CH-231-A

Algorithms and Data Structures

ADS

Lecture 14

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Spring 2024

Converting an Existing Array *A* to a Max-Heap (1)

Call MAX-HEAPIFY on which set of nodes?

- All inner nodes $\Rightarrow$ move them down if necessary (*Max-Heapify*)
- Not necessary to call on leaf nodes

Leaves of a Heap of Size n

Let $n = A.heapsize$.

Where is the parent of the last element of a heap?

- At index $n/2$.

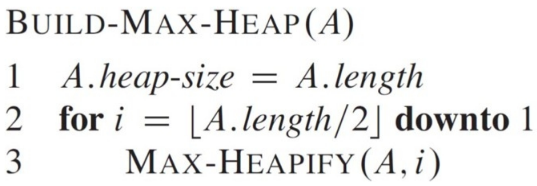
Therefore, the element at index $n/2 + 1$ does not have a child in the heap, and hence is a leaf.

In a heap, there are $n/2$ leaves:

- from index $n/2 + 1$ to n

Each leaf is the root of a valid max-heap of size 1.

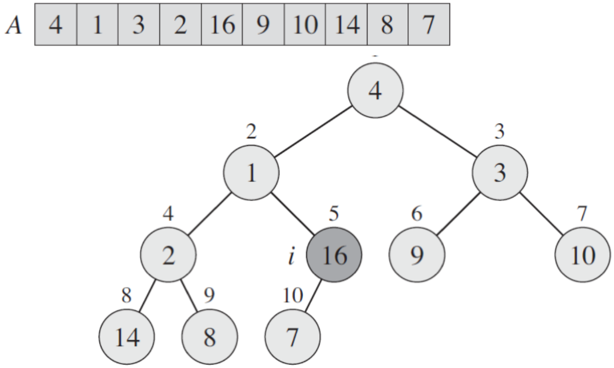
Converting an Existing Array A to a Max-Heap (2)



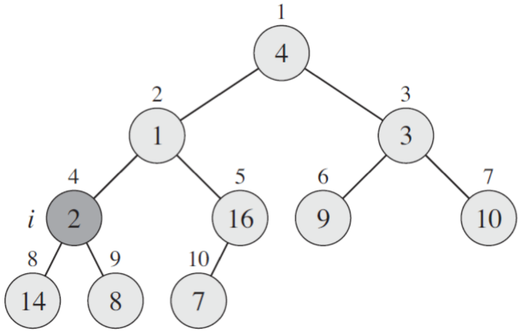
```
BUILD-MAX-HEAP(A)
  $A.heap\text{-}size = A.length$
  for $i = \lfloor A.length/2 \rfloor$ downto 1
      MAX-HEAPIFY(A, i)
```

Loop invariant:
At the start of each iteration of the for loop, each node $i + 1, \ldots, n$ is the root of a max-heap.

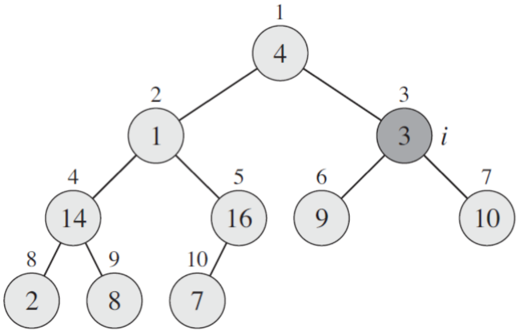
Build Max-Heap (2)



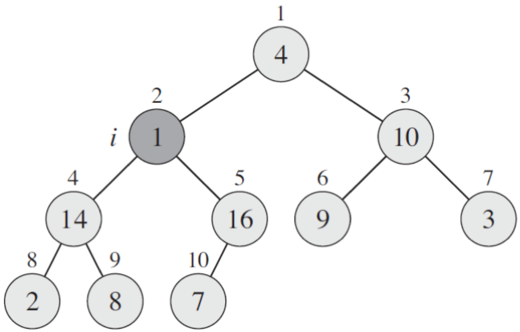
Build Max-Heap (3)



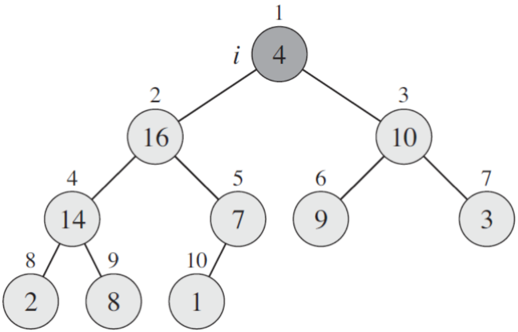
Build Max-Heap (4)



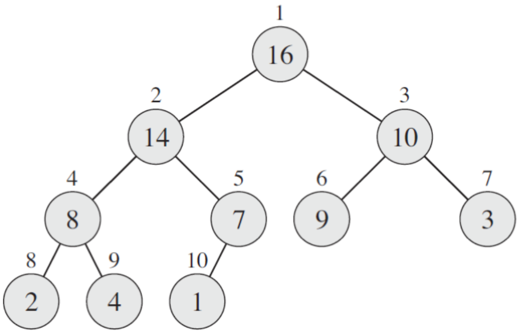
Build Max-Heap (5)



Build Max-Heap (6)



Build Max-Heap (7)



Build Max-Heap (8)

What is the time complexity of the algorithm?

Theorem:
Let m_h be the number of nodes of height h in any n element heap $A(n)$.
Then $m_h(A, n) \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$.

(Proof by induction over h)

Build Max-Heap (9)

Time complexity:

The time needed by *Max-Heapify* when called on a node of height h is O(h). Therefore, the total cost of *Build-Max-Heap*(A) is upper bounded by

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n).$$

$$\sum_{k=0}^{\infty} k\, x^k = \frac{x}{(1-x)^2} \text{ if } |x| < 1.$$

Conclusion:

We can convert an unordered array into a max-heap in linear time.

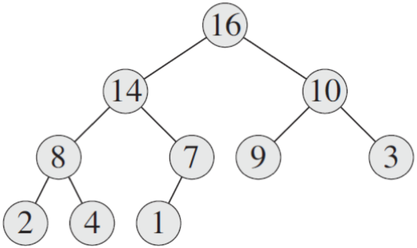
Heap Sort

- Start by generating a max-heap.
- The maximum element of a max-heap is at the root.
- Put it in its right sorted place at $n = A.heapsize$ by swapping it with the last element $A[n]$, which now becomes $A[1]$.
- Decrement the heap size to create a smaller heap and thus implicitly remove the last element (the maximum) from the heap.
- The new $A[1]$ may not satisfy the max-heap property, so move it down.
- Iterate.

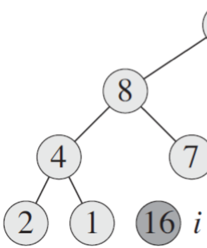
Heap Sort: Pseudocode

```
HEAPSORT($A$)
1  BUILD-MAX-HEAP($A$)
2  for $i = A.length$ downto 2
3      exchange $A[1]$ with $A[i]$
4      $A.heap\text{-}size = A.heap\text{-}size - 1$
5      MAX-HEAPIFY($A$, 1)
```

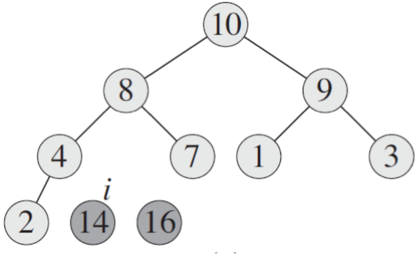
Heap Sort: Example (1)



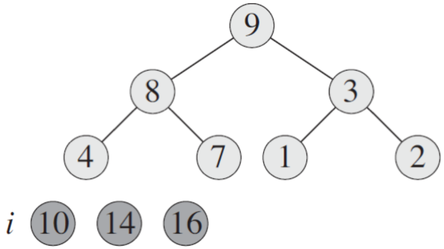
Heap Sort: Example (2)



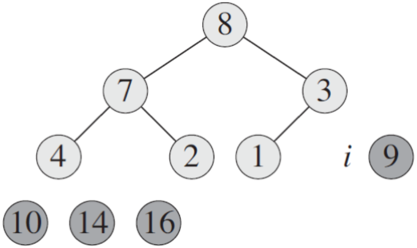
Heap Sort: Example (3)



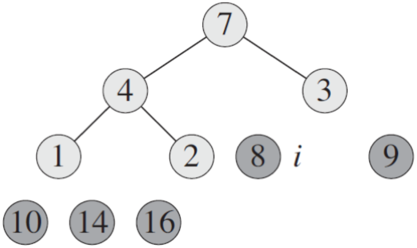
Heap Sort: Example (4)



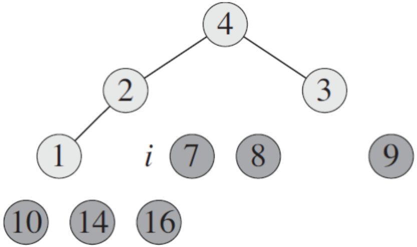
Heap Sort: Example (5)



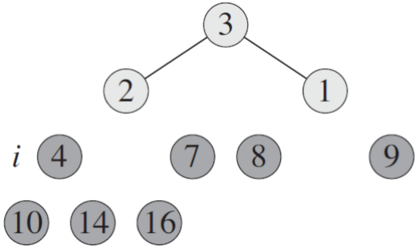
Heap Sort: Example (6)



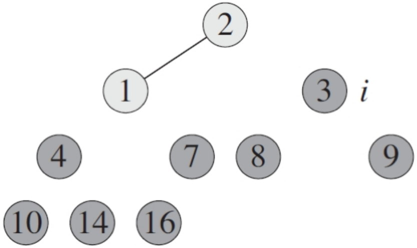
Heap Sort: Example (7)



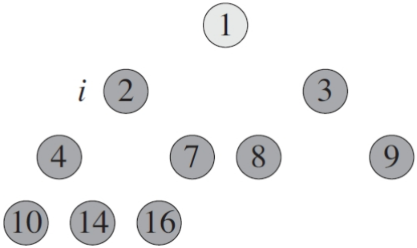
Heap Sort: Example (8)



Heap Sort: Example (9)



Heap Sort: Example (10)



Heap Sort: Runtime Analysis

```
HEAPSORT($A$)
BUILD-MAX-HEAP($A$)
for $i = A.length$ downto 2
    exchange $A[1]$ with $A[i]$
    $A.heap\text{-}size = A.heap\text{-}size - 1$
    MAX-HEAPIFY($A$, 1)
```

- Runtime costs:
  $O(n) + O(n \lg n) = O(n \lg n)$
- Memory costs:
  $O(1)$, i.e., in-situ sorting
- Visualization:
  `http://www.sorting-algorithms.com/heap-sort`