CH-231-A

Algorithms and Data Structures

ADS

Lecture 8

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Spring 2024

Divide & Conquer

- Divide & Conquer (Latin Divide et Impera) is one concept/method/technique that can produce faster algorithms.
- It is based on three steps:
  - Divide the given problem into smaller subproblems.
  - Conquer the subproblems by solving them recursively.
  - Combine the solutions of the subproblems.
- Example: Sort recursively?

Merge Sort Idea

MERGE-SORT $A[1 \,..\, n]$

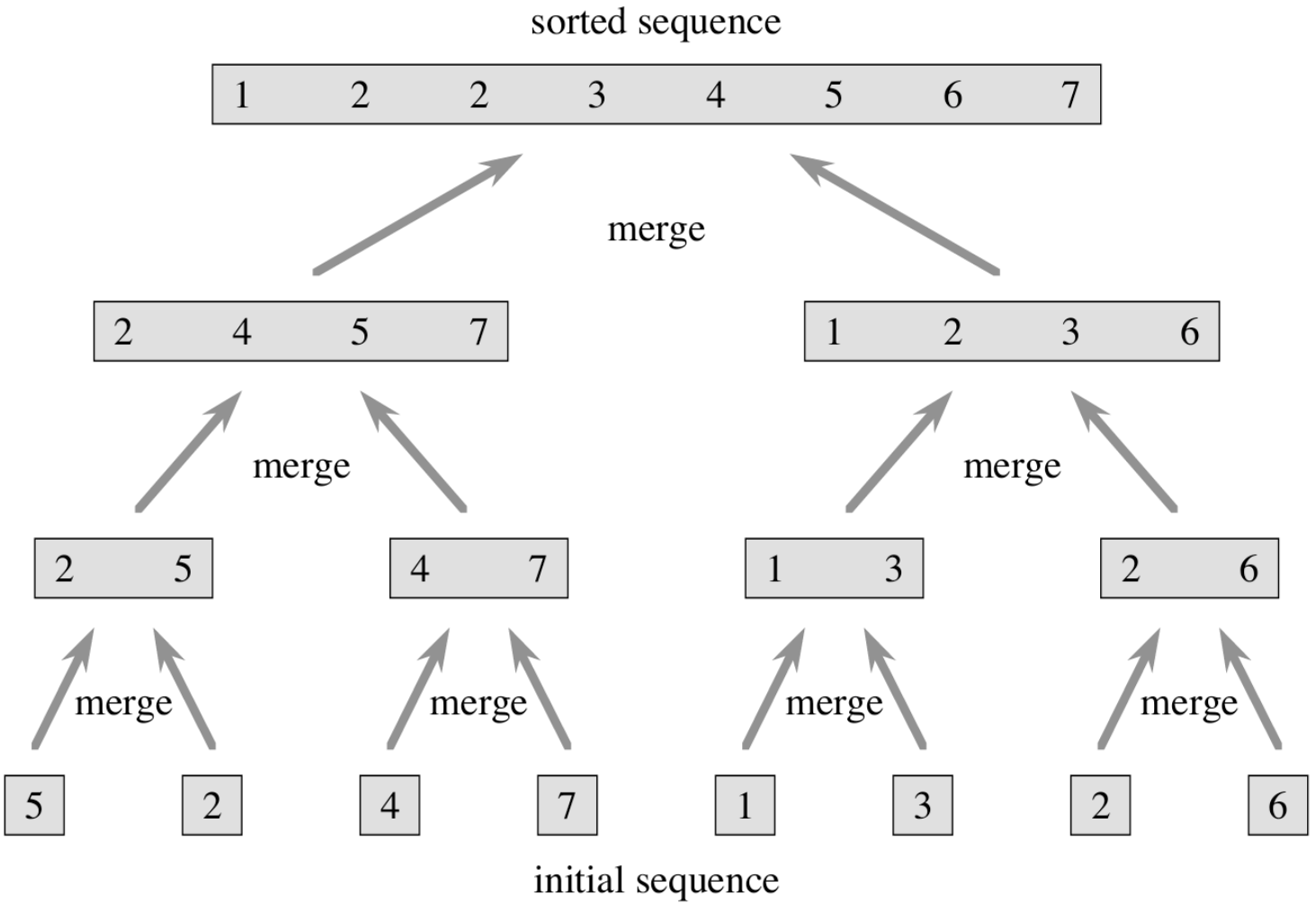
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Merge Sort as Divide & Conquer

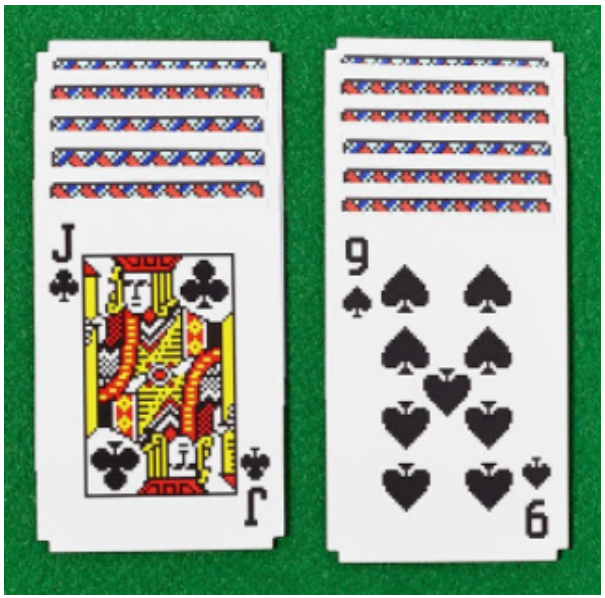
```
MERGE-SORT(A, p, r)
  if p < r                          // check for base case
      q = ⌊(p + r)/2⌋               // divide
      MERGE-SORT(A, p, q)           // conquer
      MERGE-SORT(A, q + 1, r)       // conquer
      MERGE(A, p, q, r)             // combine
```

Initial call: `Merge-Sort(A, 1, A.length)`

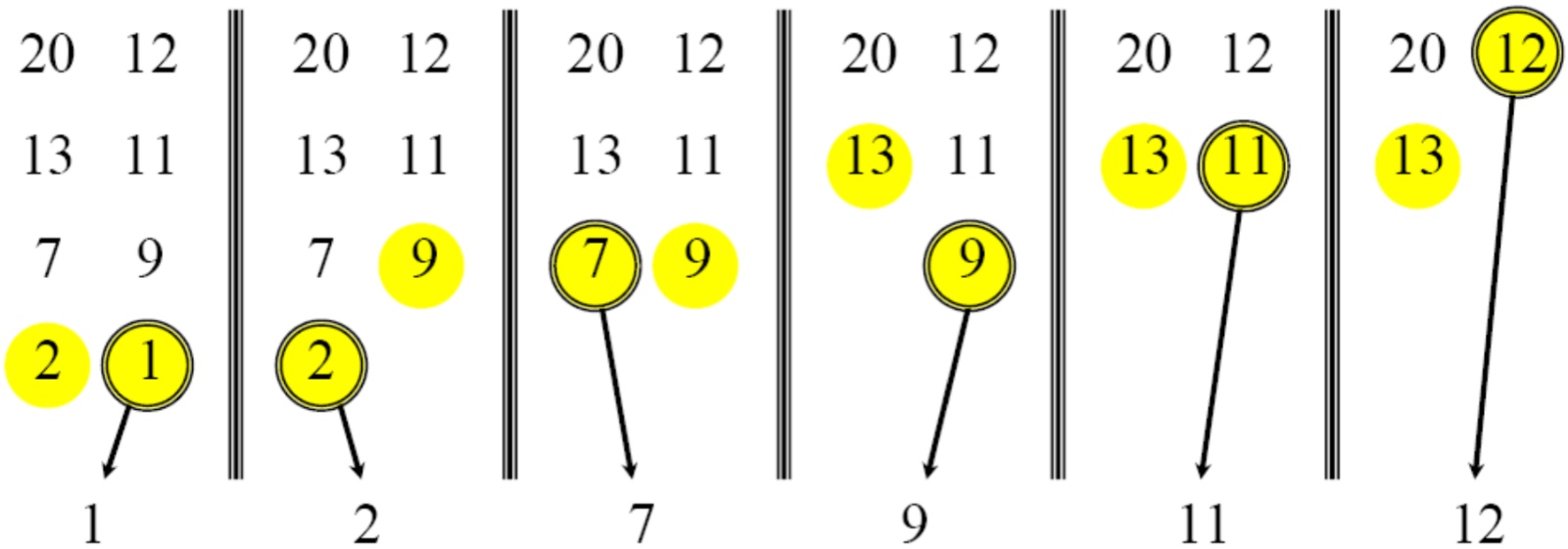
Merge Sort: Example



How Is Merging Done? (1)



How Is Merging Done? (2)



Merge

```
MERGE(A, p, q, r)
  n_1 = q − p + 1
  n_2 = r − q
  let L[1..n_1 + 1] and R[1..n_2 + 1] be new arrays
  for i = 1 to n_1
      L[i] = A[p + i − 1]
  for j = 1 to n_2
      R[j] = A[q + j]
  L[n_1 + 1] = ∞
  R[n_2 + 1] = ∞
  i = 1
  j = 1
  for k = p to r
      if L[i] ≤ R[j]
          A[k] = L[i]
          i = i + 1
      else A[k] = R[j]
          j = j + 1
```

Correctness of Merging

Loop Invariant:

- At the start of each iteration of the for k loop, the subarray $A[p..k-1]$ contains the $k-p$ smallest elements of L and R in sorted order.
- Moreover, $L[i]$ and $R[j]$ are the smallest elements of their arrays which have not been copied back into A.

Asymptotic Analysis (1)

- Merging step?
  Computation time is $\Theta(n)$.
- What is the overall computation time of Merge Sort?

Asymptotic Analysis (2)

$T(n)$	**MERGE-SORT** $A[1 \,..\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
$\Theta(n)$	3. ***"Merge"*** the 2 sorted lists

Asymptotic Analysis (3)

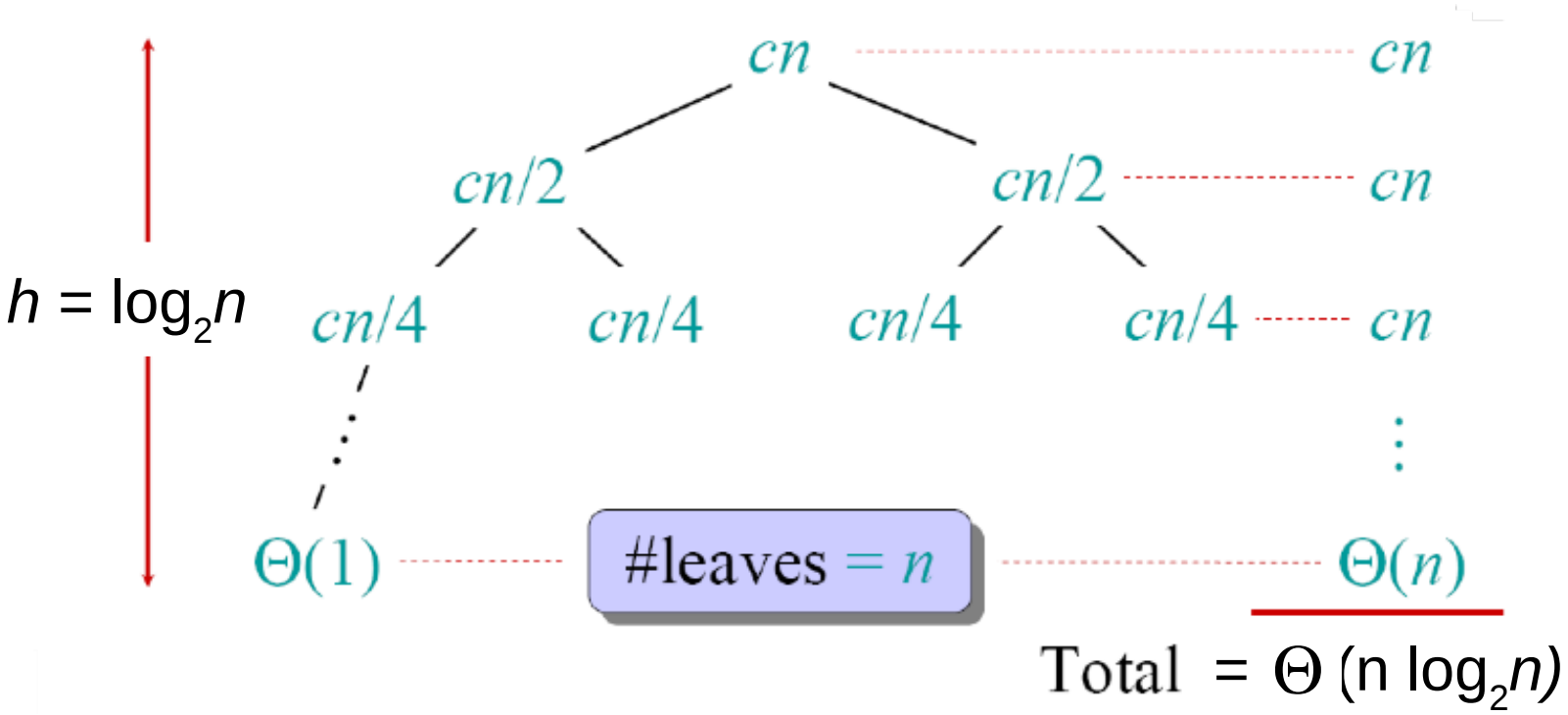
- The overall running time for Merge Sort is given by the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

- The base case may be omitted, if it is obvious that it can be neglected in the asymptotic analysis.

Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



Merge Sort vs. Insertion Sort

- $\Theta(n \log_2 n)$ grows more slowly than $\Theta(n^2)$.
- Merge Sort asymptotically beats Insertion Sort (in the average case and in the worst case).
- In practice, Merge Sort beats Insertion Sort for $n > 30$ or so.

Sorting Algorithms' Complexities

Algorithm	Data structure	Time complexity:Best	Time complexity:Average	Time complexity:Worst	Space complexity:Worst
Quick sort	Array	O(n log(n))	O(n log(n))	O(n^2)	O(n)
Merge sort	Array	O(n log(n))	O(n log(n))	O(n log(n))	O(n)
Heap sort	Array	O(n log(n))	O(n log(n))	O(n log(n))	O(1)
Smooth sort	Array	O(n)	O(n log(n))	O(n log(n))	O(1)
Bubble sort	Array	O(n)	O(n^2)	O(n^2)	O(1)
Insertion sort	Array	O(n)	O(n^2)	O(n^2)	O(1)
Selection sort	Array	O(n^2)	O(n^2)	O(n^2)	O(1)

Here the notation $log(n)$ is mathematically $log_2 n$.