CHAPTER 6

Stable Manifolds

If we look at a picture of a saddle point (for example Figure 6.1), and try to analyse what qualitative features give it its characteristic appearance, we are bound to pick out the four special orbits that begin or end at the fixed point. These, together with the fixed point itself, form the *stable* and *unstable manifolds* of the dynamical system at the fixed point. We have noted in the last chapter the importance of hyperbolicity in the theory of dynamical systems, and a hyperbolic structure always implies the presence of such manifolds. If we know the "singular elements" of a system (in some sense which must include periodic points for diffeomorphisms, and fixed points and closed orbits for flows), and if we also know the way in which their stable and unstable manifolds fit together, then we have a pretty good hold on the orbit structure of the system. In this chapter we develop a theory for such manifolds, first for hyperbolic fixed points and then for more general invariant sets.

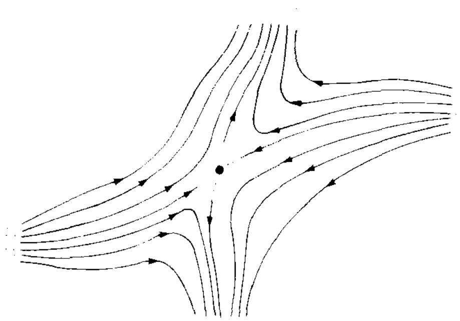

FIGURE 6.1

I. THE STABLE MANIFOLD AT A HYPERBOLIC FIXED POINT OF A DIFFEOMORPHISM

Let $f: X \to X$ be a diffeomorphism, and let p be a fixed point of f. The *stable set (or in-set) of f at p* is the set

$$\{x \in X : f^n(x) \to p \quad \text{as} \quad n \to \infty\}.$$

Notice that this set is always non-empty, since it contains p. The *unstable set (or out-set) of f at p* is the stable set of f^{-1} at p. (Thus results about stable sets can generally be restated in terms of unstable sets; we leave this to the reader.) Given any open neighbourhood U of p, we define the *local stable set of $f|U$ at p* to be the set of all $x \in U$ such that $(f^n(x): n \geq 0)$ is a sequence in U converging to p. Notice that any point of the global stable set can be taken to a point of the local stable set by a suitable power of the diffeomorphism f. Our main theorem asserts that if p is hyperbolic then the global stable set is an immersed submanifold of X which is at least as smooth as the diffeomorphism f. We call it the *stable manifold* of f at p, $W_s(p)$. It need not be an embedded submanifold, however. For example, there is nothing to prevent the situation illustrated in Figure 6.2, where the stable and unstable manifolds of p coincide. Moreover, things look far worse when the stable and

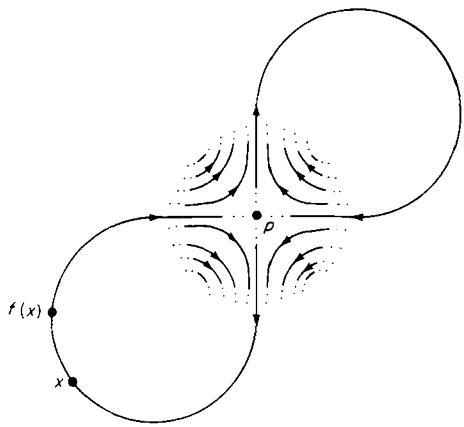

FIGURE 6.2

unstable manifolds at p have a point of transverse intersection elsewhere than at p. In such a situation, the point of intersection is said to be *transversally homoclinic*. This phenomenon is worth examining in more detail. First of all, we consider the situation (shown in Figure 6.3) where the

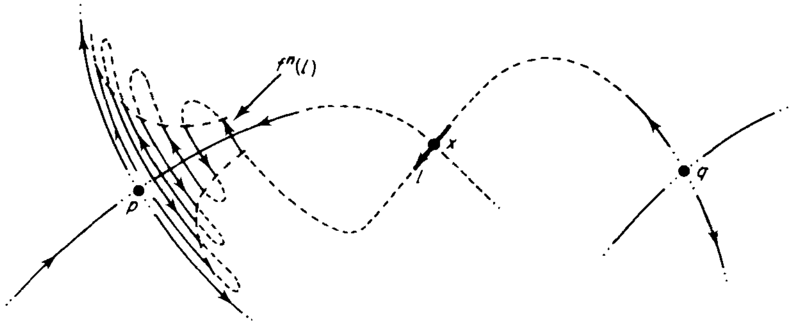

FIGURE 6.3

stable manifold at a hyperbolic fixed point p intersects the unstable manifold at another hyperbolic fixed point q transversally at some point x, say. The point x is then said to be *transversally heteroclinic*. The sequence $(f^n(x): n \geq 0)$ tends to p along the stable manifold at p, and is contained in a sequence $(f^n(l): n \geq 0)$ of images of a small segment l of the unstable manifold at q. The members of the latter sequence are transverse to the stable manifold at p and eventually "press themselves up against" any local unstable manifold at p. Thus any point of the local (and hence of the global) unstable manifold of p is a limit of points on the unstable manifold of q. Similarly every point of the stable manifold at q is a limit of points on the stable manifold at p. This is an interesting situation, but does not make any extreme demands on the topology of the submanifolds in question. However, if we now put $p = q$, then we have that each point of the stable manifold is a limit of points on other branches (that is, local connected components) of the stable manifold, and similarly for the unstable manifold. Thus these immersed submanifolds are not, globally, copies of the real line with the usual topology. The points of intersection of the stable and unstable manifold of p, other than p itself, are transversally homoclinic points. If the reader is sceptical as to whether such behaviour can actually occur, he should take another look at the toral automorphisms of Example 1.30.

We now return to the general theory. We shall deduce the global version of the stable manifold theorem from the local version. Roughly speaking, once we have planted a local stable manifold, we can easily grow it to obtain a global stable manifold; the difficulty lies in establishing it locally.

As usual, we take a chart at the point in question, and thus transfer the problem to the model space $\mathbf{E}$. Thus we obtain a local diffeomorphism with a hyperbolic fixed point at the origin, say, and the differential there is a linear approximation for the map itself nearby. We have been through this routine already in Corollary 5.20. The difference now is that we are interested in smoothness of maps, and whereas before we had no difficulty in extending a local Lipschitz map to a global one, a similar extension for smooth maps presents problems when $\mathbf{E}$ is an arbitrary Banach space. It seems a good idea, then, to give our basic results on perturbations of hyperbolic linear maps a local character.

We start out with a hyperbolic linear automorphism T of a Banach space $\mathbf{E}$. Thus $\mathbf{E}$ splits into stable and unstable summands

$$\mathbf{E} = \mathbf{E}_s(T) \oplus \mathbf{E}_u(T) = \mathbf{E}_s(T) \times \mathbf{E}_u(T).$$

We use the max $\{|\ |_s, |\ |_u\}$ norm on $\mathbf{E}$. Suppose that T has skewness a (see Theorem 4.19), and let κ be any number with $0 < \kappa < 1 - a$. Let $B = B_s \times B_u$ be the closed ball with centre 0 and radius b (possibly $b = \infty$) in $\mathbf{E}$. We have, for T, the picture on the left in Figure 6.4. We shall show that, after a

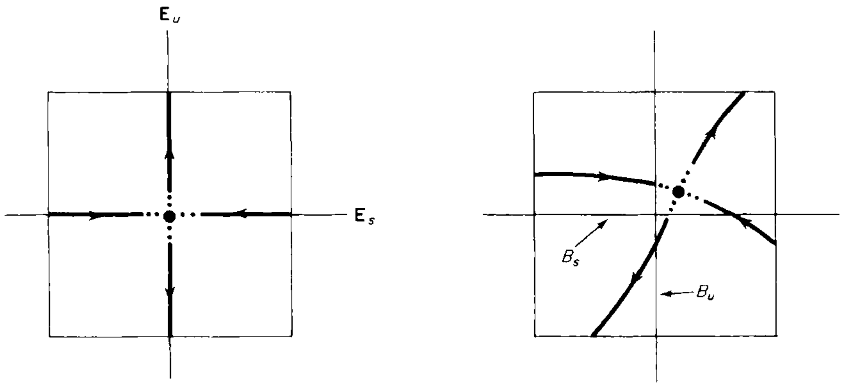

FIGURE 6.4

Lipschitz perturbation η with constant κ satisfying $|\eta|_0 \leq b(1-a)$, the local stable set of $(T+\eta)|B$ is, as in the picture on the right, the graph of a map, h say, from B_s to B_u. Moreover the map h is as smooth as the perturbation η. By Exercise 5.13, $T+\eta$ has a unique fixed point in B, and by transferring the origin to this point we may assume, for simplicity, that $\eta(0) = 0$. We may now state:

(6.5) Theorem. (*Local stable manifold theorem*) *Let $\eta: B \to \mathbf{E}$ be Lipschitz with constant κ, where $\kappa < 1 - a$. Suppose $\eta(0) = 0$. Then there is a unique map $h: B_s \to B_u$ such that graph h is the stable set of $(T+\eta)|B$ at 0. Moreover the map h is Lipschitz, and C^r when η is C^r.*

Proof. The idea is to identify in $\mathscr{S}_0(B)$, which we recall (Appendix B) is a closed subset of the Banach space $\mathscr{S}_0(\mathbf{E})$ of sequences in $\mathbf{E}$ converging to 0, those members of the form $((T+\eta)^n(x): n \geq 0)$ for some $x = (x_s, x_u) \in B$.

This is a good idea because the first members of such sequences are precisely the points of the local stable manifold. We show first that $\gamma \in \mathcal{S}_0(B)$ is of this form if and only if

(6.6) $$\gamma(n) = \chi(x_s, \gamma)(n)$$

where

(6.7) $$\chi(x_s, \gamma)(n) = \begin{cases} ((T+\eta)_s\gamma(n-1),\ T_u^{-1}(\gamma_u(n+1) - \eta_u\gamma(n))) & \text{for } n > 0 \\ (x_s,\ T_u^{-1}(\gamma_u(1) - \eta_u\gamma(0))) & \text{for } n = 0, \end{cases}$$

and $\eta = (\eta_s, \eta_u): B \to \mathbf{E}_s \times \mathbf{E}_u$. This is clear, since the relation

$$\gamma_u(n) = T_u^{-1}(\gamma_u(n+1) - \eta_u\gamma(n))$$

holding on the unstable component for $n \geq 0$ may be rewritten as

$$\gamma_u(n+1) = (T+\eta)_u\gamma(n),$$

and this combines with a similar relation on the stable component to give

$$\gamma(n+1) = (T+\eta)\gamma(n)$$

for $n \geq 0$.

The useful feature of (6.6) is that it exhibits γ as the fixed point of a contraction of $\mathcal{S}_0(B)$. For, given γ and $\gamma' \in \mathcal{S}_0(B)$,

$$\begin{aligned} &|\chi(x_s, \gamma) - \chi(x_s, \gamma')| \\ &= \sup\{|(T+\eta)_s\gamma(n) - (T+\eta)_s\gamma'(n)|, \\ &\qquad |T_u^{-1}(\gamma_u(n+1) - \gamma_u'(n+1) + \eta_u\gamma'(n) - \eta_u\gamma(n))| : n \geq 0\} \\ &\leq \sup\{(|T_s| + \operatorname{Lip}\eta)|\gamma - \gamma'|, |T_u^{-1}|(1 + \operatorname{Lip}\eta)|\gamma - \gamma'|\} \\ &\leq (a+\kappa)|\gamma - \gamma'|. \end{aligned}$$

Since $a+\kappa$ is <1, χ contracts the second factor uniformly. Putting $\gamma' = 0$ proves that $\chi(x_s, \gamma)$ takes values in B. A similar estimate on (6.7) shows that $\chi(x_s, \gamma)$ converges to 0.

We have shown that (6.7) defines a map $\chi: B_s \times \mathcal{S}_0(B) \to \mathcal{S}_0(B)$. Notice that χ is Lipschitz on the first factor, since

$$|\chi(x_s', \gamma) - \chi(x_s, \gamma)| = |x_s - x_s'|.$$

Thus by Theorems C.5 and C.7 of Appendix C, there is a unique map $g: B_s \to \mathcal{S}_0(B)$ such that, for all $x_s \in B_s$, (6.6) holds with γ replaced by $g(x_s)$, Correspondingly there is for each $x_s \in B_s$ a unique point $g(x_s)(0)$ of the local stable set with stable component x_s. Thus the map h of the theorem is given by $h(x_s) = (g(x_s)(0))_u$.

Notice that by Theorem C.7 the map g is Lipschitz. Now h is g composed with the map from $\mathscr{S}_0(B)$ to B that evaluates sequences at 0, followed by the product projection. Since the latter maps are both continuous linear, we deduce that h is Lipschitz.

Finally suppose that η is C^r. We can split χ into the sum of two functions, one of which factors through B_s (this one takes (x_s, γ) to the sequence whose only non-vanishing term is $(x_s, 0)$ in the 0th place) and the other through $\mathscr{S}_0(B)$. The first of these functions is trivially C^∞, and the second is C^r, by Lemma B.4 and Theorem B.20 of Appendix B. This is because T_s, T_u^{-1}, product projections, and maps of $\mathscr{S}_0(B)$ that move sequences to the left or right are all continuous linear. Thus χ is C^r. We deduce from Theorem C.7 that g, and hence h, is C^r. □

(6.8) Remark. The formula (6.7) defines equally well a contraction on $\mathscr{B}(B)$, the space of all bounded sequences in B, and one obtains a unique fixed point map $: B_s \to \mathscr{B}(B)$ which, by uniqueness, is the above map g. Thus the local stable manifold may be characterized as the set of all points x whose iterates $(T+\eta)^n(x)$ for $n \geq 0$ form a bounded sequence in B.

Notice that the stable manifold theorem works under substantially weaker hypotheses than Hartman's theorem. We do not necessarily assume η to be bounded when $b = \infty$, nor do we impose a condition to ensure that $T+\eta$ is a homeomorphism. In the appendix to this chapter we further investigate the dependence of h on η.

There are two other features of the local stable manifold that we would like to establish. Firstly the tangent space at 0 to the local stable manifold is the stable summand of the tangent map $T_0 f$, where we are now writing $f = T + \eta$. Secondly, iterates under f of points of the local stable manifold do not drift in gently towards 0; they approach it, and one another, exponentially.

(6.9) Theorem. *(i) If η is C^1 in Theorem 6.5, then the tangent to graph h at 0 is parallel to the stable manifold of the hyperbolic linear map $T + D\eta(0)$.*

(ii) The maps h and $f|$graph h are Lipschitz with constant λ, where $f = T+\eta$ and $\lambda = a + \kappa < 1$. For any norm $\|\ \|$ on $\mathbf{E}$ equivalent to $|\ |$, there exists $A > 0$ such that, for all $n \geq 0$ and for all x and $y \in$ graph h,

$$\|f^n(x) - f^n(y)\| \leq A\lambda^n \|x-y\|.$$

Proof. (i) Differentiating the relation $g(x_s) = \chi(x_s, g(x_s))$ at 0 gives $Dg(0) = D\chi(0,0)(id, Dg(0))$. We compute that, for all $(x_s, \gamma) \in B_s + \mathscr{S}_0(B)$, $D\chi(0,0)(x_s, \gamma)$ is the sequence

$$n \mapsto \begin{cases} ((T+D\eta(0))_s \gamma(n-1),\ T_u^{-1}(\gamma_u(n+1) - D\eta_u(0)\gamma(n))) & \text{for } n > 0 \\ (x_s,\ T_u^{-1}(\gamma_u(1) - D\eta_u(0)\gamma(0))) & \text{for } n = 0. \end{cases}$$

Thus $Dg(0):\mathbf{E}_s \to \mathscr{S}_0(\mathbf{E})$ is the fixed point map for χ when η is replaced by $D\eta(0)$. Thus the stable manifold map for $T+D\eta(0)$ is $x_s \mapsto (Dg(0)(x_s)(0))_u$. But this is the map $Dh(0)$.

(ii) Recall that we are working with the norm

$$|x| = \max\{|x_s|, |x_u|\}.$$

Let x and $y \in \operatorname{graph} h$, with $x \neq y$. Suppose that $|x-y| = |x_u - y_u|$. Consider

$$f(x)-f(y) = (T_s(x_s-y_s)+\eta_s(x)-\eta_s(y),\ T_u(x_u-y_u)+\eta_u(x)-\eta_u(y)).$$

The norm of the first component is no greater than $(a+\kappa)|x-y|$, while that of the second component is at least $((1/a)-\kappa)|x-y|$. Since $a+\kappa<1$ and

$$\frac{1}{a}-\kappa > \frac{1}{a} - \frac{\kappa}{a(a+\kappa)} = \frac{1}{a+\kappa} > 1,$$

the norm of the whole expression is the norm of its second component. Thus, by induction, for all $n \geqslant 0$,

$$|f^n(x)-f^n(y)| \geqslant (a+\kappa)^{-n}|x-y|.$$

Since $(a+\kappa)^{-n} \to \infty$ as $n \to \infty$, the sequences $(f^n(x))$ and $(f^n(y))$ are not both bounded, which contradicts the fact that x and y are on the local stable manifold. We deduce that $|x_u - y_u| < |x_s - y_s|$. But now observe that, since $f(x)$ and $f(y) \in \operatorname{graph} h$, $|f(x)-f(y)|$ is also the norm of its first component, and hence, as above, not greater than $(a+\kappa)|x-y|$. Thus $f|\operatorname{graph} h$ is Lipschitz with constant λ. The given inequality follows immediately by changing the norm in

$$|f^n(x)-f^n(y)| \leqslant \lambda^n |x-y|.$$

Finally, if $|x_u-y_u| > \lambda|x_s-y_s|$, then $|f_u(x)-f_u(y)| > ((\lambda/a)-\kappa))|x-y|$, which is a contradiction, since $(\lambda/a)-\kappa = 1+\kappa(1-a)/a > 1 > a+\kappa$. Thus h is Lipschitz with constant λ. □

(6.10) Exercise. Prove an *unstable manifold theorem* for $T+\eta$ by using the formula

$$\chi(x_u, \gamma)(n) = \begin{cases} ((T+\eta)_s\gamma(n+1),\ T_u^{-1}(\gamma_u(n-1)-\eta_u\gamma(n))) & \text{for } n>0, \\ ((T+\eta)_s\gamma(1),\ x_u) & \text{for } n=0. \end{cases}$$

(6.11) Exercise. We call T *α-hyperbolic* $(\alpha>0)$ if the circle of radius α in $\mathbf{C}$ does not intersect the spectrum of T. Thus $\alpha^{-1}T$ is hyperbolic, and we deduce from Theorem 4.19 that there is a decomposition $\mathbf{E} = \mathbf{E}_s \oplus \mathbf{E}_u$ into T-invariant summands and an equivalent norm $|\ | = \max\{|\ |_s, |\ |_u\}$ on $\mathbf{E}$ with respect to which the restrictions T_s and T_u satisfy $|T_s| < \alpha$ and $|T_u^{-1}| < 1/\alpha$. Let B be a ball with centre 0 in $\mathbf{E}$ (if $\alpha > 1$ we require $B = \mathbf{E}$), and let $\eta: B \to \mathbf{E}$ be a Lipschitz map with constant $\kappa < \max\{\alpha - |T_s|, |T_u^{-1}|^{-1} - \alpha\}$ and with $\eta(0)=0$. Replace $\mathscr{S}_0(B)$ by $\tilde{\alpha}(\mathscr{S}_0(B))$ (see Exercise B.22 of

Appendix B) in the proof of Theorem 6.5, and thus prove that there is a unique map $h: B_s \to B_u$ such that, for all $x \in B$, $x \in \text{graph } h$ if and only if $\{\alpha^{-n}(T+\eta)^n(x): n \geqslant 0\}$ is well defined and converges to 0. Prove that h is Lipschitz. Prove also that if $\alpha \leqslant 1$† and η is C^r then h is C^r. We call graph h the *local α-stable manifold of $T+\eta$ at* 0. Convince yourself that this result is not an immediate application of Theorem 6.5 to the hyperbolic automorphism $\alpha^{-1}T$.

We now prove the stable manifold theorem for fixed points of a diffeomorphism of a smooth manifold X. We start with the local version. The proof is completely straightforward, but we give it in full to establish some notation.

(6.12) Theorem. *Let p be a hyperbolic fixed point of a C^r diffeomorphism $(r \geqslant 1)$ f of X. Then, for some open neighbourhood U of p, the local stable set of $f|U$ at p is a C^r embedded submanifold of X, tangent at p to the stable summand of T_pf.*

Proof. Let $\xi: V \to V'$ be an admissible chart at p, with $\xi(p) = 0$. Let g be defined, on some small neighbourhood W of 0 by $g(y) = \xi f \xi^{-1}(y)$. Then g is C^r and has a hyperbolic fixed point at 0. Pick a norm $|\ \ |$ on $\mathbf{E}$ with respect to which $Dg(0)$ has skewness $a < 1$. Then on some open ball B with centre 0, $|Dg(y) - Dg(0)| \leqslant \kappa < 1 - a$. Thus $\eta = g - Dg(0): B \to \mathbf{E}$ is Lipschitz with constant κ. Let $h: B_s \to B_u$ be the corresponding C^r stable manifold map, given by Theorem 6.5. Then $\xi^{-1}(id, h)$ is a C^r embedding of B_s in X with image the local stable manifold of $f|U$ at p, where $U = \xi^{-1}(B)$. Tangency follows from Theorem 6.9. □

(6.13) Theorem. (*Global stable manifold theorem*) *Let p be a hyperbolic fixed point of C^r diffeomorphism f $(r \geqslant 1)$ of X. Then the global stable set Y of f at p is a C^r immersed submanifold of X, tangent at p to the stable summand of T_pf.*

Proof. We continue with the notation of the proof of Theorem 6.12. We shall give the set Y the structure of a C^r manifold by constructing a C^r atlas on Y, and then prove that the inclusion of Y in X is a C^r immersion. Let Y_0 denote the local stable manifold of $f|U$ at p, so that $Y_0 = \xi^{-1}$ (graph h). For all integers $i \geqslant 0$, let $Y_i = f^{-i}(Y_0)$, and let $\xi_i: Y_i \to B_s$ be defined by $\xi_i(y) = \pi_s \xi f^i(y)$, where $\pi_s: B \to B_s$ is projection onto the stable summand. Then the family $\{Y_i: i \geqslant 0\}$ covers Y. For all i, j with $i \geqslant j \geqslant 0$, $\xi_j(Y_i \cap Y_j) = B_s$ and $\xi_i(Y_i \cap Y_j) = \pi_s g^{i-j}$(graph h). Moreover the coordinate transformation

$$\xi_{ij} = \xi_i \xi_j^{-1} : \xi_j(Y_i \cap Y_j) \to \xi_i(Y_i \cap Y_j)$$

† The reason for the failure of this approach in the case $\alpha > 1$ is that Exercise B.22 does not cope successfully with smoothness. However there are some positive results in this case; see, for example, Irwin [2] and Hirsch, Pugh and Shub [1].

and its inverse ξ_{ji} are the C^r diffeomorphisms given by $\xi_{ij}(x) = \pi_s g^{i-j}(x, h(x))$ and $\xi_{ji}(x) = \pi_s g^{j-i}(x, h(x))$. Notice that B_s is open, and hence, by the inverse function theorem $\pi_s g^{i-j}(\text{graph } h)$ is open. Hence $\{\xi_i : i \geq 0\}$ is a C^r atlas on Y.

Let $y \in Y_i$. To show that the inclusion is a C^r immersion at y, it suffices to show that its composite with f^i is an immersion at y (since f^i is a C^r diffeomorphism). But the representative of this composite, with respect to the charts ξ_i at y and ξ at $f^i(y)$ is the C^r embedding (id, h). □

The nature of the charts ξ_i in the above theorem strongly suggests that the global stable manifold of f at p is an immersed copy of the stable manifold of the linear approximation. This is borne out by the following exercise.

(6.14) Exercise. By extending the map η (in the proof of Theorem 6.12) to the whole of $\mathbf{E}$, construct a locally Lipschitz bijection of the Banach space $\mathbf{E}_s(Dg(0))$ onto the stable manifold of f. Show that if f is C^r and if $\mathbf{E}$ admits C^r bump functions then this can be done so that the bijection gives a C^r immersion of $\mathbf{E}_s(Dg(0))$ in X.

One may extend the above theory from fixed points to periodic points of a diffeomorphism with no extra effort. If p is a periodic point of a diffeomorphism $f: X \to X$, then p is a fixed point of the diffeomorphism f^k, where k is the period of p. Notice that $d(f^m(x), f^m(p)) \to 0$ as $m \to \infty$ (where f is some admissible distance function on X) if and only if $f^{nk}(x) \to p$ as $n \to \infty$. Thus, if we define the stable set of f at p to be

(6.15) $$\{x \in X : d(f^m(x), f^m(p)) \to 0 \text{ as } m \to \infty\},$$

and say that p is *hyperbolic* if it is a hyperbolic fixed point of f^k, then Theorem 6.13 becomes:

(6.16) Theorem. *Let p be a hyperbolic periodic point of period k of a C^r diffeomorphism f of X, where $r \geq 1$. Then the stable set of f at p is a C^r immersed submanifold of X tangent at p to the stable summand of $T_p f^k$.* □

Once X is given a distance function d, our new definition of stable set makes sense for *any* point $p \in X$. We shall shortly be proving a stable manifold theorem of a more general nature that involves non-periodic as well as periodic points.

II. STABLE MANIFOLD THEORY FOR FLOWS

The stable manifold theorem for a hyperbolic fixed point of a flow is a simple corollary of the corresponding theorem for diffeomorphisms. We first give a definition of stable set which will serve for any point of the manifold. Let ϕ be a flow on a manifold X, and let d be an admissible distance function on X. The *stable set of ϕ at $p \in X$* is the set

$$\{x \in X : d(\phi(t, x), \phi(t, p)) \to 0 \text{ as } t \to \infty\}$$

and the *unstable set of ϕ at p* is the set

$$\{x \in X : d(\phi(t, x), \phi(t, p)) \to 0 \text{ as } t \to -\infty\}$$

The connection between stable manifolds of diffeomorphisms and flows is made by recalling the result of Exercise 1.39, that if p is a fixed point of ϕ then $\phi(t, x) \to p$ as $t \to \infty$ if and only if $\phi(n, x) \to p$ as $n \to \infty$, where $t \in \mathbf{R}$ and $n \in \mathbf{Z}$. Thus the stable set of the flow ϕ at p is precisely the stable set of the diffeomorphism ϕ^1 at p, or, equally, of ϕ^t for any other $t > 0$). We deduce:

(6.17) Theorem. (*Global stable manifold theorem*) *Let p be a hyperbolic fixed point of a C^r flow ($r \geq 1$) ϕ on X. Then the stable set of ϕ at p is a C^r immersed submanifold of X, tangent at p to the stable summand of $T_p\phi^1$.* □

A local version is only less trivial to deduce in that, if U is an open neighbourhood of p then the local stable manifold of the flow ϕ in U is not necessarily the whole of the local stable manifold of ϕ^1 in U because orbits may leave U at non-integer values of t and then come back again. Nevertheless the former is an open subset of the latter, and thus a submanifold of it. One can get a Lipschitz version of the theorem, corresponding to part of Theorem 6.5, and also a local version when a C^r vector field does not have an integral flow defined for all t. Both are completely straightforward to prove.

The theory for a hyperbolic closed orbit $\Gamma = \mathbf{R} \,.\, p$ is slightly more complicated. Let Γ have period τ. It is an easy consequence of continuity of ϕ and compactness of Γ that $d(\phi(t, x), \phi(t, p)) \to 0$ as $t \to \infty$ in $\mathbf{R}$ if and only if $\phi(n\tau, x) \to \phi(n\tau, p) = p$ as $n \to \infty$ in $\mathbf{Z}$ (see Exercise 1.39). Thus we are interested in the stable set of the diffeomorphism ϕ^τ at its fixed point p. Now p is not a hyperbolic fixed point of ϕ^τ; we have the splitting $T_pX = \mathbf{E}_s \oplus \langle v \rangle \oplus \mathbf{E}_u$, where $\mathbf{E}_s$ and $\mathbf{E}_u$ are the stable and unstable summands at p and v is tangent to Γ at p. If β is the spectral radius of $T_p\phi^\tau | \mathbf{E}_s$, we may choose any α with $\beta < \alpha < 1$ and deduce from Exercise 6.11 the existence of a C^r local α-stable manifold $W_s^\alpha(p)$ of ϕ^τ at p. This submanifold is independent of the choice of α, by uniqueness, and is, as above, contained in the stable set of ϕ at p. We may extend it backwards into a global C^r immersed submanifold $W_s(p)$ modelled on $\mathbf{E}_s$, just as in the proof of Theorem 6.13. The images under ϕ^t of $W_s(p)$, for all $t \in \mathbf{R}$, give a family of submanifolds, one through each point $q = \phi^t(p)$ of Γ. We write $W_s(q)$ for $\phi^t(W_s(p))$ if $q = \phi^t(p)$, and $W_s(\Gamma)$ for $\{x \in W_s(q) : q \in \Gamma\}$. Note that, since $\phi(n\tau, x) \to p$ as $n \to \infty$ for all

$x \in W_s(p)$, $\phi(n\tau, x) \to q$ as $n \to \infty$ for all $x \in W_s(q)$ (see Figure 6.18, where the double arrows indicate orbits of the diffeomorphism ϕ^τ). Notice also that $W_s(\Gamma)$ is determined by the small portion of it at p that is the image of a small neighbourhood of $(0, 0)$ in $\mathbf{R} \times \mathbf{E}_s$ under the map $\phi(id \times h)$, where h is the local α-stable manifold function at p. Since $\langle v \rangle$ does not lie in $\mathbf{E}_s$, which is tangent to the α-stable manifold at p, this map is a C^r immersion, and it follows that globally $W_u(\Gamma)$ is a C^r immersed submanifold modelled on $\mathbf{R} \times \mathbf{E}_s$ and C^r foliated (see Appendix A) by the family $\{W_s(q): q \in \Gamma\}$.

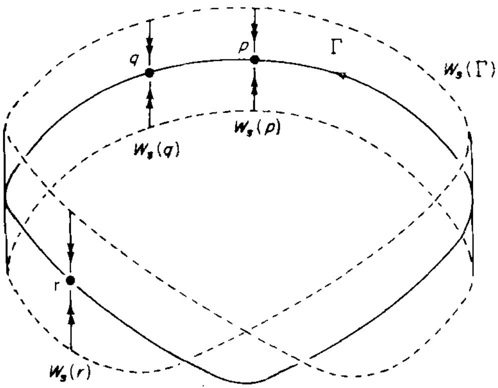

FIGURE 6.18

We call $W_s(\Gamma)$ the *stable manifold* of ϕ at Γ. We emphasize that we have not yet ruled out the possibility that

(i) $W_s(q)$ depends on choice of p, or that

(ii) for some point x, $\phi^{n\tau}(x) \to p$ as $n \to \infty$, but so slowly that x is not on the α-stable manifold for any $\alpha < 1$, or that

(iii) for some point x, the distance from $\phi(t, x)$ to Γ tends to 0 as $t \to \infty$ without $\phi^{n\tau}(x)$ tending to any particular point q of Γ.

If any of these phenomena were actually to occur, we would have points outside $W_s(\Gamma)$ that would nevertheless have a good claim to being called members of the stable set of Γ. However it is easy to see, in finite dimensions at any rate, that (i)–(iii) cannot occur. For, let dim $\mathbf{E}_s = d$, and take a cross section Y to the flow at p. Let $f: U \to Y$ be a Poincaré map at p, and let W be the (d-dimensional) local stable manifold of $f|U$ at p. If $\rho : U \to \mathbf{R}$ is the first return function, then the *positive orbit* $R^+ . W = \{t . y: t \geq 0, y \in W\}$ is a $(d+1)$-dimensional submanifold consisting of all points x in the neighbourhood $V = \{t . y: y \in U, 0 \leq t \leq \rho(y)\}$ of Γ whose positive orbits $\mathbf{R}^+ . x$ are wholly in V. All points w of $\mathbf{R}^+ . W$ have the property that the distance from $t . w$ to Γ tends to 0 as $t \to \infty$, and any orbit whose points have this property contains points of $\mathbf{R}^+ . W$. But now observe that $A = \{t . y : y \in W_s^\alpha(p), 0 \leq t \leq \tau\}$ is a $(d+1)$-dimensional submanifold with $\mathbf{R}^+ . A = A$ and, provided $W_s^\alpha(p)$ is small enough, with $A \subset V$. We deduce that A is contained in $\mathbf{R}^+ . W$, and hence, by the dimensions, that A is a neighbourhood of Γ in $\mathbf{R}^+ . W$. Therefore $W_s(\Gamma)$ is the set of points $x \in X$ such that the distance from $t . x$ to Γ tends to 0 as $t \to \infty$, the *stable set of* ϕ at Γ. This effectively rules out possibilities (i), (ii) and (iii). In particular $W_s(p)$ is precisely the stable set of ϕ at p. In the infinite dimensional case, we are still able to prove that A is a neighbourhood of Γ in $\mathbf{R}^+ . W$. It is not hard to show that the two submanifolds have the same tangent space $\mathbf{E}_s + \langle v \rangle$ at p, and the result follows from this. Our theorem, then, may be stated as follows:

(6.19) Theorem. (*Stable manifold theorem for closed orbits*) *Let* $\Gamma = \mathbf{R} . p$ *be a hyperbolic closed orbit of a* C^r *flow* ϕ $(r \geq 1)$ *on a manifold X with distance function d. Then the stable set* $W_s(p)$ *of* ϕ *at p is a* C^r *immersed submanifold tangent at p to the stable summand of* T_pX *with respect to* $T_p\phi^\tau$, *where* τ *is the period of* Γ. *If* $q = \phi^t(p)$, *then* $W_s(q) = \phi^t(W_s(p))$. *The stable set* $W_s(\Gamma)$ *of* ϕ *at* Γ *is a* C^r *immersed submanifold which is* C^r *foliated by* $\{W_s(q) : q \in \Gamma\}$. □

The submanifold $W_s(p)$ is called the *stable manifold of* ϕ *at p*. Since by the above remarks $W_s(p)$ is also the stable manifold of the diffeomorphism ϕ^τ at p, it is independent of the distance function d, as also is $W_s(\Gamma) = \bigcup_{q \in \Gamma} W_s(q)$.

III. THE GENERALIZED STABLE MANIFOLD THEOREM

Let X be a finite dimensional Riemannian manifold (see Appendix A) and let $f: X \to X$ be a diffeomorphism. We have already commented that our definition (6.15) of the stable set of f at a point p is valid for any $p \in X$, and we have seen that, when p is periodic and has a hyperbolic structure with respect to f, its stable set is a manifold. The question now arises as to whether we can extend our notion of hyperbolicity to non-periodic points p and get a stable manifold theorem for such points. The difficulty, of course, is that T_pf^n does not map T_pX to itself for $n > 0$, and so we cannot define hyperbolicity in terms of eigenvalues of this map. If we concentrate for the moment on the expanding and contracting properties of hyperbolic maps rather than their eigenvalues, we can see how a definition might go, but it will involve the whole orbit of p (as indeed does the definition of stable set). In fact, we may as well define the term hyperbolic structure for an arbitrary invariant subset of X, since many applications of stable manifold theory need this degree of generality.

Let Λ be any invariant subset of X, and let $T_\Lambda M$ be the tangent bundle of X over Λ (that is, $\{T_xX : x \in \Lambda\}$). We say that Λ has a *hyperbolic structure* (with respect to f) if there is a continuous splitting of $T_\Lambda X$ into the direct sum of Tf-invariant subbundles E_s and E_u such that, for some constants A and λ and for all $v \in E_s$, $w \in E_u$ and $n \geq 0$,

(6.20) $$|Tf^n(v)| \leq A\lambda^n|v|, \qquad |Tf^{-n}(w)| \leq A\lambda^n|w|,$$

where $0 < \lambda < 1$. Thus one may say that Tf is eventually contracting on E_s and eventually expanding on E_u. A *hyperbolic subset* of X (with respect to f) is a closed invariant subset of X that has a hyperbolic structure.

There is a temptation to make "eventually" into "immediately" by putting $A = n = 1$ in the above definition, but the result would not be invariant under differentiable conjugacy. If Λ is compact, one may always introduce a new Riemannian metric on X with respect to which (6.20) does hold with $A = n = 1$. Such a metric is said to be *adapted to f*. The proof, which resembles that of Theorem 4.47, is due to Mather [1], and also appears in Shub [4], Hirsch and Pugh [1] and Nitecki [1]. However we shall not need this result.

The idea behind the generalized stable manifold theorem is as follows. If Λ is a compact hyperbolic subset, the space $C^0(\Lambda, X)$ of all continuous maps from Λ to X may be given the structure of a smooth manifold modelled on the Banach space $\Gamma^0_\Lambda(X)$ of C^0 sections of $T_\Lambda(X)$, given the sup norm derived from the Riemannian Finsler $|\ |_x$ on X (see Appendix A). One considers the map $f^\#: C^0(\Lambda, X) \to C^0(\Lambda, X)$ defined by $f^\#(h) = fh(f|\Lambda)^{-1}$. Clearly this has a fixed point at ι (the inclusion of Λ in X). It turns out that the hyperbolic structure of Λ makes ι a hyperbolic fixed point. Thus $f^\#$ has a stable manifold at ι, consisting of $h \in C^0(\Lambda, X)$ such that $(f^\#)(h) \to \iota$ as $n \to \infty$. For such h, $f^nhf^{-n}(x) \to x$ as $n \to \infty$ for all $x \in \Lambda$, or, equivalently, by compactness of Λ, $d(f^nh(y), f^n(y)) \to 0$ as $n \to \infty$ for all $y \in \Lambda$. Here d is the distance function on X derived from the Riemannian metric (see Appendix A). So if we evaluate h at $y \in \Lambda$, we get a point on the stable set of y. If we do so for all h near ι, we get the local stable manifold at y, and properties (such as smoothness) of the stable manifold at ι yield, after evaluation at y, properties of the stable manifold at y. Moreover, if we let y vary, we can get information about how the stable manifold at y varies with y.

There is no need for us to give a full proof of all the statements in the previous paragraph. Since we are only applying the stable manifold theorem at a single point of $C^0(\Lambda, X)$ we can, provided we have a chart at ι, remove the problem to the model space $\Gamma^0_\Lambda(X)$, and, indeed, doing so makes some of our arguments easier to apply. For the manifold structure of $C^0(\Lambda, X)$ and related theory we refer the reader to Franks [2], Eliasson [1], Foster [1]. Notice, however, that for $\sigma \in \Gamma^0_\Lambda(X)$ near the origin, composing with the exponential map exp: $TX \to X$ gives an element h of $C^0(\Lambda, X)$ near the inclusion, and that all such h come about in this way. Reversing the process gives us a chart at ι. Moreover the tangent map $Tf^{\#}$ is, in terms of this chart, the map $\sigma \mapsto (Tf)\sigma(f|\Lambda)^{-1}$. The decomposition

$$\Gamma^0_\Lambda(X) = C^0(E_s) \oplus C^0(E_u)$$

is $T_\iota f^{\#}$-invariant, and, if we write T_s and T_u for the restrictions of $T_\iota f^{\#}$ to the two summands, we have that $|T_s^n| \leq A\lambda^n$ and $|T_u^{-n}| \leq A\lambda^n$. Thus, by our remarks about the spectral radius in the appendix to Chapter 4, the spectra of T_s and T_u are separated by the unit circle, and so ι is hyperbolic, as asserted.

We now make some definitions needed for the statement of our main theorem. Let $B(x, a)$ be the open ball in X with centre x and radius a, with respect to the Riemannian distance function d, and let $\Sigma(x, b)$ denote the set $\{y \in X: d(f^n(x), f^n(y)) < b \text{ for all } n \geq 0\}$. For $b \geq a \geq 0$ we define the *stable set of size* (b, a) of f at x to be $B(x, a) \cap \Sigma(x, b)$. We say that a map of an open subset of the total space of a vector bundle into a manifold is F^r (r times continuously fibre differentiable) if, with respect to admissible atlases, all partial derivatives in the fibre direction up to order r exist and are continuous as functions on the total space. One often says, in this case, that the images of the fibre vary C^r-continuously. Some of the theory of such maps is given in Appendix B. We can now state:

(6.21) Theorem. (*Generalized stable manifold theorem*) *Let f be a C^r diffeomorphism of X, and let Λ be a compact hyperbolic subset of X, with associated decomposition $T_\Lambda X = E_s \oplus E_u$. Then there exists an open neighbourhood W of the zero section in E_s and an F^r map $h: W \to X$, such that, for some $b \geq a \geq 0$ and for all $x \in \Lambda$, g restricted to the fibre W_x over x is a C^r embedding with image $W_s^{\text{loc}}(x)$, the stable set of size (b, a) at x. The tangent space to $W_s^{\text{loc}}(x)$ at x is $(E_s)_x$.*

Proof. For some neighbourhood P of the zero section in $T_\Lambda X$, we have a map $\tilde{\phi}: P \to T_\Lambda X$ defined by

$$\tilde{\phi}(v) = (\exp_{f(x)})^{-1} f \exp(v),$$

where v is in the fibre P_x. We may define a map $\phi: C_b(P) \to C_b(T_\Lambda X)$ by

$$\phi(\rho) = \tilde{\phi}\rho(f|\Lambda)^{-1},$$

where $C_b(T_\Lambda X)$ is the Banach space of bounded sections of $T_\Lambda X$ (using the sup norm) and $C_b(P)$ is the subset of sections with values in P. We observe that the map taking ρ to $(Tf)\rho(f|\Lambda)^{-1}$ is continuous linear, and that the map on sections induced by the F^r map $\tilde{\phi}(Tf^{-1})$ is C^r (see Remarks B.26 of Appendix B). Thus ϕ is C^r.

The differential of ϕ at $0 \in C_b(P)$ is the linear automorphism $\rho \mapsto (Tf)\rho(f|\Lambda)^{-1}$ of $C_b(T_\Lambda X)$. As indicated earlier, this automorphism is hyperbolic, with stable summand $C_b(E_s)$ and unstable summand $C_b(E_u)$. We may, by Theorem 4.19, give those subspaces norms (equivalent to the previous ones) with respect to which the stable component of $D\phi(0)$ and the inverse of its unstable component have operator norm <1. We may then apply Theorem 6.5, with $\eta = \phi - D\phi(0)$, and obtain a C^r stable manifold map $\psi: B_s \to B_u$, for some small balls B_s and B_u of equal radii (see Theorem 6.9 (ii)) and with centre 0 in $C_b(E_s)$ and $C_b(E_u)$ respectively. This ψ is the unique map with the property that $\tau = \psi(\sigma)$ if and only if $\phi^n(\sigma, \tau) \in B_s \times B_u$ for all $n \geqslant 0$.

To get back to the original norm on $C_b(T_\Lambda X)$, choose balls $B(0, a)$ and $B(0, b)$ with respect to that norm, and balls D_s and D_u with respect to the new norm, all with centre 0, such that

$$B(0, a) \subset D_s \times D_u \subset B(0, b) \subset B_s \times B_u.$$

We have a stable manifold function: $D_s \to D_u$ which is the restriction of $\psi: B_s \to B_u$. Let G denote its graph $\{(\sigma, \psi(\sigma)) : \sigma \in D_s\}$. If $\rho \in B(0, a) \cap G$, then $\phi^n(\rho)$ stays in $D_s \times D_u$, and hence in $B(0, b)$ for all $n \geqslant 0$, whereas if $\rho \in B(0, a) \backslash G$, then $\phi^n(\rho)$ leaves $B_s \times B_u$, and hence leaves $B(0, b)$, for some $n > 0$. Let V denote the image of $B(0, a) \cap G$ in $C_b(E_s)$ by the product projection. Clearly V is open in $C_b(E_s)$.

Now consider the restriction $\psi: V \to C_b(E_u)$. It is the unique map with the property that $\tau = \psi(\sigma)$ if and only if $(\sigma, \tau) \in B(0, a)$ and $\phi^n(\sigma, \tau) \in B(0, b)$ for all $n \geqslant 0$. Equivalently, $\tau = \psi(\sigma)$ if and only if, for all $x \in \Lambda$, $\exp(\sigma(x), \tau(x))$ is in the stable manifold of size (b, a) at x. The first thing that this characterization shows is that $\tau(x)$ is a function of $\sigma(x)$ rather than the whole section σ. By this we mean that, if two elements of V had the same value at $x \in \Lambda$ but their images under ψ had different values at x, we could interchange the latter values without disturbing the stable manifold property (here we use the fact that our sections are merely bounded). This offends uniqueness of ψ. We conclude that if any $\sigma \in V$ has $\sigma(x) = v$ for some $x \in \Lambda$, then the section σ_v defined by

$$\sigma_v(x) = v, \ \sigma_v(y) = 0 \quad \text{for } y \neq x$$

is in V, and $\psi(\sigma)(x) = \psi(\sigma_v)(x)$. Thus, if W denotes the open subset $\{\sigma(x) : x \in \Lambda, \sigma \in V\}$ of E_s, we may define a map $g: W \to E_u$ by

$$g(v) = \psi(\sigma_v)(x),$$

where $v \in W_x$, such that ψ is induced by g. That is to say, $V = C_b(W)$ and, for all $\sigma \in V$, $\psi(\sigma) = g\sigma$. In the notation of Appendix B, $\psi = g_*$.

The map h of the theorem is defined by

$$h(v) = \exp(v, g(v)).$$

Thus if g is F^r, so is h. We would like to deduce the result for g from the fact that g_* is C^r. This is not immediate. However if we now restrict our attention to the subspace $C^0(T_\Lambda M)$ of all continuous sections of $T_\Lambda M$, we obtain as before a C^r stable manifold map from $C^0(W) = V \cap C^0(E_s)$ to $C^0(E_u)$ which agrees with $\psi: W \to C_b(E_u)$ by uniqueness of the latter. Thus the C^r map $\psi: C^0(W) \to C^0(E_u)$ is induced from g, and this fact, by Theorem B.27 of Appendix B, ensures that g is F^r.

Now that we know that g is F^r, we may work with bounded sections again. By Theorem 6.9, the differential $D\psi(0)$ of ψ: $V \to C_b(E_u)$ is the zero map from $C_b(E_s)$ to $C_b(E_u)$. The fibre differential of g at $v \in W_x$ is the linear map from $(E_s)_x$ to $(E_u)_x$ whose value at w is the value at x of $D\psi(\sigma_v)(\sigma_w) \in C_b(E_u)$, where σ_v is the section defined above (see Remarks B.26 of Appendix B). Thus the fibre differential at $0 \in W_x$ is the zero map from E_s to E_u, and this gives the required tangency property. □

(6.22) Corollary. *There exists constants A and λ, with $0 < \lambda < 1$, such that, for all $x \in \Lambda$, for all y and $z \in W_s^{\text{loc}}(x)$ and for all $n \geq 0$,*

$$d(f^n(y), f^n(z)) \leq A\lambda^n d(y, z).$$

For all x and $x' \in \Lambda$, $W_s^{\text{loc}}(x) \cap W_s^{\text{loc}}(x')$ is open in $W_s^{\text{loc}}(x)$.

Proof. The inequality follows easily from Theorem 6.9 (ii) applied to the stable manifold of the map ϕ in the above proof. Now let $y \in W_s^{\text{loc}}(x) \cap W_s^{\text{loc}}(x')$, and let $z \in W_s^{\text{loc}}(x)$. By the first part

$$\begin{aligned} d(f^n(z), f^n(x')) &\leq A\lambda^n d(z, x') \\ &\leq A\lambda^n (d(z, x) + d(x, y) + d(y, x')) \\ &\leq 3aA\lambda^n, \end{aligned}$$

and so, for all $n \geq$ some n_0, $d(f^n(z), f^n(x')) < b$ for all $z \in W_s^{\text{loc}}(z)$. But, for z sufficiently near y, $d(y, z) < a - d(y, x')$ and, by continuity of f, $d(f^n(y), f^n(z)) < b - d(f^n(y), f^n(x'))$ for all n with $0 \leq n \leq n_0$. Thus $z \in W_s^{\text{loc}}(x')$. □

We shall not give a detailed account of the generalized stable manifold theorem for flows. We could obtain a version by following through the proof of Theorem 6.21, and replacing stable manifolds by α-stable manifolds (for $\alpha < 1$) as in the proof of Theorem 6.19. There would remain the difficulty of proving that the α-stable manifolds so obtained are precisely the stable sets of the points of the invariant set Λ. The generalized stable manifold theorem for flows is actually a corollary of an even more general theorem, Theorem 6.1 of Hirsch, Pugh and Shub [1], which is the reference for further reading on the subject.

Appendix 6

I. PERTURBED STABLE MANIFOLDS

We extend the proof of the local stable manifold theorem to give information about the dependence of the stable manifold function h upon the perturbation η. We continue with the notations of Theorem 6.5. In addition, let N^r $(r \geq 0)$ denote the subset of $C^r(B, \mathbf{E})$ consisting of maps η that are Lipschitz with constant κ and satisfy $\eta(0) = 0$.

(6.23) Theorem. *For all $\eta \in N^r$, the stable manifold function h^η of $T + \eta$ is C^r-bounded. The map $\theta: N^r \to C^r(B_s, B_u)$ sending η to h^η is continuous at η_0 if η_0 is uniformly C^r.*

Proof. We regard η as another independent variable on the right-hand side of (6.7), so that χ becomes a map from $N^r \times B_s \times \mathcal{S}_0(B)$ to $\mathcal{S}_0(B)$. We obtain a fixed point map $f: N^r \times B_s \to \mathcal{S}_o(B)$. Using Corollary B.11 of Appendix B, $\chi^\eta - \chi^0$ is C^0-bounded and $D\chi^\eta$ is C^{r-1}-bounded. Thus by Theorem C.10 of Appendix C, $g^\eta - g^0$ is C^r-bounded, and hence $h^\eta - h^0$ is C^r-bounded. Since h^0, the stable manifold function of T, is zero, h^η is C^r-bounded.

Now let $\eta_0 \in N^r$ be uniformly C^r, and let $\eta \in N^r$. Then the map from N^r to $C^r(B_s \times \mathcal{S}_o(B), \mathcal{S}_0(\mathbf{E}))$, taking η to $\chi^\eta - \chi^{\eta_0}$ is continuous at η_0 (this is essentially Corollary B.12). Using Theorem B.20, we see that the map χ^{η_0} is uniformly C^r. Hence, by Theorem C.10, the map from N^r to $C^r(B_s, \mathcal{S}_0(B))$ taking η to $g^\eta - g^{\eta_0}$ is continuous at η_0. Thus so is θ, which is this map composed with $(\pi_u\, ev^0)_*$, where $\pi_u: \mathbf{E} \to \mathbf{E}_u$ is the product projection. $\square$