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STRUCTURE OF TRANSVERSAL LEAVES IN MULTIDIMENSIONAL SEMIDISPERSING BILLIARDS

N. I. Chernov

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1. Introduction

An important class of dynamical systems is made up of the systems of billiard type (billiards), which are generated by the motion of a particle in a d-dimensional domain of Euclidean space or a d-dimensional Euclidean torus T^d, out of which an open subset U is excised, while the boundary of the domain is piecewise smooth. We shall denote this domain by Q. A billiard particle moves rectilinearly with unit speed inside Q and is reflected from the boundary ∂Q in accord with the law "angle of incidence equals angle of reflection."

The so-called dispersing billiards, i.e., those such that the smooth components of the boundary ∂Q are strictly convex inside the domain Q, have the strongest ergodic properties. By virtue of this property of the boundary, close trajectories of the system diverge with exponential speed, as also in the case of smooth hyperbolic systems. For such billiard systems, the properties of ergodicity and mixing, and the K property have been proved (cf. [1, 2, 10]); also, in a series of cases the rate of decrease of correlations has been investigated with the help of the method of Markov partitions (cf. [11, 12]).

In the present paper we study billiard systems for which the boundary ∂Q is not strictly convex from within, i.e., for which subspaces of flat directions with zero curvature are possible. Such billiards are called semidispersing. A typical example is the model of a gas of rigid spheres. For this system the configuration space is a dN-dimensional cube V^N (dN-dimensional torus T^{dN}), from which cylinders in R^{dN} are excised (cf. [5, 6]). Each cylinder is the direct product of a (d — 1)-dimensional sphere and a Euclidean space $R^{d(N-2)}$. One can get other examples by considering a system of rigid spheres, split up into several noninteracting groups, and also systems of rigid spheres on a torus.

The ergodic properties of semidispersing billiards can depend on properties of the boundary; more precisely, the ergodic properties are determined by the nonparallelness of the flat directions of the boundary of the billiard at its various points. For example, the system of

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two rigid spheres on the torus T^2 has a first integral (total momentum) and is nonergodic, while at the same time such a system on the square is ergodic and is a K system (cf. [1]).

Just as in the case of hyperbolic systems, the ergodic properties of semidispersing billiards can be investigated with the help of transversal foliations, which are a generalization of the well-known horospherical foliations in the case of geodesic flows in spaces of negative curvature (cf. [13, 14]).

In the case of dispersing billiards, just as in geodesic flows, the dimension of the foliations is equal to $d - 1$, where d is the dimension of the configuration space. In semidispersing billiards the dimension of the foliations can be smaller, depending on the properties of the boundary ∂Q.

The goal of the present paper is the investigation of the transversal foliations and the calculation of their dimensions for general semidispersing billiards.

We proceed to precise formulations. We denote by M the phase space and by $\{T^t\}$ the dynamics of a semidispersing billiard in some domain Q in d-dimensional Euclidean space. A point $x \in M$ is a pair $x = (q, v)$, where $q \in Q$ and $v \in S^{d-1}$ (unit velocity vector), while points $x \in Q \times S^{d-1}$, whose trajectories pass through points of intersection of smooth components of the boundary ∂Q, are excluded from the space M. We denote the natural projection of the space M onto the domain Q by π and we call the point $\pi(x) \in Q$ the carrier of the point $x \in M$. The domain Q should satisfy certain conditions, which are given in Sec. 2 (cf. also [5], p. 132).

A billiard system $\{T^t\}$ preserves the measure $d\mu(x) = dl(q) \times d\omega_q$, where $dl(q)$ is Lebesgue measure in the domain Q and $d\omega_q$ is Lebesgue measure on the $(d - 1)$-dimensional sphere $S^{d-1}(q) = \pi^{-1}(q)$.

For each point $x = (q, v) \in M$ we denote by J(x) the hyperspace in $\mathbf{R}^d$, containing the point q and orthogonal to the vector v. In the space J(x) we introduce the linear operator B(x), defined by the formula

$$B(x)=\cfrac{I}{\tau_1 I+\cfrac{I}{2\cos\varphi_1 V_1^* K_1 V_1+U_1^{-1}\cfrac{I}{\tau_2 I+\cfrac{I}{2\cos\varphi_2 V_2^* K_2 V_2+\dots}}U_1}}, \tag{1}$$

on whose right side there is written an operator-continued fraction (the values of the variables and operators occurring in this formula are explained in detail in Sec. 3). Actually, this continued fraction determines a solution of the Jacobi equation for billiard systems. In Sec. 5 it is proved that the operator B(x) defines the tangent plane to a transversal leaf in the phase space. It is proved in [8] that the fraction B(x) converges almost everywhere in M and determines a symmetric nonnegative definite operator J(x). These properties allow one to decompose J(x) into the orthogonal sum of two B(x)-invariant spaces $J_0(x)$ and $J_+(x)$, which are, respectively, null and positive for the operator B(x).

We consider the function $j(x) = \dim J_+(x)$, defined almost everywhere on M. We denote by M^C the set of points $x \in M$, in a neighborhood of which the function j(x) is constant. In Sec. 3 it will be shown that this set is open and dense in the domain of definition of the operator B(x). We denote by $\hat{M}$ the set of points $x \in M$, for which one can find a $t \geqslant 0$, such that $\hat{x}(x) = T^t x \in M^C$. It is clear that $M^C \subset \hat{M}$ and the set $\hat{M}$ is invariant with respect to the action of $\{T^t\}$.

In Secs. 2-5 we prove:

THEOREM 1. Let $x_0 = (q_0, v_0) \in \hat{M}$. In some neighborhood U_{x_0} of the point x_0 there exists a manifold $W \subset \hat{M}$ of dimension $j(\hat{x}(x_0))$ with carrier $\tilde{W} = \pi(W)$, having the following properties:

(1) $x_0 \in W$ and the trajectories of points of the manifold W under the action of $\{T^t\}$ approach one another with exponential speed as $t \to \infty$;

(2) if $\tilde{W} = j(x)$ for some point $x = (q, v) \in W$, then the tangent space $\mathcal{T}_q \tilde{W}$ to the surface $\tilde{W}$ coincides with $J_+(x)$;

(3) let $x = (q, v)$ and $x' = (q + dq, v + dv)$ be close points on W; we let the origin of the vector dv coincide with the point q; then $dv \in \mathcal{T}_q \widetilde{W}$ and $dv = B(x)dq$.

Such a manifold W is called a locally transversal leaf (LTL) (cf. [2, 8]).

We formulate two corollaries of Theorem 1:

COROLLARY 1. The entropy of a semidispersing billiard is positive.

COROLLARY 2. The group of unitary operators generated by a semidispersing billiard has a countably multiple Lebesgue component in the spectrum.

Corollary 1 is proved in Sec. 5; Corollary 2 follows from the results of Ch. 13 in [5] (cf. also [8]).

2. Properties of Typical Phase Trajectories

We shall assume that the configuration space Q of a semidispersing billiard satisfies the following conditions (cf. also [5]):

1. The boundary ∂Q of the domain Q is piecewise smooth, consisting of a finite number of regular (smooth) components $\partial Q_1^{(o)}, \partial Q_2^{(o)}, \ldots, \partial Q_r^{(o)}$.

2. At points of intersection of two or more components of the boundary $\partial Q_i^{(o)}$, the vectors normal to them are linearly independent.

3. The operator of the second quadratic form K(q) of the surface ∂Q, with respect to the normal directed into the domain Q, is nonnegative definite at regular points $q \in \partial Q$.

Property 3 means that the boundary ∂Q is convex into the domain Q. The boundary $\partial M = \partial Q \times S^{d-1}$ of the phase space M will be piecewise smooth with regular components $\partial M_i^{(o)} = \partial Q_i^{(o)} \times S^{d-1}$, $1 \leqslant i \leqslant r$. We write $\partial M^{(0)} = \bigcup\limits_{i=1}^{r} \partial M_i^{(0)}$ and $M_1 = \{x \in \partial M^{(0)}: \ (n(q), x) \geqslant 0, \ q = \pi(x)\}$, where n(q) is a unit vector normal to ∂Q at the point q, directed into the domain Q. For each $x \in M_1$ we define $T_1 x = T^{s+o} x$, where $s > 0$ is the first moment of reflection of the trajectory $T^t x$ of the point x from the boundary ∂Q. The transformation T_1 on M_1 preserves the measure $d\mu(x) = (n(q), x)\, dl(q) d\omega_q$, where $dl(q)$ is the Riemannian volume on the surface ∂Q and $d\omega_q$ is the Lebesgue measure on the $(d - 1)$-dimensional hemisphere $S_+^{d-1}(q) = \{v \in S^{d-1}: (v, n(q)) \geqslant 0\}$ (cf. [5]). We denote by

$$S = \{x \in \partial M \setminus \partial M^{(0)}\} \bigcup \{x \in \partial M^{(0)}: \ (n(q), \ x) = 0, \quad q = \pi(x)\} \tag{2}$$

the set of singular points of the boundary.

For the existence of an LTL at the point $x \in M$ it is necessary that the trajectory of the point x satisfy certain conditions which are formulated below. The set M^T of points satisfying these conditions has full measure, which allows us to call the trajectories of these points typical in the phase space M.

Condition T.1. One can find a point $\hat{x} \in M$, such that for any neighborhood $U \subset M$ of the point $\hat{x}$ one can find a positive $\rho = \rho(U)$, for which the number of returns of the trajectory $T^t x$ to the neighborhood U in the time interval (0, T) is not less than ρT for all sufficiently large T.

Condition T.2. The trajectory $T^t x$ does not approach the singular points of the boundary too closely, which means the following: For arbitrary $\hat{c} > 0$ and $i \geqslant 2$ we consider in $\mathbf{R}^d$ the cylinder $Z_i(\hat{c})$ of radius $\hat{c} i^{-2}$, whose axis passes through the segment of the trajectory $T^t x$ between the $(i - 1)$-st and i-th reflections from the ∂Q. Condition T.2 is that one can find a positive $\hat{c} = \hat{c}(x)$ such that for each $i \geqslant 2$ the connected component of the cylinder $Z_i(\hat{c})$, lying in the domain Q around the i-th segment of the trajectory $T^t x$, intersects precisely two regular components of the boundary ∂Q.

The collection of cylinders figuring in Condition T.2 will be called the system of corridors of radius $\hat{c} i^{-2}$ about the trajectory of point x. We denote this system by Θ_0.

We shall prove the fullness of the measure of the set M^T. To prove the fullness of the measure of the set of points satisfying Condition T.1, we set $\hat{x} = x$ and we consider the characteristic function χ_U of the neighborhood $U = U(x)$. By virtue of the Birkhoff ergodic theorem, the limit

$$\lim_{n\to\infty} \frac{1}{n+1} [\chi_U(y) + \chi_U(T^1 y) + \ldots + \chi_U(T^n y)] = g(y),$$

exists almost everywhere, while

$$\int_M g(y)\mu(dy) = \int_M \chi_U(y)\mu(dy) > 0.$$

We consider the set $U_0 = U \cap \{y\colon g(y) = 0\}$. From the recurrence theorem of Poincaré it is easy to show that $\mu(U_0) = 0$, i.e., $g(y) > 0$ almost everywhere on U.

To prove the fullness of the measure of the set of points satisfying Condition T.2, we pass to the transformation T_1 of the boundary $M_1 \subset \partial M$. We consider the set $S \cup T_1^{-1}S$ (cf. (2)). The intersection of this set with each regular component of the boundary $\partial M_i^{(0)}$, $1 \leqslant i \leqslant r$ consists of a finite number of smooth compact manifolds of codimension 1 with the boundary. The Riemannian volume of the ε neighborhood of such a manifold does not exceed const·ε for all sufficiently small ε (cf., e.g., [9]). Application of the Borel–Cantelli lemma gives the necessary result (for analogous estimates, cf. [1]).

3. Geometry of the Leaves in Semidissipating Billiards

In this section we give properties of smooth families of billiard trajectories that will be needed later.

Let $\tilde{\Sigma}$ be an arbitrary C^2-smooth orientable manifold of codimension 1 without boundary in the domain Q and Σ be a continuous family of unit vectors normal to $\tilde{\Sigma}$ (exactly two such families exist). The manifold $\Sigma \subset M$ is called the leaf with carrier $\tilde{\Sigma}$. Points of the leaf Σ will be denoted by (q, v(q)), where $q \in \tilde{\Sigma}$ and v(q) is a vector normal to $\tilde{\Sigma}$.

We denote by B(q) the operator of the second quadratic form of the manifold $\tilde{\Sigma}$ at the point q with respect to the normal v(q). If $B(q) \geqslant 0$ at each point $q \in \tilde{\Sigma}$, then the leaf Σ is called convex. In what follows, as a rule, we shall consider convex fibers.

For arbitrary $t > 0$ we consider the image $T^t\Sigma = \Sigma_t \subset M$ of the leaf Σ with carrier $\pi(\Sigma^t) = \tilde{\Sigma}_t$. The set $\Sigma_t \setminus \partial M$ consists of a finite number of leaves. We introduce the following notation: If $T^t(q, v(q)) \notin \partial M$, then $T^t(q, v(q)) = (q_t, v(q_t))$ is a point lying on some leaf in Σ_t, and $B(q_t)$ is the operator of the second quadratic form of this leaf at the point q_t. If $T^t(q, v(q)) \in \partial M$, then one can consider the vectors $v(q_{t+0})$ and $v(q_{t-0})$, corresponding to the motion up to and after reflection, and the operators $B(q_{t+0})$ and $B(q_{t-0})$. We denote the spaces in which the operators $B(q_t)$, $B(q_{t\pm 0})$ act by $J(q_t)$, $J(q_{t\pm 0})$, respectively.

The following two lemmas (cf. [8]) establish a connection between the operators B(q) and $B(q_t)$:

LEMMA 1. Let the point (q, v(q)) not undergo reflection from the boundary of the billiards on the time interval [0, t]. Then the vectors v(q) and $v(q_t)$ are parallel and the planes J(q) and $J(q_t)$ can be identified. Here

$$B(q_t) = B(q)(I + tB(q))^{-1}.$$

COROLLARY. Under the hypotheses of Lemma 1,

$$\|B(q_t)\| \leqslant 1/t.$$

LEMMA 2. Let the trajectory $T^t(q, v(q))$ undergo reflection from a regular component of the boundary of the billiards at time s, i.e., $q_s \in \partial Q$. Then

$$B(q_{s+0}) = U^{-1}B(q_{s-0}), U + 2(v(q_{s+0}), n(q_s))V^*K(q_s)V$$

where $n(q_s)$ is a vector normal to ∂Q at the point q_s; $K(q_s)$, operator of the second quadratic form of ∂Q, acting in the plane $\mathcal{T}_{q_s}$; U, projector of $J(q_{s+0})$ onto $J(q_{s-0})$ parallel to the vector (q_s); V, projector of $J(q_{s+0})$ onto $\mathcal{T}_{q_s}$ parallel to the vector $v(q_{s+0})$; and V^*, projector of $\mathcal{T}_{q_s}$ onto $J(q_{s+0})$ parallel to the vector $n(q_s)$.

COROLLARY. If $B(q) \geqslant 0$, then $B(q_t) \geqslant 0$ for all $t > 0$, i.e., the image of a convex leaf always consists of convex leaves.

We consider the general case. Let the point $(q, v(q))$ on the time interval $(0, t)$ undergo l reflections from the boundary ∂Q at times $0 < t_1 < t_2 < \ldots < t_l < t$. For the i-th reflection we introduce notation corresponding to the notation of Lemma 2: $q_{t_i} = q_i \in \partial Q$ is the point of reflection; $n(q_i)$, vector normal to ∂Q; K_i, operator of the second quadratic form of ∂Q at the point q_i; U_i V_i, V_i^*, corresponding projectors. Then from Lemmas 1 and 2 the formula

$$B(q_t) = \cfrac{I}{(t-t_l)I + \cfrac{I}{2\cos\varphi_l V_l^* K_l V_l + U_l^{-1} \cfrac{I}{I + \cdots + U_1^{-1} \cfrac{B(q)}{I + t_1 B(q)} U_1} U_l}}, \tag{3}$$

follows, where $\cos\varphi_i = (v(q_{t_i}+0), n(q_i))$, and the notation I/A denotes A^{-1}.

We define the operator $B(x)$, introduced in Sec. 1 by formula (1). Let $x = (q, v)$ be an arbitrary point of the set M. We fix some $t > 0$. We consider a planar area element in Q, containing the point q_t and orthogonal to the vector $v(q_t)$. From the points of this element of area close to the point q_t, we send out the pencil of trajectories $\Sigma_\delta^{(t)}$, parallel to the vector $v(q_t)$. This pencil, upon motion for time $s > 0$, passes into some convex leaf $T^s\Sigma_\delta^{(t)} = \Sigma_s^{(t)}$, containing the point $(q_{t-s}, -v_{t-s})$. We denote the convex leaf $\Sigma_t^{(t)}$ by Σ_t^*, and the leaf with opposite normal vectors by $-\Sigma_t^*$. The leaf $-\Sigma_t^*$ contains the point x for all $t > 0$.

Let the point x be reflected from the boundary of the billiards at times $0 < t_1 < t_2 < \ldots < t_n < \ldots$. For the i-th reflection we shall use the notation introduced earlier, q_i, $n(q_i)$, K_i, U_i, V_i, V_i^*, $\cos\varphi_i$, and we set $\tau_i = t_i - t_{i-1}$ $(t_0 = 0)$. The operator $B^{(t)}(q)$ of the leaf Σ_t^* at the point q, according to (3), has the form

$$B^{(t)}(q) = \cfrac{I}{\tau_1 I + \cfrac{I}{2\cos\varphi_1 V_1^* K_1 V_1 + U_1^{-1} \cfrac{I}{\tau_2 I + \cdots + \cfrac{I}{2\cos\varphi_l V_l^* K_l V_l}} U_1}},$$

where the index l is determined from the relations $t_l < t \leqslant t_{l+1}$.

In [8] it is proved that under the condition $\sum_{i=1}^{\infty} t_i^{-2} < \infty$ the operators $B^{(t)}(q)$ converge as $t \to \infty$ to the operator $B(x)$, which we represent as a continued fraction operator (1). From Condition T.1 it follows that the operator $B(x)$ is defined everywhere on M^T.

We note that all the operators $B(x)$, $B^{(t)}(q)$, $2\cos\varphi_i V_i^* K_i V_i$ are symmetric and nonnegative definite, and the projectors U_i are isometric. This allows us to estimate the degree of approximation of the continued fraction $B(x)$ by the odd convergents

$$B(x, k) = \cfrac{I}{\tau_1 I + \cfrac{I}{2\cos\varphi_1 V_1^* K_1 V_1 + U_1^{-1} \cfrac{I}{\tau_2 I + \cdots + \cfrac{1}{\tau_k} I} U_1}},$$

namely:

$$\| B(x) - B(x, k) \| \leqslant \frac{1}{\tau_1 + \tau_2 + \ldots + \tau_k}. \tag{4}$$

The continuous dependence of the operator $B(x)$ on the point x in the domain of definition follows from (4) (cf. also [6]).

We consider the decomposition

$$J(x) = J_0(x) \oplus J_+(x) \tag{5}$$

of the space $J(x)$ into two $B(x)$-invariant subspaces, introduced in Sec. 1. Using the representation (2) of the operator $B(x)$, one can deduce the following fact:

LEMMA 3. Let w be an arbitrary vector of the space $J(x)$. Then $w \in J_0(x)$ if and only if for all $i \geqslant 1$, $K_i V_i U_{i-1} U_{i-2} \ldots U_1 w = 0$.

We note that by virtue of the finite dimensionality of the space $J(x)$ it is sufficient that a finite number of these relations hold.

We consider on the set M^T the nonnegative integer-valued function $j(x) = \dim J_+(x)$. From Lemma 3 and the continuous dependence of the operator $B(x)$ on the point x follows:

LEMMA 4. All points of the set M^T are local minima of the function $j(x)$. The points of local constancy of the function form an open dense subset of M^T, on which the decomposition (5) depends continuously on the point x.

Directly from Lemma 3, there follows:

LEMMA 5. The function $j(x)$ does not increase along the trajectory of the point x, i.e., $j(T^t x) \leqslant j(x)$ for $t > 0$. Moreover, if $j(T^t x) = j(x)$, then

$$J_0(\cdot T^t x) = U_i U_{i-1} \ldots U_1 J_0(x) \quad u \quad J_+(T^t x) = U_i U_{i-1} \ldots U_1 J_+(x)$$

(here i is the number of reflections of the trajectory of the point x from the boundary ∂Q in time t), i.e., the decomposition (5) is invariant with respect to translations along the trajectory of the point x on the time interval $(0, t)$.

We consider the motion of the carrier of a convex leaf in the configuration space Q. Let Σ be a convex leaf and $\Sigma_t = T^t \Sigma$ for $t > 0$. We denote by $\tilde{S}_t$ the map of the surface $\tilde{\Sigma}$ onto the surface $\tilde{\Sigma}_t$, defined by the formula $\tilde{S}_t q = q_t$. We consider the tangent map $d\tilde{S}_t(q)$ at a point $q \in \tilde{\Sigma}$ such that $q_t \notin \partial Q$. If the point $(q, v(q))$ does not undergo reflection from the boundary ∂Q on the time interval $(0, t)$, then

$$d\tilde{S}_t(q) = \tilde{S}_t(q + dq) - \tilde{S}_t(q) = (I + tB(q))\, dq,$$

where $B(q)$ is the operator of the second quadratic form of the leaf Σ at the point q. In the general case (cf. [8]) one has

LEMMA 6. Let the point $(q, v(q)) \in \Sigma$ undergo reflection from the boundary of the billiards at times $0 < t_1 < t_2 < \ldots < t_l < t$. Then

$$d\tilde{S}_t(q) = \tilde{S}_t(q + dq) - \tilde{S}_t(q) = (I + (t - t_l) B_l)\, U_l (I + \tau_l B_{l-1}) U_{l-1} \ldots U_1 (I + \tau_1 B(q))\, dq, \tag{6}$$

where $\tau_i = t_i - t_{i-1}$ $(t_0 = 0)$, $B_i = B(q_{t_i+0})$ and U_i are the projectors introduced earlier for trajectories of the point $(q, v(q))$.

We note that in (6) the operators $B_1, B_2, \ldots, B_l$ are determined by the operator $B(q)$ by virtue of (3).

4. Properties of Tangent Mappings

We consider an arbitrary point $x_0 = (q_0, v_0) \in M$. In the preceding section, the leaves $\Sigma_0^{(t)}$, Σ_t^* and $-\Sigma_t^*$ were defined. The leaf Σ_t^* contains the point x_0, and the trajectories of its points on the time interval $(0, t)$ approach the trajectory of the point x_0. This allows us to construct the manifold W (cf. Theorem 1) as the limit of the leaves $-\Sigma_t^*$ as $t \to \infty$.

In the present section we study the tangent mappings to translations of the carrier of the leaf $\Sigma_0^{(t)}$ under the action of T^s, $0 < s \leqslant t$. The properties of these mappings allow us to deduce the existence of an LTL from the results of Katok and Strelcyn [7] (cf. Sec. 5) and to make a direct geometric construction of the LTL (cf. Sec. 6).

We shall assume that in Condition T.1, $\hat{x}(x_0) = x_0$, and we shall prove Theorem 1 in this case. The general case reduces to the case $\hat{x}(x_0) = x_0$ with the help of transport of the leaf $W(\hat{x}(x_0))$ under the action of $\{T^t\}$.

Since $\hat{x}(x_0) = x_0$, one has $x_0 \in M^C$, i.e., the trajectory $T^t x_0$ satisfies the Conditions T.1 and T.2, and the dimension of the spaces $J_0(x)$ and $J_+(x)$ is constant in some neighborhood of the point x_0.

We consider the trajectory $T^t x_0$ of the point x_0 for $t > 0$. We denote by $0 < t_1 < t_2 < \ldots < t_n < \ldots$ the times of reflection of the trajectory $T^t x_0$ from the boundary ∂Q. We call a series of reflections with indices $\{1, 2, \ldots, j\}$ sufficient, if for any $w \in J(x_0)$, from the relations (cf. Lemma 3) $K_i V_i U_{i-1} U_{i-2} \ldots U_2 U_1 w = 0$ for $i = 1, 2, \ldots, j$, these relations follow for all $i \geqslant 1$. By virtue of the finite dimensionality of the space $J(x_0)$, there exist finite sufficient series of reflections, and we can consider the minimal one of them, $\{1, 2, \ldots, p\}$.

In some neighborhood $U_{x_0} \subset M$ of the point x_0 the function $j(x)$ will be constant and the series $\{1, 2, \ldots, p\}$ is sufficient for all $x \in U_{x_0} \cap M^T$.

The decomposition (5) can be defined everywhere in the neighborhood U_{x_0}, setting for any point $x \in U_{x_0}$

$$J_0(x) = \{w \in J(x)\colon\ K_i V_i U_{i-1} U_{i-2} \ldots U_1 w = 0,\ 1 \leqslant i \leqslant p\}$$

and

$$J_+(x) = J(x) \ominus J_0(x), \tag{7}$$

where the operators K_l, V_l, U_l correspond to the trajectory of the point x. The decomposition of the space $J(x)$ introduced in this way coincides with the one already constructed in the domain of definition of the operator $B(x)$. This decomposition depends continuously on the point x everywhere in U_{x_0}, by virtue of which it will be invariant with respect to translations along trajectories of points $x \in U_{x_0}$.

From Condition T.1 it follows that the trajectory $T^t x_0$ returns to the neighborhood U_{x_0} with positive frequency $\rho = \rho(U_{x_0})$. We choose the neighborhood U_{x_0} sufficiently small that the number of reflections of the trajectory $T^t x_0$ between two consecutive returns to U_{x_0} is greater than p. Let the trajectory $T^t x_0$ undergo $p + q_1$ reflections up to the first return and $p + q_i$ reflections between the $(i-1)$-st and i-th returns to U_{x_0} for $i \geqslant 2$. By $N_m = mp + q_1 + \ldots + q_m$ we denote the total number of reflections up to the m-th return. By virtue of Condition T.1, for all sufficiently large m

$$m > \rho(\tau_1 + \tau_2 + \ldots + \tau_{N_m}). \tag{8}$$

We fix some sufficiently large T and we consider the motion of the leaf $\Sigma_0^{(T)}$ under the action of $\{T^t\}$ on the time interval $(0, T)$. In what follows we shall consider only the part of $\Sigma_0^{(T)}$, whose carrier remains, under the action of $\{T^t\}$ on the time interval $(0, T)$, inside the system of corridors Θ_0 (cf. Sec. 2). Here the points of the leaf $\Sigma_0^{(T)}$ are reflected from one of the regular components of the boundary ∂Q. We denote the total number of reflections by N and the number of returns to the neighborhood U_{x_0} by m. For convenience we shall assume the time T chosen so that $-\Sigma_0^{(T)} \subset U_{x_0}$. Consequently, the trajectories of points of the leaf $\Sigma_0^{(T)}$ under motion to time T goes through a sufficient series of reflections (here they went through this series of reflections in inverse order).

Let $0 = t_0^* < t_1^* < \ldots < t_m^* = T$ be the times at which the leaf $T^t \Sigma_0^{(T)}$ is found in the neighborhood U_{x_0}. We denote the leaf $T^{t_i^*} \Sigma_0^{(T)}$ by $\Sigma^*(i)$ and the map of the surface $\tilde{\Sigma}^*(i)$ onto the surface $\tilde{\Sigma}^*(i+1)$, generated by the translation $T^{t_{i+1}^* - t_i^*}$, by $\tilde{S}_i$, $0 \leqslant i \leqslant m-1$. Thus,

$$\tilde{\Sigma}_T^* = \tilde{\Sigma}^*(m) = \tilde{S}_{m-1} \tilde{S}_{m-2} \ldots \tilde{S}_0 \tilde{\Sigma}_0^{(T)}.$$

To estimate the deformation of the surface $\tilde{\Sigma}_0^{(T)}$ under the map $\tilde{S}_T = \tilde{S}_{m-1} \tilde{S}_{m-2} \ldots \tilde{S}_0$, it suffices to get the corresponding estimate for the tangent mapping $d\tilde{S}_i$ for $0 \leqslant i \leqslant m-1$. The needed properties of the maps $d\tilde{S}_i$ are established in Lemmas 7 and 8. In what follows we denote by $J_0(-y)$ and $J_+(-y)$ the subspaces J_0 and J_+ of the space $J(y)$, constructed for the point $-y = (q, -v)$, where $(q, v) = y$.

Let $y_0 = (q_0, v_0)$ be an arbitrary point of the leaf $\Sigma_0^{(T)}$ and $y_i = (q_i, v_i) = T^{t_i^*} y_0$ be its image on the leaf $\Sigma^*(i)$, $1 \leqslant i \leqslant m$. We consider the tangent space $J(y_i)$ to the surface $\tilde{\Sigma}^*(i)$ at the point q_i and its decomposition

$$J(y_i) = J_0(-y_i) \oplus J_+(-y_i) \tag{9}$$

for $0 \leqslant i \leqslant m$. The decomposition (9) is invariant with respect to translations along the trajectory of the point y_0. Whence and from (6) for the map $d\tilde{S}$ there follows:

LEMMA 7. For all $y_0 \in \Sigma_0^{(T)}$

$$d\tilde{S}_i(q_i)\, J_0(-y_i) = J_0(-y_{i+1}) \text{ and } d\tilde{S}_i(q_i)\, J_+(-y_i) = J_+(-y_{i+1})$$

for $i = 0, 1, 2, \ldots, m-1$.

COROLLARY. For all $y_0 \in \Sigma_0^{(T)}$

$$d\tilde{S}_T(q_0)\, J_0(-y_0) = J_0(-y_m) \text{ and } d\tilde{S}_T(q_0)\, J_+(-y_0) = J_+(-y_m).$$

LEMMA 8. Let y_0 be an arbitrary point of the leaf $\Sigma_0^{(T)}$. For all $i = 0, 1, \ldots, m-1$, the operator $d\tilde{S}_i(q_i)$ is an isometry on the space $J_0(-y_i)$ and an expansion on the space $J_+(-y_i)$, i.e., for all $w \in J_+(-y_i)$, $\|d\tilde{S}_i(q_i)w\| \geqslant D\,\|w\|$, where the constant $D > 1$ is determined only by the choice of the neighborhood U_{x_0}.

Proof. From the structure of the leaves $\Sigma^*(i)$ and (3) it follows that the operator of the second quadratic form of the leaf $\Sigma^*(i)$ at the point y_i carries the space $J_0(-y_i)$ to zero. Together with (6) this proves that the operator $d\tilde{S}_i$ is an isometry on $J_0(-y_i)$.

To prove the second assertion of Lemma 8 we consider an arbitrary point $\hat{y} = (\hat{q}, \hat{v}) \in M'$ such that $-\hat{y} \in U_{x_0}$ and $-T^{\hat{t}}\hat{y} \in U_{x_0}$ for some $\hat{t} > 0$. We denote by $0 < \hat{t}_1 < \hat{t}_2 < \ldots < \hat{t}_m < \hat{t}$ the times of reflection from the boundary of the trajectory of the point $\hat{y}$ on the time interval $(0, t)$. The reflections at times $\hat{t}' = \hat{t}_{m-p+1}, \hat{t}_{m-p+2}, \ldots, \hat{t}_m$ form a sufficient series (in inverse order).

Let the point $\hat{y}$ lie in some convex leaf Σ'. We consider the leaf $\Sigma'' = T^{\hat{t}}\Sigma'$ and we denote by $\tilde{S}'$ the map of $\tilde{\Sigma}'$ onto $\tilde{\Sigma}''$, generated by the translation $T^{\hat{t}}$. We consider the decompositions

$$J(\hat{y}) = J_+(-\hat{y}) \oplus J_0(-\hat{y}) \tag{10'}$$

and

$$J(T^{\hat{t}}\hat{y}) = J_+(-T^{\hat{t}}\hat{y}) \in J_0(-T^{\hat{t}}\hat{y}), \tag{10''}$$

which are defined by (7). The image of the decomposition (10') under translation along the trajectory of the point $\hat{y}$ in time $\hat{t}$ coincides with the decomposition (10''). We consider the point $d\hat{S}'(\hat{q})$ conserves the structure of decompositions (10') and "10"). We examine the point $T^{\hat{t}'-0}\hat{y} = \hat{y}' = (\hat{q}', \hat{v}')$ and we write

$$J(\hat{y}') = J_+(-\hat{y}') \oplus J_0(-\hat{y}') \tag{10'''}$$

for the image of the decomposition (10') under translation along the trajectory of the point $\hat{y}$ for time $\hat{t}' - 0$. We consider the leaf $\Sigma''' = T^{\hat{t}-0}\Sigma'$ and we denote by $\tilde{S}'$ the map of $\tilde{\Sigma}'$ onto $\tilde{\Sigma}'''$, generated by the translation $T^{\hat{t}'-0}$, and by $\tilde{S}_2'$ the map of $\tilde{\Sigma}'''$ onto $\tilde{\Sigma}''$, generated by the translation $T^{\hat{t}-\hat{t}'}$. Then $\tilde{S}' = \tilde{S}_2'\tilde{S}_1'$, while the maps $d\tilde{S}_1'(\hat{q})$ and $d\tilde{S}_2'(\hat{q}')$ preserve the structure of the decompositions (10'-10''').

By virtue of the convexity of the leaf Σ', the operator $d\tilde{S}_1'(q)$ is noncompressing, so it suffices to prove $\|d\tilde{S}_2'(\hat{q}')w\| \geqslant D > 1$ for all unit vectors $w \in J_+(-\hat{y}')$, where the constant D is independent of the point $\hat{y}$ and of the leaf Σ'. According to (6), for each vector $w \in J_+(\hat{y}')$ one can find at least one value $j = 1, 2, \ldots, p$, for which $K_jV_jU_{j-1}U_{j-2}\ldots U_1w \neq 0$, where the operators K_i, V_i, U_i correspond to the $(m-p+i)$-th reflections of the trajectory of the point $\hat{y}$. Together with (6) this proves that $d\tilde{S}_2'(\hat{q}')w \neq w$, whence $\|d\tilde{S}_2(\hat{q}')w\| > \|w\|$.

By virtue of (6), the operator $d\tilde{S}_2'(\hat{q}')$ depends continuously on the operator of the second quadratic form $\hat{B}(\hat{q}')$ of the leaf Σ''' at the point $\hat{y}'$. This operator is nonnegative definite, self-adjoint, and, by virtue of the Corollary to Lemma 1,

$$\|\hat{B}(\hat{q}')\| \leqslant \frac{1}{\hat{t}_{m-p+1} - \hat{t}_{m-p}}.$$

The set of operators Λ satisfying these conditions is compact, so

$$D(\hat{y}) = \inf_{\hat{B}(\hat{q}')\in\Lambda} \inf_{\substack{\|w\|=1 \\ w\in J_+(\hat{y}')}} \| d\tilde{S}_2'(\hat{q})w \| > 1.$$

Since the function $D(\hat{y})$ is continuous on U_{x_0}, for suitable choice of the neighborhood U_{x_0},

$$\inf_{\hat{y}\in U_{x_0}} D(\hat{y}) = D > 1.$$

Lemma 8 is proved.

COROLLARY. Let y_0 be an arbitrary point of the leaf $\Sigma_0^{(T)}$. The restriction of the operator $d\tilde{S}_T(q_0)$ to the subspace $J_0(-y_0)$ is an isometry. The restriction of this operator to the subspace $J_+(-y_0)$ expands any vector no less than $D^m > D^{\rho T+1}$ times.

The last inequality follows from (8).

We note that the condition $\Sigma_0^{(T)} \subset U_{x_0}$ is inessential, i.e., Lemmas 7 and 8 remain true for all sufficiently large T.

5. Nonuniform Partial Hyperbolicity of Billiard Systems

In this section we study the Lyapunov characteristic exponents of a billiard system. Lemma 8, proved in Sec. 4, allows us to establish the existence of negative exponents on the set

$$\Lambda = \hat{M} \cap \{x \in M^T:\ j(\hat{x}(x)) > 0\}.$$

We shall assume that the boundary of the billiards ∂Q satisfies the conditions:

(A.1) One can find a regular point $q \in \partial Q$, at which $K(q) \neq 0$;

(A.2) The norm of the operator $K(q)$ is uniformly bounded:

$$\sup_{q\in\partial Q} \| K(q) \| < \infty.$$

It is easy to show that in billiards with the condition (A.1) $\mu(\Lambda) > 0$, but the question of the fullness of the measure of the set Λ is open. This question is essentially the question of the fullness of the measure of the set M^C (cf. Sec. 1).

Corollary 1 follows directly from a theorem of Pesin (cf. [3]) and the following lemma:

LEMMA 9. On the set Λ there exist negative Lyapunov exponents.

Proof. Let $x_0 = (q_0, v_0)$ be an arbitrary point of the set Λ. Just as in the proof of Theorem 1, it suffices to investigate the case $x_0 \in M^C$, so Lemma 8 is applicable to the trajectory of the point x_0. We write $j(x_0) = j_0 > 0$.

We consider the tangent space $\mathcal{T}_x M$ to the manifold M at an arbitrary point $x = (q, v)$. It is clear that $\mathcal{T}_x M = \mathcal{T}_q Q \oplus \mathcal{T}_v S$, where $\mathcal{T}_q Q$ is the tangent space to Q at the point q and $\mathcal{T}_v S$ is the tangent space to the $(d-1)$-dimensional sphere S of velocity vectors at the point $v \in S$. Hence, a vector $u \in \mathcal{T}_x M$ is a pair $u = (dq, dv)$, where $dq \in \mathcal{T}_q Q$ and $dv \in \mathcal{T}_v S$ and $\| u \|^2 = \| dq \|^2 + \| dv \|^2$.

We consider the leaf $\Sigma_0^{(T)} \subset M$ and point $y_T = -x_T = (q_T, -v_T) = -T^T x_0 \in \Sigma_0^{(T)}$. The tangent space $\mathcal{T}_{y_T} \Sigma_0^{(T)} \subset \mathcal{T}_{y_T} M$ consists of pairs of the form $(dq, 0)$, where $dq \in \mathcal{T}_{q_T} \tilde{\Sigma}_0^{(T)} \subset \mathcal{T}_{q_T} Q$. In the space $\mathcal{T}_{y_T}\Sigma_0^{(T)}$ one can single out the j_0-dimensional subspace $\mathcal{T}_{y_T}^{(+)}\Sigma_0^{(T)}$, consisting of pairs $(dq, 0)$, where $dq \in J_+(x_T)$.

Let $u = (dq, 0)$ be an arbitrary vector of the space $\mathcal{T}_{y_T}^{(+)} \Sigma_0^{(T)}$. We consider its image $dT^t u = (dq_t, dv_t) \in \mathcal{T}_{T^t y_T} M$ under the action of the tangent map to the translation T^t.

Since $dT^t u \in \mathcal{T}_{T^t y_T} \Sigma_t^{(T)}$, one has $dv_t = B_t dq_t$, where B_t is the operator of the second quadratic form of the leaf $\Sigma_t^{(T)}$ at the point $T^t y_T$. By virtue of Lemma 8, $\| dq_t \| \geqslant D^{m_t} \| dq \|$, where m_t is the number of times the trajectory of the point y_T lands in the neighborhood U_{x_0} in time t. Whence it follows that

$$\| dT^t u \| = \sqrt{\| dq_t \|^2 + \| B_t dq_t \|^2} \geqslant \| dq_t \| \geqslant D^{m_t} \| dq \|. \tag{11}$$

On the other hand, $\| dT^t u \| \leqslant \| dq_t \| + \| B_t dq_t \|$.

From Conditions T.2 and (A.2) it is easy to derive that $\| B_t \| \leqslant \text{const}\ (T - t)$, whence

$$\| dT^t u \| \leqslant c_1 (T - t) \| dq_t \|. \tag{12}$$

We choose a vector $dq \in J_+(y_T)$ such that the vector $u_0 = dT^T u$ has unit length. Then from (11), $\| dq \| = \| du \| \leqslant D^{-mT} \leqslant c_2 D^{-\rho T}$. The last inequality follows from (8). From (11) and (12) it follows that for all $t < T$

$$\| dT^t u_0 \| \leqslant c_3 t D^{-\rho T}. \tag{13}$$

We note that in (13) the quantities c_3, D, and ρ do not depend on the choice of the vector $dq \in J_+(-y_T)$. For arbitrary $\rho_1 < \rho$, from (13) there follows the estimate

$$\| dT^t u_0 \| \leqslant c_4 D^{-\rho_1 t}. \tag{14}$$

In (14) the choice of u_0 depends on t, but by virtue of the compactness of the unit sphere in the space $\mathcal{T}_{x_0}M$, one can find a vector $u_0 \in \mathcal{T}_{x_0}M$, for which (14) will hold for all $t > 0$. Whence

$$\chi^+(x_0, u_0) = \overline{\lim} \frac{1}{t} \ln \| dT^t u_0 \| \leqslant -\rho \ln D < 0,$$

where χ^+ is the Lyapunov characteristic exponent (cf. [3-4]). Lemma 9 is proved.

Analogous arguments show that the space of vectors satisfying (14) is j_0-dimensional and its projection onto $\mathcal{T}_{q_0}Q$ coincides with $J_+(x_0)$. Since $dv_T = B_T dq_T$ for all T for the components of the vector $dT^T u$, one has for the components of the vector $u_0 = (dq_0, dv_0)$ that the relation $dv_0 = B dq_0$ holds.

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Remark Added in Proof. The results obtained are also applicable to dynamical systems generated by the motion of several rigid circles inside a planar polygon.

SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR WITH A POTENTIAL HAVING A SLOW FALLOFF

D. R. Yafaev

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1. Introduction

The structure of the spectrum of the operator $H_0 = -\Delta$ in the space $L_2(R^m)$ in the neighborhood of the spectral point $\lambda = 0$ is stable under perturbations of H_0 by potentials $q(x)$ which have rapid enough falloff. More precisely, if $q(x) = O(|x|^{-\rho})$, $\rho > 2$, and $|x| \to \infty$, then the spectral density $dE_\lambda/d\lambda$, $\lambda > 0$, of the operator $H = -\Delta + q(x)$ behaves (in an appropriate topology) for $\lambda \to 0$ essentially as that of the operator H_0. The explicit asymptotic formula for $dE_\lambda/d\lambda$ as $\lambda \to 0$ depends (see [1, 2]) on the dimension m, but in any case $dE_\lambda/d\lambda$ behaves as a power (or a logarithm) when $\lambda \to 0$. On the negative half axis there can arise only a finite number of eigenvalues of the operator H, each having finite multiplicity.

The change in the spectral characteristics produced by a potential q that has falloff less than $|x|^{-2}$ depends essentially upon the sign of q. As is known, when $q < 0$ an infinite negative spectrum arises for operator H. The case $q > 0$ is investigated in the present paper. It turns out that for a positive potential whose falloff is less than $|x|^{-2}$ the point $\lambda = 0$ remains in the spectrum but becomes almost regular (quasiregular). This can be observed, in particular, in the expansion of the resolvent $R(z) = (H - z)^{-1}$ in an asymptotic series (again, in an appropriate topology) in the neighborhood of the point $z = 0$. In other words, the series for $R(z)$, where $z \to 0$ in an arbitrary sector of the complex plane which does not contain the real line, is made up of terms of the same type as those of the expansion of $R(z)$ as a convergent power series in a neighborhood of any regular point. At the same time, this expansion contains only integral powers of z, while for potentials with rapid falloff the expansion of $R(z)$ also contains (see [1, 2]) half-integral powers of z (or logarithmic terms).

The basic analytic tool in our investigation is the assertion concerning the boundedness for all $n > 0$ of the operators ΩH^{-n}, where Ω is the operator of multiplication by some function which decays as $|x| \to \infty$ faster than any power of $|x|^{-1}$. The proof of this statement rests heavily upon the positivity and slow falloff property of the potential q. Here we should mention that for $4n \geq m$, the operator ΩH_0^{-n} is no longer bounded. The boundedness of ΩH^{-n} implies that the locally parabolic semigroup corresponding to H tends to zero faster than any power of t^{-1}, i.e., $\|\Omega \exp(-Ht)\| = O(t^{-n})$, $t \to \infty$, for any $n > 0$.

In order to prove the validity of the asymptotic expansion of $R(z)$ as $z \to 0$ in the entire plane with a cut along the positive semiaxis, one has to assume beforehand that the resolvent grows in the vicinity of the point $z = 0$ no faster than a power. Under this hypothesis, one also succeeds in showing that the function $d\Omega E_\lambda \Omega/d\lambda$ tends to zero faster than any power of λ as $\lambda \to 0$, independent of the dimension m. At the heuristic level, this result can be interpreted as the "washing away" at the point $\lambda = 0$ of the boundary between the continuous spectrum and the set of regular points of the operator H. In the one-dimensional case, using the technique of ordinary differential equations, one can find the exponentially decaying asymptotics of the kernel of the operator $dE_\lambda/d\lambda$ for $\lambda \to 0$. In addition, this technique permits verification of the correctness of the a priori assumption mentioned above for the case of spherically symmetric potentials.

The behavior of $d\Omega E_\lambda \Omega/d\lambda$ for $\lambda \to 0$ determines the character of the local decay, for large values of the time t, of the solutions to the corresponding time-dependent Schrödinger equation. Since, for positive potentials with slow falloff, the function $d\Omega E_\lambda \Omega/d\lambda$ is not singular at $\lambda = 0$, it is to be expected that the quantity $\| \Omega \exp(-iHt)\Omega \|$ should decay for $t \to \infty$ fas-

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V. A. Steklov Mathematics Institute, Leningrad Branch, Academy of Sciences of the USSR.
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