Alternative Finance
Study on Bitcoin modeling and derivatives pricing

Raphaël Lederman, Raphaël Seksik, Dylan Bokobza

April 26, 2018

Introduction

This paper will focus on the qualitative and quantitative aspects of Bitcoin to develop a derivatives valuation framework consistent with the cryptocurrency's fundamentals. In order to do so, it will show to what extent the Bitcoin price evolution doesn't fit the conventional Black and Scholes configuration, through the successive invalidation of its naive assumptions using both graphical and statistical tools.

This will lead to exhibit Bitcoin's singularities, from the point of view of the price evolution and the market as a whole, with a particular focus on the non-Gaussian distribution of returns and the lack of liquidity on major exchanges. It will then question the sources of such extreme features, and investigate the probable impact of very unstable demand mechanisms and behavioral biases. Thus, it will finally compare the Bitcoin to a commodity from the perspective of it's structural attributes as well as it's price evolution, and will therefore open up the path to corresponding modeling approaches.

In the second part, this paper will build on the observations made previously in order to develop a robust and consistent modeling framework, deviating from a risk-neutral based closed formula approach and moving towards an econometric study. Hence, considering and testing various models, the final optimal choice being an Exponential GARCH model based on asymmetric innovations and Student's T noises, with fine tuned parameters. Finally, this paper will give some details on the possible implementation of such a fitted model through a Monte-Carlo procedure to value derivatives on Bitcoin.

In the context of our work, we used Bitcoin traded volumes and volume-weighted prices from ten major exchange platforms from the year 2013 onward.

1 A singular alternative asset class

1.1 Invalidation of the Black & Scholes framework

Our first instinct in order to value derivatives on Bitcoin was to think of the standard Black and Scholes approach, for it is relatively convenient and intuitive. But after a closer look at Bitcoin's price behavior, we questioned the validity of the rather naive Black and Scholes assumptions in the context of crypto-currency.

Let's first recall the initial assumptions linked to this model:

- No transaction cost, perfect liquidity and no restrictions to short selling
- Asset price follows a random walk
- Normal distribution of returns
- Constant and known variance of returns
- Constant risk-free interest rate
- No dividend on the asset

We will go through the majority of these assumptions in order to underline their significant mismatch with Bitcoin's characteristics.

Black-Scholes assumptions state that there should be no transaction costs in buying the option, thus assuming perfect liquidity in the market. We can recall that liquidity is defined as the extent to which an asset may be bought or sold in the market without affecting its price. Nevertheless, historical data show a structural lack of liquidity in the Bitcoin market, and this fact has been underlined by several authors in the literature.

For instance, Trimborn, Li and Hardle, in [7] study Bitcoin's illiquidity compared to traditional assets, looking to incorporate these liquidity issues in portfolio optimization. They look at a way to optimize Bitcoin allocation in a diversified portfolio with traditional assets, taking into account liquidity constraints under a Markowitz framework. Thus, proposing a liquidity bounded risk-return optimization approach.

Before presenting their portfolio optimization approach, they looked at Bitcoin's median trading volume compared to S&P 500'. Figure 1 reproduces the results for all crypto-currencies and S&P 500 components, using the sample between Jan. 1, 2014 and Mar. 20, 2017.

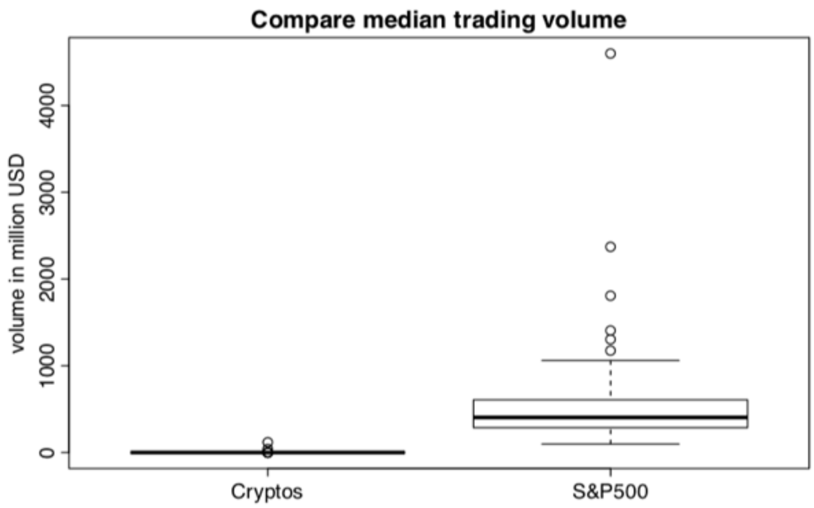

Figure 1: Boxplot of median trading volume (in US Dollar)[1]

Looking at the box-plot, we can see that the average daily trading amounts of crypto-currencies are lower than the 25% quantile of S&P 500 Stocks, therefore, it is observable that crypto-currencies in general, and Bitcoin in particular, have much lower trading volume than S&P 500 stocks.

Hio Loi, in [6], a specific paper on Bitcoin's liquidity, studies the liquidity of Bitcoin across five major Bitcoin exchanges (Bitstamp, Bitfinex, BTC-e, HitBTC, and itBit) – chosen for their available data – and further compares the liquidity of Bitcoin with different stock indices. The paper tests five liquidity measures on Bitcoin's data, underlining the fact that the liquidity of

[1] Cf. [7]

Bitcoin, although depending on the choice of the exchange, is still much lower than on stock markets. The paper states that there is not a standard way to measure the liquidity of an asset or currency in the literature. Aitken and Winn in [1], report that we can find up to 68 liquidity measures used in the research papers. However, as expressed by Hio Loi, "not all of the liquidity measurements are suitable for Bitcoin, because of the limited data". For instance, he underlines the fact that Bitcoin lacks turnover rate. The paper hence discusses five different liquidity measures, the best finally being Amihud's proxy for illiquidity, developed in [2].

The illiquidity measure, called ILLIQ, is defined as the daily ratio of absolute stock return to its dollar volume. It may be interpreted as "the daily price response associated with one dollar of trading volume, thus serving as a rough measure of price impact"[2]. The intuition behind this measure assumes that less trading volume is required to move the prices of illiquid assets compared with liquid assets. It is defined such as:

$$ILLIQ_i = \frac{1}{D_i} \sum_{t=1}^{D_i} \frac{|R_{it}|}{V_{it} P_{it}}$$

$$R_{it} = ln\left(\frac{P_{it}}{P_{it-1}}\right)$$

where D_i is the number of days for which data are available for stock i, $|R_{it}|$ is the absolute value of the daily return on stock i on day t, V_{it} is the daily traded volume, and P_{it} is the closing price of stock i on day t.

A higher Amihud's proxy value, thus a higher ILLIQ, means less liquidity on the studied stock. Hence, a very liquid stock should display a low ILLIQ value. The paper results regarding Amihud's proxy are displayed below in figure 2:

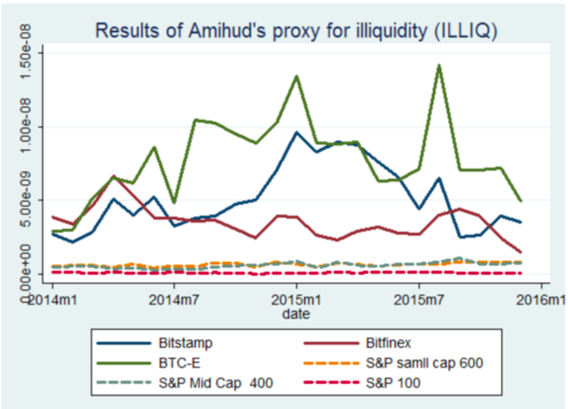

Figure 2: Results of Amihud's proxy for illiquidity (ILLIQ)

We clearly see that Bitcoin exchanges (solid lines) display much higher ILLIQ values, showing the lack of liquidity on those markets compared to stock indices (dashed lines). Moreover, it should be noted that the paper purposely chose to remove HitBTC and iBit exchanges from the graph since their ILLIQ value was too large to get a clear visual comparison.

The paper studies various company sizes, and even the smaller capitalization stocks display much lower ILLIQ values than Bitcoin exchanges. Looking at the graph, we see that the illiquidity difference between stocks and Bitcoin is far larger than the illiquidity difference observed between large and small capitalizations (red vs. yellow dashed lines).

The paper further underlines the link between liquidity and transaction costs, and looks at transactions costs of several Bitcoin exchanges concluding that those platforms take on average

[2] Cf. [2]

0,2% transaction fee. Even if it can vary depending on the choice of the exchange, this value is significantly higher than what can be observed on stock markets, thus it confirms the lack of liquidity on the Bitcoin market.

According to the Black and Scholes framework, we can make the assumption of a normal distribution of the log-returns, and a constant volatility. Just as in the case of perfect liquidity, these relatively naive assumptions seem not to be consistent with the crypto-currencies market at all.

Let's first have a look at figure 3 to have an idea of the price volatility behavior:

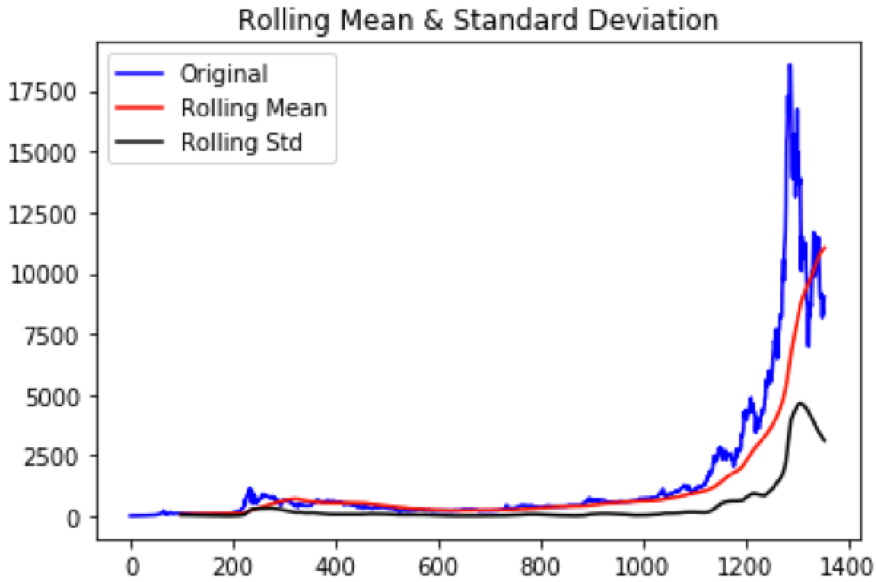

Figure 3: Rolling Mean & Standard Deviation

We can clearly see that the price evolution is explosive, with a very important upside volatility even characterized by spikes and it seems contrary to the basic assumptions cited above: the returns seem far from exhibiting a Gaussian behavior.

As a preliminary step, we performed a Dickey-Fuller Test on Bitcoin's daily log-returns and were able to justify our intuition of non-stationarity. On the results displayed in figure 4, the test statistic is strictly superior to the critical value at 1% probability. Thus, it is impossible to reject the null hypothesis of presence of unit root, and confirms the non-stationarity hypothesis.

```
Results of Dickey-Fuller Test:
Test Statistic                  -0.906407
p-value                          0.785855
#Lags Used                      23.000000
Number of Observations Used   1331.000000
Critical Value (1%)             -3.435273
Critical Value (5%)             -2.863714
Critical Value (10%)            -2.567927
```

Figure 4: Dickey-Fuller Test on the Bitcoin price time series

```python
def test_stationarity(timeseries):

    #Determing rolling statistics
    rolmean = pd.rolling_mean(timeseries, window=100)
    rolstd = pd.rolling_std(timeseries, window=100)

    #Plot rolling statistics:
    orig = plt.plot(timeseries, color='blue',label='Original')
    mean = plt.plot(rolmean, color='red', label='Rolling Mean')
    std = plt.plot(rolstd, color='black', label = 'Rolling Std')
    plt.legend(loc='best')
    plt.title('Rolling Mean & Standard Deviation')
    plt.show(block=False)

    #Perform Dickey-Fuller test:
    print ('Results of Dickey-Fuller Test:')
    dftest = adfuller(timeseries, autolag='AIC')
    dfoutput = pd.Series(dftest[0:4], index=['Test Statistic','p-value',
    for key,value in dftest[4].items():
        dfoutput['Critical Value (%s)'%key] = value
    print (dfoutput)
```

Figure 5: Python code for Rolling Mean / Std and DF Test

Moreover, as we can perceive a time-varying volatility, another of core assumption of the Black and Scholes framework seems invalidated. Focusing on the daily log-returns' distribution, we can observe the presence of heteroskedasticity. Therefore, confirming once again our assumption on the non-normality of Bitcoin returns.

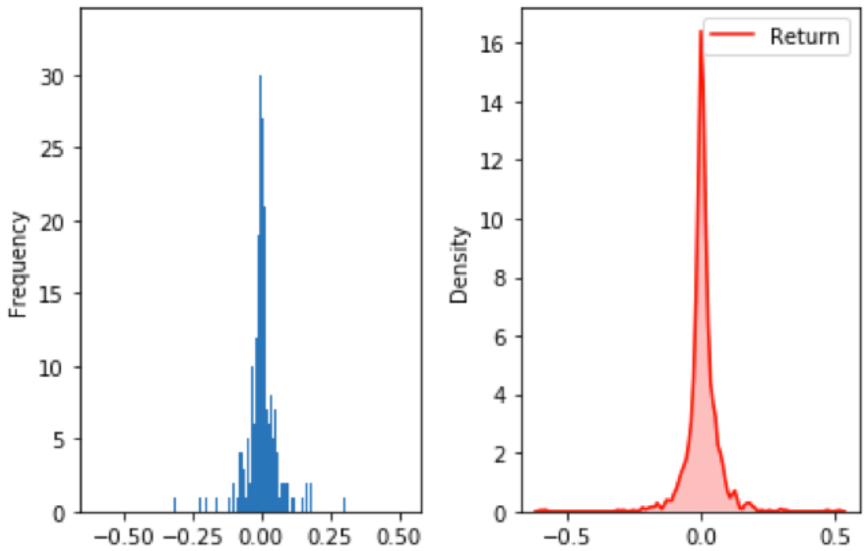

Figure 6: Bitcoin returns frequency and density

```python
fig,axes=plt.subplots(nrows=1,ncols=2)
returns.plot(kind='hist',bins=1000,ax=axes[0])
sns.kdeplot(returns,shade=True, color="r",ax=axes[1])
axes[1].set_ylabel('Density')
plt.tight_layout()
```

Figure 7: Python code for Histogram and KDE

The figure 6 displays on the left the histogram of frequencies of log-returns, and on the right a kernel density estimation. From these, we can see the presence of extreme values, characteristic of a non-normal distribution.

To further prove our intuition of non-normality, we decided to run a Jarque-Bera test on Bitcoin daily returns, a goodness-of-fit test based on the moments of the distribution. It tests whether the skew and kurtosis of the time series fit the normal distribution.

With a close to null P-value, we can reject the null hypothesis of normality at almost any confidence level.

```
Jarque-Bera test statistic = 26216.1770591
Jarque-Bera test P-value = 0.0
```

Figure 8: Jarque-Bera Test on Bitcoin price time series

```
JB=stats.jarque_bera(returns)
print("Jarque-Bera test statistic =",JB[0])
print("Jarque-Bera test P-value =",JB[1])
```

Figure 9: Python code for JB Test

The non-normality of the daily returns' distribution can also be verified using a Quantile-Quantile (QQ) plot. On the following figure 10, one can explicitly observe the presence of fat tails as we perceived in the frequency and density graphs above.

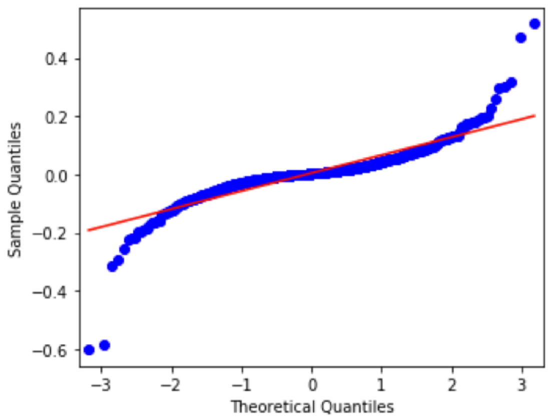

Figure 10: QQ-plot on Bitcoin price time series

```
fig = plt.figure(figsize=(10,4))
layout = (1, 2)
QQReturns_ax = plt.subplot2grid(layout, (0, 0))
QQSquared_ax = plt.subplot2grid(layout, (0, 1))
sm.qqplot(returns, line='s', ax=QQReturns_ax)
sm.qqplot(squared_returns, line='s', ax=QQSquared_ax)
plt.tight_layout()
```

Figure 11: Python code for QQ-plot

In order to put the emphasis on the moments of Bitcoin's daily log-returns, we also computed their mean, standard deviation and kurtosis. The results can be found in figure 12.

```
Mean =  1.2468774653901062
Standard Deviation =  0.993931938254779
Skew =  -0.5719624278240696
Kurtosis =  21.518294715231427
```

Figure 12: Statistics of Bitcoin price time series

```python
mean=np.mean(returns*260)
std=np.std(returns*260**0.5)
skew=skew(returns)
kurtosis=kurtosis(returns)
print("Mean = ",mean)
print("Standard Deviation = ",std)
print("Skew = ",skew)
print("Kurtosis = ",kurtosis)
```

Figure 13: Python code for Moments

As we can see, the annualized standard deviation approaches the 100%, and the returns are negatively skewed, with fat tails largely more important than in the normal case (closer to a leptokurtic distribution).

A last way of displaying these extreme behaviors of Bitcoin returns is to use violin and box plots, showing distribution quantiles and extreme values repartition.

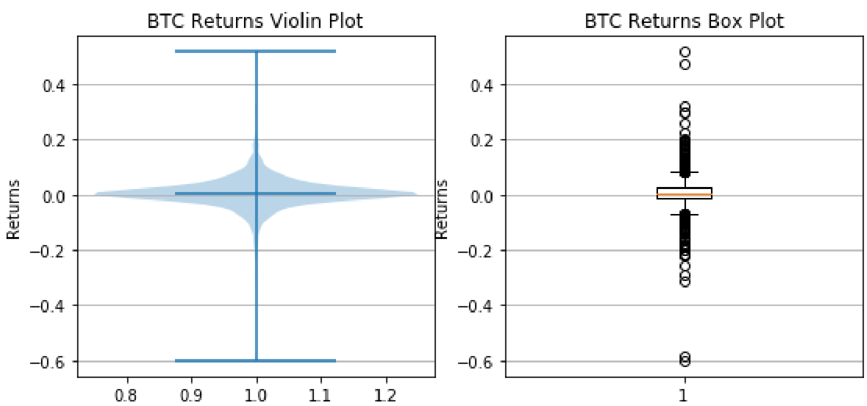

Figure 14: Violin and box plots of Bitcoin price time series

```python
fig, axes = plt.subplots(nrows=1, ncols=2, figsize=(9, 4))
axes[0].violinplot(squared_returns,showmeans=False,showmedians=True)
axes[0].set_title('BTC Squared Returns Violin Plot')
axes[1].boxplot(squared_returns)
axes[1].set_title('BTC Squared Returns Box Plot')
for ax in axes:
    ax.yaxis.grid(True)
    ax.set_ylabel('Squared Returns')
plt.show()
```

Figure 15: Python code for Violin and Box plots

As for the interest rate, there is no such thing as a “risk-free” rate in the cryptocurrency environment. The fundamentals driving the Bitcoin price lead us to think that we are not in presence of an “arbitrage-free” framework governed by conventional asset pricing dynamics such as in the Capital Asset Pricing Model (CAPM): as an alternative asset class, cryptocurrencies don't rely on the same determinants. Of course, it is possible to compute a Bitcoin implied interest-rate. But such a value would certainly not correspond to the concept of "risk-free" asset used in traditional risk-neutral frameworks.

Moreover, the risk-free rate stability assumption isn't verified even in the traditional assets' world. If the Fed Fund rate has been very stable around 0.25% from 2013 to the end of 2015, it has been increasing since then to reach the 1.5-1.75% range, with the Fed being expected to continue the rates hikes over 2018 and after. As a result, the use of the risk-free interest rate in the diffusion process of our underlying crypto-asset seems irrelevant, therefore another central assumption of the Black and Scholes framework is invalidated.

Finally, on the subject of dividends, recent innovations might be opening the road to generalized crypto-dividends in the near-future. For instance, a new ramification emerged in the Bitcoin block-chain last October in the form of Bitcoin gold, allowing holders to receive one-for-one unit for every Bictoin they owned. This implied a consequent proportional drop in the Bitcoin price, just in the way stock valuations drop with the occurrence of dividends. This concept might bring new complexities in the crypto-asset valuation process, and participate to, once again, render Black and Scholes assumptions obsolete.

1.2 Unstable demand dynamics and behavioral biases

As we were able to observe the rather extreme behaviors of Bitcoin returns, and the singularities of its market characteristics, we will now try to explore their possible sources, in order then to identify what kind of modeling might be the most appropriate. Bitcoin, and all other cryptocurrencies, make a very specific asset class: they are increasingly being sought after by many different types of investors. In order to apprehend how Bitcoin is still riding its tide of popularity and better understand the agents' sometimes irrational behavior towards it, let's recall some useful description of its features.

"A *cryptocurrency* is a digital or virtual currency that uses cryptography for security. It is difficult to counterfeit this cryptocurrency because of this security feature, making it increasingly computationally intensive to generate new Bitcoins. A defining feature of a cryptocurrency, and arguably its most endearing allure, is its organic nature; it is not issued by any central authority, rendering it theoretically immune to government interference or manipulation."[3]

Bitcoin, created in 2009, follows the ideas set out in a white paper by Satoshi Nakamoto, whose true identity has not concretely been verified yet. Bitcoin offers the promise of lower transaction costs than traditional on-line payment mechanisms and is operated by a decentralized authority, unlike government-issued currencies. It's essence is to bypass the intermediation of central banks, governments and regulatory entities: Bitcoins are not issued or backed by any banks or government.

This new asset class attracts different types of investors, and even appeals agents very far from the traditional investors on financial markets, with different investment goals and risk appetites. The quasi-ideological mindset of some of the most active crypto-currencies investors, along with the renewed enthusiasm towards automatized computational processes, participated to create a vast mania on Bitcoin. Bubble-like characteristics have emerged at some point, and cognitive biases became of importance.

Even in the case of "rational" investors, some biases linked to profit-maximizing dynamics have been well studied: first, certainly the most known bias is linked to the supply and demand relationship. In fact, the higher the demand, the higher the price, and the rational investor will react to this change in price. This creates strong momentum and might lead to explosive price paths. Besides, in a risky universe, a perfectly rational individual makes investment and consumption decisions maximizing the probabilistic profit expectation as a function of its utility. The utility conveys the satisfaction the individual will take from a consumption or a given future wealth, and the shape of the utility function can itself be skewed (asymmetric utility between gains

[3]Investopedia

and losses, attraction towards lottery-like profits, etc.). Moreover, financial investment yields being random, individuals affect occurrence probabilities to future events, and these probabilities are used to compute the expectation. Hence, there are very high biases in these anticipations.

A complementary element in the context of the study of investors' behavior, not linked to rationality, though important for the shape of the utility function, is the risk aversion hypothesis. Economic agents take risks only if they notice that they will get a sufficient reward. That is, if a risk-less investment yields 5% p.a., then a risky investment, *e.g.* a share, should on average yield more to be worth considering by the investor.

The case of the Bitcoin, and of cryptocurrencies more broadly, makes us understand that the investors are of another type.

According to a study by SEMrush, a search engine marketing agency, the current price of Bitcoin has a 91% correlation with the volume of Google search requests for Bitcoin-relation terms. Of course, we don't know yet which one impacts the second, however this relationship is higher than for any other traditional investment vehicle.[4]

Another important matter is the rationality of agents that we have defined earlier. When paying attention over time to the Bitcoin's price curve, we observe investment behaviors that are very far from rational and not linked to traditional fundamental analysis. For example, let's consider the point where the Bitcoin price was at $20,000. The more the price got closer to this threshold, the less people invested, as if investors stated *a priori* that it was a kind of resistance level, and at some point, people started to sell massively their positions. This massive turnaround in the Bitcoin market, unexpected by the majority of financial markets specialists, was not supported by any real fundamental. The central contradiction of Bitcoin remains the idea that it will one day be used as a currency for transactions, even though the investment thesis is "*hold*" and don't spend it. There is an inherent duality between its "mean of trade" essence and "speculative vehicle" reality. Since the value of the Bitcoin is finally linked to the confidence of investors in what the underlying technology would potentially be worth, and their biased anticipations of such confidence over the near future, it cannot be considered as a stock when it comes to modeling its price diffusion.

Hence, a question appears more and more on the Internet, and in discussions about Bitcoin and cryptocurrencies. Should we consider this asset class as subject to perpetual bubble cycles?

Here is the definition of a bubble given by Investopedia. "A *bubble* is an economic cycle characterized by rapid escalation of asset prices followed by a contraction. It is created by a surge in asset prices unwarranted by the fundamentals of the asset and driven by exuberant market behavior. When no more investors are willing to buy at the elevated price, a massive sell-off occurs, causing the bubble to deflate." [5] This definition of a bubble through the price curve can oddly be applied to describe the behavior of the Bitcoin price curve itself. Therefore, considering Bitcoin price evolution as closely related to bubble dynamics doesn't seem unfair.

Morgan Stanley recently suggested that it might be too early to evaluate the potential future of Bitcoin. "The market is still in price discovery mode for Bitcoin and all other cryptocurrencies. The valuation of any of these cryptocurrencies is based on whatever they can be used for. Bitcoin is a mechanism for payment, yet isn't currently used for many payments for goods and services. The current price therefore reflects a future expectation that Bitcoin and related technology may be used for payments."[6]

We can therefore state that Bitcoin price movements are far determined in great proportion by highly irrational behaviors, including the following:

- Anchoring: attachment of thoughts to an unfounded reference point
- Confirmation bias: overweighting of information confirming the investor's original idea
- Hindsight bias: past events considered obvious and predictable

[4]Business Insider UK, *The price of bitcoin has a 91% correlation with Google searches for bitcoin*, Sept. 2017
[5]Investopedia
[6]The Financial Times, *The Bitcoin collapses, charted*, March 19, 2018

- Gambler's fallacy: occurrence of a particular random event considered as less likely to happen after a certain event
- Herding: tendency of individuals to mimic actions of a broader group
- Overconfidence: overestimation of one's capacity to successfully perform a task
- Overreaction: excessively emotional reactions to newly available information
- Availability bias: decisions oriented towards interpretation of more recent information
- Prospect asymmetry: gains and losses are valued differently

These cognitive biases affecting short-term demand create explosive price mechanisms: we were able to observe such extreme price behaviors in our first part. It leads us to think of another asset class that would fit well the Bitcoin features, both in terms of fundamentals and statistical metrics: commodities.

1.3 Bitcoin: a commodity?

We previously stated that many of the core Black and Scholes assumptions for options pricing do not stand in the case of Bitcoin. One can therefore find it hard to use a Black and Scholes framework in order to model the price of cryptocurrencies. Indeed, Bitcoin displays specific statistical characteristics, particularly in terms of volatility, which make it a specific alternative asset class very distinct from stocks. When looking at the shape of the Bitcoin's price curve, one may link it to the shape of several commodities' price curves such as gold. Hence, having a closer look to the link between Bitcoin and commodities seems relevant. Even the concept of "mining" using ever increasing costly resources in order to obtain new assets, with a limited long term amount of supply, seems to fit the concept of commodity.

In terms of regulation, Bitcoin has already been classified as a commodity by the Commodity Futures Trading Commission (CFTC) in 2015, and added to the Commodity Exchange Act. Being regulated under the CFTC means that virtual currencies are treated as a "commodity" under the Commodity Exchange Act over which the SEC does not have direct oversight, and not as a "security" under the securities laws. The CFTC has limited jurisdiction over spot markets in virtual currencies — in which participants buy and sell virtual currencies for prompt delivery — while it has broad jurisdiction over derivatives markets, including futures, in such currencies.

Introduction of futures on Bitcoin has been effective as of December 1, 2017, and such products will be regulated under the CFTC. Any addition of new derivative products on Bitcoin will therefore need approval of the CFTC prior to inception.

Moreover, this legal denomination of Bitcoin as a currency has been confirmed by the US case law, as District Judge Jack Weinstein ruled on March 6, 2018 that cryptocurrencies are commodities, saying it was supported by the plain meaning of the word "commodity" and that the agency (CFTC) had "broad leeway to interpret the law" regulating commodities according to the The Commodity Exchange Act of 1936.

On this matter, Anne Haubo Dyhrberg showed in [3] that "the return on Bitcoin is more affected by the demand for Bitcoin as a medium of exchange and less by temporary shocks to the price which indicate similarities to a currency." This statement, while not taking into account the bubble-like effect on major cryptocurrencies, underlines well the commodity aspect of Bitcoin. She also reports the analysis from Hammoudeh and Yuan in [5]. Indeed, she uses their research to show the same effect exists in commodities, especially gold. According to Dyhrberg, they "have similar results for gold and identified that gold was much more affected by the demand for jewelry and recycling as it is a precious metal and not an industrial metal, and is less influenced by short term shocks. Therefore, it seems that Bitcoin and gold have similarities when it comes to the volatility of the return and what type of shocks are most influential, though currency similarities were also identified".

In this way and in the next part, we will try to chose an optimal econometric model to fit the Bitcoin time series, taking inspiration from some major models widely used on the commodity derivatives market.

2 Modeling of the Bitcoin returns

2.1 Choice of the optimal econometric model

We tried to model the non-Gaussian characteristics of the Bitcoin returns using econometric models. Indeed, in the context of cryptocurrencies, we have seen that conventional hypothesis on assets' return and market characteristics are not verified: it is impossible to state that we are in an arbitrage free framework, the price returns exhibiting complex behaviors (stochastic volatility, asymmetry in shocks, etc.). In this way, it seems infeasible to use closed formula: Monte-Carlo simulations will be more in line with the Bitcoin's singularities in order to value derivatives.

We first tried to see if an ARIMA model would yield interesting results, as the explosive tendencies and behavioral biases we observed before lead us to anticipate the presence of autocorrelation in the series' returns. We therefore applied an iterative process in order to determine the optimal ARIMA coefficients as follows.

```python
def _get_best_model(TS):
    best_aic = np.inf
    best_order = None
    best_mdl = None

    pq_rng = range(5)
    d_rng = range(2)
    for i in pq_rng:
        for d in d_rng:
            for j in pq_rng:
                try:
                    tmp_mdl = smt.ARIMA(TS, order=(i,d,j)).fit(
                        method='mle', trend='nc'
                    )
                    tmp_aic = tmp_mdl.aic
                    if tmp_aic < best_aic:
                        best_aic = tmp_aic
                        best_order = (i, d, j)
                        best_mdl = tmp_mdl
                except: continue
    p('aic: {:6.5f} | order: {}'.format(best_aic, best_order))
    return best_aic, best_order, best_mdl
best = _get_best_model(TS)
```

Figure 16: Python code to find the best ARIMA model

Nevertheless, we observed very few auto-correlations in pure daily returns, as we can see in the autocorrelation and partial autocorrelation functions (ACF and PACF) graphs in figure 17.

In order to model the complete returns characteristics, an ARMA model would be a first step, but would be insufficient: errors should exhibit more complex than Gaussian properties. This is why we focused on the modeling of the error term in order to display a better fit.

We therefore chose to focus our work on the observation of the squared-returns behavior: as the serial correlation in daily returns seemed not to yield sufficient results, we oriented our research towards stochastic volatility models. This intuition of potentially exploitable information contained in past realized variance led us first of all to draw the ACF and PACF graphs for the squared-returns in figure 19.

On the ACF and PACF graphs for daily squared returns, we can see that there is some kind of variance autocorrelation: this finding is well in line with our initial observations on the behavior

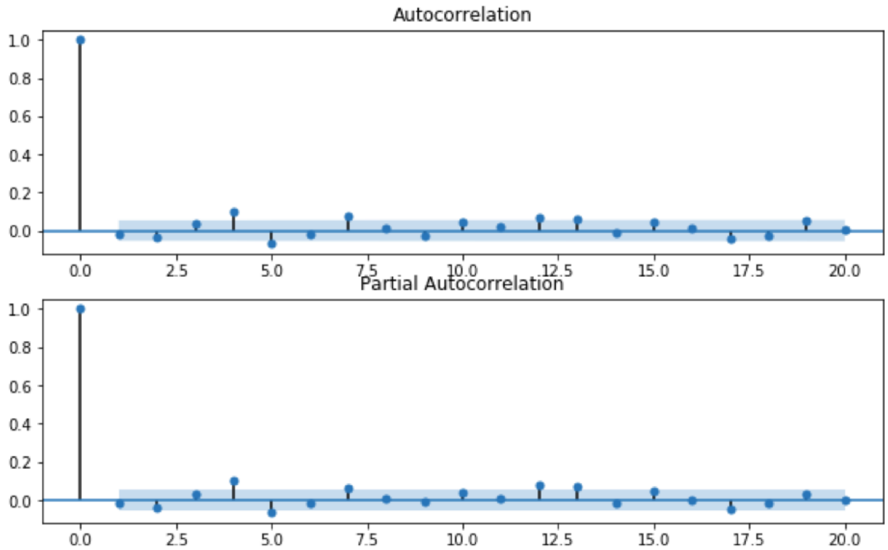

Figure 17: ACF and PACF with ARMA

```
fig = plt.figure(figsize=(10,6))
ax1 = fig.add_subplot(2,1,1)
fig = sm.graphics.tsa.plot_acf(returns, lags=20, ax=ax1)
ax2 = fig.add_subplot(2,1,2)
fig = sm.graphics.tsa.plot_pacf(returns, lags=20, ax=ax2)
```

Figure 18: Python code for ACF and PACF

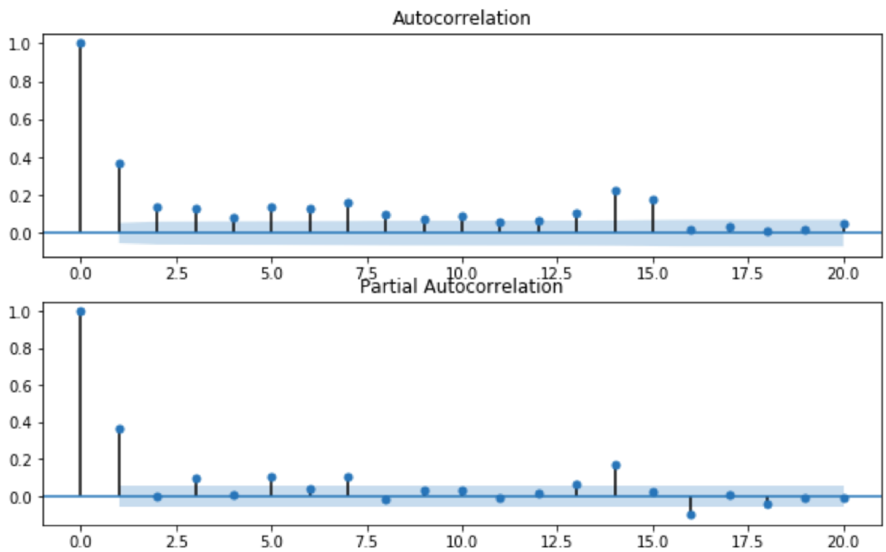

Figure 19: ACF and PCF on the squared-returns

of Bitcoin returns volatility. Indeed, we perceived some kind of regime changing volatility, with clusterings exhibiting mania or stress periods, and important spikes.

This is the reason why we chose to fit a GARCH (General Autoregressive Conditional Heteroskedastic) model on the time series in order to specify the behavior of our error terms.

In order to choose the optimal model, we tried through a recursive process different cases of the GARCH models (with different choices of volatility models such as ARCH, TARCH, EGARCH, Power GARCH and Asymmetric Power GARCH). We chose the model in order to minimize the Akaike Information Criteria and the Bayesian Information Criteria (these two criteria providing a interesting approximation of the fit of a model to a set of data, with a penalization term for the number of parameters to estimate).

$$AIC = 2k - 2ln(\hat{L})$$
$$BIC = ln(n)k - 2ln(\hat{L})$$

After testing different types of GARCH models with generic parameters, we finally opted for the APGARCH model that allowed to take into account leverage effects, using an Exponential volatility model and Student's T innovations to improve our fit and obtain more heavy tails.

We initially wanted to replace these Student's T distributed errors by more complex ones, but our statistical packages didn't allow us to go further. Our best intuition was to inject innovation terms following a Skewed Generalized Error Distribution as presented in the general approach proposed by Fernandez and Steel in [4] in order to introduce skewness in any continuous unimodal and symmetric distribution.

Keeping the Student's T residuals, our best model yielded an AIC of -1135, which was the lowest score across all models tested.

Thereafter, we optimized the parameters of the preferred model in order to minimize both information criteria. We did this through a loop iterating on the different possible levels of parameters for a pre-determined range, progressively saving as the best model the one that obtained the minimum criteria.

We finally chose the model with the following parameters (putting the emphasis on asymmetric innovations):

- p (lag order of the symmetric innovation) =1
- o (lag order of the asymmetric innovation) = 2
- q (lag oder of lagged volatility) = 1

We then fitted our optimal model to the time series of Bitcoin, on a pre-selected time window: we only took values after 2016 because it is the period where the behavior of Bitcoin has been the most explosive. We believe that in order to forecast future paths of Bitcoin, these recent information should be included with the most emphasis.

Figure 20 displays the results of the fit we obtained.

```
              Current function value: -574.533641138208
              Iterations: 34
              Function evaluations: 378
              Gradient evaluations: 34
                       Constant Mean - EGARCH Model Results
==============================================================================
Dep. Variable:                      y   R-squared:                      -0.000
Mean Model:             Constant Mean   Adj. R-squared:                 -0.000
Vol Model:                     EGARCH   Log-Likelihood:                543.378
Distribution:                  Normal   AIC:                          -1078.76
Method:            Maximum Likelihood   BIC:                          -1063.27
                                        No. Observations:                  355
Date:                Wed, Apr 25 2018   Df Residuals:                      351
Time:                        14:44:55   Df Model:                            4
                                 Mean Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
mu         6.7361e-03  2.677e-03      2.516  1.186e-02 [1.490e-03,1.198e-02]
                              Volatility Model
=============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
-----------------------------------------------------------------------------
omega         -0.0587  2.529e-02     -2.320  2.032e-02  [ -0.108,-9.118e-03]
gamma[1]       0.0325  2.296e-02      1.414      0.157 [-1.253e-02,7.746e-02]
beta[1]        0.9893  4.240e-03    233.320      0.000       [  0.981,  0.998]
=============================================================================
```

Figure 20: Optimized Exponential GARCH model results

```
garch = arch_model(returns[1000:], vol='EGARCH',p=1, o=2,q=1,dist='StudentsT')
res = garch.fit(update_freq=1)
print(res.summary())
```

Figure 21: Python code for GARCH-type fit

After fitting the model, we displayed the different following graphs of the residuals in figure 22, and found that we reduced importantly the divergence from normal residuals. In particular in the QQ-plots in figure 23. In this way, we can say that we achieved to find a model that finds relatively closely the behavior of Bitcoin price movements.

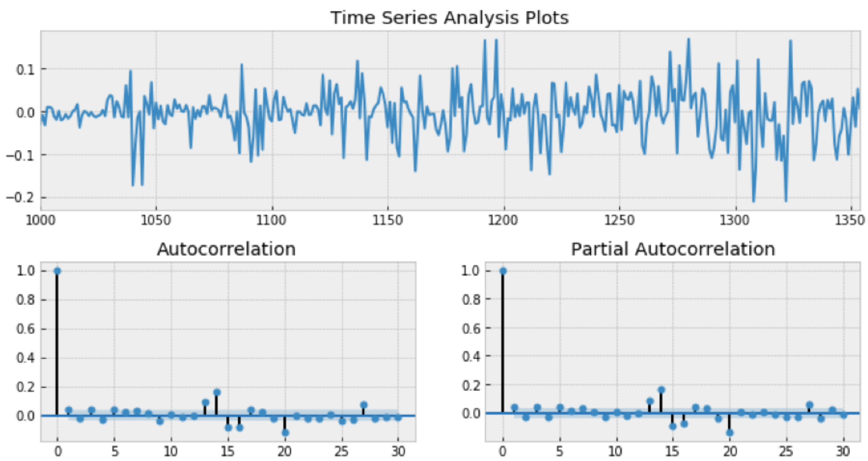

Figure 22: Times series plot with ACF and PACF

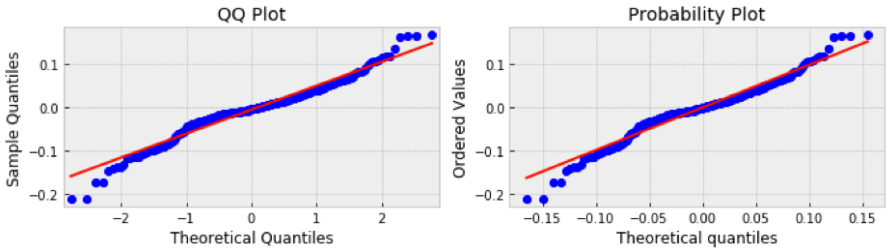

Figure 23: QQ-plot

```
def tsplot(y, lags=None, figsize=(10, 8), style='bmh'):
    if not isinstance(y, pd.Series):
        y = pd.Series(y)
    with plt.style.context(style):
        fig = plt.figure(figsize=figsize)
        #mpl.rcParams['font.family'] = 'Ubuntu Mono'
        layout = (3, 2)
        ts_ax = plt.subplot2grid(layout, (0, 0), colspan=2)
        acf_ax = plt.subplot2grid(layout, (1, 0))
        pacf_ax = plt.subplot2grid(layout, (1, 1))
        qq_ax = plt.subplot2grid(layout, (2, 0))
        pp_ax = plt.subplot2grid(layout, (2, 1))

        y.plot(ax=ts_ax)
        ts_ax.set_title('Time Series Analysis Plots')
        smt.graphics.plot_acf(y, lags=lags, ax=acf_ax, alpha=0.5)
        smt.graphics.plot_pacf(y, lags=lags, ax=pacf_ax, alpha=0.5)
        sm.qqplot(y, line='s', ax=qq_ax)
        qq_ax.set_title('QQ Plot')
        scs.probplot(y, sparams=(y.mean(), y.std()), plot=pp_ax)

        plt.tight_layout()
    return
```

Figure 24: Python code for multiple statistical plots

2.2 Use of a classical valuation tool : Monte-Carlo simulations

As we precised earlier, in the particular context of Bitcoin, there doesn't seem to be a way to derive a closed formula for derivatives pricing. The finding of analytical approximation is not time-efficient, and in order to value derivatives in a client-oriented environment, time is key. This consideration has led us to conclude that Monte-Carlo simulations would be the proper solution.

In order to value derivatives, we would then use forecasted means, conditional variances, and residual variances obtained from our model fit in order to derive different paths for our underlying Bitcoin, the most complete solution being to use the parameters of our best fitted ARMA(p,q) (or to neglect this part of the model if we consider no statistically significant serial correlation in simple returns) and to incorporate in this model our EGARCH asymmetric errors with Student's T innovations.

Following (figure 25 & 26) is an example of a regular GARCH Monte-Carlo simulation algorithm we have constructed with random parameters.

The different simulated final payoffs (depending on the type of derivative) would then be averaged and discounted, a main challenge then being to determine what discount rate to use.

A first naive assumption would be to use the conventional discount factor, considering that cryptocurrencies becoming another alternative asset class among tons of others, rational agents should discount it at prevailing risk-free interest rates. This is contrary to the elements we have discussed in the part linked to behavioral biases: crypto-currencies are part of a parallel paradigm, with its own constituents and dynamics, therefore, classical asset pricing modeling should be avoided in order to derive present values.

```
result = []
S0=1
sigma0, a0, a1, b1 = 0.01, 0.02, 0.5, 0.3
n = 100000
for j in range(1,252):
    eps = np.random.normal(0, 1, n)
    sigma = np.ones(n) * sigma0
    r = np.zeros(n)
    for i in range(1,n):
        sigsq[i] = a0 + a1*(eps[i-1]**2) + b1*sigsq[i-1]
        eps[i] = w[i] * np.sqrt(sigsq[i])
    price_list=[S0]
    for x in eps:
        price_list.append(price_list[-1]*x)
    result.append(payoff(price_list[-1]))
print(round(np.mean(result),2))'
```

Figure 25: Python code for regular GARCG MC

```
forecasts=res1.forecast(horizon=5, start=1, method='simulation', simulations=100)
forecast_variance=forecasts.variance.tail(1)
forecast_mean=forecasts.mean.tail(1)
forecast_residual_variance=forecasts.residual_variance.tail(1)
forecast_values=forecasts.simulations.values
print (forecast_variance)
print (forecast_mean)
print (forecast_residual_variance)
print (forecast_values)
```

Figure 26: Python code for regular mean, conditional variance and residual variances forecasts

A first bet would be to use the USD/Bitcoin implied Bitcoin interest rate from various different exchange platforms in order to get the discounted factors necessary to obtain present values of payoffs. An interesting research path would be to adopt a more fundamental than econometric approach by identifying key risk factors driving such an implied interest rate (from conventional macro factors to some of the ones we mentioned in our part dedicated to behavioral biases, for instance the number of research of "Bitcoin" in Google), and drawing conclusions on its forward potential dynamics. Then a mapping could be done in order to obtain a term structure of implied Bitcoin interest rate, and discount factors could be computed.

Another simpler approach would be to use the newly introduced futures market for Bitcoin derivatives in order to directly derive the futures' implied Bitcoin interest rate: a weak spot of this approach is the fact that the exchange traded futures are very recent, and haven't attracted the majority of crypto-investors yet (lack of liquidity that can lead to inconsistent information). The implied interest rate obtained might also be biased by the choice of modeling validated by the exchange regulators.

Conclusion

In this paper, we underlined the different deviations of the Bitcoin dynamics from the traditional derivatives valuation framework of Black and Scholes. We have invalidated the different naive assumptions of this model, and put the emphasis on the singular non-Gaussian features of the daily log-returns.

We then investigated in order to understand the sources of these extreme characteristics: we concluded on the important impact of behavioral biases and proximity of Bitcoin price evolution with asset bubbles.

Once this had been studied, we tried to found an alternative asset class that exhibited behaviors comparable to those of crypto-currencies: commodities were the perfect fit. Indeed, with explosive processes, varying volatilities regimes and complex supply/demand mechanisms, it seemed that the commodities environment fitted well our Bitcoin features. Moreover, not only the time series features matched, but also the fundamentals and lexical fields intrinsically linked (overall constrained supply of asset, concept on "mining" etc.). This led us on the path to narrow the type of approaches we could use in order to model the crypto-currency's price behavior.

Finally, we oriented our work toward econometric models in order to fit empirical characteristics, having acknowledged the relative difficulty of finding closed analytical formula in our context of study. Through trial and error, we fitted various types of models, and finally opted for Exponential GARCH model allowing for leverage effects with Student's T distributed errors.

We ended our work by discussing the Monte-Carlo implementation that would be required in order to concretely value derivatives on Bitcoin.

References

[1] M.J. Aitken and R. Winn. What is this thing called liquidity ? *Securities Industry Research Centre of Asia-Pacific*, 1997.

[2] Y. Amihud. Illiquidity and stock returns: cross-section and time-series effects. *Journal of Financial Markets 5, 31–56*, 2002.

[3] A.H. Dyhrberg. Bitcoin, gold and the dollar – a garch volatility analysis. *Finance Research Letters 16, 85–92*, 2016. https://doi.org/10.1016/j.frl.2015.10.008.

[4] C. Fernández and M.F.J. Steel. On bayesian modeling of fat tails and skewness. *Journal of the American Statistical Association, 93: 359–371*, 1998.

[5] S. Hammoudeh and Y. Yuan. Metal volatility in presence of oil and interest rate shocks. *Energy Economics 30, 606–620*, 2008. https://doi.org/10.1016/j.eneco.2007.09.004.

[6] L. Hio. The liquidity of bitcoin. *International Journal of Economics and Finance; Vol. 10, No. 1*, December 2018.

[7] S. Trimborn, M. Li, and W.K. Hardle. Investing with cryptocurrencies - a liquidity constrained investment approach. *SFB 649 "Economic Risk"*, 2017. https://ssrn.com/abstract=2999782.