CHAPTER 4

Linear systems

In a naive approach to the theory of dynamical systems on Euclidean space $\mathbf{R}^n$, we might hope to obtain a complete classification of all systems, starting with the very simplest types and gradually building up an understanding of more complicated ones. The simplest conceivable diffeomorphisms of R^n are *translations*, maps of the form $x \mapsto x + p$ for some constant $p \in R^n$. It is an easy exercise to show that two such maps are *linearly conjugate* (topologically conjugate by a linear automorphism of $\mathbf{R}^n$) providing the constant vectors involved are non-zero. Similarly, any two non-zero constant vector fields on $\mathbf{R}^n$ are linearly flow equivalent. The next simplest systems are linear ones, discrete dynamical systems generated by linear automorphisms of $\mathbf{R}^n$ and vector fields on $\mathbf{R}^n$ with linear principal part. Rather surprisingly, the classification problem for such systems is far from trivial. It has been completely solved for vector fields (Kuiper [2]) but not as yet for diffeomorphisms, although Kuiper and Robbin [1] have made a great deal of progress in this latter case.

The difficulties that one encounters even at the linear stage underline the complexity and richness of the theory of dynamical systems, and this is part of its attraction. They also indicate the need for a less ambitious approach to the whole classification problem. Instead of aiming at a complete solution, we attempt to classify a "suitably large" class of dynamical systems. In the case of linear systems on $\mathbf{R}^n$, it is easy to give a precise definition of what we mean by "suitably large", as follows.

The set $L(\mathbf{R}^n)$ of linear endomorphisms of $\mathbf{R}^n$ is a Banach space with the *induced norm* given by

$$|T| = \sup\{|T(x)|: x \in \mathbf{R}^n, |x| = 1\}.$$

A "suitably large" subset $\mathscr{S}$ of linear vector fields on $\mathbf{R}^n$ is one that is both open and dense in $L(\mathbf{R}^n)$ (from now on we shall not usually bother to distinguish between a vector field on $\mathbf{R}^n$ and its principal part). The density of such a set $\mathscr{S}$ implies that we can approximate any element of $L(\mathbf{R}^n)$ arbitrarily closely by an element of $\mathscr{S}$; the fact that $\mathscr{S}$ is open implies that any element of $\mathscr{S}$ remains in $\mathscr{S}$ after any sufficiently small perturbation. In general, of these two desirable properties we tend to regard density as essential and openness as negotiable.

The set $GL(\mathbf{R}^n)$ of linear automorphisms of $\mathbf{R}^n$ inherits the subspace topology from $L(\mathbf{R}^n)$. "Suitably large" in the context of discrete linear systems means open and dense in $GL(\mathbf{R}^n)$.

How, then, do we select a candidate for the set $\mathscr{S}$? There are very good reasons for our choice, as follows. Suppose that X is a topological space and that $\sim$ is an equivalence relation on X. We say that a point $x \in X$ is *stable* with respect to $\sim$ if x is an interior point of its $\sim$ class. The *stable set* Σ of $\sim$ is the set of all stable points in X. Of course Σ is automatically an open subset of X. Moreover, if the topology of X has a countable basis, then Σ contains points of only countably many equivalence classes. Thus there is some chance that Σ will be classifiable. The snag is that Σ may fail to be dense in X. In the present context, $X = GL(\mathbf{R}^n)$ (or $L(\mathbf{R}^n)$), $\sim$ is topological conjugacy (or topological equivalence), and Σ is called the set of *hyperbolic* linear automorphisms (or the *hyperbolic* linear vector fields). In each case the stable set Σ is dense and is easily classifiable, as we shall see.

There is another vital reason for considering these particular subsets: in the next chapter we shall find that hyperbolic linear automorphisms (and hyperbolic linear vector fields) are stable not only under small linear perturbations but also, more generally, under small smooth perturbations. Now the basic idea of differential calculus is one of approximating a function locally by a linear function. Similarly, when discussing the local nature of a dynamical system we consider its "linear approximation", which is a certain linear system. The important question is whether this linear system is a good approximation in the sense that it is (locally) qualitatively the same as the original system. This question is obviously bound up with the stability (in the wider sense) of the linear system. In fact, we shall obtain a satisfactory linear approximation theory precisely when the linear system is stable.

We begin with a review of linear systems on $\mathbf{R}^n$. The bulk of the chapter, however, deals with stable linear systems on a general Banach space. For this we need a certain amount of spectral theory, which we include in an appendix to the chapter. The material of $\mathbf{R}^n$ provides a background of concrete examples against which the more general results on Banach spaces may be tested and placed in perspective.

I. LINEAR FLOWS ON $\mathbf{R}^n$

Let T be any linear endomorphism of $\mathbf{R}^n$. Then we may think of T as a vector field on $\mathbf{R}^n$, in which case we call it a *linear vector field.* The corresponding ordinary differential equation is

$$x' = T(x), \tag{4.1}$$

where $x \in \mathbf{R}^n$. One may immediately write down the integral flow. Since $L(\mathbf{R}^n)$ is a Banach space, it makes sense to talk of the infinite series

$$\exp(tT) = id + tT + \frac{1}{2}t^2T^2 + \cdots + \frac{1}{n!}t^nT^n + \cdots$$

where id is the identity map on $\mathbf{R}^n$ and $t \in \mathbf{R}$. This series converges for all t and T, and the integral flow ϕ of T is given by

$$\phi(t, x) = \exp(tT)(x), \tag{4.2}$$

(term-by-term differentiation makes this plausible; another proof is indicated later in the chapter). We call a flow ϕ *linear* if ϕ^t is a linear automorphism varying smoothly with t. Thus ϕ is linear if and only if its velocity vector field is linear.

(4.3) Example. If $T = a(id)$ for some $a \in R$ then

$$\begin{aligned}\exp(tT) &= id + at(id) + \cdots + \frac{1}{n!}(at)^n id^n + \cdots \\ &= (1 + at + \cdots + \frac{1}{n!}(at)^n + \cdots)id \\ &= e^{at}id.\end{aligned}$$

(4.4) Example. If T has matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{then } T^2 \text{ has matrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so the series for $\exp(tT)$ terminates as $id + tT$.

We may define a notion of *linear equivalence* for linear flows, as follows. Let ϕ and ψ be linear flows with velocities S and $T \in L(\mathbf{R}^n)$ respectively. Then ψ is *linearly equivalent* to ϕ, written $\psi \sim_L \phi$, if for some $\alpha \in \mathbf{R}$, $\alpha > 0$, and some linear automorphism $h \in GL(\mathbf{R}^n)$ of $\mathbf{R}^n$, the diagram

(4.5)

$$\begin{tikzcd} \mathbf{R}\times\mathbf{R}^n \arrow[r, "\phi"] \arrow[d, "\alpha\times h"'] & \mathbf{R}^n \arrow[d, "h"] \\ \mathbf{R}\times\mathbf{R}^n \arrow[r, "\psi"'] & \mathbf{R}^n \end{tikzcd}$$

commutes. That is, $h\phi(t, x) = \psi(\alpha t, h(x))$ for all $(t, x) \in \mathbf{R}\times\mathbf{R}^n$. Equivalently $\psi \sim_L \phi$ if and only if $S = \alpha(h^{-1}Th)$. The problem of classifying linear flows up to linear equivalence is therefore the same as the problem of classifying $L(\mathbf{R}^n)$ up to similarity (linear conjugacy). Now the latter is solved by the theory of real Jordan canonical form (see Hirsch and Smale [1] or Gantmacher [1]) which we recall briefly.

Every $T \in L(\mathbf{R}^n)$ is similar to a direct sum $T_1 \oplus \cdots \oplus T_q$ of linear endomorphisms $T_j \in L(V_j)$, $j = 1, \ldots, q$, where $V_1 \oplus \cdots \oplus V_q$ is some direct sum decomposition of $\mathbf{R}^n$, and, for some basis of V_j, the matrix of T_j takes the form

$$M_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_j & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \lambda_j & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \lambda_j \end{bmatrix}$$

where $\lambda_j \in \mathbf{C}$. For each M_j there are two possibilities:

(i) $\lambda_j \in \mathbf{R}$, in which case the entries of M_j are real numbers, and T_j is associated with dim V_j repeated eigenvalues λ_j of T.

(ii) $\lambda_j = a + ib$, where $a, b \in \mathbf{R}$, $b > 0$, in which case the entries of M_j are real 2×2 submatrices

$$\lambda_j = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and T_j is associated with $\frac{1}{2}\dim V_j$ pairs of repeated complex conjugate eigenvalues λ_j and $\bar{\lambda}_j$ of T.

Notice that there may be more than one T_j having the same eigenvalue λ_j. The set of numbers dim V_j and the corresponding eigenvalues λ_j (or λ_j, $\bar{\lambda}_j$) determine the similarity class of T.

It follows that the integral flow ϕ of T can be decomposed as

$$\phi_1 \oplus \cdots \oplus \phi_q, \qquad \text{where} \qquad \phi_j(t, x_j) = \exp(tT_j)(x_j),$$

and $x = (x_1, \ldots, x_q) \in V_1 \oplus \cdots \oplus V_q = \mathbf{R}^n$. With respect to the above basis of V_j, $(\phi_j)^t \colon V_j \to V_j$ has matrix

$$N_j = e^{at} \begin{bmatrix} e^{ibt} & t\,e^{ibt} & \cdot & \cdot & \cdot & \cdot & \dfrac{t^{m-1}\,e^{ibt}}{(m-1)!} \\ 0 & e^{ibt} & t\,e^{ibt} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & e^{ibt} & t\,e^{ibt} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & e^{ibt} \end{bmatrix},$$

where $\lambda_j = a + ib$, $a, b \in \mathbf{R}$, and, if $b > 0$, $(t^r/r!)\,e^{ibt}$ must be interpreted as the 2×2 block

$$\frac{t^r}{r!}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

Thus, for example if $\dim V_j = 4$ and $\lambda_j = i$, then

$$M_j = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$N_j = \begin{bmatrix} \cos t & -\sin t & t\cos t & -t\sin t \\ \sin t & \cos t & t\sin t & t\cos t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix}.$$

Let us examine the case $a < 0$, $b = 0$, $\dim V_j = 2$. The phase portrait in the plane V_j is illustrated in Figure 4.6. Every orbit has the point 0 as its unique ω-limit point. This leads us to suspect that any two negative values of a give

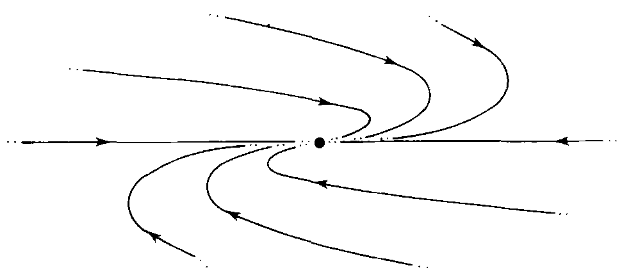

FIGURE 4.6

topologically equivalent flows. How can we prove this? Suppose it were true that for some value of a the distance $|t.x|$ decreased with t. Then we could define a map $h: V_j \to V_j$ as the identity on 0 and on the unit circle S^1 and, for all $x \in S^1$ taking $t.x$ to $e^{-t}x$ (see Figure 4.7). It is not hard to show that h is a

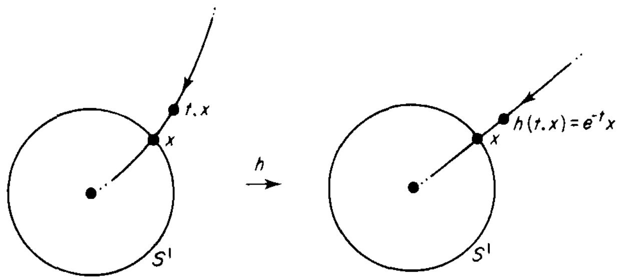

FIGURE 4.7

homeomorphism, and hence that it is a flow equivalence from the given vector field to the vector field $-id$ on V_j. Now $|t.x|$ decreases if and only if the scalar product of x and $T_j(x)$ is negative definite. This quadratic form is $a(x_1^2+x_2^2)+x_1x_2$ which is certainly *not* negative definite for small negative a. However we now note that, for any $\varepsilon > 0$, a linear conjugacy with

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \text{ changes } \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \text{ to } \begin{bmatrix} a & \varepsilon \\ 0 & a \end{bmatrix},$$

and, for given a, we can always take ε small enough for the form

$$a(x_1^2+x_2^2)+\varepsilon x_1x_2$$

to be negative definite. Thus

$$\begin{bmatrix} a & \varepsilon \\ 0 & a \end{bmatrix} \text{ is flow equivalent to } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and hence so is } \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},$$

since linear conjugacy trivially implies flow equivalence. Thus, as we suspected, negative values of a give topologically equivalent vector fields.

We may go considerably further than this, using similar techniques. We may take the direct sum V_- of all subspaces corresponding to eigenvalues ξ_j with negative real part, and show that the vector field on this subspace is flow equivalent to the vector field $-id$. Thus any two linear *contractions* on a subspace of R^n are flow equivalent. Similarly for positive eigenvectors we get a vector field on a subspace V_+ which is flow equivalent to the vector field id. We do not go into details at this stage, since we prove a more general version later in the chapter.

There remains the linear subspace V_0 obtained by combining those summands V_j for which the real part of λ_j is zero. If $V_0 = \{0\}$, then $\mathbf{R}^n = V_+ \oplus V_-$ and the vector field T is flow equivalent to the vector field on $\mathbf{R}^n$ with matrix $I_r \oplus -I_s = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$, where $r = \dim V_+$ and $s = \dim V_-$. Such a linear vector field T is called a *hyperbolic linear vector field* and its integral flow ϕ is called a *hyperbolic linear flow*. Hyperbolic linear vector fields are sometimes referred to as *elementary* linear vector fields, which explains our notation $EL(\mathbf{R}^n)$ for the set of all hyperbolic linear vector fields on $\mathbf{R}^n$.

We do not give a detailed analysis of the flow on V_0, since it would involve a fair amount of work and the hyperbolic case is of more fundamental importance. Instead we state the main conclusion reached by Kuiper. The fact is that if ϕ and ψ are linear flows given by linear endomorphisms S and T of $\mathbf{R}^n$ whose eigenvalues all have zero real part then ϕ is topologically equivalent to ψ if and only if ϕ is linearly equivalent to ψ. More generally, two linear flows ϕ and ψ on $\mathbf{R}^n$ with decompositions $U_+ \oplus U_- \oplus U_0$ and $V_+ \oplus V_- \oplus V_0$ are topologically equivalent if and only if $\phi | U_0$ is linearly equivalent to $\psi | V_0$, $\dim U_+ = \dim V_+$ and $\dim U_- = \dim V_-$. For full details see Kuiper [2].

(4.8) Example. *Linear flows on* $\mathbf{R}$. Any linear endomorphism of the real line $\mathbf{R}$ is of the form $x \mapsto ax$ for some real number a, which is the eigenvalue of the map. Thus the map is a hyperbolic vector field if and only if $a \neq 0$. The integral flow is $t.x = x e^{at}$, and there are exactly three topological equivalence classes, corresponding to the cases $a < 0$, $a = 0$ and $a > 0$ (see Figure 4.8).

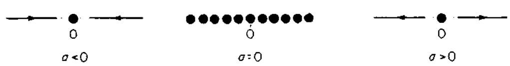

FIGURE 4.8

(4.9) Example. *Linear flows on* $\mathbf{R}^2$. The real Jordan form for a real 2×2 matrix is one of the following three types:

$$\text{(i)} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where λ, μ, a, b are real numbers and $b > 0$. The corresponding eigenvalues are λ, μ and $a \pm ib$. It follows that a linear vector field on $\mathbf{R}^2$ is hyperbolic if and only if $\lambda\mu \neq 0$ (case (i)), $\lambda \neq 0$ (case (ii)) or $a \neq 0$ (case (iii)). We have three topological equivalence classes of hyperbolic flows, and Figure 4.9

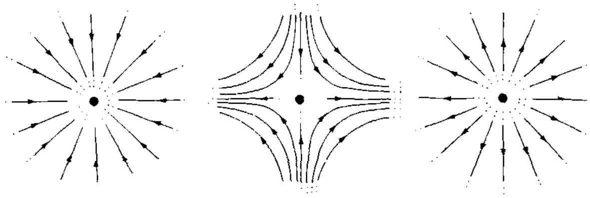

FIGURE 4.9

illustrates representatives

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of these classes. (N.b. Many classical texts on ordinary differential equations employ a finer classification than ours, giving rise to terminology such as *proper node*, *improper node* and *focus*.) There are also five topological equivalence classes of non-hyperbolic linear flows. Case (i) with $\mu = 0$ gives three classes corresponding to $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$ we have the trivial flow; the other two cases are illustrated in Figure 4.10. These three are, of course, product flows. The remaining two, which are irreducible, are case (ii) with $\lambda = 0$ and case (iii) with $a = 0$ (see Figure 4.11).

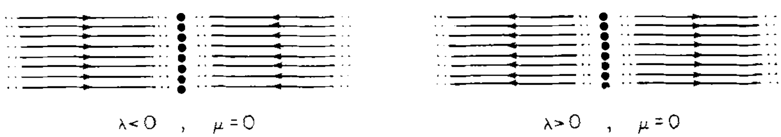

FIGURE 4.10

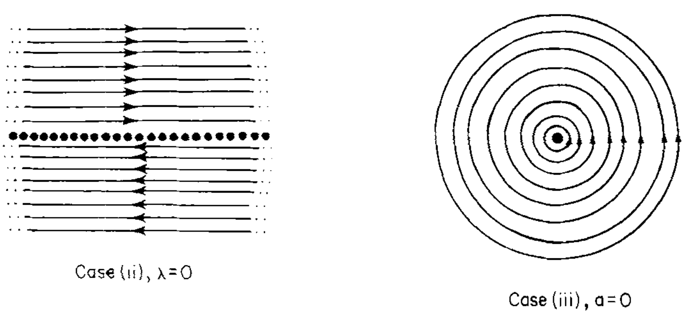

FIGURE 4.11

(4.12) Exercise. Show that there are seventeen topological equivalence classes of linear flows on $\mathbf{R}^3$. Show also that, for all $n > 3$, there is a continuum of topological equivalence classes of linear flows on $\mathbf{R}^n$.

(4.13) Exercise. Classify up to topological equivalence affine vector fields on $\mathbf{R}^n$ for $n = 1, 2, 3$.

II. LINEAR AUTOMORPHISMS OF $\mathbf{R}^n$

Discrete linear dynamical systems on $\mathbf{R}^n$ are determined by linear automorphisms of $\mathbf{R}^n$. There are two obvious equivalence relations to compare, topological conjugacy and *linear conjugacy*, which is similarity in the ordinary sense of linear algebra. Recall that if S and $T \in GL(\mathbf{R}^n)$, S is similar to T if for some $P \in GL(\mathbf{R}^n)$, $T = PSP^{-1}$. The relation between linear and topological conjugacy is more difficult to pin down than that between linear and topological equivalence for flows. Kuiper and Robbin [1] have made a detailed study of the problem. We content ourselves here with the observation that there is an open, dense subset $HL(\mathbf{R}^n)$ of $GL(\mathbf{R}^n)$ analogous to $EL(\mathbf{R}^n)$ in the flow case. Elements of $HL(\mathbf{R}^n)$ are called *hyperbolic linear automorphisms* of $\mathbf{R}^n$. For any $T \in GL(R^n)$, $T \in HL(\mathbf{R}^n)$ if and only if none of its eigenvalues lies on the unit circle S^1 in $\mathbf{C}$. Note that T is a hyperbolic vector field if and only if $\exp T$ is a hyperbolic automorphism. It is unfortunate that the term "hyperbolic" must carry two meanings, but the context should always make clear which is to be taken. If $T \in HL(\mathbf{R}^n)$ then we find that $\mathbf{R}^n$ decomposes into two direct summands corresponding to eigenvalues of T inside S^1 and eigenvalues outside S^1. The classification of $HL(\mathbf{R}^n)$ up to topological conjugacy is bound up with the dimensions of these invariant subspaces, and so resembles the classification of $EL(\mathbf{R}^n)$ up to topological equivalence. In fact, elements S and T of $EL(\mathbf{R}^n)$ are topologically equivalent if and only if $\exp S$ and $\exp T$ are topologically conjugate.

(4.14) Example. *Automorphisms of* $\mathbf{R}$. As we have seen, any element T of $GL(\mathbf{R})$ is of the form $T(x) = ax$, where a is a non-zero real number. There are exactly six topological conjugacy classes, given by (i) $a < -1$, (ii) $a = -1$, (iii) $-1 < a < 0$, (iv) $0 < a < 1$, (v) $a = 1$, (vi) $a > 1$. Of these, all but (ii) and (v) are hyperbolic.

(4.15) Example. *Automorphisms of* $\mathbf{R}^2$. Let $T \in GL(\mathbf{R}^2)$. Referring to Example 4.9, we can suppose that the matrix of T is in Jordan form (i), (ii) or (iii). Now T is hyperbolic if and only if $\lambda^2 \neq 1 \neq \mu^2$, in cases (i) and (ii), or $a^2 + b^2 \neq 1$, in case (iii). There are eight topological conjugacy classes ol hyperbolic maps. All are product of automorphisms of $\mathbf{R}$, all occur in case (i) and six occur only there. Case (i) yields also eleven classes of non-hyperbolic maps. These are the products of the six classes of $GL(\mathbf{R})$ with the identity map id on $\mathbf{R}$, and of five of them (id being excluded) with the antipodal map $-id$ on $\mathbf{R}$. We get two further classes from case (ii) with $\lambda = \pm 1$. The non-hyperbolic automorphisms of case (iii) give a continuum of different topological conjugacy classes, as we now show.

When $a^2 + b^2 = 1$ in case (iii), T is a rotation through an angle θ, say, where $0 < \theta < \pi$. Rotation through θ is topologically conjugate to rotation through $-\theta$, by conjugating with a reflection. The question is: Can rotations T_1 and T_2 through distinct angles θ_1 and θ_2 be topologically conjugate if $0 < \theta_i < \pi$, $i = 1, 2$? We first observe that if $\theta_1 = 2\pi p/q$, with $q > 0$ and p coprime integers, then every point $x \neq 0$ is a periodic point of T_1 of period q. Thus T_2 cannot be topologically conjugate to T_1 unless $\theta_2 = 2\pi p'/q$, where p' and q are coprime. Suppose, then, that θ_2 has this form, and let $h: \mathbf{R}^2 \to \mathbf{R}^2$ be a topological conjugacy from T_1 to T_2. Let S_r denote the circle with centre 0 and radius r. Choose r large enough for S_r to contain the compact set $h(S_1)$ in its interior. By a highly non-trivial theorem of topology, there is a homeomorphism, f say, from the closed region D between $h(S_1)$ and S_r to the closed annulus A between S_1 and S_2. We may assume that f takes S_r to S_2 and preserves the standard orientation. Let g be the homeomorphism of A induced by f from $T_2|D$ (i.e. $g = f(T_2|D)f^{-1}$). Then g is pointwise periodic of period q, since $T_2|D$ is. Let $C = [1, 2] \times \mathbf{R}$, and let $\pi: C \to A$ be the covering defined by $\pi(t, u) = te^{2\pi i u}$. We may lift g to a homeomorphism $G: C \to C$ such that $\pi G = g\pi$, by a similar technique to that used in Exercise 2.4. By the periodicity of g, $G^q(t, u) = (t, u + n(t, u))$, where $n(t, u)$ is an integer that varies continuously with (t, u). Now $g|S_2$ has rotation number $[p'/q]$, since it is topologically conjugate to $T_2|S_r$ by $f|S_r$. Thus $G^q(2, u) = (2, u + mq + p')$ for some fixed integer m and all $u \in \mathbf{R}$. One easily deduces that since $n(t, u)$ is a continuously varying integer, $n(t, u) = mq + p'$ for all (t, u). In particular the rotation number of $g|S_1$ is $[p'/q]$. But $g|S_1$ is topologically conjugate to $T_1|S_1$ by $gh|S_1$. Hence $[p/q] = [p'/q]$.

We have now proved that if $0 < \theta_i < \pi$ $(i = 1, 2)$, $\theta_1/2\pi$ is rational, and T_1 and T_2 are topologically conjugate then $\theta_1 = \theta_2$. This leaves the case when $\theta_1/2\pi$ and $\theta_2/2\pi$ are irrational, and, fortunately, this is much easier than the rational case. If h is a conjugacy between T_1 and T_2 and if $x \in S^1$ then h maps the closure of the T_1-orbit of x onto the closure of the T_2-orbit of $h(x)$. But the first of these sets is S_1 and the second is S_r for some $r > 0$. Thus h restricts to a conjugacy between $T_1|S_1$ and $T_2|S_r$. Therefore, $\theta_1/2\pi = \theta_2/2\pi$, since these are the rotation numbers of the two circle homeomorphisms.

As one might expect from the above details, it is rotations, and, in particular, periodic rotations that cause the most trouble when one attempts to deal with automorphisms of $\mathbf{R}^n$ for large n. In fact, Kuiper and Robbin in their paper [1] reduced the classification of automorphisms of $\mathbf{R}^n$ to the so called *periodic rotation conjecture*, that any two periodic orthogonal maps of

$\mathbf{R}^n$ are topologically conjugate if and only if they are linearly conjugate. It is this conjecture that is unsolved at the time of writing, although some results are known in the positive direction.

III. THE SPECTRUM OF A LINEAR ENDOMORPHISM

We now embark on a study of linear dynamical systems on a Banach space $\mathbf{E}$ (real or complex). As we have seen above, when $\mathbf{E} = \mathbf{R}^n$ the eigenvalues of the system (vector field or automorphism) play a vital role in the theory. In infinite dimensions, the analogue of the set of eigenvalues of a linear endomorphism T is called the *spectrum* $\sigma(T)$ of T. In the complex case, it is the set of all complex numbers λ for which $T - \lambda(id)$ is *not* an automorphism. It is a compact subset of $\mathbf{C}$, and is only empty in the trivial case $\mathbf{E} = \{0\}$. We shall need a certain amount of spectral theory for our investigation of linear systems. For those unacquainted with this theory, most of the necessary material is gathered in the appendix to this chapter. We give here a brief summary.

Recall that the set $L(\mathbf{E})$ of (continuous) linear endomorphisms of $\mathbf{E}$ is a Banach space, with norm defined by

$$|T| = \sup\{|T(x)|: x \in \mathbf{E}, |x| \leq 1\}.$$

The spectral radius $r(T)$ of T is the radius of the smallest circle in the Argand diagram with centre 0 containing the spectrum $\sigma(T)$. That is to say

$$r(T) = \sup\{|\lambda|: \lambda \in \sigma(T)\}.$$

It measures the eventual size of T under repeated iteration, in the sense that it equals $\lim_{n\to\infty} |T^n|^{1/n}$. One may always pick a norm of $\mathbf{E}$ equivalent to the given one, such that, with respect to the corresponding new norm of $L(\mathbf{E})$, T has norm precisely $r(T)$. This is of especial interest to us when $r(T) < 1$. In this case we call T a *linear contraction*, and with respect to the new norm T is a contraction in the sense of metric space theory (Appendix C). If T is an automorphism, and T^{-1} is a contraction, we call T an *expansion*. Notice that T is an automorphism if and only if $0 \notin \sigma(T)$, and that

$$r(T^{-1}) = (\inf\{|\lambda|: \lambda \in \sigma(T)\})^{-1},$$

since, in fact, $\sigma(T^{-1}) = \{\lambda^{-1}: \lambda \in \sigma(T)\}$.

Now let D be a contour in the Argand diagram, symmetrical about the real axis if $\mathbf{E}$ is a real Banach space, such that $\sigma(T) \cap D$ is empty. Suppose that $0 \notin \sigma(T)$ and that D separates $\sigma(T)$ into two subsets $\sigma_s(T)$ and $\sigma_u(T)$. Then, by a corollary of Dunford's spectral mapping theorem, $\mathbf{E}$ splits uniquely as a direct sum $\mathbf{E}_s \oplus \mathbf{E}_u$ of T-invariant closed subspaces, such that, if $T_s: \mathbf{E}_s \to \mathbf{E}_s$ and $T_u: \mathbf{E}_u \to \mathbf{E}_u$ are the restrictions of T, then $\sigma(T_s) = \sigma_s(T)$ and $\sigma(T_u) = \sigma_u(T)$. Moreover, the splitting of $\mathbf{E}$ depends analytically on $T \in L(\mathbf{E})$. More precisely, the subspace $\mathbf{E}_s$, for example, is obtained as the image of a projection $S \in L(\mathbf{E})$ (*projection* means $S^2 = S$), and S varies in $L(\mathbf{E})$ as an analytic function of T in $\{T \in L(\mathbf{E}): \sigma(T) \cap D \text{ empty}\}$.

IV. HYPERBOLIC LINEAR AUTOMORPHISMS

Let $T \in GL(\mathbf{E})$, the open subset of $L(\mathbf{E})$ consisting of all linear automorphisms of $\mathbf{E}$. We say that T is a *hyperbolic linear* automorphism if $\sigma(T) \cap S^1$ is empty, where S^1 is the unit circle in the Argand diagram $\mathbf{C}$. The set of all hyperbolic linear automorphisms of $\mathbf{E}$ is denoted by $HL(\mathbf{E})$.

(4.16) Example. The map $T: R^2 \to R^2$ given by

$$T(x, y) = (\tfrac{1}{2}x, 2y)$$

is in $HL(\mathbf{R}^2)$. Its spectrum is the set $\{\frac{1}{2}, 2\}$ of eigenvalues of T and neither of these points of $\mathbf{C}$ is on S^1. The effect of T is suggested in Figure 4.16, where

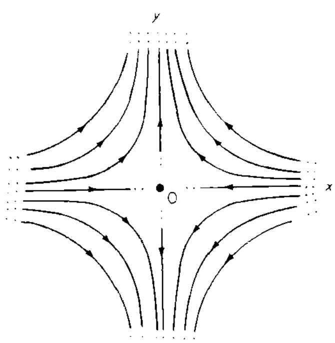

FIGURE 4.16

the hyperbolae and their asymptotes are invariant submanifolds of T. The x-axis is stable in the sense that positive iterates of T take its points into bounded sequences (in fact they converge to the origin 0). The y-axis is unstable (meaning, in this context, stable with respect to T^{-1}). These axes together generate $\mathbf{R}^2$, of course. A direct sum decomposition into stable and unstable summands (one of which may, however, be $\{0\}$) is typical of hyperbolic linear automorphisms.

(4.17) Exercise. Prove that if $T \in GL(\mathbf{R}^n)$ and if $a \in \mathbf{R}$ is sufficiently near but not equal to 1, then aT is hyperbolic. Deduce that $HL(\mathbf{R}^n)$ is dense in $GL(\mathbf{R}^n)$.

(4.18) Exercise. Let $T \in HL(\mathbf{E})$. Prove that $0 \in \mathbf{E}$ is the unique fixed point of T.

Let $T \in HL(\mathbf{E})$ and let $\sigma_s(T)$ and $\sigma_u(T)$ be the parts of $\sigma(T)$ inside and outside the unit circle in the Argand diagram. By the spectral decomposition theorem (Corollaries 4.56 and 4.58 of the appendix) $\mathbf{E}$ splits as a direct sum $\mathbf{E}_s \oplus \mathbf{E}_u$ of T-invariant subspaces, where the spectra of the restrictions $T_s \in GL(\mathbf{E}_s)$ and $T_u \in GL(\mathbf{E}_u)$ of T are respectively $\sigma_s(T)$ and $\sigma_u(T)$. Thus T_s is a contraction and T_u is an expansion. Let b be any number less than 1 but greater than the larger of the spectral radii of T_s and T_u^{-1}. By Theorem 4.47 of the appendix we have:

(4.19) Theorem. *If $T \in HL(\mathbf{E})$ then there is an equivalent norm $\| \ \|$ such that*

(i) *for all $x = x_s + x_u \in E$, $\|x\| = \max\{\|x_s\|, \|x_u\|\}$,*

(ii) $\max\{\|T_s\|, \|T_u^{-1}\|\} = a \leq b$. □

Thus, with respect to the new norm $\| \ \|$, T_s is a metric contraction and T_u is a metric expansion. We call a the *skewness* of T with respect to $\| \ \|$. The *stable summand* $\mathbf{E}_s$ *of* $\mathbf{E}$ *with respect to* T is equivalently characterized as $\{x \in \mathbf{E}\colon T^n(x) \text{ is bounded as } n \to \infty\}$ and $\{x \in \mathbf{E}\colon T^n(x) \to 0 \text{ as } n \to \infty\}$. We also call $\mathbf{E}_s$ the *stable manifold* of 0 with respect to T and T_s the *stable summand of* T. A similar characterization holds for the *unstable summand* $\mathbf{E}_u$, with T^{-n} replacing T^n.

The main aim of this section is to prove that hyperbolic linear automorphisms of $\mathbf{E}$ are stable in $L(\mathbf{E})$ with respect to topological conjugacy. A first step in this direction is:

(4.20) Theorem. *$HL(\mathbf{E})$ is open in $GL(\mathbf{E})$, and hence in $L(\mathbf{E})$.*

Proof. Let $T \in HL(\mathbf{E})$. Choose a norm $\| \ \|$ as in Theorem 4.19. Let T have skewness a. We assert that, for all $T' \in GL(\mathbf{E})$ with $\|T' - T\| < 1 - a$, $T' \in HL(\mathbf{E})$. Let $\lambda \in S^1$. Then $\lambda \notin \sigma(T)$, so $T - \lambda(id)$ is an automorphism. In fact we may explicitly write down the components of $(T - \lambda(id))^{-1}$ in the $\mathbf{E}_s$ and $\mathbf{E}_u$ directions; they are $-\sum_{r=0}^{\infty} \lambda^{-r-1}(T_s)^r$ and $\sum_{r=0}^{\infty} \lambda^r (T_u)^{-r-1}$. From these expressions it is easy to see that $\|(T - \lambda(id))^{-1}\| \leq (1-a)^{-1}$. Thus, if T' satisfies the given inequality, then the series

$$\sum_{r=0} (T - \lambda(id))^{-1}((T - T')(T - \lambda(id))^{-1})^r$$

converges and is an inverse of $T' - \lambda(id)$. Thus $\lambda \notin \sigma(T')$, and hence T' is hyperbolic. □

Our next step is to study the direct sum decomposition of $\mathbf{E}$ associated with a hyperbolic linear automorphism. We say that $T \in HL(\mathbf{E})$ is *isomorphic* to $T' \in HL(\mathbf{E}')$ if there exists a (topological) linear isomorphism from $\mathbf{E}$ to $\mathbf{E}'$ taking $\mathbf{E}_s(T)$ onto $\mathbf{E}'_s(T')$ and $\mathbf{E}_u(T)$ onto $\mathbf{E}_u(T')$. Equivalently, T is isomorphic to T' if and only if there are linear isomorphisms of $\mathbf{E}_s(T)$ onto $\mathbf{E}'_s(T')$ and of $\mathbf{E}_u(T)$ onto $\mathbf{E}'_u(T')$. Thus, for example, there are exactly $n+1$ isomorphism classes in $HL(\mathbf{R}^n)$. We prove:

(4.21) Theorem. *Any $T \in HL(\mathbf{E})$ is stable with respect to isomorphism.*

Proof. By the spectral theory of the appendix to the chapter, there is a continuous (in fact, analytic) map $f: HL(\mathbf{E}) \to L(\mathbf{E})$ such that for all $T \in HL(\mathbf{E})$, $f(T)$ is a projection with image $\mathbf{E}_s(T)$. Let $T \in HL(\mathbf{E})$. Then $f(T)f(T') \to f(T)^2$ as $T' \to T$ in $HL(\mathbf{E})$. Since the restriction of $f(T)^2 (= f(T))$ to $\mathbf{E}_s(T)$ is the identity, we deduce that if T' is sufficiently near T then $f(T)f(T')$ restricts to a (topological linear) automorphism of $\mathbf{E}_s(T)$. Thus $f(T')$ maps $\mathbf{E}_s(T)$ injectively into $\mathbf{E}_s(T')$. Similarly, since $f(T')f(T) - f(T')^2 \to 0$ as $T' \to T$ and since $f(T')^2$ is the identity on $\mathbf{E}_s(T')$, $f(T')f(T)$ is an automorphism of $\mathbf{E}_s(T')$ for T' sufficiently close to T. Thus $f(T')$ maps $\mathbf{E}_s(T)$ surjectively to $\mathbf{E}_s(T')$. Thus $\mathbf{E}_s(T)$ is isomorphic to $\mathbf{E}_s(T')$. A similar argument applies to the unstable summands. Thus T' is isomorphic to T. □

(4.22) Exercise. Let $T \in GL(R^n) \backslash HL(R^n)$. By comparing aT with bT, for $0 < a < 1 < b$, show that any neighbourhood of T in $GL(\mathbf{R}^n)$ contains elements of more than one isomorphism class.

The foregoing theory enables us to make a connection between the stability of a hyperbolic automorphism and the stability of its restriction to its stable and unstable summands. So let us now suppose, for the time being, that T is a linear contraction of $\mathbf{E}$, and that a norm has been chosen so that T is a metric contraction. Then T maps the unit sphere S_1 in $\mathbf{E}$ into the interior of the unit ball B_1. The closed annulus A between S_1 and $T(S_1)$ is called a *fundamental domain* of T. Its images under positive and negative powers of T cover $\mathbf{E}\backslash\{0\}$ (see Figure 4.23). Moreover, for $r > s$, $T^r(A)$ intersects $T^s(A)$ only when $s = r+1$, in which case the intersection is $T^{r+1}(S_1)$.

It is easy to show that any topological conjugacy $h: \mathbf{E} \to \mathbf{E}'$ between T and some homeomorphism f of a Banach space $\mathbf{E}'$ is completely determined by its values on A. For, to find $h(x)$ for any $x \in E\backslash\{0\}$, we map x into A by some power T^n, then map by h and finally map by f^{-n}. Since h is a conjugacy, we have now reached $h(x)$, since $h = f^{-n}hT^n$. (Because h is a conjugacy, $fh = hT$ and so $f^rh = hT^r$ for all integers r.)

Conversely, let T and T' be contracting linear homeomorphisms of $\mathbf{E}$ and $\mathbf{E}'$ respectively, and let A and A' be corresponding fundamental domains.

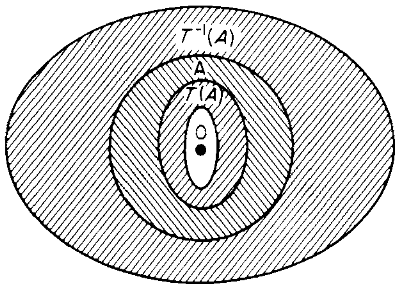

FIGURE 4.23

Suppose that $h: A \to A'$ is a homeomorphism such that

(4.23) $$\text{for all } x \in S_1, \qquad hT(x) = T'h(x).$$

Then we may extend h to a conjugacy from T to T' by putting $h(0) = 0$ and, for all $x \in \mathbf{E} \backslash \{0\}$, $h(x) = (T')^{-n} h T^n(x)$, where $T^n(x) \in A$. By (4.23), if $T^n(x) \in S_1$ then $(T')^{-n} h T^n(x) = (T')^{-n-1} h T^{n+1}(x)$, and so h is well-defined. In like fashion we can construct an inverse $h^{-1}: \mathbf{E} \to \mathbf{E}$ from $h^{-1}: A' \to A$. Continuity of h (and of h^{-1}) is in doubt only at 0. To deal with this, observe that h maps $\{T^n(B_1): n \in \mathbf{Z}\}$, which is a basis of neighbourhoods of 0 in $\mathbf{E}$, onto a similar basis $\{(T')^n(B_1): n \in \mathbf{Z}\}$ in $\mathbf{E}'$. The result that we have now proved may be formulated, rather loosely, as follows:

(4.24) Theorem. *Contracting linear homeomorphisms are topologically conjugate if and only if they are topologically conjugate on fundamental domains.* □

In the following important case, we may explicitly construct a conjugacy between fundamental domains.

(4.25) Theorem. *Let T and T' belong to the same path component of $GL(\mathbf{E})$ and have spectral radii <1. Then T and T' are topologically conjugate.*

Proof. Let T and T' be metric contractions with respect to norm $|\ |$ and $|\ |'$ respectively on $\mathbf{E}$, and let A and A' be the corresponding fundamental domains (between S_1 and $T(S_1)$ and between S_1' and $T'(S_1')$). Pick ε with $0 < \varepsilon < \min\{1 - |T|, 1 - |T'|'\}$, and let $I = [1-\varepsilon, 1]$. Since $T'T^{-1}$ and the identity map *id* belong to the same path component of $GL(\mathbf{E})$, there exists a continuous map $g: I \to GL(\mathbf{E})$ with $g(1-\varepsilon) = T'T^{-1}$ and $g(1) = id$. Identifying $I \times S_1$ and $I \times S_1'$ with annuli in $\mathbf{E}$, by putting $(t, x) = tx$, we have a homeomorphism $h: I \times S_1 \to I \times S_1'$ defined by

$$h(tx) = \frac{tg(t)(x)}{|g(t)(x)|'}.$$

We extend to a homeomorphism $h: A \to A'$, by mapping the line segment $[(1-\varepsilon)x, x/|T^{-1}(x)|]$ linearly onto the line segment

$$\left[(1-\varepsilon)\frac{T'T^{-1}(x)}{|T'T^{-1}(x)|'}, \frac{T'T^{-1}(x)}{|T^{-1}(x)|'}\right]$$

for each $x \in S_1$. Notice that for all $y \in S_1$,

$$y = \frac{T^{-1}(x)}{|T^{-1}(x)|}$$

for some $x \in S_1$, and that

$$hT(y) = h\left(\frac{x}{|T^{-1}(x)|}\right) = \frac{T'T^{-1}(x)}{|T^{-1}(x)|'} = T'h\left(\frac{T^{-1}(x)}{|T^{-1}(x)|}\right) = T'h(y).$$

Thus (4.23) holds, and we can apply Theorem 4.24 to complete the argument. We leave the details of checking that $h: A \to A'$ is a homeomorphism to the reader. □

This result enables us to classify hyperbolic linear automorphisms up to topological conjugacy, in finite dimensional spaces. We concern ourselves particularly with $\mathbf{R}^n$. In the first place, we have

(4.26) Corollary. *For each $n > 0$ there are exactly two topological conjugacy classes of contracting linear homeomorphisms of $\mathbf{R}^n$, consisting respectively of orientation preserving and orientation reversing maps.*

Proof. $GL(\mathbf{R}^n)$ has exactly two path components, consisting of orientation preserving and orientation reversing maps (distinguished by the sign of the determinant). These give distinct topological conjugacy classes, since any topological conjugate of an orientation preserving homeomorphism is orientation preserving. This is most easily proved by extending such homeomorphisms to the one-point compactification S^n of $\mathbf{R}^n$, and using the fact that if $f: \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism, and $f: S^n \to S^n$ is its extension to S^n, then the induced isomorphism of homology, maps a generator 1 of $H_n(S^n) \cong \mathbf{Z}$ to 1 or -1 according as f is orientation preserving or reversing (see Greenberg [1] for details). □

(4.27) Corollary. *Two hyperbolic linear homeomorphisms of $\mathbf{R}^n$ are topologically conjugate if and only if*

(i) *they are isomorphic,*

(ii) *their stable components are either both orientation preserving or both orientation reversing, and*

(iii) *their unstable components are either both orientation preserving or both orientation reversing.*

Proof. Suppose that T, $T' \in HL(\mathbf{R}^n)$ satisfy conditions (i) to (iii). For simplicity of notation, we put $\mathbf{R}^n = \mathbf{E}$. Let $L \in GL(\mathbf{E})$ map $\mathbf{E}_s(T)$ onto $\mathbf{E}_s(T')$ and $\mathbf{E}_u(T)$ onto $\mathbf{E}_u(T')$. By Corollary 4.26, the restrictions of T and $L^{-1}T'L$ to $\mathbf{E}_s(T)$ are topologically conjugate, and so are their restrictions to $\mathbf{E}_u(T)$. This enables us to construct a topological conjugacy from T to $L^{-1}T'L$, and hence to T'.

Conversely, let h be a topological conjugacy from $T \in HL(\mathbf{E})$ to $T' \in HL(\mathbf{E})$. Then h preserves 0, the unique fixed point of T and T'. By the continuity of h and of h^{-1}, $T^n(x) \to 0$ as $n \to \infty$ if and only if $hT^n(x) = (T')^n h(x) \to 0$ as $n \to \infty$. Thus h maps $\mathbf{E}_s(T)$ onto $\mathbf{E}_s(T')$, and similarly $\mathbf{E}_u(T)$ onto $\mathbf{E}_u(T')$. Since dimension is a topological invariant (§ 1 of Chapter 4 of Hu [2]) T and T' are isomorphic. Conditions (ii) and (iii) now follow by Corollary 4.26. □

It follows immediately from this theorem that there are exactly $4n$ topological conjugacy classes of hyperbolic linear automorphisms of $\mathbf{R}^n$ ($n \geqslant 1$). (See Examples 4.14 and 4.15.)

(4.28) Exercise. Prove that, for any $n > 0$, $GL(\mathbf{R}^n)$ has, as asserted above, precisely two path components. (*Hint*: move $T \in GL(\mathbf{R}^n)$ into the subspace $GL(\mathbf{R}) \times GL(\mathbf{R}^{n-1})$.)

(4.29) Exercise. Classify hyperbolic linear automorphisms of $\mathbf{C}^n$ up to topological conjugacy.

For infinite dimensional Banach spaces, the situation is less clear. Homeomorphic Banach spaces need not be linearly homeomorphic, so that topological conjugacy does not immediately imply isomorphism. Furthermore, the connectivity properties of $GL(\mathbf{E})$ are not fully understood for an arbitrary Banach space $\mathbf{E}$. In several important cases (for example if $\mathbf{E}$ is a Hilbert space—see Kuiper [1]) $GL(\mathbf{E})$ is path connected, and so *any* two linear contractions are topologically conjugate. However our main stability theorem, which we now prove, is, happily, valid for all Banach spaces.

(4.30) Theorem. *For any Banach space* $\mathbf{E}$, *any* $T \in HL(\mathbf{E})$ *is stable in* $L(\mathbf{E})$ *with respect to topological conjugacy.*

Proof. Consider the ball B in $L(\mathbf{E})$ with centre T and radius $d > 0$. Let $T^1 \in B$ and, for each $t \in [0, 1]$, define T^t by $T^t = (1-t)T + tT^1$. As in the proof of Theorem 4.21, we choose d so small that for all T^1 and $t \in [0, 1]$, the restrictions $P^t : \mathbf{E}_s(T) \to \mathbf{E}_s(T^t)$ and $Q^t : \mathbf{E}_s(T^t) \to \mathbf{E}_s(T)$ of the projections $f(T^t)$ and $f(T)$ respectively are isomorphisms. We now compare T_s and $(P^1)^{-1}(T^1)_s P^1$, both of which are in $GL(\mathbf{E}_s(T))$ and have spectral radius <1. We show that these two maps are in the same path component of $GL(\mathbf{E}_s(T))$. In fact we assert that $t \mapsto (P^t)^{-1}(T^t)_s P^t$ is a path joining the one to the other. The only difficulty is whether this map of $[0, 1]$ into $GL(\mathbf{E}_s(T))$ is continuous. Since f is continuous the map from $[0, 1]$ to $L(\mathbf{E}_s(T))$ taking t to $Q^t(T^t)_s P^t$ (which is the restriction of $f(T)(T^t)f(T^t)$) is continuous. Similarly the map taking t to $Q^t P^t$ is continuous. Hence the map taking t to $(Q^t P^t)^{-1} Q^t(T^t)_s P^t$ is continuous, as stated. To complete the proof of the theorem, we apply Theorem 4.25, giving that T_s is topologically conjugate to $(P^1)^{-1}(T^1)_s P^1$, and hence to $(T^1)_s$. A similar argument holds for the unstable components. Thus T is topologically conjugate to T^1. □

(4.31) Corollary. *Let* $\mathbf{E}$ *be finite dimensional. Then* $HL(\mathbf{E})$ *is the stable set of* $GL(\mathbf{E})$ *with respect to topological conjugacy. Furthermore* $HL(\mathbf{E})$ *is an open dense subset of* $GL(\mathbf{E})$.

Proof. By Theorem 4.30 any hyperbolic map is stable with respect to topological conjugacy. If $T \in GL(\mathbf{E})$ is not hyperbolic, every neighbourhood of T meets two different isomorphism classes of $HL(\mathbf{E})$ (by Exercise 4.22), and hence meets two different topological conjugacy classes (by Corollary 4.27). Thus T is unstable. This proves that $HL(\mathbf{E})$ is the stable set of $GL(\mathbf{E})$, and also that $HL(\mathbf{E})$ is dense in $GL(\mathbf{E})$. It is open in $GL(\mathbf{E})$ by Theorem 4.20. □

In the above theory we have not bothered to analyse the size of perturbation that a given hyperbolic map admits without change to its topological conjugacy class. We shall, in the next chapter, return to Theorem 4.30 and give it a new proof, which is better designed to deal with this point.

V. HYPERBOLIC LINEAR VECTOR FIELDS

We now turn to a study of linear vector fields. We are interested in determining which ones are stable in $L(\mathbf{E})$ with respect to topological equivalence, and once again we find we can give a complete description for finite dimensional $\mathbf{E}$. Let $T \in L(\mathbf{E})$. We say that T is a *hyperbolic linear vector field* if its spectrum $\sigma(T)$ does not intersect the imaginary axis of $\mathbf{C}$. We denote the set of all hyperbolic linear vector fields on $\mathbf{E}$ by $EL(\mathbf{E})$. As we have commented earlier, the notion of hyperbolicity depends on whether we are regarding T as a vector field or a diffeomorphism. It will, we hope, always be clear from the context in which sense we are using the term. The connection between the two notions is expressed in the following propositions, whose proofs are straightforward exercises in spectral theory (see Theorem 4.55 of the appendix). The special case $\mathbf{E} = \mathbf{R}^n$ has already been discussed above. The linear automorphism $\exp(tT)$ may be defined either by its power series or by a contour integral, as in (4.50) of the appendix.

(4.32) Proposition. *A linear vector field T is hyperbolic if and only if, for some (and hence all) non-zero real t,* $\exp(tT)$ *is a hyperbolic automorphism.* □

(4.33) Proposition. *The linear vector field T has integral flow* $\phi : \mathbf{R} \times \mathbf{E} \to \mathbf{E}$ *given by* $\phi(t, x) = \exp(tT)(x)$. □

(4.34) Example. The linear vector on $\mathbf{R}^2$ defined by $T(x, y) = (-x, y)$ is hyperbolic. The integral flow is illustrated by Figure 4.16, branches of the hyperbolae now being orbits of T, as are the positive and negative semi-axes and the point 0.

Suppose that $T \in EL(\mathbf{E})$. Let $\sigma_s(T) = \{\lambda \in \sigma(T) : \text{re}\, \lambda < 0\}$ and $\sigma_u(T) = \{\lambda \in \sigma(T) : \text{re}\, \lambda > 0\}$. Let D be a contour in the Argand diagram consisting of the line segment $[-ir, ir]$ and the semi-circle

$$\left\{ r e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\},$$

where r is large enough for D to enclose σ_s. Then the spectral decomposition theorem gives us, correspondingly, a T-invariant direct sum splitting $\mathbf{E} = \mathbf{E}_s \oplus \mathbf{E}_u$ such that $\sigma(T_s) = \sigma_s(T)$ and $\sigma(T_u) = \sigma_u(T)$, where $T_s \in L(\mathbf{E}_s)$ and $T_u \in L(E_u)$ are the restrictions of T. Once again we call $\mathbf{E}_s$ and T_s *stable summands*, and $\mathbf{E}_u$ and T_u *unstable summands*. Similarly $\mathbf{E}_s$ and $\mathbf{E}_u$ are again called *stable* and *unstable manifolds* at 0.

(4.35) Exercise. Prove that, for all $t > 0$, the stable summand of the hyperbolic automorphism $\exp(tT)$ is $\exp(tT_s)$. Similarly for unstable summands.

The map $\exp : L(\mathbf{E}) \to L(\mathbf{E})$ is continuous (see the appendix), and thus, since $HL(\mathbf{E})$ is open in $L(\mathbf{E})$, Proposition 4.32 implies that $EL(\mathbf{E})$ is open in $L(\mathbf{E})$. We define the term *isomorphic* for hyperbolic vector fields exactly as for hyperbolic automorphisms. Since the stable and unstable summands of $\mathbf{E}$ are the same with respect to $T \in EL(\mathbf{E})$ as they are with respect to $\exp T \in HL(\mathbf{E})$ (by Exercise 4.35), the fact that $T \in EL(\mathbf{E})$ is stable with respect to isomorphism follows immediately from the corresponding result for $HL(\mathbf{E})$. We sum up these observations in the following proposition.

(4.36) Theorem. *Let* $T \in EL(\mathbf{E})$. *Then* $T = T_s \oplus T_u$, *where* $\sigma(T_s) = \sigma_s(T)$ *and* $\sigma(T_u) = \sigma_u(T)$. *The set* $EL(\mathbf{E})$ *is open in* $L(\mathbf{E})$, *and its elements are stable with respect to isomorphism.* □

Let $T \in EL(\mathbf{E})$. Since, for any $t \neq 0$, $\exp(tT) \in HL(\mathbf{E})$, we may choose (by Theorem 4.19) a norm on $\mathbf{E}$ with respect to which the stable and unstable summands of $\exp(tT)$ are respectively a metric contraction and a metric expansion. We need a stronger result than this, however. We may deal with the stable and unstable manifolds separately.

(4.37) Theorem. *Let $T \in EL(\mathbf{E})$ have spectrum $\sigma(T) = \sigma_s(T)$. Then there exists a norm $\| \ \|$ on $\mathbf{E}$ equivalent to the given one such that, for any non-zero $x \in E$, the map from $\mathbf{R}$ to $]0, \infty[$ taking t to $\|\exp(tT)(x)\|$ is strictly decreasing and surjective. There exists $b > 0$ such that, for all $t > 0$, $\|\exp(tT)\| \leq e^{-bt}$.*

Proof. Let d be the distance from $\sigma(T)$ to the imaginary axis. By the spectral mapping theorem the spectral radius $r(\exp(tT))$ of $\exp(tT)$ is e^{-dt} for all $t \in \mathbf{R}$. Choose b and c with $0 < b < c < d$, and a norm $| \ |$ on $\mathbf{E}$ equivalent to the given one, such that $|\exp T| \leq e^{-c}$ (this is possible by Theorem 4.19). We first show that $\int_0^\infty |e^{bt} \exp(tT)| \, dt$ converges. Let $|\exp(tT)|$ be bounded by $A > 0$ for $t \in [0, 1]$. Then, for all integers m and n with $0 \leq m < n$,

$$\begin{aligned} \int_m^n |e^{bt} \exp(tT)| \, dt &\leq \sum_{r=m}^{n-1} e^{b(r+1)} e^{-rc} A \\ &\leq A e^b \sum_{r=m}^{n-1} e^{-r(c-b)} \\ &\leq A e^b e^{-m(c-b)} (1 - e^{b-c})^{-1}, \end{aligned}$$

which tends to zero as $m \to \infty$.

One easily checks that the relation

$$\|x\| = \int_0^\infty |e^{bt} \exp(tT)(x)| \, dt$$

defines a norm $\| \ \|$ on $\mathbf{E}$. Moreover, for all $x \in \mathbf{E}$,

$$\|x\| < |x| \int_0^\infty |e^{bt} \exp(tT)| \, dt.$$

By the interior mapping theorem, $\| \ \|$ is equivalent to $| \ |$.

Finally, for all $t \in \mathbf{R}$,

$$\begin{aligned} \|\exp(tT)(x)\| &= \int_0^\infty |e^{bu} \exp((u+t)T)(x)| \, du \\ &= e^{-bt} \int_t^\infty |e^{bu} \exp(uT)(x)| \, du, \end{aligned}$$

which shows that $\|\exp(tT)(x)\|$ decreases from ∞ to 0 as t increases from $-\infty$ to ∞ (for fixed $x \neq 0$). Also, for $t > 0$,

$$\|\exp(tT)(x)\| \leq e^{-bt} \|x\|,$$

as required. □

Theorem 4.37 gives a very precise hold on the phase portrait of T. Let S_1 be the unit sphere in E with respect to the norm $\| \ \|$ of Theorem 4.37. Then, with the exception of the fixed point 0, every orbit of T crosses S_1 in precisely one point. Thus S_1 is a sort of cross section to the integral flow of T. We may sharpen this remark as follows:

(4.38) Corollary. *If* $\phi: \mathbf{R} \times \mathbf{E} \to \mathbf{E}$ *is the integral flow of* T, *then* ϕ *maps* $\mathbf{R} \times S_1$ *homeomorphically onto* $\mathbf{E} \backslash \{0\}$.

Proof. We know that ϕ is continuous, and Theorem 4.37 shows that it maps $\mathbf{R} \times S_1$ bijectively onto $\mathbf{E} \backslash \{0\}$. Thus it remains to prove that the inverse of the restricted map is continuous. Consider, in $\mathbf{E} \backslash \{0\}$, a sequence (x_n) converging to a point x. Then $x_n = \phi(t_n, y_n)$ for some $t_n \in \mathbf{R}$ and $y_n \in S_1$. The sequence (t_n) is bounded, since $\|x_n\| \leq e^{-bt_n}$ for $t_n > 0$ and $(\|x_n\|)$ converges to the non-zero number $\|x\|$. Let (t_{n_k}) be a convergent subsequence of (t_n), converging to t, say. Then, as $k \to \infty$, $y_{n_k} = \phi(-t_{n_k}, x_{n_k})$ converges to $y = \phi(-t, x)$, which is thus some point of S_1. If the sequence (t_n) has some other cluster point t', we similarly deduce that $\phi(t', x)$ is in S_1, contrary to Theorem 4.37. Thus (t_n) converges to t, and hence $y_n = \phi(-t_n, x_n)$ converges to y. This proves continuity of the inverse map at x. □

Now suppose that $T, T' \in EL(\mathbf{E})$ both have spectra in re $z < 0$. We can pick equivalent norms on $\mathbf{E}$ such that the unit spheres with respect to them are cross sections of the integral flows of T and T' restricted to $\mathbf{E} \backslash \{0\}$. This situation is illustrated in Figure 4.39, in which S_1 and S_1' are the unit spheres.

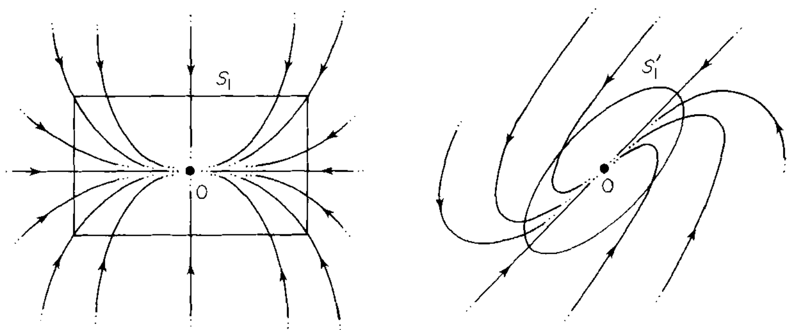

FIGURE 4.39

Since S_1 and S_1' are homeomorphic, there is an obvious way of constructing a flow equivalence from T to T'. Once the proposed flow equivalence is given on the unit spheres (mapping S_1 to S_1') its values elsewhere are determined up to the speeding up factor α in the definition of flow equivalence.

(4.40) Theorem. *Let* $T, T' \in HL(\mathbf{E})$ *have spectra in* re $z < 0$. *Then* T *and* T' *are flow equivalent.*

Proof. We show that there is a homeomorphism h of $\mathbf{E}$ such that, for all $(t, x) \in \mathbf{R} \times \mathbf{E}$,

(4.41) $$h\phi(t, x) = \psi(t, h(x))$$

where ϕ and ψ are the integral flows of T and T' respectively. Let $\| \ \|$ be as in the statement of Theorem 4.37, and let $\| \ \|'$ have the same property relative to T'. Let S_1 and S_1' be the unit spheres with respect to these norms. We define a homeomorphism $\bar{h}: S_1 \to S_1'$ by $\bar{h}(y) = y/\|y\|'$. Let $\bar{\phi}: \mathbf{R} \times S_1 \to \mathbf{E} \backslash \{0\}$ and $\bar{\psi}: \mathbf{R} \times S_1' \to \mathbf{E} \backslash \{0\}$ be the restrictions of ϕ and ψ. By Corollary 4.38, $\bar{\phi}$ and $\bar{\psi}$ are homeomorphisms. We define $h: \mathbf{E} \to \mathbf{E}$ by putting $h(0) = 0$ and letting h be the homeomorphism $\bar{\psi}(id \times \bar{h})\bar{\phi}^{-1}$ on $\mathbf{E} \backslash \{0\}$. Then h extends $\bar{h}$.

Also (4.41) holds trivially for $x = 0$. For $x \neq 0$, $x = \bar{\phi}(t', y)$ for some $t' \in \mathbf{R}$ and $y \in S_1$, and so, for all $t \in \mathbf{R}$,

$$\begin{aligned} h\phi(t, x) &= h\phi(t + t', y) \\ &= \bar{\psi}(id \times \bar{h})(t + t', y) \\ &= \bar{\psi}(t + t', \bar{h}(y)) \\ &= \psi(t, \bar{\psi}(t', \bar{h}(y)) \\ &= \psi(t, \bar{\psi}(id \times \bar{h})\bar{\phi}^{-1}(\bar{\phi}(t', y)) \\ &= \psi(t, h(x)). \end{aligned}$$

Finally we show that h is continuous at 0 (the proof for h^{-1} being similar). Let W be any neighbourhood of 0 in $\mathbf{E}$. By Theorem 4.37, there exists $t_0 > 0$ such that, for all $s \geq t_0$, $\psi^s(S_1') \subset W$. For all $x \in E$ with $\|x\| < \|\phi^{-t_0}\|^{-1}$, $\|\phi(-t_0, x)\| \leq 1$ and so, by Theorem 4.37, the number s such that $\phi(-s, x) \in S_1$ satisfies $s > t_0$. Thus, since $h(x) = \psi(s, h\phi(-s, x))$ and $h\phi(-s, x) \in S_1'$, $h(x) \in W$. □

By applying this theorem to stable and unstable summands separately, we have:

(4.42) Corollary. *If T and $T' \in EL(\mathbf{E})$ are isomorphic, then they are flow equivalent (and thus topologically equivalent).* □

By Theorem 4.36, we now deduce our main stability result:

(4.43) Corollary. *Any $T \in EL(\mathbf{E})$ is stable in $L(\mathbf{E})$ with respect to flow equivalence (and thus with respect to topological equivalence).* □

(4.44) Exercise. Let $\mathbf{E}$ be finite dimensional. Prove

(i) $EL(\mathbf{E})$ is dense in $L(\mathbf{E})$,

(ii) for all T, $T' \in EL(\mathbf{E})$ the following statements are equivalent:
   (a) T is flow equivalent to T',
   (b) T is topologically equivalent to T',
   (c) T is isomorphic to T',

(iii) $EL(\mathbf{E})$ is the stable set of $L(\mathbf{E})$ with respect to
   (a) flow equivalence, and
   (b) topological equivalence.

Thus, for $\mathbf{E} = \mathbf{R}^n$ there are precisely $n + 1$ flow equivalence classes in $EL(\mathbf{E})$, classified by the dimension of the stable summand. (See Examples 4.8 and 4.9.)

Appendix 4

I. SPECTRAL THEORY

In this brief survey of spectral theory we omit a number of proofs that can be found in our main reference for the subject, which is Dunford and Schwartz [1]. We shall also assume without comment easy generalizations of certain parts of the theory of functions of one complex variable from $\mathbf{C}$-valued to $\mathbf{E}$-valued functions, where $\mathbf{E}$ is any complex Banach space. For these we refer to § 14 of Chapter 3 of Dunford and Schwartz [1].

Let $\mathbf{E}$ be a complex Banach space, with $\mathbf{E} \neq \{0\}$, and let $T \in L(\mathbf{E})$. The *resolvent* $\rho(T)$ of T is the set $\{\lambda \in \mathbf{C}: T - \lambda(id)$ is a homeomorphism$\}$. Clearly $\rho(T)$ is an invariant of the linear conjugacy class of T. The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbf{C}$. For finite dimensional $\mathbf{E}$, $\sigma(T)$ is the set of zeros of det $(T - \lambda(id))$, the eigenvalues of T. Spectral theory links properties of the map T with properties of its spectrum. For example, we shall find that the size of $\sigma(T)$ reflects the size of T itself (as an element of the Banach space $L(\mathbf{E})$), and that, when $\sigma(T)$ is disconnected, T splits into factors having the components of $\sigma(T)$ as spectra.

We define a map $R: \rho(T) \to L(\mathbf{E})$ by $R(\lambda) = (T - \lambda(id))^{-1}$. For fixed $\lambda \in \rho(T)$ and small $\eta \in \mathbf{C}$, the inverse of $T - (\lambda + \eta)id$ is the map

$$(T - \lambda(id))^{-1} \sum_{n=0}^{\infty} \eta^n (T - \lambda(id))^{-n}.$$

Thus $\rho(T)$ is open, and R is analytic. Notice the relation

(4.45) $\quad (T - \lambda(id))^{-1} - (T - \mu(id))^{-1} = (\lambda - \mu)(T - \lambda(id))^{-1}(T - \mu(id))^{-1}$

resulting from the trivial manipulation

$$L^{-1} - M^{-1} = M^{-1}ML^{-1} - M^{-1}LL^{-1} = M^{-1}(M - L)L^{-1}.$$

We may continue manipulating to obtain

$$(L^{-1}-M^{-1})(id+(M-L)L^{-1})=L^{-1}(M-L)L^{-1}$$

and then, for fixed L and small $|M-L|$,

$$L^{-1}-M^{-1}=\sum_{r=0}^{\infty}(-1)^r L^{-1}((M-L)L^{-1})^{r+1}.$$

We use this in the particular case

(4.46) $$(T-\lambda(id))^{-1}-(T'-\lambda(id))^{-1}$$
$$=\sum_{r=0}^{\infty}(-1)^r(T-\lambda(id))^{-1}((T'-T)(T-\lambda(id))^{-1})^{r+1}$$

showing that $R(\lambda)$ depends analytically on T, for fixed λ.

We define a non-negative real number $r(T)$ by

$$r(T)=\limsup_{n\to\infty}|T^n|^{1/n}.$$

Notice that $r(T)\leqslant|T|$ (where $|\ |$ denotes both the given norm on $\mathbf{E}$ and the induced norm on $L(\mathbf{E})$). The series $-\sum_{n=0}^{\infty}\lambda^{-n-1}T^n$ converges to $R(\lambda)$ for $|\lambda|>r(T)$, and diverges for $|\lambda|<r(T)$. Since this is a Laurent series for R centred on 0, we deduce that $r(T)$ is precisely the *spectral radius*

$$\sup\{|\lambda|: \lambda\in\sigma(T)\}$$

of T. Since $\sigma(T)$ is bounded and closed it is compact. Moreover, since $\mathbf{E}\neq\{0\}$, $\sigma(T)$ is non-empty. For, otherwise, the domain of R is $\mathbf{C}$, so the above Laurent series is a Taylor series, which is impossible, since the coefficient of λ^{-1} is *id*.

It is possible to simplify the formula for $r(T)$. We observe that, for all $n>0$, $\lambda\in\sigma(T)$ implies $\lambda^n\in\sigma(T^n)$, because $T-\lambda(id)$ is a factor of $T^n-\lambda^n(id)$. In this case, $|\lambda^n|\leqslant|T^n|$, and so $|\lambda|\leqslant|T^n|^{1/n}$. Thus $r(T)\leqslant\lim_{n\to\infty}\inf|T^n|^{1/n}$, and it follows that $\lim_{n\to\infty}|T^n|^{1/n}$ exists and equals $r(T)$. We now give a very useful characterization of $r(T)$, due to Holmes [1].

(4.47) Theorem. *Let N be the set of all norms on $L(\mathbf{E})$ induced by norms on $\mathbf{E}$ equivalent to the given norm $|\ |$. Then*

$$r(T)=\inf\{\|T\|: \|\ \|\in N\}.$$

Proof. It follows from the definition of $r(T)$ that $r(T)\leqslant\|T\|$ for all $\|\ \|\in N$. Suppose that $r(T)<\kappa<|T|$. Then, for some integer $m>0$ and all $n\geqslant m$, $|T^n|\leqslant\kappa^n$. We define a norm $\|\ \|$ on $\mathbf{E}$ by

$$\|x\|=\max\{|x|, |\kappa^{-1}T(x)|, \ldots, |\kappa^{-m+1}T^{m-1}(x)|\}.$$

It is easy to check that $\| \ \|$ is a norm. Moreover, $|x| \leqslant \|x\|$, so, by the interior mapping theorem, $| \ |$ is equivalent to $\| \ \|$. Now

$$\begin{aligned}\|T(x)\| &= \max\{|T(x)|, |\kappa^{-1}T^2(x)|, \ldots, |\kappa^{-m+1}T^m(x)|\}\\ &= \kappa \max\{|\kappa^{-1}T(x)|, \ldots, |\kappa^{-m+1}T^{m-1}(x)| \ |\kappa^{-m}T^m(x)|\}\\ &\leqslant \kappa\|x\|,\end{aligned}$$

since $|\kappa^{-m}T^m| \leqslant 1$. Thus $\|T\| \leqslant \kappa$. □

In particular:

(4.48) Corollary. *The spectral radius $r(T)$ of T is strictly less than 1 if and only if there is an equivalent norm on $\mathbf{E}$ with respect to which T is a metric contraction.* □

(4.49) Exercise. Prove that if T is a linear automorphism then $\sigma(T^{-1}) = \sigma(T)^{-1}(=\{\lambda^{-1}: \lambda \in \sigma(T)\})$. Deduce that $\sigma(T)$ lies outside the unit circle S^1 of $\mathbf{C}$ if and only if T is a metric expansion with respect to some equivalent norm on $\mathbf{E}$. Deduce that if $\mathbf{E}$ has a T-invariant direct sum decomposition $\mathbf{E} = \mathbf{E}_s \oplus \mathbf{E}_u$, and if T is a metric contraction on $\mathbf{E}_s$ and a metric expansion on $\mathbf{E}_u$, then T is hyperbolic (i.e. $\sigma(T) \cap S^1$ is empty).

If $p(z)$ is a complex polynomial, say $p(z) = \sum_{n=0}^{s} \alpha_n z^n$, where $\alpha_n \in \mathbf{C}$, then we can define a function $p: L(\mathbf{E}) \to L(\mathbf{E})$ (by an abuse of notation) by $p(T) = \sum_{n=0}^{s} \alpha_n T^n$, and we can handle power series in the same way. The technique of contour integration allows us to carry this process one step further. Let $f: U \to \mathbf{C}$, where U is open in $\mathbf{C}$, be an analytic function. Suppose that $\sigma(T) \subset K \subset U$, where the boundary ∂K of the compact set K is a union of contours (K need not be connected). We define $f(T) \in L(\mathbf{E})$ by

$$\textbf{(4.50)} \qquad f(T) = -\frac{1}{2\pi i}\int_{\partial K} f(\lambda)R(\lambda)\, d\lambda,$$

where ∂K is positively oriented with respect to K in the usual way. Note that $f(T)$ is well defined (by (4.45)), is independent of the choice of K, and that we obtain an analytic (by (4.46)) function f from the open subset $\{T \in L(\mathbf{E}): \sigma(T) \subset U\}$ into $L(\mathbf{E})$. The proofs of the following propositions appear in § 3 of Chapter 7 of Dunford and Schwartz [1]).

(4.51) Proposition. *For all $\alpha, \beta \in \mathbf{C}$ and all analytic functions $f, g: U \to \mathbf{C}$, $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$.* □

(4.52) Proposition. *For all analytic functions $f, g: U \to \mathbf{C}$, let $g.f: U \to \mathbf{C}$ be the map whose value at z is the product $g(z)f(z)$. Then $(g.f)(T)$ is the composite $g(T)f(T)$ of the maps $g(T)$ and $f(T)$.* □

(4.53) Proposition. *If $f(z)=\sum_{r=0}^{\infty}\alpha_r z^r$ converges on a neighbourhood of $\sigma(T)$, then $f(T)=\sum_{r=0}^{\infty}\alpha_r T^r$.* □

(4.54) Proposition. *If $\sigma(T)$ is contained in the domain of the composite gf of analytic maps f and g then $(gf)(T)=g(f(T))$.* □

(4.55) Theorem. (*Dunford's spectral mapping theorem*). *For all analytic maps $f: U \to \mathbf{C}$, $f(\sigma(T))=\sigma(f(T))$.* □

(4.56) Corollary. (*Spectral decomposition theorem*). *Let $T\in GL(\mathbf{E})$, and let D be a contour in the Argand diagram $\mathbf{C}$ such that $\sigma(T)\cap D$ is empty. Let $\sigma_s(T)$ be the part of $\sigma(T)$ inside D and let $\sigma_u(T)$ be the part outside D. Then there is a direct sum decomposition $\mathbf{E}=\mathbf{E}_s\oplus\mathbf{E}_u$ into T-invariant closed subspaces such that, if $T_s\in L(\mathbf{E}_s)$ and $T_u\in L(\mathbf{E}_u)$ are the restrictions of T, then $\sigma(T_s)=\sigma_s(T)$ and $\sigma(T_u)=\sigma_u(T)$.*

Proof. We define the analytic complex function f on the complement of D in C by

$$f(\lambda)=\begin{cases}1 & \text{if } \lambda \text{ is inside } D,\\ 0 & \text{if } \lambda \text{ is outside } D.\end{cases}$$

Let K_s (resp. K_u) be a compact subset inside (resp. outside) D and containing $\sigma_s(T)$ (resp. $\sigma_u(T)$), and let $K=K_s\cup K_u$. We suppose further that ∂K is a union of contours, so that $f(T)$ may be defined by the formula (4.50). By Proposition 4.52 since $f.f=f$, $f(T)^2=f(T)$. That is to say, $f(T)$ is a projection. Thus $\mathbf{E}=\mathbf{E}_s(T)\oplus\mathbf{E}_u(T)$, where $\mathbf{E}_s(T)$ is the kernel of $id-f(T)$ (or, equivalently, the image of $f(T)$), and $\mathbf{E}_u(T)$ is the kernel of $f(T)$. Since T commutes with $f(T)$ (by Proposition 4.52 *any* two functions of T commute), $\mathbf{E}_s(T)$ and $\mathbf{E}_u(T)$ are invariant under T. Moreover $Tf(T)=T_s\oplus 0=h(T)$, where, by Proposition 4.52,

$$h(\lambda)=\begin{cases}\lambda & \text{if } \lambda \text{ is inside } D,\\ 0 & \text{if } \lambda \text{ is outside } D.\end{cases}$$

By Theorem 4.55,

$$\sigma(Tf(T))=h(\sigma(T))=\sigma_s(T)\cup\{0\}$$

(or $\sigma_s(T)$, if $\sigma_u(T)$ is empty). Since T_s is an automorphism $0\notin\sigma(T_s)$, and hence $\sigma(T_s)=\sigma_s(T)$. Similarly $\sigma(T_u)=\sigma_u(T)$. □

The above theory is for a complex Banach space. The corresponding theory for real spaces can be extracted easily from it, as follows.

Suppose that $\mathbf{E}$ is a real Banach space and that $T\in GL(\mathbf{E})$, the space of all (topological) $\mathbf{R}$-linear automorphisms of $\mathbf{E}$. We form the space $\tilde{\mathbf{E}}=\mathbf{E}\otimes_{\mathbf{R}}\mathbf{C}$ (any point of which can be written uniquely as $x\otimes 1+y\otimes i$, where $x, y\in\mathbf{E}$).

The space $\tilde{\mathbf{E}}$ has a natural complex vector space structure (in which scalar multiplication acts on the second factor $\mathbf{C}$ of $\tilde{\mathbf{E}}$) and we may give it a norm $|x \otimes 1 + y \otimes i| = \max\{|x|, |y|\}$, which makes it into a Banach space. Then T induces a $\mathbf{C}$-linear automorphism $\tilde{T} \in L(\tilde{\mathbf{E}})$ by its action on the first factor $\mathbf{E}$. We define the spectrum $\sigma(T)$ of T to be $\sigma(\tilde{T})$. The crucial fact that emerges is that, if the spectrum of $\tilde{T}$ decomposes as in Corollary 4.56 and if D is symmetric about the real axis, then the projection $f(\tilde{T})$ in the proof of the Corollary acts independently on the real and imaginary parts of $\tilde{\mathbf{E}}$, and so decomposes the real space $\mathbf{E}$. Once we have proved this, the real version of Corollary 4.56 follows immediately.

(4.57) Theorem. *Let the spectrum of $\tilde{T}$ decompose as in Corollary 4.56. Suppose that D is symmetric about the real axis. Then the projection $\tilde{P} = f(\tilde{T})$ is of the form $\tilde{P}(x \otimes 1 + y \otimes i) = P(x) \otimes 1 + P(y) \otimes i$, for all $x, y \in \mathbf{E}$, where P is a projection of $\mathbf{E}$.*

Proof. Any $R \in L(\tilde{E})$ may be written in matrix form as

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the entries are in $L(\mathbf{E})$. Let $\bar{R}$ denote the matrix

$$\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}.$$

Note that, for all $z \in \tilde{\mathbf{E}}$, the complex conjugate $\overline{Rz} = \bar{R}\bar{z}$, and hence, for all $R_1, R_2 \in L(\mathbf{E})$, $\overline{R_1 R_2} = \bar{R}_1 \bar{R}_2$.

Now recall that $\tilde{P} = f(\tilde{T})$ is defined by (4.50), where K is as described in the proof of Corollary 4.56. Moreover we may choose K so that ∂K is symmetric about the real axis. Now, in the formula (4.50), $R(\lambda)$ denotes the resolvent $(\tilde{T} - \lambda(id))^{-1}$. By conjugating the relation $(\tilde{T} - \lambda(id))R(\lambda) = id$, we see that $\overline{R(\lambda)} = R(\bar{\lambda})$. Let ∂K_+ and ∂K_- be the portions of ∂K in the upper and lower half planes of the Argand diagram, with orientation induced by that of ∂V. Then by the symmetry, $\partial K_- = -\overline{\partial K_+}$. Thus

$$\begin{aligned} \tilde{P} &= \frac{1}{2\pi i}\left(\int_{\partial K_+} R(\lambda)\, d\lambda + \int_{\partial K_-} R(\lambda)\, d\lambda\right) \\ &= \frac{1}{2\pi i}\left(\int_{\partial K_+} R(\lambda)\, d\lambda - \int_{\partial K_+} R(\bar{\lambda})\, \overline{d\lambda}\right) \\ &= \frac{1}{2\pi} \int_{\partial K_+} \left(\frac{1}{i} R(\lambda)\, d\lambda + \overline{\frac{1}{i} R(\lambda)\, d\lambda}\right). \end{aligned}$$

Since the term in the bracket acts separately on the real and complex parts of $\tilde{\mathbf{E}}$, so does $\tilde{P}$. Thus $\tilde{P}$, being a projection, has the form

$$\tilde{P}(x \otimes 1 + y \otimes i) = P(x) \otimes 1 + Q(y) \otimes i.$$

But since $\tilde{P}$ is $\mathbf{C}$-linear $\tilde{P}(iz) = i\tilde{P}(z)$, and this implies that $P = Q$. □

It follows immediately that the stable summand $\tilde{T}_s : \mathbf{E}_s \to \mathbf{E}_s$ of $\tilde{T}$ is the complexification of an $\mathbf{R}$-linear automorphism $T_s : \mathbf{E}_s \to \mathbf{E}_s$ restricting T. Similarly for the unstable summand. Summing up:

(4.58) Corollary. (*Real spectral decomposition theorem*) *Let* $\mathbf{E}$ *be a real Banach space, let* $T \in GL(\mathbf{E})$ *and let* D *be a contour in the Argand diagram, symmetric about the real axis, such that* $\sigma(T) \cap D$ *is empty. Let* $\sigma_s(T)$ *be the part of* $\sigma(T)$ *inside* D *and let* $\sigma_u(T)$ *be the part outside* D. *Then there is a direct sum decomposition* $\mathbf{E} = \mathbf{E}_s \oplus \mathbf{E}_u$ *into* T*-invariant closed subspaces such that if* $T_s \in L(\mathbf{E}_s)$ *and* $T_u \in L(\mathbf{E}_u)$ *are the restrictions of* T, *then* $\sigma(T_s) = \sigma_s(T)$ *and* $\sigma(T_u) = \sigma_u(T)$. □