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Markov partitions and shadowing for non-uniformly hyperbolic systems with singularities

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Abstract. We show the existence of countable Markov partitions for a large class of non-uniformly hyperbolic systems with singularities including dispersing billiards in any dimension.

0. *Introduction*

Many physical dynamical systems can be modeled by discrete topological Markov chains. Such a modeling, called symbolic dynamics, has many uses, one of the most important being the study of invariant measures. For such systems the statistical physical behavior is governed by the Sinai–Bowen–Ruelle measure (SBR), which for ergodic systems is the unique distribution that is the asymptotic limit of an arbitrary absolutely continuous initial condition. Another measure of interest is the measure of maximal entropy, which governs the growth rate of periodic orbits. Both these measures are specific examples of Gibbs States. Other uses are in deriving statistical properties such as the central limit theorem and estimating decay of auto-correlation functions.

Symbolic representations of physical systems first arose in works of Hadamard (1898) and Morse (1921) in the study of geodesic flows on negatively curved surfaces. In 1965 Smale constructed his celebrated horseshoe and showed that it was conjugate to the full 2-shift. Also in 1967 Adler and Weiss constructed Markov partitions for toral automorphisms. Sinai (1968) constructed them for all Anosov diffeomorphisms and Bowen (1970) extended this to Axiom A systems. In 1975 Bowen gave a much simpler construction for Axiom A systems using a shadowing lemma.

The Markov partition for Axiom A systems is finite. However, hyperbolic dynamical systems with singularities usually have countable Markov partitions. The first example of a countable Markov partition is due to Bunimovich and Sinai for certain classes of dispersive billiards in two-dimensional domains (1980). (See also Bunimovich and Sinai, 1986; Levy, 1986). Recently Bunimovich *et al.*, (1990) have extended this to a wider class of two-dimensional billiards.

In this paper we show how to construct countable Markov partitions for a large class of hyperbolic dynamical systems with singularities. Examples include n-dimensional dispersive billiards, the induced map for Bunimovich type billiards including higher dimensional examples (Bunimovich, 1990; Bunimovich and Krüger, 1992)†, hard core billiards studied by Kubo (1976) and non-billiard systems such as the Lozi maps (Misiurewicz, 1980), geodesic flow on S^2 with the Donnay–Burns–Gerber metric (Donnay, 1988a, b; Burns and Gerber, 1989), Hamiltonian flows on the two-torus with the Donnay–Liverani potentials (1991) and discontinuous hyperbolic toral automorphisms such as

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1+\varepsilon^2 \end{pmatrix} \quad \text{for } 0<\varepsilon\leq 1.$$

In the smooth uniformly hyperbolic case our construction gives rise to finite Markov partitions.

For uniformly hyperbolic systems Anosov (Katok, 1981; Bowen, 1975) has shown a strong shadowing property: every ε-pseudo-orbit is δ-shadowed. For systems with singularities we introduce the weaker notion of essential ε_i-pseudo-orbits and prove that each essential ε_i-pseudo-orbit is δ_i-shadowed by a unique point. We use this fact to construct our Markov partitions. Our construction is inspired by Bowen's construction for the Axiom A case (1975), but several new ideas are needed. In particular Bowen starts with a cover of his space by ε-balls. The cover needed for our case is much more complicated in structure. Because our cover is countable our procedure also differs from Bowen's in the cutting stage, however if we apply our procedure to the uniformly hyperbolic case our cutting reduces to Bowen's cutting.

The dynamical systems we consider must satisfy some simple topological and hyperbolic properties. In particular an a.e. 1-step contraction condition must hold. Such conditions hold for dispersing billiards. We also have a condition on the measure of small neighborhoods of the set of points where our system is discontinuous, has no contraction or infinite contraction. At the cost of a bit more work it should be possible to extend this to a more general contraction condition analogous to some semi-dispersing billiards.

1. *Description of the dynamical systems considered and several definitions*

Let M be a compact n-dimensional $(n>1)$ Riemannian manifold with metric d. Suppose an injective mapping $f: M^+ \to M$ and its inverse $f^{-1}: M^- \to M$ are given where $M^+, M^- \subset M$. Let $S = \overline{M \backslash M^+ \cap M^-}$ and $M^* = M\backslash S$. Since M^* is open we assume that f is a C^2 diffeomorphism on M^*. Furthermore suppose M^* is a dense subset of M. Assume that μ is a f invariant Borel probability measure on M with the property:

$$\exists C>0 \text{ and } a>0 \quad \text{such that } \mu(\mathcal{U}_\varepsilon(S))<C\varepsilon^a \quad \forall \varepsilon>0 \tag{P1}$$

where $\mathcal{U}_\varepsilon(S)$ is an open ε neighborhood of the set S. Furthermore we assume

$$\|D^2 f_x\| < C'\, d(x,S)^{-b} \quad \text{for some } C'>0,\, b>0. \tag{P2}$$

Here for convenience we have assumed that $\operatorname{diam}(M)\leq 1$. For $i\in\mathbb{Z}$ let $S_i =$

† In this case the billiard transformation is represented as a suspension over a countable topological Markov chain.

$\{x \in M^*: f^i x \in S\}$ and let

$$M^*_{s,k} = M^* \Big\backslash \bigcup_{i=0}^{k} S_i, \quad M^*_{u,k} = M^* \Big\backslash \bigcup_{i=0}^{-k} S_i,$$

$$M^{**}_s = M^*_{s,\infty}, \quad M^{**}_u = M^*_{u,\infty}, \quad M^*_k = M^*_{s,k} \cap M^*_{u,k} \quad \text{and} \quad M^{**} = M^{**}_s \cap M^{**}_u.$$

We will consider systems with the following hyperbolic structure: $\forall x \in M^*$, $\exists$ non-empty subspaces $E^{s,1}_x \subset T_xM$, $E^{u,1}_x \subset T_xM$ and $\lambda^s_1(x)$, $\lambda^u_1(x) \in (0, 1)$ such that

$$T_xM = E^{s,1}_x \oplus E^{u,1}_x$$

and

$$\begin{aligned} \|(Df_x)v\| &\le \lambda^s_1(x)\|v\| \quad \forall v \in E^{s,1}_x \\ \|(Df^{-1}_x)v\| &\le \lambda^u_1(x)\|v\| \quad \forall v \in E^{u,1}_x \end{aligned} \tag{+}$$

where $\inf \lambda^{s(u)}_1(x)$ for which equation (+) holds is strictly positive $\forall x \in M^*$. The union $\mathcal{S}^{s,1}_x$, $\mathcal{S}^{u,1}_x$ of the maximal-dimensional 1-step contraction and expansion subspaces $E^{s,1}_x$, $E^{u,1}_x$ form cones ($\mathcal{S}$ is a cone if $x, y \in \mathcal{S} \to ax + by \in \mathcal{S}$ $\forall a, b$ such that $a \cdot b \ge 0$). We additionally require that

$$\mathcal{S}^{s,k}_x = \bigcap_{i=0}^{k-1} Df^{-i} \mathcal{S}^{s,1}_{f^i x} \ne \{0\} \quad \forall x \in M^*_{s,k}$$

$$\mathcal{S}^{u,k}_x = \bigcap_{i=0}^{k-1} Df^{i} \mathcal{S}^{u,1}_{f^{-i} x} \ne \{0\} \quad \forall x \in M^*_{u,k}.$$

Equivalently we could assume the existence of an abstract cone field on M^*_1 which is strictly contracted. $\mathcal{S}^{s,k}_x$ and $\mathcal{S}^{u,k}_x$ are cones consisting of the maximal-dimensional k-step contraction and expansion subspaces $E^{s,k}_x$, $E^{u,k}_x$ of T_xM. That is $\forall x \in M^*_{s,k} \exists E^{s,k}_x \subset T_xM$ and $\exists \lambda^s_1(f^j x) \in (0, 1)$ $0 \le j \le k-1$ such that $\|(Df^j_x)v\| \le \prod_{i=0}^{j-1} \lambda^s_1(f^i x)\|v\|$ $\forall v \in E^{s,k}_x$. An analogous statement holds for k-step contraction along the backward orbits for $x \in M^*_{u,k}$. For $x \in M^*_{s,k} \cap M^*_{u,k}$ we have $T_xM = E^{s,k}_x \oplus E^{u,k}_x$.

For points $x \in M^{**}_s$ there are unique $E^{s,\infty}_x \subset T_xM$ which we denote by E^s_x such that $(Df_x)E^s_x \subset E^s_{fx}$ and for points $x \in M^{**}_u$ the same is true with E^u_x for f^{-1}. The uniqueness of E^s_x and E^u_x follows from Pesin theory (Katok and Strelcyn, 1986).

For fixed subspaces $E^{s,1}_x$, $E^{u,1}_x$ define $\lambda^{s(u)}_{E^{s(u),1}_x}(x) = \inf\{\lambda^{s(u)}_1(x)\}$ where the infimum is taken over all $\lambda^{s(u)}_1(x)$ satisfying equation (+). For $x \in M^{**}$ $\lambda^{s(u)}_2(x) = \lambda^{s(u)}_{E^{s(u)}_x}(x)$ and for $x \in M^*$ we define $\lambda^{s(u)}_2(x)$ by continuity. Finally set $\lambda^{s(u)}(x) = (1 + \lambda^{s(u)}_2(x))/2$. In the following we will always consider subspaces $E^{s,1}_x$, $E^{u,1}_x$ that satisfy equation (+) for $\lambda^s(x)$, $\lambda^u(x)$.

Note, that the ergodic theorem implies that the contraction, $\Pi^n_{i=0} \lambda^s(f^i x)$ along a forward orbit of $x \in M^*_{s,n}$ is dominated by $C(x) \cdot \gamma^n$ for μ-a.e. x where $C(x) > 0$ and $0 < \gamma < 1$. For backward orbits the same holds for $x \in M^*_{u,n}$. The union $\mathcal{S}^{s,k}_x(\mathcal{S}^{u,k}_x)$ of the k-step maximal-dimension contraction spaces $E^{s,k}_x(E^{u,k}_x)$ can be locally smoothly parametrized by a parameter $\alpha(x) \in S^{\operatorname{codim} E^{s,k}_x}(\alpha(x) \in S^{\operatorname{codim} E^{u,k}_x})$ (here S^n denotes the n-sphere). For $\mathcal{U}$ an open neighborhood of x such that $\mathcal{U} \cap (\bigcup^k_{i=0} S_i) = \emptyset$, $E^{s,k}_x(\alpha(x))$ is continuous in x and α. This enables us to define k-stable (unstable) manifolds. The set $\{E^{s,k}_x(\alpha(x))\}$ induces a canonical dim-$E^{s,k}_x$-dimensional foliation of $M^*_{s,k}$ which is denoted by $\mathcal{F}^{s,k}_\alpha$ (respectively $\mathcal{F}^{u,k}_\alpha$ for the foliation induced by

$\{E_x^{u,k}(\alpha(x))\}$). Note that this is not necessarily an f-invariant foliation except for $k=\infty$. Call the element of $\mathcal{F}_\alpha^{s,k}(\mathcal{F}_\alpha^{u,k})$ which contains the point x by $W^{s,k}(x,\alpha)$, $(W^{u,k}(x,\alpha))$. It is clear from the definition that

$$d(x, \partial\overline{W^{s,k}(x,\alpha)}) \geq \min_{0\leq i\leq k-1} d(x, S_i). \tag{$*$}$$

Here ∂ is the boundary of the differentiable part of the stable manifold containing x. For a given choice of $\alpha(x)$ let $[\cdot,\cdot]_\xi^k$ be the mapping from $M_{s,k}^*\times M_{u,k}^*$ into the set of subsets of $M_{s,k+1}^*\cap M_{u,k+1}^*$ given by

$$[x,y]_\xi^k=\left\{z\colon z\in M_{s,k+1}^*\cap M_{u,k+1}^* \text{ s.t. } z\in \bigcup_{\substack{\tilde{x}\in B(x,\xi)\cap M_{s,k}^*\\ \tilde{y}\in B(y,\xi)\cap M_{u,k}^*}} W_{\varepsilon_0(x)}^{s,k}(\tilde{x},\alpha)\cap W_{\varepsilon_0(y)}^{u,k}(\tilde{y},\alpha)\right\}$$

where $B(x,\xi)$ is the ball with center x, diameter ξ greater than or equal to zero and $W_{\varepsilon_0(x)}^{s,k}(\tilde{x},\alpha)$ are all points on $W^{s,k}(\tilde{x},\alpha)$ whose distance from $\tilde{x}$ in the induced metric on $W^{s,k}(\tilde{x},\alpha)$ is less than $\varepsilon_0(x)>0$. Note, that $k\in\{0,1,\ldots,\infty\}$. We introduced the small ξ-balls around x and y to enable the jump from $M_{s(u),k}^*$ to $M_{s(u),k+1}^*$. See Figure 1.

Remark. A slightly different definition which could also be used is

$$[x,y]_\xi^k=\bigcup_{\substack{\tilde{x}\in B(x,\xi)\cap M_{s,k+1}^*\\ \tilde{y}\in B(y,\xi)\cap M_{u,k+1}^*}} W_{\varepsilon_0(x)}^{s,k+1}(\tilde{x},\alpha)\cap W_{\varepsilon_0(y)}^{u,k+1}(\tilde{y},\alpha).$$

Lemma 1. (Local product structure.)

(1) *For all $x\in M_k^*$ there exists $\nu(x,k)>0$ such that*

$$[x,y]_\xi^k\neq\varnothing$$

for all $y\in M_k^*$ satisfying $d(x,y)<\nu(x,k)$ and for all positive ξ less than a constant times $\nu(x,k)$. Here $\varepsilon_0(x)$ can be chosen to be a constant times $\nu(x,k)$.

(2) *For all $x\in M_k^*$, $y\in M_k^*\cap B(x,\nu(x,k))$ where the previous bracket is defined, there exists $\xi_0>0$ such that for all $\xi<\xi_0$*

$$\operatorname{card}(W_{\varepsilon_0(x)}^{s,k}(\tilde{x},\alpha)\cap W_{\varepsilon_0(y)}^{u,k}(\tilde{y},\alpha))=1$$

$$\forall\tilde{x}\in B(x,\xi)\cap M_{s,k+1}^* \quad\text{and}\quad \forall\tilde{y}\in B(y,\xi)\cap M_{u,k+1}^*.$$

The function $\nu(x,k)$ depends on the constants chosen.

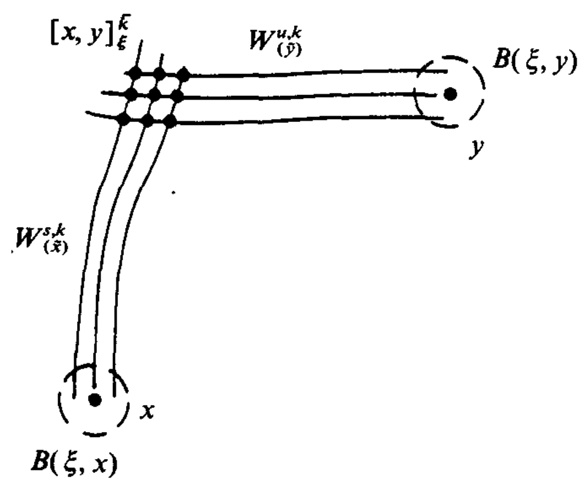

FIGURE 1

Remark. Lemma 1 follows from formula (*) and the existence of a lower bound on the angle between the tangent subspace $E_x^{s,k}(\alpha)$ and $E_x^{u,k}(\alpha)$ for $x \in M^*_{s,k} \cap M^*_{u,k} \cap B(x, \nu)$, the continuity of $\alpha(x)$ and the local continuity of the foliations $\mathscr{F}_\alpha^{s,k}$ and $\mathscr{F}_\alpha^{u,k}$.

LEMMA 2. (Absolute continuity of approximate fibers.) *For all k and $x \in M_k^*$ the following is true: $\forall A > 1\ \exists \eta_0(x, k)$ such that $\forall \eta < \eta_0(x, k) \exists \xi(\eta, k)$ such that $\forall y \in M_k^*$ with $\eta < d(x, y) < \eta_0$ and $\forall \omega \in [x, y]_\xi^k$, $\forall \omega' \in [y, x]_\xi^k$*

$$\frac{1}{A} \leq \frac{d_{w^s}(\omega, x')}{d_{w^s}(\omega', y')} \leq A$$

$$\frac{1}{A} \leq \frac{d_{w^u}(\omega, y'')}{d_{w^u}(\omega', x'')} \leq A$$

$$\forall x' \in B(x, \xi) \cap W^{s,k+1}_{\varepsilon_0(x)}(\omega), \quad \forall y' \in B(y, \xi) \cap W^{s,k+1}_{\varepsilon(y)}(\omega')$$

$$\forall x'' \in B(x, \xi) \cap W^{u,k+1}_{\varepsilon(x)}(\omega'), \quad \forall y'' \in B(x, \xi) \cap W^{u,k+1}_{\varepsilon(y)}(\omega)$$

where $d_{w^{u(s)}}(u, v)$ is the induced distance on $W^{u(s),k}_{\text{loc}}(u)$. (See Figure 2.)

Remark. This lemma is a slight reformulation of the standard absolute continuity property. See for example for the statement and proof in Katok and Strelcyn (1986). For the special case of dispersing billiards Gallavotti (1975) has a formulation close to ours.

Definition. Fix a constant c such that $c > b$ and $c \geq 1$ and a monotone increasing function $g: (0, 1] \to (0, 1]$. We call a sequence of pairs $\{x_i, \varepsilon_i\}_{i \in \mathbb{Z}}$ with $x_i \in M^*$ and $\varepsilon_i > 0$ an essential ε_i-pseudo-orbit (EPO) iff

$$d(fx_i, x_{i+1}) < g(\varepsilon_{i+1})\varepsilon_{i+1}, \quad d(x_i, f^{-1}x_{i+1}) < g(\varepsilon_i)\varepsilon_i, \tag{1}$$

$$\lambda^s_{i,i+1} < \frac{\varepsilon_{i+1}}{\varepsilon_i} < [\lambda^u_{i,i+1}]^{-1} \tag{2}$$

and

$$\varepsilon_i < 1/7\, d(x_i, S)^c. \tag{3}$$

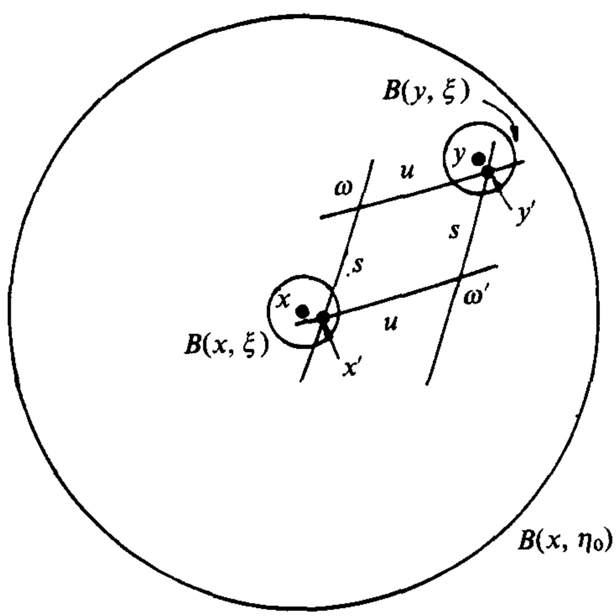

FIGURE 2

Here

$$\lambda^s_{i,i+1}=\frac{1+\max(\lambda^s_i,\lambda^s_{i+1})}{2},\quad \lambda^u_{i,i+1}=\frac{1+\max(\lambda^u_i,\lambda^u_{i+1})}{2},$$

with

$$\lambda^s_i=\lambda^s_i(x_i,\varepsilon_i)=\sup_{x\in B(x_i,\varepsilon_i)}\lambda^s(x)\quad\text{and}\quad \lambda^u_i=\lambda^u_i(x_i,\varepsilon_i)=\sup_{x\in B(x_i,\varepsilon_i)}\lambda^u(x).$$

For a given sequence $\{\delta_i\}$ we say the orbit $\{f^ix\}_{i\in\mathbb{Z}}$ of $x\in M^{**}$ δ_i-shadows an EPO iff:

$$d(x_i,f^ix)<\delta_i\quad\text{and}\quad \delta_i<d(x_i,S).$$

The definition of $\lambda^{s(u)}_{i,i+1}$ guarantees that it is larger than $\frac{1}{2}$ and that $\lambda^{s(u)}_{i,i+1}>\lambda^{s(u)}(x_i)$ and $\lambda^{s(u)}_{i,i+1}>\lambda^{s(u)}(x_{i+1})$. The special function $(1+x)/2$ used can be replaced by any other function that satisfies these properties. Condition (2) implies the following four inequalities

$$\varepsilon_i\lambda^s_i<\varepsilon_{i+1},\quad \varepsilon_i>\varepsilon_{i+1}\lambda^u_i,\quad \varepsilon_i\lambda^u_i<\varepsilon_{i-1},\quad \varepsilon_i>\varepsilon_{i-1}\lambda^s_i. \tag{4}$$

Remark. For billiards there is no known metric for which the billiard map satisfies the conditions of this section. However, the billiard map does satisfy the conditions for a pseudo-metric. This pseudo-metric is given by the projection of the billiard phase space onto the underlying manifold. Nonetheless all the results of this paper are true, when we replace (except in conditions (P1), (P2)) the notion of metric by this special pseudo-metric. The only result which needs a bit of discussion is the uniqueness in Lemma 4.

For two-dimensional dispersing billiards with smooth boundaries there is a constant $0<C_3<1$ such that $\lambda^{s(u)}(x)<C_3\forall x\in M^*$. If we restrict our attention to such systems all the theorems in this article can be proven if we replace condition (2) by the weaker condition (4).

2. *Statement of the results*

LEMMA 3. (Existence of EPOs.) *For all $\varepsilon>0$, for all monotonically increasing functions $g\colon(0,1]\to(0,1]$ and for μ a.e. $x\in M^{**}$ there exists an EPO with the properties that $f^ix\in B(x_i,\varepsilon_i)$ and $\sup\{\varepsilon_i\}<\varepsilon$.*

LEMMA 4. (Shadowing lemma.)

(a) *If μ is an ergodic probability measure then there exists a constant $\varepsilon_{\max}>0$ and a monotone increasing function $g:(0,\varepsilon_{\max}]\to(0,1]$ such that every EPO with $\sup\{\varepsilon_i\}\le\varepsilon_{\max}$ is shadowed by the orbit of a unique point $x\in M^{**}$ with $\delta_i\le\varepsilon_i$.*

(b) *When μ is not necessarily ergodic if $\exists\delta>0$, $C_1>0$ and $p>0$ such that $c>p-1+b$ and $\forall x\in\mathcal{U}_\delta(S^1)\backslash S$*

$$\|D^2fx\|>C_1d(x,S^1)^p$$

then part (a) holds and furthermore the function $g(\varepsilon)$ can be chosen to be $a_1\cdot\varepsilon^{a_2}$ for some positive constants a_1, a_2. Here $S^1=\{x\in S\colon\exists$ a seqeunce $\{x_i\}$, $x_i\in M^*\forall i\in\mathbb{N}$ s.t. $x_i\to x$ and $\lambda^s(x_i)\to 1$ or $\lambda^u(x_i)\to 1\}$.

(c) *If* $\exists C_3<1$ *such that* $\lambda^{s(u)}(x)<C_3 \forall x\in M^*$, *then* $g(\varepsilon)$ *can be chosen to be identically equal to a constant.*

LEMMA 5. (Closing lemma.) *Periodic points are dense in* supp μ.

A subset $R\subset M^{**}$ of poistive μ-measure will be called a rectangle if for every pair of points $x, y\in R$, $[x, y]_0^\infty$ is also in R. Rectangles are sometimes referred to in the literature as parallelograms. Note, that rectangles are generically disconnected sets (products of Cantor sets) which have grid-structure. Suppose η is a finite or countable partition of M^{**} (mod 0) whose elements are rectangles. If $x\in R_i$ we write $W^s_{R_i}(x)=W^{s,\infty}_{\mathrm{loc}}\cap R_i$ and similarly we define $W^u_{R_i}(x)$. η is called a Markov partition if for a.e. $x\in M^{**}$, if $x\in R_i$, $fx\in R_j$ and $f^{-1}x\in R_k$ then the conditions

$$f(W^s_{R_i}(x))\subset W^s_{R_j}(fx),\quad f^{-1}(W^u_{R_i}(x))\subset W^u_{R_k}(f^{-1}x)$$

hold. The main result of our paper is the following theorem.

THEOREM 1. *For the non-uniformly hyperbolic dynamical systems with singularities defined in* § 1 *and satisfying either of the additional assumptions in Lemma* 4 *there exist at most countable Markov partitions of arbitrarily small diameter. Furthermore*

(i) card $\{R_i: f(R_i)\cap R_j\neq\emptyset$ or $f^{-1}(R_j)\cap R_j\neq\emptyset\}<\infty\ \forall j$;

(ii) card $\{R_i:$ diam $(R_i)\geq\delta\}<\infty\ \forall\delta>0$; and

(iii) $\forall i\ \exists U_i$ open and simply connected s.t. $R_i\subset U_i$ and $U_i\cap S=\emptyset$.

Remark. Even if the system does not satisfy the additional assumption of Lemma 4(b), the ergodicity of μ is not needed to prove Theorem 1. We will discuss this in a forthcoming publication on the Local Ergodicity properties of the systems in § 1.

3. *Proofs of Lemma* 3 *and* 4

Proof of Lemma 3. For any $x\in M^{**}$ let $x_i=f^ix$ $i\in\mathbb{Z}$. Without loss of generality let μ be ergodic. For almost all x we will construct an EPO of the form $\{x_i, \varepsilon_i\}$. ε_i will be of the form $\beta_i d_i$ for some $\beta_i\in(0,1)$, where d_i denotes $\frac{1}{7}d(x_i, S)^c$. The ratios $\beta_{i+1}d_{i+1}/(\beta_i d_i)$ must satisfy inequality (2)

$$\lambda^s_{i,i+1}\leq\frac{\varepsilon_{i+1}}{\varepsilon_i}\leq(\lambda^u_{i,i+1})^{-1}.$$

We will construct a half infinite EPO $\{x_i, \varepsilon_i\}$, $i\geq 0$. Using the same procedure we will construct a 'backward' EPO, $\{x_i, \tilde{\varepsilon}_i\}$, $i\leq 0$. Then rescalling the one with larger time zero ε-ball and glueing the two half infinite EPO together gives a whole EPO.

To construct a half infinite forward EPO fix β_0 small enough. Iterating inequality (2) $n-1$ times we get

$$\prod_{i=0}^{n-1}\lambda^s_{i,i+1}\leq\frac{\beta_n d_n}{\beta_0 d_0}\leq\prod_{i=0}^{n-1}(\lambda^u_{i,i+1})^{-1}.$$

Choose $\beta_n=(\prod_{i=0}^{n-1}\gamma^s_{i,i+1})\beta_0 d_0/d_n$ where $\gamma^s_{i,i+1}=(1+\lambda^s_{i,i+1})/2$. This clearly satisfies (2) for all i, and if β_0 is small enough $\beta_i\in(0,1)$ for $0\leq i\leq n$. It remains to show that $\beta_0>0$ can be fixed such that for all $i\geq 1\beta_i\in(0,1)$. This follows immediately from Proposition 1 which is formulated and proven below. Conditions (1) and (3) clearly hold for such EPOs. □

PROPOSITION 1.

$$\inf_n \frac{d_n}{\prod_{i=0}^{n-1} \gamma_{i,i+1}^s} \overset{\text{def}}{=} V(x) > 0 \quad \text{for } \mu\text{-a.e. } x.$$

Proof. C and a are the constants introduced in (P1) and c was introduced in (3). Choose α s.t. $a\alpha > 1$. Then

$$\mu\left(x\colon x \in M^{**}, d(f^n x, S)^c < \frac{1}{n^\alpha}\right) \le C\frac{1}{n^{a\alpha}}$$

and since $1/n^{a\alpha}$ is summable, we can apply the Borel-Cantelli lemma to deduce that for μ-a.e. x $\exists n_0(x)$ s.t. $\forall n > n_0$ $d(f^n x, S)^c > 1/n^\alpha$.

Then for $\gamma_0 < 1$ set $G_{\gamma_0} = \{x \in M^{**}\colon \gamma_{i,i+1}^s < \gamma_0\}$. Fix $\varepsilon > 0$. For γ_0 sufficiently close to 1 we have $\mu(G_{\gamma_0}) > 1 - \varepsilon$. For μ-a.e. x, by the ergodic theorem x visits G_{γ_0} with frequency greater than $1-\varepsilon$. Hence for G_{γ_0} generic x, $\exists n_1(x) > 0$ s.t. $\forall n > n_1(x) \prod_{i=0}^{n-1} \gamma_{i,i+1}^s < (\gamma_0^{1-\varepsilon})^n$. Then for $n > n_2(x) = \max(n_0(x), n_1(x))$

$$\frac{d_n(x)}{\prod_{i=0}^{n-1} \gamma_{i,i+1}^s(x)} > \frac{1/n^\alpha}{(\gamma_0^{1-\varepsilon})^n}. \qquad \square$$

Remark. The EPOs constructed in the proof of Lemma 3 have the property, that

$$\omega^+(\{\varepsilon_i\}) = \omega^-(\{\varepsilon_i\}) = \{0\},$$

when $\omega^\pm$ denote forward and backward limit sets. Such EPO are somewhat pathological, but by glueing together finite segments of different EPOs we can easily construct EPOs with the property that $\omega^+(\{\varepsilon_i\})$ and $\omega^-(\{\varepsilon_i\})$ contain non-zero elements.

More precisely, for a given x and a given n_j we construct an EPO $\{x_i, \varepsilon_i^{n_j}\}$ as above, but starting the construction at the point $f^{n_j}x$. For convenience of notation we have shifted our notation so that $x_0 = x$ for all j. Choose an infinite sequence $\{n_j\}$ with $V(f^{n_j}x) > \varepsilon$. Notice that almost every point x will by the ergodic theorem visit the set $V_\varepsilon = \{y\colon V(y) < \varepsilon\}$ with frequency $\le C\varepsilon^a$. The statement $\mu(V_\varepsilon) < C\varepsilon^a$ follows from assumption (P1) and the ergodic theorem applied to the set G_{γ_0} to get an eventual contraction estimate.

For convenience assume $\varepsilon_0^{n_j} = \varepsilon$ for all j. Notice $\varepsilon_i^{n_j} > \varepsilon_{i+1}^{n_j}$ for $i \ge n_j$ and $\varepsilon_i^{n_j} < \varepsilon_{i+1}^{n_j}$ for $i < n_j$. Set m to be the unique m for which $\varepsilon_m^{n_j} \ge \varepsilon_m^{n_{j+1}}$ and $\varepsilon_{m+1}^{n_j} < \varepsilon_{m+1}^{n_{j+1}}$. Now we form a single EPO by glueing the jth and $(j+1)$st EPOs at m. That is $\tilde{\varepsilon}_i = \varepsilon_i^{n_j}$ for $i \le m$ and $\tilde{\varepsilon}_i = \varepsilon_i^{n_{j+1}}$ for $i > m$. See Figure 3. Glueing all the EPOs $\{x_i, \varepsilon_i^{n_j}\}$ will form an EPO $\{x_i, \varepsilon_i\}$ for which $\omega^+(\{\varepsilon_i\})$ and $\omega^-(\{\varepsilon_i\})$ will contain a number larger than or equal to ε.

The EPOs constructed have another peculiar property, they are centered along an orbit (i.e. $x_i = f^i x$). However, noncentered EPOs also clearly exist. In fact for any given EPO, a whole collection of 'close' EPOs exist, which we call a 'shell'. That is, if $\{x_i, \varepsilon_i\}$ is a given EPO, then $\exists a_i, b_i > 0$ s.t.

$$\forall y_1 \in B(x_i, a_i) \cap M_1^* \quad \text{and} \quad \forall \tilde{\varepsilon}_i \in (\varepsilon_i - b_i, \varepsilon_i + b_i),$$

$\{y_i, \tilde{\varepsilon}_i\}$ is an EPO shadowed by the *same* point that shadows $\{x_i, \varepsilon_i\}$. This statement follows from the openness of conditions (1), (2) and (3) and the local continuity of $\lambda_{i,i+1}^s$ and $\lambda_{i,i+1}^u$.

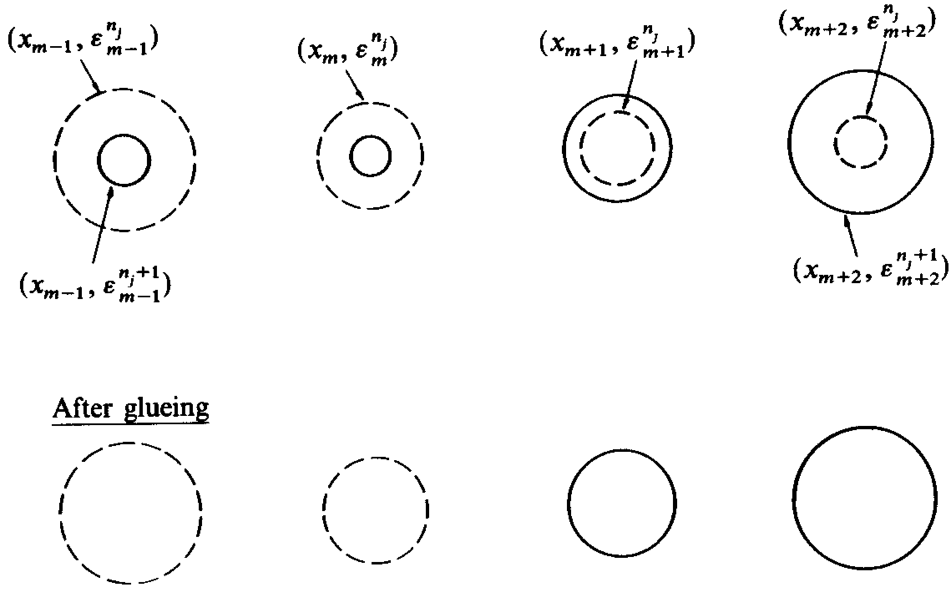

FIGURE 3

Furthermore this property is reflexive in the sense that the EPO $\{x'_i, \varepsilon'_i\}$ is in the shell of the EPO $\{x_i, \varepsilon_i\}$ iff $\{x_i, \varepsilon_i\}$ is in the shell of $\{x'_i, \varepsilon'_i\}$. This follows from the local continuity of a_i and b_i. From now on we call this property the weak shell property.

Proof of Lemma 4.

(a) For the proof of the Lemma we first assume that for each EPO $\exists \delta_0 > 0$ so that:

$$\limsup_{m,n\to\infty} \frac{1}{n+m-1} \sum_{i=-m}^{+n} 1_{\mathcal{U}_\delta(S)}(x_i) \leq C_2 \mu(\mathcal{U}_\delta(S)) \quad \forall \delta < \delta_0. \tag{5}$$

Let $\{x_i^{(0)}, \varepsilon_i\}$ be such an EPO with $\sup_i \varepsilon_i < \varepsilon_{\max}$. The constant $\varepsilon_{\max}$ will be determined later. Introduce a mapping F defined on sequences $\{a_i\}_{i\in\mathbb{Z}}$ of elements of M^* given by $(F\{a_j\})_i = fa_{i-1}\ \forall i \in \mathbb{Z}$. For shortness let $Fa_i = (F\{a_j\})_i$. We will construct a sequence of sequences $\{x_i^{(n)}\}_{i\in\mathbb{Z},n\in\mathbb{N}}$ that uniformly converge in the sup norm to a sequence $\{x_i^{(\infty)}\}_{i\in\mathbb{Z}}$ such that $F\{x_i^{(\infty)}\} = \{x_i^{(\infty)}\}$ and $d(x_i^{(0)}, x_i^{(\infty)}) \leq \varepsilon_i$.

Given the sequence $\{x_i^{(n)}\}$ we inductively define the sequence $\{x_i^{(n+1)}\}$. For each i and n choose a point $\tilde{x}_i^{(n+1)}$ from the set $[x_i^{(n)}, Fx_i^{(n)}]^k_{\xi(x_i^{(n)})}$ and a point $Fx_i^{(n+1)}$ from $[\tilde{x}_i^{(n+1)}, F\tilde{x}_i^{(n+1)}]^k_{\xi(x_i^{(n)})}$ where $k = k(n)$ is the largest integer for which the bracket is defined in the sense of Lemma 1 and $\xi(x_i^{(n)}) > 0$ is small enough and goes sufficiently fast to zero as $n \to \infty$. See Figure 4. Formulas (1) and (3) imply that there exist k and ξ for which the above brackets are defined. Clearly $x_i^{(n+1)} = f^{-1}Fx_{i+1}^{(n+1)}$.

For the proof of the convergence of the sequences $\{x_i^{(n)}\}$ we need to use Lemma 2. For the applicability of Lemma 2 here we need that for any fixed A, $\exists \varepsilon_{\max}$ such that every EPO with $\sup \varepsilon_i < \varepsilon_{\max}$ satisfies $\varepsilon_i < \eta_0(x_i, k)\ \forall k \geq 1$. This follows immediately from combining property (P2) with condition (3) and by observing that

$$\sup_{v\in E_x^{s,k}} \operatorname*{var}_x \|Df_x(v)\| \leq \|D^2 f_x\|.$$

The proof of Lemma 4 proceeds as follows.

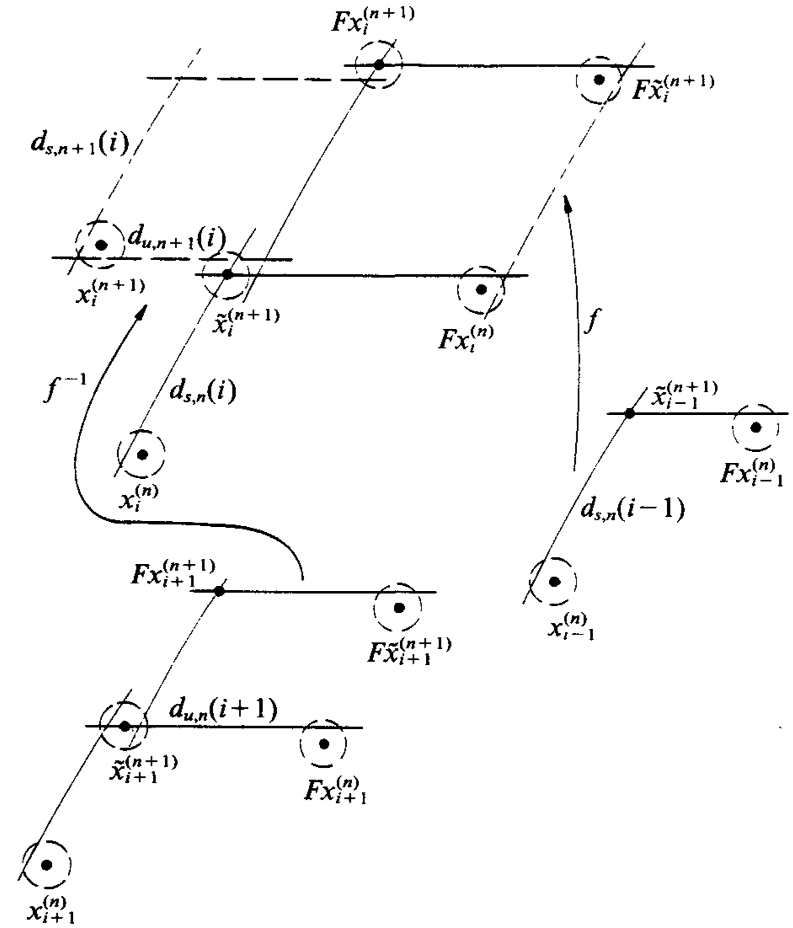

FIGURE 4

We introduce the notion

$$d_s(x, y) = \sup_{\substack{x' \in B(x,\xi) \cap W^{s,k}_{\mathrm{loc}}(\omega) \\ \omega \in [x,y]^k_\xi}} d_{w^s}(x', \omega)$$

(in the notation of $d_s(x, y)$ we supress the obvious ξ, k and α dependence).

For the proof of the Lemma it is convenient to introduce for each given EPO the following abstract dynamical system defined on a countable number of copies of $\mathbb{R}^n$. For each $i \in \mathbb{Z}$ we consider the ball $B_i \overset{\text{def}}{=} B(x_i^{(0)}, \frac{1}{7}d(x_i, S))$ embedded in the ith copy of $\mathbb{R}^n$. If $x \in B_i$ and $fx \in B_{i+1}$ then f is defined in the embedded system as before (respectively f^{-1}). For points $x \in B_i$ such that $fx \notin B_{i-1}$ (respectively $f^{-1}x \notin B_{i-1}$) we extend f continuously to $\mathbb{R}^n$ so that it has uniform hyperbolicity with $\lambda^u = \lambda_i^u$ and $\lambda^s = \lambda_i^s$. The proof proceeds in the artificially constructed system. The reason for introducing this system is that the points $x_i^{(n)}$ and $x_i^{(0)}$ have the same upper bound on their hyperbolicity.

Define A_i^{n+1} to be the infimum of A such that the ball with center $x_i^{(n+1)}$ and radius $\mathrm{const}\,(\varepsilon_i) \cdot d(x_i^{(n+1)}, F\tilde{x}_i^{(n+1)})$ satisfies Lemma 2 for the number A. The $\mathrm{const}\,(\varepsilon)$ appears in the previous statement for the following reason. For $y \in W^s_{\mathrm{loc}}(x)$ we compare $d_{w^s}(x, y)$ to $d(x, y)$. If both points are in an ε-ball satisfying EPO condition (3) then these metrics are equivalent with equivalence constant $\mathrm{const}\,(\varepsilon)$. The following estimates are clear from figure 4 and Lemma 2.

$$d_{s,n+1}(i) \overset{\text{def}}{=} d_s(x_i^{(n+1)}, Fx_i^{(n+1)}) < A_i^{n+1} \cdot \lambda_{i-1}^s d_{s,n}(i-1)$$

$$d_{u,n+1}(i) \overset{\text{def}}{=} d_u(x_i^{(n+1)}, Fx_i^{(n+1)}) < A_{i+1}^{n} \cdot \lambda_{i+1}^u d_{u,n}(i+1)$$

and

$$d(x_i^{(n)}, x_i^{(n+1)}) < A(d_{s,n}(i) + d_{u,n+1}(i)).$$

Note that the above inequalities are not necessarily *a priori* true for the original system, which is the (only) reason we introduced the abstract system.

Let $D_s(i) = \sum_{n=0}^{\infty} d_{s,n}(i)$ and $D_u(i) = \sum_{n=0}^{\infty} d_{u,n}(i)$ then we have $d(x_i^{(\infty)}, x_i^{(0)}) \leq A(D_s(i) + D_u(i))$. Combining the above estimates we see, that

$$D_s(i) \leq \sum_{n=1}^{\infty} \prod_{m=1}^{n} A_{i-m+1}^{m} \lambda_{i-m}^{s} d_{s,0}(i-n) + d_{s,0}(i) \stackrel{\text{def}}{=} \tilde{D}_s(i)$$

and

$$D_u(i) \leq \sum_{n=1}^{\infty} \prod_{m=1}^{n} A_{i+m}^{m-1} \lambda_{i+m}^{u} d_{u,0}(i+n) + d_{u,0}(i) \stackrel{\text{def}}{=} \tilde{D}_u(i).$$

Taking into account $A_i^n \leq A$, $d_{s,0}(i) < \varepsilon_i$ and $\prod_{m=1}^{n} (1 + \lambda_{i-m}^{s})/2)\varepsilon_{i-n} < \varepsilon_i$ (see inequality 2) we get

$$\tilde{D}_s(i) \leq \sum_{n=1}^{\infty} A^n \prod_{m=1}^{n} \frac{2\lambda_{i-m}^{s}}{1+\lambda_{i-m}} \varepsilon_i + \varepsilon_i. \qquad (**)$$

A similar inequality holds for $\tilde{D}_u(i)$. To see that the sum over the products converges we use a standard ergodic theoretic argument. The argument is not needed for the case of smooth two-dimensional dispersing billiards and any other examples for which there is a constant $C_3 < 1$ s.t.

$$\lambda^{s(u)}(x) < C_3 \quad \text{for all } x \in M^*.$$

To prove the convergence of $(**)$ we need to make an estimate (in the original system) along the EPO $\{x_i^{(0)}, \varepsilon_i\}$. Set $G_{\lambda_0}^s = \{x \in M^{**}: 2\bar{\lambda}^s(x)/(1+\bar{\lambda}^s(x)) \leq \lambda_0\}$ with $\bar{\lambda}^s(x) = \sup\{\lambda^s(y): y \in B(x, d^c(x, S)/7\}$. Noting that $M \backslash G_{\lambda_0}^s \subset \mathcal{U}_\delta(S) (\text{mod } 0)$ for some $\delta = \delta(\lambda_0)$ and $\delta \to 0$ as $\lambda_0 \to 1$, we say that an EPO satisfies property $W_N^s(\delta)$ if $\forall n \geq N$

$$\frac{1}{n} \sum_{i=0}^{n-1} \chi_{G_{\lambda_0}^s}(x_i) \geq 1 - C_2 \mu(\mathcal{U}_\delta(S)).$$

Using property (5) every EPO satisfies property $W_N^s(\delta)$ for some N and a sufficiently small δ. Note that the shifted EPO $(\sigma\{x_i, \varepsilon_i\})_n = (\{x_i, \varepsilon_i\})_{n+1}$ satisfies this property for the same δ but for a possibly different N. If μ is an ergodic measure then we regard the set of EPOs which satisfy property (5) for a fixed $\delta_0 = \delta(\lambda_0)$. Suppose δ is so small that $\beta = C_2 \mu(\mathcal{U}_\delta(S)) < 1$. Fix A small enough such that $\lambda_0 A \cdot \lambda_0^{-\beta} < 1$. This determines as discussed previously $\varepsilon_{\max}$. Fix an EPO with $\sup_j \varepsilon_j < \varepsilon_{\max}$. Suppose this EPO has property $W_N^s(\delta(\lambda_0))$ for some N.

Notice that

$$\frac{\lambda_{-m}^{s}}{\lambda_{-m,-m+1}^{s}} < \frac{2\bar{\lambda}^s(x_{-m})}{1+\bar{\lambda}^s(x_{-m})}.$$

Thus by property $W_N^s(\delta)$ and equation $(**)$ we see that

$$\tilde{D}_s(0) \leq \varepsilon_0 \left[\sum_{j=0}^{N} A^j + \sum_{n>N}^{\infty} A^n \lambda_0^{(1-\beta)n} \right] \stackrel{\text{def}}{=} C^s(0)\varepsilon_0. \qquad (***)$$

By an identical argument there exists a constant $C^u(0)$ s.t. $\tilde{D}_u(0) \le C^u(0)\varepsilon_0$. It is clear from formula (***) that the constant $C^s(0)$ only depends on N and $\delta(\lambda_0)$.

From the above it is clear that $d(x_i^{(n)}, Fx_i^{(n)})$ and $d(\tilde{x}_i^{(n)}, F\tilde{x}_i^{(n)})$ both go to zero as $n \to \infty$. Thus the function $k(n)$ in the brackets defining $\tilde{x}_i^{(n)}$ and $x_i^{(n)}$ increases to infinity.

Formal manipulation of the definition $\tilde{D}_s(i)$ shows that

$$\tilde{D}_s(i+1) = A_{i+1}^1 \lambda_i^s \tilde{D}_s(i) + d_{s,0}(i+1).$$

We define a sequence $\{C^s(i)\}$ recursively by

$$C^s(i+1) = A_{i+1}^1\left(\frac{\lambda_i^s}{\lambda_{i,i+1}^s}\right) C^s(i) + \frac{d_{s,0}(i+1)}{\varepsilon_{i+1}}.$$

We need to show that we can choose a function g, such that $\{C^s(i)\}$ is bounded. Notice that there exists a constant only depending on ε, such that $d_{s,0}(i) < \text{const}(\varepsilon_i) g(\varepsilon_i)\varepsilon_i \stackrel{\text{def}}{=} g'(\varepsilon_i)\varepsilon_i$. Define a new sequence $\{\bar{C}^s(i)\}$ by $\bar{C}^s(0) = C^s(0)$ and

$$\bar{C}^s(i+1) = A_{i+1}^1\left(\frac{\lambda_i^s}{\lambda_{i,i+1}^s}\right) \bar{C}^s(i) + g'(\varepsilon_{i+1}).$$

Clearly $C^s(i) \le \bar{C}^s(i)$ for all $i \ge 0$. If

$$0 < g'(\varepsilon_{i+1}) < 1 - A_{i+1}^1\left(\frac{\lambda_i^s}{\lambda_{i,i+1}^s}\right) \tag{****}$$

then $\bar{C}^s(i+1) \le \bar{C}^s(i)$ for all i for which $\bar{C}^s(i) \ge 1$.

Actually g can be chosen to satisfy the stronger inequality:

$$0 < g'(\varepsilon_{i+1}) < \inf\left(1 - A_{i+1}'\left(\frac{\lambda_i^s}{\lambda_{i,i+1}^s}\right)\right)$$

where the infimum is taken over all pairs of balls $\hat{x}_i$, $\hat{x}_{i+1}$ with $\varepsilon(\hat{x}_i) = \varepsilon_i$, $\varepsilon(\hat{x}_{i+1}) = \varepsilon_{i+1}$ that appear in any EPO $\hat{x}$. This follows since $\sup \lambda_i^s/\lambda_{i,i+1}^s < 1$ and $\sup d_{s,0}(i) \to 0$ as $\varepsilon_i \to 0$ where the sup is taken over the same set as above.

The EPO satisfies property $W_N^s(\delta)$ for a bi-infinite sequence of shifts $\{\sigma^{n_i}\}_{i\in\mathbb{Z}}$. This follows from our ergodic theoretic assumption (5) and the fact that $M - G_{\lambda_0}^s \subset \mathcal{U}_\delta(S)$ (mod 0) for some $\delta > 0$. Thus it follows that $C^s(i) \le C^s(0)\ \forall i \in \mathbb{Z}$.

We now show that by a proper choice of the function g, $K_1 = C^s(0) + C^u(0)$ is less than or equal to 1 and thus independent of the N and δ for $\delta < \delta_0$ for which is satisfies property $W_N^s(\delta)$. Once we have shown this it is clear that in the abstractly defined space each EPO is ε_i-shadowed by the orbit $\{f^i x_0^{(\infty)}\}$.

For any L, $0 < L \ll 1$ we can choose g so that $K_1 \le L$. To see this let $\Gamma_{i+1} = \frac{1}{2}(1 + \lambda_i^s/\lambda_{i,i+1}^s)$. We can choose g so that (additionally) $A_{i+1}^1(\lambda_i^s/\lambda_{i,i+1}^s) < \Gamma_{i+1}$ and $g'(\varepsilon_{i+1}) < L/4(1 - \Gamma_{i+1})$ for balls of radius ε_i which satisfy EPO condition (3). Then if $C^s(i) \ge L/2$ it follows that $\bar{C}^s(i+1) \le \bar{C}^s(i)((1+\Gamma_j)/2)$. Thus $\bar{C}^s(i+1) \le C^s(0)\prod_{j=0}^{1}((1+\Gamma_j)/2)$. This shows that for some i_0, $\bar{C}^s(i) \le L/2$ for all $i \ge i_0$. We now show that this happens for a bi-infinite sequence of i_0. Consider a time $n_0 < 0$ for which the given EPO satisfies property $W_N^s(\delta)$ (thus $C^s(n_0) = C^s(0)$). The exact same argument as above shows that $C^s(i) \le L/2$ for all $i \ge i_0 + n$. This proves that

$\bar{C}^s(i) \leq L/2$ for all $i \in \mathbb{Z}$. The proof works for any $\lambda_0 < 1$ where the given EPO has property W_N^s for some N with respect to the set G_{λ_0}.

It follows that the EPO's in the original system are also ε_i shadowed since the geometric construction in the proof took place in the region where the original system and the abstract system coincide and in this region the local stable and unstable fibers used in the proof coincide for the two systems.

We now discuss how the pseudo-metric affects this proof in a case of billiards. The only difference is that $x_i^{(n)}$ converges to $x_i^{(\infty)}$ in the pseuod-metric; and thus $x_i^{(\infty)}$ is a point in the manifold and not in the phase space. However the sequences $\{x_i^{(\infty)}\}$ uniquely determines a sequence $\{x_i'\}$ in the phase space such that $f^i x_0' = x_i'$.

We now show the uniqueness of the point that shadows a given EPO. According to the assumptions in the statement of the shadowing lemma the following holds: for all $i \in \mathbb{Z}$ the point $x_i^{(\infty)}$ has stable $W^s(x_i^{(\infty)})$ and unstable $W^u(x_i^{(\infty)})$ fibers which completely stretch through the ball $B(x_i^{(0)}, \tilde{A}\varepsilon_i/2)$ for some small positive constant $\tilde{A}$ and in this ball there only intersection is the points $x_i^{(\infty)}$. To see that this is the case we show that $\forall \tilde{A} > 0\ \exists \varepsilon_{\max} > 0$ s.t. for any $x, y \in M^{**}$ and any $\varepsilon, 0 < \varepsilon < \varepsilon_{\max}$ where x, y are points inside the EPO ball of radius ε and $y \in W^s_{\text{loc}}(x)$ the following inequality holds

$$d_{w^s}(x, y) < \tilde{A} d(x, y). \tag{$\triangle$}$$

To enable us to compare $d_{w^s}(x, y)$ to $d(x, y)$ we need to use the following estimate

$$\sup \|D^2 f x\| \cdot \varepsilon \leq \text{const}\,(\varepsilon_{\max}) \quad x \in B(x_0, \varepsilon)$$
$$\forall x_0, \varepsilon \quad \text{s.t.} \quad \varepsilon < \tfrac{1}{7} d(x_0 S)^c \quad \text{and} \quad \varepsilon < \varepsilon_{\max}.$$

To see that the estimate is true we use $\|D^2 f x\| < C' d(x, S)^{-b}$ and $\varepsilon < \frac{1}{7} d(x_0, S)^c$. Furthermore const $(\varepsilon_{\max}) \to 0$ as $\varepsilon_{\max} \to 0$. Now fix $K_1 = \tilde{A}/2$, for small enough $\tilde{A}$. Then inequality ($\triangle$) implies that the forward images of

$$W^s_{\text{loc}}(x_i^{(\infty)}) \cap B\left(x_i^{(0)}, \frac{\tilde{A}\varepsilon_i}{2}\right) \quad \text{stay inside } B(x_j^{(0)}, \varepsilon_j) \quad \forall j \geq i.$$

Here by $W^s_{\text{loc}}(x_i^{(\infty)})$ we mean the smooth part of the local stable manifold containing the point $x_i^{(\infty)}$. If our given EPO would be shadowed by more than one point these points would necessarily have a local homoclinic intersection in the balls $B(x_j^{(0)}, \varepsilon_j)$. The above argument shows that the only type of local homoclinic intersection which could happen would be between the smooth local stable and unstable manifolds of $x_i^{(\infty)}$. This is clearly excluded and it shows that the local product structure lemma for $k = \infty$ holds exactly on the Cantor set of points whose local smooth stable and unstable manifolds stretch through $B(x_i^{(0)}, \varepsilon_i)$.

Now we show that EPOs that do not satisfy the additonal condition (5) are also uniquely ε_i shadowed. Fix $\{x_i, \varepsilon_i\}$ an arbitrary EPO. Suppose we are given EPOs $\{y_i^{(n)}, \varepsilon_i^{(n)}\}$, $n = 1, 2, \ldots$, which satisfy condition (5) and $\{y_i^{(n)}, \varepsilon_i^{(n)}\} = \{x_i, \varepsilon_i\}$ for $i \in (-m_n, m_n)$ where $m_n \to \infty$ as $n \to \infty$. Let $y^{(n)}$ be the unique point which shadows the EPO $\{y_i^{(n)} \cdot \varepsilon_i^{(n)}\}$. Choose any limit point y of the sequence $y^{(n)}$. For each $-m_n \leq i \leq m_n$, $f^i y^{(n)} \in B(x_i, \varepsilon_i)$ and thus $f^i y \in B(x_i, \varepsilon_i)$ for all i. Thus the point y

ε_i-shadows the EPO $\{x_i, \varepsilon_i\}$. The uniqueness of y follows as before because of the impossibility of local homoclinic intersections.

(b) Let the class of λ-EPOs be EPOs which satisfy Condition 5 for $\delta(\lambda)$. Let $\{\lambda_i\}$ be a sequence with $\lambda_i \uparrow 1$. Clearly the union of λ-EPOs includes all EPOs which satisfy Condition 5. Suppose $A = A(\lambda_i) = A(\varepsilon_{\max}(\lambda_i))$. Below we will show (in a more delicate computation) that

$$A(\lambda_i) \cdot \lambda_i < 1 + \text{const}\,(d(x, S^1)^{c-b} - d(x, S)^{p-1}) - \text{const}\, d(x, S)^{c-b+p-1}$$

thus if $c > b+p-1$, for $d(x, S)$ small enough $A(\lambda_i)\lambda_i < 1$. This shows that the $\varepsilon_{\max}$ defined by $A(\lambda_0)$ can be used for all λ_i.

The function g needed to satisfy two constraints. The first was

$$A^1_{i+1}\left(\frac{\lambda^s_i}{\lambda^s_{i,i+1}}\right) < \frac{1}{2}\left(1 + \frac{\lambda^s_i}{\lambda^s_{i,i+1}}\right). \tag{i}$$

Define $\Lambda^s(\varepsilon_{i+1}) = \sup\{2\lambda^s_i/(1+\lambda^s_i)\}$ where the sup is taken over all balls $B(x_i, \varepsilon_i/2)$ satisfying $d(x_i, S) < (\varepsilon_i/2)^c$. By monotonicity of $(1+x)/2x$ and the fact that $\lambda^s_i/\lambda^s_{i,i+1} \le \Lambda^s(\varepsilon_{i+1}) < 1$, equation (i) holds if $A^1_{i+1}\Lambda^s(\varepsilon_{i+1}) < \frac{1}{2}(1+\Lambda^s(\varepsilon_{i+1}))$. Using our additional assumption $\|D^2 f(x)\| > C_1 d(x, S^1)^p$, and assuming (WLOG) that $p > 1$ we see that $\lambda^{s(u)}(x) < 1 - C'_1 d(x, S^1)^{p-1}$. Thus

$$\frac{2\lambda^{s(u)}(x)}{1+\lambda^{s(u)}(x)} < 1 - \frac{C^1}{2}\, d(x, S^1)^{p-1}.$$

It follows that

$$\Lambda^s(\varepsilon_{i+1}) < 1 - \frac{C_1}{2}\left(\frac{\varepsilon_{i+1}}{2}\right)^{(p-1)/c}.$$

A straightforward calculation shows the inequality (i) becomes true if

$$A^1_{i+1} < 1 + \frac{C_1}{4}\left(\frac{\varepsilon_{i+1}}{2}\right)^{(p-1)/c}.$$

Using the inequality

$$\|D^2 f(x_{i+1})\|\, g(\varepsilon_{i+1})\varepsilon_{i+1} \le C' d(x_{i+1}, S)^{c-b} g(\varepsilon_{i+1})$$

we see that $A^1_{i+1} - 1$ is bounded by a constant times $g(\varepsilon_{i+1}) d(x_{i+1}, S)^{c-b}$. Thus $g(\varepsilon_i)$ can be chosen to be

$$\frac{C_1}{4}\left(\frac{\varepsilon_{i+1}}{2}\right)^{(p-1)/c}$$

for equation (i) to hold.

The second constraint on the function g was

$$g'(\varepsilon_i) < \tfrac{1}{4}(1-\Gamma). \tag{ii}$$

Since

$$\Gamma_i < \Lambda^s(\varepsilon_i) < 1 - \frac{C_1}{2}\left(\frac{\varepsilon_i}{2}\right)^{(p-1)/c}$$

it follows that if

$$g'(\varepsilon_i) < \frac{C_1}{8}\left(\frac{\varepsilon_i}{2}\right)^{(\rho-1)/c}$$

then inequality (ii) is satisfied. We now claim that the constant relating $g(\varepsilon)$ and $g'(\varepsilon)$ is also polynomial. This constant depends only on the variation in the angle between stable and unstable fibers in an ε-ball satisfying EPO condition (3) and thus can be bounded from above by $C'd(x, S)^{-b}$.

Putting together the two estimates, we see that

$$g(\varepsilon) = \text{const}\ \varepsilon^{(\rho-1)/c+b}.$$

(c) This part follows trivially. □

To prove Lemma 5 we need the following auxiliary Lemma.

LEMMA 6. *For almost every x there exists an EPO which is shadowed by x with the property that each pair (x_i, ε_i) in the EPO occurs infinitely often.*

Proof of Lemma 5. (Closing Lemma.) Let $\mathcal{U}(x)$ be an arbitrary open neighborhood of x in supp μ. Choose a point y in $\mathcal{U}(x) \cap M^{**}$ and construct an EPO satisfying Lemma 6 shadowed by y which has the property that $B(x_0, \varepsilon_0)$ is contained in $\mathcal{U}(x)$. Let n be a recurrence time of (x_0, ε_0) in the EPO. Construct now an EPO by periodically repeating the string $\{(x_0, \varepsilon_0), \ldots, (x_{n-1}, \varepsilon_{n-1})\}$. If n is large enough then (using formula (***)) this EPO is shadowed by a periodic point z in $M^{**} \cap \mathcal{U}(x)$. □

Proof. (Lemma 6.) Let $\{x_i, \varepsilon_i\}$ be a centered EPO shadowed by a point x (that is $x_i = f^i x$). We show that we can construct another EPO $\{x'_i, \varepsilon'_i\}$ which is also shadowed by x and has the property that for each $i \in \mathbb{Z}$, there is a sequence $n_j(i), j \in \mathbb{Z}$, so that $B(x'_{n_j}, \varepsilon'_{n_j}) = B(x'_i, \varepsilon'_i)$. For each i consider a bi-infinite sequence $m_j(i) \in \mathbb{Z}$ defined by $x_{m_j(i)} \in \mathcal{U}(x_i)$ where $\mathcal{U}(x_i)$ are sufficiently small neighborhoods of x_i which are contained in $B(x_i, a_i)$ where a_i was defined by the weak shell argument. These sequences are bi-infinite by the recurrence of x. Now $\omega^+(\{\varepsilon_k\}) = \omega_i^+(\{\omega_j^+(\varepsilon_{m_j(i)})\})$ where ω_j^+ is the ω-limit set as $j \to \infty$, thus for some fixed i^+, $\omega_j^+(\{\varepsilon_{m_j(i^+)}\})$ contains a non-zero element. But, then by the EPO property (2) $\omega_j^+(\{\varepsilon_{m_j(i)}\})$ contains a non-zero element $\tilde{\varepsilon}_i^+$ for all $i \in \mathbb{Z}$. Similarly $\omega_j^-(\{\varepsilon^-_{m_j(i)}\})$ contains a non-zero element $\tilde{\varepsilon}_i^-$ for all $i \in \mathbb{Z}$. Suppose without loss of generality that $\tilde{\varepsilon}_0^+ \geq \tilde{\varepsilon}_0^-$ and set $\beta = \tilde{\varepsilon}_0^+/\tilde{\varepsilon}_0^-$. We first defined a new EPO $\{x_i, \varepsilon_i^1\}$ which is still a centered EPO by $\varepsilon_i^1 = \varepsilon_i$ for $i \leq 0$ and $\varepsilon_i^1 = \beta^{-1}\varepsilon_i$ for $i > 0$. For ease of notation let $m_j = m_j(0)$. Clearly $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0^- = \beta^{-1}\tilde{\varepsilon}_0^+ \in \omega^+(\{\varepsilon_{m_j}\}) \cap \omega^-(\{\varepsilon_{m_j}\})$. Now define a new EPO $\{x_i^2, \varepsilon_i^2\})$ from $\{x_i, \varepsilon_i^1\}$ by using the weak shell property to replace a bi-infinite subsequence of the balls $(x_{m_j}, \varepsilon^1_{m_j})$ by the fixed ball $(x_0, \tilde{\varepsilon}_0)$.

Choose sufficiently small neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_{m_1-1}$ of the orbit string $\{fx, \ldots, f^{m_1-1}x\}$. By recurrence the orbit of $(x, fx, \ldots, f^{m_1}x)$ under $f \times \cdots \times f$ recurs infinitely often in both directions to the neighborhood $\mathcal{V}_1 = \mathcal{U}(x_0) \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{m_1-1} \times \mathcal{U}(x_0)$. Whenever the orbit recurs through this finite string of neighborhoods we can replace the corresponding string by

$$(x_0^2, \varepsilon_0^2), (x_1^2, \varepsilon_1^2), \ldots, (x_{m_1-1}^2, \varepsilon_{m_1-1}^2), (x_{m_1}^2, \varepsilon_{m_1}^2)$$

remembering $(x_0^2, \varepsilon_0^2) = (x_{m_1}^2, \varepsilon_{m_1}^2) = (x_0, \tilde{\varepsilon}_0)$. Call the resulting EPO $\{x_i^3, \varepsilon_i^3\}$.

Now we continue inductively. Suppose and EPO $\{x_i^{k-1}, \varepsilon_i^{k-1}\}$ was constructed on the previous step. Consider the orbit string $\{f^{m_k+1}x, \ldots, f^{m_{k+1}-1}x\}$. If this string was already replaced in one of the previous steps proceed to $k+1$. Otherwise we can place this string infinitely often into the EPO $\{x_i^{k-1}, \varepsilon_i^{k-1}\}$. The times the orbit of $(x, fx, \ldots, f^{m_{k+1}-m_k}x)$ recurs to the set $\mathcal{V}_k = \mathcal{U}(x_0) \times \mathcal{U}_{m_k+1} \times \cdots \times \mathcal{U}_{m_{k+1}-1} \times \mathcal{U}(x_0)$ must be disjoint from the recurrence times to the $\mathcal{V}_j$, $1 \le j < k$, otherwise the replacement would have occurred at a previous step. The inductive procedure produces the desired EPO. □

4. *Proof of the Theorem* 1

Before proving the theorem we will introduce several new notions and prove certain facts about them.

Introduce the following convenient notation: $\tilde{x}$ is an EPO and $\pi(\tilde{x}) = x$ is the point that shadows it. For the construction of the Markov-partition we need two further restrictions on the class of EPOs. First we define the class of Markov EPOs (MEPOs) which have a more stringent requirement on the lower and upper bounds of the ratio's $\varepsilon_{i+1}/\varepsilon_i$. Second we restrict to MEPOs which are built from a special ball-set. The existence and properties of this special MEPO-ball-set is stated in Lemma 7. MEPOs are EPOs that satisfy a certain uniformized contraction condition. This uniformization enables us to prove the existence of MEPOs, all of whose balls are in some sense large. This is contained in Propositon 2, which is the key technical proposition which enables us to construct the Markov Partition.

Let $M_i = \{x \in M^*: d(x, S) \ge (7r_i)^{1/c}\}$, $\tilde{M}_i = \{x \in M^*: d(x, M_i) < 4r_i\}$, $\Lambda_i = \inf\{\lambda_2^s(x), \lambda_2^u(x): x \in \tilde{M}_i\}$ and $\lambda_i = (1 + \sup\{\lambda^s(x), \lambda^u(x): x \in \tilde{M}_i\})/2$ where r_0 is a fixed sufficiently small number. For $i \in \mathbb{N}$ set

$$r_{i+1} = \inf\{r^*: r^* \ge r_i\sqrt{\lambda_i}\}.$$

An EPO (x_i, ε_i) is called an MEPO if it additionally satisfies

$$h(x_i, x_{i+1}) < \varepsilon_{i+1}/\varepsilon_i < h(x_i, x_{i+1})^{-1} \tag{2'}$$

where $h(x_i, x_{i+1}) = \lambda_j$ if j is the smallest integer such that

$$B(x_i, \varepsilon_i) \subset \tilde{M}_j \quad B(x_{i+1}, \varepsilon_{i+1}) \subset \tilde{M}_j.$$

LEMMA 7. $\exists \mathcal{B} = \{B_i\}_{i \in \mathbb{N}} \subset M^* \times (0, \varepsilon_{\max}]$ *satisfying* $\forall \beta > 0$ card $\{B_i : \varepsilon(B_i) \ge \beta\} < \infty$ s.t. *for* μ-a.e. $x \in M^{**}$ $\exists$MEPO $\tilde{x} \in \pi^{-1}(x)$ s.t. $\tilde{x}_i \in \mathcal{B}$ $\forall i \in \mathbb{Z}$.

Proof of Lemma 7. Choose $\gamma_i < g(r_i)r_i/2$ so small that for $x, y \in \tilde{M}_i$ and $d(x, y) < \gamma_i$ we have $d(fx, fy) < \Lambda_i g(r_i/2)r_i/4$ and $d(f^{-1}x, f^{-1}y) < \Lambda_i g(r_i/2)r_i/4$. Choose further γ_i-dense subsets in M_i and let $\{B_j^i\}_{1 \le j \le N_i} = \mathcal{B}_i$ the set of balls of radius r_i with centers in this γ_i-dense set. The set $\mathcal{B} = \{B_j^i\}_{i \in \mathbb{N}, 0 < j \le N_i}$ clearly forms a cover of M^*. The stringent conditions imposed on the density of this cover will be needed in Proposition 2 to enable us to approximate certain MEPOs.

The proof of the existence of MEPOs restricted to $\mathcal{B}$ will be along the same lines as the proof of Lemma 3. As before we will construct forward half infinite MEPOs with monotonically decreasing ε_i. Glueing this with backward MEPOs will give MEPOs and these can be glued with each other to remove the monotonicity.

We now introduce some notation: let $\lceil a \rceil = r_i$ if $r_{i+1} < a \leq r_i$ and denote by $c(B)$ the center of a ball B. Let $B(x) = \{B \in \mathcal{B}: x \in B$ and $d(x, c(B)) < \gamma_i$ for $B \in \mathcal{B}_i\}$. It is clear that for all $x \in M^{**}$ $\exists i_{\min}(x)$ s.t. $B(x) \cap \mathcal{B}_i \neq \emptyset$ $\forall i \geq i_{\min}(x)$. For each $i \geq i_{\min}(x)$ choose one $B_i \stackrel{\text{def}}{=} B_i^x \in B(x) \cap \mathcal{B}_i$.

Fix $i_0 \geq \max\{i_{\min}(x), i_{\min}(fx)\}$. Define i_{k+1} by $\varepsilon_{i_{k+1}} = \lceil \lambda_{j_k} \varepsilon_{j_k} \rceil$ with $\varepsilon_{i_0} = r_{i_0}$ where j_k is the smallest integer such that $B_{i_k}^{f^k(x)} \subset \tilde{M}_{j_k}$ and $B_{i_{k+1}}^{f^{k+1}(x)} \subset \tilde{M}_{j_k}$. For fixed k, if i_0 is sufficiently large, then $f^k(x) \in M_{i_k}$. Note that in the following i denotes the layer index and l the orbit index. We show now, that the balls $B_{i_l}^{f^l(x)}$ for $0 \leq l \leq k$ satisfy the MEPO-conditions. MEPO-condition (2′) for the MEPO (x_l, ε_{i_l}) follows from the inequalities

$$\lambda_{j_l}\varepsilon_{i_l} \leq \lceil \lambda_{j_l}\varepsilon_{i_l} \rceil = \varepsilon_{i_{l+1}} \leq \varepsilon_{i_l}.$$

For ease of notation set $\tilde{B}_l^x = B_{i_l}^{f^l(x)}$. MEPO-condition (1) follows from the inequality

$$\begin{aligned} d(f(c(\tilde{B}_l^x)), c(\tilde{B}_{l+1}^x)) &\leq d(f(c(\tilde{B}_l^x)), f^{l+1}(x)) + d(f^{l+1}(x), c(\tilde{B}_{l+1}^x)) \\ &\leq g\left(\frac{\varepsilon_{i_l}}{2}\right)\frac{\varepsilon_{i_l}}{4} + g(\varepsilon_{i_{l+1}})\frac{\varepsilon_{i_{l+1}}}{2} \leq g(\varepsilon_{i_{l+1}})\varepsilon_{i_{l+1}}. \end{aligned}$$

MEPO-condition (3) follows from the definition of the cover.

Analogously to what was shown in Lemma 3 there exists $i_0 > 0$ such that the above holds for $k = \infty$. The proof of this fact is an exact copy of the proof in Lemma 3 with the only changes being that now the contractions are considered over the $\tilde{M}_{j_k}$. In an analogous fashion we can construct a backwards half-infinite MEPO. Glueing the forward and backwards MEPOs together the lemma is proved. □

Define $\varepsilon^M(x) = \max\{\varepsilon(B_i): \exists \tilde{x} \in \pi^{-1}(x)\ \tilde{x}_0 = B_i\}$ and $\lfloor a \rfloor = r_{i+1}$ if $r_{i+1} \leq a < r_i$. Set

$$T(\tilde{x}_0) = \{z \in M^{**}: z \in \pi^{-1}(\tilde{z}) \text{ with } \tilde{z}_0 = \tilde{x}_0\}$$

and

$$W^s_{T(\tilde{x}_0)}(x) = W^s_{\text{loc}}(x) \cap T(\tilde{x}_0).$$

PROPOSITION 2.

(a) *For* μ-a.e. $x \in M^{**}$ $\forall$MEPO $\hat{x} \in \pi^{-1}(x)$ s.t. $\varepsilon(\hat{x}_0) \geq \lfloor \frac{1}{2}\varepsilon^M(x) \rfloor$ $\exists$MEPO $\tilde{x} \in \pi^{-1}(x)$ s.t.

$$\tilde{x}_i \in \mathcal{B}, \tilde{x}_0 = \hat{x}_0 \quad \text{and } \varepsilon(\tilde{x}_i) \geq \lfloor \tfrac{1}{2}\varepsilon^M(f^i x) \rfloor \quad \forall i \in \mathbb{Z}.$$

(b) $\forall \tilde{x} \in \pi^{-1}(x)$ s.t. $\tilde{x}_i \in \mathcal{B}$ and $\varepsilon(\tilde{x}_i) \geq \lfloor \frac{1}{2}\varepsilon^M(f_x^i) \rfloor$ $\forall i \in \mathbb{Z}$, $\forall y \in W^s_{T(\tilde{x}_0)}(x)$ and $\forall \hat{y} \in \pi^{-1}(y)$ s.t. $\hat{y}_0 \cap \tilde{x}_0 \neq \emptyset$, $\hat{y}_1 \cap \tilde{x}_1 \neq \emptyset$ and $\varepsilon(\hat{y}_0) \geq \lfloor \frac{1}{2}\varepsilon^M(x_0) \rfloor$

$$\exists \tilde{y} \in \pi^{-1}(y) \text{ s.t.}$$

$$\tilde{y}_0 = \hat{y}_0, \quad \tilde{y}_i \cap \tilde{x}_i \neq \emptyset, \quad y_i \in \mathcal{B}$$

and

$$\varepsilon(\tilde{y}_i) \geq \lfloor \tfrac{1}{2}\varepsilon^M(f_x^i) \rfloor, \quad i \geq 0.$$

Remark. The same holds for $y \in W^u_{T(\tilde{x}_0)}(x)$ and $i \leq 0$.

Proof. (a) Let $\mathscr{P}(x)=\{B_i\in\mathscr{B}_i: \exists\text{MEPO } \tilde{x}\in\pi^{-1}(x) \text{ with } \tilde{x}_0=B_i \text{ and } \varepsilon(B_i)\geq \lfloor\varepsilon^M(x)/2\rfloor\}$.

All MEPOs in this proof will be shadowed by the same point x. First we will construct a MEPO $\bar{x}$ s.t. $\varepsilon(\bar{x}_n)\geq\lfloor\varepsilon^M(f^nx)/2\rfloor$ for all n, but $\bar{x}_n$ is not necessarily from the collection $\mathscr{B}$. This MEPO is constructed inductively. Set $\bar{x}_0=\hat{x}_0$. The centers $c(\bar{x}_i)$ are chosen to be equal to the centers $c(\hat{x}_i)$. The radii are defined inductively. Assume $\varepsilon(\bar{x}_0),\ldots,\varepsilon(\bar{x}_n)$ are already given.

Consider $n+1$. If $\hat{x}_{n+1}\in\mathscr{P}(f^{n+1}x)$ then set $\bar{x}_{n+1}=\hat{x}_{n+1}$. If not then $\varepsilon(\hat{x}_{n+1})<\lfloor\varepsilon^M(f^{n+1}x)/2\rfloor=r_j$ for some j. We claim that the pair of balls $\{(c(\hat{x}_{n+1}),r_j),\hat{x}_n\}$ is a legal MEPO transition. EPO condition (1) is trivially satisfied. To see that MEPO condition (2′) (and thus EPO condition (2)) is satisfied we note that the ball $(c(\hat{x}_{n+1}),r_j)$ is contained in $\tilde{M}_k$ where k is given by $\varepsilon^M(f^{n+1}x)=r_k$. See Figure 5. Similarly the ball $\hat{x}_n$ belongs to $\tilde{M}_{k'}$ where $\varepsilon^M(f^nx)=r_{k'}$. Thus

$$\max(\lambda_k,\lambda_{k'})\leq\frac{\varepsilon(\hat{x}_{n+1})}{\varepsilon(\hat{x}_n)}\leq\frac{r_j}{\varepsilon(\hat{x}_n)}\leq\frac{\lfloor\varepsilon^M(f^{n+1}x)/2\rfloor}{\lfloor\varepsilon^M(f^nx)/2\rfloor}.$$

If $\varepsilon^M(f^{n+1}x)/\varepsilon^M(f^nx)\leq 1$ then $\lfloor\varepsilon^M(f^{n+1}x)/2\rfloor/\lfloor\varepsilon^M(f^nx)/2\rfloor\leq 1$ and MEPO condition (2′) holds. If $\varepsilon^M(f^{n+1}x)/\varepsilon^M(f^nx)>1$ then $\lfloor\varepsilon^M(f^{n+1}x)/2\rfloor/\lfloor\varepsilon^M(f^nx)/2\rfloor<\varepsilon^M(f^{n+1}x)/\varepsilon^M(f^nx)<(\max(\lambda_k,\lambda_{k'}))^{-1}$ and MEPO condition (2′) also holds. EPO condition (3) in this case can be written $\lfloor\varepsilon^M(f^{n+1}x)/2\rfloor<\frac{1}{7}d(c(\hat{x}_{n+1}),S)$. From our definitions (see figure 5) it is clear that

$$d(c(\hat{x}_{n+1}),S)>d(x^M_{n+1},S)-\tfrac{3}{2}\varepsilon^M(f^{n+1}x)$$

where x^M_{n+1} is the center of some ball of radius $\varepsilon^M(f^{n+1}x)$ in $\mathscr{P}(f^{n+1}x)$. EPO condition (3) follows from the equation

$$d(x^M_{n+1},S)>7\varepsilon^M(f^{n+1}x)$$

which follows from EPO condition (3) for the maximal ball.

The backwards half of the MEPO is constructed by a similar induction and thus the proof of the existence of $\bar{x}$ is finished.

Now we show that $\bar{x}$ can be well approximated by a MEPO $\tilde{x}$ using balls from the cover $\mathscr{B}$ and with $\varepsilon(\tilde{x}_i)=\varepsilon(\bar{x}_i)$ for all i. This will be shown inductively. $\tilde{x}_0=\bar{x}_0\in\mathscr{B}$

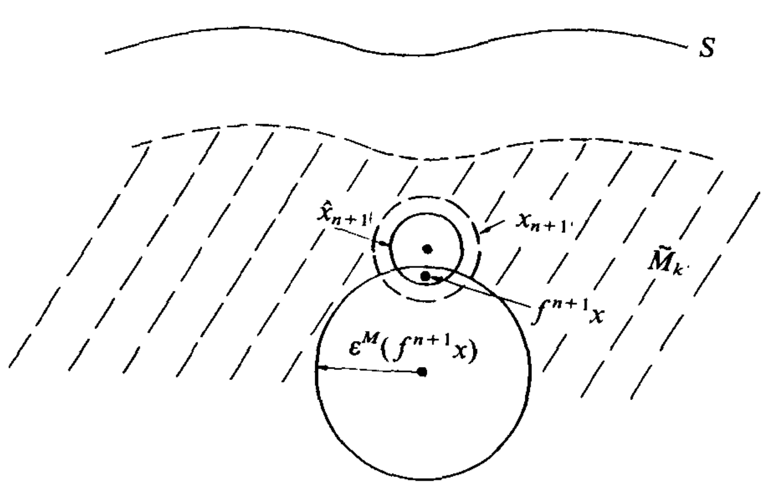

FIGURE 5

by assumption. Assume $\tilde{x}_0, \ldots, \tilde{x}_n \in \mathcal{B}$ and $\ldots \bar{x}_{-1}\tilde{x}_0, \ldots, \tilde{x}_{n-1}, \tilde{x}_n, \bar{x}_{n+1}, \bar{x}_{n+2}, \ldots$ is a MEPO. We show that $\bar{x}_{n+1}$ can be replaced by a ball $\tilde{x}_{n+1} \in \mathcal{B}$. Set $r_j = \varepsilon(\bar{x}_{n+1})$. Let $g \cdot \bar{x}_{n+1} = B(c(\bar{x}_{n+1}), g(\varepsilon(\bar{x}_{n+1})) \cdot \varepsilon(\bar{x}_{n+1}))$. Consider the set

$$H = \{y \in M^*: y \in g \cdot \bar{x}_{n+1}, f^{-1}y \in g \cdot \tilde{x}_n \text{ and } fy \in g \cdot \bar{x}_{n+2}\}.$$

By our contraction and expansion EPO assumptions it is clear that there exists a ball U of radius γ_j such that

$$c(\bar{x}_{n+1}) \in U \subset H.$$

By the density assumption of the cover $\mathcal{B}$ (Lemma 7) there is a ball $\tilde{x}_{n+1} \in \mathcal{B}_j$ with $c(\tilde{x}_{n+1}) \in \mathcal{U}$.

(b) The choice of constant $\frac{1}{7}$ in EPO condition 3 and of constant 4 in the definition of $\tilde{M}_i$ are needed in their full force here. The structure of the proof of (b) is the same as the proof of (a), thus we only give a sketch. Again we inductively define a MEPO $\bar{y}_n$ with

$$c(\bar{y}_n) = c(\hat{y}_n), \bar{y}_n \cap \tilde{x}_n \neq \varnothing, \quad \varepsilon(\bar{y}_n) \geq \lfloor \tfrac{1}{2}\varepsilon^M(f^n_x) \rfloor$$

but $\bar{y}_n$ is not necessarily from the collection $\mathcal{B}$. Using the triangle inequality and the constants $\frac{1}{7}$ and 4 mentioned above ensures that the ball $B\ (c(\bar{y}_n), \lfloor \frac{1}{2}\varepsilon^M(f^n x) \rfloor)$ satisfies EPO condition (3) on the distance to the singularities. The condition $\bar{y}_n \cap x_n \neq \varnothing$ follows because the MEPO $\tilde{z}$ defined by $\tilde{z}_i = \hat{y}_i\ i \leq 0$ and $\tilde{z}_i = \tilde{x}_i\ i \geq 0$ is shadowed by y. The approximation of $\bar{y}$ by $\tilde{y}$ with all balls from the collection $\mathcal{B}$ is completely analogous to part (a).

Proof of Theorem 1. Fix $\varepsilon_{\max} > 0$ very small. Using Lemma 7 we get a cover $\mathcal{B} = \{B^i_j\}_{i \in \mathbb{N}, 1 \leq j \leq N_i} = \{B_k\}_{k \in \mathbb{N}}$ of M^* by balls satisfying $\varepsilon(B^i_j) = r_i =$ radius of ball B^i_j and $\varepsilon_{\max} \geq r_0 > r_1 > r_2 \ldots$. Let $\Sigma(\mathcal{B}) = \{\tilde{b} = \{b_i\}^{-\infty}_{i=-\infty}, b_i \in \mathcal{B}, \tilde{b}$ is an MEPO$\}$. For each $\tilde{b} \in \Sigma(\mathcal{B})$ there is a unique point $\pi(\tilde{b}) \in M^{**}$ which shadows $\tilde{b}$ and for a.e. $x \in M^{**}$ there are $\tilde{x} \in \Sigma(\mathcal{B})$ with $x = \pi(\tilde{x})$. For $\tilde{b}, \tilde{c} \in \Sigma(\mathcal{B})$ with $b_0 = c_0$ we define $[\tilde{b}, \tilde{c}] \in \Sigma(\mathcal{B})$ by

$$[\tilde{b}, \tilde{c}]_j = \begin{cases} b_j & \text{for } j \geq 0 \\ c_j & \text{for } j \leq 0 \end{cases}.$$

Observe that $\pi[\tilde{b}, \tilde{c}] = [\pi\tilde{b}, \pi\tilde{c}]^\infty_0$. This follows because

$$d(f^j\pi([\tilde{b}, \tilde{c}]), f^j\pi(\tilde{b})) \leq \delta(b_j) \leq \varepsilon(b_j) \quad \text{for } j \geq 0$$

and

$$d(f^j\pi([\tilde{b}, \tilde{c}]), f^j\pi(\tilde{c})) \leq \delta(c_j) \leq \varepsilon(c_j) \quad \text{for } j \leq 0.$$

From here on $\tilde{x}, \tilde{y}$ will always denote MEPOs such that $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$. Let $T_k = \{\pi(\tilde{b}): \tilde{b} \in \Sigma(\mathcal{B}), \tilde{b}_0 = B_k\}$. For $x, y \in T_k$, the brackets $[x, y]^\infty_0$ and $[y, x]^\infty_0$ are defined and belong to T_k. To see this consider MEPOs $\tilde{x}$, $\tilde{y}$ with $\tilde{x}_0 = \tilde{y}_0 = B_k$. It follows that $[x, y]^\infty_0 = \pi[\tilde{x}, \tilde{y}] \in T_k$. Similarly $[y, x]^\infty_0 \in T_k$ and thus T_k is a rectangle. T_k is in fact closed and since μ is a Borel measure is thus measurable. To see this first consider any convergent sequence $y_i \in T_k$ which satisfies $y_i \in W^s(y_0, T_k) = W^s_{\text{loc}}(y_0) \cap T_k$. Clearly $y = \lim y_i \in W^s_{\text{loc}}(y_0)$. We claim that $y \in M^{**}$. Suppose not, then $y \in S_{-j}$ for some $j > 0$. Now for all i, $d(f^{-j}y_i, S) \geq \varepsilon(T_k)/2^j$ by EPO condition

2. Thus since $\lim_{i\to\infty} f^{-j}y_i = f^{-j}y$ we have a contradiction. Thus we have shown that $y \in M^{**}$. Next we show that $y \in T_k$. Clearly y has a forward half MEPO, we can use the forward half MEPO for any of the y_i. To construct the backwards half MEPO consider the set

$$G_j^i = \{b_{-j} \cdots b_0 : b_l \in \mathcal{B},\ b_0 = B_k,\ \exists \text{MEPO } \tilde{y}_i \in \pi^{-1}(y_i) \text{ s.t. } (\tilde{y}_i)_l = b_l \text{ for } -j \le l \le 0\}.$$

G_j^i is finite and for large enough $i_0 = i_0(j)$ $G_j^i = G_j^{i_0}$ by the properties of our cover $\mathcal{B}$. Clearly if

$$b_{-(j+1)} b_{-j} \cdots b_0 \in G_{j+1}^{i_0(j+1)}$$

then $b_{-j} \cdots b_0 \in G_j^{i_0(j)}$. Because the $G_j^{i_0(j)}$ can be considered as cylinder sets in $\mathbb{N}^{\mathbb{Z}}$, the infinite intersection is non-empty and contains MEPOs shadowed by y. Similarly any limit point of a sequence $y_i \in W^s(y_0, T_k)$ must also be in T_k. Now consider any sequence $x_i \in T_k$ which converges to a point x. We have shown that $x_i^s = [x_i, x_0]$ and $x_i^u = [x_0, x_i]$ converge to points $x^s \in T_k$ and $x^u \in T_k$ respectively. However $x = [x^s, x^u]$ which must be in T_k since T_k is a rectangle.

$\mathcal{T} = \{T_k : k \in \mathbb{N}\}$ is a cover of M^{**} by rectangles. We now show that $\mathcal{T}$ satisfies a semi-Markov condition. Suppose $\tilde{x}$ is an MEPO with $\tilde{x}_0 = B_i$ and $\tilde{x}_1 = B_j$. Consider $y \in W^s(x, T_i) = W^s_{\text{loc}}(x) \cap T_i$ and an MEPO $\tilde{y}$ with $\tilde{y}_0 = B_i$. Then $y = [x, y]_0^\infty = \pi[\tilde{x}, \tilde{y}]$ and thus $fy = \pi(\sigma[\tilde{x}, \tilde{y}]) \in T_j$ since $\sigma[\tilde{x}, \tilde{y}]$ is an MEPO having B_j in its zeroth position. Since $fy \in W^s_{\text{loc}}(fx)$ we have $fy \in W^s(fx, T_j)$. We have proven (i) $fW^s(x, T_i) \subset W^s(fx, T_j)$. A similar proof shows (ii) $fW^u(x, T_i) \supset W^u(fx, T_j)$. We call this a semi-Markov condition because it is not necessarily true that if $x \in T_i$ and $fx \in T_j$ then there exists an MEPO $\tilde{x}$ with $\tilde{x}_0 = B_i$ and $\tilde{x}_1 = B_j$.

The rectangles are likely to overlap. Define $\varepsilon(T_i) = \varepsilon(B_i)$. For $T_j \cap T_k \neq 0$ define $T_{j,k}^n$, $n = 0, 1, 2, 3$ as follows:

$$T_{j,k}^0 = \{x \in T_j : W^u(x, T_j) \cap T_k \neq \emptyset;\ W^s(x, T_j) \cap T_k \neq \emptyset\} = T_j \cap T_k$$
$$T_{j,k}^1 = \{x \in T_j : W^u(x, T_j) \cap T_k \neq \emptyset;\ W^s(x, T_j) \cap T_k = \emptyset\}$$
$$T_{j,k}^2 = \{x \in T_j : W^u(x, T_j) \cap T_k = \emptyset;\ W^s(x, T_j) \cap T_k \neq \emptyset\}$$
$$T_{j,k}^3 = \{x \in T_j : W^u(x, T_j) \cap T_k = \emptyset;\ W^s(x, T_j) \cap T_k = \emptyset\}.$$

See Figure 6. If $x, y \in T_j$ then $W^s([x, y]_0^\infty, T_j) = W^s(x, T_j)$ and $W^u([x, y]_0^\infty, T_j) = W^u(y, T_j)$. This implies that $T_{j,k}^n$ are rectangles. Define $\mathcal{T}(x) = \{T_i \in \mathcal{T} : x \in T_i$ and

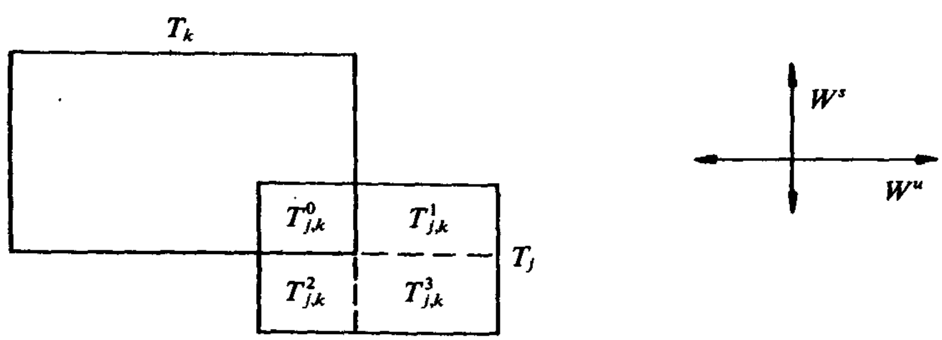

FIGURE 6

$\varepsilon(T_i) \geq \lfloor \varepsilon^M(x)/2 \rfloor\}$, that is the collection of nearly maximal rectangles containing x. Now define

$$R(x) = \bigcap \left\{ T^n_{j,k} : x \in T^n_{j,k},\ T_j \in \mathcal{T}(x),\ \varepsilon(T_k) \geq \left\lfloor \frac{\varepsilon^M(x)}{2} \right\rfloor \text{ and } \mu(T^n_{j,k}) > 0 \right\}.$$

It is clear that $R(x)$ is a rectangle and the collection $\mathcal{R} = \{R(x) : x \in M^{**}\}$ is a partition of M^{**} (mod 0). $\mathcal{R}$ is countable because each rectangle T_i was cut only a finite number of times.

First we show that if $y \in R(x)$ (mod 0) then $\mathcal{T}(y) = \mathcal{T}(x)$. To see this note that $R(x) \subset \bigcap\{T_i : T_i \in \mathcal{T}(x)\}$. If $T_j \in \mathcal{T}(y) \setminus \mathcal{T}(x)$ and $\varepsilon(T_j) \geq \lfloor \varepsilon^M(x)/2 \rfloor$ then $\mu(R(x) \cap T_j) = 0$, a contradiction. The case $\varepsilon(T_j) < \lfloor \varepsilon_M(x)/2 \rfloor$ cannot occur since $y \in R(x)$ implies that $\varepsilon^M(y) \geq \varepsilon^M(x)$. Now if $y \in R(x)$ (mod 0), $T_j \in \mathcal{T}(x) = \mathcal{T}(y)$ and T_k satisfies $\mu(T_k \cap T_j) \neq 0$, and $\varepsilon(T_k) \geq \lfloor \varepsilon^M(x)/2 \rfloor$ then y lies in the same $T^n_{j,k}$ as x does, since $T^n_{j,k} \supset R(x)$; hence $R(y) = R(x)$.

Now we are ready to verify the Markov condition for the partition $\mathcal{R}$. Suppose $R(x) = R(y)$ and $y \in W^s(x, R)$. We must show $R(fx) = R(fy)$. To see that $\mathcal{T}(fx) = \mathcal{T}(fy)$ assume that $T_j \in \mathcal{T}(fx)$. Let $fx = \pi(\sigma\tilde{x})$ with $\tilde{x}_0 = B_i$ and $\tilde{x}_1 = B_j$. We can assume that $T_i \in \mathcal{T}(x)$ (and thus also in $\mathcal{T}(y)$) by Proposition 2. Then by the semi-Markov property (i) we have $fy \in fW^s(x, T_i) \subset W^s(fx, T_j)$ and thus $fy \in T_j$. Thus we have shown that $\mathcal{T}(fx) \subset \mathcal{T}(fy)$. If we interchange the roles of x and y it follows that $\mathcal{T}(fx) = \mathcal{T}(fy)$.

Next we need to show that fx, fy belong to the same $T^n_{j,k}$ when $fx, fy \in T_j$, $\mu(T_k \cap T_j) \neq 0$ and $\varepsilon(T_k) \geq \lfloor \varepsilon^M(fx)/2 \rfloor$. Clearly $W^s(fy, T_j) = W^s(fx, T_j)$. Assume that fx and fy are not in the same $T^n_{j,k}$ (mod 0) that is without loss of generality $W^u(fx) \cap T_k \neq \varnothing$ and $W^u(fy) \cap T_k = \varnothing$. Choose $fz \in W^u(fx, T_j) \cap T_k \subset T^0_{j,k}$. Choose a MEPO $\tilde{x}$ with $\tilde{x}_0 = B_i$ with $\tilde{x}_1 = B_j$. By Proposition 2 we can assume that $B_i \in \mathcal{T}(x)$. By the semi-Markov property (ii), $fz \in W^u(fx, T_j) \subset fW^u(x, T_i)$ thus we can conclude that $z \in W^u(x, T_i)$. By Proposition 2(b) we can choose a MEPO $\tilde{z}$ with $\tilde{z}_0 = B_m$, $\tilde{z}_1 = B_k$ and $\varepsilon(B_m) \geq \lfloor \varepsilon^M(x)/2 \rfloor$. Now because $T_i \in \mathcal{T}(x)$ (and thus in $\mathcal{T}(y)$), it follows that $z \in T_m \cap T_i$ and that $z \in W^u(x, T_j) \cap T_m$. Then $z' = [z, y]_0^\infty \in W^s(z, T_m) \cap W^u(y, T_k)$. Therefore $fz' = [fz, fy]_0^\infty \in W^s(fz, T_k) \cap W^u(fy, T_j)$. We can find a set of positive measure of such z', a contradiction. Thus $R(fx) = R(fy)$. This completes the forward half of the Markov condition. The other half of the Markov property is proved similarly. □

Remark. We have shown the following. For a.e. point $x \in M^{**}$ there exists a unique corresponding symbolic sequence. The set of points $x \in M^{**}$ which have no corresponding symbolic sequence are of two types. Either no MEPO exists for this point, that is the point approaches the singularity set too quickly as discussed before or the point was dropped in our cutting stage.

Thus (M^{**}, f) is conjugate to (Σ_R, σ) where $\Sigma_R \subset \mathbb{N}^{\mathbb{Z}}$ is a topological Markov chain in the sense that there is a set $B \subset M^{**}$ of μ-measure 1 such that the conjugating map $\phi : (B, f) \to (\phi B, \sigma)$ satisfies ϕ^{-1} is continuous.

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