CH-231-A

Algorithms and Data Structures

ADS

Lecture 7

Dr. Kinga Lipskoch

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Correctness

```
INSERTION-SORT(A, n)
 for j = 2 to n
     key = A[j]
     // Insert A[j] into the sorted sequence A[1 .. j - 1].
     i = j - 1
     while i > 0 and A[i] > key
         A[i + 1] = A[i]
         i = i - 1
     A[i + 1] = key
```

Loop invariant: is a property of a program loop that is true before (and after) each iteration

Example of loop invariant: at the start of each iteration of the for loop, the subarray $A[1 ... j-1]$ consists of elements originally in $A[1 ... j-1]$, but in sorted order.

Running Time

- The running time depends on the input: an already sorted sequence is easier to sort
- Parameterize running time by the size of the input: short sequences are easier to sort than long ones
- Generally, we seek upper bounds on the running time: we would like to have a guarantee

Types of Analyses

- Worst case (usually)
  $T(n) =$ maximum time of algorithm on any input of size n
- Average case (sometimes)
  $T(n) =$ expected time of algorithm over all inputs of size n
  (Need assumption of statistical distribution of inputs)
- Best case (almost never)
  Does not make much sense, e.g., we can start with the solution

Asymptotic Analysis

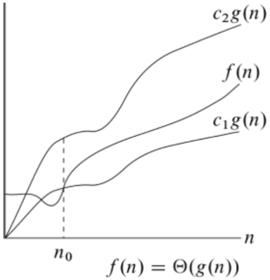
- What is Insertion Sort's worst-case time?
  - It depends on the speed of our computer: relative speed (on the same machine), absolute speed (on different machines)
- Idea
  - Ignore machine-dependent constants
  - Look at growth of $T(n)$ as $n \to \infty$

Asymptotically Tight Bound: Θ-Notation

For a given asymptotically non-negative function $g(n)$, we define
$$\Theta(g(n)) = \{f(n) | \exists \text{ positive constants } c_1, c_2 \text{ and } n_0, \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}$$

We often write $f(n) = \Theta(g(n))$ (not an equation, also not an assignment) instead of $f(n) \in \Theta(g(n))$.
The same is meant by both notations.

Example:
$f(n) = 3n^3 + 90n^2 - 5n + 6046$
$3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$



Example

$$c_1 n^3 \leq 3n^3 + 90n^2 - 5n + 6046 \leq c_2 n^3$$

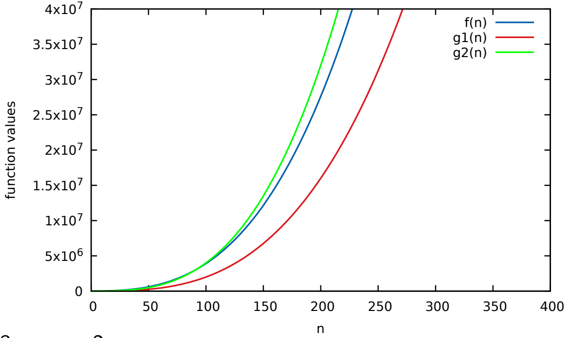
$$c_1 \leq 3 + \frac{90}{n} - \frac{5}{n^2} + \frac{6046}{n^3} \leq c_2$$

Try $c_1 = 2; c_2 = 4; n_0 = 100; \Rightarrow f_{div}(n_0) = 3.906546$

Intuitively:

- set c_1 to a value smaller than the coefficient of the highest-order term
- and c_2 to a value that is slight larger

Plotting Functions Using Gnuplot

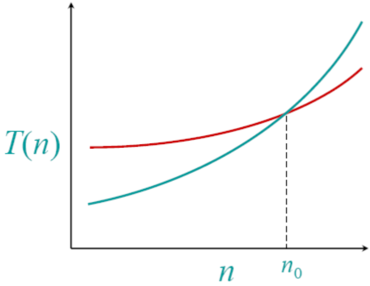


$f(n) = 3n^3 + 90n^2 - 5n + 6046$
$g1(n) = 2n^3$
$g2(n) = 4n^3$

`Gnuplot script`

Asymptotic Performance

- When n gets large enough, a $\Theta(n^2)$ algorithm always beats a $\Theta(n^3)$ algorithm
- Informal notion:
  - throw away lower-order terms
  - ignore the leading coefficient of the highest-order term



$$3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$$

Asymptotic Analysis

- We should not ignore asymptotically slower algorithms
- Real-world design situations often call for a careful balancing of engineering objectives
- Asymptotic analysis is a useful tool to help to structure our thinking

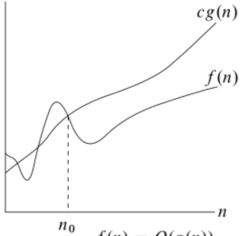
Asymptotically Upper Bound: O-Notation

For a given asymptotically non-negative function $g(n)$, we define

$$\mathrm{O}(g(n)) = \{f(n) | \exists \text{ positive constants } c \text{ and } n_0, \\ \text{such that } 0 \leq f(n) \leq cg(n), \forall n \geq n_0\}$$

Example:

We say that $f(n)$ is polynomially bounded if $f(n) = \mathrm{O}(n^k)$ for some constant k



$f(n) = O(g(n))$

Examples

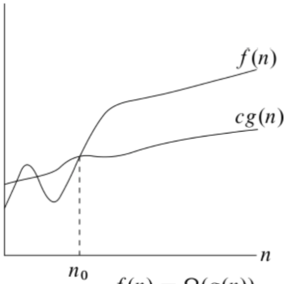
- $f(n) = 3n^3 + 90n^2 - 5n + 6046$
  $\implies f(n) = O(n^3)$

- $f(n) = n$
  $\implies f(n) = O(n^3)$???
  $\implies f(n) = O(n^2)$???
  $\implies f(n) = O(n)$ also true

Asymptotically Lower Bound: Ω-Notation

For a given asymptotically non-negative function $g(n)$, we define

$$\Omega(g(n)) = \{f(n) | \exists \text{ positive constants } c \text{ and } n_0, \\ \text{such that } 0 \leq cg(n) \leq f(n), \forall n \geq n_0\}$$



$f(n) = \Omega(g(n))$

For tight bounds, we get
$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

Non-tight Upper Bound: o-Notation

For a given asymptotically non-negative function $g(n)$, we define
$$\mathrm{o}(g(n)) = \{f(n) | \text{ for any constant } c > 0, \exists n_0 > 0, \\ \text{such that } 0 \leq f(n) < cg(n), \forall n \geq n_0\}$$

$f(n) = \mathrm{o}(g(n))$ implies $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

Examples

- $2n = o(n^2)$
- $2n^2 \neq o(n^2)$???
- $n^b = o(a^n)$ for $a > 1$

Non-tight Lower Bound: ω-Notation

For a given asymptotically non-negative function $g(n)$, we define $f(n) \in \omega(g(n))$ iff $g(n) \in o(f(n))$

$f(n) = \omega(g(n))$ implies $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

Asymptotic Analysis: Computation with Limits

$f \in o(g)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f \in O(g)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$
$f \in \Omega(g)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$
$f \in \Theta(g)$	$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$
$f \in \omega(g)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Asymptotic Analysis of Insertion Sort (1)

INSERTION-SORT(A, n)	*cost*	*times*
for $j = 2$ **to** n	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n}(t_j-1)$
$i = i-1$	c_7	$\sum_{j=2}^{n}(t_j-1)$
$A[i+1] = key$	c_8	$n-1$

- c_k is the number of steps a computer needs to perform instruction k once (e.g., a Random Access Machine)
- t_j is the number of times the while loop is executed in the for iteration j

Asymptotic Analysis of Insertion Sort (2)

- Best case:
  Input series is ordered.
- Then, $t_j = 1$.
- $T(n) = \Theta(n)$

Asymptotic Analysis of Insertion Sort (3)

- Worst case:
  Input series was ordered in reverse.
- Then, $t_j = j$.
- With the arithmetic series
  $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
  the worst-case asymptotic complexity is
  $$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

Asymptotic Analysis of Insertion Sort (4)

- Average case:
  All permutations are equally likely.
- Then, t_j is expected to be $\dfrac{j}{2}$ on average.
- Hence, the average-case asymptotic complexity is
  $$T(n) = \sum_{j=2}^{n} \Theta\left(\frac{j}{2}\right) = \Theta(n^2)$$

Asymptotic Analysis of Insertion Sort (5)

- Is Insertion Sort fast?
- For small n, it is moderately fast.
- For large n, it is slow.

Summary: Asymptotic Analysis

- O-notation – Asymptotically upper bound
- Ω-notation – Asymptotically lower bound
- Θ-notation – Asymptotically tight bound
- o-notation – Non-tight upper bound
- ω-notation – Non-tight lower bound

- $f(n) \in \mathrm{O}(g(n))$ is like $a \leq b$,
- $f(n) \in \Omega(g(n))$ is like $a \geq b$,
- $f(n) \in \Theta(g(n))$ is like $a = b$,
- $f(n) \in \mathrm{o}(g(n))$ is like $a < b$,
- $f(n) \in \omega(g(n))$ is like $a > b$.