1 Question 1

Let's compute the complete log-likelihood for the observations $\{y_{i,j}\}_{i,j}$. The parameters are $\theta = (\overline{t_0}, \overline{v_0}, \sigma_\xi, \sigma_\tau, \sigma) \in \mathbb{R}^5$. The latent variable is $Z = \{(z_{\text{pop}}, z_i)\}_i = \{((t_0, v_0), (\alpha_i, \tau_i))\}_i$. We want the result in the form

$$\log q(y, Z, \theta) = -\Phi(\theta) + < S(y, z), \Psi(\theta) >_{\mathbb{R}^5} + \text{cste}.$$

Since $\{z_i\}_i$ and z_{pop} are independant, and $z_1, \dots, z_n$ are independant, we have

$$\begin{aligned}\log q(y, Z, \theta) &= \log q(y|Z, \theta) q(Z|\theta) p(\theta) \\ &= \log q(y|Z, \theta) + \log q(z_{\text{pop}}|\theta) + \sum_{i=1}^{N} \log q(\{z_i\}|\theta) + \log p(\theta)\end{aligned}$$

Let's compute this term by term.

- We have $\xi_{i,j} \sim \mathcal{N}(0, \sigma^2) \implies y_{i,j} \sim \mathcal{N}(d_i(t_{i,j}), \sigma^2)$ and then

$$\log q(y, Z, \theta) = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{K} \left(\frac{y_{i,j} - d_i(t_{i,j})}{\sigma^2} \right)^2 + KN \log \sigma + \text{cste}.$$

- We have $\xi_i \sim \mathcal{N}(0, \sigma_\xi^2)$ and $\tau_i \sim \mathcal{N}(0, \sigma_\tau^2)$. Since $(\xi_i)_i$ and $(\tau_i)_i$ are independant, $(\alpha_i)_i$ and $(\tau_i)_i$ are independant. Plus $\alpha_i = \exp(\xi_i)$. Let's consider the change of variable $T : x \mapsto \exp(x)$ from $\mathbb{R}$ to $\mathbb{R}_+^*$. It's a diffeomorphism and its density is given by

$$p(\alpha_i|\theta) = \mathcal{N}(T^{-1}(\alpha_i); 0, \sigma_\xi^2) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp(-\frac{\log(\alpha_i)}{2\sigma_\xi^2}) |\det J[\log](\alpha_i)|$$

  where $\det J[\log](\alpha_i)| = 1/\alpha_i$. Thus we have

$$\begin{aligned}\log q(z_i|\theta) &= \log q(\alpha_i|\theta) + \log q(\tau_i|\theta) \\ &= -\frac{1}{2} \left[\left(\frac{\xi_i}{\sigma_\xi} \right)^2 + \log(\alpha_i) + \left(\frac{\tau_i}{\sigma_\tau} \right)^2 \right] - \log(\sigma_\xi) - \log(\sigma_\tau) + \text{cste} \\ &= -\frac{1}{2} \left[\left(\frac{\xi_i}{\sigma_\xi} \right)^2 + \left(\frac{\tau_i}{\sigma_\tau} \right)^2 \right] - \log(\sigma_\xi) - \log(\sigma_\tau) + \text{cste}.\end{aligned}$$

- We have $t_0 \sim \mathcal{N}(\overline{t_0}, \sigma_{t_0}^2)$ and $v_0 \sim \mathcal{N}(\overline{v_0}, \sigma_{v_0}^2)$. Conditionality to θ, the parameters of the normals are deterministic and

$$\begin{aligned}\log q(z_{\text{pop}}|\theta) &= -\frac{1}{2} \left(\frac{t_0 - \overline{t_0}}{\sigma_{t_0}} \right)^2 - \log(\sigma_{t_0}) - \frac{1}{2} \left(\frac{v_0 - \overline{v_0}}{\sigma_{v_0}} \right)^2 - \log(\sigma_{v_0}) + \text{cste} \\ &= -\frac{1}{2} \left(\frac{t_0 - \overline{t_0}}{\sigma_{t_0}} \right)^2 - \frac{1}{2} \left(\frac{v_0 - \overline{v_0}}{\sigma_{v_0}} \right)^2 + \text{cste}.\end{aligned}$$

- We assume the following *a priori* on $\theta = (\overline{t_0}, \overline{v_0}, \sigma_\xi, \sigma_\tau, \sigma)$:
  $\overline{t_0} \sim \mathcal{N}\left(\overline{\overline{t_0}}, s_{t_0}^2\right)$; $\quad \overline{v_0} \sim \mathcal{N}\left(\overline{\overline{v_0}}, s_{v_0}^2\right)$; $\quad \sigma_\xi^2 \sim \mathcal{W}^{-1}(v_\xi, m_\xi)$; $\quad \sigma_\tau^2 \sim \mathcal{W}^{-1}(v_\tau, m_\tau)$; $\quad \sigma^2 \sim \mathcal{W}^{-1}(v, m)$.
  Thus the prior log-probability is given by

$$\begin{aligned}\log p(\theta) = &-\frac{1}{2} \left(\frac{\overline{t_0} - \overline{\overline{t_0}}}{s_{t_0}} \right)^2 - \frac{1}{2} \left(\frac{\overline{v_0} - \overline{\overline{v_0}}}{s_{v_0}} \right)^2 - \log(s_{t_0}) - \log(s_{v_0}) - (m_\xi + 2) \log(\sigma_\xi) - \frac{v_\xi^2}{2\sigma_\xi^2} \\ &- (m_\tau + 2) \log(\sigma_\tau) - \frac{v_\tau^2}{2\sigma_\tau^2} - (m + 2) \log(\sigma) - \frac{v^2}{2\sigma^2} + \text{cste}.\end{aligned}$$

Recall that we want to find Φ mapping to $\mathbb{R}$, and Ψ and S mapping to $\mathbb{R}^5$ such that

$$\log q(y,z,\theta) = -\Phi(\theta) + < S(y,z), \Psi(\theta) >_{\mathbb{R}^5} + \text{cste}.$$

Here we have

$$\begin{aligned}
\log q(y,z,\theta) = & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{K}\left(\frac{y_{i,j}-d_i(t_{i,j})}{\sigma^2}\right)^2 + KN\log\sigma \\
& -\frac{1}{2}\left(\frac{t_0-\overline{t_0}}{\sigma_{t_0}}\right)^2 - \frac{1}{2}\left(\frac{v_0-\overline{v_0}}{\sigma_{v_0}}\right)^2 \\
& -\frac{1}{2}\sum_{i=1}^{N}\left[\left(\frac{\xi_i}{\sigma_\xi}\right)^2 + \left(\frac{\tau_i}{\sigma_\tau}\right)^2\right] - N\log(\sigma_\xi) - N\log(\sigma_\tau) \\
& -\frac{1}{2}\left(\frac{\overline{t_0}-\overline{\overline{t_0}}}{s_{t_0}}\right)^2 - \frac{1}{2}\left(\frac{\overline{v_0}-\overline{\overline{v_0}}}{s_{v_0}}\right)^2 - \log(s_{t_0}) - \log(s_{v_0}) - (m_\xi+2)\log(\sigma_\xi) - \frac{v_\xi^2}{2\sigma_\xi^2} \\
& -(m_\tau+2)\log(\sigma_\tau) - \frac{v_\tau^2}{2\sigma_\tau^2} - (m+2)\log(\sigma) - \frac{v^2}{2\sigma^2} + \text{cste}.
\end{aligned}$$

We take $\psi(\theta) = \begin{bmatrix} \frac{\overline{t_0}}{\sigma_{t_0}^2} \\ \frac{\overline{v_0}}{\sigma_{v_0}^2} \\ -\frac{N}{2\sigma_\xi^2} \\ -\frac{N}{2\sigma_\tau^2} \\ -\frac{NK}{2\sigma^2} \end{bmatrix}$; $\quad S(y,z) = \begin{bmatrix} t_0 \\ v_0 \\ \frac{1}{N}\sum_{i=1}^N \xi_i^2 \\ \frac{1}{N}\sum_{i=1}^N \tau_i^2 \\ \frac{1}{NK}\sum_{i=1}^N\sum_{j=1}^K (y_{i,j}-d_i(t_{i,j}))^2 \end{bmatrix}$ and

$$\begin{aligned}
\Psi(\theta) = & \frac{1}{2}\frac{\overline{t_0}^2}{s_{t_0}^2} + \frac{1}{2}\frac{\overline{t_0}^2}{\sigma_{t_0}^2} - \frac{\overline{t_0}\overline{\overline{t_0}}}{s_{t_0}^2} \\
& + \frac{1}{2}\frac{\overline{v_0}^2}{s_{v_0}^2} + \frac{1}{2}\frac{\overline{v_0}^2}{\sigma_{v_0}^2} - \frac{\overline{v_0}\overline{\overline{v_0}}}{s_{v_0}^2} \\
& + (N+m_\tau+2)\log(\sigma_\tau) \\
& + (N+m_\xi+2)\log(\sigma_\xi) \\
& + (NK+m+2)\log(\sigma)
\end{aligned}$$

We assumed $K = k_i$ for all individual i.

2 Question 2

Let's compute the *a posteriori* distribution. It verifies $q(z|y,\theta) = \frac{q(y,z,\theta)}{q(y,\theta)} =\propto q(y,z,\theta)$. Then, **up to a constant** we have

$$\begin{aligned}
\log q(z|y,\theta) &= \log q(y,z,\theta) \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{K}\left(\frac{y_{i,j}-d_i(t_{i,j})}{\sigma^2}\right)^2 - \frac{1}{2}\left(\frac{t_0-\overline{t_0}}{\sigma_{t_0}}\right)^2 - \frac{1}{2}\left(\frac{v_0-\overline{v_0}}{\sigma_{v_0}}\right)^2 \\
&\quad -\frac{1}{2}\sum_{i=1}^{N}\left[\left(\frac{\xi_i}{\sigma_\xi}\right)^2 + \log(\alpha_i) + \left(\frac{\tau_i}{\sigma_\tau}\right)^2\right].
\end{aligned}$$

We removed terms that do not depend of z.

References