of daily correlation can be seen as a sum of both:

$$(34) \quad \rho_S = \frac{\int_0^{\Delta-\delta} \rho(t) \sigma_1(t) \sigma_2(t) dt}{\sqrt{\int_0^{\Delta} \sigma_1^2(t) dt \int_0^{\Delta} \sigma_2^2(t) dt}} \quad \rho_A = \frac{\int_{\Delta-\delta}^{\Delta} \rho(t) \sigma_1(t) \sigma_2(t) dt}{\sqrt{\int_0^{\Delta} \sigma_1^2(t) dt \int_0^{\Delta} \sigma_2^2(t) dt}}$$

Figure 12: Example of daily returns on asynchronous securities, *Source Correlations in Asynchronous Markets*



$$(35) \quad \hat{\rho} = \rho_S + \rho_A$$

This correlation using the asynchronicity assumption can be compared to the usual n-days picking frequency “close-to-close” correlation as follows:

$$(36) \quad \hat{\rho}_n = \rho_S + \frac{n-1}{n} \rho_A$$

In particular, for a 3-days correlation:

$$(37) \quad \hat{\rho}_3 = \rho_S + \frac{2}{3} \rho_A$$

Once correlations among n variables are computed, they can be expressed in a square matrix \mathbf{M}_ρ .

Considering $\rho_{i,j}$ the correlation between assets i and j , the fact that correlation between a stock and itself is equal to 1 and the fact that correlation between asset j and i is the same as correlation between asset i and j , we can write the correlation matrix as follows:

$$\mathbf{M}_\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \dots \\ \rho_{21} & \rho_{22} & \dots \\ \vdots & \vdots & \dots \end{pmatrix} = \begin{pmatrix} 1 & \rho_{21} & \dots \\ \rho_{12} & 1 & \dots \\ \vdots & \vdots & \dots \end{pmatrix}$$

From this correlation, it is possible to obtain the realized basket correlation according to the following formula, knowing that the sum of respective weights $\sum_{i=1}^n w_i = 1$ and $0 \leq w_i \leq 1$ for all $i=1,2,\dots,n$:

$$(38) \quad \rho_{Basket} = \frac{\sum_{1 \leq i < j \leq n} w_i w_j \rho_{i,j}}{\sum_{i < j}^n w_i w_j}$$

This weighted average of the realized correlation matrix between stocks excludes the diagonal terms as they always have the same value (1).

It is important to note that correlation between underlying assets has a non-negligible impact on the variance of a portfolio, and more precisely on its covariance term. This is why there exist a very important link between volatility and correlation as we can see with the below formula:

$$(39) \quad \sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i \sigma_j \rho_{i,j} = \sum_{1 \leq i \leq n} w_i^2 \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

Nevertheless, even if these two parameters are intrinsically linked to each others, the method used to quote correlation differs from the one used in the case of implied volatilities as we will now see.

3. 2. 2. The implied correlation quoting process

Major investment banks compute implied correlations on a monthly basis for massively traded pairs of stocks, and share them through a system called Totem. Once every participating institution has posted its implied correlation

evaluations, extreme values are deleted in order to keep only relatively homogenous data. Then, “fair” values of implied correlations between liquid pairs of stocks are computed through an averaging process, and banks can use them their systems for valuation and hedging purposes without further treatment. But in the case of illiquid pairs of stock whose implied correlation is not provided by Totem, traders have to calculate a crucial parameter: λ . This parameter, which is computed at SG every month for all geographical areas, is used to establish a mathematical relationship between historical correlation and implied correlation for illiquid pairs of stocks (that is to say not often traded as underlyings in the same product). This process is used for a simple reason: it is more difficult to infer specific implied correlations by pairs of stocks from tradable derivatives than in the case of implied volatility. Indeed, in the case of single-asset derivatives, a lot of tradable options exist and they are very liquid: it is easy to obtain individual implied volatilities from market quotes. Nevertheless, there is no liquid listed market for such multi-asset products; this is why traders rely on the value of the λ parameter in order to quote implied correlations.

The main product used by major investment banks to compute and calibrate implied correlations before sending it in the Totem system is the Calls vs. Call (it will be described more precisely in the section dedicated to correlation hedging). The quotes on this product, whose payoff depends on the dispersion between underlying stocks, give a good idea of the market consensus on implied correlation. This implied correlation can be inferred indirectly from the product’s prices, using the formula of the basket variance. The implied correlation is the value that, when used in place of the $n(n - 1)$ individual correlations $\rho_{i,j}$, results in the same basket variance. This reasoning can also be applied to an index instead of a basket of stocks, as it is an even more liquid instrument:

$$(40) \quad \sigma_{Index}^2 = \sum_{1 \leq i \leq n} w_i^2 \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n} w_i w_j \sigma_i \sigma_j \rho_{Index \text{ implied}}$$

Once this index implied correlation is computed, it is possible to solve for λ in the following equation:

$$(41) \quad \rho_{Index \text{ implied}} = \rho_{Index \text{ realized}} + \lambda (1 - \rho_{Index \text{ realized}})$$

It is then possible, using historical correlations between pairs of stocks and the λ parameter, to obtain specific implied correlation quotes. Therefore, the λ parameter is computed through the establishment of a relationship between Totem implied correlations and historical correlations, and is then used to infer specific implied correlations from realized correlations for relatively illiquid pairs of stocks. This implied correlation is very often greater than realized correlation, as major investment banks in the sell-side are structurally short correlation (through there structured products activities) and have to keep a certain degree of conservatism in there parameters: this is why the λ parameter is most of the time superior to 1.

Then, the question of the existence of a correlation skew arises. If we can plot an implied volatility skew for indices, as well as for their individual components, we might also be able to infer implied correlations at different strikes. Such a correlation skew could be explained for instance by the fact that in lower strike regions, we expect volatility and correlation between stocks to go up. But this concept is highly questionable and major researchers (including SG’s global head of quantitative research Lorenzo Bergomi) have come to the conclusion that it can’t be considered as a market parameter to the same extent than the implied volatility skew. Nevertheless, even if there is no consensus on the question of modeling or not the correlation skew in pricing models, it seems different for the implied correlation term structure. Indeed, the exotic derivatives desk on indices at SG uses a λ that doesn’t have a single fixed value for one geographical area: their parameter $\lambda(t)$ is expressed as a function of time. Therefore they infer implied correlations for different maturities, and compute $\lambda(t)$ such that their quotes on implied correlations take into account the term structure dynamics.

2. 3. Succinct presentation of interest rate, foreign-exchange and dividend models

Other parameters play a preponderant role in the valuation of structured products, but in a matter of conciseness, we chose a particular focus on volatility and correlation models. This choice was made considering the fact that volatility and correlation are the most crucial variables in the daily risk management process of an exotic derivatives trader at SG (apart from the dynamic Delta hedging process of course).

Interest rates model

Interest rate models are also key in order to value structured products, as their payoff are directly linked to interest rates. Nevertheless, because of time-constraint, almost all structured products based on stocks are priced with a deterministic rate model at SG. In order to value the impact of correlation between stock returns and interest rates on autocallable structures prices, a simple fee is computed and added to the original price. The question there arises as to whether we should use stochastic interest rate models in order to value autocallable structures, instead of computing an external cost to take into account the impact of interest rates/equity correlation on the product's price. Several studies have been done on this subject, and the main conclusion is that using stochastic rate models doesn't provide additional information on the way to hedge interest rates/equity correlation, therefore simply deciding this parameter's level and computing its impact as a supplementary cost seems sufficient. The extra fee, whose magnitude is a function of the product's maturity, is evaluated using the Ho and Lee one factor and one currency model, which assumes the following process for interest rates:

$$(42) \quad dr_t = h_t dt + \sigma dW_t$$

with h_t the calibration factor and W_t a standard Brownian motion.

The calibration factor is here a bit more complex than in regular stock models in order to satisfy the no-arbitrage assumption. It consists in an upstate and a downstate perturbation function derived from the implied forward function. This model, whose initial purpose was to price callable bonds, models a binomial tree showing the evolution of the discount factor (and not interest rates *per se*), considering interest rates as path-independent. A non-constant volatility term is finally plugged into the model in order to reflect a whole interest rate volatility surface and match the available market data on term structure dynamics.

Foreign exchange rates model

Foreign exchange rate models also play an important role in the pricing of structured products, but as it doesn't represent a crucial element in the context of our study, we chose not to develop these models. At SG most the structured products are valued using a "forward" foreign exchange model: the forward value of the FX rate is simply plugged into the drift component of the underlying assets. It is also possible to use more complete ones that use stochastic foreign exchange rates volatilities. Such advanced models are used for the valuation of derivatives that have a preponderant exposition to foreign exchange rates, but in the case of structured products on stocks, there is no need to have this level of complexity (these models rather concern hybrid derivatives desks).

Dividends model

Concerning dividends, the current convention at SG is the following: they are modeled as a function of the expected cash dividend, the future underlying asset level and time. First, future dividends are based on the announced future dividends: it represents the "cash" (or determinist) part of the model (comparable to the intercept of a linear

function). Then, future dividends are considered to be an increasing function of the underlying stock's level: if the stock reaches very low levels, it is more likely that it will not distribute the totality of the announced dividends, whereas if it grows considerably, it might increase its expected dividends. Therefore, we add the product of the future stock level (obtained through its diffusion) and a certain coefficient to the fixed announced dividends: it adds a yielded component to the model. Then we consider that the level of future dividends is increasingly uncertain with time: this is why the further in time the ex-dividend date, the higher the weight given to the yielded (uncertain) part of the model. Equivalently, for closer ex-dividend dates, the "cash" part (corresponding to the fixed announced dividend) is overweighted, whereas the uncertain part (the yield component) is underweighted.

Lastly, a floor and a cap are input in the model in order to avoid extreme inconsistent values. Such a model has various implications in terms of pricing: for instance, forwards on stocks don't have a Delta equal to one anymore, as the sum of future dividends in the exponential term becomes itself a function of the stock, which is diffused with a volatility term (that can whether be constant, local or stochastic depending on the product and the model as seen before). This adds volatility exposure for products that shouldn't generally generate Vega, and impacts the valuation and hedging process for all types of derivatives.

4. Analysis of the challenges linked to the hedging process

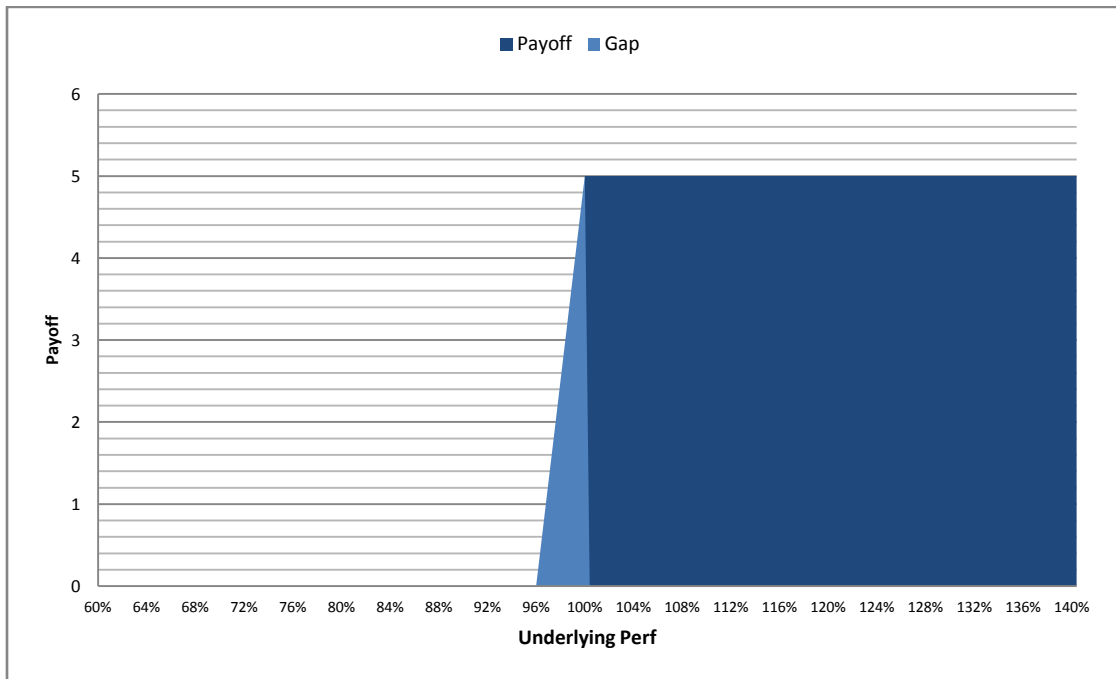
4. 1. The crucial smoothing of discontinuities: the impact of gaps

Hedging a multi-underlying Autocallable on Worst-Of implies a first major difficulty: discontinuities, linked to digitals embedded in the payoff of the product. Indeed, there are different digitals in the payoff, for instance around the barrier of the Down-and-In Put but also around the recall limit. Such discontinuities in the payoff have important consequences in terms of risk management and hedging as they imply potentially unstable and explosive Greeks.

With our autocallable product, as the Delta can jump or fall from a level to another when the worst performing stock of the basket crosses a barrier, it can reach infinite levels, which is not desirable for sensitivities management and even regulatory matters. Moreover, on a practical point of view, there always is a limit on the number of shares available on the market: liquidity is a constraint, and it has to be taken into account when elaborating a consistent hedging strategy.

Therefore, gaps are used in order to mitigate the impact of digital risk. Thanks to the use of gaps, it is possible to smooth the Gamma and therefore have a more efficient dynamic Delta hedging through the life of the product. Gaps represent an add-on (extra "technical cost") to the initial payoff that allows shifting the barrier and determining the continuous payoff that is the closest possible to the original one. This add-on implies a higher fair price for the product, and hence has a negative impact on the margin (but this information is not available for the investor, it is an "invisible cost").

Figure 13: A European digit and a triangular gap



In this previous graph, we can see that the discontinuity disappeared, making the risk management easier to control.

Gaps have different characteristics, and can be defined as follows:

- They can have whether a triangular or a rectangular shape.
- Their value can vary, depending on the “digit size”, that is to say the value of the price jump once a barrier is crossed.

This gap value is calibrated in a way such that the maximum Delta on a particular underlying of the product stays under a pre-determined limit: the goal is to cap the first order sensitivity of the product. This limit is based on the average daily traded volume of the stock: liquidity is a decisive criterion.

4. 1. 1. European digitals and triangular gaps

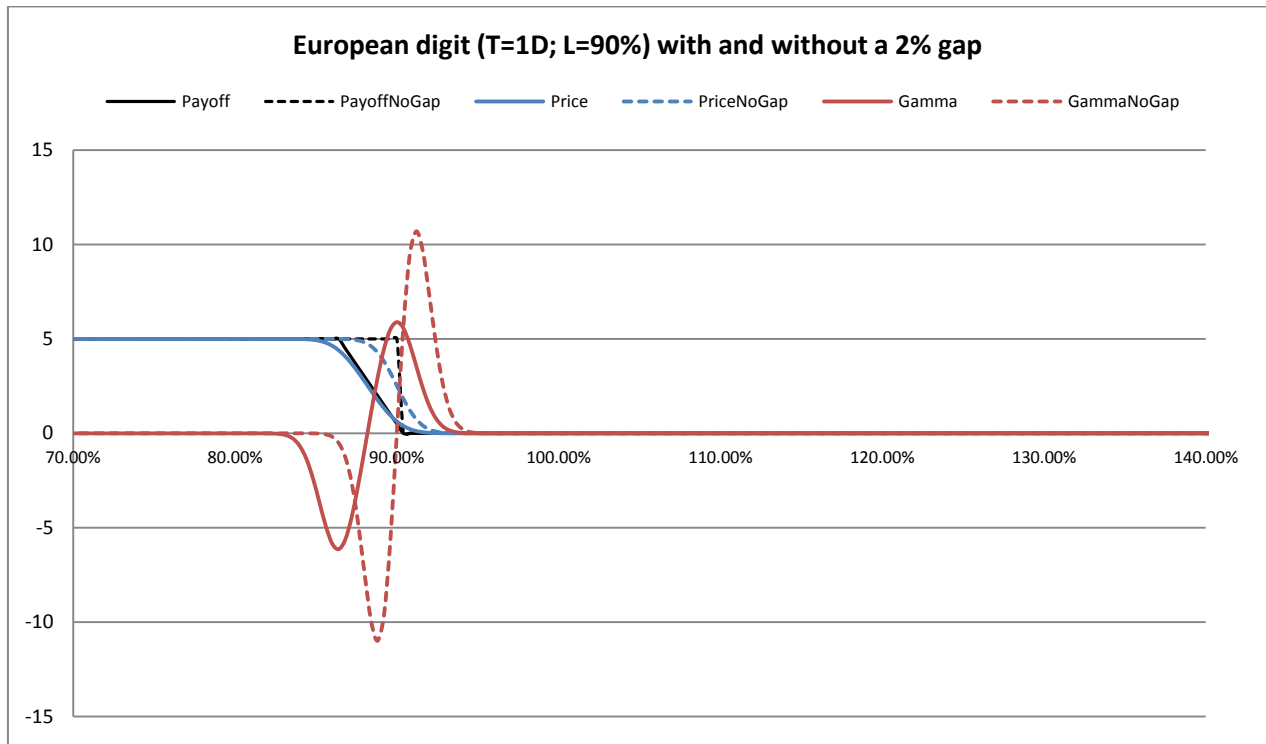
Let's first take the case of the triangular gap, mainly used for European limits that imply a discontinuity concentrated at a single date. This strategy relies on shifting the barrier to the left or to the right (depending on which way is the most conservative) by a certain value g called Gap. The original discontinuous payoff is therefore replaced by a linear one on the interval $[L-g; L]$ where L stands for the limit, or barrier. This triangular gap shape, combined with the original payoff, gives a horizontal spread option (bearish or bullish, depending on the way). The payoff of this gap can be written as follows:

$$(43) \quad C(t) = D(t) * Slope(t)$$

where:

- $C(t)$ is the coupon payed at time t
- $D(t)$ is the digit size
- $Slope(t)$ is the spread option type of function between the limit and the gap.

Figure 14: the impact of a triangular gap on the Gamma of a European digit



4. 1. 2. American digitals and rectangular gaps

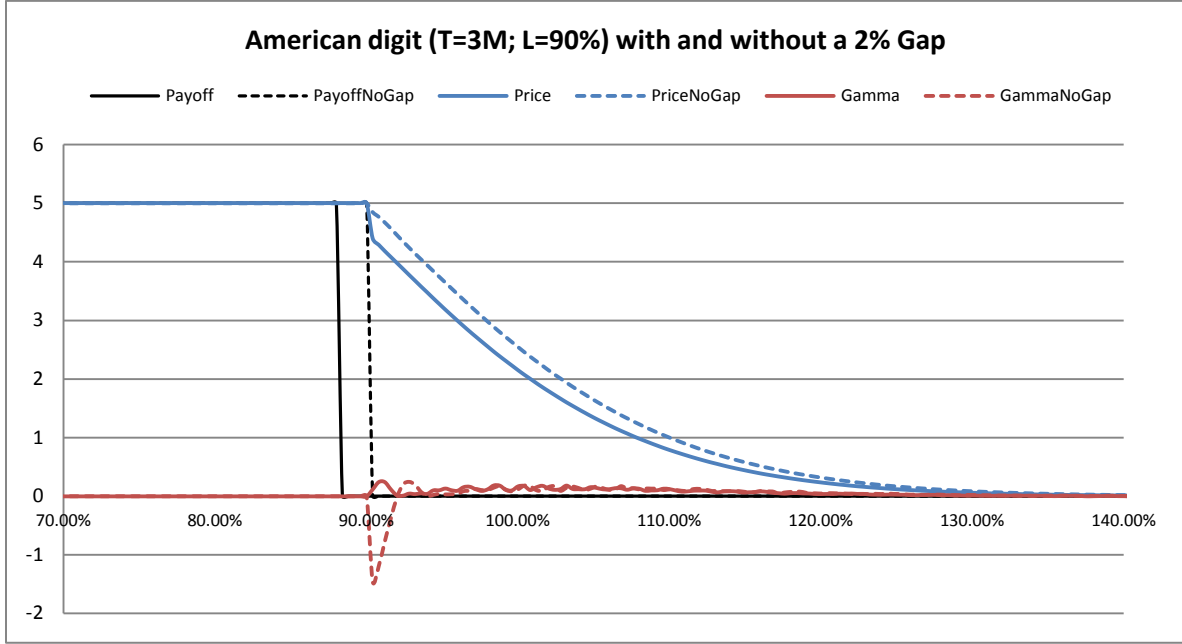
Rectangular gaps are based on the same concept: shifting the barrier in the most conservative way. The main difference is that instead of replacing the original payoff by a linear payoff, we replace it by a rectangular payoff on the interval $[L-g;L]$. Thanks to this strategy, the jump in the payoff and the Greeks associated to it only happens once the effective barrier has been crossed, therefore annihilating the entire residual Gamma. This strategy is mainly used on American and Bermudean barriers. The payoff of this gap can be written as follows:

$$(44) \quad C(t) = D(t) * Rect(t)$$

where:

- $C(t)$ is the coupon payed at time t
- $D(t)$ is the digit size
- $Rect(t)$ is the indicator function of the crossing or not of the barrier

Figure 15: the impact of a rectangular gap on the Gamma of an American digit



These two different gap shapes are based on the same concept: they both represent an add-on that inflates the product's price proportionally to each gap's area. It is therefore possible to establish an approximate relationship between the two kinds of add-ons on the initial product:

$$(45) \quad \text{Rectangular Gap} = 2 * \text{Triangular Gap}$$

4. 1. 3. Autocallables and intelligent gaps

As most of the autocallable products are in fact a combination of European digits (for instance with conditional intermediary coupons during the life of the product), it is often impossible to determine *ex-ante* the way of the discontinuity in the payoff. Therefore, it is impossible in these cases to simply apply a classical gap: we wouldn't now in what direction we should shift the barrier. The idea is then to apply an adaptive "intelligent gap". Such a gap is in some cases a long gap (the barrier is lowered), and in other cases a short gap (the barrier is shifted up). This can happen if in case of realization of the considered event, the digit given to the client in fact becomes negative (the product's value falls if this barrier is crossed). In the precise case of our Autocallable, this "intelligent gap" methodology consists in adding, at every single potential recall date, a gap on the limit that fits the way of the discontinuity. This type of gap can therefore be written like:

$$(46) \quad \sum_{i=1}^N \underbrace{CS_i * (1 - \text{Rect}_i) * [P_i^- - P]^+}_{\text{GapLong}} + \underbrace{PS_i * (\text{Rect}_i) * [P_i^+ - P]^+}_{\text{GapShort}}$$

where:

- N is the number of potential recall dates of the product
- P is the un-gapped product
- P_i^- is the product which barrier on the potential recall date i has been shifted down
- P_i^+ is the product which barrier on the potential recall date i has been shifted up
- CS_i (for Call Spread) and PS_i (for Put Spread) are spread option type of functions between the limit and the gap on the potential recall date i
- $Rect_i$ is the indicator function of the crossing or not of the barrier on the potential recall date i

Now that we understand the concept of gap, it is important to know how to evaluate its value: by how many percentage points of the limit should the barrier be shifted in order not to cross a certain maximum level of Delta. For instance, in the case of a triangle gap on a European autocall limit, modifying the gap value is equivalent to modifying the slope of the Call spread, and therefore the maximum Delta. From this point we clearly understand that gap values vary from a deal to another, depending on which underlying stocks are concerned (as they are not equally liquid), and on the issuance nominal.

Let's first describe how to define the digit size. It represents the observed price jump once a digital event has happened. In the particular case of our product, we can define two digit sizes, one for the PDI limit which is equal to PDI Strike – DI Limit, and one for the recall limit which is equal to the Rebate coupon (the “bonus” paid to the investor when the autocall barrier is crossed). For some products, the determination of the exact digit size is not that simple: simulations have to be done in order to find the highest observable price jump across all trajectories. It is equivalent to:

$$(47) \text{ DigitSize}(i) = \text{Abs}[P^-(\text{constat}(i) - 3D) - P^+(\text{constat}(i))]$$

where:

- $P^-(t)$ is the price at time t with a spot level at 99% of the barrier
 - $P^+(t)$ is the price at time t with a spot level at 101% of the barrier
- ($3D$ stands for the regular number of settlement days)

In the particular case of a multi-underlying product, there are digits on the basket of stocks: the digit size is split between the components of the basket; it therefore has to be divided by the number of stocks. Once the digit size is known and the maximum Delta is determined it is possible to solve the problem and fix a gap that satisfies all constraints. It is also possible to take into account other inputs like a floor/cap, the underlying's volatility or even toxicity.

In the next part, we will provide a detailed presentation of the algorithm that allows valuing efficiently the “intelligent gaps”. We chose to introduce this algorithm in this part because it is crucial for all the hedging process: it determines the characteristics of gaps, and therefore impacts the sensitivities management all through the product's life.

4. 2. Longstaff-Schwartz Monte Carlo: the essential algorithm used to value autocallable structures and gaps

There exist different ways to price products with variable maturities (autocallable, callable, putable), but the one that is mainly used at SG is the Longstaff-Schwartz Monte-Carlo algorithm. It is a numerical resolution methodology initially introduced in the article *Valuing American Options by Simulation: A Simple Least-Squares Approach*. Generally called Least-Squares Monte Carlo (LSMC), it is based on both a regular Monte Carlo simulation and a regression. The fact that this methodology mainly relies on a basic Monte Carlo simulation allows to price products with a wide range of features (multi-underlying, path dependant...) using all the existing models on volatility or interest rates.

4. 2. 1. Presentation of the algorithm's purpose and method

The core purpose of this algorithm consists in estimating the expected value of a product's future cash-flows at a given exercise date in case of non-exercise. This value is then compared to the cash-flows that would have been received in case of an immediate exercise. There lies the specificity of this methodology: in order to compute the expected value of future cash-flows, we determine a regression function between the spot level at the exercise date and the future cash-flows obtained in case of non-exercise. This regression function then allows to define the optimal exercisability rule, and to determine whether or not the product should be recalled. The rule therefore influences directly the diffusion of the underlying assets and the option price obtained is the averaged present value of the optimal exercises at every exercise dates. In the original article where this algorithm was initially presented, the different steps were described as follows.

The first step consists in the generation of thousands of random paths representing the possible values that the underlying asset can take throughout the life of the product. This diffusion depends of course on the chosen model, for instance for volatility (constant volatility, local volatility, stochastic models...). The goal is then to find the moment in time that maximizes the option value in every single path, to determine the optimal exercise date. This is done through a recurrence starting at the expiration date of the product.

At the expiration date T of the option, it is exercised if it is In-the-Money, or it vanishes if it is Out-of-the-Money just like a regular European option. At every date before expiration, the option buyer has the choice whether to exercise or to wait, hoping that the option will be worth more in the future. The optimal date will be the one when the immediate exercise is more profitable for the option buyer than the future expected value. Therefore, at every moment, we look at the immediate exercise value and expected value if the option is not exercised. This second value (the expected value of the option if it is not exercised) is computed using the least square approach. At the moment n , it is the solution of the following problem. Determine the regression function f that minimizes the spread:

$$\mathbb{E} \left[(Y_{n+1} - f(S_n))^2 \right].$$

To simplify, we choose a polynomial form for f for which we chose the degree, for instance:

$$f(X) = a + bX + cX^2$$

The authors also specify that all Laguerre polynomials can be used for the regression function. The regression is only done on all In-the-Money trajectories: other trajectories are not interesting as the question to know whether or not we

should exercise is not relevant. This regression is done on prices at time n and future cash flows observed on the generated paths. For each exercise date n , we then have a new polynomial f_n .

Thanks to the regressions computed at each exercise date n , we finally obtain a model capable of predicting the future expected payoff. It is then possible to take a decision on whether to exercise or not the option, as we know if the expected value is bigger than the actual value. Doing this backwards throughout all the life of the option, it is possible to find the optimal exercise date for each trajectory. The option price is finally given by the present expected value of the maximum payoff, obtained through the mean of maxima on each trajectory. This price reflects well the purpose of the algorithm: on a single trajectory, the price will be equal to the biggest payoff possible.

4. 2. 2. Numerical illustration

Let's consider an American Put of maturity 3 years on a stock with no dividend. It is possible to exercise the Put at time $t=1, 2, 3$ at a strike equal to 1.10 , the initial stock price being 1.0 . The continuous risk free rate is 6% .

First of all, we generate 8 random paths under the risk free probability.

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The goal is then to find the exercise moment that maximizes the option value on each path. In order to complete this task, one must look at the payoffs at the last time period (they are identical in both the European and the American case).

Path	t=1	t=2	t=3
1	-	-	0.00
2	-	-	0.00
3	-	-	0.07
4	-	-	0.18
5	-	-	0.00
6	-	-	0.20
7	-	-	0.09
8	-	-	0.00

We then place ourselves at time $t=2$: if the option is In-the-Money, we must decide whether we exercise immediately or we wait until expiration at time $t=3$. In our matrix, there are only 5 paths that are In-the-Money at time $t=2$, let's write:

X : the value of the underlying at time $t=2$.

Y : the present value of payoffs Z_{n+1} received at time $t=3$ if the Put is not exercised at time $t=2$.

Path	Y	X
1	$0.00 \times e^{-0.06}$	1.08
2	-	-
3	$0.07 \times e^{-0.06}$	1.07
4	$0.18 \times e^{-0.06}$	0.97
5	-	-
6	$0.20 \times e^{-0.06}$	0.77
7	$0.09 \times e^{-0.06}$	0.84
8	-	-

In order to estimate the expected payoffs for the rest of the option life, we do a regression on Y based on polynomials and defined by a constant, X and X^2 . One could also choose Laguerre polynomials as mentioned above.

The goal is to find a_0 , a_1 and a_2 that minimize:

$$\sum_{i=1}^5 (Y_i - a_0 - a_1 X_i - a_2 X_i^2)^2$$

We obtain:

$$\mathbb{E}(Y|X) = -1.070 + 2.983X - 1.813X^2$$

This conditional expected value is then compared to the immediate exercise payoff at time $t=2$.

Path	Exercise	Future
1	0.02	0.0369
2	-	-
3	0.03	0.0461
4	0.13	0.1176
5	-	-
6	0.33	0.1520
7	0.26	0.1565
8	-	-

The payoff obtained by the immediate exercise is equal to $1.10 - X$ for In-the-Money paths; the expected payoff for the next period is obtained by replacing X by its value in the expression of the expected value.

Here, we would exercise for paths 4, 6 and 7, and we would obtain the following payment matrix:

Path	t=1	t=2	t=3
1	-	0.00	0.00
2	-	0.00	0.00
3	-	0.00	0.07
4	-	0.13	0.00
5	-	0.00	0.00
6	-	0.33	0.00
7	-	0.26	0.00
8	-	0.00	0.00

We proceed backwards, and now focus on which paths should be exercised at time $t=1$. We find 5 paths to take into account, we therefore define X and Y accordingly.

Path	Y	X
1	$0.00 \times e^{-0.06}$	1.09
2	-	-
3	-	-
4	$0.13 \times e^{-0.06}$	0.93
5	-	-
6	$0.33 \times e^{-0.06}$	0.77
7	$0.26 \times e^{-0.06}$	0.84
8	$0.00 \times e^{-0.06}$	0.88

On deduce the new least squares relation:

$$\mathbb{E}(Y|X) = 2.038 - 3.335X - 1.356X^2$$

This relation allows us to build the following table in order to take the decision to exercise or to wait:

Path	Exercise	Future
1	0.01	0.0139
2	-	-
3	-	-
4	0.17	0.1092
5	-	-
6	0.34	0.2866
7	0.26	0.1175
8	0.22	0.1533

An early exercise is obviously optimal for trajectories 4, 6, 7 and 8. We sum up our results for each step in the following tables:

Path	t=1	t=2	t=3
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

Path	t=1	t=2	t=3
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

We have therefore identified the optimal exercise moment for each of the trajectories. It is possible that a path does not have any optimal exercise moment (like path 2) or that the regression does not allow to spot such an optimal moment (like path 1). We can now evaluate the option price by computing the mean of the present value of the chosen payoffs. Here, we have the mean of the payoffs equal to *0.1144*, it is the price at which the option will be sold. It is approximately twice the price of the European option (*=0.564*) obtained by computing the present value of final payoffs at time $t=3$.

This short numerical example illustrates well how least squares use the information from simulated paths in order to estimate the conditional expected value: this conditional expected value is the key for the determination of the optimal exercise moment.

4. 3. Products and strategies available to hedge volatility exposition

In this section, we will study different products and strategies that are used on a daily basis by exotic traders to return the risk they accumulate through the selling of autocallable structures like the one we have been studying. It is important to note that traders manage risk on a large scale, and they deal with various types of clients (from private banks to hedge funds): part of the risk generated by one particular business can be compensated by another business. The variety of investor profiles implies a variety of products with different expositions, and a trader's book is composed of these heterogeneous components. They rarely focus on the hedging of one particular product, as they want to optimize their risk management on a global scale. Therefore, the analysis we are going to lead thereafter (centered on the neutralization of sensitivities linked to a single autocallable structure), while being a good way to study hedging challenges, doesn't perfectly correspond to a trader's day-to-day book management.

4. 3. 2. Main issues linked to the hedging of a regular Down-and-In Put option

Let's first present the main difficulties linked to the hedging of a single-underlying PDI, as it represents one fundamental optional component of our structure. The main risk with such a discontinuous payoff is mainly located around the barrier: the monitoring around this level has to be very careful as we saw before with the use of smoothing gaps for instance. The sensitivities implied by this option can vary a lot depending on whether it is American or European. In the context of our work we chose to study a structure with an option observable at maturity, but one could have had chosen a continuous observation throughout the product's life. This feature can have a particularly important impact on the volatility exposition: due to the fact that continuously monitored barrier options can "knock-in" at any moment until maturity, the Vega sensitivity is spread across the different time-buckets (the implied volatility term structure then becomes crucial). Moreover, no matter if it is American or European, a PDI always is very sensitive to volatility across different strikes: the skew is also an important parameter to take into account as it has a positive impact on the PDI price. More precisely in the case of a barrier that is only monitored at maturity (European PDI), traders must identify what will be the skew level at the end of the product's life in order to establish a consistent Vega hedge strategy. Such a strategy would typically imply Vanilla options (Out-of-the-Money Put options for instance) on the same underlying asset. For this strategy to be efficient, the trader must take into account the "downside skew cost" linked to the Vega hedging with low strike Vanilla options in order to calibrate his pricing and cost calculation model. These particular sensitivities to the whole implied volatility surface give a justification to the necessary use of volatility models taking into account both the skew and the term structure: even with a single-underlying PDI, the Vega hedging process requires many subtleties.

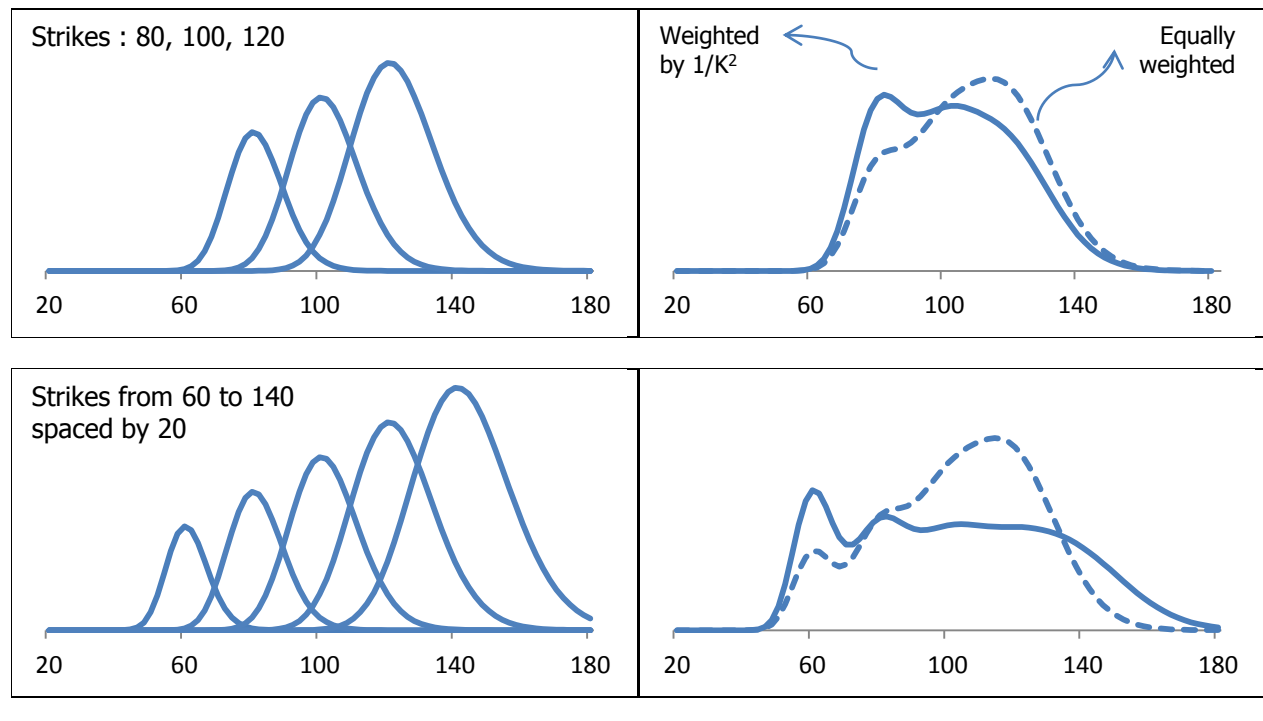
In the case of a WOF PDI, the dispersion effect implies a much higher potential payoff, hence the higher price. But on a general point of view, the risk management stays relatively equivalent except for the fact that sensitivities are split across the different underlying assets. Traders are no longer buying volatility on a single underlying stock, they are now long volatility on each component of the basket. The Vega hedging strategy can therefore still be executed with Out-of-the-Money Put options, but on several underlyings: transaction costs can grow substantially. Moreover, the individual Vega on each stock can vary greatly depending on the individual volatilities, as the price of the option is more sensitive to very volatile underlyings (because they have a higher probability to become the worst performing one). This is why different amounts must be used for each single-asset Out-of-the-Money Put options to Vega hedge the WOF PDI.

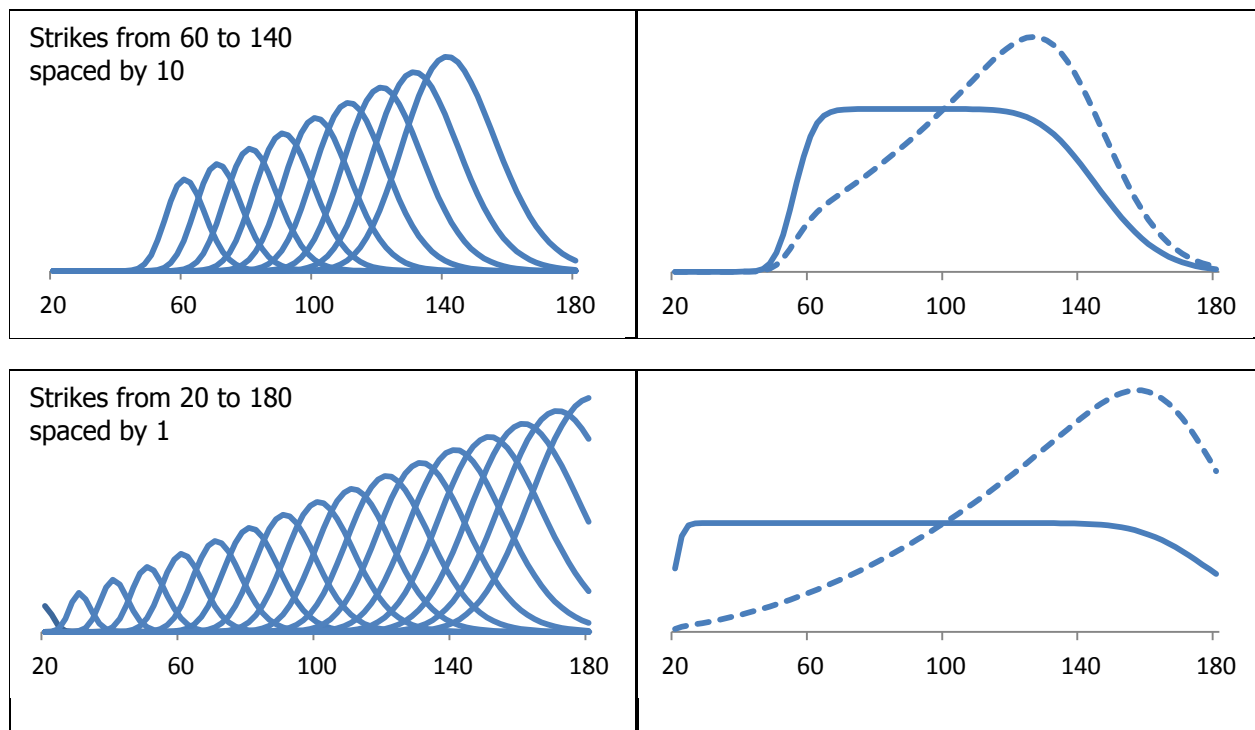
4. 3. 2. Volatility hedging with Variance and Volatility Swaps

Different types of products are used to hedge exotic derivatives volatility, from plain Vanilla options to volatility and variance swaps. The use of regular individual Call and Put options is common as it can help to neutralize local volatility expositions while taking advantage of a high liquidity on the market. Classical structures like straddles or a strangles (respectively composed of a long Put and a long Call with the same or different strike prices) represent widely used volatility strategies for instance. Nevertheless it seems legitimate to analyze whether it is relevant or not to only use such Vanilla strategies. With these products, it is possible to capture the underlying's volatility, but exclusively around a predetermined strike level. If throughout the life of the product, the underlying ends up at a very inferior level, the product is not exposed to its volatility in the same way. Indeed, far from the strike price, Greeks are diminished, and the sensitivity to volatility is almost null. There we understand the usefulness of volatility and variance swaps as they allow obtaining an exposition to the underlying's volatility all across the range of possible strikes.

Some years ago, variance swaps were more often traded in the European market than volatility swaps because of the possibility to replicate them, and therefore to hedge them. Indeed, it has been proven that it is theoretically possible to perfectly replicate a variance swap, using an infinite number of Put and Call options weighted in a way inversely proportional to their squared strike price, that is to say weighted by $\frac{1}{K^2}$ (we will not detail the calculations as variance swaps don't represent our main study subject). Thanks to this weighting it is possible to obtain a large and smooth Gamma for the replicating portfolio of Vanilla options. As we can see thereafter, the more options across all possible strikes are used, the more efficient the replicating portfolio will be (the only limit being the availability of deep In-the-Money and Out-of-the-Money options):

Figure16: impact of the number of Vanilla options in the replicating portfolio on the Gamma





This product is therefore a perfect way to obtain a linear exposition to the underlying's variance. Nevertheless, the fact that it is sensitive to the squared volatility makes it potentially explosive: this is why a lot of major Investment Banks went through severe losses. Even if nowadays variance swaps are usually capped at $2.5 * Strike$, the European market isn't as liquid as it used to be, and there is a growing interest for volatility swaps.

Indeed, volatility swaps have recently gained popularity in the market as they allow a direct exposition to the underlyings' volatility. However, such a product comes with its drawbacks: it is not as easy to replicate and hedge it as variance swaps. It is possible to represent the exposition of the volatility swap to the underlying's variance as a square root function. It is then possible to approximate this curve by a piece-wise affine function composed of Call Spreads on the variance level between each point. Through this angle it seems possible to replicate the product to a certain extent.

But the concave shape of the function leads us to another issue: the sensitivity to a higher order parameter, the volatility of volatility. As the volatility swap payoff is a concave function of variance, an increase of the "vol of vol" would have a negative impact on the product. By buying a "vol swap", a trader is instantaneously "short vol of vol". This particular exposition requires the use of stochastic volatility models: pricing a volatility swap with a deterministic volatility model would result in an inflated price compared to the fair one, because it wouldn't take into account the concavity exposed before. Volatility and variance swaps therefore represent efficient Vega hedging strategies, but they tend to be less liquid, and such products are not available for all possible baskets of underlying stocks: in order to hedge our typical autocallable structure, the use of Vanilla options seems more appropriate.

4. 3. 3. Volatility hedging with Vanilla options

We chose to lead a short study on the volatility hedging of the bank's short position on the autocallable structure with Vanilla options. In order to do so, we used the simulation tools available in SG's systems. First, we chose a generic script used to price a multitude of autocallable structures: such a script consists in approximately 1200 lines of code in Vegas (SG's proprietary programming language) and contains highly confidential information. We then filed the different parameters (underlying stocks, schedules, barriers...) and market data that corresponded to our typical structure in order to model its payoff. The next step was to create a simulation file based on this pricing script and payoff: the purpose was to automate the pricing with the script according to different scenarios through a simulation algorithm. Here is a screenshot of the simulation script we wrote in order to iterate prices according to the simulation data we will show after:

```
1 FINSCEANRIO scenario
2 {
3     DATETIME[] PricingDates,
4     DOUBLE[][] PricingSpots,
5     DOUBLE[] InitialSpots,
6     INTEGER[] IndexInBasket,
7     BOOLEAN SimulsCroisees,
8     BOOLEAN ShiftQuotes
9 }
10 {
11     /*****/
12     // Initialisations
13     /*****/
14     IF (SimulsCroisees == FALSE) DO {
15         SET_RESULT_KEY({"PricingDate"});
16     }
17     ELSE {
18         SET_RESULT_KEY({"PricingDate", "Scenar"});
19     }
20
21
22     /*****/
23     // Simulation loop
24     /*****/
25     numSimuls = 0;
26     maxSimuls = 100000;
27     index = 1;
28     FOR date FROM PricingDates DO
29     {
30         PRICINGSETTINGS.pricingDate = date;
31         PRICINGSETTINGS.paramDate = date;
32
33         //
34         //-- Cas des simuls synchrones --
35         IF (SimulsCroisees == FALSE) DO
36         {
37             // Remplacement des spots
38             spots = PricingSpots[index];
39             IF (ShiftQuotes == FALSE) DO {
40                 FOR i FROM [1; spots.size] DO {
41                     OVERRIDE_QUOTE("Spot", PAYOFF.Basket[IndexInBasket[i]], spots[i]);
42                 }
43             }
44         }
45     }
46 }
```



```

50 // Pricing
51 IF (numSimuls <= maxSimuls) DO {
52     PRICE({TOSTRING(TODATE(date))}, PAYOFF, PRICINGSETTINGS);
53     numSimuls = numSimuls + 1;
54 }
55 ELSE {
56     PRICE_AND_WAIT({TOSTRING(TODATE(date))}, PAYOFF, PRICINGSETTINGS);
57     numSimuls = 0;
58 }
59 index = index + 1;
60 }
61
62 //
63 //-- Cas des simuls croisées --
64 ELSE {
65     FOR k FROM [1; PricingSpots.size] DO {
66
67         // Remplacement des spots
68         spots = PricingSpots[k];
69         IF (ShiftQuotes == FALSE) DO {
70             FOR i FROM [1; spots.size] DO {
71                 OVERRIDE_QUOTE("Spot", PAYOFF.Basket[IndexInBasket[i]], spots[i]);
72             }
73         }
74         ELSE {
75             FOR i FROM [1; spots.size] DO {
76                 SHIFT_QUOTE("Spot", PAYOFF.Basket[IndexInBasket[i]], spots[i] - InitialSpots[i], TRUE);
77             }
78         }
79
80         // Pricing
81         IF (numSimuls <= maxSimuls) DO {
82             PRICE({TOSTRING(TODATE(date)), TOSTRING(k)}, PAYOFF, PRICINGSETTINGS);
83             numSimuls = numSimuls + 1;
84         }
85         ELSE {
86             PRICE_AND_WAIT({TOSTRING(TODATE(date)), TOSTRING(k)}, PAYOFF, PRICINGSETTINGS);
87             numSimuls = 0;
88         }
89     }
90 }
91 }
92 }

```

Once we had designed this pricing algorithm in Vegas, we created a scripted simulation data file that gathered the parameters we wanted to implement in our simulations. For our simulation, we chose to study the specific case where only one of the three underlying stocks of the basket drops dramatically: this scenario seemed interesting as it would underline and amplify the sensitivities linked to the emergence of a worst performing asset. We therefore created another programming sequence that contained the spot levels consistent with the drop of one of the underlying stock. Here is a screenshot of part of our simulation data script:

```

1
2 FINSIMULATION
3 {
4 SimulsCroisees = TRUE;
5 ShiftQuotes = FALSE;
6 IndexInBasket = {1, 2, 3};
7 InitialSpots = {0.0, 0.0, 0.0};
8 PricingDates =
9 {
10 2017:04:07:23:59:59
11 };
12
13 PricingSpots =
14 {
15 {5832.01, 78.99, 11.23},
16 {5832.01, 78.99, 11.01},
17 {5832.01, 78.99, 10.79},
18 {5832.01, 78.99, 10.57},
19 {5832.01, 78.99, 10.36},
20 {5832.01, 78.99, 10.15},
21 {5832.01, 78.99, 9.95},
22 {5832.01, 78.99, 9.75},
23 {5832.01, 78.99, 9.55},
24 {5832.01, 78.99, 9.36},
25 {5832.01, 78.99, 9.18},
26 {5832.01, 78.99, 8.99},
27 {5832.01, 78.99, 8.81},
28 {5832.01, 78.99, 8.64},
29 {5832.01, 78.99, 8.46},
30 {5832.01, 78.99, 8.29},
31 {5832.01, 78.99, 8.13},
32 {5832.01, 78.99, 7.97},
33 {5832.01, 78.99, 7.81},
34 {5832.01, 78.99, 7.65},
35 {5832.01, 78.99, 7.5},
36 {5832.01, 78.99, 7.35},
37 {5832.01, 78.99, 7.2},
38 {5832.01, 78.99, 7.06},
39 {5832.01, 78.99, 6.92},
40 {5832.01, 78.99, 6.78},
41 {5832.01, 78.99, 6.64},
42 {5832.01, 78.99, 6.51},
43 {5832.01, 78.99, 6.38},

```

As we can see, we chose to simulate the scenarios where the two first underlying stocks stayed at their initial level, while the last one dropped by 2% at each step.

We were then able to obtain the Autocallable price and Greeks for each of the simulation's step through the use of the Longstaff-Schwartz Monte-Carlo algorithm presented before. For each scenario step, we extracted the Autocallable price and the individual underlying stocks' Delta and Vega.

We then tested 3 different volatility hedging strategies at the product's inception using Delta hedged Vanilla options on each underlying stock expiring at the Autocallable's maturity. The first one was with At-the-Money Call options, the second one with 85% Out-of-the-Money Put options, and the last one with 70% Out-of-the-Money Put options.

It is important to note that we only considered a "Vega-weighted" approach at inception in order to neutralize the volatility exposition on each underlying stock at the Autocallable's start date. It is also possible to consider a dynamic Vega hedging strategy consisting in the permanent rebalancing of the individual Vanilla options portfolio through the product's life: this is actually what traders do on the scale of the whole book. We thought that implementing a dynamic Vega hedging strategy for a single structured product was inconsistent with the actual hedging process of exotic derivatives traders. Indeed, they daily accumulate dozens of products, and the purpose of the dynamic rebalancing of their hedging strategy is to adapt to the constantly evolving composition and sensitivities of their book, not to neutralize local expositions for specific trades. Thus we chose to focus on volatility hedging

methods at inception: it seemed more appropriate in the context of massive derivatives books distorted by a permanent incoming flow.

In order to obtain the prices and sensitivities linked to the single-asset Vanilla options, we used simulation scripts and scenarios analogous to the ones described for the Autocallable, but based on single-asset Call and Put payoffs. For each scenario, we once again retrieved the Vanilla options prices and the Delta and Vega values. We then began testing the consistency of our hypothetical hedging methods. We will present our main conclusion for each strategy, and then describe a little more precisely our process for the case of the 70% Put options

Using At-the-Money single-asset Call options in order to hedge the Vega exposure of the Autocallable seemed at first sight to be a decent strategy, as Call options represent the most basic and liquid derivatives. We therefore computed the particular quantities of individual Call options to sell in order to neutralize the Vega exposure of our position at inception. The impact of this strategy was satisfying in terms of volatility sensitivity: it achieved to reduce considerably the Vega exposure of the position on each underlying stock. But this strategy had a major inconvenient: the dynamic Delta hedging adjustments implied by the short Call position increased greatly the quantities of stocks needed to remain Delta neutral. Indeed, the bank's short position on the Autocallable implied a negative exposition to the underlying stocks' changes, that is to say that we initially had to buy stocks in order to remain Delta neutral. Adding a short position on Call options implied an even more negative Delta exposition, and it was clear that it was not the effect we sought. We therefore chose to reject this strategy, as it required too many transaction costs.

We therefore tried to hedge the volatility exposure with the selling of single-asset 85% Put options. The intuition behind this was that Put and Call options have equivalent sensitivities to volatility, but opposite expositions to spot changes. We therefore thought that the strategy would remain interesting from the perspective of the Vega hedge, while not incurring additional transaction costs due to dynamic Delta rebalancing. As the Autocallable is partly composed of a WOF PDI, Vanilla Put options rather seemed like a decent way to hedge the position. We chose the 85% strike level because it is exactly between the WOF PDI knock-in threshold and the automatic recall barrier. We were satisfied by this strategy as it showed the desired results: the volatility exposure was reduced around the inception of the Autocallable, and the Delta hedging adjustments linked to the short Put positions lowered the overall Delta hedging volumes. Moreover, the overall cost of buying Out-of-the-Money Put options was cheaper than At-the-Money Call options even if we had to buy fewer Call options in order to be Vega neutral at inception.

But then we realized that the sensitivity to volatility of our autocallable structure was centered on the 70% Knock-In threshold. Indeed, it is around this activation barrier that all the sensitivities of the WOF PDI are the most explosive. This is why we implemented a last hedging strategy using exactly the same reasoning as the second one, but with 70% strikes instead of 85%. The impact on the volatility exposure after the inception of the Autocallable finally was more satisfying than with the 85% Put s. This is why we considered this strategy as the most efficient one between all three. The only drawback was that the 70% Put options being further Out-of-the-Money than the 85% ones, Delta values were smaller, and the impact on the dynamic Delta hedging process was less positive than with the 85% Puts. Nevertheless, as these 70% Puts were cheaper than the 85% ones, the overall effect was negligible, and we considered that the efficiency of the Vega neutralization closely after inception was the most important.

On the following screenshot is an extract of the different prices and sensitivities extracted from our simulations. On the left we can see the Autocallable prices and the Delta and Vega on each individual stock of the underlying basket. Please note that here we expressed the Delta in terms of the number of underlying stocks to sell (-) or buy (+) in order to be Delta neutral for a 100 position in the product.

Scenario	Autocall Price	vega (TOYOTA)	delta (TOYOTA)	vega (FORD)	delta (FORD)	vega (RENAULT)	delta (RENAULT)		70% Put Toyota	vega (TOYOTA)	delta (TOYOTA)
1	76.59604889	-0.276037715	0.4558646	-0.40919	3.027334	-0.415139	0.4310474		3.71542153	0.364934	-0.2946
2	75.88316695	-0.269760926	0.4455008	-0.40948	3.433573	-0.402989	0.4095467		3.71542153	0.364934	-0.2946
3	75.11457411	-0.258314926	0.4301109	-0.41778	3.740637	-0.395596	0.3942912		3.71542153	0.364934	-0.2946
4	74.19163601	-0.253602514	0.4007328	-0.41347	4.276979	-0.38605	0.3770079		3.71542153	0.364934	-0.2946
5	73.28523049	-0.245131587	0.3878411	-0.40301	4.565315	-0.381645	0.3667389		3.71542153	0.364934	-0.2946
6	72.31378864	-0.241990522	0.3684088	-0.39659	4.847089	-0.370654	0.3498355		3.71542153	0.364934	-0.2946
7	71.30841163	-0.232526292	0.3557251	-0.38182	4.894142	-0.359321	0.3255785		3.71542153	0.364934	-0.2946
8	70.30090558	-0.2233744	0.3377647	-0.36328	5.132131	-0.348643	0.3189497		3.71542153	0.364934	-0.2946
9	69.25403376	-0.207333499	0.3120755	-0.38602	5.467406	-0.339076	0.3051642		3.71542153	0.364934	-0.2946
10	68.17738145	-0.195581011	0.2949112	-0.34734	5.961392	-0.32882	0.2913969		3.71542153	0.364934	-0.2946
11	67.06402565	-0.189947579	0.2821172	-0.33992	6.371242	-0.313441	0.2756433		3.71542153	0.364934	-0.2946
12	65.83868538	-0.180210099	0.2551208	-0.30895	6.333332	-0.3062	0.2652274		3.71542153	0.364934	-0.2946
13	64.67141345	-0.170568715	0.2270793	-0.29096	6.668847	-0.296105	0.2504797		3.71542153	0.364934	-0.2946
14	63.51025646	-0.164907857	0.2239886	-0.27978	6.93189	-0.286998	0.2390937		3.71542153	0.364934	-0.2946
15	62.26642284	-0.160776265	0.2154829	-0.2664	6.816504	-0.274449	0.2254963		3.71542153	0.364934	-0.2946
16	61.03720805	-0.15345524	0.1982874	-0.22683	7.122626	-0.261365	0.2075042		3.71542153	0.364934	-0.2946
17	59.91825524	-0.146394127	0.1847942	-0.25777	7.725573	-0.248678	0.1897431		3.71542153	0.364934	-0.2946
18	58.69140157	-0.142703088	0.1750346	-0.20103	7.520947	-0.240036	0.181281		3.71542153	0.364934	-0.2946
19	57.42645702	-0.136956747	0.1643599	-0.14686	7.948009	-0.228018	0.1689243		3.71542153	0.364934	-0.2946
20	56.14751375	-0.129063752	0.1539349	-0.15078	8.1963	-0.217401	0.1622661		3.71542153	0.364934	-0.2946
21	54.94380054	-0.121152602	0.1420388	-0.15548	7.952109	-0.208831	0.152928		3.71542153	0.364934	-0.2946
22	53.73413653	-0.114923124	0.1316424	-0.14554	8.117163	-0.196283	0.1432589		3.71542153	0.364934	-0.2946
23	52.54538358	-0.109512006	0.1241011	-0.12484	7.758793	-0.187228	0.1359414		3.71542153	0.364934	-0.2946
24	51.40925558	-0.102411036	0.1136903	-0.10085	8.565283	-0.180158	0.127535		3.71542153	0.364934	-0.2946
25	50.19960813	-0.097975737	0.1105986	-0.08149	8.097085	-0.170341	0.1166325		3.71542153	0.364934	-0.2946
26	49.0375853	-0.092107922	0.1038785	-0.06856	8.640525	-0.162167	0.1078012		3.71542153	0.364934	-0.2946
27	47.82645478	-0.086341525	0.0973632	-0.03283	8.55333	-0.149123	0.0959476		3.71542153	0.364934	-0.2946
28	46.7297502	-0.080480602	0.0873112	-0.05223	8.907957	-0.141604	0.0891919		3.71542153	0.364934	-0.2946
29	45.56497971	-0.074665305	0.0791969	-0.0104	8.804232	-0.133367	0.0828602		3.71542153	0.364934	-0.2946
30	44.43646201	-0.068873682	0.0734228	-0.0359	8.693463	-0.126496	0.0768312		3.71542153	0.364934	-0.2946
31	43.38572988	-0.06469738	0.0650679	-0.01696	8.919494	-0.119236	0.0720201		3.71542153	0.364934	-0.2946
32	42.21283708	-0.060858708	0.06166	0.006369	8.89666	-0.112335	0.0671294		3.71542153	0.364934	-0.2946
33	41.14093371	-0.057715301	0.0590405	0.019979	8.811072	-0.106462	0.0633066		3.71542153	0.364934	-0.2946
34	40.20158095	-0.055654755	0.0568128	-0.01142	8.455568	-0.101305	0.0601645		3.71542153	0.364934	-0.2946
35	39.16730031	-0.05335871	0.0517701	0.025745	8.947725	-0.094947	0.0552683		3.71542153	0.364934	-0.2946
36	38.16944195	-0.048079671	0.0452243	0.045648	9.000954	-0.088426	0.0496603		3.71542153	0.364934	-0.2946
37	37.18198204	-0.041859937	0.0371083	0.042783	8.759912	-0.083978	0.0461677		3.71542153	0.364934	-0.2946
38	36.21814806	-0.037270906	0.0333218	0.031396	9.117412	-0.079127	0.0430083		3.71542153	0.364934	-0.2946
39	35.19422609	-0.035069986	0.0306356	0.071824	8.986811	-0.073854	0.0394579		3.71542153	0.364934	-0.2946
40	34.30446746	-0.032633009	0.0287172	0.061068	8.9631	-0.070538	0.0374785		3.71542153	0.364934	-0.2946
41	33.40097149	-0.030064013	0.0263627	0.071053	9.184389	-0.067099	0.0357287		3.71542153	0.364934	-0.2946
42	32.47997673	-0.026236061	0.0220857	0.087689	9.316412	-0.063349	0.0328446		3.71542153	0.364934	-0.2946
43	31.56372893	-0.023170438	0.0200854	0.079862	9.01509	-0.056841	0.0282871		3.71542153	0.364934	-0.2946
44	30.68069079	-0.021272669	0.0185771	0.045155	8.934863	-0.052249	0.025686		3.71542153	0.364934	-0.2946

To calculate the amount of single-asset Vanilla options to buy on each stock in order to be Vega hedged at inception, we determined the quantity x_i of Vanilla options on the underlying stock i to buy to neutralize the position's sensitivity to moves in σ_i . In the case of our three underlyings basket, we therefore computed x_i such that:

$$(48) \quad \forall i \in \{1,2,3\}, \frac{\partial A_t^B}{\partial \sigma_i} = x_i \frac{\partial V_i}{\partial \sigma_i}$$

where A_t^B is the Autocallable price on the underlying basket B at time t , and V_i is the Vanilla option price on the underlying stock i .

We therefore obtained the optimal Vega hedge ratio $\frac{Vega_{Autocall}(S_i)}{Vega_{Vanilla}(S_i)}$ at the start date of our Autocallable for each stock. As an example, following are the optimal initial Vega hedge ratios we obtained for the 70% Put options:

Underlying	Toyota	Ford	Renault
Vega Hedge Ratio	-0.756403769	-0.888388864	-0.927587985

We then computed the overall position's Vega exposition in each scenario considering both the initial short position on the Autocallable and the additional positions in Vanilla options. In this way we were able to compare the Vega exposure in the case of the "naked" short Autocallable position, and in the case of the hedged position. Following is a screenshot of Vega exposures throughout the first scenarios (the left table corresponds to the Vega hedged position at inception and the right one to the "naked" short Autocallable position):

Vega Toyota	Vega Ford	Vega Renault		Vega Toyota	Vega Ford	Vega Renault	
0	0	0		-0.27604	-0.40919	-0.41514	
0.006277	0.009546	0.01215		-0.26976	-0.40948	-0.40299	
0.017723	-0.00011	0.019543		-0.25831	-0.41778	-0.3956	
0.022435	0.004502	0.029088		-0.2536	-0.41347	-0.38605	
0.030906	0.013331	0.033493		-0.24513	-0.40301	-0.38165	
0.034047	0.021186	0.044484		-0.24199	-0.39659	-0.37065	
0.043511	0.041089	0.055817		-0.23253	-0.38182	-0.35932	
0.052663	0.05505	0.066495		-0.22337	-0.36328	-0.34864	
0.068704	0.025109	0.076062		-0.20733	-0.38602	-0.33908	
0.080457	0.066603	0.086319		-0.19558	-0.34734	-0.32882	
0.08609	0.068611	0.101697		-0.18995	-0.33992	-0.31344	
0.095828	0.092906	0.108938		-0.18021	-0.30895	-0.3062	
0.105469	0.106261	0.119034		-0.17057	-0.29096	-0.29611	
0.11113	0.112604	0.128141		-0.16491	-0.27978	-0.287	
0.115751	0.116032	0.140680		-0.16078	-0.2654	-0.27445	

Finally, we computed the dynamic Delta hedging by summing the initial Delta hedging requirements (for the short Autocallable position) and the Delta adjustments linked to the Vanilla options. We therefore obtained the number of stocks required to have an overall Delta neutral position, and were able to analyze each hedging strategy's impact on the Delta rebalancing process.

4. 4. Correlation exposition hedging and dispersion strategies

In order to hedge correlation, two main products are available in the market: the correlation swap and the Calls vs. Call. Let's study the second one, as it is currently the most liquid on the European market, and evaluate how it represents a good way to neutralize part of the exposition to correlation of a book.

4. 4. 1. A widely used correlation trade: the Calls vs. Call

This product is structured as follows: the buyer buys Call options on each underlying stock of a basket, and sells a multi-underlying Call option on the basket. By doing so, a trader bets on the correlation between the different components of the basket. Indeed, the asymmetric nature of the Call options implies that the buyer benefits from very un-correlated stock performances: extremely positive performances induce deep In-the-Money Call options, while extremely negative ones won't have impact. On the other side, by selling a Call on the basket of underlyings,

the buyer of the product bets against the correlation between the stocks. Indeed, a Call option on basket is intrinsically “long correlation”. The payoff of the option is calculated based on the mean stock performance of the basket: stocks moving in different directions would lower the payoff as their opposite performances would be offset through the averaging process. All in, the buyer of a Calls vs. Call is therefore short correlation. However, in order for this product to be traded as an efficient correlation hedge, it has to be neutral regarding other parameters, like volatility (otherwise, the hedge process would finally add sensitivities to the book). Nevertheless, this product in itself has a positive sensitivity to volatility: the Vega of a Call on basket is lower than the Vega of the sum of the individual Calls because of the impact of correlation on the covariance term of the Call on basket. In order to trade a Calls vs. Call “Vega Flat” it is then important to add a multiplier in front of the individual Calls leg in order to obtain a neutral exposition to volatility moves. Usually, this multiplier is around 0.8.

While it is possible to hedge volatility and correlation separately with the particular products mentioned before, there also exist ways to neutralize both expositions at the same time: dispersion trades.

4. 4. 2. The potential benefits of dispersion strategies

These trades are based on taking a position on mono-underlying options, while taking the inverse position on basket options (the basket being composed of the same underlying stocks as the individual options): the structure is quite similar to the Calls vs. Call, but is not traded “Vega flat” in order to keep a volatility exposition. For instance, by selling Puts on individual underlyings and buying a Put on the basket, one would be “selling” individual volatilities, and “buying” correlation between stocks. From this we clearly understand that such a product would nicely return the risk implied by the selling of the autocallable structure. Nevertheless it is important to note that by selling the individual volatilities and buying the basket volatility, the trader would convert the individual “stock Vega” created by the Autocallable into a “basket Vega”. As the sensitivity to the basket volatility doesn’t seem more comfortable to manage than the one to individual stocks volatilities, it doesn’t seem as a satisfying solution. Dispersion strategies in the context of our single autocallable product therefore don’t seem efficient.

Moreover, it is important to bring a few precisions on the subtleties linked to basket derivatives risk management. The basket is a linear combination of log-normal random variables, which is not log-normal. Therefore, when the basket value is calculated with an arithmetic mean, it makes the classical Black-Scholes model unfit. One solution would be to replace the arithmetic mean by a geometric mean, the advantage being that the product of two log-normal random variables is log-normal. From there, it would be possible to deduce a “Black-Scholes-like” explicit formula: the arithmetic basket option value would then be approximated by the geometric one using an adjusted strike price for a better precision (the strike price difference corresponding to the spread between the arithmetic and the geometric mean expected value). Through this method it is possible to obtain a closed-formula, and to compute Greeks easily.

Another method would be to consider that the linear combination of log-normal random variables is also log-normal. This approach implies a bias, but allows using the classical Black-Scholes formula to compute the basket option price, and to quote the implied volatility of the basket. This bias is mainly linked to the fact that the weighting of the different stocks of the basket is not constant through time: it is going to progressively evolve in parallel to the spots changes. The main question would then be to determine what implied volatility to plug into the formula: the idea is to find the basket implied volatility that would correct the bias implied by the model. We will now give more detailed calculations on how to obtain such value.

Let’s first write the expression of α_i the weight in the index I of the stock S_i , where n_i is the number of stocks issued by the company:

$$(49) \quad \alpha_i = \frac{n_i S_i}{\sum_{j=1}^N n_j S_j}$$

We now suppose the following dynamics for the stocks and the index (considering the hypothesis of log-normal returns for both the stocks and the index as mentioned before):

$$(50) \quad \frac{dS_i}{S_i} = r dt + \sigma_i dW_t^i$$

$$(51) \quad \frac{dI}{I} = r dt + \sigma_I dW_t^I$$

Consequently:

$$(52) \quad S_i(t) = S_i(0) \exp \left(\left(r - \frac{\sigma_i^2}{2} \right) t + \sigma_i W_t^i \right)$$

$$(53) \quad I(t) = I(0) \exp \left(\left(r - \frac{\sigma_I^2}{2} \right) t + \sigma_I W_t^I \right)$$

We therefore obtain:

$$(54) \quad \alpha_i(t) = \alpha_i(0) \exp \left(\left(\frac{\sigma_I^2 - \sigma_i^2}{2} \right) t + \sigma_i W_t^i - \sigma_I W_t^I \right)$$

As the weights of the underlying stocks appear in the variance formula through their products, we can write:

$$(55) \quad \alpha_i(t) \alpha_j(t) = \alpha_i(0) \alpha_j(0) \exp \left(\left[\sigma_I^2 - \frac{\sigma_i^2 - \sigma_j^2}{2} \right] t + \sigma_i W_t^i + \sigma_j W_t^j - 2\sigma_I W_t^I \right)$$

Let's write $K_{ij}(t) = \exp \left(\left[\sigma_I^2 - \frac{\sigma_i^2 - \sigma_j^2}{2} \right] t \right)$ (deterministic)

And $u_{ij}(t) = \sigma_i W_t^i + \sigma_j W_t^j - 2\sigma_I W_t^I$ (random)

Thus:

$$(56) \quad \alpha_i(t) \alpha_j(t) = \alpha_i(0) \alpha_j(0) K_{ij}(t) \exp(u_{ij}(t))$$

Let's now compute the second-order expansion of $\exp(u_{ij}(t))$:

$$(57) \quad \exp(u_{ij}(t)) \approx 1 + u_{ij}(t) + \frac{u_{ij}(t)^2}{2} + O(u_{ij}(t)^3)$$

As $u_{ij}(t)$ is a centered random variable, we can compute the expected value of the weights product:

$$(58) \quad E(\alpha_i(t) \alpha_j(t)) \approx \alpha_i(0) \alpha_j(0) K_{ij}(t) \left(1 + \frac{1}{2} E(u_{ij}(t)^2) \right)$$

With:

$$(59) \quad E(u_{ij}(t)^2) = [\sigma_i^2 + \sigma_j^2 + 4\sigma_I^2 + 2\sigma_i \sigma_j \rho_{i,j} - 4\sigma_i \sigma_I \rho_{i,I} - 4\sigma_j \sigma_I \rho_{j,I}] t$$

Lastly, let's note $\sigma_I^F(t)$ the expression obtained for our estimation in $t=0$ of the index instantaneous volatility in t .

$$(60) \quad \sigma_I^2(t) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i(t) \alpha_j(t) \rho_{i,j} \sigma_i \sigma_j$$

$$(61) \quad \sigma_I^{F^2}(t) = E(\sigma_I^2(t)) = \sum_{i=1}^N \sum_{j=1}^N E(\alpha_i(t) \alpha_j(t)) \rho_{i,j} \sigma_i \sigma_j$$

$$(62) \quad \sigma_I^{F^2}(t) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i(0) \alpha_j(0) K_{ij}(t) \left(1 + \frac{1}{2} [\sigma_i^2 + \sigma_j^2 + 4\sigma_i^2 + 2\sigma_i \sigma_j \rho_{i,j} - 4\sigma_i \sigma_I \rho_{i,I} - 4\sigma_j \sigma_I \rho_{j,I}] t\right) \rho_{i,j} \sigma_i \sigma_j$$

We finally obtain the following result: the implied volatility to use for an option of maturity T is the mean of forward volatilities between $t=0$ and $t=T$:

$$(63) \quad \sigma_I^*(T) = \sqrt{\frac{1}{M} \sum_{i=0}^{M-1} \sigma_I^F\left(i \cdot \frac{T}{M}\right)^2}$$

$$(64) \quad \sigma_I^*(T) = \sigma_I^F\left(\frac{T}{2}\right)$$

In reality, as traders have to manage the volatility and correlation sensitivities for hundreds of different underlying stocks. The aggregation of all these individual volatilities expositions makes it possible to enter into more interesting dispersion deals at the scale of the book: trading individual options against index options, and not against specific baskets. Instead of converting stock volatility risk into basket volatility risk, it makes it possible to convert it into index volatility exposition. This conversion has a very positive impact on the management of the book Vega, as it allows the using of index options to dynamically hedge volatility and correlation risk. This index options are more liquid, and easier to trade than a multiplicity of individual options: these dispersion structures reduce the number of hedging trades, and lower transaction costs (as for instance the bid-ask spread is tighter on for more liquid derivatives).

5. Conclusion

Typical autocallable structures represent attractive products for different investor profiles as they imply expositions to a very wide range of variables while being a relatively conservative alternative to regular long equity investments. As their payoff is based on a basket of underlying assets and involves several optional and digital components, the use of particular valuation models is crucial: in order to provide competitive and consistent prices, traders must integrate a multitude of parameters, from non-constant volatilities to asynchronous correlations. Indeed, quotes are derived from models, and traders must fully master the use and calibration of these so as to monitor the degree of aggressiveness or conservatism of their position according to constantly evolving market conditions. But exotic derivatives are not only challenging on the valuation aspect as they also involve potentially explosive expositions: while investors are willing to bear the risks intrinsically embedded in these products, traders have the duty to neutralize them. This is why we studied the concept of triangle, rectangle and “intelligent gaps”, their crucial role in the smoothing of sensitivities, and the methodology to price them through the use of the Longstaff-Scwhartz Monte Carlo algorithm. As traders have the task to counterbalance the various risks created by the sale of structured products, they use a vast scope of other derivatives to implement relevant hedging schemes. Through our study, we have been able to analyze different methodologies involving both Vanilla and exotic options. We saw on the one hand that plain Vanilla options represent the most efficient way to hedge the Vega exposure linked to the Autocallable, and on the other hand that Calls vs. Call trades allow returning nicely the sensitivity to correlation. Finally, we came to the conclusion that dispersion strategies represent a relevant way to hedge volatility and correlation exposure at the same time.

We must keep in mind that traders have to simultaneously execute dynamic hedging strategies for thousands of heterogeneous derivatives. Therefore, at the scale of a whole book, there is no such thing as a perfect hedge: it is impossible to fully annihilate the risk embedded in the products. It is all a matter of using optimally the budget in order to mitigate part of the sensitivities. Even if proprietary trading is not anymore authorized in major Investment Banks, some choices have to be made: they may not be bold gambles, yet they still represent probabilistic directional bets.

6. References

- [1] M. BOUZOUBAA and A. OSSEIRAN. Exotic Options and Hybrids – A Guide to Structuring, Pricing and Trading. Chichester, Wiley Finance, 2010.
- [2] DERMAN, Emanuel. Spring 2008 Lecture 7: Local Volatility Continued, Copyright Emanuel Derman.
- [3] DERMAN, Emanuel. Spring 2008 Lecture 9: Patterns of Volatility Change, Copyright Emanuel Derman.
- [4] DUPIRE, Bruno. Pricing with a Smile, Risk, January 1994.
- [5] BERGOMI, Lorenzo. Correlations in Asynchronous Markets, Société Générale Paper, January 2010.
- [6] F. A. LONGSTAFF and E. S. SCHWARTZ. Valuing American Options by Simulation: A Simple Least-Squares Approach, The Review of Financial Studies, Vol. 14 No. 1. (Spring, 2001), pp. 113-147.
- [7] K. DEMETERFI, E. DERMAN, M. KAMAL, J. ZOU. More Than You Ever Wanted To Know About Volatility Swaps, Copyright 1999 Goldman, Sachs & Co., March 1999.
- [8] S. BOSSU, E. STRASSER, R. GUICHARD. Just What You Need To Know About Variance Swaps, Copyright 2005 J. P. Morgan Chase & Co., February 2005.
- [9] G. DENG, J. MALLET, C. MCCANN. Modeling Autocallable Structured Products, Copyright Securities Litigation and Consulting Group, Inc., August 16, 2011.
- [10] GRANDCHAMP DES RAUX, Nicolas. Pricing and Risk Managin Autocalls, Copyright HSBC Quantitative Research, May 2010.
- [11] J. GATHERAL, N. N. TALEB. The Volatility Surface: A Practitioner's Guide, Wiley Finance, September 2006.

7. Appendix

7. 1 Some precisions regarding the pricing of PDI barrier options

Let's give the closed formula for the price of a continuously monitored Down-and-In Put in the case where the down-and-in barrier H is less than the strike K . We note that the following formulas assume that the underlying asset's price follows a log-normal distribution (Black-Scholes assumption).

$$(65) \quad DI\ PUT_{Price} = -SN(-x_1)e^{-qT} + Ke^{-rT}N(-x_1 + \sigma\sqrt{T}) + Se^{-qT}\left(\frac{H}{S}\right)^{2\lambda}[N(y) - N(y_1)] - Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2}[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})]$$

where

$$(66) \quad x_1 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$(67) \quad y_1 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

and

$$(68) \quad \lambda = \frac{r-q+\sigma^2/2}{\sigma^2}$$

$$(69) \quad y = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

7. 2 Empirical study: implementation of a new implied correlation Bid quoting methodology

Let's explain precisely how we managed to design a consistent implied correlation quote methodology in order to feed SG's systems and provide parameters for derivatives valuation. As the implied correlation Ask quotes methodology was already implemented and efficient, we chose to focus on the Bid quotes generation. The idea was to infer market quotes on implied correlation from past Calls vs. Call data in order to define target levels, and then, through the use of relevant indicators, to determine the optimal estimator for implied correlation Bids.

The first step was to use massive amounts of data from past Calls vs. Call trades: the choice was made to calibrate our quoting methodology on this type of product because of its very close link to correlation (as we have explained before it represents both a relatively "pure" correlation trade and a sufficiently liquid product). Other products like Correlation Swaps, Puts on Worst-Of or Calls on Best-Of could also have been chosen, but market data linked to these products was insufficient. We therefore extracted Calls vs. Call Bid and Ask prices from the last 6 years' database. For products that actually traded, Bid and Ask levels were naturally equivalent, whereas in the case where no trade occurred, there was a certain Bid-Ask spread: this database had been filled with information obtained by derivatives brokers. As a Calls vs. Call is a product that is intrinsically selling correlation (through the buying of the

individual legs and the selling of the basket leg as explained before), we inferred market correlation Bid levels implied from Calls vs. Call Ask prices and Ask levels implied from Calls vs. Call Bid prices . Moreover, in order to obtain a quoting methodology that would not generate too tight implied correlation quotes Bid/Ask spreads, we chose to consider Bid levels 6 points inferior to the ones inferred from market prices. We thus determined the target levels for our new quoting methodology to *Market Bid - 6pts*.

Then, we decided to use three specific indicators in order to build a robust estimator that would fit our pre-determined target levels. These indicators have been chosen based on empirical econometric studies led with the index exotics trading team: the daily average of 3 months correlations through the last 3 years (Avg3m3y), the daily average of correlation's 3 months standard deviations through the last 3 years (StDev3m3y) and correlations during the last 6 months (Last6m). Moreover, these indicators were the ones already used for the implied correlation Ask quoting methodology. We then had to compute the values of these three indicators for each Calls vs. Call trade in the database through the following process. For every single Calls vs. Call deal in the database, we adjusted the trade date and the basket components (taking into account each stock pair in the underlying basket). Knowing these elements, we were able to obtain Avg3m3y, StDev3m3y and Last6m values as of the trade date for each pair of stocks in correlation matrices format. We then applied the proper λ coefficients in order to “convert” historical correlations into implied correlation. After that, we computed each indicator's average value at the scale of the whole basket using only values in the strict superior triangle of each indicator's matrix. In the case where there was no sufficient historical market data to compute the indicators for a particular stock pair, this pair was not taken into account in the computations.

At this stage of the study, we had on the one hand the target levels used to calibrate our methodology, and on the other hand the indicators used to construct the quotes: the last task was to optimize the parameters in order to fit our targets in the best way. For this optimization process, we chose to develop a fully manual tool without using generic solver functions. We iterated parameter levels for each of the three indicators inside a pre-determined range arbitrarily fixed at [-1; 1] in order to obtain all possible estimators, and computed the resulting implied correlation quote for each estimator. Each estimator was therefore the following form, corresponding to the weighted sum of chosen indicators:

$$CorrelBidEstimator = \beta_1 * Last6m + \beta_2 * StDev3m3y + \beta_3 * Avg3m3y$$

We then calculated the residual sum of squares, average gap (in absolute value), and standard deviation of each estimator based on the implied correlation quotes inferred from Calls vs. Call deals as follows:

$$RSS = \sum_{i=1}^N (CorrelBidMarket_i - CorrelBidEstimator_i)^2$$

$$AverageGap = \frac{1}{N} \sum_{i=1}^N |CorrelBidMarket_i - CorrelBidEstimator_i|$$

$$StandardDeviation = \frac{1}{N} \sum_{i=1}^N [(CorrelBidMarket_i - CorrelBidEstimator_i) - AverageGap]^2$$

Where:

- $CorrelBidMarket_i$ is the implied correlation Bid quote given by the estimator for the i th Call vs Call deal
- $CorrelBidEstimator_i$ is the implied correlation Bid quote inferred from market data for the i th Call vs Call deal
- N is the total number of Call vs Call deals in our database

We finally calculated the geometric mean of the residual sum of squares, the mean error and the standard deviation for each estimator in order to be able to rank them: the smallest the geometric mean, the fittest the estimator compared to the implied correlation market quotes.

$$GeometricMean = RSS^{1/3} * AverageGap^{1/3} * StandardDeviation^{1/3}$$

Following is the VBA code that iterated the different coefficients levels to test all estimators, and computed the measurements detailed above.

```
Sub Optim_coeffs2()
'
' Computes the difference between the implied correlation obtained with estimators and
' the one given by market Bid for different parameters.
' This aims at finding optimal estimators coefficients.
Dim n As Integer
n = 1
Dim last6m As Double, stDev3m3y As Double, avg3m3y As Double
For last6m = -1 To 1 Step 0.01
    For stDev3m3y = -1 To 1 Step 0.01
        For avg3m3y = -1 To 1 Step 0.01

            'Compute estimator
            Dim correls(1554) As Double
            Dim numLigne As Integer
            For numLigne = 2 To 1556
                If Sheets("Model").Cells(numLigne, 41).value <> 1 Then
                    correls(numLigne - 2) = last6m * Sheets("Model").Cells(numLigne, 31) + stDev3m3y * Sheets("Model").Cells(numLigne, 30) _
                        + avg3m3y * Sheets("Model").Cells(numLigne, 32)
                Else
                    correls(numLigne - 2) = -1
                End If
            Next numLigne

            'Compute distance with Bid
            Dim dBid(1554) As Double
            Dim i As Integer
            For i = 0 To 1554
                If Sheets("Model").Cells(i + 2, 41).value <> 1 Then
                    dBid(i) = Abs(Sheets("Model").Cells(i + 2, 27).value - correls(i))
                Else
                    dBid(i) = -1
                End If
            Next i

            'Display results on sheet
            With Sheets("Optimisation")
                .Cells(n + 2, 1).value = last6m
                .Cells(n + 2, 2).value = stDev3m3y
                .Cells(n + 2, 3).value = avg3m3y
                .Cells(n + 2, 4).value = ComputeRSS(dBid)

                Dim avgBid As Double
                avgBid = ComputeAvgGap(dBid)
                .Cells(n + 2, 5).value = avgBid
                .Cells(n + 2, 6).value = ComputeStDev(dBid, avgBid)
            End With
            n = n + 1
        Next avg3m3y
    Next stDev3m3y
Next last6m

End Sub
```

```

Function ComputeRSS(distance() As Double) As Double

    Dim RSS As Double
    RSS = 0
    Dim i As Integer
    For i = 0 To UBound(distance)
        If distance(i) <> -1 Then
            RSS = RSS + distance(i) ^ 2
        End If
    Next i

    ComputeRSS = RSS

End Function

```

```

Function ComputeAvgGap(distance() As Double) As Double

    Dim avgGap As Double
    Dim count As Integer
    count = 0
    avgGap = 0
    Dim i
    For i = 0 To UBound(distance)
        If distance(i) <> -1 Then
            avgGap = avgGap + distance(i)
            count = count + 1
        End If
    Next i

    avgGap = avgGap / count
    ComputeAvgGap = avgGap

End Function

```

```

Function ComputeStDev(distance() As Double, avg As Double) As Double

    Dim stDev As Double
    Dim count As Integer
    count = 0
    stDev = 0
    Dim i
    For i = 0 To UBound(distance)
        If distance(i) <> -1 Then
            stDev = stDev + (distance(i) - avg) ^ 2
            count = count + 1
        End If
    Next i

    stDev = Sqr(stDev / (count - 1))
    ComputeStDev = stDev

End Function

```

The *GeometricMean* was then computed as an Excel formula on the same worksheet, and estimators were ranked according to this value in order to obtain the best fitting ones. The following screenshot shows how results were displayed on the worksheet. Please note that parameter levels have been here replaced by “x.xx” in order to avoid any confidentiality issues).

Parameters			Computations			GeomMean(RSS, EcartMoy, StDev)	Min(RSS)	Min(AverageGap)	Min(StDev)	Best Estimators		
Last6m	StDev3m3y	Avg3m3y	RSS	AverageGap	StDev					Last6m	StDev3m3y	Avg3m3y
x.xx	x.xx	x.xx	19562.60434	2.844861734	2.555861	52.20054185	19562.6	2.844861734	2.554639595	x.xx	x.xx	x.xx
x.xx	x.xx	x.xx	19564.2684	2.846176088	2.55464	52.20174317				x.xx	x.xx	x.xx
x.xx	x.xx	x.xx	19624.97403	2.848539288	2.560883	52.31268301				x.xx	x.xx	x.xx
x.xx	x.xx	x.xx	19629.96622	2.853115858	2.556509	52.3153078						
x.xx	x.xx	x.xx	19751.37748	2.859580262	2.56702	52.53430411						
x.xx	x.xx	x.xx	19759.6978	2.86598335	2.561079	52.54027282						
x.xx	x.xx	x.xx	19941.81469	2.878023496	2.574127	52.86436121						
x.xx	x.xx	x.xx	19953.46314	2.883690047	2.569468	52.87739505						
x.xx	x.xx	x.xx	20196.28566	2.900111312	2.586259	53.30722571						
x.xx	x.xx	x.xx	20211.26223	2.907591075	2.580013	53.32320075						
x.xx	x.xx	x.xx	20514.79038	2.926627091	2.602373	53.86020472						
x.xx	x.xx	x.xx	20533.09509	2.935858131	2.594586	53.87895993						
x.xx	x.xx	x.xx	20897.32886	2.958850165	2.620815	54.51927144						
x.xx	x.xx	x.xx	20918.9617	2.970574178	2.61061	54.53904053						
x.xx	x.xx	x.xx	21343.9011	2.997509166	2.640501	55.28069931						
x.xx	x.xx	x.xx	21368.86207	3.010697993	2.628995	55.30266874						
x.xx	x.xx	x.xx	21854.5071	3.040087765	2.663998	56.14619351						
x.xx	x.xx	x.xx	21882.79619	3.056397878	2.649251	56.16666311						
x.xx	x.xx	x.xx	22429.14686	3.086567722	2.691041	57.11317273						
x.xx	x.xx	x.xx	22460.76408	3.106628528	2.672268	57.13005115						
x.xx	x.xx	x.xx	23067.82037	3.138459121	2.719582	58.17571434						
x.xx	x.xx	x.xx	23102.76572	3.16051702	2.698744	58.19172614						
x.xx	x.xx	x.xx	23770.52764	3.194600432	2.750618	59.33284965						
x.xx	x.xx	x.xx	23808.80112	3.220159375	2.725885	59.34363276						
x.xx	x.xx	x.xx	24537.26867	3.255032716	2.78376	60.58118821						
x.xx	x.xx	x.xx	24578.87027	3.283862891	2.755318	60.5860839						
x.xx	x.xx	x.xx	25368.04345	3.321038522	2.817145	61.91416936						
x.xx	x.xx	x.xx	25412.97319	3.351853104	2.78642	61.91497246						
x.xx	x.xx	x.xx	26311.10986	3.42436593	2.818539	63.32586439						
x.xx	x.xx	x.xx	26262.852	3.391293434	2.851955	63.33104304						
x.xx	x.xx	x.xx	27273.28029	3.502325598	2.850159	64.81183372						
x.xx	x.xx	x.xx	27221.6943	3.466168374	2.887381	64.82705839						
x.xx	x.xx	x.xx	28299.48448	3.583850515	2.883195	66.37443234						
x.xx	x.xx	x.xx	28244.57036	3.545424661	2.923336	66.39886772						
x.xx	x.xx	x.xx	29389.72242	3.670106794	2.915833	68.00574794						
x.xx	x.xx	x.xx	29331.48018	3.628083359	2.96065	68.04550067						
x.xx	x.xx	x.xx	30543.99413	3.761766921	2.946811	69.69848636						
x.xx	x.xx	x.xx	30487.47375	3.713889548	2.999305	69.76476447						

Once we found the optimal estimator (that is to say the one with the lowest *GeometricMean*, we were able to compare the new Bid methodology (NEW) to the one previously used in the systems (PROD). We compared each methodology to the available market quotes on implied correlation through the computation of residual sums of squares, mean errors, and standard deviations.

In the following screenshots, we can see how the previous and the new methodology have been compared. For each Bid methodology (PROD and NEW), gaps and squared gaps compared to Bid quotes inferred from Calls vs. Call deals have been computed, and in the case where there was insufficient market data available to obtain indicators, the deal hasn't been taken into account.

TD Date	BID	StDev3m3y	Last 6m	Avg3m3y	PROD	GAP: ABS(BID-PROD)	SQUARED GAP: (BID - PROD)^2	NEW	GAP: ABS(BID - NEW)	SQUARED GAP: (BID - NEW)^2	Deleted Values
10/02/2011	52.885	15.79994948	42.5697414	46.4069	40.3907	12.4940	156.1007	51.8281	1.0567	1.1166	
14/02/2011	60.293	14.58208925	48.30825638	53.0734	47.2195	13.0734	170.9126	57.1516	3.1413	9.8677	
15/02/2011	64.179	14.37074487	51.17138393	57.2109	51.0476	13.1314	172.4348	60.6581	3.5209	12.3968	
15/02/2011	60.225	14.3096571	52.80233299	53.0769	48.7949	11.4303	130.6522	57.7575	2.4677	6.0893	
15/02/2011	62.855	13.38991384	47.74976808	55.6441	49.1767	13.6784	187.0978	58.3070	4.5481	20.6851	
15/02/2011	55.240	16.43564927	36.30671179	49.1645	39.9893	15.2504	232.5754	53.2342	2.0055	4.0222	
16/02/2011	62.119	16.25569514	39.14182472	44.4338	37.7674			50.0147			1
17/02/2011	60.679	17.14479484	43.1274226	48.7713	41.7387	18.9403	358.7352	54.5133	6.1658	38.0168	
17/02/2011	61.885	14.31577668	46.79697733	53.7093	47.2411	14.6434	214.4286	57.2209	4.6636	21.7495	
17/02/2011	42.819	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
18/02/2011	51.919	15.82059598	33.12600165	45.0576	36.3499	15.5687	242.3844	49.2093	2.7093	7.3403	
22/02/2011	59.503	15.52770207	35.27526143	46.0094	37.8031	21.7003	470.9034	50.1252	9.3781	87.9491	
22/02/2011	62.634	15.09652363	41.18878723	48.7032	41.7286			52.9235			1
22/02/2011	51.462	15.37122409	37.0764068	43.9722	37.0638	14.3983	207.3107	48.7909	2.6712	7.1354	
22/02/2011	51.462	15.37122409	37.0764068	43.9722	37.0638	14.3983	207.3107	48.7909	2.6712	7.1354	
22/02/2011	63.563	15.05812815	44.7340086	54.4941	46.8490	16.7137	279.3479	57.9046	5.6581	32.0145	
22/02/2011	49.866	15.37122409	37.0764068	43.9722	37.0638	12.8021	163.8938	48.7909	1.0750	1.1557	
23/02/2011	49.576	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
23/02/2011	49.576	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
23/02/2011	63.844	16.73075263	45.8895929	53.5481	46.0352	17.8091	317.1631	58.3690	5.4753	29.9785	
23/02/2011	60.294	17.13539761	42.77832998	48.8163	41.6572	18.6366	347.3236	54.4826	5.8113	33.7710	
23/02/2011	62.995	14.85681497	46.6102097	54.0153	47.2090	15.7858	249.1918	57.7409	5.2540	27.6040	
23/02/2011	64.451	16.13433074	38.61014956	47.3555	39.6188	24.8320	616.6269	52.0732	12.3776	153.2043	
23/02/2011	60.294	17.13539761	42.77832998	48.8163	41.6572	18.6366	347.3236	54.4826	5.8113	33.7710	
24/02/2011	58.378	16.8802043	42.72906117	48.5732	41.5599	16.8184	282.8574	54.1389	4.2394	17.9722	
24/02/2011	58.378	16.8802043	42.72906117	48.5732	41.5599	16.8184	282.8574	54.1389	4.2394	17.9722	
28/02/2011	49.039	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
28/02/2011	51.629	15.62913617	36.43377336	54.5733	43.9754	7.6533	58.5736	56.8906	5.2619	27.6879	
28/02/2011	61.516	15.69573012	48.84437733	50.2033	45.0773	16.4388	270.2353	55.7186	5.7976	33.6121	
28/02/2011	49.039	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
01/03/2011	58.313	16.43866353	39.70002006	47.6422	40.0729	18.2399	332.6938	52.6559	5.6570	32.0011	
01/03/2011	44.622	20.76392939	11.54331106	25.3839	14.2182	30.4037	924.3839	33.5048	11.1171	123.5896	
02/03/2011	58.235	16.46200095	37.0986992	47.6169	39.1896	19.0454	362.7279	52.2082	6.0268	36.3225	
02/03/2011	54.396	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
01/03/2011	60.238	15.69369042	48.18217665	50.2048	44.8604	15.3774	236.4659	55.6059	4.6320	21.4556	
02/03/2011	58.931	15.5570963	39.78556249	53.6529	44.4794	14.4520	208.8610	56.7184	2.2130	4.8972	
02/03/2011	66.577	15.18566804	43.41151657	58.7887	49.2908	17.2859	298.8030	61.0189	5.5579	30.8897	
02/03/2011	54.327	ERROR	ERROR	ERROR	#VALUE!			#VALUE!			1
02/03/2011	57.945	14.80333774	33.32745234	52.9300	42.1053	15.8400	250.9049	54.6264	3.3189	11.0151	
02/03/2011	58.832	15.09395098	36.42108195	49.8834	40.9587	17.8730	319.4429	53.0084	5.8232	33.9100	
02/03/2011	58.832	15.09395098	36.42108195	49.8834	40.9587	17.8730	319.4429	53.0084	5.8232	33.9100	
02/03/2011	58.832	15.09395098	36.42108195	49.8834	40.9587	17.8730	319.4429	53.0084	5.8232	33.9100	
03/03/2011	62.202	15.34419532	40.25851008	54.8104	45.4928	16.7092	279.1971	57.5530	4.6491	21.6138	
03/03/2011	63.202	15.34419532	40.25851008	54.8104	45.4928	16.7092	279.1971	57.5530	4.6491	21.6138	

Then, from these calculations we were able to obtain the following tabs:

PROD	Last6m	x.xx	NEW	Last6m	x.xx
	Stdev3m3y	x.xx		Stdev3m3y	x.xx
	Avg3m3y	x.xx		Avg3m3y	x.xx
	RSS	165922.015299		RSS	16872.942624
	AverageGap	10.130204		AverageGap	2.808200
	StDev	4.636301		StDev	2.176594
Without absurd values					
	RSS	31668.6		RSS	11455.7
	AverageGap	4.32		AverageGap	2.46
	Stdev	2.622039399		StDev	1.783649032

Note that here parameter levels have been replaced again by “x.xx” in order to avoid any confidentiality issue.

These tables allow comparing easily the previous and the new methodologies: the results showed that with the optimized methodology, RSS was divided by 9.83, average gap by 3.61 and standard deviation by 2.13. Moreover, we chose to re-compute the RSS, average gap and standard deviation based only on relevant values by not taking into account values with more than 10 correlation points difference between the market Bid and one of the estimators (PROD or NEW). With these new computations, we still find satisfying results for the new methodology.

Following is the VBA code that computed the RSS, average gap and standard deviation without taking into account absurd values.

```
Sub ComputeRSS()
' Compute RSS without absurd values ==> extract values with more than 10 points of correl difference between Bid and one of the estimators (new or old)

Dim RSS_Old As Double
Dim RSS_New As Double
RSS_Old = 0
RSS_New = 0
Dim numLigne
For numLigne = 2 To 1556
    If (Sheets("Model").Cells(numLigne, 34).value <= 10 And Sheets("Model").Cells(numLigne, 37).value <= 10) Then
        RSS_Old = RSS_Old + Sheets("Model").Cells(numLigne, 35).value
        RSS_New = RSS_New + Sheets("Model").Cells(numLigne, 38).value
    End If
Next numLigne

Sheets("Model").Cells(12, 45).value = RSS_Old
Sheets("Model").Cells(12, 49).value = RSS_New

End Sub
```

```
Sub ComputeEcartMoy()
' Compute Average Diff without absurd values

Dim EM_Old As Double
Dim EM_New As Double
Dim count As Integer
count = 0
EM_Old = 0
EM_New = 0
Dim numLigne
Sheets("Model").Select
For numLigne = 2 To 1562
    If Cells(numLigne, 34).value <= 10 And Cells(numLigne, 37).value <= 10 And Not IsEmpty(Cells(numLigne, 34)) Then
        EM_Old = EM_Old + Sheets("Model").Cells(numLigne, 34).value
        EM_New = EM_New + Sheets("Model").Cells(numLigne, 37).value
        count = count + 1
    End If
Next numLigne

EM_Old = EM_Old / count
EM_New = EM_New / count

Sheets("Model").Cells(13, 45).value = EM_Old
Sheets("Model").Cells(13, 49).value = EM_New

End Sub
```

```

Sub ComputeStDev()
' Compute Standard Dev without absurd values

    Dim StDev_Old As Double
    Dim StDev_New As Double
    Dim count As Integer
    count = 0
    Dim avg_Old As Double
    Dim avg_New As Double
    avg_Old = Sheets("Model").Cells(13, 45).value
    avg_New = Sheets("Model").Cells(13, 49).value
    StDev_Old = 0
    StDev_New = 0
    Dim numLigne
    Sheets("Model").Select
    For numLigne = 2 To 1562
        If Cells(numLigne, 34).value <= 10 And Cells(numLigne, 37).value <= 10 And Not IsEmpty(Cells(numLigne, 34)) Then
            StDev_Old = StDev_Old + (Sheets("Model").Cells(numLigne, 34).value - avg_Old) ^ 2
            StDev_New = StDev_New + (Sheets("Model").Cells(numLigne, 37).value - avg_New) ^ 2
            count = count + 1
        End If
    Next numLigne

    StDev_Old = Sqr(StDev_Old / (count - 1))
    StDev_New = Sqr(StDev_New / (count - 1))

    Sheets("Model").Cells(14, 45).value = StDev_Old
    Sheets("Model").Cells(14, 49).value = StDev_New

End Sub

```

Lastly, we chose to compute the ratio Bid-Ask Spread / StDev3m3y both for the quotes inferred from Calls vs. Call market data and the quotes obtained with the Ask methodology actually in use and the new Bid methodology. This computation was made in order to make sure that with the new Bid methodology, implied correlation Bid-Ask spreads were tighter than before, but not too much: a certain degree of conservatism had to be taken into account. Unfortunately, with the new methodology the spread had become way too tight: it was undesirable to implement such aggressive Bid quotes in our systems.

This is why we finally chose to keep fixed parameter levels (equal to the ones previously in use) for the Last6m and the Avg3m3y, and to find the optimal parameter level only for the StDev3m3y indicator. Thanks to this new solution, we were able to reduce consequently the errors compared to implied correlation Bid market quotes while keeping a certain level of Bid-Ask spread.

Market Bid/Ask Spread	StDev3m3y	Market Spread/StDev					Bid NEW	Ask PROD	Methodology Bid/Ask Spread	Methodology Spread/StDev
6.58556743	15.7999495	0.416809398	Ask PROD	$x.xx * Last6m + x.xx * StDev3m3y + x.xx * Avg3m3y$			52.8848	54.83008	1.945311966	0.123121404
8.20442826	14.5820893	0.562637364	Bid NEW	$x.xx * Last6m + x.xx * StDev3m3y + x.xx * Avg3m3y$			60.2929	59.92219		
6.00000000	14.3707449	0.417514893					64.1790	63.38856		
6.00000000	14.3096571	0.419297259					60.2252	60.47637	0.251179116	0.017553119
7.71056138	13.3899138	0.501165389		Average			62.8551	60.85108		
7.09859842	16.4356493	0.431902525	Market Spread/StDev	0.505501194			55.2398	56.357	1.117236467	0.067976412
6.00000000	16.2556951	0.36910141	Methodology Spread/StDev	0.292447787			62.1191	53.10325		
6.79893994	17.1447948	0.396560006					60.6790	57.77078		
7.47048881	14.3157767	0.521836082					61.8845	59.94088		
6.00000000	ERROR						42.8186	#VALUE!	#VALUE!	
6.32884972	15.820596	0.400038641					51.9186	52.21526	0.296610018	0.018748347
7.24492952	15.5277021	0.46658092					59.5034	53.07551		
8.58140686	15.0965236	0.568435957					62.6340	55.79183		
7.59618965	15.3712241	0.494182481					51.4621	51.71143	0.249321164	0.016219994
7.59618965	15.3712241	0.494182481					51.4621	51.71143	0.249321164	0.016219994
6.00000000	15.0581281	0.3984559					63.5627	60.76566		
6.00000000	15.3712241	0.390339765					49.8659	51.71143	1.845510814	0.12006271
6.00000000	ERROR						49.5763	#VALUE!	#VALUE!	
6.00000000	ERROR						49.5763	#VALUE!	#VALUE!	
8.08488997	16.7307526	0.483235282					63.8442	61.5478		
7.32489650	17.1353976	0.427471639					60.2939	57.73831		
8.02154591	14.856815	0.53992366					62.9948	60.56365		
6.83683945	16.1343307	0.423744843					64.4507	55.13868		
7.32489650	17.1353976	0.427471639					60.2939	57.73831		
7.53996235	16.8802043	0.446674828					58.3783	57.34615		
7.53996235	16.8802043	0.446674828					58.3783	57.34615		
6.84412854	ERROR						49.0393	#VALUE!	#VALUE!	
7.35566549	15.6291362	0.470638006					51.6287	59.86016	8.231460477	0.526674052
6.00000000	15.6957301	0.38226957					61.5162	58.70076		
6.84412854	ERROR						49.0393	#VALUE!	#VALUE!	
7.13583087	16.4386635	0.434088261					58.3128	55.77922		
6.00000000	20.7639294	0.288962647					44.6219	37.44997		
6.85047292	16.462001	0.416138532					58.2350	55.336		
7.20442348	ERROR						54.3956	#VALUE!	#VALUE!	
6.84916721	15.6936904	0.436428082					60.2379	58.58767		
7.53121189	15.5570963	0.484101387					58.9314	59.6743	0.742896893	0.047752928

On this screenshot we can see the left side the Call vs Call implied correlation quotes' Bid/Ask Spreads for deals in the database, and on the right side the Bid/Ask spreads obtained when using the NEW Bid methodology and the Ask methodology actually used in the systems. By dividing each spread by the StDev3m3y previously calculated for each deal and computing the average ratio for both market data and our methodology's quotes we were able to draw our final conclusion. Indeed, we saw that we achieved to fit better our targeted market data, while at the same time reducing our implied correlation quotes average spread and keeping a certain degree of conservatism.