An Introduction to Ergodicity

Rareș Cristian

Big O Theory Club

November 11, 2019

Overview

Dynamical Systems

A Dynamical System describes the evolution of some state space according to a fixed law.

Dynamical Systems

A Dynamical System describes the evolution of some state space according to a fixed law.

1. A phase space X, whose elements represent the possible states of the system.

Dynamical Systems

A Dynamical System describes the evolution of some state space according to a fixed law.

1. A phase space X, whose elements represent the possible states of the system.
2. The notion of Time, either discrete or continuous.

Dynamical Systems

A Dynamical System describes the evolution of some state space according to a fixed law.

1. A phase space X, whose elements represent the possible states of the system.
2. The notion of Time, either discrete or continuous.
3. The time evolution law, $\phi : X \to X$

What are we Studying?

- The *orbit* of a point $x \in X$ is the sequence

$$x, \phi(x), \phi^2(x), \ldots, \phi^k(x), \ldots$$

The theory of dynamical systems focuses on *asymptotic* behavior of orbits.

Attractor States

- An *attractor* is a subset $A \subset X$ invariant under X (i.e. $\phi(A) \subset A$) such that for a neighborhood A' of A, $\lim_{n\to\infty} \phi^n(A') \subset A$.

Attractor States

- An *attractor* is a subset $A \subset X$ invariant under X (i.e. $\phi(A) \subset A$) such that for a neighborhood A' of A, $\lim_{n\to\infty} \phi^n(A') \subset A$.

  Trivial Examples of Invariant Sets:

Attractor States

- An *attractor* is a subset $A \subset X$ invariant under X (i.e. $\phi(A) \subset A$) such that for a neighborhood A' of A, $\lim_{n\to\infty} \phi^n(A') \subset A$.

  Trivial Examples of Invariant Sets:
- A point $x \in X$ is called a *fixed point* if $\phi(x) = x$.

Attractor States

- An *attractor* is a subset $A \subset X$ invariant under X (i.e. $\phi(A) \subset A$) such that for a neighborhood A' of A, $\lim_{n\to\infty} \phi^n(A') \subset A$.

Trivial Examples of Invariant Sets:

- A point $x \in X$ is called a *fixed point* if $\phi(x) = x$.
- $x \in X$ is *periodic* if there exists $n > 0$ such that $\phi^n(x) = x$. The minimal such n is the *period* of x.

Example of Attractors: Logistic Family of Maps

Consider the family of logistic maps parameterized by θ:

$$\phi_\theta(x) = \theta x(1 - x)$$

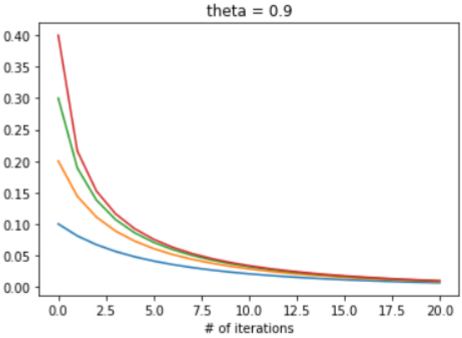
Example of Attractors: Logistic Family of Maps

Consider the family of logistic maps parameterized by θ:

$$\phi_\theta(x) = \theta x(1-x)$$

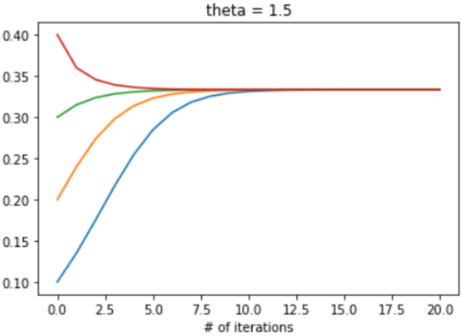
Attractors $\theta \leq 1$

For $0 < \theta \leq 1$, every $x \in [0, 1)$ converges to 0 under the map ϕ_θ.



Attractors $1 < \theta \leq 2$

For $1 < \theta \leq 2$, every $x \in [0, 1)$ converges to $\frac{\theta-1}{\theta}$ under the map ϕ_θ.

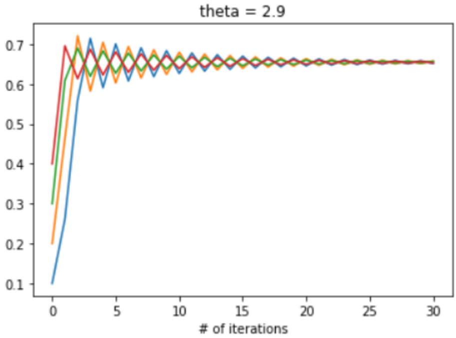


Attractors $2 < \theta \leq 3$

For $2 < \theta \leq 3$, every $x \in [0, 1)$ converges again to $\frac{\theta - 1}{\theta}$. But, different convergence pattern...

Attractors $2 < \theta \leq 3$

For $2 < \theta \leq 3$, every $x \in [0, 1)$ converges again to $\frac{\theta - 1}{\theta}$. But, different convergence pattern...

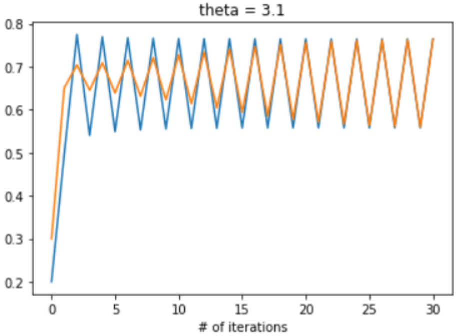


Periodic Attractors

For $3 < \theta \leq 1 + \sqrt{6}$, we approach periodic orbits rather than fixed points:

Periodic Attractors

For $3 < \theta \leq 1 + \sqrt{6}$, we approach periodic orbits rather than fixed points:



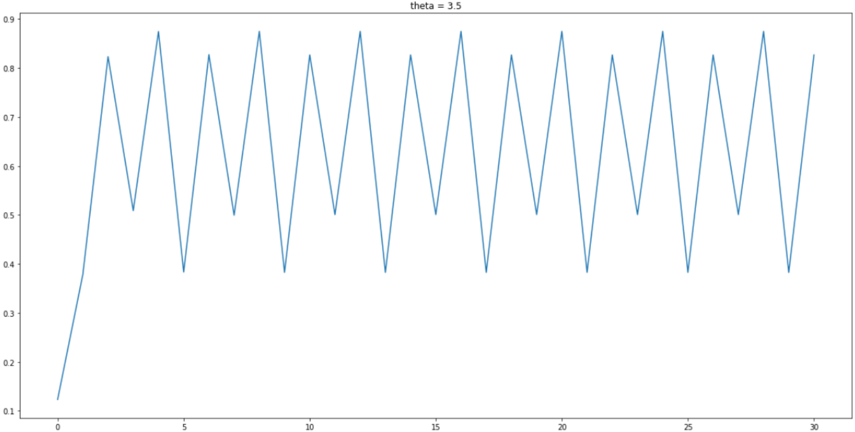
More Periodicity

Almost all initial conditions will approach oscillations among 4 values, then 8, 16, etc.

More Periodicity

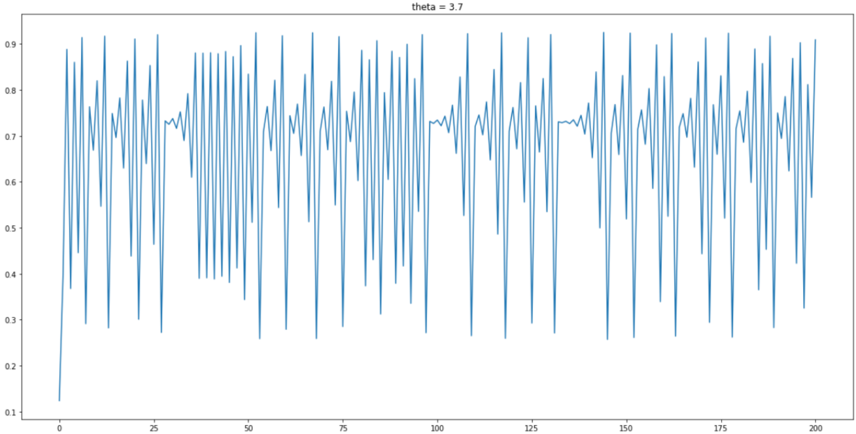
Almost all initial conditions will approach oscillations among 4 values, then 8, 16, etc.

$$\theta = 3.5$$



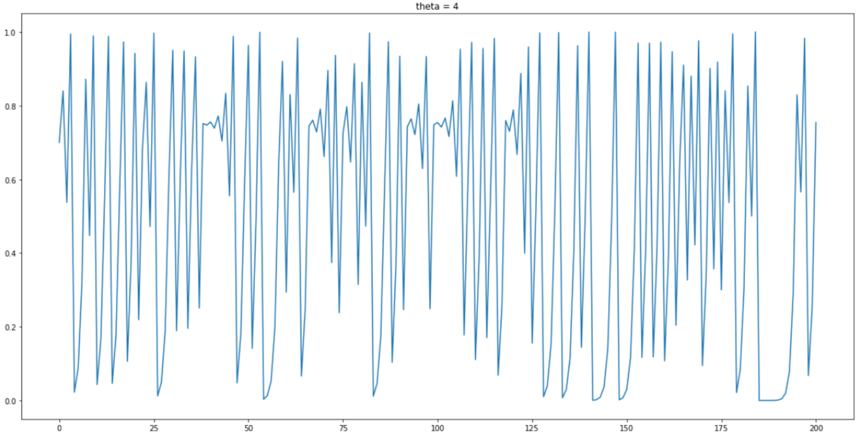
It gets Weirder

$$\theta = 3.7$$



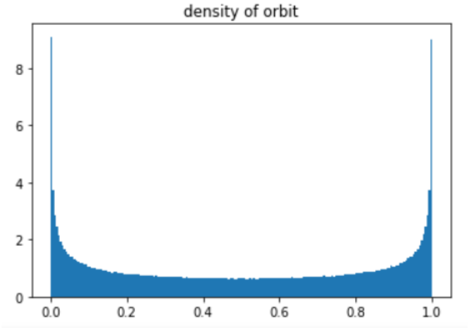
It gets Weirder

$$\theta = 4$$



Asymptotic Distribution

For $\theta = 4$, all patterns seem to have disappeared. Where does the orbit spend its time?

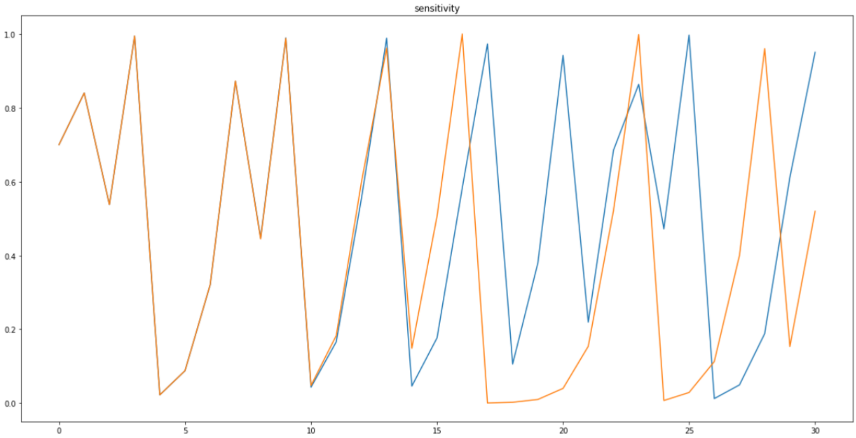


Chaos

Chaotic Dynamical Systems: Small perturbations in initial conditions yield widely diverging outcomes:

Chaos

Chaotic Dynamical Systems: Small perturbations in initial conditions yield widely diverging outcomes:



Expanding Maps

A map ϕ is *expanding* if there exists $\epsilon > 0, L > 1$ such that for all $x, y \in X$ with $d(x, y) < \epsilon$,

$$d(\phi(x), \phi(y)) \geq L \cdot d(x, y)$$

Expanding Maps

A map ϕ is *expanding* if there exists $\epsilon > 0, L > 1$ such that for all $x, y \in X$ with $d(x, y) < \epsilon$,

$$d(\phi(x), \phi(y)) \geq L \cdot d(x, y)$$

We say ϕ is a *contraction* if there exists a constant $0 \leq \tau < 1$ such that

$$d(\phi(x), \phi(y)) \leq \tau d(x, y), \ \forall x, y \in X$$

Contractions - Fixed Points

Banach Fixed Point Theorem

Let (X, d) be a complete metric space with contraction mapping $\phi : X \to X$ with contraction constant τ. Then, ϕ admits a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = \phi(x_{n-1})$ converges to x^* at the following rate:

$$d(x^*, x_n) \leq \frac{\tau^n}{1-\tau} d(x_1, x_0)$$

Stability of Fixed Points

From now on, let ϕ be a diffeomorphism, and $X \subset \mathbb{R}^n$ an open set.

How to describe behavior of neighborhood of a fixed point $p \in X$?

Stability of Fixed Points

From now on, let ϕ be a diffeomorphism, and $X \subset \mathbb{R}^n$ an open set.

How to describe behavior of neighborhood of a fixed point $p \in X$? By looking at a linear approximation of ϕ at p - the derivative $D_p\phi$.

Stability of Fixed Points

From now on, let ϕ be a diffeomorphism, and $X \subset \mathbb{R}^n$ an open set.

How to describe behavior of neighborhood of a fixed point $p \in X$? By looking at a linear approximation of ϕ at p - the derivative $D_p\phi$.
Let the Generalized Eigenvectors of $D_p\phi$ be $\alpha_1, \ldots, \alpha_r$ with corresponding $D_p\phi$-invariant Generalized Eigenspaces $E_1, \ldots, E_r$.

Stability of Fixed Points

From now on, let ϕ be a diffeomorphism, and $X \subset \mathbb{R}^n$ an open set.

How to describe behavior of neighborhood of a fixed point $p \in X$? By looking at a linear approximation of ϕ at p - the derivative $D_p\phi$.
Let the Generalized Eigenvectors of $D_p\phi$ be $\alpha_1, \ldots, \alpha_r$ with corresponding $D_p\phi$-invariant Generalized Eigenspaces $E_1, \ldots, E_r$.

If $v_i \in E_i, D_p\phi^n v_i = a_i^n v_i$, hence $\log|D_p\phi^n v_i| = n\log|a_i| + \log|v_i|$. If $v_i \neq 0$ and $|a_i| > 1$,

$$\lim_{n\to\infty} \frac{1}{n}\log|D_p\phi^n v_i| = \log|a_i|$$

Stability of Fixed Points

From now on, let ϕ be a diffeomorphism, and $X \subset \mathbb{R}^n$ an open set.

How to describe behavior of neighborhood of a fixed point $p \in X$? By looking at a linear approximation of ϕ at p - the derivative $D_p\phi$.
Let the Generalized Eigenvectors of $D_p\phi$ be $\alpha_1, \dots, \alpha_r$ with corresponding $D_p\phi$-invariant Generalized Eigenspaces $E_1, \dots, E_r$.

If $v_i \in E_i, D_p\phi^n v_i = a_i^n v_i$, hence $\log|D_p\phi^n v_i| = n\log|a_i| + \log|v_i|$. If $v_i \neq 0$ and $|a_i| > 1$,

$$\lim_{n\to\infty} \frac{1}{n}\log|D_p\phi^n v_i| = \log|a_i|$$

Hence, for $|a_i| > 1$, the vector $D_p\phi^n$ grows exponentially fast, and if $|a_i| < 1$ it shrinks exponentially fast.

Lyapunov Exponents

The *Lyapunov Exponent*, denoted by λ_i, associated with E_i is $\log |a_i|$.

Lyapunov Exponents

The *Lyapunov Exponent*, denoted by λ_i, associated with E_i is $\log|a_i|$.

Let $E^s = \bigoplus_{\lambda_i<0} E_i$ and $E^u = \bigoplus_{\lambda_i>0} E_i$. Vectors in E^s are contracted by forward iterations of $D_p f$ while vectors in E^u are expanded.

If no Lyapunov exponents are equal to 0, or equivalently, $E^s \bigoplus E^u = \mathbb{R}^d$, then we say p is a *hyperbolic* fixed point.

(Un)Stable Manifolds

Notice E^s, E^u are in the tangent space.

(Un)Stable Manifolds

Notice E^s, E^u are in the tangent space.

There correspond ϕ-invariant stable and unstable submanifolds $W^s(p), W^u(p) \subset X$ such that

1. $T_p W^s(p) = E^s, T_p W^u(p) = E^u$
2. $\mathrm{dist}(\phi^n(y), p) \leq Ce^{-\lambda n}$ for all $y \in W^s(p)$
3. $\mathrm{dist}(\phi^{-n}(y), p) \leq Ce^{-\lambda n}$ for all $y \in W^u(p)$

where $\lambda = \min_i |\lambda_i|$.

Aperiodic Orbits

- Lyapunov Exponents: $\lambda_i = \lim_{n\to\infty} \frac{1}{n} \log |D_p \phi^n v_i|$ for $v_i \neq 0$.
- (un)stable manifolds still exist for almost every point $p \in X$, with

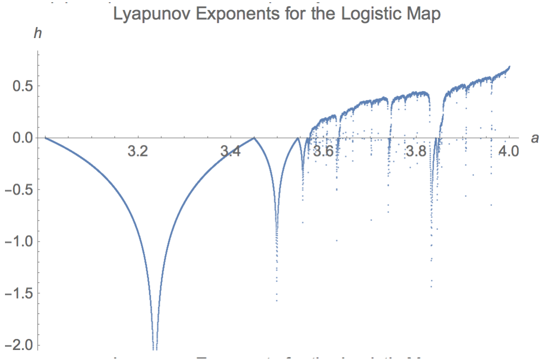
$$\mathrm{dist}(\phi^n(y), \phi^n(p)) \leq Ce^{-\lambda n} \qquad y \in W^s(p)$$

and

$$\mathrm{dist}(\phi^{-n}(y), \phi^{-n}(p)) \leq Ce^{-\lambda n} \qquad y \in W^u(p)$$

For any point $p' \in X$ near p but not on either $W^s(p)$ or $W^u(p)$, the trajectory of p' separates from that of p both in the future and the past.

Lyapunov Exponents for Logistic Map



Chaos: Hyperbolicity

How can we formalize this view point of chaos? For expanding maps, $\|D\phi^n v\| \geq CL^n \|v\|$ for all tangent vectors v. While for contractions, $\|D\phi^n v\| \leq C\tau^n \|v\|$.

Chaos: Hyperbolicity

How can we formalize this view point of chaos? For expanding maps, $\|D\phi^n v\| \geq CL^n \|v\|$ for all tangent vectors v. While for contractions, $\|D\phi^n v\| \leq C\tau^n \|v\|$.

Hyperbolicity is the presence of expanding and contracting directions for the derivative.

Chaos: Hyperbolicity

How can we formalize this view point of chaos? For expanding maps, $\|D\phi^n v\| \geq CL^n \|v\|$ for all tangent vectors v. While for contractions, $\|D\phi^n v\| \leq C\tau^n \|v\|$.

Hyperbolicity is the presence of expanding and contracting directions for the derivative.

ϕ is *uniformly hyperbolic* if for all $p \in X$, there is a splitting of the tangent space $T_p M = E^s(x) \bigoplus E^u(x)$ and constants $C > 0, \lambda \in (0,1)$ such that

$$\begin{aligned} \|D_p\phi^n v\| &\leq C\lambda^n \|v\|, \quad \forall v \in E^s(p) \\ \left\|D_p\phi^{-n} v\right\| &\leq C\lambda^n \|v\|, \quad \forall v \in E^u(p) \end{aligned}$$

Numerical Imprecision

Rounding errors in numerical computation yield widely diverging outcomes for such dynamical systems, rendering long-term prediction of their behavior impossible in general

Numerical Imprecision

Rounding errors in numerical computation yield widely diverging outcomes for such dynamical systems, rendering long-term prediction of their behavior impossible in general

A ϵ-*pseudo-orbit* is a sequence (x_n) of points such that

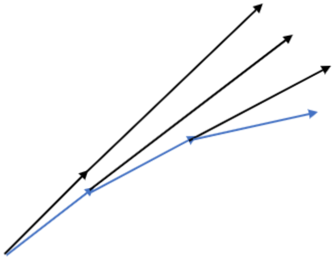
$$d(\phi(x_n), x_{n+1}) \leq \epsilon$$

Numerical Imprecision

Rounding errors in numerical computation yield widely diverging outcomes for such dynamical systems, rendering long-term prediction of their behavior impossible in general

A ϵ-*pseudo-orbit* is a sequence (x_n) of points such that

$$d(\phi(x_n), x_{n+1}) \leq \epsilon$$



Shadowing

Any pseudo-orbit approximately follows some orbit! That is, a pseudo-orbit is *shadowed* by a true orbit.

A sequence $\mathbf{x}$ is δ-shadowed by the orbit of p if $d(x_n, \phi^n(p)) \leq \delta$.

Shadowing

Any pseudo-orbit approximately follows some orbit! That is, a pseudo-orbit is *shadowed* by a true orbit.

A sequence $\mathbf{x}$ is δ-shadowed by the orbit of p if $d(x_n, \phi^n(p)) \leq \delta$.

Shadowing Lemma: Suppose ϕ is hyperbolic. Then, for any $\delta > 0$ there is an $\epsilon > 0$ so that every ϵ-pseudo-orbit is δ-shadowed by a unique orbit of ϕ.

Shadowing Lemma Proof (for Contractions)

Suppose ϕ is a contraction with constant τ.

Shadowing Lemma Proof (for Contractions)

Suppose ϕ is a contraction with constant τ.

Consider the set of sequences which are point-wise within δ of $\mathbf{x}$. Define the metric space $(\mathbf{E}, D)$ by

$$\mathbf{E} = \{\mathbf{y} : \mathbf{y} = \{y_n\}_{n \in \mathbb{Z}}, d(x_n, y_n) \leq \delta\}$$

and

$$D(\mathbf{x}, \mathbf{y}) = \sup\{d(x_n, y_n)\}$$

Shadowing Lemma Proof continued...

If there is a true orbit $\mathbf{y}^* \in \mathbf{E}$, then $\mathbf{x}$ is δ-shadowed by $\mathbf{y}^*$.

Let $\mathbf{z} \in \mathbf{E}$. Consider applying the map ϕ to each element. This produces a new sequence. Define

$$\theta(\mathbf{z})_n = \phi(z_{n-1})$$

Shadowing Lemma Proof continued...

If there is a true orbit $\mathbf{y}^* \in \mathbf{E}$, then $\mathbf{x}$ is δ-shadowed by $\mathbf{y}^*$.

Let $\mathbf{z} \in \mathbf{E}$. Consider applying the map ϕ to each element. This produces a new sequence. Define

$$\theta(\mathbf{z})_n = \phi(z_{n-1})$$

A fixed point of θ is a true orbit of ϕ!

Shadowing Lemma Proof continued...

If there is a true orbit $\mathbf{y}^* \in \mathbf{E}$, then $\mathbf{x}$ is δ-shadowed by $\mathbf{y}^*$.

Let $\mathbf{z} \in \mathbf{E}$. Consider applying the map ϕ to each element. This produces a new sequence. Define

$$\theta(\mathbf{z})_n = \phi(z_{n-1})$$

A fixed point of θ is a true orbit of ϕ!

Let's show that θ is a contraction for $(\mathbf{E}, D)$. By the Banach Fixed Point Theorem, θ must have a unique fixed point.

Shadowing Lemma Proof continued...

If there is a true orbit $\mathbf{y}^* \in \mathbf{E}$, then $\mathbf{x}$ is δ-shadowed by $\mathbf{y}^*$.

Let $\mathbf{z} \in \mathbf{E}$. Consider applying the map ϕ to each element. This produces a new sequence. Define

$$\theta(\mathbf{z})_n = \phi(z_{n-1})$$

A fixed point of θ is a true orbit of ϕ!

Let's show that θ is a contraction for $(\mathbf{E}, D)$. By the Banach Fixed Point Theorem, θ must have a unique fixed point.

Also need to show that $\mathbf{E}$ is θ-invariant.

Shadowing Lemma Proof - Invariance

To show $\theta(\mathbf{z}) \in \mathbf{E}$, we want show that $d(\phi(z_{n-1}), x_n) \leq \delta$.

Shadowing Lemma Proof - Invariance

To show $\theta(\mathbf{z}) \in \mathbf{E}$, we want show that $d(\phi(z_{n-1}), x_n) \leq \delta$.

By the triangle inequality

$$d(\phi(z_{n-1}), x_n) \leq d(\phi(z_{n-1}), \phi(x_{n-1})) + d(\phi(x_{n-1}), x_n)$$

Shadowing Lemma Proof - Invariance

To show $\theta(\mathbf{z}) \in \mathbf{E}$, we want show that $d(\phi(z_{n-1}), x_n) \leq \delta$.

By the triangle inequality

$$d(\phi(z_{n-1}), x_n) \leq d(\phi(z_{n-1}), \phi(x_{n-1})) + d(\phi(x_{n-1}), x_n)$$

Since ϕ is a contraction, and $\mathbf{x}$ is a ϵ-pseudo-orbit

$$d(\phi(z_{n-1}), x_n) \leq \tau d(z_{n-1}, x_{n-1}) + \epsilon$$

Shadowing Lemma Proof - Invariance

To show $\theta(\mathbf{z}) \in \mathbf{E}$, we want show that $d(\phi(z_{n-1}), x_n) \leq \delta$.

By the triangle inequality

$$d(\phi(z_{n-1}), x_n) \leq d(\phi(z_{n-1}), \phi(x_{n-1})) + d(\phi(x_{n-1}), x_n)$$

Since ϕ is a contraction, and $\mathbf{x}$ is a ϵ-pseudo-orbit

$$d(\phi(z_{n-1}), x_n) \leq \tau d(z_{n-1}, x_{n-1}) + \epsilon$$

Since $\mathbf{z} \in \mathbf{E}$, $d(z_{n-1}, x_{n-1}) \leq \delta$. Let $\epsilon = (1-\tau)\delta$. Then,

$$d(\phi(z_{n-1}), x_n) \leq \tau\delta + (1-\tau)\delta = \delta$$

Shadowing Lemma Proof - θ is a Contraction

Take $\mathbf{y}, \mathbf{z} \in \mathbf{E}$. Then,

$$\begin{aligned} D(\theta(\mathbf{y}), \theta(\mathbf{z})) &= \sup\{d(\phi(y_{n-1}), \phi(z_{n-1})) : n \in \mathbb{Z}\} \\ &\leq \tau \cdot \sup\{d(y_{n-1}, z_{n-1}) : n \in \mathbb{Z}\} \\ &\leq \tau D(\mathbf{y}, \mathbf{z}) \end{aligned}$$

Invariance

A set $A \subset X$ is ϕ-invariant if $\phi^{-1}(A) = A$.

Ergodicity

Invariance
A set $A \subset X$ is ϕ-invariant if $\phi^{-1}(A) = A$.

An invariant measure μ is a measure that is preserved by some function. That is, μ is ϕ-invariant if for all $A \subset X$,

$$\mu(\phi^{-1}(A)) = \mu(A)$$

We say ϕ is measure-preserving.

Invariance
A set $A \subset X$ is ϕ-invariant if $\phi^{-1}(A) = A$.

An invariant measure μ is a measure that is preserved by some function. That is, μ is ϕ-invariant if for all $A \subset X$,

$$\mu(\phi^{-1}(A)) = \mu(A)$$

We say ϕ is measure-preserving.

Ergodicity

Invariance
A set $A \subset X$ is ϕ-invariant if $\phi^{-1}(A) = A$.

An invariant measure μ is a measure that is preserved by some function. That is, μ is ϕ-invariant if for all $A \subset X$,

$$\mu(\phi^{-1}(A)) = \mu(A)$$

We say ϕ is measure-preserving.

A function f is ϕ-invariant if $f \circ \phi = f$.

Ergodicity

Ergodicity

Invariance
A set $A \subset X$ is ϕ-invariant if $\phi^{-1}(A) = A$.

An invariant measure μ is a measure that is preserved by some function. That is, μ is ϕ-invariant if for all $A \subset X$,

$$\mu(\phi^{-1}(A)) = \mu(A)$$

We say ϕ is measure-preserving.

A function f is ϕ-invariant if $f \circ \phi = f$.

Ergodicity
Let ϕ be a measure-preserving map on a probability space. ϕ is ergodic if every ϕ-invariant set has measure 0 or 1.

Cute Lemma

If a measurable function $f : X \to \mathbb{R}$ is invariant under an ergodic map T, then f is constant almost everywhere.

Proof: Define the level sets $A_c = \{x \in X : f(x) \leq c\}$. We first show that A_c is T-invariant. Suppose $x \in A_c$. Then $f(x) \leq c$, and by invariance, $f(T(x)) \leq c$. Finally $T(x) \in A_c$ and so $A_c \subset T^{-1}(A_c)$. We can similarly show $T^{-1}(A_c) \subset A_c$ and hence $T^{-1}(A_c) = A_c$. By the ergodicity of T, $\mu(A_c) = 0$ or $\mu(A_c) = 1$. Let $p = \inf\{c : \mu(A_c) = 1\}$. Then, since $\mu(A_{p-1/n}) = 0$, $f(x) \geq p$ a.e. and since $\mu(A_p) = 1$, $f(x) \leq p$ a.e. The claim follows.

Orbit Distribution

Let $A \subset X$, $x \in X$. The limit

$$\tau(x, A) = \lim_{n \to \infty} \frac{1}{n} \text{card}\{0 \leq m < n : T^m(x) \in A\}$$

is defined as the *frequency of returns* of the point x to the set A.

Orbit Distribution

Let $A \subset X$, $x \in X$. The limit

$$\tau(x, A) = \lim_{n\to\infty} \frac{1}{n}\text{card}\{0 \leq m < n : T^m(x) \in A\}$$

is defined as the *frequency of returns* of the point x to the set A. Equivalently,

$$\tau(x, A) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(T^j(x))$$

where χ_A denotes the indicator function of A. $\chi_A(x) = 1$ if $x \in X$ and 0 otherwise.

Orbit Distribution

Birkhoff Ergodic Theorem
Let $(X, \mathcal{A}, \mu)$ be a probability space, and $T : X \to X$ a measure-preserving map. If $f : X \to \mathbb{R}$ is an integrable function, the limit

$$\tilde{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x))$$

exists for almost everywhere $x \in X$, and the function $\tilde{f}$ is T-invariant, integrable, and

$$\int_X \tilde{f} d\mu = \int_X f d\mu$$

The function $\tilde{f}$ is called the *time average* of f.

Orbit Distribution

Letting $f(x) = \chi_A(x)$, we find that $\tilde{f} = \tau(x, A)$. By the Birkhoff Ergodic Theorem,

$$\begin{aligned}
\int_X \tau(x, A) d\mu(x) &= \int_X \tilde{f}(x) d\mu(x) \\
&= \int_X \chi_A(x) d\mu(x) \\
&= \mu(A)
\end{aligned}$$

Distribution of Ergodic Orbits

Now, suppose that ϕ is ergodic. Note that $\tau(x, A)$ is clearly ϕ-invariant. So, using the above "cute" lemma,

$$\begin{aligned} \mu(A) &= \int_X \tau(x, A) d\mu(x) \\ &= \tau(x, A) \int_X d\mu(x) \\ &= \tau(x, A)\mu(X) \\ &= \tau(x, A) \end{aligned}$$

The fraction of time that an orbit spends in a subset is equal to the measure of the set!

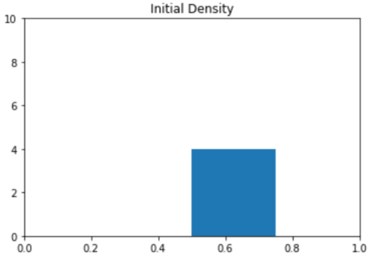
Hunting for Invariant Measure

Hunting for Invariant Measure

Let's take another look at the Logistic map $\phi(x) = 4x(1 - x)$.

Let's take another look at the Logistic map $\phi(x) = 4x(1-x)$.

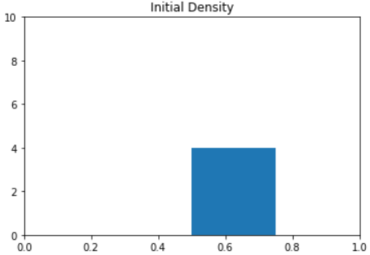
Let's start with some density ρ_0



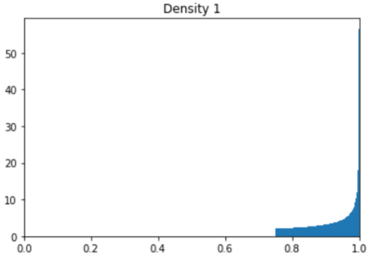
Hunting for Invariant Measure

Let's take another look at the Logistic map $\phi(x) = 4x(1-x)$.

Let's start with some density ρ_0



Apply ϕ to transform ρ_0 to ρ_1

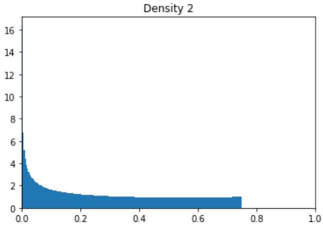


Hunting for Invariant Measure

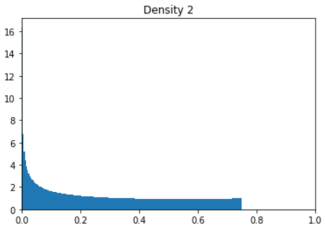
And Keep Applying ϕ.

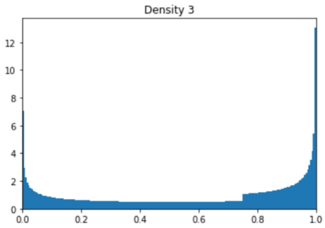
Hunting for Invariant Measure

And Keep Applying ϕ.



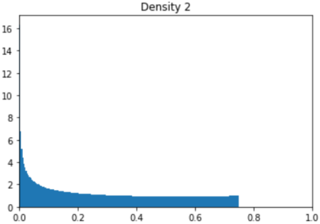
And Keep Applying ϕ.

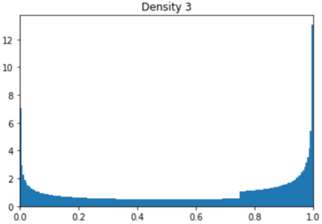


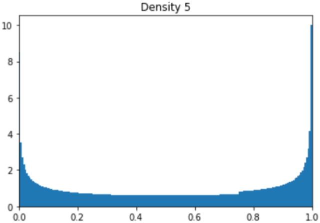


Hunting for Invariant Measure

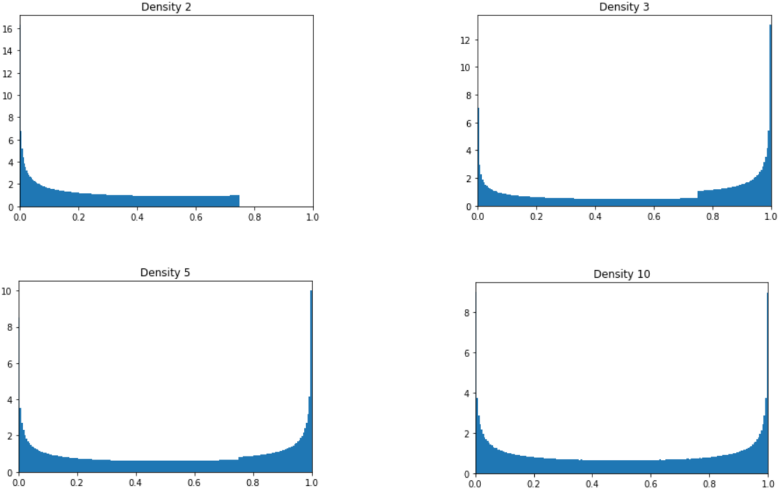
And Keep Applying ϕ.







And Keep Applying ϕ.



Transferring Densities

Let ρ_0 be some initial density, and ρ_n the resulting density after applying ϕ^n. For any $A \subset X$,

$$\int_A \rho_n(x)dx = \int_{\phi^{-n}(A)} \rho_0(x_0)dx_0$$

Transferring Densities

Let ρ_0 be some initial density, and ρ_n the resulting density after applying ϕ^n. For any $A \subset X$,

$$\int_A \rho_n(x)dx = \int_{\phi^{-n}(A)} \rho_0(x_0)dx_0$$

Change of coordinates: $x = \phi^n(y)$ (suppose ϕ invertible)

$$\int_{\phi^{-n}(A)} \rho_n(\phi^n(y)) \left|\det J^n(y)\right| dy = \int_{\phi^{-n}(A)} \rho_0(x_0)dx_0$$

Transferring Densities

Let ρ_0 be some initial density, and ρ_n the resulting density after applying ϕ^n. For any $A \subset X$,

$$\int_A \rho_n(x)dx = \int_{\phi^{-n}(A)} \rho_0(x_0)dx_0$$

Change of coordinates: $x = \phi^n(y)$ (suppose ϕ invertible)

$$\int_{\phi^{-n}(A)} \rho_n(\phi^n(y)) \left|\det J^n(y)\right| dy = \int_{\phi^{-n}(A)} \rho_0(x_0)dx_0$$

So,

$$\rho_n(\phi^n(y)) \left|\det J^n(y)\right| = \rho_0(y)$$

Perron-Frobenius Operator

The *Transfer Operator*, $\mathcal{L}$, takes ρ_0 to ρ_n. So, $\mathcal{L} \circ \rho_0 = \rho_n$ where

$$(\mathcal{L} \circ \rho_0)(x) = \frac{\rho_0(y)}{|\det J^n(y)|}, \quad y = \phi^{-n}(x)$$

Perron-Frobenius Operator

The *Transfer Operator*, $\mathcal{L}$, takes ρ_0 to ρ_n. So, $\mathcal{L} \circ \rho_0 = \rho_n$ where

$$\left(\mathcal{L} \circ \rho_0\right)(x) = \frac{\rho_0(y)}{|\det J^n(y)|}, \quad y = \phi^{-n}(x)$$

If ϕ not invertible?

$$\left(\mathcal{L} \circ \rho_0\right)(x) = \sum_{y \in \phi^{-n}(x)} \frac{\rho_0(y)}{|\det J^n(y)|}$$

Fixed Points of $\mathcal{L}$

What is a fixed point of the transfer operator?

$$\mathcal{L}\rho = \rho$$

Fixed Points of $\mathcal{L}$

What is a fixed point of the transfer operator?

$$\mathcal{L}\rho = \rho$$

An Invariant Measure!

Fixed Points of $\mathcal{L}$

What is a fixed point of the transfer operator?

$$\mathcal{L}\rho = \rho$$

An Invariant Measure!

The limit of the first n density functions converges to a density function $\rho^*(x)$,

$$\rho^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho_k(x)$$

More on the Transfer Operator

$\mathcal{L}$ is a *linear* operator taking one density function to another.

More on the Transfer Operator

$\mathcal{L}$ is a *linear* operator taking one density function to another.

So, why does this converge?

More on the Transfer Operator

$\mathcal{L}$ is a *linear* operator taking one density function to another.

So, why does this converge?

Sinai, Ruelle, Bowen proved that for hyperbolic maps, $\mathcal{L}$ admits 1 as a simple eigenvalue, and the rest of the spectrum is contained in a disk of radius < 1.

So, in the limit, the behavior is dominated by "eigenvector" corresponding to an eigevalue of 1. This is precisely the invariant measure.

Mixing

A measure-preserving map ϕ on a probability space $(X, \mathcal{A}, \mu)$ is said to be *mixing* if for any $A, B \in \mathcal{A}$,

$$\lim_{n \to \infty} \mu(\phi^{-n}(A) \cap B) = \mu(A)\mu(B)$$

Mixing implies Ergodic

Proposition Any mixing map is ergodic.

Suppose $A \subset X$ is a ϕ-invariant set. Since $\phi^{-n}(A) = A$, $\lim_{n\to\infty} \mu(\phi^{-n}(A) \cap B) = \mu(A \cap B)$. Since ϕ is mixing, $\mu(A \cap B) = \mu(A)\mu(B)$. Take $A = B$. Then, $\mu(A) = \mu(A)^2$ and hence $\mu(A) = 0$ or $\mu(A) = 1$.

Ergodic implies Mixing?

Ergodic implies Mixing?

No!

Ergodic implies Mixing?

No!

Take the map

$$\phi(x) = x + \theta \mod 1$$

with θ irrational.

Ergodic implies Mixing?

No!

Take the map

$$\phi(x) = x + \theta \mod 1$$

with θ irrational. Note that ϕ is indeed ergodic. Suppose that $A \subset X$ is invariant. Consider the fourier transform of the indicator of A, say f. Note that the map on the unit circle is represented by multiplication with $e^{2\pi i\theta}$. So,

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \cdot nx} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \cdot n(x+\theta)}$$

Since fourier series are unique, $a_n = a_n e^{2\pi i\theta}$. Since θ irrational, we must have $a_n = 0$ for $n \neq 0$. Hence, f constant. So, ϕ is ergodic.

Decay of Correlations

Intuitively, the Birkhoff Ergodic Theorem can be thought of as the law of large numbers in a dynamical systems setting: time average converges to space average.

Decay of Correlations

Intuitively, the Birkhoff Ergodic Theorem can be thought of as the law of large numbers in a dynamical systems setting: time average converges to space average.

Are there any other relations to probability theory? For instance, does $f(T^n(x))$ effectively become independent of $f(x)$?

Decay of Correlations

Intuitively, the Birkhoff Ergodic Theorem can be thought of as the law of large numbers in a dynamical systems setting: time average converges to space average.

Are there any other relations to probability theory? For instance, does $f(T^n(x))$ effectively become independent of $f(x)$?

We denote by C_f the time correlation function of dynamical system T where

$$C_{f,g}(n) = \int_X g(T^n(x))f(x)d\mu - \int_X g(x)d\mu \int_X f(x)d\mu$$

Decay of Correlations and Mixing

A map is mixing if and only if $C_{f,g}$ converges to 0.

We say that correlations decay exponentially if

$$|C_{f,g}(n)| \leq Ce^{-an}$$

A Generalized Operator

$$\int (g \circ T) \cdot f \, dx = \int g \cdot (\mathcal{L}_T f) dx$$

with

$$(\mathcal{L}_T f)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}$$

Example: Doubling Map

Let's show that the map

$$\phi(x) = 2x \mod 1$$

enjoys exponential decay of correlations.

Example: Doubling Map

Let's show that the map

$$\phi(x) = 2x \mod 1$$

enjoys exponential decay of correlations.

Back to the transfer operator,

$$(\mathcal{L} \circ f)(x) = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f\left(\frac{x+1}{2}\right)\right]$$

Note that Lebesgue measure μ is invariant under $\mathcal{L}$. Let's restrict ourselves to $\mathcal{L}$: Lip $\to$ Lip (subspace of Lipschitz continuous functions on the interval [0,1]).

Example: Doubling Map

Consider the semi-norm (which only fails since it vanishes for constant functions)

$$|f|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

Example: Doubling Map

The key property is that $|\cdot|_{\mathrm{Lip}}$ is contracted by the operator $\mathcal{L}$:
$|\mathcal{L}f|_{\mathrm{Lip}} \leq \frac{1}{2}|f|_{\mathrm{Lip}}$

Example: Doubling Map

The key property is that $|\cdot|_{\text{Lip}}$ is contracted by the operator $\mathcal{L}$: $|\mathcal{L}f|_{\text{Lip}} \leq \frac{1}{2}|f|_{\text{Lip}}$

$$
\begin{aligned}
|\mathcal{L}f|_{\text{Lip}} &= \sup_{x \neq y} \frac{|(\mathcal{L}f)(x) - (\mathcal{L}f)(y)|}{|x-y|} \\
&= \sup_{x \neq y} \frac{1}{2} \frac{|f(x_1) + f(x_2) - f(y_1) - f(y_2)|}{|x-y|}
\end{aligned}
$$

where $x_1 = \frac{x}{2}, x_2 = \frac{x+1}{2}$. Note that $|x_i - y_i| = \frac{1}{2}|x-y|$.

Example: Doubling Map

The key property is that $|\cdot|_{\text{Lip}}$ is contracted by the operator $\mathcal{L}$: $|\mathcal{L}f|_{\text{Lip}} \leq \frac{1}{2}|f|_{\text{Lip}}$

$$\begin{aligned} |\mathcal{L}f|_{\text{Lip}} &= \sup_{x \neq y} \frac{|(\mathcal{L}f)(x) - (\mathcal{L}f)(y)|}{|x-y|} \\ &= \sup_{x \neq y} \frac{1}{2} \frac{|f(x_1) + f(x_2) - f(y_1) - f(y_2)|}{|x-y|} \end{aligned}$$

where $x_1 = \frac{x}{2}, x_2 = \frac{x+1}{2}$. Note that $|x_i - y_i| = \frac{1}{2}|x-y|$. So,

$$|\mathcal{L}f|_{\text{Lip}} \leq \sup_{x \neq y} \frac{1}{4} \left[\frac{|f(x_1) - f(y_1)|}{|x_1 - y_1|} + \frac{|f(x_2) - f(y_2)|}{|x_2 - y_2|} \right] \leq \frac{1}{2}|f|_{\text{Lip}}$$

The End