DISPLACEMENT-OPERATOR SQUEEZED STATES. I. EXISTENCE OF A HOLSTEIN-PRIMAKOFF/BOGOLIUBOV TRANSFORMATION

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ABSTRACT

As a start towards the goal of finding a *displacement operator* definition of squeezed states for arbitrary systems, we investigate the properties of systems where there is a Holstein-Primakoff or Bogoliubov transformation. In these cases the *ladder-operator* definition of squeezed states is equivalent to an extent displacement-operator definition. We exemplify this in a setting where there are operators satisfying $[A, A^\dagger] = 1$, but the A's are not necessarily the Fock space $a's$.

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1 Introduction

As has now been known and studied for some time, there are three equivalent, widely-used definitions of the coherent states of the harmonic oscillator [1]-[7]. These are (1) the minimum-uncertainty, (2) annihilation- (or, more generally, ladder-) operator, and (3) displacement-operator methods. These methods have been extended to the squeezed states of the harmonic oscillator. Further, with one exception, general coherent and squeezed states have been obtained for general systems by these three methods. That exception is a general definition of squeezed states by the displacement-operator method.

   With an aim towards obtaining such a general method, we are studying systems where such a definition works. Specifically, after reviewing the coherent and squeezed states for the harmonic oscillator and more general systems, we focus on why displacement-operator squeezed states can not be obtained by a naive generalization of the harmonic-oscillator case: there is, in general, no Bogoliubov transformation.

   This problem does not exist in certain systems. In particular, we here study the multi-boson formalism of Brandt and Greenberg [8], where the multi-boson operators obey canonical commutation relations, and hence one can proceed with calculations in the standard way. In following papers [9], we will study time-dependent systems which have isomorphic symmetry algebras.

2 The Coherent and Squeezed States of the Harmonic Oscillator

2.1 Coherent states

Given the canonical commutation relations

$$[a, \ a^\dagger] = 1 \ , \qquad [a, \ a] = 0 \ , \tag{1}$$

where we adopt the realization

$$a = \frac{1}{\sqrt{2}}(x + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - ip). \tag{2}$$

In paper II, we will use time-dependent generalizations of these operators. The definitions of displacement-operator and ladder-operator coherent states are well known. They are

$$D(\alpha)|0\rangle = |\alpha\rangle \tag{3}$$

and

$$a|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{4}$$

where

$$D(\alpha) = \exp[\alpha a^\dagger - \bar{\alpha} a] = \exp\left[-\frac{1}{2}|\alpha|^2\right]\exp[\alpha a^\dagger]\exp[-\bar{\alpha} a] \tag{5}$$

and

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle .$$ (6)

The last equality in Eq. (5) comes from using a Baker-Campbell-Hausdorff relation. Observe that the definition (4) follows from the definition (3) by

$$[a, \, D(\alpha)] = \alpha D(\alpha) .$$ (7)

The coherent-state wave functions are $(\hbar = 1)$

$$\psi_{cs}(x) = [2\pi\sigma_0^2]^{-1/4} \exp\left[-\left(\frac{x-x_0}{2\sigma_0}\right)^2 + ip_0 x\right] ,$$ (8)

$$\sigma_0 = [2m\omega]^{-1/2} , \qquad x_0 = \langle x\rangle , \qquad p_0 = \langle p\rangle ,$$ (9)

$$Re(\alpha) = [m\omega/2]^{1/2} x_0 , \qquad Im(\alpha) = p_0/[2m\omega]^{1/2} .$$ (10)

That is, the states are Gaussians with the width being that of the ground state.

2.2 Squeezed states

Squeezed states [10]-[14] can be defined by the displacement-operator method as the product of a unitary displacement operator and a unitary squeeze operator acting on the ground state:

$$D(\alpha)S(z)|0\rangle \equiv |\alpha, z\rangle , \qquad z \equiv re^{i\theta} .$$ (11)

θ is a phase which defines the starting time, $t_0 = (\theta/2\omega)$. $S(z)$ is given by

$$S(z) = \exp\left[\frac{1}{2}za^\dagger a^\dagger - \frac{1}{2}\bar{z}aa\right]$$ (12)

$$= \exp\left[\frac{1}{2}e^{i\theta}(\tanh r)a^\dagger a^\dagger\right]\left(\frac{1}{\cosh r}\right)^{(\frac{1}{2}+a^\dagger a)} \exp\left[-\frac{1}{2}e^{-i\theta}(\tanh r)aa\right]$$ (13)

$$= \exp\left[\frac{1}{2}e^{i\theta}(\tanh r)a^\dagger a^\dagger\right](\cosh r)^{-1/2} \sum_{n=0}^{\infty} \frac{(sechr - 1)^n}{n!}(a^\dagger)^n(a)^n$$

$$\times \exp\left[-\frac{1}{2}e^{-i\theta}(\tanh r)aa\right] ,$$ (14)

where Eqs. (13) and (14) are obtained from BCH relations. Observe that

$$D(\alpha)S(z) = S(z)D(\gamma) , \qquad \gamma = \alpha\cosh r - \bar{\alpha}e^{i\theta}\sinh r .$$ (15)

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Therefore, the ordering of D and S is only a convention.

The squeezed-state wave functions are given by by the form of Eq. (8), but with

$$\sigma_0 \rightarrow \sigma = s\sigma_0 , \qquad r = \ln s . \tag{16}$$

These wave functions are Gaussians which, in general, do not have the width of the ground state; i.e., they are squeezed by the squeeze parameter s:

$$\psi_{ss}(x) = [2\pi s^2 \sigma_0^2]^{-1/4} \exp\left[-\left(\frac{x - x_0}{2s\sigma_0}\right)^2 + ip_0 x\right] , \tag{17}$$

The elements involved in S actually are an SU(1,1) group defined by

$$K_+ = \frac{1}{2}a^\dagger a^\dagger , \qquad K_- = \frac{1}{2}aa , \qquad K_0 = \frac{1}{2}\left(N + \frac{1}{2}\right) , \tag{18}$$

where $N = a^\dagger a$. The operators $K_0, K_\pm$ satisfy the commutation relations

$$[K_0, \ K_\pm] = \pm K_\pm , \quad [K_+, \ K_-] = -2K_0 . \tag{19}$$

Therefore, S can be given by

$$S(z) = \exp[zK_+ - \bar{z}K_-] \tag{20}$$

$$= \exp[e^{i\theta}(\tanh r)K_+]\left(\frac{1}{\cosh r}\right)^{2K_0} \exp[-e^{-i\theta}(\tanh r)K_-] . \tag{21}$$

The commutation relations (1) and (19) close with

$$[K_+, \ a^\dagger] = 0 , \qquad [K_-, \ a^\dagger] = a , \qquad [K_+, \ a] = -a^\dagger ,$$

$$[K_-, \ a] = 0 , \qquad [K_0, \ a^\dagger] = \tfrac{1}{2} a^\dagger , \qquad [K_0, a] = -\tfrac{1}{2} a. \tag{22}$$

The ladder-operator definition of the squeezed states is that

$$[\mu a - \nu a^\dagger]|\alpha, z\rangle = \beta|\alpha, z\rangle . \tag{23}$$

Again this follows from the displacement-operator definition because

$$b \equiv S(z)^{-1}aS(z) = (\cosh r)a + e^{i\theta}(\sinh r)a^\dagger ,$$
$$b^\dagger \equiv S(z)^{-1}aS(z) = (\cosh r)a^\dagger + e^{-i\theta}(\sinh r)a . \tag{24}$$

where

$$[b, \ b^\dagger] = 1 , \qquad b \equiv \mu a + \nu a^\dagger , \qquad |\mu|^2 - |\nu|^2 = 1 . \tag{25}$$

Eq. (24) is a Holstein-Primakoff [15] or Bogoliubov [16] transformation. When such a transformation exits, such as for the harmonic oscillator and for some other cases [22]-[24], there is no problem defining a displacement-operator squeezed states. However, such a transformation does not always exist, and that is at the crux of the problem of finding a general definition of displacement-operator squeezed states.

Lastly, we note the time-dependent uncertainties in x and p. They are [17]

$$[\Delta x(t)]^2_{(\alpha,z)} = \tfrac{1}{2} \left[s^2 \cos^2 \omega t \ + \ \frac{1}{s^2} \sin^2 \omega t \right] \ , \tag{26}$$

$$[\Delta p(t)]^2_{(\alpha,z)} = \tfrac{1}{2} \left[\frac{1}{s^2} \cos^2 \omega t \ + \ s^2 \sin^2 \omega t \right] \ , \tag{27}$$

$$[\Delta x(t)]^2_{(\alpha,z)}[\Delta p(t)]^2_{(\alpha,z)} = \frac{1}{4} \left[1 \ + \ \frac{1}{4} \left(s^2 - \frac{1}{s^2} \right)^2 \sin^2[\omega t] \right] \ . \tag{28}$$

3   Generalized Coherent and Squeezed States

As discussed in Ref. [18], generalizations of the displacement-operator and ladder-operator coherent states have been widely discussed and studied [3, 19, 20, 21]. Also, a generalization of the minimum-uncertainty coherent states was found [25, 26], and this method turned out to also yield the generalized squeezed states as a byproduct.

Recently, we gave a generalized ladder operator method to define squeezed states for arbitrary systems [18], and there we pointed out the problem which is at the crux of the present study. In general there is no Bogoliubov transformation and hence no connection between the ladder-operator and displacement-operator methods for defining squeezed states.

This can be exemplified by considering the ordinary squeeze operator acting on the ground state, with no displacement operator:

$$S(z)|0\rangle = |z\rangle \ . \tag{29}$$

In this form, $S(z)$ is the SU(1,1) displacement operator, and hence the states $|z\rangle$ are the SU(1,1) coherent states. Note that these coherent states have only even occupation numbers in the number basis. (Indeed, recall that one of the early names for the squeezed states was "two-photon coherent states" [10].)

But if S is the displacement operator for SU(1,1), what is the SU(1,1) squeeze operator? A first guess would be to square the elements of S, i.e., to square aa and $a^\dagger a^\dagger$ to yield operators exponentiated to the fourth power. But this leads to operators that are not well-defined [27, 28]; that is, the operators

$$U_j = exp[\hat{z}_j(a^\dagger)^j - \hat{\bar{z}}_j(a)^j] \ , \quad j = 3, 4, 5, \ldots . \tag{30}$$

So, there is no naive higher-order squeezing. Another way to state this is that one cannot find simple operators which obey

$$\hat{S}(y)^{-1} aa \hat{S}(y) = \mu aa + \nu a^\dagger a^\dagger \ . \tag{31}$$

That is, there is no Bogoliubov transformation for the SU(1,1) elements. Hence, there is no obvious way to define the SU(1,1) squeezed states by the displacement-operator method.

4 Multiboson Operators

In a program to circumvent the problems with naive multiboson squeezing, a productive collaboration [29]-[34] proposed using the generalized Bose operators of Brandt and Greenberg [8]. These latter two observed that if one defines the operators

$$A_j = \sum_{k=0}^{\infty} \alpha_{jk} (a^\dagger)^k a^{k+j} \ , \quad j \geq 2 \ , \tag{32}$$

$$\alpha_{jk} = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k-l)!} \left[\frac{1 + [[l/j]]}{l!(l+j)!} \right]^{1/2} e^{i\rho_l} \ , \tag{33}$$

where we denote the greatest-integer function by $[[y]]$, and the ρ_l are arbitrary phases. Then, we have

$$[A_j^\dagger, \ A_j] = 1 \ . \tag{34}$$

That is, these functions satisfy the canonical commutation relations even though they are not the ordinary boson operators. They also satisfy

$$[N, A_j] = [a^\dagger a, A_j] = -j A_j \ , \tag{35}$$

and

$$A_j |jn+k\rangle = \sqrt{n} |j(n-1)+k\rangle \ , \tag{36}$$

$$A_j^\dagger |jn+k\rangle = \sqrt{(n+1)} |j(n+1)+k\rangle \ , \qquad 0 \leq k < j \ . \tag{37}$$

Note that for a given j we have j different sets of states. Each of them starts at a different lowest state $|k\rangle$, where $0 \leq k < j$; i.e., $|0\rangle, |1\rangle, |2\rangle, \ldots |j-1\rangle$.

If one acts on eigenstates of N, then from the normal-ordering theorems of Wilcox [35], a very useful form of A_j can be given [36]

$$A_j^\dagger = \left[[[\tilde{N}/j]] \frac{(\tilde{N}-j)!}{\tilde{N}!} \right]^{1/2} (a^\dagger)^j \ , \tag{38}$$

where $\tilde{N}$ is the eigenvalue of the operator N in the number operator basis.

The collaboration of Refs. [29]-[34] concentrated on investigating the properties of the states defined by

$$D(\alpha)V(z)|0\rangle = D(\alpha)\exp[z A_j^\dagger - \bar{z} A_j]|0\rangle = |\alpha, z_j\rangle \ . \tag{39}$$

In other words, they took an ordinary coherent state and then squeezed this state by the j-photon operators of A_j and $A_j^\dagger$. (Also, they studied [33] the properties of a states obtained from a generalized set of Weyl-Heisenberg operators, A_j^η.)

5 Generalized Coherent and Squeezed States for the Multi-Boson Systems

5.1 Coherent states

Now, from our point of view, of finding general and consistent methods of obtaining coherent and squeezed states, another path suggests itself. Since the A_j's obey the canonical commutation relations of Eq. (34), which are identical in form to Eq. (1), this means one can use *these* operators in displacement operators. That is, we consider the the operator V of equation (39) not to be a multi-boson squeeze of a coherent state, but rather a multi-boson displacement operator:

$$D_j(\alpha) = \exp[\alpha A_j^\dagger - \bar{\alpha} A_j] = \exp\left[-\frac{1}{2}|\alpha|^2\right] \exp[\alpha A_j^\dagger] \exp[-\bar{\alpha} A_j] \; . \qquad (40)$$

Therefore, the multi-boson coherent states are

$$|\alpha(j,k)\rangle = D_j(\alpha)|k\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|jn + k\rangle \; . \qquad (41)$$

Again observe that for a given j we have j different sets of (coherent) states. Each of them again starts at a different lowest state $|k\rangle$, where $0 \le k < j$; i.e., $|0\rangle$, $|1\rangle$, $|2\rangle$, ... $|j-1\rangle$. That is why we label the states by the couple (j,k). [The states $|\alpha(j,0)\rangle$ were studied in Ref. [32].]

These coherent states are, of course, consistent with the ladder-operator definition,

$$A_j|\alpha(j,k)\rangle = \alpha|\alpha(j,k)\rangle \; . \qquad (42)$$

By using the number-state basis of the wave functions,

$$\psi_n = \left(\frac{a_o}{\pi^{1/2}2^n n!}\right)^{1/2} \exp\left[-\frac{1}{2}a_0^2 x^2\right] H_n(a_0 x) \; , \qquad (43)$$

where $a_0^2 = (m\omega/\hbar)$ will be set to 1 and the H are the Hermite polynomials, one can write the normalized coherent state wave functions as

$$\psi_{cs}(j,k)(x) = \pi^{-1/4} \exp\left[-\frac{1}{2}\left(|\alpha|^2 + x^2\right)\right] I_{(j,k)}(\alpha,x) \; , \qquad (44)$$

where I is the sum

$$I_{(j,k)}(\alpha,x) = \sum_{n=0}^{\infty} \frac{\alpha^n H_{jn+k}(x)}{[n!(jn+k)!2^{jn+k}]^{1/2}} \; . \qquad (45)$$

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Note that for $(j, k) = (1, 0)$, we obtain the usual generating function [37] for the ordinary coherent states result,

$$I_{(1,0)}(x) = \exp[\sqrt{2}\alpha x - \alpha^2/2] . \tag{46}$$

The "natural quantum operators" for this system are [25, 26] (in dimensionless units)

$$X_j \equiv \frac{1}{\sqrt{2}}[A_j + A_j^\dagger] , \qquad P_j \equiv \frac{1}{i\sqrt{2}}[A_j - A_j^\dagger] , \tag{47}$$

But then, the Heisenberg-Weyl algebra tells us immediately that these are the operators directly connected to the minimum-uncertainty method. Therefore, we have that [18]

$$(\Delta X_j)^2_{(j,k)} = 1/2 , \qquad (\Delta P_j)^2_{(j,k)} = 1/2 . \tag{48}$$

We can also obtain information for the uncertainties of the physical position and momentum, x and p. We immediately observe that

$$\langle x \rangle_{(j,k)} = \langle p \rangle_{(j,k)} = 0 , \qquad j > 1 . \tag{49}$$

(For j=1 we have the ordinary harmonic oscillator). For $j > 2$, we have, then, that

$$\begin{aligned}
\langle x^2 \rangle_{(j,k)} &= (\Delta x)^2_{(j,k)} = \langle p^2 \rangle_{(j,k)} = (\Delta p)^2_{(j,k)} \\
&= \exp[-|\alpha|^2] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}[jn + k + \tfrac{1}{2}] \\
&= \tfrac{1}{2} + k + j|\alpha|^2 , \qquad j > 2 .
\end{aligned} \tag{50}$$

The case $j = 2$ is slightly more complicated because the operators x^2 and p^2 connect different number states in the expectation values. In particular,

$$\langle x^2 \rangle_{(2,k)} = (\Delta x)^2_{(2,k)} = \tfrac{1}{2} + k + 2|\alpha|^2 + C_{(2,k)} \tag{51}$$

$$\langle p^2 \rangle_{(2,k)} = (\Delta p)^2_{(2,k)} = \tfrac{1}{2} + k + 2|\alpha|^2 - C_{(2,k)} , \tag{52}$$

where

$$C_{(2,k)} = \tfrac{1}{2}[\langle a^2 \rangle_{(2,k)} + \langle (a^\dagger)^2 \rangle_{(2,k)}] , \tag{53}$$

which evaluates to

$$C_{(2,k)} = (\alpha + \bar{\alpha}) \exp[-|\alpha|^2] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \left[\frac{(n + 1 + k/2)(n + 1/2 + k/2)}{n + 1} \right]^{1/2} . \tag{54}$$

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5.2   Squeezed states

Because the A_j's define a Heisenberg-Weyl algebra, one can therefore define an SU(1,1) squeeze algebra in the normal way:

$$K_{j+} = \frac{1}{2} A_j^\dagger A_j^\dagger \ , \qquad K_{j-} = \frac{1}{2} A_j A_j \ , \qquad K_{j0} = \frac{1}{2}\left(A_j^\dagger A_j + \frac{1}{2}\right) \ . \qquad (55)$$

Then all these A_j's and K_j's again have the same commutation relations as before, and so all the results of the ordinary harmonic oscillator coherent and squeezed states goes through in the same manner, only with the a's being changed into the A_j's. That is, The squeeze operators are

$$S_j(z) \;=\; \exp[z K_{j+} - \bar{z} K_{j-}] \qquad\qquad (56)$$

$$\;=\; \exp[e^{i\theta}(\tanh r)K_{j+}]\left(\frac{1}{\cosh r}\right)^{2K_{j0}} \exp[-e^{-i\theta}(\tanh r)K_{j-}] \ , \qquad (57)$$

where

$$z = r e^{i\theta} \ , \qquad\qquad (58)$$

meaning the squeezed states are

$$D_j(\alpha)S_j(z)|k\rangle = |\alpha, z(j,k)\rangle \ . \qquad\qquad (59)$$

Furthermore, all the mathematics of the ordinary squeezed states follows automatically, just changing notation. For example, there is a Bogoliubov transformation:

$$B_j \equiv S_j(z)^{-1} A_j S_j(z) \;=\; (\cosh r)A_j + e^{i\theta}(\sinh r)A_j^\dagger \ , \qquad (60)$$

$$B_j^\dagger \equiv S_j(z)^{-1} A_j S_j(z) \;=\; (\cosh r)A_j^\dagger + e^{-i\theta}(\sinh r)A_j \ . \qquad (61)$$

where

$$[B_j, \ B_j^\dagger] = 1 \ , \qquad B_j \equiv \mu A_j + \nu A_j^\dagger \ , \qquad |\mu|^2 - |\nu|^2 = 1 \ . \qquad (62)$$

This means, of course, that there is an equivalent ladder-operator definition of these squeezed states:

$$[\mu A_j - \nu A_j^\dagger]|\alpha, z(j,k)\rangle = \beta|\alpha, z(j,k)\rangle \ . \qquad (63)$$

Again, from the the Heisenberg-Weyl algebra, it follows that

$$(\Delta X_j)_{ss}^2 (\Delta P_j)_{ss}^2 = 1/4 \ . \qquad\qquad (64)$$

Of course, being squeezed states the above equality holds at $t = 0$ and oscillates, and the uncertainty in each quadrature also oscillates.

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