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Homework # 4
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We aim to sample on $\mathbb{R}^2$ the target distribution given by

$$d\pi(x,y) \propto \exp\left(-\frac{x^2}{a^2} - y^2 - \frac{1}{4}\left(\frac{x^2}{a^2} - y^2\right)^2\right)$$

where $a > 0$. We consider a Markov transition kernel P defined by

$$P = \frac{1}{2}(P_1 + P_2)$$

where $P_i((x,y),(\mathrm{d}x',\mathrm{d}y'))$ is the Markov transition kernel which only updates the i-th component: this update follows a symmetric random walk proposal mechanism and uses a Gaussian distribution with variance σ_i^2.

1 Exercise 1

1.1 Q1

Let's show that P is a Markov transition kernel.

$$P((x,y),(\mathrm{d}x',\mathrm{d}y')) = \frac{1}{2}P_1((x,y),(\mathrm{d}x',\mathrm{d}y')) + \frac{1}{2}P_2((x,y),(\mathrm{d}x',\mathrm{d}y'))$$

where $P_i((x,y),(\mathrm{d}x',\mathrm{d}y')) \propto \begin{cases} \delta_y(\mathrm{d}y')\exp(-\frac{1}{2\sigma_1^2}(x-x')^2)\mathrm{d}x' & \text{if } i=1, \\ \delta_x(\mathrm{d}x')\exp(-\frac{1}{2\sigma_2^2}(y-y')^2)\mathrm{d}y' & \text{if } i=2. \end{cases}$

It means that if $i = 1$, y remains constant during the update and new value x' is proposed from a gaussian centered in x. We have the reverse for i_2, no changement is brought to x so $x' = x$, and y' is proposed from a gaussian centered in y.

$$\begin{aligned}
\mathbb{E}_{P(X_0,.)}[h(X_1)|X_0] &= \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)(1_{B=0} + 1_{B=1})|X_0] \\
&= \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)1_{B=0}|X_0] + \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)1_{B=1}|X_0] \\
&= \delta_y(A)\int_A \exp(-\frac{1}{2\sigma_1^2}(x-x')^2)\mathrm{d}x'
\end{aligned}$$

$P_1((x,y),A) = \mathbb{P}(X_1 \in A|X_0 = (x,y)) \propto \delta_y(A)\int_A \exp(-\frac{1}{2\sigma_1^2}(x-x')^2)\mathrm{d}x'$

$$\begin{aligned}
\mathbb{E}_{P(X_0,.)}[h(X_1)|X_0] &= \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)(1_{B=0} + 1_{B=1})|X_0] \\
&= \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)1_{B=0}|X_0] + \mathbb{E}_{B\sim\mathcal{B}(\frac{1}{2})}[h(X_1)1_{B=1}|X_0]
\end{aligned}$$

We consider the logarithm of the target distribution

$$\log d\pi(x,y) = -\frac{x^2}{a^2} - y^2 - \frac{1}{4}\left(\frac{x^2}{a^2} - y^2\right)^2 + \text{cste}$$

References