MARKOV PARTITIONS FOR AXIOM *A* DIFFEOMORPHISMS.

By Rufus Bowen.

1. Introduction. In this paper we shall study the homeomorphism $f = f^* \mid \Omega_s$ where f^* is a diffeomorphism satisfying S. Smale's Axiom A and Ω_s is a basic set for f^* given by his Spectral Decomposition Theorem (see [25]). Our approach is *symbolic dynamics.* We find a covering $\mathscr{E} = \{E_1, \cdots, E_n\}$ of Ω_s by closed sets and, for each $x \in \Omega_s$, consider certain *symbolic sequences* $(E_{i_k})_{k=-\infty}^{+\infty}$ such that $f^k(x) \in E_{i_k}$ for all k. One then attempts to study the dynamics of f by using that of these symbolic sequences. This approach is an old one, originally used to study geodesics on surfaces of negative curvature; see [5], [12], [13], [16], [17] and [18] (warning: the formalism above is not quite the right one for contiuous time examples, e.g. geodesic flows).

The key to success is to find a good covering $\mathscr{E}$. The right notion seems to be that of *Markov partition.* This is defined in Section 3. Most of the work of this paper is in proving:

Theorem. *f has Markov partitions.*

With a Markov partition the space of all symbolic sequences used can be taken to be a subshift of finite type (Section 4) and one obtains

Theorem. *f is the quotient of a subshift of finite type. One then studies f by studying the structure of this quotient; this is what we mean by symbolic dynamics.*

In [2] Adler and Weiss constructed Markov partitions for hyperbolic automorphisms of the 2-torus and used the associated symbolic dynamics to study the measure theory of these examples. Ya. Sinai then defined and constructed Markov partitions for Anosov maps in [22] and [23]. We use Sinai's methods, with some modification because Ω_s is not generally connected.

There are a considerable number of other papers as well that suggested symbolic dynamics for Axiom A diffeomorphisms. Smale studied in [24] certain examples where $f = f^* \mid \Omega_s$ is a full shift. In [25] he showed how

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to obtain other subshifts of finite type for Ω_s as well, and in [6] we saw that in fact every zero-dimensional Ω_s is a subshift of finite type. The papers of R. F. Williams [26], [27] and J. Guckenheimer [11] study various one-dimensional Ω_s and produce Markov partitions for them. M. Hirsch [14] showed that any expanding map (the present paper applies to these examples by easy modification) is a quotient of a full one-sided shift. Finally, H. Keynes and Robertson [28] showed that any expansive homeomorphism is the quotient of some subshift (not necessarily of finite type).

In Section 5 we study the entropy theory of f. For any f-invariant Borel measure ν defined on Ω_s there is an entropy $h_\nu(f)$ (see [20] for the definition); there is also a topological entropy $h(f)$ (see [1]). It is a general fact that $h_\nu(f) \leqq h(f)$ (see [9]). We prove

THEOREM. *There is a unique normalized f-invariant Borel measure μ_f on Ω_s with $h_{\mu_f}(f) = h(f)$; (f, μ_f) is an ergodic Markov chain.*

This is derived from W. Parry's theorem [19] that this fact is true for subshifts of finite type. B. Gurevic used Sinai's results to prove this theorem for transitive Anosov maps. K. Berg [4] has shown that, for an ergodic automorphism of a torus, Haar measure is the only invariant measure which maximizes entropy.

THEOREM. *If f is a hyperbolic automorphism of a nilmanifold N, then μ_f equals Haar measure μ_N on N. If f_1 and f_2 are two such maps, then (f_1, μ_{N_1}) and (f_2, μ_{N_2}) are measure theoretically isomorphic iff they have the same entropy.*

For the 2-torus this theorem was proved by Adler and Weiss [2]. The first part was proved for any torus by Sinai [22]. The second part is merely an application of the theorem of Friedman and Ornstein [29] that entropy is complete for mixing Markov chains.

2. Canonical coordinates and rectangles.

For $x \in \Omega_s$ and $\delta > 0$ we define the closed stable and unstable sets of size δ as follows.

$$W_\delta{}^s(x) = \{y \in \Omega_s \colon d(f^n(x), f^n(y)) \leqq \delta \text{ for all } n \geqq 0\}$$
$$W_\delta{}^u(x) = \{y \in \Omega_s \colon d(f^{-n}(x), f^{-n}(y)) \leqq \delta \text{ for all } n \geqq 0\}.$$

The metric d we use on Ω_s is an "adapted" one as in Hirsch and Pugh [15]. We recall some properties of f proved in [15] and Smale [25].

Fact 1. There are positive numbers $\lambda < 1$, ϵ and γ for which the following statements are true. For $n \geqq 0$,

$$d(f^n(y), f^n(z)) \leqq \lambda^n d(y,z) \text{ if } y, z \in W_\gamma{}^s(x)$$

and

$$d(f^{-n}(y), f^{-n}(z)) \leqq \lambda^n d(y,z) \text{ if } y, z \in W_\gamma{}^u(x).$$

If $d(x,y) \leqq \epsilon$, then $W_\gamma{}^s(x) \cap W_\gamma{}^u(y)$ consists of a single point, which we denote by $[x,y]$. The map

$$[\cdot,\cdot] : \{(x,y) \in \Omega_s \times \Omega_s : d(x,y) \leqq \epsilon\} \to \Omega_s$$

is continuous.

Problem. Does every topologically transitive homeomorphism satisfying the above occur as $f^* \mid \Omega_s$ with f^* an Axiom A diffeomorphism?

$B_\delta(x)$ will denote the closed δ-ball in Ω_s centered at x. For $X \subset \Omega_s$, we write $W_\delta{}^u(X) = \bigcup_{x \in X} W_\delta{}^u(x)$.

LEMMA 2. (a) $W_{\delta_1}{}^u W_{\delta_2}{}^u(x) \subset W_{\delta_1+\delta_2}{}^u(x)$.

(b) *If* $y \in W_\gamma{}^u(x)$, *then* $W_\gamma{}^u(x) \cap B_\rho(y) \subset W_\rho{}^u(y)$.

(c) *If* $\delta_1 \leqq \delta_2 \leqq \gamma$, *then* $W_{\delta_1}{}^u(x) = W_{\delta_2}{}^u(x) \cap B_{\delta_1}(x)$.

(d) *Let* $\delta_1 < \delta_2 \leqq \gamma$. *Then* $U \subset W_{\delta_1}{}^u(x)$ *is open as a subset of* $W_{\delta_2}{}^u(x)$ *iff for every* $y \in U$ *there is a* $\rho > 0$ *with* $W_\rho{}^u(y) \subset U$.

(e) *If* $Y \subset W_\delta{}^u(X)$ *with* $\delta \leqq \gamma$, *then* $f^{-1}Y \subset W_{\lambda\delta}{}^u(f^{-1}X)$.

Proof. (a) Use the triangle inequality.

(b). If $z \in W_\gamma{}^u(x) \cap B_\rho(y)$, then, for $n \geqq 0$,

$$d(f^{-n}(z), f^{-n}(y)) \leqq \lambda^n d(z,y) \leqq \rho$$

and $z \in W_\rho{}^u(y)$.

(c). Clearly $W_{\delta_1}{}^u(x) \subset W_{\delta_2}{}^u(x) \cap B_{\delta_1}(x)$. The reverse inclusion follows from (b) with $y = x$ and $\rho = \delta_1$.

(d). U is open in $W_{\delta_2}{}^u(x)$ iff, for each $y \in U$, $U \supset B_\rho(y) \cap W_{\delta_2}{}^u(x)$ for sufficiently small $\rho > 0$. Now (a) shows that $W_{\delta_1}{}^u(x) \supset W_{\delta_2-\delta_1}{}^u(y)$. For $\rho < \delta_2 - \delta_1$, (c) shows

$$W_\rho{}^u(y) = W_{\delta_2-\delta_1}{}^u(y) \cap B_\rho(y) \subset W_{\delta_2}{}^u(x) \cap B_\rho(y);$$

by (b), $W_\rho{}^u(y) \supset W_{\delta_2}{}^u(x) \cap B_\rho(y)$; thus $W_\rho{}^u(y) = D_{\delta_2}{}^u(x) \cap B_\rho(y)$ and we have the openness condition we want.

(e). If $y \in W_\delta{}^u(x)$, then fact 1 shows that, for $n \geqq 0$,

$$d(f^{-n}f^{-1}(x), f^{-n}f^{-1}(y)) \leqq \lambda^{n+1} d(x,y) \leqq \lambda\delta.$$

Thus $f^{-1}(y) \in W_{\lambda\delta}{}^u(f^{-1}(x))$.

LEMMA 3. *γ is an expansive constant for f, i.e. if $x \neq y$, then* $d(f^n(x), f^n(y)) > \gamma$ for some $n \in Z$.

Proof. Otherwise $\{x, y\} \subset W_\gamma{}^s(x) \cap W_\gamma{}^u(x)$.

LEMMA 4. *For any $\zeta > 0$ there is a positive integer $D(\zeta)$ such that $d(x, y) < \zeta$ whenever $d(f^n(x), f^n(y)) \leqq \gamma$ for all $|n| \leqq D(\zeta)$.*

Proof. This is a property of expansive constants [10].

LEMMA 5. *For every $0 < \delta \leqq \gamma$, there is an $\epsilon(\delta) > 0$ such that $[x, y] \in W_\delta{}^s(x) \cap W_\delta{}^u(y)$ whenever $d(x, y) \leqq \epsilon(\delta)$.*

Proof. Clearly $[x, x] = x$. Since $[\cdot, \cdot]$ is uniformly continuous, there is an $\epsilon(\delta) > 0$ such that $d([x, y], x) < \delta$ and $d([x, y], y) < \delta$ when $d(x, y) \leqq \epsilon(\delta)$. By Lemma 2(c)

$$[x, y] \in W_\gamma{}^u(y) \cap B_\delta(y) = W_\delta{}^u(y).$$

The W^s analogue of 2(c) gives $[x, y] \in W_\delta{}^s(x)$.

Definition. Choose a descending sequence of positive numbers $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ with $\alpha_0 = \gamma$ and $\alpha_{n+1} < \frac{1}{3}\min(\frac{1}{3}\alpha_n, \epsilon(\frac{1}{3}\alpha_n))$ so small that $d(f(x), f(y)) < \alpha_n$ and $d(f^{-1}(x), f^{-1}(y)) < \alpha_n$ whenever $d(x, y) \leqq \alpha_{n+1}$.

LEMMA 6. (a). *If* $\operatorname{diam} X \leqq 3\alpha_{n+1}$, then $\operatorname{diam}[X, X] \leqq \alpha_n$.
(b) *If* $\operatorname{diam}\{w, x, y, z\} \leqq 3\alpha_2$, *then*

$$[x, z] = [x, [y, z]] = [[x, y], z] = [[x, w], [y, z]].$$

Proof. (a). Suppose $w, x, y, z \in X$. Then $[w, x] \in W_{\alpha_n/3}{}^s(w) \subset B_{\alpha_n/3}(w)$ since $d(w, x) \leqq \epsilon(\alpha_n/3)$. Similarly $[y, z] \in B_{\alpha_n/3}(y)$. As $d(w, y) \leqq \alpha_n/3$, $d([w, x], [y, z]) \leqq \alpha_n$.

(b). Now $[y, z] \in W_{\alpha_1}{}^u(z)$. Since $d(x, [y, z]) = d([x, x], [y, z]) \leqq \alpha_1$, $[x, [y, z]] \in W_{\gamma/3}{}^s(x) \cap W_{\gamma/3}{}^u([y, z])$. Because

$$W_{\gamma/3}{}^u([y, z]) \subset W_{\gamma/3}{}^u W_{\alpha_1}{}^u(z) \subset W_\gamma{}^u(z),$$

we have $[x, [y, z]] \in W_\gamma{}^s(x) \cap W_\gamma{}^u(z) = [x, z]$.

The other parts are similar.

We now recall a result of Smale on the existence of "canonical coordinates." Manfred Denker has shown us how to prove this directly from fact 1; this is relevant to the problem mentioned earlier.

LEMMA 7 (Smale [25]). *If* $U \subset W_{\alpha_2}{}^u(x)$ *is open in* $W_\gamma{}^u(x)$ *and* $V \subset W_{\alpha_2}{}^s(x)$ *is open in* $W_\gamma{}^s(x)$, *then* $[U, V]$ *is open in* Ω_s *and*

$$[\cdot, \cdot]: U \times V \to [U, V]$$

is a homeomorphism.

LEMMA 8. *Suppose* $d(x,y) \leqq \alpha_3$ *and* $U \subset W_{\alpha_2}{}^u(x)$ *is open in* $W_\gamma{}^u(x)$. *Then* $[U, y]$ *is open in* $W_\gamma{}^u(y)$ *and* $[\cdot, y]: U \to [U, y]$ *is a homeomorphism. The corresponding statement for* W^s*'s is likewise true.*

Proof. By 6(a), $z = [x, y] \in W_{1/3\alpha_2}{}^s(y)$. Let $V \subset W_{\alpha_2}{}^s(x)$ be an open neighborhood of z in $W_\gamma{}^s(x)$. Using Lemma 7 then,

$$W = [U, V] \cap W_\gamma{}^u(y) \text{ is open in } W_\gamma{}^u(y).$$

Since $[U, y] \subset W_\gamma{}^u(y)$, $W \supset [U, y]$. Furthermore, if $w = [u, v] \in W$, then $[w, y] = W_\gamma{}^s(w) \cap W_\gamma{}^u(y) = w$ because $w \in W_\gamma{}^u(y)$. Hence, by Lemma 6(b),

$$w = [w, y] = [[u, v], y] = [u, y] \in [U, y].$$

Thus $W = [U, y]$.

Using 6(b) one checks that the continuous map $[\cdot, x]: [U, y] \to U$ is the inverse of $[\cdot, y]: U \to [U, y]$.

Definition. A nonempty set $A \subset \Omega_s$ is a *rectangle* if $\operatorname{diam} A \leqq \alpha_3$, $A = \overline{\operatorname{int} A}$ and $[x, y] \in A$ whenever $x, y \in A$. For each $x \in A$ we define

$$W^s(x, A) = W_\gamma{}^s(x) \cap A \subset W_{\alpha_3}{}^s(x)$$

and

$$W^u(x, A) = W_\gamma{}^u(x) \cap A \subset W_{\alpha_3}{}^u(x).$$

Let

$$\partial^s A = \{x \in A : x \notin \operatorname{int} W^u(x, A) \text{ in } W_\gamma{}^u(x)\}$$
$$\partial^u A = \{x \in A : x \notin \operatorname{int} W^s(x, A) \text{ in } W_\gamma{}^s(x)\}.$$

Remark. $A = \overline{\operatorname{int} A}$ implies that $\operatorname{int} \partial A = \emptyset$ (∂A denotes the boundary of A).

LEMMA 9. $\partial A = \partial^s A \cup \partial^u A$. *If* $x \in A$, *then* $W^u(x, A)$ *has dense interior in* $W_\gamma{}^u(x)$ *and* $W^s(x, A)$ *has dense interior in* $W_\gamma{}^s(x)$.

Proof. If $x \in \operatorname{int} A$, then $W^u(x, A) = A \cap W_\gamma{}^u(x)$ is a neighborhood of x in $W_\gamma{}^u(x)$ since $W_\gamma{}^u$ has the subspace topology. Thus $x \notin \partial^s A$; similarly $x \notin \partial^u A$. If $x \notin \partial^s A \cup \partial^u A$, let $U \subset W^u(x, A)$ be an open neighborhood of x in $W_\gamma{}^u(x)$ and $V \subset W^s(x, A)$ be an open neighborhood of x in $W_\gamma{}^s(x)$. As A is a rectangle, $[U, V] \subset A$; by Lemma 7, $[U, V]$ is open. Hence $x = [x, x] \in \operatorname{int} A$. Thus $\partial A = A \backslash \operatorname{int} A = \partial^s A \cup \partial^u A$.

Now $[W^u(x, A), W^s(x, A)] \subset A$ since A is a rectangle. If $z \in A$, then

$$[x, z] \in W_\gamma{}^s(x) \cap A = W^s(x, A)$$

and

$$[z, x] \in W_\gamma{}^u(x) \cap A = W^u(x, A).$$

By 6(b), $z = [[z, x], [x, z]]$; so $A = [W^u(x, A), W^s(x, A)]$. Let U be an open neighborhood of $W^u(x, A)$ in $W_{\alpha_2}{}^u(x)$ and let V be an open neighborhood of $W^s(x, A)$ in $W_{\alpha_2}{}^s(x)$. Since $\overline{\operatorname{int} A} = A$, Lemma 7 shows that $W^u(x; A) \times W^s(x, A)$ has dense interior as a subset of $U \times V$. From this it follows that $W^u(x, A)$ has dense interior in U and $W^s(x, A)$ has dense interior in V.

Lemma 10. *Suppose* $C \subset W_{\alpha_4}{}^u(x)$ *with* $C = \overline{\operatorname{int} C}$ *in* $W_\gamma{}^u(x)$ *and* $D \subset W_{\alpha_4}{}^s(x)$ *with* $D = \overline{\operatorname{int} D}$ in $W_\gamma{}^s(x)$. *Then* $A = [C, D]$ *is a rectangle with* $\partial^s A = [\partial C, D]$ *and* $\partial^u A = [C, \partial D]$. *For* $x \in A$, $W^u(x, A) = [C, x]$ *and* $W^s(x, A) = [x, D]$.

Proof. Since $[\cdot, \cdot]$ is continuous, $[C, D] = \overline{[\operatorname{int} C, \operatorname{int} D]}$; hence $A = \overline{\operatorname{int} A}$ as $[\operatorname{int} C, \operatorname{int} D] \subset \operatorname{int} A$ by 7. Suppose $x = [c_1, d_1]$, $y = [c_2, d_2]$ with $c_i \in C$, $d_i \in D$. Then

$$[x, y] = [[c_1, d_1], [c_2, d_2]] = [c_1, d_2] \in A$$

by 6(b). $\operatorname{diam} A \leqq \alpha_3$ by 6(a) since $\operatorname{diam} C \cup D \leqq 2\alpha_4$. Thus A is a rectangle. The boundary statements follow from 8.

Lemma 11. (a). *If* $d(f^j(x), f^j(y)) \leqq \alpha_2$ *for all* $0 \leqq j \leqq m$, *then* $f^m[x, y] = [f^m(x), f^m(y)]$.

(b). *Let* $g = f^m$, $m > 0$. *Suppose* $V \subset W_{\alpha_3}{}^u(z)$, $y \in W_{\alpha_3}{}^s(z)$ *and* $g(V) = \bigcup_k V_k$ *where* $V_k \subset W_{\alpha_3}{}^u(g(z_k))$ *and* $z_k \in W_{\alpha_3}{}^u(z)$. *Then*

$$g[V, y] = \bigcup_k [V_k, g([z_k, y])].$$

Proof. (a). Let $w_j = [f^j(x), f^j(y)]$. Then $w_j \in W_{\alpha_1}{}^u(f^j(x)) \cap W_{\alpha_1}{}^u(f^j(y))$. Clearly $f(w_j) \in W_\gamma{}^s(f^{j+1}(x))$. Since $d(w_j, f^j(y)) < \alpha_1$, $d(f(w_j), f^{j+1}(x)) < \gamma$;

hence $f(w_j)$ is in $W_\gamma{}^u(f^{j+1}(y))$. Thus

$$f(w_j) \in W_\gamma{}^s(f^{j+1}(x)) \cap W_\gamma{}^u(f^{j+1}(y)) = w_{j+1}.$$

Inductively, $f^j(w_0) = w_j$.

(b). Since $[V, y] = \bigcup_k [g^{-1}(V_k), y]$, it is enough to show that

$$g[g^{-1}(V_k), y] = [V_k, g[z_k, y])].$$

For $w \in V_k$, we show

$$g[g^{-1}(w), y] = [w, g([z_k, y])].$$

First, $[g^{-1}(w), y] = [g^{-1}(w), [z_k, y]]$ by 6(b). As $w \in W_{\alpha_3}{}^u(g(z_k))$,

$$d(f^j(g^{-1}(w)), f^j(z_k)) \leqq \alpha_3 < \tfrac{1}{3}\alpha_2$$

for $0 \leqq j \leqq m$; since $d(z_k, y) \leqq 2\alpha_3 < \epsilon(\frac{1}{3}\alpha_2)$, $[z_k, y] \in W_{1/3\alpha_2}{}^s(z_k)$ and

$$d(f^j(z_k), f^j([z_k, y])) \leqq \tfrac{1}{3}\alpha_2$$

for $0 \leqq j \leqq m$. Thus $d(f^j(g^{-1}(w)), f^j([z_k, y])) < \alpha_2$ for $0 \leqq j \leqq m$; by 11(a),

$$[w, g([z_k, y])] = g[g^{-1}(w), [z_k, y]].$$

3. Constructing Markov partitions.

Definition. A finite cover $\mathscr{C} = \{A_1, \cdots, A_r\}$ of Ω_s by rectangles is a *rectangle partition* provided that $A_i \cap A_j \subset \partial A_i \cap \partial A_j$ for $i \neq j$. $\mathscr{C}$ is a *Markov partition* if, in addition,

$$fW^u(x, A_i) \supset W^u(f(x)\, A_j) \text{ and } fW^s(x, A_i) \subset W^s(f(x), A_j)$$

whenever $x \in \operatorname{int} A_i \cap f^{-1} \operatorname{int} A_j$.

We spend this section giving a proof of the following theorem.

THEOREM 12. *f has Markov partitions.*

Let $\mathcal{A} = \{A_1{}^0, \cdots, A_r{}^0\}$ be a family of rectangles whose interiors cover Ω_s. For this we use Lemma 10, taking $A_i{}^0 = [C_i{}^0, D_i{}^0]$ where $C_i{}^0 \subset W_{\alpha_7}{}^u(x_i)$ with $C_i{}^0 = \overline{\operatorname{int} C_i{}^0}$ in $W_\gamma{}^u(x_i)$, $D_i{}^0 \subset W_{\alpha_7}{}^u(x_i)$ with $D_i{}^0 = \overline{\operatorname{int} D_i{}^0}$ in $W_\gamma{}^s(x_i)$ and $x_i \in A_i{}^0$. The proof of Lemma 10 shows that $\operatorname{diam} A_i{}^0 < \frac{1}{2}\alpha$ where $\alpha = \alpha_5$ throughout this section.

LEMMA 13. *There is an $a > 0$ and a map $F\colon \Omega_s \to \{1, \cdots, r\}$ so that*

$x \in A_{F(x)}{}^0$, $W_a{}^s(z) \subset A_{F(x)}{}^0$ for all $z \in W^u(x, A_{F(x)}{}^0)$, and $W_a{}^u(y) \subset A_{F(x)}{}^0$ for all $y \in W^s(x, A_{F(x)}{}^0)$.

Proof. Let $b > 0$ be a Lebesgue number for $\mathcal{A}$. Since $[\cdot, \cdot]$ is continuous, choose a so small that

$$d([x_1, y_1], [x_2, y_2]) < b$$

whenever $\max(d(x_1, x_2), d(y_1, y_2)) \leqq a$. For $x \in \Omega_s$ choose $F(x)$ so that $B_b(x) \subset A_{F(x)}{}^0$. Suppose $z \in W^u(x, A_{F(x)}{}^0)$ and $w \in W_a{}^s(z)$. Then

$$d([x, w], x) = d([x, w], [x, z]) < b,$$

so $[x, w] \in A_{F(x)}{}^0$. As $A_{F(x)}{}^0$ is a rectangle, 6(b) gives

$$w = [z, [x, w]] \in A_{F(x)}{}^0.$$

Now choose m so large that

$$\sum_{j=1}^{\infty} \lambda^{mj} < \min(1, a/\gamma) = a/\gamma.$$

Set $g = f^m \colon \Omega_s \to \Omega_s$ and $\beta = \lambda^m$.

LEMMA 14. *Let $1 \leqq i \leqq r$. We can find points $y_{i,1}, \cdots, y_{i,s_i}$ in $W_\alpha{}^u(x_i)$ and integers $T_{i,j} \in \{1, \cdots, r\}$ such that*

(a) $g(y_{i,j}) \in W^s(x_{T_{i,j}}, A_{T_{i,j}}{}^0) = D_{T_{i,j}}{}^0$

(b) $g(C_i{}^0) \cap [C_{T_{i,j}}{}^0, g(y_{i,j})] \neq \emptyset$

(c) $g(C_i{}^0) \subset \bigcup_{1 \leqq j \leqq s_i} [C_{T_{i,j}}{}^0, g(y_{i,j})]$

and

(d) *$W_a{}^s(z) \subset A_{T_{i,j}}{}^0$ for all $z \in W^u(g(y_{i,j}), A_{T_{i,j}}{}^0)$.*

Proof. For each $z \in C_i{}^0$ let

$$Y_z = W^u(g(x), A_{F(g(z))}{}^0) = [C_{F(g(z))}{}^0, g(z)].$$

Then $\{g^{-1} Y_z \colon z \in C_i{}^0\}$ is a family of closed subsets of $W_{\frac{1}{2}\alpha}{}^u W_{\frac{1}{2}\alpha}{}^u(x_i) \subset W_\alpha{}^u(x_i)$ whose interiors cover $C_i{}^0$. Let $g^{-1} Y_{z_1}, \cdots, g^{-1} Y_{z_{s_i}}$ cover $C_i{}^0$. Set

$$y_{i,j} = g^{-1}[x_{F(g(z_j))}, g(z_j)].$$

Then $[C_{T_{i,j}}{}^0, g(y_{i,j})] = Y_{z_j}$ where $T_{i,j} = F(g(z_j))$. Since $\operatorname{diam} A_i{}^0 < \frac{1}{2}\alpha$, Lemma 2(c) gives $z_j \in W_{\frac{1}{2}\alpha}{}^u(x_i)$, $g(y_{i,j}) \in W_{\frac{1}{2}\alpha}{}^u(g(z_j))$; from 2(e) and 2(a) one obtains $y_{i,j} \in W_\alpha{}^u(x_i)$. Statements (a)-(d) are all clear from the choice of the $y_{i,j}$.

We now define $C_i^1 = \bigcup_{1 \leqq j \leqq s_i} g^{-1}[C_{T_{i,j}}{}^0, g(y_{i,j})]$ and recursively

$$C_i^n = \bigcup_{1 \leqq j \leqq s_i} g^{-1}[C_{T_{i,j}}{}^{n-1}, g(y_{i,j})].$$

One thing the next lemma shows is that this definition make sense, i.e. $C_{T_{i,j}}{}^{n-1} \subset B_\epsilon(g(y_{i,j}))$.

LEMMA 15. *For* $n \geqq 1$ *and* $y \in W_{2\alpha}{}^s(x_i)$, $[C_i^n, y]$ *has dense interior in* $W_\gamma{}^u(y)$ *and*

$$[C_i^{n-1}, y] \subset [C_i^n, y] \subset W_{\beta^n\alpha}{}^u[C_i^{n-1}, y] \subset W_{(1+\cdots+\beta^n)\alpha}{}^u(y).$$

Proof. We check this first for $n=1$. $gC_i^0 \subset gC_i^1$ is just 14(c). Hence $C_i^0 \subset C_i^1$ and $[C_i^0, y] \subset [C_i^1, y]$. Since $[y_{i,j}, y] \in W_\gamma{}^s(y_{i,j})$,

$$g[y_{i,j}, y] \in W_{\beta\gamma}{}^s(y_{i,j}) \subset W_a{}^s(y_{i,j}) \subset W^s(x_{T_{i,j}}, A_{T_{i,j}}{}^0)$$

because of the choice of m, 14(d) and 14(a). By Lemma 11(b),

$$[C_i^1, y] = \bigcup_{1 \leqq j \leqq s_i} g^{-1}[C_{T_{i,j}}{}^0, g([y_{i,j}, y])].$$

By 14(b) there is a $g(z) \in gC_i^0 \cap [C_{T_{i,j}}{}^0, g(y_{i,j})]$. Then

$$\begin{aligned} v = g([z, y]) = g([z, [y_{i,j}, y]]) &= [g(z), g([y_{i,j}, y])] \\ &\in g[C_i^0, y] \cap [C_{T_{i,j}}{}^0, g([y_{i,j}, y])]. \end{aligned}$$

Since diam $A_{T_{i,j}}{}^0 < \alpha$, Lemma 2(c) shows

$$[C_{T_{i,j}}{}^0, g([y_{i,j}, y])] = [C_{T_{i,j}}{}^0, v] \subset W_\alpha{}^u(v) \subset W_\alpha{}^u(g[C_i^0, y]).$$

Varying j,

$$g[C_i^1, y] \subset W_\alpha{}^u(g([C_i^0, y]).$$

Lemma 2(e) gives $[C_i^1, y] \subset W_{\beta\alpha}{}^u[C_i^0, y]$. Since $[C_i^0, y] \subset W_\alpha{}^u(y)$, $W_{\beta\alpha}{}^u[C_i^0, y] \subset W^u{}_{(1+\beta)\alpha}(y)$ by 2(a).

We now consider $n \geqq 2$ and assume the result for $n-1$. Since $g[y_{i,j}, y]] \in W_\alpha{}^s x_{T_{i,j}})$, we have

$$\begin{aligned} [C_{T_{i,j}}{}^{n-2}, g([y_{i,j}, y])] &\subset [C_{T_{i,j}}{}^{n-1}, g([y_{i,j}, y])] \\ &\subset W_{\beta^{n-1}\alpha}{}^u[C_{T_{i,j}}{}^{n-2}, g([y_{i,j}, y])] \subset W^u{}_{(1+\cdots+\beta^n)\alpha}(y) \\ &\subset W_{2\alpha}{}^u(y) \subset W_\epsilon{}^u(y). \end{aligned}$$

Lemma 11(b) shows

$$g[C_i^n, y] = \bigcup_{1\leqq j\leqq s_i} [C_{T_{i,j}}{}^{n-1}, g[y_{i,j}, y]]$$
$$g[C_i^{n-1}, y] = \bigcup_{1\leqq j\leqq s_i} [C_{T_{i,j}}{}^{n-2}, g[y_{i,j}, y]].$$

Thus

$$g[C_i^{n-1}, y] \subset g[C_i^n, y] \subset W_{\beta^{n-1}\alpha} g[C_i^{n-1}, y].$$

Applying g^{-1} and using 2(e),

$$[C_i^{n-1}, y] \subset [C_i^n, y] \subset W_{\beta^n\alpha}{}^u[C_i^{n-1}, y].$$

Since $[C_i^{n-1}, y] \subset W_{(1+\cdots+\beta^{n}+1)\alpha}{}^u(y)$, 2(a) gives

$$W_{\beta^n\alpha}{}^u[C_i^{n-1}, y] \subset W_{(1+\cdots+\beta^n)\alpha}{}^u(y).$$

Now consider the property of having dense interior. It is true for the C_i^0 because they were so chosen; for $[C_i^0, y]$ it follows by Lemma 8. As dense interior is preserved by taking finite union and application of g^{-1}, inductively one sees that $[C_i^n, y]$ has dense interior.

Define $C_i = \overline{\bigcup_{n\geqq 0} C_i^n} \subset W_{2\alpha}{}^u(x_i)$. Notice that $C_i = \overline{\operatorname{int} C_i}$ in $W_\gamma{}^u(x_i)$.

LEMMA 16. *If $z \in [C_i, W_{2\alpha}{}^s(x_i)]$, then, for some $[C_j, D_j^0]$ containing $g(z)$,*

$$g[C_i, z] \supset [C_j, g(z)].$$

Proof. Let $y = [x_i, z] \in W_{2\alpha}{}^s(x_i)$; $[C_i, z] = [C_i, y]$.

$$\begin{aligned} g[C_i, y] &= \overline{\bigcup_{n>0} g[C_i^n, y]} = \overline{\bigcup_{n>0}\bigcup_{1\leqq j\leqq s_i} [C_{T_{i,j}}{}^{n-1}, g[y_{i,j}, y]]} \\ &= \bigcup_{1\leqq j\leqq s_i} \overline{\bigcup_{n>0} [C_{T_{i,j}}{}^{n-1}, g[y_{i,j}, y]]} \\ &= \bigcup_{1\leqq j\leqq s_i} [C_{T_{i,j}}, g[y_{i,j}, y]] \end{aligned}$$

where Lemma 8 justifies the last step. Since $z \in [C_i, y]$, for some j,

$$g(z) \in [C_{T_{i,j}}, g[y_{i,j}, y]] \subset [C_{T_{i,j}}, D_j^0].$$

Then

$$g[C_i, z] = g[C_i, y] \supset [C_{T_{i,j}}, g[y_{i,j}, y]] = [C_{T_{i,j}}, g(z)].$$

Similarly, working with g^{-1} and the D_i^0's we can find $D_i \subset W_{2\alpha}{}^s(x_i)$ with $D_i^0 \subset D_i = \overline{\operatorname{int} D_i}$ and

LEMMA 17. *If* $z \in [W_{2\alpha}^u(x_i), D_i]$, *then, for some* $[C_j^0, D_j]$ *containing* $g^{-1}(z)$,

$$g^{-1}[z, D_j] \supset [g^{-1}(z), D_j].$$

Set $A_i = [C_i, D_i]$, a rectangle by Lemma 10 since $2\alpha = 2\alpha_5 < \alpha_4$. Lemmas 16 and 17 combine to give:

LEMMA 18. *Let* $z \in A_i$. *Then, for some* A_j *containing* $g(z)$, $g[C_i, z] \supset [C_j, g(z)]$. *For some* A_k *containing* $g^{-1}(z)$, $g^{-1}[z, D_i] \supset [g^{-1}(z), D_k]$.

LEMMA 19. *Let* $U_1, \cdots, U_m$ *be closed with dense interior. Then so is* $\bar{V}$ *where* $V = \bigcap_{i=1}^{k} \operatorname{int} U_i \setminus \bigcup_{i=k+1}^{m} U_i$. *Also* $\operatorname{int} \bar{V} = V$.

Proof. Since V is open, $V \subset \operatorname{int} \bar{V}$. We need only to show that $\operatorname{int} \bar{V} \subset V$. Since $V \subset \operatorname{int} U_i$ for $1 \leqq i \leqq k$,

$$\operatorname{int} \bar{V} \subset \operatorname{int} \overline{\operatorname{int} U_i} = \operatorname{int} U_i.$$

We have left to show that $x \in \operatorname{int} \bar{V}$ lies in no U_i with $k < i \leqq m$. Suppose otherwise. Since $\operatorname{int} U_i \cap V = \emptyset$, we have $\operatorname{int} U_i \cap \bar{V} = \emptyset$. Hence $\operatorname{int} U_i \cap \operatorname{int} \bar{V} = \emptyset$; but this is impossible since $\operatorname{int} \bar{V}$ is an open set containing x and $x \in U_i = \overline{\operatorname{int} U_i}$.

For each A_i define $R_i = \{j : A_i \cap A_j \neq \emptyset\}$ and $V_i = \bigcup_{j \in R_i} A_j$. For $y \in \Omega_s$ let $R(y) = \bigcup_{A_i \ni y} R_i$. Notice that $j \in R(y)$ if and only if $y \in V_j$.

Now diam $A_j < \alpha_4$. Thus, if $j \in R_i$, then $A_j \subset B_{\alpha_4}(A_i) \subset B_{2\alpha_4}(x_i)$. For $y \in V_i$, $d(y, x_i) < 2\alpha_4$ and so

$$[x_i, y] \in W_{1/3\alpha_3}^s(x_i) \text{ and } [y, x_i] \in W_{1/3\alpha_3}^u(x_i).$$

By 6(b), $y = [[y, x_i], [x_i, y]] \in [W_{1/3\alpha_3}^u(x_i), W_{1/3\alpha_3}^s(x_i)]$. Find W_i so that $W_{1/3\alpha_3}^u(x_i) \subset W_i \subset W_{2/3\alpha_3}^u(x_i)$ and W_i is open in $W_\gamma^u(x_i)$. Let $C_i^* = W_i \setminus C_i$. Then, using Lemma 19, $C_i^* = \overline{\operatorname{int} C_i^*}$ in $W_\gamma^u(x_i)$, $\operatorname{int} C_i^* \cap \operatorname{int} C_i = \emptyset$ and $C_i \cup C_i \supset W_{1/3\alpha_3}^u(x_i)$. Similarly, find $D_i^* \subset W_{2/3\alpha_3}^s(x_i)$ so that $D_i^* = \overline{\operatorname{int} D_i^*}$ in $W_\gamma^s(x_i)$, $\operatorname{int} D_i^* \cap \operatorname{int} D_i = \emptyset$ and $D_i \cup D_i^* \supset W_{1/3\alpha_3}^s(x_i)$. Using Lemmas 10 and 7, we see that $\mathfrak{D}_i = \{[C_i, D_i], [C_i^*, D_i^*], [C_i, D_i^*], [C_i^*, D_i]\}$ is a family of rectangles which intersect each other only in their boundaries.

$$V_i \subset [C_i \cup C_i^*, D_i \cup D_i^*] = \bigcup \mathfrak{D}_i.$$

Let $Z = \Omega_s \setminus \bigcup_{\substack{K \in \mathfrak{D}_i \\ 1 \leqq i \leqq m}} \partial K$. Since rectangles have nowhere dense boundaries, Z is a dense open set. For $z \in Z \cap U_i$ let $K_i(z)$ be the unique member of $\mathcal{D}_i$ containing z; $z \in \operatorname{int} K_i(z)$. For $z \in Z$ let $B(z) = \bigcap_{i \in R(y)} \operatorname{int} K_i(z)$.

LEMMA 20. (a). *If $z \in Z$, then $B(z)$ is a rectangle.*

(b). *If $y, z \in Z$ and $B(y) \cap B(z) \neq \emptyset$, then $B(y) = B(z)$.*

Proof. (a). As $z \in B(z)$, $B(z) \neq \emptyset$. By Lemma 19, $B(z) = \operatorname{int} \overline{B(z)}$. We must show that $[x, y] \in \overline{B(z)}$ whenever $x, y \in \overline{B(z)}$. As $[\cdot, \cdot]$ is continuous, it is enough to check this for $x, y \in B(z)$. Then, for each $i \in R(z)$, $x, y \in \operatorname{int} K_i(z)$. As $K_i(z)$ is a rectangle, it follows that $[x, y] \in \operatorname{int} K_i(z)$ (using Lemmas 7, 8 and 9). Hence $[x, y] \in B(z)$.

(b). Since $B(z)$ and $B(y)$ are open and Z is dense, if $B(y) \cap B(z) \neq \emptyset$ then there is a $w \in B(y) \cap B(z) \cap Z$. If $z \in A_i$, then $i \in R(z)$ and $K_i(z) = A_i$; hence $B(z) \subset K_i(z) \subset A_i$ and $w \in A_i$. Pick $A_{i_0} \ni z$. If $w \in A_i$, then

$$w \in A_i \cap B(z) \subset A_i \cap A_{i_0};$$

so $i \in R_{i_0} \subset R(z)$. If we had $z \notin A_i$, then $A_i \cap \operatorname{int} K_i(z) = \emptyset$;

$$w \in A_i \cap B(z) \subset A_i \cap \operatorname{int} K_i(z) = \emptyset,$$

a contradiction. We have shown that w and z belong to the same A_i's; it follows that $R(w) = R(z)$. For $i \in R(z)$, $w \in B(z) \subset K_i(z)$ and so $K_i(w) = K_i(z)$. From this one obtains $B(w) = B(z)$. Similarly, $B(w) = B(y)$.

LEMMA 21. *$\mathcal{B} = \{\overline{B(z)} : z \in Z\}$ is a rectangle partition.*

Proof. Since $B(z)$ is determined by $R(z)$ and the $K_i(z)$, there can be only finitely many different $B(z)$. Since Z is covered by $\mathcal{B}$ and Z is dense in Ω_s, $\mathcal{B}$ covers Ω_s. $\overline{B(z)}$ is a rectangle by 20(a). Finally, if

$$B(y) \cap B(z) = \operatorname{int} \overline{B(y)} \cap \operatorname{int} \overline{B(z)} \neq \emptyset,$$

then $B(y) = B(z)$ by 20(b); hence $\overline{B(y)} \cap \overline{B(z)} \subset \partial \overline{B(y)} \cap \partial \overline{B(z)}$ if $\overline{B(y)} \neq \overline{B(z)}$.

LEMMA 22. *Suppose $z, g(z) \in Z$. Then*

$$gW^u(z, \overline{B(z)}) \supset W^u(g(z), \overline{B(g(z))})$$

and

$$gW^s(z, \overline{B(z)}) \subset W^s(g(z), \overline{B(g(z))}).$$

Proof. Suppose $u \in W_{\gamma}{}^s(v)$. Then, by 2(e).

$$gW_{\gamma}{}^s(v) \subset W_{\beta\gamma}{}^s(g(v)) \subset W_a{}^s(g(v))$$

and by 13 there is an

$$A_{F(g(v))} \supset A_{F(g(v))}{}^0 \supset \{g(u), g(v)\}.$$

Hence, if $g(u) \in A_j$, then $j \in R_{F(g(v))} \subset R(g(v))$.

We now proceed to prove the first statement; the second statement can be proved similarly. Consider $i \in R(z)$ such that $K_i(z) = [C_i, D_i]$ or $[C_i, D_i{}^*]$, i.e. $z \in [C_i, z]$. Let $y = [z, x_i] \in [[C_i, z], x_i] = C_i$. By Lemma 18 pick j so that $g(y) \in A_j$ and

$$g(C_i) = g[C_i, y] \supset [C_j, g(y)].$$

By the first paragraph of this proof, since $y \in W_{\gamma}{}^s(z)$, $j \in R(g(z))$. Since

$$g(z) = [g(y), g(z)] \in [[C_j, g(y)], g(z)] = [C_j, g(z)],$$

we must have $W^u(g(z), K_j(z)) = [C_j, g(z)]$. Now $d(z, x_i) < 2\alpha_4$ because $z \in V_i$ (as was shown earlier, where V_i was defined). Hence $z \in W_{\alpha_3}{}^s(y)$ and 11(b) gives

$$\begin{aligned} gW^u(z, K_i(z)) &= g[C_i, z] \supset g[g^{-1}[C_j, g(y)], z] \\ &= [[C_j, g(y)], g(z)] = [C_j, g(z)] = W^u(g(z), K_j(g(z))). \end{aligned}$$

Taking interiors and intersecting:

$$\begin{aligned} g\Big(\bigcap_{\substack{i \in R(z) \\ z \in [C_i, z]}} \operatorname{int} W^u(z, K_i(z)\Big) &\supset \bigcap_{j \in R(g(z))} \operatorname{int} W^u(g(z), K_j(g(z))) \\ &\supset \operatorname{int} W^u(g(z), \overline{B(z)}). \end{aligned}$$

Suppose we did *not* in fact have

$$gW^u(z, \overline{B(z)}) \supset W^u(g(z), \overline{B(g(z))}).$$

Then, for some $g(w) \in \operatorname{int} W^u(g(z), \overline{B(g(z))}$ and some $i \in R(z)$ with $z \notin [C_i, z]$ we have $w \notin \operatorname{int} W^u(z, K_i(z))$. Let A_{i_0} z. By the preceding paragraph, $w \in W^u(A_{i_0}, z)$. As $z \notin [C_i, z]$, $W^u(z, K_i(z)) = [C_i{}^*, z]$. Since $w \in W^u(A_{i_0}, z) \backslash [C_i{}^*, z] \subset [C_i, z]$,

$$v = [w, x_i] \in [[C_i, z], x_i] = [C_i, x_i] = C_i \subset A_i.$$

By Lemma 18, $gC_i = g[C_i, v] \supset [C_j, g(v)]$ for some $A_j \ni g(v)$. Since $v \in W_{\gamma}{}^s(w)$, the first paragraph of this proof shows $j \in R(g(w))$. Since

$g(w) \in B(g(z))$ the proof of 21(b) shows $R(g(w)) = R(g(z))$; hence $j \in R(g(z))$. Lemma 11(b) gives us $g[C_i, w] \supset [C_j, g(w)]$. Since $z \notin [C_i, z] = [C_i, w]$,

$$g(z) \notin [C_j, g(w)] = [C_j, g(z)].$$

Hence $K_j(g(z))$ is $[C_j^*, D_j]$ or $[C_j^*, D_j^*]$; in either case, $g(w) \in [C_j, g(w)]$ cannot lie in $K_j(z)$—a contradiction.

For $z_0, \cdots, z_{m-1} \in Z$ set $E(z_0, \cdots, z_{m-1}) = \bigcap_{k=0}^{m-1} f^{-k} B(z_k)$. Let

$$\mathscr{E} = \{\overline{E(z_0, \cdots, z_{m-1})} : E(z_0, \cdots, z_{m-1}) \neq \emptyset\}.$$

LEMMA 23. *$\mathscr{E}$ is a rectangle partition.*

Proof. As there are only finitely many possibilities for $B(z_0), \cdots, B(z_{m-1})$, $\mathscr{E}$ is finite. Because $\bigcup_{z \in Z} B(z)$ is open and dense, one sees that

$$\bigcup E(z_0, \cdots, z_{m-1})$$

is also open and dense; hence $\mathscr{E}$ covers Ω_s. Since two $B(z)$'s are disjoint or equal, one sees that two $E(z_0, \cdots, z_{m-1})$'s are also disjoint or equal. By Lemma 19, $\operatorname{int} \overline{E(z_0, \cdots, z_{m-1})} = E(z_0, \cdots, z_{m-1})$; so members of $\mathscr{E}$ have dense interior. Now, either

$$\overline{E(z_0, \cdots, z_{m-1})} \cap \overline{E(z'_0, \cdots, z'_{m-1})} \subset \partial \overline{E(z_0, \cdots, z_{m-1})} \cap \partial \overline{E(z'_0, \cdots, z'_{m-1})}$$

or

$$E(z_0, \cdots, z_{m-1}) \cap E(z'_0, \cdots, z'_{m-1}) = \operatorname{int} \overline{E(z_0, \cdots, z_{m-1})} \cap \operatorname{int} \overline{E(z'_0, \cdots, z'_{m-1})} \neq \emptyset;$$

in this second case $E(z_0, \cdots, z_{m-1}) = E(z'_0, \cdots, z'_{m-1})$.

Finally we check that $[x, y] \in \overline{E(z_0, \cdots, z_{m-1})}$ whenever

$$x, y \in \overline{E(z_0, \cdots, z_{m-1})}.$$

As $[\cdot, \cdot]$ is continuous, we may assume $x, y \in E(z_0, \cdots, z_{m-1})$. Then

$$f^k(x), f^k(y) \in B(z_k).$$

By 20(a) $B(z_k)$ is the interior of the rectangle $\overline{B(z_k)}$, and so (using Lemmas 8 and 9) $[f^k(x), f^k(y)] \in B(z_k)$. By Lemma 11(a), $f^k[x, y] = [f^k(x), f^k(y)]$; so $[x, y] \in E(z_0, \cdots, z_{m-1})$.

LEMMA 24. *Suppose $E, F \in \mathscr{R}$, $z \in \operatorname{int} E \cap f^{-1} \operatorname{int} F$ and $z, f(z), \cdots, f^m(z) \in Z$. Then $fW^u(z,E) \supset W^u(f(z),F)$ and $fW^s(z,E) \subset W^s(f(z),F)$.*

Proof. Since $z \in \operatorname{int} E \cap E(z, f(z), \cdots, f^{m-1}(z))$ and the interiors of the elements of $\mathscr{R}$ are disjoint,

$$E = \overline{E(z, f(z), \cdots, f^{m-1}(z))}.$$

Similarly,

$$F = \overline{E(f(z), f^2(z), \cdots, f^m(z))}.$$

Now $\operatorname{int} W^u(z,E) = \bigcap_{k=0}^{m-1} f^{-k} \operatorname{int} W^u(f^k(z), B(f^k(z)))$ by Lemma 19; as well

$$\begin{aligned}\operatorname{int} W^u(f(z),F) &= \bigcap_{k=0}^{m-1} f^{-k} \operatorname{int} W^u(f^{k+1}(z), B(f^{k+1}(z))) \\ &= \bigcap_{k=1}^{m} f^{-k+1} \operatorname{int} W^u(f^k(z), B(f^k(z))).\end{aligned}$$

So

$$f \operatorname{int} W^u(z,E) = \bigcap_{k=0}^{m-1} f^{-k+1} \operatorname{int} W^u(f^k(z), B(f^k(z))).$$

We will have $f \operatorname{int} W^u(z,E) \supset \operatorname{int} W^u(f(z),F)$ (and so be done) provided that

$$f \operatorname{int} W^u(z, B(z)) \supset f^{-m+1} \operatorname{int} W^u(f^m(z), B(f^m(z))).$$

Applying f^{m-1}, we need

$$f^m \operatorname{int} W^u(z, B(z)) \supset \operatorname{int} W^u(f^m(z), B(f^m(z))),$$

but this comes from Lemma 22 (remember $g = f^m$).

Since $z, f(z), \cdots, f^m(z) \in Z$ is true for an open dense set of z, that $\mathscr{R}$ is a Markov partition (and hence Theorem 12) follows from 24 and the following:

LEMMA 25. *Suppose A and B are rectangles and for some $x \in A \cap f^{-1}B$ we have $fW^u(x,A) \supset W^u(f(x),B)$ and $fW^s(x,A) \subset W^s(f(x),B$. Then this is true for all $x \in A \cap f^{-1}B$.*

Proof. Suppose it is true for $x_0 \in A \cap f^{-1}B$. Consider $x \in A \cap f^{-1}B$ and $f(v) \in W^u(f(x),B)$. Then

$$[f(v), f(x_0)] \in W^u(f(x_0),B) \subset fW^u(x_0,A).$$

By Lemma 11(a), $f[v,x_0] = [f(v), f(x_0)]$; so $[v,x_0] \in W^u(x_0,A)$. As $x \in A$ and A is a rectangle,

$$v=[v,x]=[[v,x_0],x]\in W_\gamma{}^u(x)\cap A=W^u(x,A).$$

Hence $fW^u(x,A)\supset W^u(f(x),B)$.

4. Symbolic representation. Let be any Markov partition for f. For $E,F\in\mathscr{B}$ define

$$t(E,F)=\begin{cases}1 \text{ if } f(\operatorname{int}E)\cap\operatorname{int}F\neq\emptyset\\ 0 \text{ otherwise.}\end{cases}$$

Let $\Sigma=\Sigma(\mathscr{B})$ be the set of all doubly-infinite sequences $\boldsymbol{E}=(E_i)_{i=-\infty}^{+\infty}$ of elements of $\mathscr{B}$ such that

$$t(E_i,E_{i+1})=1 \text{ for all } i\in Z.$$

Giving $\mathscr{B}$ the discrete topology $\prod_Z\mathscr{B}$ is a compact space; it has a metric defined by

$$d(\boldsymbol{E},\boldsymbol{F})=\sum_{i\in Z}2^{-|i|}\omega(E_i,F_i)$$

where

$$\omega(E_i,F_i)=\begin{cases}0 \text{ if } E_i=F_i\\ 1 \text{ if } E_i\neq F_i.\end{cases}$$

Now Σ is a closed subset of this compact metric space and one has the shift homeomorphism $\sigma\colon\Sigma\to\Sigma$ defined by $(\sigma(\boldsymbol{E}))_i=E_{i+1}$. Soon (Proposition 30) we show that σ is topologically transitive; σ is then called a subshift of finite type (see [25, p. 787]). One can check that σ satisfies fact 1 given earlier; actually any subshift of finite type occurs as Ω for some "local" Axiom A diffeomorphism [6]. We restate Lemma 25:

LEMMA 25(a). *If $x\in E\cap f^{-1}F$ with $E,F\in\mathscr{B}$ and $t(E,F)=1$, then $fW^s(x,E)\subset W^s(f(x),F)$ and $fW^u(x,E)\supset W^u(f(x),F)$.*

Definition. Suppose A and B are rectangles. A is a *u-subrectangle* of B if $A\subset B$ and $W^u(x,B)\subset A$ whenever $x\in A$.

LEMMA 26. *Let $E,F\in\mathscr{B}$ with $t(E,F)=1$. If A is a u-subrectangle of E, then $f(A)\cap F$ is a u-subrectangle of F.*

Proof. Choose $w\in\operatorname{int}E\cap f^{-1}\operatorname{int}F$. Let $D=W^s(w,E)\cap A$. Since A is a u-subrectangle of E,

$$A=\bigcup_{z\in D}W^u(z,E)=[W^u(w,E),D].$$

Since A is a rectangle, $D=\overline{\operatorname{int}D}$ in $W_\gamma{}^u(w)$ and $D\neq\emptyset$. We claim

$f(A) \cap F = [W^u(f(w),F),f(D)]$. Since $D \subset W^s(w,E)$ and $fW^s(w,E) \subset F$, $f(D) \subset F$. If $z \in D$, then by 25(a)

$$f(A) \supset fW^u(z,E) \supset W^u(f(z),F) = [W^u(f(w),F),f(z)].$$

Thus $f(A) \cap F \supset [W^u(f(w),F),f(D)]$. On the other hand, if $f(y) \in f(A) \cap F$, then $[w,y] \in D$ and

$$f(y) \in W_\gamma{}^u(f([w,y])) \cap F = W^u(f([w,y]),F) \subset [W^u(f(w),F),f(D)].$$

Since f maps $W_\gamma{}^s(w)$ homeomorphically into $W_\gamma{}^s(f(w))$, $f(D) \neq \emptyset$ has dense interior in $W_\gamma{}^s(f(w))$. Thus Lemma 10 shows that $f(A) \cap F$ is a rectangle. It is clearly a u-subrectangle of F.

LEMMA 27. *If* $\boldsymbol{E} = (E_i)_{i=-\infty}{}^{+\infty} \in \boldsymbol{\Sigma}$. *then* $\bigcap_{i \in Z} f^{-i}E_i$ *consists of a single point.*

Proof. Let $A_n = \bigcap_{i=-n}^{n} f^{-i}E_i$ and $B_{n,k} = \bigcap_{i=-n}^{n+k} f^{k-i-n}E_i$ for $0 \leqq k \leqq 2n$. Since $B_{n,k+1} = f(B_{n,k}) \cap E_{k+1-n}$, one uses the preceding lemma to see inductively that $B_{n,k}$ is a u-subrectangle of E_{k-n}. In particular $f^n(A_n) = B_{n,2n}$ is nonempty. Hence A_n is nonempty. Since $A_0 \supset A_1 \supset \cdots$ is a decreasing sequence of compact sets, $\bigcap_{i \in Z} f^{-i}E_i = \bigcap_{n>0} A_n$ is nonempty.

Suppose x, y are both in this set. Then $f^i(x), f^i,y) \in E_i$ and $d(f^i(x),f^i(y)) < \gamma$ for all $i \in Z$. By Lemma 3, $x = y$.

Definition. The map $\pi\colon \boldsymbol{\Sigma} \to \Omega_s$ is given by

$$\pi(\boldsymbol{E}) = \bigcap_{i \in Z} f^{-i}E_i.$$

THEOREM 28. π *is a continuous surjective map and* $f \circ \pi = \pi \circ \sigma$.

Proof. For $\zeta > 0$ let $D(\zeta)$ be as in Lemma 4. If $\boldsymbol{E}$ and $\boldsymbol{F}$ agree in places $-D(\zeta)$ to $+D(\zeta)$, then, for $|n| \leqq D(\zeta)$,

$$f^n(\pi(\boldsymbol{E})), f^n(\pi(\boldsymbol{F})) \in E_n = F_n$$

and

$$d(f^n(\pi(\boldsymbol{E})), f^n(\pi(\boldsymbol{F}))) \leqq \gamma;$$

so $d(\pi(\boldsymbol{E}),\pi(\boldsymbol{F})) < \zeta$. This shows that π is continuous.

Since $Y = \bigcup \{\operatorname{int} E \colon E \in \mathscr{E}\}$ is an open dense set, $X = \bigcap_{i \in Z} f^{-i}Y$ is dense in Ω_s. If $x \in X$, define $E_i(x) \in$ by $f^i(x) \in E_i(x)$. Then

$$f^i(x) \in \operatorname{int} E_i(x) \cap f^{-1} \operatorname{int} E_{i+1}(x),$$

and so $\boldsymbol{E}(x)=(E_i(x))\in\Sigma$. Since $x\in\bigcap f^{-i}E_i(x)$, $x=\pi(\boldsymbol{E}(x))$. Thus the compact set $\pi(\Sigma)$ contains the dense set X; hence $\pi(\Sigma)=\Omega_s$.

Finally, $f\circ\pi=\pi\circ\sigma$ is clear from the definitions of π and σ.

LEMMA 29. *If $x=\pi(\boldsymbol{E})$ and $f^i(x)\in\operatorname{int}F$, then $E_i=F$.*

Proof. $f^i(x)\in E_i\cap\operatorname{int}F$; but $E_i=F$ or $E_i\cap F\subset\partial F$.

PROPOSITION 30. *σ is topologically transitive. If $f:\Omega_s\to\Omega_s$ is topologically mixing, then so is σ.*

Proof. Let $U,V\subset\Sigma$ be nonempty open sets. For some n and some strings $(F_{-n},\cdots,F_n)$ and $(G_{-n},\cdots,G_n)$ of elements of one has

$$U\supset U_1=\{\boldsymbol{E}\in\Sigma: E_i=F_i \text{ for all } |i|\leqq n\}\neq\emptyset$$

and

$$V\supset V_1=\{\boldsymbol{E}\in\Sigma: E_i=G_i \text{ for all } |i|\leqq n\}\neq\emptyset.$$

Let $U_2=\bigcap_{k=-n}^{n}f^{-k}\operatorname{int}F_k$ and $V_2=\bigcap_{k=-n}^{n}f^{-k}\operatorname{int}G_k$, open sets in Ω_s. Using Lemma 26 as we did in the proof of Lemma 27, one can see that $W=\bigcap_{k=-n}^{n}f^{-k}F_k$ is a rectangle; since $W=\overline{\operatorname{int}W}$ and $W\neq\emptyset$, $\emptyset\neq\operatorname{int}W\subset\bigcap_{k=-n}^{n}f^{-k}\operatorname{int}F_k=U_2$. Similarly, $V_2\neq\emptyset$.

Using Lemma 29, $\pi^{-1}(U_2)\subset U_1$ and $\pi^{-1}(V_2)\subset V_1$. For any m then,

$$\sigma^mU\cap V\supset\sigma^mU_1\cap V_1\supset\sigma^m(\pi^{-1}U_2)\cap\pi^{-1}V_2\supset\pi^{-1}(f^mU_2\cap V_2).$$

Since f is topologically transitive (Smale's Spectral Decomposition Theorem [25]), $f^mU_2\cap V_2\neq\emptyset$ for some m. Then $\sigma^mU\cap V\neq\emptyset$ and σ is topologically transitive.

f topologically mixing means that $f^mU_2\cap V_2\neq\emptyset$ for all sufficiently large m; this implies $\sigma^mU\cap V\neq\emptyset$ for these m also.

Definition. $\partial^s\mathscr{R}=\bigcup\{\partial^sE: E\in\mathscr{R}\}$ and $\partial^u\mathscr{R}=\bigcup\{\partial^uE: E\in\mathscr{R}\}$.

PROPOSITION 31. *$f(\partial^s\mathscr{R})\subset\partial^s\mathscr{R}$ and $f^{-1}(\partial^u\mathscr{R})\subset\partial^u\mathscr{R}$.*

Proof. Consider $x\in\partial^sF$, $F\in\mathscr{R}$. By Lemma 29

$$U_1=\{\boldsymbol{E}\in\Sigma: E_0=F\}\supset\pi^{-1}(\operatorname{int}F).$$

Since $\pi(U_1)$ is compact and $\pi(U_1)\supset\operatorname{int}F$, $\pi(U_1)\supset F$. Thus let $x=\pi(\boldsymbol{E})$

with $E_0 = F$. Since $\mathbf{E} \in \Sigma$, $t(F, E_1) = 1$ and Lemma 25(a) gives $fW^u(x,F) \supset W^u(f(x), E_1)$. If we had $f(x) \notin \partial^s E_1$, then $W^u(f(x), E_1)$ would be a neighborhood of $f(x)$ in $W_\gamma{}^u(f(x))$ and so $W^u(x,F) \supset f^{-1}W^u(f(x),E_1)$ would be a neighborhood of x in $W_\gamma{}^u(x)$, contradicting $x \in \partial^s F$. Hence $f(x) \in \partial^s E_1 \subset \partial^s \mathscr{C}$.

5. Entropy and measures. Let $N_n(f)$ be the number of fixed points of $f^n\colon \Omega_s \to \Omega_s$. The topological entropy (defined in [1]) of f is given by the formula

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(f),$$

as was shown in [6] and [7]. Since $\sigma\colon \Sigma \to \Sigma$ satisfies the same hypotheses,

$$h(\sigma) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma).$$

We recall Theorem 4.7 of [7]:

LEMMA 32. *Suppose $A \subset \Omega_s$ is closed, $f(A) \subset A$ and $A \neq \Omega_s$. Then* $\limsup_{n\to\infty} \frac{1}{n} \log N_n(f \mid A) \leqq h(f \mid A) < h(f)$.

THEOREM 33. $h(f) = h(\sigma)$.

Proof. Since $f\colon \Omega_s \to \Omega_s$ is a quotient of $\sigma\colon \Sigma \to \Sigma$, $h(f) \leqq h(\sigma)$ (see [1]). Let $Y = \bigcup \{\operatorname{int} E\colon E \in \mathscr{C}\}$ and $X = \bigcap_{i\in Z} f^{-i}Y$. Then $f(X) = X$ and π gives a bijection between $\pi^{-1}(X)$ and X; for π is surjective and Lemma 29 shows that $x \in X$ can have only one inverse image under π. Since $f\circ\pi = \pi\circ\sigma$, it follows that $N_n(f \mid X) = N_n(\sigma \mid \pi^{-1}(X))$. Let $A_s = \pi^{-1}(\partial^s \mathscr{C})$ and $A_u = \pi^{-1}(\partial^u \mathscr{C})$. Then $\sigma(A_s) \subset A_s$ and $\sigma^{-1}(A_u) \subset A_u$. Since $\partial \mathscr{C} = \partial^s \mathscr{C} \cup \partial^u \mathscr{C}$,

$$\Omega_s = X \cup \bigcup_{i\in Z} f^i(\partial^s \mathscr{C}) \cup \bigcup_{i\in Z} f^i(\partial^u \mathscr{C}).$$

From this one gets

$$\Sigma = \pi^{-1}(X) \cup \bigcup_{i\in Z} \sigma^i(A_s) \cup \bigcup_{i\in Z} \sigma^i(A_u).$$

Suppose that $p \in \bigcup_{i\in Z} \sigma^i(A_s)$ is periodic. Then $\sigma^n(p) \in A_s$ for some n, and for some $m > 0$

$$p = \sigma^{n+m}(p) \in \sigma^m(A_s) \subset A_s.$$

If $p \in \bigcup \sigma^i(A_u)$ is periodic, then $\sigma^n(p) \in A_u$ for some n, and for some $m > 0$

$$p = \sigma^{n-m}(p) \in \sigma^{-m}(A_u) \subset A_u.$$

Thus the periodic points of σ lie in $\pi^{-1}(X) \cup A_s \cup A_u$.

$$N_n(\sigma \mid \pi^{-1}(X)) \geqq N_m(\sigma) - N_n(\sigma \mid A_s) - N_n(\sigma^{-1} \mid A_u)$$

Applying Lemma 32 to σ and σ^{-1}:

$$\limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma \mid A_s) < h(\sigma)$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma^{-1} \mid A_u) < h(\sigma^{-1}) = h(\sigma).$$

Since

$$\limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma) = h(\sigma),$$

we get

$$\limsup_{n\to\infty} \frac{1}{n} \log N_n(\sigma \mid \pi^{-1}(X)) \geqq h(\sigma).$$

As $N_n(f) \geqq N_n(f \mid X) = N_n(\sigma \mid \pi^{-1}(X))$,

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(f) \geqq h\sigma).$$

We now recall theorems of W. Parry and L. Goodwyn.

THEOREM (Parry). *σ has a unique normalized invariant Borel measure μ_σ such that $h_{\mu_\sigma}(\sigma) = h(\sigma)$. (σ, μ_σ) is a finite ergodic Markov chain.*

Proof. [19], because σ is a subshift of finite type.

THEOREM (Goodwyn). *Let $g\colon W \to W$ be a homeomorphism of a compact space and ρ a g-invariant normalized Borel measure on W. Then $h_\rho(g) \leqq h(g)$.*

We now generalize Parry's theorem. Let $\mu_f = \pi^*(\mu_\sigma)$, i.e. μ_f is the measure on Ω_s given by $\mu_f(S) = \mu_\sigma(\pi^{-1}(S))$ for Borel sets S.

THEOREM 34. *μ_f is the unique normalized f-invariant Borel measure on Ω_s with entropy $h(f)$. (f, μ_f) is measure theoretically isomorphic to the Markov chain (σ, μ_σ). If $f\colon \Omega_s \to \Omega_s$ is C-dense, then (f, μ_f) is a K-automorphism.*

Proof. Let X, A_s, A_u be as in the proof of Theorem 33. Assume ρ is f-invariant on Ω_s with $h_\rho(f) = h(f)$. Suppose $\rho(\partial^s \mathscr{C}) = a > 0$. As ρ is f-invariant and countably additive, setting $W = \bigcap_{n\geqq 0} f^n(\partial^s \mathscr{C})$ we find that $\rho(W) = a$ and $\rho(\bigcup f^i(\partial \mathscr{C}) \backslash W) = 0$. Define μ_1 on W by

$$\mu_1(S) = \frac{1}{a}\rho(S) \text{ for } S \subset W,$$

and μ_2 on Ω_s by

$$\mu_2(T) = \frac{1}{1-a}\rho(T \backslash W).$$

Then $\rho = a\mu_1 + (1-a)\mu_2$; μ_1 and μ_2 are f-invariant and have disjoint support. From the definition of entropy (see [20])

$$h_\rho(f) = ah_{\mu_1}(f) + (1-a)h_{\mu_2}(f).$$

By Goodwyn's theorem, $h_{\mu_1}(f) \leqq h(f)$ and $h_{\mu_2}(f) \leqq h(f \mid W) < h(f)$ (by Lemma 32). Hence $h_\rho(f) < h(f)$, a contradiction. Thus we must have $\rho(\partial^s \mathscr{C}) = 0$; likewise $\rho(\partial^u \mathscr{C}) = 0$.

This proof applied to σ, A_s and A_u instead shows that $\mu_\sigma(A_s) = \mu_\sigma(A_u) = 0$. Hence $\mu_\sigma(\pi^{-1}(X)) = 1$. Since π is injective on $\pi^{-1}(X)$ it follows that (σ, μ_σ) and $(f, \mu_f) = (f, \pi^*\mu_\sigma)$ are measure theoretically isomorphic. In particular

$$h_{\mu_f}(f) = h_{\mu_\sigma}(\sigma) = h(\sigma) = h(f).$$

If ρ is any invariant measure on Ω_s with $h_\rho(f) = h(f)$, we saw that $\rho(X) = 1$. Define ν on Σ by $\nu(\Sigma \backslash \pi^{-1}(X)) = 0$ and $\nu(S) = \rho(\pi(S))$ for $S \subset \pi^{-1}(X)$. The (σ, ν) and (f, ρ) are measure theoretically isomorphic; in particular

$$h_\nu(\sigma) = h_\rho(f) = h(f) = h(\sigma)$$

and Parry's theorem says that $\nu = \mu_\sigma$. One now sees that

$$\rho = \pi^*\nu = \pi^*\mu_\sigma = \mu_f.$$

If $f \colon \Omega_s \to \Omega_s$ is C-dense, then f is topologically mixing (see [7]). By Proposition 30, σ is also topologically mixing. This implies (see [19]) that (σ, μ_σ) is a mixing Markov chain, and hence a K-automorphism.

Remark. In [7] we showed that the periodic points of f were equidistributed with respect to an invariant measure and that this measure had entropy $h(f)$. The above theorem shows that this measure in μ_f. In [7] we proved some ergodic properties and the K-automorphism statement was conjectured.

COROLLARY 35. *If f is a hyperbolic automorphism of a nilmanifold N, then μ_f equals Haar measure μ_N on N.*

Proof. This was proved in [7] (remember the remark above). Another proof is given by seeing directly that $h(f) = h_{\mu_N}(f)$. This was done in [8]. The K-automorphism statement for this example is well-known [3].

COROLLARY 36. *Let f_1, f_2 be hyperbolic automorphisms of nilmanifolds N_1, N_2. Then (f_1, μ_{N_1}) is measure theoretically isomorphic to (f_2, μ_{N_1}) if and only if they have the same entropy.*

Proof. Since N_i is connected, f_i is C-dense by the C-density decomposition theorem of [7]. Hence (f_i, μ_{N_i}) is a mixing Markov chain. Friedman and Ornstein [29] have shown that entropy is a complete invariant for these.

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