1 Question 1

Let $G = (V, E)$ be an undirected graph of n nodes without self-loops. The sum of the degree sequence is twice the number of edges because each degree counts the number of times the node appears as an endpoint of an edge, and any edge has two endpoints. The number of edge is maximal if G is complete, so each degree is equal to $(n-1)$ and

$$n \times (n-1) = 2|E|$$

which leads to

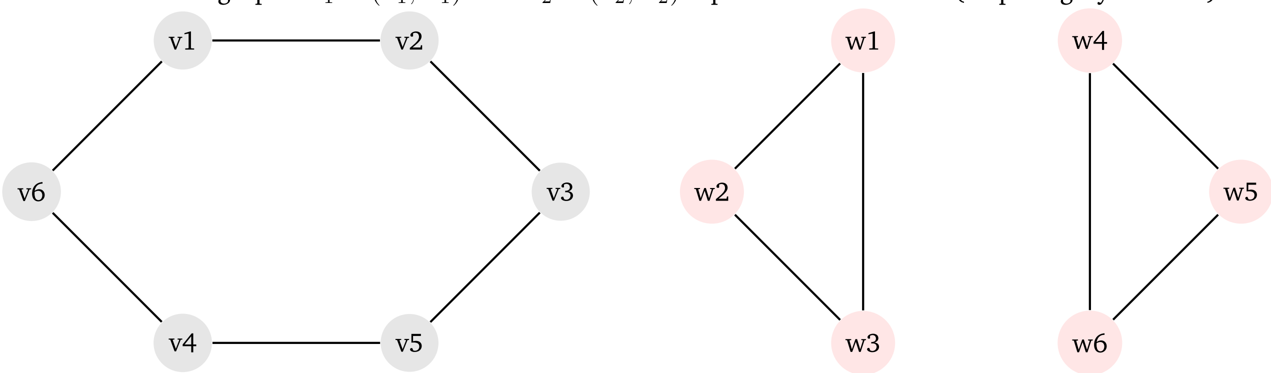
$$|E| = \frac{n(n-1)}{2}.$$

The maximum number of triangles is the number of ways to choose 3 distinct nodes from a set of n ones. Thus, we have

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

2 Question 2

Let's consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ represented as follows (resp. in grey and red).



Both share the same degree sequence $(2, 2, 2, 2, 2, 2)$. However they are not isomorphic to each other. Indeed, G_2 has two 3-cycles, whereas G_1 has none.

3 Question 3

Let C_n denote a cycle graph on n vertices, and α_n its corresponding global clustering coefficient. Then $\alpha_3 = 1$ because C_3 is a triangle. And $\forall k \geq 4, \alpha_k = 0$ because they have no closed triplet.

4 Question 4

We show that $u_1 = \mathbb{1}_n = (1 \dots 1)^T$. Since $\sum_j A_{ij} = d_i$ (where d_k is the degree of node k),
$L_{rw}\mathbb{1}_n = (I - D^{-1}A)\mathbb{1}_n = \mathbb{1}_n - D^{-1}A\mathbb{1}_n = \mathbb{1}_n - D^{-1}(d_1 \dots d_n)^T = \mathbb{1}_n - \mathbb{1}_n = 0$
So the smallest eigenvalue is 0 and its corresponding eigenvector is $u_1 = \mathbb{1}_n$.
We deduce that $\sum_{i,j} A_{ij}([u_1]_i - [u_1]_j)^2 = 0$.

5 Question 5

$$Q = \sum^{n_c} \left[\frac{l_c}{m} - \left(\frac{d_c}{2m} \right)^2 \right] \tag{1}$$

where, $m = |E|$ is the total number of edges in the graph, n_c is the number of communities in the graph, l_c is the number of edges within the community c and d_c is the sum of the degrees of the nodes that belong to community c.

5.1 First graph

$m = 14$; $n_c = 2$;
For the both communities, $l_c = 6$ and $d_c = 3 + 3 + 4 + 4 = 14$.
So $Q_1 = \left[\frac{6}{14} - \left(\frac{14}{2\times 14}\right)^2\right] \times 2 = \frac{5}{14}$.

5.2 Second graph

$m = 14$; $n_c = 2$;
For the blue community, $l_c = 5$ and $d_c = 3 + 3 + 4 + 4 + 3 = 17$.
For the red community, $l_c = 2$ and $d_c = 4 + 4 + 3 = 11$.
So $Q_2 = \left[\frac{5}{14} - \left(\frac{17}{2\times 14}\right)^2\right] + \left[\frac{2}{14} - \left(\frac{11}{2\times 14}\right)^2\right] = -\frac{9}{392}$.

6 Question 6

The kernel maps the graphs into a feature space where each feature corresponds to the shortest path distance and the value is the frequency of that distance. We compute the degree sequence of P_4 and S_4.
$\phi(P_4) = [3, 2, 1, 0, \ldots, 0]$.
$\phi(S_4) = [4, 6, 0, 0, \ldots, 0]$.
Thus

$$k_{\text{shortest path}}(P_4, S_4) =< \phi(P_4), \phi(S_4) >= 3 \times 4 + 2 \times 6 = 24$$
$$k_{\text{shortest path}}(P_4, P_4) =< \phi(P_4), \phi(P_4) >= 3 \times 3 + 2 \times 2 + 1 \times 1 = 14$$
$$k_{\text{shortest path}}(S_4, S_4) =< \phi(S_4), \phi(S_4) >= 4 \times 4 + 6 \times 6 = 52$$

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7 Question 7

Let's assume $k(G, G') = f_G^T f_{G'} = 0$. The i-th entry of f_G is the number of sampled subraphs from G that are isomorphic to graphlet$_i$. So there exist no graphlet$_i$ such that G and G' are both isomorphic to graphlet$_i$ at the same time.
We can take for example two graphs G and G' with 4 nodes. Consider G a "square" (a cycle graph) and G' an "empty graph" (4 nodes and no edge). Then samples of G' can only be isomorphic to G_4, whereas samples of G are never isomorphic to G_4. Consequently $f_G^T f_{G'} = 0$.

References