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# Dynamical systems with elastic reflections

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# DYNAMICAL SYSTEMS WITH ELASTIC REFLECTIONS

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In this paper we consider dynamical systems resulting from the motion of a material point in domains with strictly convex boundary, that is, such that the operator of the second quadratic form is negative-definite at each point of the boundary, where the boundary is taken to be equipped with the field of inward normals. We prove that such systems are ergodic and are $K$-systems. The basic method of investigation is the construction of transversal foliations for such systems and the study of their properties.

## Contents



## Introduction

This paper is concerned with the problem of investigating the statistical properties of conservative dynamical systems of classical mechanics. This problem has been discussed many a time and in various connections, for example, in connection with the second law of thermodynamics or the foundation of the Gibb's distribution, the interpretation of quantum mechanics, and others.

The idea that the dynamical systems of classical mechanics are devoid of statistical properties goes back to Laplace and rests on the fact that the motion of such a system is uniquely determined once the initial conditions are given. However, for a dynamical system the practical significance is in the motion of a complete cell of the phase space in which the points move according to given differential equations (of motion). Regularity in the motion of such cells must be of a totally different character to the regularity of the motion of individual points.

For example, even in fairly simple conservative systems cells having initially a regular form become distorted in time, take on a very intricate form, and are distributed in extremely complex shapes of the phase space. This phenomenon may be regarded as an irreversibility property of the dynamical systems of classical mechanics: domains of regular form are eventually transformed into domains of extremely complex form. The idea of such irreversibility was very clearly expressed by Krylov [8]. A similar irreversibility already occurs in systems with a small number of degrees of freedom.

The character of the motion of a single cell in phase space is connected with a stability property. The trajectory of a point $x$ is called stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that the $\delta$-neighbourhood of $x$ eventually stays in the $\varepsilon$-neighbourhood of the point $x_t$. Clearly for the spreading of the cell described above the motion must be unstable, that is, arbitrarily close points eventually diverge and move independently. So we arrive at the idea that irreversibility and statistical properties are connected with instability of motion. This idea has often been expressed by various authors. We refer, for example, to the article by M. Born "Is classical mechanics really deterministic?" in the book [6] and to Krylov's book [8] mentioned already.

Mathematical methods for the study of the statistical properties of unstable systems have emerged since the publication of the paper by Kolmogorov [7]. Kolmogorov introduced two new ideas into ergodic theory – the idea of entropy of a dynamical system and the idea of a $K$-system. Both ideas are connected with Shannon's information theory and with stationary stochastic processes, but we shall not dwell on this connection in any more detail. We note only that a $K$-system has very good statistical properties, in a certain sense the best.

In a whole series of problems it has turned out that it is more natural first to establish that a dynamical system is a $K$-system and then to obtain as corollaries all the necessary ergodicity and mixing properties. The way to clarify when a dynamical system is a $K$-system lies in the construction and study of the so-called transversal foliations. Transversal foliations were first used for the study of statistical properties of dynamical systems by E. Hopf in studying geodesic flows on manifolds of negative curvature [18]. These geodesic flows are ideal examples of unstable systems. But only the notion of a $K$-system revealed the true role of transversal foliations.

Roughly, the situation is the following. Cells in unstable systems do spread, but in a short time, depending on the dimensions and form of the cells, such spreading occurs equally in different cells. It could be said that the spreading has a Markov character. Transversal foliations allow us to establish exactly this basic property. Using transversal foliations we may study the relaxation time, the rate of convergence of time averages to space averages, etc.

In the papers of Anosov [2], [3] smooth unstable systems, which he calls $U$-systems, are defined axiomatically. For such systems he constructs transversal foliations and studies some of their properties. In the author's papers [11], [12], [13] transversal foliations were used systematically to prove that a given dynamical system is a $K$-system. The geodesic flows mentioned above are basic amongst the examples in question.

The present paper studies more "physical" examples of unstable systems, which we call scattering billiard balls. The motion of a spherical molecule in the torus or square under elastic reflections from other fixed spherical molecules is an example of such a system. To avoid minor complications of differential geometry we first consider the two-dimensional case and at the end we show how to obtain the generalization to many dimensions. An intuitive explanation of the analogy between billiards and geodesic flows was given by Arnol'd [5]. In such systems the convex boundary plays the scattering role of negative curvature.

Systems of billiard type are discontinuous: uneven changes in velocity occur at the boundary relection points. In addition, the instability property is unequal near different trajectories. For example, there may be trajectories where the moving molecule is never reflected by the fixed molecules. Such exceptional trajectories possess stable directions. Naturally these trajectories are not typical and have probability 0, but in a detailed analysis it is necessary to select trajectories with enough instability properties.

We construct transversal foliations for the systems in question. Because of the singularities mentioned above they exist almost everywhere. In addition, individual fibres have singularities. All this, of course, makes the investigation difficult.

Formally this paper may be read independently of the other papers mentioned on transversal foliations and $K$-systems, in particular, the part concerning ergodicity. However, for a clearer understanding of the logical connections an aquaintance with the papers [4], [11] is desirable.

This paper contains a detailed account of some of the results announced in [14] (see also [15]).

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## §1. Description of the dynamical system and its simplest properties

Let $Q$ be a closed domain obtained from a two-dimensional torus by cutting out a finite number of pairwise disjoint convex domains. Assume that each of these domains is bounded by a convex curve of class $C^3$ so that we can speak of the curvature, being a differentiable function of the point. We assume that the curvature at each point is non-zero. The union of these curves forms the boundary $\partial Q$ of $Q$. The individual curves are called the regular components of the boundary and are denoted by $\partial Q_i (i = 1, 2, \ldots, k)$.

$M$ denotes the restriction to $Q$ of the unit tangent bundle of the two-dimensional torus. It is clear that $M$ is a foliation with base $Q$ and with a unit circle $S^1(q)$ as fibre over $q \in Q$. Let $\pi$ denote the natural projection of $M$ onto $Q$. The points of $M$ are called line elements.

$M$ is a three-dimensional manifold with boundary. $\partial M$ is made up of line elements $x$ for which $\pi(x) \in \partial Q$. We call $\partial M_i = \pi^{-1}(\partial Q_i)$ a regular component of $\partial M$. It is obvious that $\partial M_i$ is topologically a torus.

There is a natural coordinate system $(r, \varphi)$ on $\partial M_i$ : $r$ is the arc length in $\partial Q_i$, $\varphi$ is the natural angular parameter.

REMARK 1.1. We take $\partial Q$ as equipped with the field of inward normals $n(q)$ (relative to $Q$). The angle $\varphi$ is counted from $n(q)$ and varies in $0 \leqslant \varphi < 2\pi$.

Put $S_- = \{ x \in \partial M:$ the scalar product $(x, n(q)) \leqslant 0$, where $q = \pi(x) \}$, $S_+ = \{ x \in \partial M: (x, n(q)) \geqslant 0$, where $q = \pi(x) \}$. The intersection $S = S_- \cap S_+$ consists of line elements tangent to the boundary. $S$ is called the set of singular points of the boundary. Its existence is unavoidable and greatly complicates the analysis. The reader will have more than one opportunity to convince himself of this.[1]

Consider a curve $\tilde{l} \in Q$ of class $C^2$. By an equipment $l$ of $\tilde{l}$ we mean a continuous section of the tangent bundle over $\tilde{l}$ such that at each point $q \in \tilde{l}$ the line element $\pi^{-1}(q) \cap l$ is orthogonal to the line element tangent to $\tilde{l}$ at $q$. Each curve $\tilde{l}$ has two equipments. After an equipment has been chosen, we can talk of the curvature of $\tilde{l}$ having a sign. If the curvature is nowhere zero, then its value at a single point uniquely determines an equipment.

The Euclidean metric in $Q$ determines a metric in $M$. The latter in its turn induces a metric on $\partial M$. We denote the distance determined by this metric by $d$. Without loss of generality we may assume that $d(x, y) \leqslant 1$ for $x, y \in M$. We also introduce a measure $\mu$ in $M$ putting $d\mu = dq d\omega$ where $dq$ is the measure in $Q$ induced by the metric, and $d\omega$ is the natural measure on the fibre $S^1(q)$. We take $\mu$ to be normalized.

DEFINITION 1.1. The dynamical system generated by the motion of a material point of mass 1 with unit velocity along straight lines inside $Q$ is called *billiards in* $Q$. On reaching the boundary $\partial Q$ the point is reflected by the law "the angle of incidence is equal to the angle of reflection". The latter means that at the boundary the tangential component of the velocity is preserved and the normal component changes sign.

$M$ is the phase space of billiards. Billiards in domains $Q$ whose boundaries have everywhere positive curvature (with respect to the equipment chosen at the boundary) are called scattering. Clearly the domains we consider are of this type.

We denote by $\{ S_t \}$, $-\infty < t < \infty$, the one-parameter group of displacements along the trajectories of billiards. It is not difficult to see that it preserves $\mu$ (see [4]). Therefore $\{ S_t \}$ is a flow in the sense of ergodic

[1] Later we shall study vector fields on $M$ of a special form. In the usual theory of vector fields on manifolds with boundary it is assumed that at the boundary the vector field is transversal to the boundary. In our case this will not be so. In $S$ the vector field is tangent to the boundary.

theory (see [9]), and it is permitted to neglect sets of trajectories of measure zero. In particular, trajectories that are at least once tangent to the boundary have measure zero.

A convenient means of studying flows are special representations by an automorphism and a function (see [9]). In the case of billiards there is a natural special representation.

Put $M_1 = S_-$ and let $\tau(x)$ be the nearest negative moment of a boundary reflection of the trajectory of $x$. It is easily seen that $\tau(x) > -\infty$ and that the transformation $Tx = S_{\tau(x)-0}x \in M_1$ is defined for every $x \in M_1$. We introduce a measure $\nu$ in $M_1$, putting for any $A \subset M_1$

$$\nu(A) = \int_{\widetilde{A}} \frac{d\mu(y)}{F(y)}, \tag{1}$$

where $\widetilde{A} = \{y \in M\colon y = S_t x \text{ for } 0 \geqslant t > \tau(x),\ x \in A\}$, $F(y) = \tau(x)$. Then $T$ preserves $\nu$ and is called a derived automorphism of the flow $\{S_t\}$.

The modulus of the function $\tau$ is bounded below: $|\tau| \geqslant \tau_0 = \text{const}$ and, in general, is not bounded above. It is not bounded above in a neighbourhood of closed trajectories that are always tangent to $\partial Q$, and where $\partial Q$ at points of contact lies on one side of the trajectory. The number of such trajectories is finite.

Let us find $\nu$ in terms of the variables $(r, \varphi)$. Let $x$ be an interior point of $M$ and let $y \in M_1$ be such that $x = S_t y$ for some $t$, $0 > t > \tau(y)$. In a neighbourhood of $y$ in $M_1$ there are natural coordinates $(r, \varphi)$ (see above). Then $(t, r, \varphi)$ serve as coordinates in a neighbourhood of $x$, where the point $x'$ with the coordinates $(t, r, \varphi)$ has the form $S_t y(r, \varphi)$, $y(r, \varphi) \in M_1$ being the point with the coordinates $(r, \varphi)$. It can be checked directly that $\mu$ in the variables $(t, r, \varphi)$ has the form

$$d\mu = -\operatorname{const} \cos\varphi \, dt \, dr \, d\varphi. \tag{2}$$

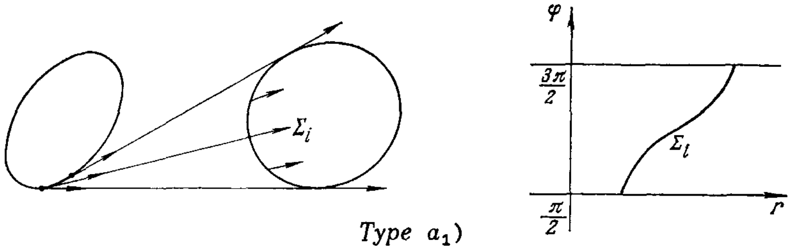
Consequently, $\nu$ has the form

$$d\nu = -\operatorname{const} \cos\varphi \, dr \, d\varphi. \tag{3}$$

We recall that $\cos\varphi \leqslant 0$ for $x \in M_1$.

In the remainder of this section we investigate the smoothness properties of $T$. In general, $T$ is discontinuous. It is easy to see that it has singularities on the set $M_2 = T^{-1}S \cup S$. We now examine in detail the structure of the set $M_2$.

REMARK 1.2. In the estimates and constructions to be carried out later, and throughout the paper, there will be many constants that depend on the properties of $Q$, more precisely, on $\partial Q$. The explicit form of these constants is not essential. Throughout we denote such constants by the word const without additional indices.



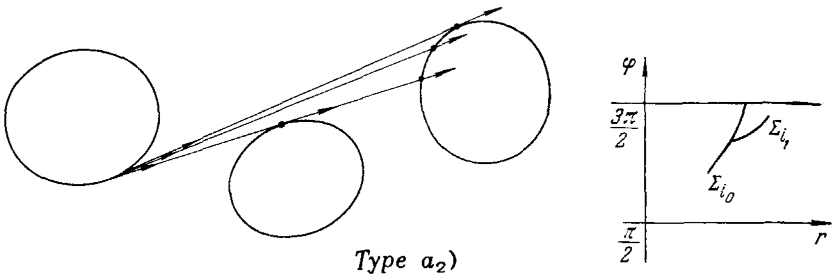

Type $a_1$)

Fig. 1.

If $x \in M_2$ and $x \notin S$, then in a neighbourhood of $x$ the set $T^{-1}S$ is a smooth curve given by a function $\varphi = \varphi(r)$. It can be verified directly (see also §2: it follows easily from the lemmas there) that $\varphi(r)$ satisfies the differential equation

$$\frac{d\varphi}{dr} = k^{(0)}(r) + \frac{\cos\varphi}{\tau(r, \varphi)},$$

where $k^{(0)}(r)$ is the curvature of $\partial Q$ at $r$. If we continue this curve as far as possible in both directions, preserving regularity, then we obtain a smooth curve. The number of such curves is, in general, infinite. Denote them by $\Sigma_1, \Sigma_2, \ldots$




Type $a_2$)

Fig. 2.

We call the $\Sigma_i$ the discontinuity curves of $T$. It is easy to see that the ends $z_1$, $z_2$ of a curve $\Sigma_i$ must be of one of the following two types (see fig. 1, 2)

$$a_1)\ z_j \in S, \quad a_2)\ z_j \in T^{-1}S \cap T^{-2}S.$$

In case $a_2$) $z_j$ defines a trajectory for which at least the first two boundary reflections for $t < 0$ degenerate to tangency. The point $z_j$ is both an end of one $\Sigma_i$ and an interior point of another $\Sigma_{i_1}$. Such points are called double points.

The number of curves $\Sigma_i$ can be infinite. The limit points for these curves are points $\hat{x}_i$ in a neighbourhood of which $\tau$ is unbounded. As was said above, such points define trajectories whose boundary reflections all degenerate to tangency. The number of such points is finite and we denote them by $\hat{x}_1, \ldots, \hat{x}_I$. For example, if $Q$ is a torus with a circle of sufficiently large radius $\rho$ removed then $I = 8$. It is also not difficult to construct a domain $Q$ for which $I = 0$ and then the modulus of $\tau$ is bounded above. In this case the number of curves $\Sigma_i$ is finite.

Later we need information on the structure of the $\Sigma_i$ in a neighbourhood of the $\hat{x}_r$. For this purpose we take the trajectory defined by $\hat{x}_r$ which for all reflections is tangent to the boundary and, at all tangency points, $\partial Q$ lies on one side of this trajectory. The $\Sigma_i$ in a neighbourhood of $\hat{x}_r$ are drawn in Fig. 3.

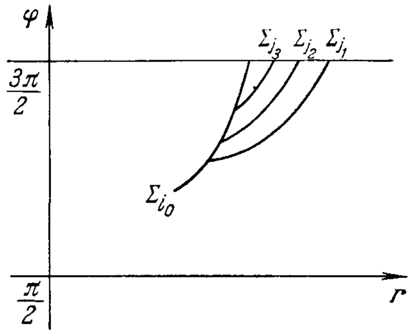



Fig. 3.

Put $\tau(\Sigma_j) = \min_{x \in \Sigma_j} \tau(x)$. It is easy to see that all the discontinuity curves can be renumbered so that in the new numbering

$$\text{const} \leqslant \frac{j}{|\tau(\Sigma_j)|} \leqslant \text{const}.$$

Also it is not difficult to deduce from geometrical arguments that the function $\varphi_j(r)$ giving the curve $\Sigma_j$ satisfies the differential equation $\frac{d\varphi_j}{dr} = k^{(0)}(r) + \frac{\cos\varphi_j}{\tau(r, \varphi_j)}$ and is defined on a segment $\Delta r_j \sim \frac{\text{const}}{\tau^2(\Sigma_j)}$. In addition, one end of $\Sigma_j$ has the coordinate $\varphi = \frac{3\pi}{2}$ (or $\frac{\pi}{2}$) and the other end is a double point. The double points occurring here lie on a fixed $\Sigma_{j_0}$, and the other end of the $\Sigma_j$ (see fig. 3) have the coordinate $\varphi = \frac{3\pi}{2}\left(\text{ or } \frac{\pi}{2}\right)$.

As above, $d$ is the distance on $M_1$ generated by the metric $ds^2 = dr^2 + d\varphi^2$. Put $d(x) = d(x, M_2)$. Denote by $\varphi(x)$ the angle between $x$ and $n(q)$, where $q = \pi(x)$. Since $S \subset M_2$, we have $-\cos\varphi(x) \sim \varphi(x) - \frac{\pi}{2}$ if $\varphi(x)$ is near $\frac{\pi}{2}$ and the right-hand side is small; $-\cos\varphi(x) \sim \frac{3\pi}{2} - \varphi(x)$ if $\varphi(x)$ is near $\frac{3\pi}{2}$ and the right-hand side is small. Therefore

$$|\cos\varphi(x)| \geqslant \text{const}\, d(x). \tag{4}$$

We turn to the function $\tau(x)$. From what we have done so far in the analysis of the discontinuity curves $\Sigma_j$ it follows easily that

$$|\tau(x)| \leqslant \frac{\text{const}}{d(x)}. \tag{5}$$

Consider the function $\cos\varphi(Tx)$. Take a half-open curve $\gamma$ consisting of line elements parallel to $Tx$ and going out from $Tx$ in the direction of decrease of $\cos\varphi(z)$, $z \in \gamma$, such that $T^{-1}|\gamma$ is smooth. One end of $\gamma' = T^{-1}\gamma$ is $x$ and the other is a point $y \in M_2$. Here the length $l(\gamma') \geqslant d(x)$. It is not difficult to see that $l(\gamma') \leqslant \text{const}\,|\cos\varphi(Tx)|$, and therefore

$$|\cos\varphi(Tx)| \geqslant \text{const}\, d(x). \tag{6}$$

Later we need an estimate of the logarithmic derivatives of $\cos\varphi(x)$, $\cos\varphi(Tx)$ and $\tau(x)$. Assume that coordinates $(r, \varphi)$ are chosen in a neighbourhood of $x \in \partial M$, where $r$ is the natural parameter on $\partial Q$ counted from $\pi(x)$, and $\varphi$ is the angle, $\frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}$.

The following lemma holds.

LEMMA 1.1. *There is a constant* const *such that the logarithmic derivatives of* $\cos\varphi(x)$, $\cos\varphi(Tx)$ *and* $\tau(x)$ *do not exceed in modulus* const $d^{-1}(x)$.

PROOF. We use the following inequality:

$$d(x) \leqslant \frac{\text{const}\cos^2\varphi(Tx)}{|\tau(x)|}. \tag{7}$$

To obtain this inequality it is simplest to go over to the two-dimensional Euclidean plane and to make all constructions in it. Then $\pi(x)$ and $\pi(Tx)$ are points on two smooth convex curves $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ in the plane. Further, we take it that these are the only curves in the plane and consider line elements $y \in S^1(\pi(x))$ which give lines intersecting $\tilde{\gamma}_2$. The set of such $y$ defines an arc $\gamma \subset S^1(\pi(x))$. One of the ends of this arc is a point $y_0$ such that the line defined by this point touches $\tilde{\gamma}_2$. It is easy to see that the length $\omega(\gamma)$ of $\gamma$ is not more than const $\frac{\cos^2\varphi(Tx)}{|\tau(x)|}$, where const depends only on the curvature of $\partial Q$ in a neighbourhood of $\pi(Tx)$, and therefore $d(x) \leqslant \omega(\gamma) \leqslant \text{const}\,\frac{\cos^2\varphi(Tx)}{|\tau(x)|}$.

By means of (4) we obtain

$$\frac{\partial \ln|\cos\varphi(x)|}{\partial r} = 0, \quad \left|\frac{\partial \ln|\cos\varphi(x)|}{\partial\varphi}\right| = |\operatorname{tg}\varphi(x)| \leqslant \frac{\text{const}}{d(x)}.$$

It is not difficult to check that

$$\frac{\partial \ln|\tau|}{\partial\varphi} = \frac{1}{|\tau|}\frac{\partial|\tau|}{\partial\varphi} = |\operatorname{tg}\varphi(Tx)|,$$

and therefore $\left|\frac{\partial \ln|\tau|}{\partial\varphi}\right| \leqslant \frac{\text{const}}{d(x)}$. From geometrical arguments it is easy to obtain that

$$\frac{\partial \ln|\tau|}{\partial r} = \frac{\sin\varphi(x)}{\tau(x)} + \frac{\cos\varphi(x) - k^{(0)}(\pi(x))\,\tau(x)}{\tau(x)}\cdot\operatorname{tg}\varphi(Tx),$$

where $k^{(0)}(q)$ is the curvature of $\partial Q$ at $q$. Since $|\tau(x)| \geqslant \tau_0 = \text{const}$, we have

$$\left|\frac{\partial \ln|\tau|}{\partial r}\right| \leqslant \frac{\text{const}}{d(x)}.$$

We turn to $|\cos\varphi(Tx)|$. From (7) and (6)

$$\left|\frac{\partial \ln|\cos\varphi(Tx)|}{\partial\varphi}\right| = \left|\operatorname{tg}\varphi(Tx)\cdot\frac{k^{(0)}(\pi(Tx))\,\tau(x)+\cos\varphi(Tx)}{\cos\varphi(Tx)}\right| \leqslant \frac{\text{const}}{d(x)}.$$

Finally, $\left|\frac{\partial \ln|\cos\varphi(Tx)|}{\partial r}\right| = \left|\operatorname{tg}\varphi(Tx)\cdot\frac{\partial\varphi(Tx)}{\partial r}\right|$. As is not difficult to see (see also §2),

$$\frac{\partial\varphi(Tx)}{\partial r} = k^{(0)}(\pi(x)) + k^{(0)}(\pi(Tx))\,\frac{\cos\varphi(x) - k^{(0)}(\pi(x))\,\tau(x)}{\cos\varphi(Tx)}.$$

Since $|\tau| \geqslant \tau_0 = \text{const}$, from (7) we have $\tau(x)\,(\cos\varphi(Tx))^{-1} \leqslant \text{const}\; d^{-1}(x)$ and

$$\left|\frac{\partial \ln|\cos\varphi(Tx)|}{\partial r}\right| \leqslant \frac{\text{const}}{d(x)}.$$

The assertion of the lemma follows from the estimates obtained.

## §2. Lemmas on reflections and the construction of the functions $\kappa^{(c)}(x)$, $\kappa^{(e)}(x)$

The scattering character of billiards is exhibited in this section.

Let $\tilde{\gamma} \subset Q$ be an arbitrary smooth curve without self-intersections, and $\gamma$ its equipment. We assume that $\tilde{\gamma}$ with the equipment $\gamma$ has strictly negative curvature. We assume further that $s$ is the arc length in $\tilde{\gamma}$ counted from some interior point. We may also take $s$ as parameter on $\gamma$. Let $k(s)$ denote the curvature of $\tilde{\gamma}$ at the point with the coordinate $s$. Then $k = k(s)$ is the natural equation of $\tilde{\gamma}$. Choose $t < 0$ so that in the time from $t$ to 0 no point $x \in \gamma$ reaches the boundary.

LEMMA 2.1. *The natural equation of* $\pi(\gamma_1) = \tilde{\gamma}_1$, $\gamma_1 = S_t\gamma$, *is*

$$k_1(s_1) = \frac{k(s)}{1+tk(s)},$$

*where* $s_1$ *is the arc length in* $\tilde{\gamma}_1$, *naturally consistent with* $s$ *on* $\tilde{\gamma}$, *and*

$$\frac{ds_1}{ds} = 1 + tk(s).$$

PROOF. In a neighbourhood of any $q' = q'(s) \in \tilde{\gamma}$ the elements $ds$ of $\tilde{\gamma}$ may be replaced, to within smallness of higher order by the elements of length of a circle of radius $|k(s)|^{-1}$. Up to the same accuracy the corresponding element $ds_1$ of $\tilde{\gamma}_1$ is the element of a circle of radius $|t| + |k(s)|^{-1}$ with the same centre (Fig. 4). Consequently,

$$ds_1 : (|t| + |k(s)|^{-1}) = ds : |k(s)|^{-1} \quad \text{or} \quad \frac{ds_1}{ds} = 1 + tk(s).$$

For the same reason it follows that the radius of curvature at $s_1$ is $|t| + |k(s)|^{-1}$ and, hence, the curvature is $\frac{k(s)}{1+tk(s)}$. The lemma is now proved.

We now assume that between $t$ and 0 each point of $\gamma$ has exactly one boundary reflection and that for all $x \in \gamma$ this reflection occurs in one regular component of the boundary. Hence it follows that at the time $t$ all $x \in \gamma$ have been reflected from the boundary and that $\gamma_1 = S_t\gamma$ is in the interior of $M$. For the point with the coordinate $s$ let $t(s)$, $\varphi(s)$, $k^{(0)}(s)$ denote the time of the boundary reflection, the angle of reflection, and the boundary curvature at the point of reflection, respectively.

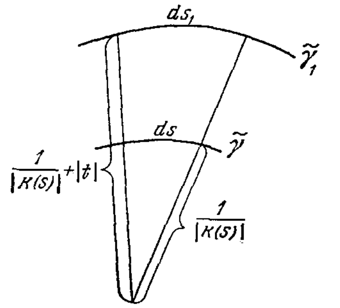



Fig. 4.

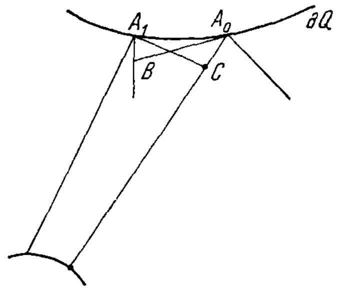



Fig. 5.

LEMMA 2.2. *The natural equation of* $\tilde{\gamma}_1 = \pi(S_t\gamma)$ *is*

$$k_1(s_1) = \cfrac{1}{t - t(s) + \cfrac{1}{\cfrac{k^{(0)}(s)}{\cos\varphi(s)} + \cfrac{1}{t(s) + \cfrac{1}{k(s)}}}},$$

*where*

$$\frac{ds_1}{ds} = 1 + tk(s) - \frac{k^{(0)}(s)}{\cos\varphi(s)}(1 + t(s)k(s)).$$

PROOF. Lemma 2.2 like Lemma 2.1 is essentially of local character. Therefore it is sufficient to prove it for $s = 0$. Again we take $\gamma$ as the equipment of negative curvature of an arc of a circle of radius $|k(0)|^{-1}$. Then by Lemma 2.1, at $t(0)$ we obtain from $ds$ an element of length $ds(1 + t(0)k(0))$ of a circle of radius $(|t(0)| + |k(0)|^{-1})$. For the subsequent argument it is convenient to regard $ds$ as one-sided lying in $\{s < 0\}$ and so that the boundary reflections of its points occur for $t(s) > t(0)$. Let (Fig. 5) $A_0$ be the point of reflection of the right end and $A_1$ the point of reflection of the left end of $ds$. If $dt$ is the difference of the reflection times of the ends, then $dt = ds' \tan\varphi$ where $\varphi$ is the angle of incidence and $ds' = A_1A_0$. At $t(0)$ the left end of the segment is at $B$. If $C$ is the position of the right end at $t(0) - dt$, then up to small terms of higher order $A_0B = A_1C = ds(1 + t(0)k(0))$. Thus, $S_t\gamma$ at $t = t(0) + 0$ is the equipment of $A_0B$, and our problem is to calculate its curvature.

From Fig. 5 it is clear that this curvature depends on the curvature $(t(0) + |k(0)|^{-1})^{-1}$ at the moment of reflection and on the quantity $\frac{k^{(0)}(0)}{\cos\varphi(0)}$, characterizing the curving of the boundary. Thus, at $t(0) + 0$ the curvature of the equipped segment $A_0B$ is

$$\frac{k^{(0)}(0)}{\cos\varphi(0)}+\frac{1}{t(0)+\frac{1}{k(0)}}.$$

The final formula follows on putting $t$ equal to $t-t(0)$ in Lemma 2.1. Our lemma is now proved.

COROLLARY 2.1. *If $\tilde{\gamma}=\pi(\gamma)$ has negative curvature, then so has $\pi(S_t\gamma)$.*

We now prove a lemma from which a similar assertion for $T$ follows. Having the curve $\gamma$ of Lemma 2.2 we consider for each $x(s)\in\gamma$ the least $\tilde{t}(s)>0$ such that $S_{\tilde{t}(s)+0}x(s)\in M_1$ and we assume that $\gamma_1=\{S_{t(s)+0}x(s)\}$ is a curve in $M_1$ belonging to one regular component. Let $k(s)\,t(s)>-1$. Further we assume that in the natural coordinate system $(r,\varphi)$ the curve $\gamma_1$ is given by a function $\varphi=\varphi(r)$. The condition $k(s)\,t(s)>-1$ together with Lemma 2.1 shows that $\frac{d\varphi}{dr}=\frac{k(s)\cos\varphi(s)}{1+k(s)\,t(s)}$, where $s$ and $r$ are connected by the relation $\pi(S_{t(s)}x(s))=r$. Thus, curves with negative curvature correspond to curves on the boundary for which $\frac{d\varphi}{dr}<0$. Similarly curves for which $\frac{d\varphi}{dr}>0$ arise from curves with positive curvature.

LEMMA 2.3. *Let $\gamma_1\in M_1$ be given by the equation $\varphi=\varphi(r)$ and $\frac{d\varphi}{dr}(r)\leqslant 0$. Let $T|\gamma_1$ be smooth. Then $\gamma_1'=T\gamma_1$ is a curve in $M_1$. If $\gamma_1'$ is given by a function $\varphi'=\varphi'(r')$, where $r$ and $r'$ are naturally consistent, then*

$$\frac{d\varphi'}{dr'}=-k^{(0)}(r')+\frac{\frac{d\varphi}{dr}\cos\varphi(r)\cdot\cos\varphi'(r')}{1+\tau(r)\cos\varphi(r)\frac{d\varphi}{dr}},\quad \frac{dr'}{dr}=\frac{\cos\varphi(r)}{\cos\varphi'(r')}\left(1-\frac{d\varphi}{dr}\,\frac{\tau(r)}{\cos\varphi(r)}\right).$$

The proof is easily obtained by a combination of Lemmas 2.1 and 2.2.

Curves $\varphi=\varphi(r)$ for which $\frac{d\varphi}{dr}\leqslant 0$, will be called decreasing and curves for which $\frac{d\varphi}{dr}\geqslant 0$, increasing. From the form of the domain $Q$ we deduce:

COROLLARY 2.2. *If $\gamma_1$ is decreasing and $T\,|\,\gamma_1$ is smooth, then $T\gamma_1$ is also decreasing. If $\gamma_2$ is increasing and $T^{-1}\,|\,\gamma_2$ is smooth, then $T^{-1}\gamma_2$ is increasing.*

We now construct two vector fields in $M$, which will play an important rôle later. These vector fields are everywhere discontinuous. Therefore the usual existence and uniqueness theorems for differential equations are not applicable. Nevertheless, as we shall see in the following section, through a typical point $x$ there pass segments of trajectories of these vector fields. Moreover, these trajectory segments form equipments $\gamma(x)$ of positive and negative curvature of curves $\tilde{\gamma}(x)$. In this case the vector at

$x$ of the vector field is uniquely determined by the curvature of $\tilde{\gamma}(x)$ at $\pi(x)$. With this in view we write out the value of this curvature.

Let $0 < t_1 < t_2 < \ldots$ be the sequence of boundary reflection times of $x$. The interval[1] between two successive reflections is $\tau_i = t_i - t_{i-1}$ ($i = 1, 2, \ldots, t_0 = 0$). We denote by $k_i^{(0)}(x)$ the curvature of the boundary at the $i$-th reflection point and by $\varphi_i(x)$ the angle of incidence for the $i$-th reflection $\frac{\pi}{2} \leqslant \varphi_i \leqslant \frac{3\pi}{2}$.

We assume that the required curvature is an infinite continued fraction

$$\varkappa^{(c)}(x) = \cfrac{1}{-\tau_1 + \cfrac{1}{\cfrac{k_1^{(0)}(x)}{\cos\varphi_1(x)} + \cfrac{1}{-\tau_2 + \cfrac{1}{\cfrac{k_2^{(0)}(x)}{\cos\varphi_2(x)} + \cfrac{1}{-\tau_3 + \ddots}}}}}. \tag{8}$$

All the coefficients of (8) are non-positive.

LEMMA 2.4. *The continued fraction* (8) *converges for each* $x$.

PROOF. Since our continued fraction has negative elements, by a theorem of Seidel and Stern (see [17] §5) a necessary and sufficient condition for convergence is that $\sum \left(-\tau_i + \frac{k_i^{(0)}(x)}{\cos\varphi_i(x)}\right) = -\infty$. By our conditions there are finitely many boundary reflections in any finite time interval from each point of a trajectory, and so the equality is satisfied. This proves the lemma.

LEMMA 2.5. *Let $x$ be such that the positive semitrajectory of $x$ is never tangent to $\partial Q$. Then $\varkappa^{(c)}(x)$ is continuous at $x$.*

PROOF. Since the continued fraction (8) converges, its value lies between even and odd convergents. Hence for any $\varepsilon > 0$ we can find $i = i(\varepsilon)$ such that

$$\varkappa^{(c)}(x) - \varkappa_i(x) \leqslant \varepsilon/3, \qquad \varkappa^{(c)}(x) - \varkappa_i'(x) \geqslant -\varepsilon/3,$$

where

$$\varkappa_i(x) = \cfrac{1}{-\tau_1 + \cfrac{1}{\cfrac{k_1^{(0)}(x)}{\cos\varphi_1(x)} + \cfrac{1}{- \ddots + \cfrac{1}{-\tau_i}}}},$$

$$\varkappa_i'(x) = \cfrac{1}{-\tau_1 + \cfrac{1}{\cfrac{k_1^{(0)}(x)}{\cos\varphi_1(x)} + \cfrac{1}{- \ddots + \cfrac{1}{-\tau_i + \cfrac{1}{\cfrac{k_i^{(0)}(x)}{\cos\varphi_i(x)}}}}}}.$$

[1] If $x \in S_-$, then we take $\tau_1 = 0$. If $x \in S_+$, then we take $\tau_1 > 0$.

We now choose a small neighbourhood $U$ of $x$ so that the change in $\tau_i(y)$, $\frac{k_i^{(0)}(y)}{\cos\varphi_i(y)}$ is so small that the $i$-th convergents $\kappa_i$, $\kappa_i'$ for $y \in U$ differ by not more than $\varepsilon/3$ from $\kappa_i(x)$, $\kappa_i'(x)$. This is possible because the trajectory of $x$ is not tangent to the boundary. Then for $y \in U$

$$|\varkappa^{(c)}(y)-\varkappa^{(c)}(x)| \leqslant |\varkappa^{(c)}(y)-\varkappa_i(y)|+|\varkappa_i(y)-\varkappa_i(x)|+|\varkappa_i(x)-\varkappa^{(c)}(x)| \leqslant \varepsilon.$$

The lemma is now proved.

If the trajectory of $x$ is tangent to the boundary at some time, then $\cos\varphi_i(x) = 0$ for the corresponding $i$. Therefore

$$\varkappa^{(c)}(x)=\cfrac{1}{-\tau_1+\cfrac{1}{\cfrac{k_1^{(0)}(x)}{\cos\varphi_1(x)}+\cfrac{1}{-\tau_2+\ddots+\cfrac{1}{-\tau_i}}}}\ .$$

It is not difficult to show that the function $\kappa^{(c)}(x)$ is discontinuous at such an $x$ and that the discontinuity is of the first kind if the semi-trajectory is only once tangent to the boundary.

Finally we write out our vector field $\mathfrak{A}^{(c)}$. Choose coordinates $q_1$, $q_2$, $\varphi$ in a neighbourhood of $x = (q, \varphi)$, where $\varphi$ is taken to be away from the $q_1$-axis. Then the components of the tangent vector have the form $(\delta q_1, \delta q_2, \delta\varphi)$, where $\delta q_1 = -\sin\varphi$, $\delta q_2 = \cos\varphi$, $\delta\varphi = \kappa^{(c)}(x)$. From Lemmas 2.1 and 2.2 it follows that $\mathfrak{A}^{(c)}$ is invariant under $\{S_t\}$ in the following sense: if $\{\hat{S}_t\}$ is the one-parameter group of transformations of the tangent space of $M$ induced by our flow, then the line $\mathfrak{A}(x)$, going in the direction of $\mathfrak{A}^{(c)}$ at $x$ is taken under the action of $\hat{S}_t$ to $\mathfrak{A}(S_t x)$.

Let us project our vector field onto $M_1$. If $x = (r, \varphi) \in M_1$, then we assume that $\frac{d\varphi}{dr} = -\kappa^{(c)}(x)\cos\varphi(x)$. The vector field $\mathfrak{A}_T^{(c)}$ so constructed is invariant under $T$ in the sense described above.

We construct a second vector field $\mathfrak{A}^{(e)}$ (the meaning of the indices $c$, $e$ will become clear later). Let $I$ be the involutory automorphism that consists in replacing $x \in M$ by the linear element with the same carrier and the opposite direction. Now put $\kappa^{(e)}(x) = -\kappa^{(c)}(Ix)$. It is easy to see that the vector field defined by means of $\kappa^{(e)}(x)$ is also invariant. Its projection onto $M_1$ has the form $\frac{d\varphi}{dr} = -\kappa^{(e)}(x)\cos\varphi(x)$.

In the first case $\frac{d\varphi}{dr} < 0$, in the second $\frac{d\varphi}{dr} > 0$.

## §3. The construction of local transversal fibres

Let $U$ be a neighbourhood of $x \in M$. By a locally contracting transversal fibre (l.c.t.f.) of $x$ in $U$ we mean a curve in $U$ consisting of points $y \in U$ such that $d(S_t x, S_t y) \to 0$ for $t \to \infty$. If we let $t \to -\infty$, we come to the definition of a locally expanding transversal fibre (l.e.t.f.). Such fibres play a key rôle in the study of $U$-systems (see [2], [3], [4]). We shall see that such fibres exist in the billiards under discussion. However, in contrast to $U$-systems, in our problems they are defined almost everywhere rather than everywhere, and each fibre has singularities of break point type. A typical l.c.t.f. in the phase space is shown in Fig. 6.

In this section we construct an l.c.t.f. for an individual point $x$. More precisely, we construct the smooth component of the l.c.t.f. containing $x$ and estimate its size. Our construction is a variant of the proof of the Hadamard-Perron theorem on manifolds (see [3]). The complication arises from the discontinuous character of $\{S_t\}$. In the following section we shall show that this construction is applicable to almost each point.

Fig. 6.

First we construct local transversal fibres for $T$ on $M_1$, and then we use them to construct the required fibres for $\{S_t\}$. The definition of transversal fibres for $T$ is the same as for $\{S_t\}$, except that we write $T^n$ instead of $S_t$ and take $n \to \pm\infty$.

It is sufficient to construct an l.c.t.f. Let $x_0 \in M$ be fixed and $x_i = T^{-i} x_0$. We assume that $x_i \notin M_2 (i = 0, 1, 2, \dots)$ and introduce the following quantities:

$$d_i = d(x_i), \quad \cos\varphi_i = \cos\varphi(x_i), \quad \tau_i = \tau(x_i), \quad \varkappa_i^{(c)} = \varkappa^{(c)}(x_i),$$

$$R_i = \frac{\cos\varphi_0}{\cos\varphi_i} \prod_{k=1}^{i} (1 + \varkappa_k^{(c)}\tau_k)^{-1} = \prod_{k=1}^{i} \frac{\cos\varphi_{k-1}(1+\varkappa_k^{(c)}\tau_k)^{-1}}{\cos\varphi_k},$$

$$l_i = R_i \sqrt{1 + (\varkappa_i^{(c)})^2 \cos^2\varphi_i}.$$

THEOREM 3.1. *Assume that for* $x_0$

$$\sum_{i=0}^{\infty} \frac{l_i}{d_i} = D < \infty, \qquad \sum_{i=0}^{\infty} \frac{il_i}{d_i} = D_1 < \infty. \tag{9}$$

*Then there exists a curve* $\gamma$ *of class* $C^1$, *passing through* $x_0$ *and belonging to the l.c.t.f. of* $x_0$, *given by an equation* $\varphi = \varphi(r)$, $r_1 \leqslant r \leqslant r_2$, *where an estimate for* $\Delta r = r_2 - r_1$ *is given below in formula* (14),

$$\frac{d\varphi}{dr} = -\varkappa^{(c)}(x(r, \varphi))\cos\varphi \quad \textit{and}$$

$$\exp(-\delta^{(0)}) \leqslant \frac{d\varphi}{dr} \;:\; \overline{\frac{d\varphi}{dr}}\Big|_{r=0} \leqslant \exp(\delta^{(0)}),$$

$\delta^{(0)}$ *being given below in formula* (15) (*on the right*)

PROOF. $\gamma$ is constructed by the method of successive approximations. Let $\gamma_i^{(i)}$ be a curve passing through $x_i$ and given by an equation[1] $\varphi = \varphi_{ii}(r)$ with the following properties:

---

[1] As always, we take it that coordinates $(r, \varphi)$ are chosen in a neighbourhood of $x_i$. Then $x_i$ has the coordinates $(0, \varphi_i)$.

(1) $\frac{d\varphi}{dr} = -\cos\varphi \cdot \varkappa_i^{(c)}$;

(2) the parameter $r$ varies in $\bar{r}_{ii} \leqslant r \leqslant \bar{\bar{r}}_{ii}$, where $\Delta r_{ii} = \bar{\bar{r}}_{ii} - \bar{r}_{ii} = \alpha_{ii} R_i$, and the value of $\alpha_{ii}$ will be indicated shortly. We now assume that $\alpha_{ii}$ is such that $T|\gamma_i^{(i)}$ is smooth and we assume inductively that for all $j(i \geqslant j > k)$ it has already been proved that $\gamma_i^{(j)} = T^{i-j}\gamma_i^{(i)}$ has the following properties: $\gamma_i^{(j)}$ is given by a function $\varphi = \varphi_{ij}(r)$ (in the coordinate system of $x_j$) and

a$_1$) $\exp(-\delta_i^{(j)}) \leqslant \frac{d\varphi_{ij}}{dr} : \left.\frac{d\varphi_{ij}}{dr}\right|_{r=0} \leqslant \exp(\delta_i^{(j)})$;

a$_2$) the parameter $r$ varies in $\bar{r}_{ij} \leqslant r \leqslant \bar{\bar{r}}_{ij}$, where $\Delta r_{ij} = \alpha_{ij} R_j$.

We find recurrence inequalities for $\alpha_{ik}$ and $\delta_i^{(k)}$. From a$_1$) and a$_2$) it follows that the length $l(\gamma_i^{(k+1)})$ of $\gamma_i^{(k+1)}$ satisfies the inequality

$$l(\gamma_i^{(k+1)}) = \int \sqrt{1 + \left(\frac{d\varphi}{dr}\right)^2}\, dr \leqslant \sqrt{1 + (\varkappa_{k+1}^{(c)})^2 \cos^2\varphi_{k+1}}\, \exp(\delta_i^{(k+1)}) \cdot \Delta r_{i,k+1}.$$

From (9) we have

$$l(\gamma_i^{(k+1)}) \leqslant D\alpha_{i\,(k+1)} \exp(\delta_i^{(k+1)}) \cdot d_{k+1}.$$

Therefore, if $D\alpha_{i(k+1)}\exp(\delta_i^{(k+1)}) \leqslant \frac{1}{2}$ or $\alpha_{i(k+1)}\exp(\delta_i^{(k+1)}) \leqslant \frac{1}{2D}$, then for all $y \in \gamma_i^{(k+1)}$

$$d(y) \geqslant d_{k+1} - l(\gamma_i^{(k+1)}) \geqslant \frac{d_{k+1}}{2}.$$

Consequently $T$ is smooth on $\gamma_i^{(k+1)}$, and by Lemma 1.1 the logarithmic derivatives of $\cos\varphi(y)$, $\cos\varphi(Ty)$, $\tau(y)$ do not exceed $2\,\mathrm{const}\, d_{k+1}^{-1}$. We use Lemma 2.3, according to which

$$\Delta r_{i,k} = \int_{\bar{r}_{i,k+1}}^{\bar{\bar{r}}_{i,k+1}} \frac{dr'}{dr} dr = \int_{\bar{r}_{i,k+1}}^{\bar{\bar{r}}_{i,k+1}} dr\, \frac{\cos\varphi(r)\left(1 + \frac{\tau(r)}{\cos\varphi(r)} \cdot \frac{d\varphi}{dr}\right)}{\cos\varphi(r')} \leqslant$$

$$\leqslant \frac{\cos\varphi_{k+1}}{\cos\varphi_k}(1 + \varkappa_{k+1}^{(c)}\tau_{k+1})\,\Delta r_{i,k+1} \cdot \exp\left[\frac{8\,\mathrm{const}}{d_{k+1}} R_{k+1}\alpha_{i\,(k+1)} + \delta_i^{(k+1)}\right].$$

Similarly we obtain an inequality for $\Delta r_{ik}$ in the other direction. Thus,

$$\alpha_{i\,(k+1)} \cdot \exp\left[-\alpha_{i\,(k+1)} \cdot \frac{8\,\mathrm{const}\, R_{k+1}}{d_{k+1}} - \delta_i^{(k+1)}\right] \leqslant \alpha_{ik} \leqslant$$
$$\leqslant \alpha_{i\,(k+1)} \exp\left[+\alpha_{i\,(k+1)} \frac{8\,\mathrm{const}\, R_{k+1}}{d_{k+1}} + \delta_i^{(k+1)}\right]. \quad (10)$$

From Lemma 2.3 it also follows that

$$\frac{d\varphi'}{dr'} = -k^{(0)}(r') + \frac{\frac{d\varphi}{dr}\cos\varphi(r)}{1-\tau(r)\frac{d\varphi}{dr}\cos\varphi(r)} = -k^{(0)}(r') + \frac{\cos\varphi(r)}{\tau(r) - \frac{1}{\frac{d\varphi}{dr}\cos\varphi(r)}}.$$

Therefore, if $L = \max\left|\frac{\partial \ln k^{(0)}(r')}{\partial r'}\right|$, then

$$\frac{d\varphi'}{dr'} : \left.\frac{d\varphi'}{dr'}\right|_{r'=0} \leqslant \exp[L\cdot R_k\alpha_{ik} + \delta_i^{(k+1)} + 6\,\mathrm{const}\,R_{k+1}\,d_{k+1}^{-1}\cdot\alpha_{i\,(k+1)}].$$

Similarly we obtain an inequality in the other direction. Thus,

$$\delta_i^{(k)} \leqslant \delta_i^{(k+1)} + LR_k\alpha_{ik} + 6\,\mathrm{const}\,R_{k+1}d_{k+1}^{-1}\alpha_{i\,(k+1)}. \tag{11}$$

We now "solve" these recurrence inequalities. Let $A$ be a solution of

$$A\exp[A(L+3\,\mathrm{const}\,D)] = (2D)^{-1}.$$

Put $u_k = R_kd_k^{-1}$, $U_k = \sum_{s=k}^{\infty} u_s$, $V_k = \sum_{s=k}^{\infty} U_s$. Then under the assumption that the $\alpha_{is}$ $(s = i, \ldots, k+1)$ do not exceed $A$ we obtain from (11) and the fact that $d_i \leqslant 1$,

$$\delta_i^{(k)} \leqslant \delta_i^{(k+1)} + ALu_k + 6\,\mathrm{const}\,Au_{k+1}.$$

Consequently, for those $k$ for which $\alpha_{is} \leqslant A(s = i, \ldots, k+1)$

$$\delta_i^{(k)} \leqslant ALU_k + 6\,\mathrm{const}\,U_{k+1}. \tag{12}$$

Write $\alpha_{ii} = A_0$. If $\alpha_{is}$ does not exceed $A$ for the same $s$, then from (10) and (12) we have

$$\alpha_{ik} \leqslant \alpha_{i\,(k+1)}\exp[8\,\mathrm{const}\,Au_{k+1} + LAU_{k+1} + 6\,\mathrm{const}\,AU_{k+2}]$$

or

$$\alpha_{ik} \leqslant A_0\exp[8\,\mathrm{const}\,AU_{k+1} + ALV_{k+1} + 6\,\mathrm{const}\,AU_{k+2}]. \tag{13}$$

Let $A_0$ be a solution of the equation

$$A = A_0\exp[14\,\mathrm{const}\,AD + AL\cdot D_1]$$

or

$$A_0 = A\exp[-A(14\,\mathrm{const}\,D + L\cdot D_1)].$$

For such an $A_0$ it is not difficult to see that $\alpha_{ik} \leqslant A$, and consequently we may continue our estimates one step further. Thus, $\alpha_{ik} \leqslant A$ for all $k = i, i-1, \ldots, 1$ and so (12) and (13) are valid. On the other hand,

$$\Delta r\cdot R_0^{-1} \geqslant \alpha_{i0} \geqslant A_0\exp[-A(14\,\mathrm{const}\,D + LD_1)], \tag{14}$$

which gives a lower estimate for the variation of $r$ along the curves $\gamma_i^{(0)}$.

For $\delta_i^{(0)}$ we have the estimate

$$\delta_i^{(0)} \leqslant (AL + 6\,\mathrm{const}\,A)\,D. \tag{15}$$

We note further that by Lemma 2.3 for all $i$ the derivative $\left.\frac{d\varphi_{i0}}{dr}\right|_{r=0} = -\varkappa^{(c)}(x_0)\cos\varphi_0$ is independent of $i$.

Now we show that for $i \to \infty$ the $\gamma_i^{(0)}$ converge to a curve $\gamma^{(0)}$, which is the required l.c.t.f.

As before, we assume the coordinates $(r, \varphi)$ fixed in a neighbourhood of $x_0$, where $x_0$ corresponds to $(0, \varphi_0)$. Let $\Delta = \frac{1}{2}\delta\exp(-16\,\mathrm{const}\cdot D\delta)$, where $\delta$ is a parameter we shall vary later, taking it to be sufficiently small, and let $\mathfrak{B}$ be the space of continuous functions $\varphi = \varphi(r)$, $|r| \leqslant \Delta$ with the uniform metric $\|\cdot\|$.

We estimate $\|\varphi_{i0} - \varphi_{(i+1)0}\|$. Let $r$, $|r| \leqslant \Delta$, be fixed. In the $(r, \varphi)$-plane choose a vertical segment $\nu_i(r)$ joining $\varphi_{i0}(r)$ to $\varphi_{(i+1)0}(r)$. We may regard $\nu_i(r)$ as a curve in $M$. Its length $l(\nu_i(r))$ is equal to $|\varphi_{i0}(r) - \varphi_{(i+1)0}(r)| \leqslant \|\varphi_{i0} - \varphi_{(i+1)0}\| = \max\limits_r l(\nu_i(r))$. From our estimates it easily follows that $T^{-1}, T^{-2}, \ldots, T^{-i}$ are smooth on $\nu_i(r)$. By Lemma 2.3, $T^{-i}\nu_i(r)$ is a curve, given by $\varphi = \varphi(r)$ in the coordinates around $x_i$, where $\frac{d\varphi}{dr} \geqslant 0$, and this curve joins a point of $\gamma_i^{(i)}$ to a point of $\gamma_{i+1}^{(i)}$. It is easy to see that the length of $T^{-i}\nu_i(r)$ does not exceed $\Delta r_{ii} + \Delta\varphi$, where $\Delta\varphi$ is the max of the increments $\Delta\varphi$ for $\gamma_{i+1}^{(i)}$ and $\gamma_i^{(i)}$. From the proven properties of the $\gamma_i^{(j)}$ it easily follows that this maximum does not exceed const $l_i$, where const depends on $\delta$ but not on $i$. Consequently $l(T^{-i}\nu_i(r))$ does not exceed const $l_i$. Now note that by the same Lemma 2.3 and the inequalities (10) and (12)

$$l(\nu_i(r)) \leqslant \mathrm{const}\,\frac{\cos\varphi_0}{\cos\varphi_i}\,l(T^{-i}\nu_i(r)) \leqslant \mathrm{const}\,l_i\cdot\frac{\cos\varphi_0}{\cos\varphi_i} \leqslant \mathrm{const}\,\frac{l_i}{d_i}\,.$$

We do not write out the explicit form of the constants, because they are not essential for us. It is important that on the right we have a number independent of $r$, which is a term of a convergent series, that is, $\|\varphi_{i0} - \varphi_{(i+1)0}\| \leqslant$ const $l_i d_i^{-1}$, and hence $\varphi_{i0}$ is a fundamental sequence. We put $\varphi_0(r) = \lim\limits_{i\to\infty}\varphi_{i0}(r)$.

We now clarify properties of $\varphi_0(r)$. Since for all $i$ the function $\varphi_{i0}(r)$ is monotone: $\frac{d\varphi_{i0}(r)}{dr} < 0$, the function $\varphi_0(r)$ is also monotone. Next, from our estimates it is easy to deduce that $\varphi_0(r)$ is differentiable at $r = 0$ and that its derivative is $-\kappa^{(c)}(x_0)\cos\varphi_0$. For if we produce estimates in an interval of length $\Delta_1 < \Delta$, then we see that $\exp(-\mathrm{const}\,\Delta_1) \leqslant \frac{d\varphi_{i0}}{dr} : \left.\frac{d\varphi_{i0}}{dr}\right|_{r=0} \leqslant \exp(\mathrm{const}\,\Delta_1)$ for all $i$, and the assertion follows easily.

We now show that the points of $\gamma^{(0)} = \lim_{i\to\infty} \gamma_i^{(0)}$ given by $\varphi = \varphi_0(r)$ belong to the l.c.t.f. of $x_0$. For since $\gamma^{(0)} = \lim_{i\to\infty} \gamma_i^{(0)}$, we have $T^{-k}\gamma^{(0)} = \lim_{i\to\infty} \gamma_i^{(k)}$. But from the estimates for the values in $a_1$) and $a_2$) it follows that $\operatorname{diam}\gamma_i^{(k)} \leqslant l(\gamma_i^{(k)}) \leqslant$ const $l_k \to 0$ as $k\to\infty$, by the conditions of the theorem. From this we deduce that $\gamma^{(0)}$ is differentiable at all points. We note that for $y \in \gamma^{(0)}$ we have $d(T^{-i}y) > 0$, by construction. From the expressions for $\kappa^{(c)}(T^{-i}y)$ and $\kappa^{(c)}(T^{-i}x_0)$ it follows that $\frac{\varkappa^{(c)}(T^{-i}y)}{\varkappa^{(c)}(T^{-i}x_0)} \to 1$ as $i\to\infty$ and also (9) for $x_0$ implies (9) for $y$, possibly with a different constant $D'$. The above arguments for the estimation of the norm allow us to show that the limit curve for $y$ coincides with $\gamma^{(0)}$ in the corresponding neighbourhood of $y$. But we have already shown that such a limit must be differentiable at $y$ and has the derivative $-\kappa^{(c)}(y)\cos\varphi(y)$. It can be checked directly that $-\kappa^{(c)}\cos\varphi(y)$ is a continuous function on $\gamma^{(0)}$. Hence Theorem 3.1 is proved.

Using this theorem we now construct an l.c.t.f. for $x \in M$. Let $x$ be such that for $x_0 = S_{\hat{t}-0}x$, where $\hat{t} = \inf\{t\colon\ t > 0,\ S_t x$ lies strictly inside $M\}$, the conditions of Theorem 3.1 are satisfied, and let $\gamma^{(0)}(x_0)$ be the l.c.t.f. constructed in that theorem. We construct a curve $\tilde{l}$ in a neighbourhood of $\pi(x)$ such that if $l$ is its equipment of negative curvature, then the image of $l$ arising by shifting each point $y \in l$ by $\hat{t}(y)$ up to its first moment of reaching the boundary, coincides with $\gamma^{(0)}(x_0)$ in some neighbourhood of $x_0$. In other words, if we take a pencil of trajectories meeting $\gamma^{(0)}(x_0)$, then $l$ is an equipment of an orthogonal trajectory to this pencil. We have to show that $l$ is in fact part of an l.c.t.f. of $x$. This follows easily from the fact that if $-\sum_{k=0}^{i}\tau_k(x_0) < t < -\sum_{k=0}^{i+1}\tau_k(x_0)$, then $S_t l$ is the equipment of negative curvature of an orthogonal trajectory to the pencil of trajectories meeting $T^{-i}\gamma^{(0)}(x_0)$, and its length is not greater than the length of $T^{-i}\gamma^{(0)}(x_0)$, which as we have proved tends to zero. Consequently $l$ is a part of an l.c.t.f. of $x$.

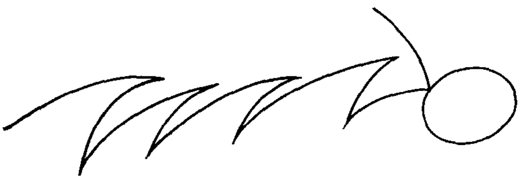


Fig. 7.

We now construct the complete contracting fibre. Again we start with the derived transformation $T$. For the points $x_i = T^{-i}x_0 (i = 0, 1, \ldots)$ we construct an l.c.t.f. $\gamma^{(i)}$ containing $x_i$, as above. Put $\Gamma^{(c)}(x_0) = \bigcup_{i=0}^{\infty} T^i\gamma^{(i)}$.

Similarly in the continuous time case we introduce $\gamma^{(t)}$, an l.c.t.f. for $S_t x$, $(t \geqslant 0)$ and put

$$\Gamma^{(c)}(x)=\bigcup_{t\geqslant 0} S_{-t}\gamma^{(t)}.$$

In both cases we call $\Gamma^{(c)}(x)$ the complete contracting fibre of $x$.

We now investigate the form of $\Gamma^{(c)}(x)$ for $\{S_t\}$. We shall see that $\Gamma^{(c)}(x)$ is the equipment of negative curvature of a curve $\tilde{\Gamma}^{(c)}(x)$ consisting of infinitely many components and singularities of cusp type at the ends of the components. The form of $\tilde{\Gamma}^{(c)}(x)$ for a typical $x$ is given in Fig. 7. We shall see that the singularities on $\tilde{\Gamma}^{(c)}$ appear because of trajectories tangent to the boundary.

Thus, let $\tilde{\gamma}\in Q$ be given with equipment of negative curvature $\gamma$ and $x_0\in\gamma$ so that the first reflection from $\partial Q$, at time $t_0<0$ of the trajectory defined by $x_0$ occurs with incidence angle $\pi/2$, that is, the trajectory is tangent to the boundary. Let $\gamma$ and $\tilde{\gamma}$ be small and let $t$ be such that in the time between 0 and $t<0$ there is not more than one boundary reflection for the trajectory of $x\in\gamma$. Then on one side of $x_0$ there lie trajectories having one boundary reflection in the time from $t$ to 0 and on the other side trajectories having no boundary reflections in the time from $t$ to 0 (in Fig. 8, on the left- and right-hand sides of $x_0$, respectively). Lemma 2.1 is applicable to the second part $\gamma_2$ and we conclude that $S_t\gamma_2$ is an equipment of the convex curve $\pi(S_t\gamma_2)$ whose curvature can be found by the formula of Lemma 2.1.

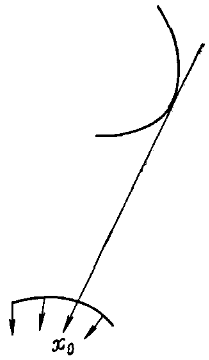



Fig. 8.

We examine the first part $\gamma_1$. Lemma 2.2 applies to $\gamma_1$ except at $x_0$ and we obtain that $S_t\gamma_1$, without $x_0$, is an equipment of the convex curve $\pi(S_t\gamma_1)$. We look at the structure of $S_t\gamma_1$ near $S_tx_0$. We may take $t=t_0$. Let $s$ be the parameter in $\tilde{\gamma}_1=\pi(\gamma_1)$ counted from $q_0=\pi(x_0)$, and $x(s)\in\gamma$ the point with the coordinate $s$. Denote by $k(s)$ the curvature of $\tilde{\gamma}_1$ and by $t(s)$ the time of reflection, $t(s)>t_0$.

We obtain immediately (Fig. 9) that $t(s)=t_0-\sqrt{(1+|t_0k_0|)s+o(s)}$ for $s\to 0$, where $k_0$ is the curvature of $\tilde{\gamma}_1$ at $q_0$. Also the angle of incidence is $\varphi(s)=\frac{\pi}{2}-k^{(0)}\sqrt{(1+|t_0k_0|)s+o(s)}$, where $k^{(0)}$ is the curvature of $\partial Q$ at $\pi(S_{t_0}x_0)$. Hence it follows easily that the carrier $\pi(S_{t_0}x(s))$ is at a distance $k^{(0)}(1+|t_0k_0|)s+o(s)$ from $\pi(S_{t_0}x_0)$, where the increase of angle is $d\varphi=k^{(0)}\sqrt{(1+|t_0k_0|)s+o(s)}$. Hence we conclude that $\pi(S_{t_0}\gamma_1)$ has infinite curvature at $\pi(S_{t_0}x_0)$.

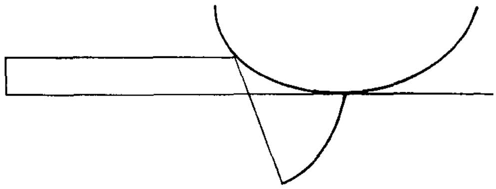


Fig. 9.

It is not difficult to see that the curvature at $\pi(S_{t_0}x(s))$ is asymptotic to $\frac{\text{const}}{\sqrt{s}}$. In particular, it follows that $S_t\gamma_1$ is a curve of finite length in $M$. Thus, for $t=t_0$ the part of $\pi(S_{t_0}\gamma)$ that is being reflected has infinite curvature at $\pi(S_{t_0}x_0)$. Consequently $\pi(S_t Y_1)$ has the curvature $\frac{1}{t-t_0}$ at $\pi(S_{t_0}x_0)$.

At the points of intersection with $\partial Q$ the fibre $\pi(\Gamma^{(c)}(x))$ has singularities of break type.

We denote by $\Pi^+$ the set of points $x$ whose trajectories are tangent to the boundary at least once for $t \geqslant 0$, and similarly by $\Pi^-$ the set of points $x$ whose trajectories are tangent to the boundary for at least one $t < 0$. We note that the cusps of $\Gamma^{(c)}(x)$ lie in the intersection $\Gamma^{(c)}(x) \cap \Pi^-$.

Replacing $t$ by $-t$ we may construct the complete expanding fibre $\Gamma^{(e)}(x)$ of $x$, which is an equipment of positive curvature of a curve with infinitely many regular components. The singular points of $\Gamma^{(e)}(x)$ are the points of the intersection $\Gamma^{(e)}(x) \cap \Pi^+$.

From the construction it easily follows that $\Gamma^{(c)}(S_t x) = S_t\Gamma^{(c)}(x)$, $\Gamma^{(e)}(S_t x) = S_t\Gamma^{(e)}(x)$.

## §4. Transversal foliations. Investigation of ergodic properties

We show first that the conditions of Theorem 3.1 are satisfied for almost all $x \in M_1$. The quantities we meet later are determined before the statement of Theorem 3.1.

THEOREM 4.1. *For almost all $x \in M_1$ there exist positive numbers $C(x) < \infty$, $\alpha(x) < \infty$, $\lambda(x) < 1$ such that for all $m > 0$*

$$d_m \geqslant \frac{C(x)}{m^{\alpha(x)}}, \quad |\cos\varphi_m| \geqslant \frac{C(x)}{m^{\alpha(x)}}, \quad \prod_{k=1}^{m} (1+\tau_k(x)\varkappa_k^{(c)}(x))^{-1} \leqslant \lambda^m(x).$$

PROOF. The last inequality is obvious, because the functions $|\tau_k(x)|$, $|\kappa_k^{(c)}(x)|$ are positive for almost all $x$ and by the Birkhoff-Khinchin ergodic theorem. The second inequality follows from the first by Lemma 1.1. Therefore we need only prove the first inequality.

It is easy to see that the first inequality follows from the following two propositions: for $i \geqslant i_0(x)$ and some constant $\alpha_1$

$$|\cos\varphi_i| \geqslant \frac{1}{i^{\alpha_1}}, \quad |\tau_i(x)| \leqslant i^{\alpha_1}.$$

All these are proved in a standard way by means of the Borel-Cantelli lemma (see [16]). We estimate the measure of the set of points for which the corresponding inequality is not satisfied for a given $i$ and check that the series of these measures converges. Then by the Borel-Cantelli lemma these inequalities are satisfied for almost every $x$ beginning with $i = i_0(x)$. The number $\alpha_1$ can be chosen independent of $x$.

It follows from (2) and (3) that on each regular component $\partial M_i$ the invariant measure $d\nu$ has the form $d\nu = \text{const}\,|\cos\varphi|\,dr\,d\varphi$ in the variables $(r, \varphi)$. Hence

$$\nu\left\{x\colon |\cos\varphi_i| < \frac{1}{i^{\alpha_1}}\right\} \leqslant \frac{\text{const}}{i^{2\alpha_1}}.$$

To obtain the corresponding inequalities for $\tau(x)$ we recall (see §1) that $\tau(x)$ is unbounded in a neighbourhood of finitely many points $\hat{x}_1, \ldots, \hat{x}_I$. From the analysis of §1 it follows that in a neighbourhood of $\hat{x}_s$

$$|\tau(x)| \leqslant \frac{\text{const}}{d(x, \hat{x}_s)} \qquad (s = 1, \ldots, I),$$

where $d(x, \hat{x}_s)$ is the distance between $x$ and $\hat{x}_s$. Therefore

$$\nu\{x: |\tau(x)| > i^{\alpha_1}\} = \nu\left\{x: d(x, \hat{x}_s) \leqslant \frac{\text{const}}{i^{\alpha_1}} \quad \text{for some } s \right\} \leqslant \frac{\text{const}}{i^{2\alpha_1}}.$$

Choosing $\alpha_1$ so that the series on the right-hand side of the inequalities converges we obtain the assertion of the theorem, which is now proved.

Thus, complete contracting and expanding transversal fibres $\Gamma^{(c)}(x_0)$ and $\Gamma^{(e)}(x_0)$ have been constructed for almost all $x_0 \in M_1$ and hence for almost all $x \in M$. We could now use the theory of measurable foliations in [11], for which we would have to construct local bases as introduced there. But here we prefer a more "local" and immediate form of argument.

By Fubini's theorem for almost all $q \in Q$ the set of $x \in S^1(q)$ for which a contracting fibre $\gamma^{(c)} = \gamma^{(c)}(x)$ can be constructed is a subset $N(q)$ of full measure in $S^1(q)$. Fixing a $q_0$ with this property we consider $x \in N(q)$ such that $\varepsilon_0(x) \geqslant \varepsilon_0$, where $\varepsilon_0(x)$ is the distance along $\Gamma^{(c)}(x)$ from $x$ to the nearest singular point of the fibre if $\Gamma^{(c)}(x)$ is defined, and $\varepsilon_0(x) = 0$ otherwise. If $\varepsilon_0$ is sufficiently small, then the set $N_1(q)$ of such $x$ has positive measure. Let $\widetilde{U}_{\varepsilon_0}(q_0)$ denote the $\varepsilon_0$-neighbourhood of $q_0$ and now take for $x \in N_1(q)$ the connected component in $\gamma^{(c)}(x)$ of the intersection $\gamma^{(c)}(x) \cap \pi^{-1}(\widetilde{U}_{\varepsilon_0}(q_0))$, containing the point $x \in N_1(q_0)$.

Put $A = \bigcup_{x \in N_1} \bigcup_{|t| < \tau} S_t \gamma^{(c)}(x)$ for $\tau < \min\left(\varepsilon_0, \frac{1}{4} d(q_0, \partial Q)\right)$. This set $A$ admits a partition $\xi_c(A)$ whose elements $C_{\xi_c}$ have the form $S_t\, \gamma^{(c)}(x)$ for $x \in N_1(q_0)$, $|t| < \tau$. We show that

1) $A$ has positive measure;

2) the partition $\xi_c(A)$ is measurable and in almost all $C_{\xi_c}$ the conditional measure is equivalent to the length; more precisely, if $s$ is the arc length in $C_{\xi_c} = S_t \gamma^{(c)}(x)$, then there exists a positive measurable function $\rho(s \mid C_{\xi_c})$ such that for any measurable set $V$

$$\mu(V \mid C_{\xi_c}) = \int_{V \cap C_{\xi_c}} \rho(s \mid C_{\xi_c})\, ds.$$

Taking 1) and 2) as proved we obtain as a corollary: consider for fixed $x$ the l.c.t.f. $\gamma^{(c)}(x)$ and the set of $y \in \gamma^{(c)}(x)$ for which an expanding local fibre $\gamma^{(e)}(y)$ can be constructed: then for almost all $x \in N_1(q)$ the set of such $y$ has full measure in $\gamma^{(c)}(x)$ (in the measure generated by the arc length on $\gamma^{(c)}(x)$).

Now put for $x \in N_1(q_0)$ and $y \in \gamma^{(c)}(x)$

$$B(x) = \bigcup_{y} \bigcup_{|t| < \tau} S_t \gamma^{(e)}(y),$$

where $\tau$ is so small that $B(x) \cap \partial M = \varnothing$. As before, we assume that $\pi(\gamma^{(e)}(y))$ lies in a fixed neighbourhood of $\pi(x)$ by considering the restriction $\gamma^{(e)}(y)$ whose projection belongs to this neighbourhood. We show that

1) for almost all $x$ the set $B(x)$ has positive measure;

2) consider the mapping $\Psi$: $B(x) \to \gamma^{(c)}(x)$ sending a point $z \in S_t\gamma^{(e)}(y)$ to the point $y$; this mapping induces a measure $\tilde{\mu}$ in $\gamma^{(c)}(x)$, which is the inverse image of the natural measure in $B(x)$; then $\tilde{\mu}$ is absolutely continuous with respect to the arc length measure.

We prove properties 1) and 2) of $A$ and $B(x)$ simultaneously.[1] The proof is almost verbatim a repeat of the proof of absolute continuity of transversal foliations of $U$-systems which was carried out in detail in [4]. Therefore we omit some of the details here. It is convenient to split the account into separate points.

1°. Having $A$ we consider for each $q \in \pi(A)$ the partial mapping $\Psi_q$: $S^1(q_0) \to S^1(q)$ sending $x \in S^1(q_0)$ to the point $z \in \bigcup\limits_{|t|<\tau} S_t\gamma^{(c)}(x)$ such that $\pi(z) = q$. This mapping is obviously defined on $N_1(q_0)$ if $q \in U_{\varepsilon_0/2}(q_0)$. A little later we shall show that this mapping is absolutely continuous in the following sense: if $\Psi_q$ is defined on a subset $\Phi_q \subset S_1(q_0)$ of positive measure, then $\Psi_q(\Phi_q)$ is a subset of positive measure in $S^1(q)$, and the natural measure in $\Phi_q$ carries over ($\Psi_q$ being one-to-one on its domain of definition) to a measure equivalent to the natural measure on $S^1(q)$. We denote the density of this measure relative to the natural measure on $S^1(q)$ by $\rho(q, x)$, where $x \in S^1(q_0)$. For fixed $q$ the function $\rho(q, x)$ is defined on a subset of $S^1(q)$, and for fixed $x$ is defined on $\pi(\bigcup\limits_{|t|<\tau} S_t\gamma^{(c)}(x))$ or, what is the same thing, on $\bigcup\limits_{|t|<\tau} S_t\gamma^{(c)}(x)$, which we call a local contracting leaf. We now have

$$\mu(A) = \int\limits_Q dq \int\limits_{\Psi_q(\Phi_q)} d\omega(y).$$

Since $\Phi_q \supset N_1(q_0)$ for $q \in \widetilde{U}_{\frac{\varepsilon_0}{2}}(q_0)$, we have $\int\limits_{\Psi_q(\Phi_q)} d\omega > 0$ for such $q$ and hence $\mu(A) > 0$. Also, for any measurable $V$ by Fubini's theorem

$$\mu(A \cap V) = \int\limits_Q dq \int\limits_{\Psi_q(\Phi_q)\cap V} d\omega(y) =$$

$$= \int\limits_Q dq \int\limits_{S^1(q_0)\cap\Psi_q^{-1}(V)} \rho(q, x)\, d\omega(x) = \int\limits_{S^1(q_0)} d\omega(x) \int dq \rho(q, x). \tag{16}$$

The inner integral, whose domain of integration is not explicitly written, is taken for fixed $x \in S^1(q_0)$, that is, over the intersection of $V$ with the local contracting leaf $\bigcup\limits_{|t|<\tau} S_t\gamma^{(c)}(x)$. Since $\bigcup\limits_{|t|<\tau} S_t\gamma^{(c)}(x)$ is a smooth local section of the unit tangent bundle, $q$ may be taken as a smooth

[1] The union of properties 1) and 2) is called the absolute continuity of the foliations $\{\Gamma^{(c)}\}$ and $\{\Gamma^{(e)}\}$ (see [3], [4], [11]).

parameter in it and from (16) we see that in $\bigcup_{|t|<\tau} S_t\gamma^{(c)}(x)$ the conditional measure is given by the density $\rho(q, x)$ with respect to $dq$. By the invariance of the measure the conditional measures are connected by the equality

$$\mu\{S_{t_1}V \mid S_{t_1}\gamma^{(c)}(x)\} = \mu\{S_{t_2}V \mid S_{t_2}\gamma^{(c)}(x)\}$$

for any $V$ and any $t_1$, $t_2$, $|t_i| < \tau$ $(i = 1, 2)$. And so the second property of $A$ is proved.

$2^{\circ}$. A similar analysis can be made for $B(x)$. Together with the mapping $\Psi_q$ we consider the partial mapping $\overline{\Psi}_q$: $\gamma^{(c)}(x) \to S^1(q)$ defined in the following way: for $y \in \gamma^{(c)}(x)$ its image is $z \in \bigcup_{|t|<\tau} (S_t\gamma^{(p)}(y) \cap S^1(q))$. It will be proved later that the mapping $\overline{\Psi}_q$ is also absolutely continuous, that is, it transfers the arc length measure in $\gamma^{(c)}(x)$ to a measure equivalent to the natural measure on $S^1(q) \cap \overline{\Psi}_q(\gamma^{(c)}(x))$. Let us show that the required properties of $B(x)$ follow from this.

Let $\overline{\varepsilon}_0(y)$ be the distance of $\pi(y)$ from the ends of $\pi(\gamma^{(e)}(y))$. For almost all $x \in M$ take a constant $\overline{\varepsilon}_0 > 0$ such that for any sufficiently small neighbourhood $G \subset \gamma^{(c)}(x)$ of $x$ the measure (length) of those $y \in G$ for which $\overline{\varepsilon}_0(y) \geqslant \overline{\varepsilon}_0$ is positive. Hence, for the neighbourhood $\widetilde{U}_{\frac{\overline{\varepsilon}_0}{2}}(q_0)$, $q_0 = \pi(x)$, it easily follows that $\omega(\overline{\Psi}_q(\gamma^{(c)}(x))) > 0$ for every $q \in \widetilde{U}_{\frac{\overline{\varepsilon}_0}{2}}(q_0)$. Consequently,

$$\mu(B(x)) \geqslant \int\limits_{\widetilde{U}_{\frac{\overline{\varepsilon}_0}{2}}(q_0)} dq \int\limits_{\overline{\Psi}_q(\gamma^{(c)}(x))} d\omega > 0.$$

Property 2 of $B(x)$ is proved in the same way as property 2 of $A$.

$3^{\circ}$. In this point we consider the problem of absolute continuity of the mappings $\Psi_q$, $\overline{\Psi}_q$. As already explained, the proof of these assertions almost entirely repeats the proof of absolute continuity of transversal foliations of $U$-systems given in [4], Lecture 5. We consider the absolute continuity of $\Phi_q$. From heuristic arguments we write out the derivative of one measure with respect to the other.

Let $y_1 \in S^1(q_0)$ and $y_2 = \Psi_q(y_1) \in S^1(q)$. It follows from the definition of $\Psi_q$ that $y_1$ and $y_2$ lie in one locally contracting leaf. As follows from the preceding section, $S_t(S^1(q))$ for almost all $y_1$ and large $t > 0$ in a neighbourhood of $S_t y_1$ is near to a local expanding fibre of $S_t y_1$. We choose a small neighbourhood $\Delta$ of $S_t y_1$ in $S_t(S^1(q_0))$ and its inverse image $\Delta_t = S_{-t}\Delta \in S^1(q_0)$. Using the lemmas of §2 we may write down the ratio of their lengths

$$\frac{l(\Delta)}{l(\Delta_t)} \approx \rho_0 \prod_{i=1}^{k} (1 + |\tau_i| r_i)\, \rho_t,$$

where $\rho_0 = \tau_0$ is the moment of the first boundary reflection of the trajectory of $y_1$, where $k$ is the number of boundary reflections from 0 to $t$, the numbers $r_i$ are connected by the recurrence relation

$$r_i = \frac{k^{(0)}(q_i)}{|\cos\varphi_i|} + \frac{1}{-\tau_i + \frac{1}{r_{i-1}}},$$

$q_i$ is the point of $i$-th reflection in $\partial Q$, $\varphi$ the angle of incidence at this reflection $\left(\frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}\right)$, $|\tau_i| = -\tau_i$ the time interval between the $(i-1)$-th and the $i$-th reflections

$$\rho_t = (1 + (t - t_k) r_k), \qquad t_k = \sum_{i=0}^{k} |\tau_i|.$$

By construction it follows that for small $\Delta$ we may represent each $z \in \Delta$ in the form

$$z = S_{t(z)} w,$$

where $w \in S^1(q)$ and $t(z)$ differs slightly from $t$. The collection of points $w$ forms a neighbourhood $\Delta_t'$ of $y_2$, and we may write similarly

$$\frac{l(\Delta)}{l(\Delta_t')} \approx \rho_0' \prod_{i=1}^{k} (1 + |\tau_i'| r_i') \rho_t',$$

where $\rho_0'$ is the moment of the first boundary reflection of $y_2$, the $r_i'$ are connected by the same recurrence relation as the $r_i$ except that we have the values corresponding to the trajectory of $y_2$, and similarly the $\rho_t'$. We note that $k$ is the same in both cases, and therefore we may write

$$\frac{l(\Delta_t)}{l(\Delta_t')} \approx \frac{\rho_0'}{\rho_0} \prod_{i=1}^{k} \frac{(1 + |\tau_i'| r_i')}{(1 + |\tau_i| r_i)} \cdot \frac{\rho_t'}{\rho_t}.$$

Since the trajectories of $y_1$ and $y_2$ converge exponentially as $t \to \infty$, the ratio of the corresponding factors converges exponentially to 1 and so the right-hand side has a limit as $t \to \infty$. This limit is the required density (the function $f(x)$ in the notation of [4]).

To prove that the function so constructed is in fact the density for almost all $y$, we have to construct the areas $\Delta_t$ and $\Delta_t'$ "compatibly", that is, as elements of some sequence of increasing partitions. The technique for such a construction was described in detail in [4], and so we omit it here. And so the absolute continuity of the mapping $\Psi_q$ is proved.

The absolute continuity of $\overline{\Psi}_q$ is proved analogously. We note that absolute continuity can also easily be deduced from the arguments of §§5, 6, 7.

Thus, the properties 1) and 2) of $A$ and $B$ are established.

We now derive the following important proposition.

PROPOSITION 1. *With the exception of an invariant set of measure zero, the ergodic components of the system $\{S_t\}$ have positive measure.*

For the proof it is sufficient to establish for almost all $x$ that $B(x)$

is contained in the ergodic component of $x$. The arguments that follow are a generalisation of a theorem of Hopf (see [3], [4], [11], [18]).

Let $h$ be a continuous function, $\bar{h}_-(x)=\lim_{t\to\infty}\frac{1}{t}\int_0^t h(S_{-\tau}x)\,d\tau$, $\bar{h}_+(x)=\lim_{t\to\infty}\frac{1}{t}\int_0^t h(S_\tau x)\,d\tau$. Then $\bar{h}_-(x)=\bar{h}_+(x)$ almost everywhere. We show that $\bar{h}_-(y)$ is constant mod 0 on $B(x)$ if $x$ has the following property:

The set $P(x)$ of those $y\in\gamma^{(c)}(x)$ for which $\bar{h}_-(y)=\bar{h}_+(y)$ is a set of full measure in $\gamma^{(c)}(x)$.

From properties 1) and 2) of $A$ and their corollary (see p. 157) it follows that this holds for points $x$ of full measure.

Let $\Psi$ be the mapping introduced in the description of property 2) of $B(x)$. From the absolute continuity of $\Psi$ it follows that $\Psi^{-1}(P(x))$ is a subset of full measure in $B(x)$. We show that for any $z_1$, $z_2\in\Psi^{-1}(P(x))$

$$\bar{h}_-(z_1)=\bar{h}_-(z_2).$$

Let $z_1\in S_{t_1}\gamma^{(e)}(y_1)$, $z_2\in S_t\gamma^{(e)}(y_2)$, $y_1\in\gamma^{(c)}(x)$, $y_2\in\gamma^{(c)}(x)$. Then by the contraction of $S_t\gamma^{(e)}(y)$ for $t\to-\infty$ and by the uniform continuity of $h$

$$h(S_{-t}y_1)-h(S_{-t-t_1}z_1)\to 0,\qquad h(S_{-t}y_2)-h(S_{-t-t_2}z_2)\to 0$$

as $t\to\infty$, hence we obtain $\bar{h}_-(z)=\bar{h}_-(y_1)$, $\bar{h}_-(z_2)=\bar{h}_-(y_2)$. Since $y_1$, $y_2\in P(x)$, by the contraction of $S_t\gamma^{(c)}(x)$ for $t\to\infty$

$$\bar{h}_-(z_1)=\bar{h}_-(y_1)=\bar{h}_+(y_1)=\bar{h}_+(y_2)=\bar{h}_-(y_2)=\bar{h}_-(z_2).$$

And so Proposition 1 is proved.

We consider a measurable partition $\zeta^-$ of a subset of full measure in $M$, where the element $C_{\zeta^-}(x)$ is the l.c.t.f. containing $x$ and such that the cusp points of $\Gamma^{(c)}(x)$ nearest to $x$ are the ends of $C_{\zeta^-}(x)$ and the interior of $C_{\zeta^-}(x)$ has no cusps. Clearly,

a$_1$) $S_t\zeta^-\geqslant\zeta^-$ for $t>0$, a$_2$) $\bigvee_t S_t\zeta^-=\varepsilon$, a$_3$) $\bigwedge_t S_t\zeta^-=\nu^{(c)}$, where $\nu^{(c)}$ is the measurable hull of the partition of $M$ into complete fibres $\Gamma^{(c)}(x)$.

Taking expanding fibres instead of contracting ones we obtain a similar partition $\zeta^+$ into l.e.t.f., where

b$_1$) $S_t\zeta^+\geqslant\zeta^+$ for $t\geqslant 0$; b$_2$) $\bigvee_t S_t\zeta^+=\varepsilon$, b$_3$) $\bigwedge_t S_t\zeta^+=\nu^{(p)}$, where $\nu^{(e)}$ is the measurable hull of the partition of $M$ into complete fibres $\Gamma^{(e)}(x)$.

Using the convexity of the fibres $\pi(\Gamma^{(c)}(x))$, $\pi(\Gamma^{(e)}(x))$, it is not difficult to show that the transversal foliations we have constructed, with fibres $\Gamma^{(c)}$ and $\Gamma^{(e)}$, are non-integrable in the sense of [11]. The methods of [11] also show that $\bigwedge_t S_t\zeta^-=\bigwedge_t S_t\zeta^+=\nu_e$, where $\nu_e$ is the partition of $M$ into ergodic components. Hence it follows that $S$ is a $K$-system in each ergodic component. It is not difficult to check that the conditional entropy is $H(S_t\zeta^-|\zeta^-) = t\int_M |\varkappa^{(c)}(x)|\,d\mu(x)$. As in [11], it turns out that in fact $H(S_1\zeta^-|\zeta^-)$ gives the entropy of $\{S_t\}$. In the following sections we shall show that the number of ergodic components of $\{S_t\}$ is 1, and, consequently, that $\{S_t\}$ is a $K$-system.

We construct analogous partitions $\zeta^+$, $\zeta^-$ for the transformation $T$. In a certain sense this is simpler than for the flow $\{S_t\}$. Namely, for $C_{\zeta^-}(x)$ $(x \in M_1)$ we choose a regular arc of the l.c.t.f. of $x$ whose ends are singular points of the complete fibre $\Gamma^{(c)}(x)$. The analysis at the end of §3 shows that these singular points are break points. The partition $\zeta^+$ is constructed with regular arcs of l.e.t.f. For such partitions the same assertions as in the case of $\{S_t\}$ hold.

## §5. An estimate of the number of ergodic components. Auxiliary geometric constructions and estimates

In this section we carry out geometric constructions and give a number of estimates that are necessary in the proof of the basic theorem in §6. The reader may read the basic definitions of this section and then go on to §6, appealing to the estimates of this section as references for §6.

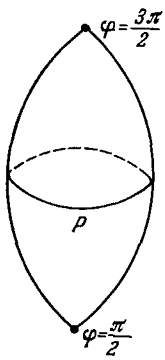


Fig. 10.

Let $M_{1i}$ be a connected component of $M_1$. In this component apart from the coordinates $(r, \varphi)$ we introduce coordinates $(p, \varphi)$, leaving $\varphi$ unchanged and putting $p = -r\cos\varphi$ (mod $(-R_i\cos\varphi)$), where $R_i$ is the length of the corresponding component of $\partial Q_i$. In the coordinates $(p, \varphi)$ the component $M_{1i}$ is the surface of revolution of the curve $p = R_i\cos\varphi$, $\frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}$ (Fig. 10).

The invariant measure $\nu$ in the variables $(p, \varphi)$ has the form $d\nu = dp\,d\varphi$ (see §1). Later we shall use frequently the notation $x(p, \varphi)$, $y(p, \varphi)$, $z(p, \varphi)$ for the linear elements with the coordinates $(p, \varphi)$. In the coordinates $(p, \varphi)$ the differential equations for l.e.t.f. and l.c.t.f. are simplified:

$$\frac{d\varphi}{dp} = \varkappa^{(e)}(p, \varphi), \qquad \frac{d\varphi}{dp} = \varkappa^{(c)}(p, \varphi).$$

Consequently, from the expressions for $\varkappa^{(e)}(p, \varphi)$ and $\varkappa^{(c)}(p, \varphi)$ we have, $k^{(0)}_{\min} = \min_{q\in\partial Q} k^{(0)}(q)$

$$-\frac{k^{(0)}(p)}{\cos\varphi} + \frac{1}{-\tau(p, \varphi) + \frac{1}{k^{(0)}_{\min}}} \leqslant \frac{d\varphi}{dp} \leqslant -\frac{k^{(0)}(p)}{\cos\varphi} - \frac{1}{\tau(p, \varphi)} \quad \text{for l.e.t.f.} \tag{17'}$$

$$\frac{k^{(0)}(p)}{\cos\varphi} + \frac{1}{\tau(T^{-1}x(p, \varphi))} \leqslant \frac{d\varphi}{dp} \leqslant \frac{k^{(0)}(p)}{\cos\varphi} + \frac{1}{\tau(T^{-1}x(p, \varphi)) - \frac{1}{k^{(0)}_{\min}}} \quad \text{for l.c.t.f.}, \tag{17''}$$

Restricting the definition of §2 somewhat we introduce

DEFINITION 5.1. Suppose that $\gamma$ lies in one regular component $M$ and is given by a function $\varphi = \varphi(p)$. We call $\gamma$ *increasing (or decreasing)* if $\varphi(p)$ is piecewise differentiable and $\frac{d\varphi}{dp}$ satisfies (17′) (or (17″)).

An increasing curve is called *maximally increasing (minimally increasing)* if

$$\frac{d\varphi}{dp} = -\frac{k^{(0)}(p)}{\cos\varphi} - \frac{1}{\tau(p,\varphi)} \tag{18′}$$

$$\left(\frac{d\varphi}{dp} = -\frac{k^{(0)}(p)}{\cos\varphi} - \frac{1}{\tau(p,\varphi) - \frac{1}{k^{(0)}_{\min}}}\right). \tag{18″}$$

Such curves are denoted by max.i.c. (min.i.c.). A max.i.c. has a simple geometric meaning. Namely, there is a $q \in \partial Q$ and an arc of $S^1(q)$ such that a regular component of the max.i.c. is the image under $T$ of this arc. This easily follows from the lemmas of §2.

The expression on the right in (18′) has discontinuities where $\tau(p, \varphi)$ has discontinuities, that is, on the curves $\Sigma_i$ described in §1. Inside the domain bounded by these curves $\tau(p, \varphi)$ is smooth. Therefore there is a unique max.i.c. through each interior point of that domain. Since a max.i.c. is given by a monotonically increasing function, it may be continued up to the boundary of the domain, so that the continued curve intersects the boundary in a definite point $z$. It is not difficult to see that if $T$ is continuous from within at $z$, then the curve is tangent to the boundary. In the opposite case it approaches the boundary transversally. A min.i.c. always approaches the boundary transversally. We also note that each discontinuity curve $\Sigma_i$ is a max.i.c. It is the envelope of the max.i.c. starting in the domain where the function $\tau$ is continuous up to the boundary. In what follows we shall frequently construct max.i.c. and min.i.c. From what we have said it follows that there is such a curve through any point. If we wish to draw curves through points of non-uniqueness, then we always assume that we have a max.i.c. different from $\Sigma_i$.

Let $\gamma$ be a decreasing curve one of whose ends is the point $\varphi = 3\pi/2$. From the differential equation (17″) it follows that there is a meridian that $\gamma$ touches at $\varphi = 3\pi/2$. We choose this meridian as the origin for $p$. Then if $\varphi(p)$ is the function giving $\gamma$, it easily follows from (17″) that

$$\text{const}\sqrt{p} \leqslant \frac{3\pi}{2} - \varphi(p) \leqslant \text{const}\sqrt{p}. \tag{19}$$

From this inequality it follows that

$$|\cos\varphi(p)| \leqslant \text{const}\sqrt{p}. \tag{20}$$

A similar observation is true for decreasing curves ending in $\varphi = \pi/2$ and for increasing curves.

In such cases we say that the origin of $p$ is compatible with $\gamma$.

Let $\gamma$ be any decreasing curve. Assume for definiteness that it lies in the upper half of the regular component $M_{1i}$, that is $\varphi \geqslant \pi$. Clearly we may always construct, in fact in many ways, a piecewise differentiable decreasing curve $\gamma'(\gamma \subset \gamma')$, ending at $\varphi = 3\pi/2$. Every such curve determines an origin on the $p$-axis. For two distinct curves $\gamma'$, $\gamma''$

$$\frac{p'(x)}{p''(x)} \leqslant \text{const}, \qquad (x \in \gamma). \tag{21}$$

Similarly if $x$ is contained in an increasing curve $\gamma'$ and a decreasing curve $\gamma''$, ending at $3\pi/2$, then for the associated coordinates

$$\frac{|p'(x)|}{|p''(x)|} \leqslant \text{const},$$

where $p'$, $p''$, are the coordinates obtained by the choice of origin using $\gamma'$ and $\gamma''$.

The following property of decreasing and increasing curves is basic.

If $\gamma$ is a decreasing curve, then $T\gamma$ is a decreasing curve and

$$\frac{p(T\,d\gamma)}{p(d\gamma)} = 1 + \tau(p, \varphi)\frac{d\varphi}{dp}, \tag{22'}$$

where $p(d\gamma)$ is the projection of the element $d\gamma$ of $\gamma$ onto the $p$-axis, and analogously $p(Td\gamma)$.

If $\gamma$ is an increasing curve, then $T^{-1}\gamma$ is an increasing curve and

$$\frac{p(T^{-1}\,d\gamma)}{p(d\gamma)} = 1 - \tau(T^{-1}(p, \varphi))\frac{d\varphi}{dp}. \tag{22''}$$

Both equalities follow from the lemmas of §2.

Put $\lambda = \left[1 + \min|\tau(T^{-1}(p,\varphi))| \cdot \min\frac{d\varphi}{dp}\right]^{-1}$, where the first min is taken over all $x$ and the second over all increasing curves. Then

$$\frac{p(T^{-1}\,d\gamma)}{p(d\gamma)} \geqslant \lambda^{-1}. \tag{23}$$

DEFINITION 5.2. A *quadrilateral* is a domain $G$ whose boundary consists of four piecewise continuously differentiable curves, one pair of opposite curves being increasing and the other being decreasing (Fig.11).

We call the decreasing curves the left and right sides of $G$ and the increasing curves the upper and lower sides of $G$ (see Fig. 11) and we denote them by $\gamma_{rt}$, $\gamma_{lt}$, $\gamma_u$, $\gamma_l$, respectively. The functions giving these curves are denoted by $\varphi_G^{(rt)}(p)$, $\varphi_G^{(lt)}(p)$, $\varphi_G^{(u)}(p)$, $\varphi_G^{(l)}(p)$.

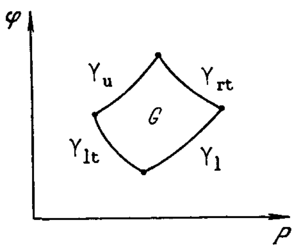


Fig. 11.

DEFINITION 5.3. An *R-quadrilateral* is a quadrilateral whose upper side is a max.i.c. and whose lower side is a min.i.c.

For an increasing or decreasing curve $\gamma$ given by a function $\varphi(p)$, let $p(\gamma)$ be equal to the length of the segment on the $p$-axis, where $\varphi(p)$ is defined.

For an $R$-quadrilateral $G$ put $p_g(G) = \sup p(\gamma)$, where the sup is taken over all increasing curves $\gamma$.

We now establish a number of properties of quadrilaterals that will be used in §6. These properties are quoted in the form we need for §6, although in some cases they are true in a more general setting.

PROPERTIES OF QUADRILATERALS.

$1^\circ$. Let $G$ and $\tilde{G}$ be quadrilaterals such that for some $m > 0$ each $T^{-i} \mid \tilde{G}$ is a smooth mapping $(i = 1, \ldots, m)$, $T^{-m}\tilde{G} \subset G$ and $T^{-m}\tilde{G}$ is a quadrilateral.

Also let $\gamma_1, \gamma_2 \subset G$ be arbitrary increasing curves whose ends lie on the decreasing sides of $\tilde{G}$.

We fix a smooth partition $\xi_0$ of $G$ whose elements $C_{\xi_0}$ are decreasing curves with ends on the increasing part of the boundary and $\gamma_{1t}(G)$, $\gamma_{rt}(G)$ are elements $C_{\xi_0}$. Let $\xi_s$ be the image under $T^s$ of $\xi_0 \mid T^{-m}\tilde{G}$. It is clear that $\xi_s$ is a smooth partition of $T^{-m+s}\tilde{G}$ and that its elements are decreasing curves.

Consider the mapping $\Psi_m\colon \gamma_1 \to \gamma_2$, putting $y = \Psi_m(x) = \gamma_2 \cap C_{\xi_m}$ for $x = \gamma_1 \cap C_{\xi_m}$. Then $\Psi_s = T^{-s}\Psi_m T^s$ is the similar mapping $T^{-s}\gamma_1 \to T^{-s}\gamma_2$ constructed by means of $\xi_s$. Put

$$\tau_i' = \tau(T^{-i-1}x), \quad \tau_i'' = \tau(T^{-i-1}y), \quad \varkappa_0' = \left.\frac{d\varphi_1}{dp}\right|_x, \quad \varkappa_0'' = \left.\frac{d\varphi_2}{dp}\right|_y$$

for the functions $\varphi_1$, $\varphi_2$ giving $\gamma_1$, $\gamma_2$

$$\varkappa_i' = -\frac{k^{(0)}(\pi(T^{-i}(x))}{\cos\varphi(T^{-i}x)} + \frac{1}{-\tau_{i-1}' + (\varkappa_{i-1}')^{-1}},$$

$$\varkappa_i'' = -\frac{k^{(0)}(\pi(T^{-i}y))}{\cos\varphi(T^{-i}y)} + \frac{1}{-\tau_{i-1}'' + (\varkappa_{i-1}'')^{-1}}.$$

These values will be needed later. For the present we note only that since $|p(T^{-s}x) - p(T^{-s}y)| \leqslant \text{const}\, \lambda^s$, we have

$$\exp\{-\text{const}\,\lambda^{\frac{i}{2}}\} \leqslant \frac{\tau_i''}{\tau_i'} \leqslant \exp\{\text{const}\,\lambda^{\frac{i}{2}}\}, \tag{24'}$$

$$\exp\{\text{const}\,\lambda^{\frac{i}{2}}\} \leqslant \frac{k^{(0)}(\pi(T^{-i}x))}{k^0(\pi(T^{-i}y))} \leqslant \exp\{\text{const}\,\lambda^{\frac{i}{2}}\}. \tag{24''}$$

Put

$$\exp\{c_i\} = \max_{x\in\gamma_1} \frac{\cos\varphi(T^{-i}x)}{\cos\varphi(T^{-i}y)} \qquad (i = 0, \ldots, m).$$

The following inequality holds: for some absolute constant $q_1$ $(0 \leqslant q_1 \leqslant 1)$

$$\frac{p(\gamma_1)}{p(\gamma_2)} \leqslant K_0 \prod_{i=1}^{m} (1 + \text{const}\,(1 + c_0)\, q_1^i + c_i) = K_0 D,$$

where the constant $K_0$ depends on $G$, but not on $\tilde{G}$ and $m$.

PROOF. We may write

$$\frac{p(\Psi_m(d\gamma_1))}{p(d\gamma_1)} = \frac{p(\Psi_0(T^{-m}d\gamma_1))}{p(T^{-m}d\gamma_1)} \cdot \frac{\dfrac{p(\Psi_m(d\gamma_1))}{p(\Psi_0(T^{-m}d\gamma_1))}}{\dfrac{p(d\gamma_1)}{p(T^{-m}d\gamma_1)}}.$$

By the lemmas of §2

$$\frac{p\,(d\gamma_1)}{p\,(T^{-m}\,d\gamma_1)}=\prod_{i=1}^{m}(1-\tau_i'\varkappa_i')^{-1},\quad \frac{p\,(\Psi_m\,(d\gamma_1))}{p\,(T^{-m}\Psi_m\,(d\gamma_1))}=\prod_{i=1}^{m}(1-\tau_i''\varkappa_i'')^{-1}.$$

$T^{-m}\gamma_1$, $T^{-m}\gamma_2$ are increasing curves inside $G$. We can obviously find a constant $K_0$ for $G$ so that for any two increasing curves $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ with ends on the decreasing boundary of $G$

$$\frac{p\,(d\tilde{\gamma}_1)}{p\,(\psi_0\,(d\tilde{\gamma}_1))}\leqslant K_0.$$

Now assume that we have already shown for some constant $D'$ and for all $x\in\gamma_1$ that

$$\frac{\prod\limits_{i=1}^{m}(1-\tau_i'\varkappa_i')}{\prod\limits_{i=1}^{m}(1-\tau_1''\varkappa_1'')}\leqslant D'.$$

Then

$$\frac{p\,(\gamma_1)}{p\,(\gamma_2)}=\frac{\int\limits_{\gamma_1}p\,(d\gamma_1)}{\int\limits_{\gamma_2}p\,(d\gamma_2)}=\frac{\int\limits_{T^{-m}\gamma_1}p\,(T^{-m}\,d\gamma_1)\cdot\frac{p\,(d\gamma_1)}{p\,(T^{-m}\,d\gamma_1)}}{\int\limits_{T^{-m}\gamma_2}p\,(T^{-m}\,d\gamma_2)\cdot\frac{p\,(d\gamma_2)}{p\,(T^{-m}\,d\gamma_2)}}=$$

$$=\frac{\int\limits_{T^{-m}\gamma_1}p\,(T^{-m}\,d\gamma_1)\cdot\prod\limits_{i=1}^{m}(1-\tau_i'\varkappa_i')^{-1}}{\int\limits_{T^{-m}\gamma_2}p\,(T^{-m}\,d\gamma_2)\cdot\prod\limits_{i=1}^{m}(1-\tau_i''\varkappa_i'')^{-1}}=\frac{\int\limits_{\tilde{\gamma}_1}p\,(d\tilde{\gamma}_1)\prod\limits_{i=1}^{m}(1-\tau_i'\varkappa_i')^{-1}}{\int\limits_{\tilde{\gamma}_2}p\,(d\tilde{\gamma}_2)\prod\limits_{i=1}^{m}(1-\tau_i''\varkappa_i'')^{-1}}\leqslant K_0D'.$$

Thus, the proof of our assertion reduces to an estimate of $D'$.

Put $\frac{\varkappa_i''}{\varkappa_i'}=\exp(\rho_i)$. From the recurrence relations for $\varkappa_i'$, $\varkappa_i''$ it follows that

$$\exp(\rho_i)\leqslant\max\left[\frac{k^{(0)}\,(\pi\,(T^{-i}y))\cos\varphi\,(T^{-i}x)}{k^0\,(\pi\,(T^{-i}x))\cos\varphi(T^{-i}y)},\ \frac{\tau_i'}{\tau_i''}\cdot\frac{-\tau_i''}{-\tau_i''+(\varkappa_{i-1}'')^{-1}}+e^{\rho_{i-1}}\frac{(\varkappa_{i-1}'')^{-1}}{-\tau_i''+(\varkappa_{i-1}'')^{-1}}\right].$$

The first expression under the max sign does not exceed $(1+c_i)\exp\{\text{const}\,\lambda^{i/2}\}$ (see (24″)). As regards the second expression, $[\varkappa_{i-1}'']^{-1}\leqslant\text{const}$, $\tau_i''\,|\geqslant\text{const}$, and therefore $\frac{(\varkappa_{i-1}'')^{-1}}{|\,\tau_{i-1}''\,|+(\varkappa_{i-1}'')^{-1}}\leqslant q<1$, where $q$ is an absolute constant. Choose $q_1$, $\max\left(\sqrt{\lambda},\,q\right)<q_1<1$ and put $\tilde{q}=\frac{1-q}{q_1-q}$.

It is easy to see that the second expression is either less than $\exp\{\text{const}\,\lambda^{i/2}\}$ or less than $\{q_1\rho_{i-1}\}$. Consequently

$$\rho_i \leqslant \max\left[q_1\rho_{i-1},\ \text{const}\ \lambda^{\frac{i}{2}},\ c_i + \text{const}\ \lambda^{\frac{i}{2}}\right].$$

Solving these recurrence inequalities we obtain

$$\rho_i \leqslant \text{const}\ q_1^i \cdot \rho_0 + c_i$$

and

$$\frac{1+\varkappa_i'\tau_i'}{1+\varkappa_i''\tau_i''} \leqslant 1 + \text{const}\ q_1^i \cdot \rho_0 + c_i.$$

Since $\exp\{\rho_0\} = \frac{\varkappa_0'}{\varkappa_0''} \leqslant \text{const}\,\frac{\cos\varphi(x)}{\cos\varphi(y)} \leqslant \text{const}\exp\{c_0\}$, we have $\rho_0 \leqslant \text{const} + c_0$ and

$$\prod_{i=1}^{m} \frac{1-\varkappa_i'\tau_i'}{1-\varkappa_i''\tau_i''} \leqslant \prod_{i=1}^{m}\left(1 + \text{const}\ (1+c_0)\, q_1^i + c_i\right) = D.$$

This is the required estimate for $D$. Our assertion is now proved.

2°. Under the conditions of the preceding point let $\tilde{\tilde{G}}$ be a quadrilateral $\tilde{\tilde{G}} \subset \tilde{G}$, and assume that the decreasing sides of $\tilde{\tilde{G}}$ lie on the corresponding decreasing sides of $\tilde{G}$. We use the partition $\xi_0$ described in the preceding point to construct mappings $\psi_i$.

Assume that for some number $\Phi$

$$\sup_{x',\,x''\in\tilde{G}} \frac{\cos\varphi(x')}{\cos\varphi(x'')} = 1 + \Phi \tag{25}$$

and for any two points $z'$, $z''$ lying in one element of $\xi_m$, $z', z'' \in \tilde{G}$,

$$\frac{\cos\varphi(T^{-s}z')}{\cos\varphi(T^{-s}z'')} \leqslant \exp\{c_s\}.$$

We show that for the ratio of areas we have the estimate

$$\frac{\nu(\tilde{\tilde{G}})}{\nu(\tilde{G})} \leqslant \text{const}\ (1+\Phi)\, K_0 \cdot D \cdot \frac{p(\gamma_{\text{rt}}(\tilde{\tilde{G}})) + 2p_g(\tilde{G})}{p(\gamma_{\text{rt}}(\tilde{G})) - 2p_g(\tilde{G})}.$$

PROOF. It easily follows from (25) and the definitions that for two arbitrary increasing curves inside $\tilde{G}$ given by functions $\tilde{\varphi}(p)$, $\tilde{\tilde{\varphi}}(p)$ the ratio $\frac{d\tilde{\varphi}}{dp} : \frac{d\tilde{\tilde{\varphi}}}{dp} \leqslant \text{const}\ (1+\Phi)$. Let $\varphi^{(\text{rt})}$, $\varphi^{(\text{lt})}$ be functions giving $\gamma_{\text{rt}}(\tilde{G})$, $\gamma_{\text{lt}}(\tilde{G})$. Using the remark just made we estimate the ratio

$$\frac{\varphi^{(\text{rt})}(p_1) - \varphi^{(\text{lt})}(p_1)}{\varphi^{(\text{rt})}(p_2) - \varphi^{(\text{lt})}(p_2)}$$

for those $p_1$ and $p_2$ for which it makes sense.

We construct increasing curves $\tilde{\gamma}$, $\tilde{\tilde{\gamma}}$, one of whose ends coincides with $\varphi^{(\text{rt})}(p_1)$, $\varphi^{(\text{rt})}(p_2)$, respectively, and the other end lies on $\gamma^{(\text{lt})}(\tilde{G})$ at the points with the coordinates $\tilde{p}$, $\tilde{\tilde{p}}$. Let $\tilde{\gamma}$, $\tilde{\tilde{\gamma}}$ be given by functions $\tilde{\varphi}$, $\tilde{\tilde{\varphi}}$ defined in $\widetilde{\Delta p}$, $\widetilde{\widetilde{\Delta p}}$, respectively. From the previous point we have $\widetilde{\Delta p} : \widetilde{\widetilde{\Delta p}} \leqslant K_0 D$. Therefore

$$\frac{\varphi^{(rt)}(p_1)-\tilde{\varphi}(\tilde{p})}{\varphi^{(rt)}(p_2)-\tilde{\tilde{\varphi}}(\tilde{\tilde{p}})}=\frac{\tilde{\varphi}(p_1)-\tilde{\varphi}(p)}{\tilde{\tilde{\varphi}}(p_2)-\tilde{\tilde{\varphi}}(\tilde{\tilde{p}})}=\frac{\int\limits_{\widetilde{\Delta p}}\frac{d\tilde{\varphi}}{dp}\cdot dp}{\int\limits_{\widetilde{\widetilde{\Delta p}}}\frac{d\tilde{\tilde{\varphi}}}{dp}\cdot dp}\leqslant \mathrm{const}\,(1+\Phi)\,K_0\cdot D.$$

From the same arguments

$$\frac{\tilde{\varphi}(\tilde{p})-\varphi^{(lt)}(p_1)}{\tilde{\tilde{\varphi}}(\tilde{\tilde{p}})-\varphi^{(lt)}(p_2)}=\frac{\varphi^{(lt)}(\tilde{p})-\varphi^{(lt)}(p_1)}{\varphi^{(lt)}(\tilde{\tilde{p}})-\varphi^{(lt)}(p_2)}\leqslant \mathrm{const}\,(1+\Phi)\,K_0\cdot D.$$

Since

$$\varphi^{(rt)}(p_1)-\varphi^{(lt)}(p_1)=\varphi^{(rt)}(p_1)-\tilde{\varphi}(p)+\tilde{\varphi}(\tilde{p})-\varphi^{(lt)}(p_1),$$

$$\varphi^{(rt)}(p_2)-\varphi^{(lt)}(p_2)=\varphi^{(rt)}(p_2)-\tilde{\tilde{\varphi}}(\tilde{\tilde{p}})+\tilde{\tilde{\varphi}}(\tilde{\tilde{p}})-\varphi^{(lt)}(p_2),$$

we have

$$\frac{\varphi^{(rt)}(p_1)-\varphi^{(lt)}(p_1)}{\varphi^{(rt)}(p_2)-\varphi^{(lt)}(p_2)}\leqslant \mathrm{const}\,(1+\Phi)\,K_0 D. \tag{26}$$

We split the $p$-axis into sets $\Delta p_1$, $\Delta p$, $\Delta p_2$ where $\varphi^{(u)}_{\tilde{G}}$ and $\varphi^{(lt)}_{\tilde{G}}$, $\varphi^{(rt)}_{\tilde{G}}$ and $\varphi^{(lt)}_{\tilde{G}}$, $\varphi^{(rt)}_{\tilde{G}}$ and $\varphi^{(l)}_{\tilde{G}}$ are defined simultaneously for $\tilde{G}$ and similarly into sets $\overline{\Delta p_1}$, $\overline{\Delta p}$, $\overline{\Delta p_2}$ for $\tilde{\tilde{G}}$. Then

$$\nu(\tilde{G})=\int\limits_{\Delta p_1}[\varphi^{(u)}_{\tilde{G}}(p)-\varphi^{(lt)}_{\tilde{G}}(p)]\,dp+\int\limits_{\Delta p}[\varphi^{(rt)}_{\tilde{G}}(p)-\varphi^{(lt)}_{\tilde{G}}(p)]\,dp+\int\limits_{\Delta p_2}[\varphi^{(rt)}_{\tilde{G}}(p)-\varphi^{(l)}_{\tilde{G}}(p)]\,dp,$$

$$\nu(\tilde{\tilde{G}})=\int\limits_{\overline{\Delta p_1}}[\varphi^{(u)}_{\tilde{\tilde{G}}}(p)-\varphi^{(lt)}_{\tilde{\tilde{G}}}(p)]\,dp+\int\limits_{\overline{\Delta p}}[\varphi^{(rt)}_{\tilde{\tilde{G}}}(p)-\varphi^{(lt)}_{\tilde{\tilde{G}}}(p)]\,dp+\int\limits_{\overline{\Delta p_2}}[\varphi^{(rt)}_{\tilde{\tilde{G}}}(p)-\varphi^{(l)}_{\tilde{\tilde{G}}}(p)]\,dp.$$

where $\varphi^{(u)}_{\tilde{\tilde{G}}}$, $\varphi^{(l)}_{\tilde{\tilde{G}}}$ are the functions giving the corresponding sides of $\tilde{\tilde{G}}$. From (26) we have

$$\frac{\int\limits_{\overline{\Delta p}}[\varphi^{(rt)}_{\tilde{\tilde{G}}}(p)-\varphi^{(lt)}_{\tilde{\tilde{G}}}(p)]\,dp}{\int\limits_{\Delta p}[\varphi^{(rt)}_{\tilde{G}}(p)-\varphi^{(lt)}_{\tilde{G}}(p)]\,dp}\leqslant \mathrm{const}\,(1+\Phi)\,K_0\cdot D\,\frac{p(\gamma_{(rt)}(\tilde{\tilde{G}}))}{p(\gamma_{(rt)}(\tilde{G}))-2p_g(\tilde{G})}$$

Therefore

$$\frac{\nu(\widetilde{\widetilde{G}})}{\nu(\widetilde{G})} \leqslant \frac{\int\limits_{\Delta p} [\varphi^{(\mathrm{rt})}_{\widetilde{\widetilde{G}}}(p) - \varphi^{(\mathrm{lt})}_{\widetilde{\widetilde{G}}}(p)]\, dp}{\int\limits_{\Delta p} [\varphi^{(\mathrm{rt})}_{\widetilde{G}}(p) - \varphi^{(\mathrm{lt})}_{\widetilde{G}}(p)]\, dp} +$$

$$+ \frac{\int\limits_{\Delta p_1} [\varphi^{(\mathrm{u})}_{\widetilde{\widetilde{G}}}(p) - \varphi^{(\mathrm{lt})}_{\widetilde{\widetilde{G}}}(p)]\, dp}{\int\limits_{\Delta p} [\varphi^{(\mathrm{rt})}_{\widetilde{G}}(p) - \varphi^{(\mathrm{lt})}_{\widetilde{G}}(p)]\, dp} + \frac{\int\limits_{\Delta p_2} [\varphi^{(\mathrm{rt})}_{\widetilde{G}}(p) - \varphi^{(\mathrm{u})}_{\widetilde{\widetilde{G}}}(p)]\, dp}{\int\limits_{\Delta p} [\varphi^{(\mathrm{rt})}_{\widetilde{G}}(p) - \varphi^{(\mathrm{lt})}_{\widetilde{G}}(p)]\, dp} \leqslant$$

$$\leqslant \mathrm{const}\,(1+\Phi)\, K_0 \cdot D\, \frac{p(\gamma_{\mathrm{rt}}(\widetilde{\widetilde{G}})) + 2p_g(\widetilde{G})}{p(\gamma_{\mathrm{rt}}(\widetilde{G})) - 2p_g(\widetilde{G})} .$$

## §6. An estimate of the number of ergodic components. The main theorem

THEOREM 6.1. *Let $x_0 \in M$ be such that $T^i x_0$, for $i = 0, 1, \ldots$ never hits the boundary in a singular point. Then for each $\alpha (0 \leqslant \alpha \leqslant 1)$ and any $C (0 < C < \infty)$ there exists an $\varepsilon = \varepsilon(x_0, \alpha, C)$ such that the $\varepsilon$-neighbourhood $U_\varepsilon$ of $x_0$ has the following property: for any increasing curve $\gamma_0 \subset U_\varepsilon$, $p(\gamma_0) = \delta_0$, there is a quadrilateral $G$ whose left side is $\gamma_0$, and if $G' = \{x\colon x \in G$, there is a regular segment of l.e.t.f. $\gamma^{(e)}(x)$ through $x$ joining the left and right sides of $G$ and $p(\gamma^{(e)}(x)) \geqslant C\delta_0\}$, then $\nu(G') \geqslant (1-\alpha)\,\nu(G)$. Similarly we can construct a quadrilateral $\widetilde{G}$ whose right side is $\gamma_0$ and such that the corresponding set $\widetilde{G}'$ satisfies the inequality: $\nu(\widetilde{G}') \geqslant (1-\alpha)\,\nu(\widetilde{G})$.*

The proof of this theorem resembles the proof of the theorem of absolute continuity of transversal foliations given in [4]. In any case it is fairly long.

Let us explain the idea in general terms. We include $\gamma_0$ in a quadrilateral $G$ where the increasing curves joining the left and right sides of $G$ have $p$-length not exceeding $C\delta_0$. If the neighbourhood of $x_0$ is sufficiently small, then we can take $T^{k_0}G$, with sufficiently large $k_0$, so that $T^{k_0}G$ is a very narrow and streched-out quadrilateral. The quadrilateral $T^{k_0}G$ is broken up into components by the curves $\Sigma_i$. The inverse images of points lying in a small neighbourhood of $\Sigma_i$ do not belong to $G'$. We apply $T$ to the connected components obtained by intersecting $T^{k_0}G$ with the $\Sigma_i$, and consider the intersection of the image with the curves $\Sigma_i$. Then the inverse images of points in a still smaller neighbourhood of the $\Sigma_i$ do not belong to $G'$. We apply $T$ to the connected components arising from the intersection with the $\Sigma_i$ and repeat the argument, etc. Since there is compression in the direction of increasing curves, at each successive stage we can take smaller and smaller neighbourhoods of the $\Sigma_i$. Finally we see that the total area of the sets thrown out is relatively small.

To realize this idea we have to take account of a whole series of details, which explains the number of estimates and arguments to be carried out later. We split the account into separate parts and points. We only prove the first assertion of the theorem, the second is proved similarly.

PART 1. CONSTRUCTION OF THE QUADRILATERAL $G$ AND OF THE SEQUENCE OF SETS $\Pi_m$, $\bigcap_m T^{-m}\Pi_m \subset G'$.

Let $k_0$ be an integer whose value depends on a number of absolute constants which appear in the course of the proof, and on the numbers $\alpha$ and $C$ in the theorem. In the course of the proof we indicate all the requirements on $k_0$.

I.1. CONSTRUCTION OF $G$. Choose $\varepsilon = \varepsilon(x_0, \alpha, C)$ so small that $T, T^2, \ldots, T^{k_0}$ are smooth mappings of $U_{2\varepsilon}$. This is possible by the conditions on $x_0$. Let $x_1$ and $x_2$ be the left and right ends of an arbitrarily given curve $\gamma_0 \subset U_\varepsilon$. We now construct a quadrilateral $G$ so that

$a_1$) $\gamma_0$ is its left side;

$a_2$) $p(\gamma_0) \geqslant C\delta_0$ for any increasing curve $\gamma$ with ends on the left and right sides of $G$;

$a_3$) $T^{k_0}(\gamma_u(G))$ $(T^{k_0}(\gamma_1(G)))$ is a max.i.c. (min.i.c.) passing through $T^{k_0}x_1(T^{k_0}x_2)$;

$a_4$) for any increasing curve $\gamma \subset T^{k_0}G$

$$p(\gamma) \ll \delta_0\lambda^{\frac{k_0}{2}}.$$

From $a_3$) it follows that $T^{k_0}G = G_{k_0,0}$ is an $R$-quadrilateral.

I.2. CONSTRUCTION OF $\Pi_m$. To investigate $G'$ we construct a system of sets $\Pi_m(m \geqslant k_0)$, where each $\Pi_m$ is a union of $R$-quadrilaterals $G_{m,s}(s = 0, 1, \ldots, N_m)$, $T^{-i}\Pi_m \subset \Pi_{m-1}$ and $\bigcap_{m \geqslant k_0} T^{-m}\Pi_m \subset G'$. The proof of the theorem then reduces to the estimation of the measure of $\bigcap_{m \geqslant k_0} T^{-m}\Pi_m$.

Putting $\Pi_{k_0} = G_{k_0,0}$, we assume inductively that we have constructed $\Pi_{k_0}, \ldots, \Pi_{m-1}$, $\Pi_i = \bigcup_{s=0}^{N_s} G_{i,s}$, and we construct $\Pi_m$. We add to the inductive hypothesis the assumption that for some constant $\lambda_2(0 < \lambda_2 < 1)$, to be calculated later, $p_g(G_{t,s}) \leqslant \lambda^t p(\gamma_{rt}(G_{t,s}))(s = 0, 1, \ldots, N_t;\ t = k_0, \ldots, m-1)$.

For $m = k + 1$ this assumption is satisfied with $\lambda_2 = \frac{1+\lambda}{2}$, provided that $k_0$ is sufficiently large.

Put $V_{n,\alpha} = \left\{(p, \varphi) : |\cos\varphi| \leqslant \frac{\delta_0}{n^\alpha}\right\}$. From the $R$-quadrilaterals $G_{m-1,s}$ we consider only those $G_{m-1,s}$ for which either

$$G_{m-1,s} \cap V_{m-1,4} = \varnothing, \quad \text{or} \quad G_{m-1,s} \cap V_{m-1,4} \neq \varnothing,$$

but $p(\gamma_{\bar{r}t}(G_{m-1,s})) \geqslant \frac{A_0\delta_0^2}{(m-1)^3}$, $A_0$ to be defined a little later. If

$p(\gamma_{rt}(G_{m-1,s})) < \frac{A_0\delta_0^2}{(m-1)^3}$ and $G_{m-1,s} \cap V_{m-1,4} \neq \varnothing$, then $G_{m-1,s} \subset V_{m-1,\frac{3}{2}}$. This condition explains the choice of $A_0$ in the previous inequality for $p(\gamma_{rt}(G_{m-1,s}))$.

In the quadrilaterals for which $G_{m-1,s} \cap V_{m-1,4} \neq \varnothing$, choose a point $z \in \gamma_{rt}(G_{m-1,s})$ such that $|\cos\varphi(z)| = \frac{\delta_0}{(m-1)^4}$, $\varphi(z) > \pi$. We pass a max.i.c. through $z$ inside $G_{m-1,s}$ up to the first intersection with $\partial G_{m-1,s}$. Since $p_g(G_{m-1,s}) \leqslant \lambda_2^{m-1} p(\gamma_{rt}(G_{m-1,s}))$, this intersection occurs at some $z' \in \gamma_{lt}(G_{m-1,s})$. If $G_{m-1,s}$ intersects $V_{m,4} \cap \{x\colon \varphi(x) < \pi\}$, then we make a similar construction substituting left for right and taking a min.i.c. instead of a max.i.c. A new $R$-quadrilateral $G'_{m-1,s}$ is then formed lying outside $V_{m-1,4}$. If $G_{m-1,s} \cap V_{m-1,4} = \varnothing$, then we put $G'_{m-1,s} = G_{m-1,s}$.

We consider the intersection of $G'_{m-1,s}$ with the discontinuity curves $\Sigma_i$ of $T$ (see §1). These curves divide $G'_{m-1,s}$ into open connected components $O_{m-1,s,l}$. The boundary of such components is formed from the boundary of $G'_{m-1,s}$ and the $\Sigma_i$. Write $\Sigma = \bigcup_i \Sigma_i$.

CLASSIFICATION OF THE $O_{m-1,s,l}$.

1. $O_{m-1,s,l}$ is called a component of the first type if $\partial O_{m-1,s,l} \cap \gamma_{rt}(G'_{m-1,s})$, $\partial O_{m-1,s,l} \cap \gamma_{lt}(G'_{m-1,s})$ are both segments of non-zero length. In this case $O_{m-1,s,l}$ is, obviously, a quadrilateral. A component not of first type is said to be of second type.

If $O_{m-1,s,l}$ is a component of the first type and $\Sigma \cap \partial O_{m-1,s,l} \neq \varnothing$, then either part of the right side or part of the lower side, or both, of $O_{m-1,s,l}$ belong to $\Sigma$.

2. Let $O_{m-1,s,l}$ be a component of the first type and let at least one of the sides $\gamma_u(O_{m-1,s,l})$, $\gamma_l(O_{m-1,s,l})$ intersect $\Sigma$. The component $O_{m-1,s,l}$ is called complete if $T$ is continuous from within $O_{m-1,s,l}$ on one of these sides. Otherwise $O_{m-1,s,l}$ is called an incomplete component.

If $O_{m-1,s,l}$ is a complete component, then the part of its boundary belonging to $\Sigma$, where $T$ is continuous from within, passes under the action of $T$ into the vertex $\varphi = 3\pi/2$ or into the vertex $\varphi = \pi/2$.

3. An incomplete component of the first type is called interior if

$$\partial O_{m-1,s,l} \cap (\gamma_u(G'_{m-1,s}) \cup \gamma_l(G'_{m-1,s})) = \varnothing.$$

For such components we show that $p(\gamma_{rt}(O_{m-1,s,l})) \geqslant$ const. For the trajectories defined by the interior points of $\gamma_{rt}(O_{m-1,s,l})$ have first reflection for $t < 0$ in a fixed $\partial Q_i$. The trajectories of the ends of $\gamma_{rt}(O_{m-1,s,l})$ touch other components of the boundary before reaching $\partial Q_i$ (Fig. 12). A similar situation, as is easily seen, is impossible in a sufficiently small neighbourhood of points where the function $\tau$ is unbounded. Hence our assertion follows.

An incomplete component of the first type is called bounding if it is not interior. The number of such components does not exceed two. If the number of incomplete bounding components is two, then inside $G'_{m-1,\,s}$ there is a complete component for which both ends of $\gamma_{rt}$ belong to $\Sigma$ (Fig. 13). Consequently in this case $p(\gamma_{rt}(G_{m-1,\,s})) \geqslant \text{const}$.

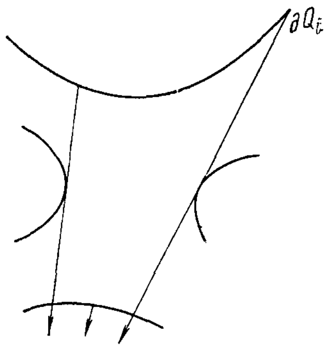



Fig. 12.

REMARK 6.1. If $\overline{G'_{m-1,\,s}} \cap \Sigma = \varnothing$, then $G'_{m-1,\,s}$ is an incomplete bounding component of the first type.

REMARK 6.2. $G_{k_0,0}$ is by definition a complete component.

NOTATION. The set of $G'_{m-1,\,s}$ for which $T^{-i}G'_{m-1,\,s}$ is contained in a complete component is denoted by $\Pi'_{m-1}$. The set of $G'_{m-1,\,s}$ that are not contained in $\Pi'_{m-1}$ but which either have an incomplete interior component of the first type or two bounding incomplete components is denoted by $\Pi''_{m-1}$. The remaining set is denoted by $\Pi'''_{m-1}$.

4. A component $O_{m-1,\,s,\,l}$ of the first type is called essential if

$$p(\gamma_{rt}(O_{m-1,\,s,\,l})) \geqslant \frac{\delta_0}{(m-1)^{222}} \quad \text{for} \quad G'_{m-1,\,s} \subset \Pi'_{m-1};$$

$$p(\gamma_{rt}(O_{m-1,\,s,\,l})) \geqslant \frac{\delta_0}{(m-1)^{100}} \quad \text{for} \quad G'_{m-1,\,s} \subset \Pi''_{m-1};$$

$$p(\gamma_{rt}(O_{m-1,\,s,\,l})) \geqslant \frac{\delta_0}{(m-1)^{172}} \quad \text{for} \quad G'_{m-1,\,s} \subset \Pi'''_{m-1}.$$

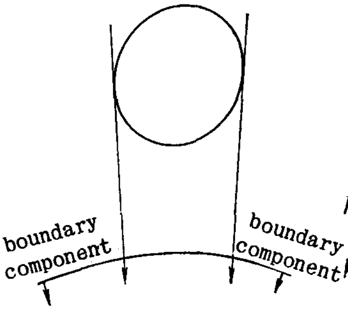
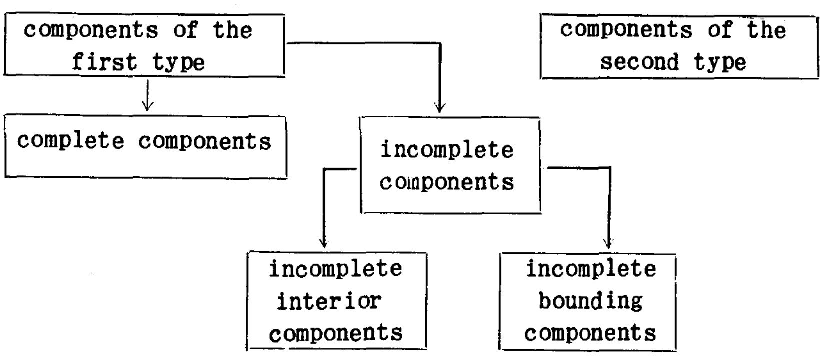



Fig. 13.

It is convenient to represent the classification described in the form of the following scheme:




In addition, the components of the first type divide into essential and inessential.

For an essential component $O_{m-1,\,s,\,l}$, let the ends of $\gamma_{rt}(O_{m-1,\,s,\,l})$ be $z_1(p_1, \varphi_1)$, $z_2(p_2, \varphi_2)$, $p_1 < p_2$. We pass a min.i.c. through $z_2$ inside $O_{m-1,\,s,\,l}$ to the intersection with $\partial O_{m-1,\,s,\,l}$. Since $O_{m-1,\,s,\,l}$ is essential and the induction assumption is satisfied, this curve intersects $\partial O_{m-1,\,s,\,l}$

at some point of $\gamma_{lt}(O_{m-1,\,s,\,l})$. Similarly a max.i.c passing through $z_1$ inside $O_{m-1,\,s,\,l}$ intersects $\partial O_{m-1,\,s,\,l}$ in some point of $\gamma_{lt}(O_{m-1,\,s,\,l})$. So there arises an $R$-quadrilateral $G_{m-1,\,s,\,l}$.

We consider the $R$-quadrilateral $G_{m-1,\,s,\,l}$ in one of the following four cases:

$c_1$) $O_{m-1,\,s,\,l}$ is a complete essential component;

$c_2$) $O_{m-1,\,s,\,l}$ is an incomplete interior essential component;

$c_3$) $O_{m-1,\,s,\,l}$ is an incomplete bounding essential component; and $G'_{m-1,\,s,\,l}$ contains two such components;

$c_4$) $O_{m-1,\,s,\,l}$ is an incomplete bounding essential component; and $G'_{m-1,\,s}$ contains one such component and

$$\nu\{T^{-r}O_{m-1,\,s,\,l} \mid G'_{m-1-r,\,s_r}\} \geqslant \frac{1}{(m-1-r)^2}$$

for the smallest $r(0 < r \leqslant m-k_0)$ for which $T^{-r}G_{m-1,\,s,\,l}$ is contained in a component satisfying one of $c_1$), $c_2$), $c_3$).

Taking any such $R$-quadrilateral $G_{m-1,\,s,\,l}$ we consider $TG_{m-1,\,s,\,l}$ and take inside $TG_{m-1,\,s,\,l}$ a max.i.c. $\gamma'$ and a min.i.c. $\gamma''$ through the image of its left upper and left lower vertices, respectively. We assert that $\gamma'$ and $\gamma''$ first intersect $\partial(TG_{m-1,\,s,\,l})$ in the decreasing curve $T\gamma_{rt}(G_{m-1,\,s,\,l})$. For if this is not so, say for $\gamma'$, then applying $T^{-1}$ to it we obtain an increasing curve passing through the upper left vertex of $G_{m-1,\,s,\,l}$ and intersecting the increasing part of the boundary, which is clearly impossible.

Thus, the curves $\gamma'$ and $\gamma''$ together with the images of the corresponding parts of the decreasing boundary of $G_{m-1,\,s,\,l}$ form $R$-quadrilaterals which we denote $G_{m,\,s'}$ $(s' = 0, 1, \ldots, N_m)$. The union of these quadrilaterals is the required set $\Pi_m$.

It is clear from the construction that $T^{-1}\Pi_m \subset \Pi_{m-1}$. The remaining unproved inductive hypothesis will be established later. In the arguments of the following point it is assumed that this has been done. We commit no logical error because in proving the inductive hypothesis we do not use the results of the next point.

I.3. PROOF OF THE INCLUSION $\bigcap\limits_{m \geqslant k_0} T^{-m}\Pi_m \subset G'$. We note that $T^{-m}G_{m,\,s}$ is a quadrilateral whose decreasing sides lie in the corresponding decreasing sides of $G$. Therefore, if $x \in \bigcap\limits_{m \geqslant k_0} T^{-m}\Pi_m$, we can apply the process described in §3 of constructing an l.e.t.f., taking an increasing curve $\gamma_m^{(m)}$ through $T^m x$ up to the intersection with the decreasing boundary of $G_{m,\,s}$, $T^m x \in G_{m,\,s}$, and taking its inverse image $T^{-m}\gamma_m^{(m)}$. As in §3, it turns out that $\gamma_m^{(0)} = T^{-m}\gamma_m^{(m)}$ converges, as $m \to \infty$, to an l.e.t.f. of $x$, which is an increasing curve with ends on the decreasing part of the boundary of $G$. By property $a_2$) of $G$ we deduce the required inclusion $\bigcap\limits_{m \geqslant k_0} T^{-m}\Pi_m \subset G'$.

PART II. PROOF OF THE INDUCTIVE HYPOTHESIS. We substitute the following inductive hypothesis for the one made earlier: for all $n \leq m-1$

$$p(\gamma_{\mathrm{rt}}(G_{n,s})) \geqslant \frac{\delta_0}{n^{200}},$$

$$|\cos\varphi(x)| \leqslant \mathrm{const}\sqrt{p(\gamma_{\mathrm{rt}}(G_{n,s}))},\quad x \in \gamma_{\mathrm{rt}}(G_{n,s}) \text{ for } G_{n,s} \subset \Pi_n';$$

$$p(\gamma_{\mathrm{rt}}(G_{n,s})) \geqslant \mathrm{const} \qquad \text{for } G_{n,s} \subset \Pi_n'';$$

$$p(\gamma_{\mathrm{rt}}(G_{n,s})) \geqslant \frac{\delta_0}{n^{150}} \qquad \text{for } G_{n,s} \subset \Pi_n'''.$$

This is obviously satisfied for $n = k_0$. It is also clear that the original hypothesis follows from the new one if $k_0$ is sufficiently large. The validity of the inductive hypothesis for $G_{n,s} \subset \Pi_n''$ follows from point 3 of the classification of the $O_{m-1,s,l}$.

II. 1. PROOF OF THE INDUCTIVE HYPOTHESIS FOR $G_{m,s} \subset \Pi_m'$. Let $G_{m,s} \subset \Pi_m'$ and $T^{-1}G_{m,s} \subset G_{m-1,s_1,l_1} \subset O_{m-1,s_1,l_1} \subset G'_{m-1,s_1} \subset G_{m-1,s_1}$. From the inductive hypothesis and the fact that $O_{m-1,s_1,l_1}$ is essential, for sufficiently large $k_0$ and $m \geqslant k_0$ we have

$$\begin{aligned}
p(\gamma_{\mathrm{rt}}(G_{m,s})) &\geqslant p(T^{-1}\gamma_{\mathrm{rt}}(G_{m,s})) \geqslant \\
&\geqslant p(\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))\left(1-\frac{2p_g(G_{m-1,s_1})}{p(\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))}\right) \geqslant \\
&\geqslant p(\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))\left(1-2(m-1)^{22}\frac{p_g(G_{m-1,s})}{p(\gamma_{\mathrm{rt}}(C_{m-1,s_1}))}\right) \geqslant \\
&\geqslant p(\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))(1-2(m-1)^{22}\lambda_2^{m-1}) \geqslant \\
&\geqslant \frac{\mathrm{const}}{(m-1)^{22}}\, p(\gamma_{\mathrm{rt}}(G_{m-1,s_1})). \qquad (27)
\end{aligned}$$

So far we have not used that $G_{m,s} \subset \Pi_m'$. If $G_{m-1,s_1} \subset \Pi_{m-1}'' \cup \Pi_{m-1}'''$, then by the inductive hypothesis

$$p(\gamma_{\mathrm{rt}}(G_{m,s})) \geqslant p(T^{-1}\gamma_{\mathrm{rt}}(G_{m,s})) \geqslant \frac{\mathrm{const}}{(m-1)^{22}}\, p(\gamma_{\mathrm{rt}}(G_{m-1,s_1})) \geqslant \frac{\mathrm{const}\,\delta_0}{(m-1)^{172}} \geqslant \frac{\delta_0}{m^{200}}$$

for sufficiently large $k_0$.

If $G_{m-1,s_1} \subset \Pi_{m-1}'$, since $\frac{p(T\,d\gamma)}{p(d\gamma)} = 1+\tau\frac{d\varphi}{dp} \geqslant \frac{\mathrm{const}}{\sqrt{p(\gamma_{\mathrm{rt}}(G_{m-1,s_1}))}}$ (see (20), (17′) and the inductive hypothesis), we have for the element $d\gamma$ of $T^{-1}\gamma_{\mathrm{rt}}(G_{m,s})$, by (27):

$$\begin{aligned}
p(\gamma_{\mathrm{rt}}(G_{m,s})) &\geqslant \frac{\mathrm{const}\, p(T^{-1}\gamma_{\mathrm{rt}}(G_{m,s}))}{\sqrt{p(\gamma_{\mathrm{rt}}(G_{m-1,s_1}))}} \geqslant \\
&\geqslant \frac{\mathrm{const}}{(m-1)^{22}}\sqrt{p(\gamma_{\mathrm{rt}}(G_{m-1,s_1}))} \geqslant \frac{\mathrm{const}\,\delta_0}{(m-1)^{122}} \geqslant \frac{\delta_0}{m^{200}}
\end{aligned}$$

It remains to show that $|\cos\varphi(x)| \leqslant \mathrm{const}\sqrt{p(\gamma_{\mathrm{rt}}(G_{m,s}))}$. Since $O_{m-1,s,l}$ is a complete component, $T\gamma_{\mathrm{rt}}(O_{m-1,s,l_1})$ is a decreasing curve one of whose ends is either $\varphi = \pi/2$ or $\varphi = 3\pi/2$. For definiteness we consider the latter case.

From (20) we have

$$|\cos\varphi(z)| \leqslant \mathrm{const}\sqrt{p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))}, \tag{28}$$

and our problem reduces to showing that $p(\gamma_{\mathrm{rt}}(G_{m,s}))$ differs by relatively little from $p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))$. Now $T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}) - \gamma_{\mathrm{rt}}(G_{m,s}) = T(\gamma' \cup \gamma'')$, where $\gamma'$ and $\gamma''$ are the connected parts of $\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}) - \gamma_{\mathrm{rt}}(T^{-1}G_{m,s})$ adjoining the left and right boundaries of $\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1})$, respectively. Let $\Delta' = [p_1', p_2']$, $\Delta'' = [p_1'', p_2'']$ be the projections of $\gamma'$ and $\gamma''$ onto the $p$-axis. Then

$$p(T\gamma') = \int_{p_1'}^{p_2'} \left[1 + \frac{d\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p)}{dp}\tau(p,\varphi)\right]dp \leqslant$$

$$\leqslant (p_2'-p_1') + \tau_{\min}[\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_1') - \varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_2')],$$

$$p(T\gamma'') = \int_{p_1''}^{p_2''} \left[1 + \frac{d\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p)}{dp}\tau(p,\varphi)\right]dp \leqslant$$

$$\leqslant (p_2''-p_1'') + \tau_{\min}[\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_2'') - \varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_1'')],$$

$$p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l})) = \int_{p_1'}^{p_2''} \left[1 + \frac{d\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p)}{dp}\tau(p,\varphi)\right]dp \geqslant$$

$$\geqslant (p_2''-p_1') + \tau_{\max}[\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_1') - \varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_2'')] \geqslant$$

$$\geqslant (p_2''-p_1') + \mathrm{const}\,|\tau_{\max}|\,\frac{(p_2''-p_1')}{\sqrt{p_2''}}.$$

From the inductive hypothesis and the fact that $O_{m-1,s_1,l_1}$ is essential, it follows that

$$\frac{p_2'-p_1'}{p_2''-p_1'} \leqslant m^{22}\lambda_2^{m-1}, \qquad \frac{p_2''-p_1''}{p_2''-p_1'} \leqslant m^{22}\lambda_2^{m-1}.$$

Next assume for definiteness that $G_{m-1,s}$ lies in the region $\varphi \geqslant \pi$ (the other case is analysed similarly). It is not difficult to see from the properties of decreasing curves that

$$\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_1') - \varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_2') \leqslant \mathrm{const}(\sqrt{p_2'} - \sqrt{p_1'}) \leqslant$$

$$\leqslant \mathrm{const}\sqrt{p_2'}\left(1 - \sqrt{1 - \frac{p_2'-p_1'}{p_2'}}\right) = \frac{\mathrm{const}(p_2'-p_1')}{\sqrt{p_2'}\left(1+\sqrt{1-\frac{p_2'-p_1'}{2}}\right)} \leqslant \mathrm{const}\frac{p_2'-p_1'}{\sqrt{p_2'}},$$

$$\varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_1'') - \varphi^{(\mathrm{rt})}_{G_{m-1,s_1}}(p_2'') \leqslant \mathrm{const}(\sqrt{p_2''} - \sqrt{p_1''}) \leqslant \frac{\mathrm{const}(p_2''-p_1'')}{\sqrt{p_2''}}$$

and since $\frac{\tau_{\min}}{\tau_{\max}} \leqslant \text{const}$, it follows that

$$\frac{p(T\gamma')+p(T\gamma'')}{p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))} \leqslant 2m^{22}\lambda_2^{m-1} + $$
$$+ \text{const}\left[\frac{(p_2'-p_1')\sqrt{p_2''}}{\sqrt{p_2'}(p_2''-p_1')} + \frac{(p_2''-p_1'')\sqrt{p_2''}}{\sqrt{p_2''}(p_2''-p_1')}\right] \leqslant$$
$$\leqslant 2m^{22}\lambda_2^{m-1} + \text{const}\left[\sqrt{\frac{p_2'-p_1'}{p_2''-p_1'}} + \sqrt{\frac{p_2''-p_1''}{p_2''-p_1'}}\right] \leqslant \text{const}\, m^{22}\lambda_2^{m/2}.$$

Consequently,

$$p(\gamma_{\mathrm{rt}}(G_{m,s})) = \left(1 - \frac{p(T\gamma')+p(T\gamma'')}{p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1}))}\right) p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1})) \geqslant$$
$$\geqslant \text{const}\, p(T\gamma_{\mathrm{rt}}(O_{m-1,s_1,l_1})).$$

Substituting this in (28) we obtain the required assertion.

II.2. Proof of the inductive hypothesis for $G_{m,s} \subset \Pi_m'''$. Let $T^{-1}G_{m,s} \subset O_{m-1,s_1,l_1}$ and let $O_{m-1,s_1,l_1}$ belong to case c$_4$) of 1.2.

Choose $r$, fixed in c$_4$), and put $\approx G = T^{-r-1}G_{m,s}$. For definiteness let $T^{-r}\gamma_{\mathrm{rt}}(G_{m,s})$ belong to part of some component $M_{1i}$, where $\varphi > \pi$. For the $R$-quadrilateral $G_{m-r,s_r} \supset T^{-r}G_{m,s}$ we include $\gamma_{\mathrm{rt}}(G_{m-r,s_r})$ in a decreasing curve finishing at $\varphi = \frac{3\pi}{2}$ and assume that the origin on the $p$-axis is consistent with this curve. From (22'), (17') and (20)

$$p(T^{-r}\gamma_{\mathrm{rt}}(G_{m,s})) = p(\gamma_{\mathrm{rt}}(T^{-r}G_{m,s})) \leqslant$$
$$\leqslant \text{const} \max_{x\in\gamma_{\mathrm{rt}}(T^{-r}G_{m,s})} |\cos\varphi(x)|\, p(\gamma_{\mathrm{rt}}(G_{m,s})) \leqslant \text{const}\sqrt{p_{\mathrm{rt}}}\cdot p(\gamma_{\mathrm{rt}}(G_{m,s})),$$

where $p_{\mathrm{rt}}$ is the $p$-coordinate of the right end of $\gamma_{\mathrm{rt}}(T^{-r}G_{m,s})$.

Assume that the inductive hypothesis is not satisfied for $G_{m,s}$, that is, $p(\gamma_{\mathrm{rt}}(G_{m,s})) < \frac{\delta_0}{m^{150}}$. In this case

$$p(T^{-r}\gamma_{\mathrm{rt}}(G_{m,s})) \leqslant \frac{\text{const}\sqrt{p_{\mathrm{rt}}}\cdot\delta_0}{m^{150}}.$$

Choose $\gamma''$, $\gamma_{\mathrm{rt}}(T^{-r}G_{m,s}) \subset \gamma'' \subset \gamma_{\mathrm{rt}}(G_{m-r,s_r})$, $p(\gamma'') = \frac{\delta_0\sqrt{p_{\mathrm{rt}}}}{m^{115}}$. Obviously this can be done if and only if $p(\gamma'') \leqslant p(\gamma_{\mathrm{rt}}(G_{m-r,s_r}))$. We show that this condition is satisfied in our case. If $T^{-r-1}G_{m,s}$ lies in a component of case c$_2$) or c$_3$), then the condition is satisfied by (27) and the fact that $O_{m-r-1,s_{r+1},l_{r+1}} \supset T^{-r-1}G_{m,s}$ is essential. If $O_{m-r-1,s_{r+1},l_{r+1}}$ is in case c$_1$), then by (21')

$$p_{\mathrm{rt}} \leqslant \text{const}\, p(T\gamma_{\mathrm{rt}}(O_{m-r-1,s_{r+1},l_{r+1}})) \leqslant \text{const}\, p(\gamma_{\mathrm{rt}}(G_{m-r,s_r})).$$

If $k_0$ is sufficiently large, then by the inductive hypothesis

$$\frac{\delta_0 \sqrt{p_{rt}}}{m^{115}} \leqslant \frac{\mathrm{const}\ \delta_0 \sqrt{p(\gamma_{rt}(G_{m-r,\,s_r}))}}{m^{115}} \leqslant$$
$$\leqslant \frac{\mathrm{const}\sqrt{\delta_0}}{(m-r)^{115}} \sqrt{p(\gamma_{rt}(G_{m-r,\,s_r}))} \leqslant p(\gamma_{rt}(G_{m-r,\,s_r})),$$

that is, our condition is also satisfied.

We construct a quadrilateral $\tilde{G} \subset G_{m-r,\,s_r}$, $\gamma_{rt}(\tilde{G}) = \gamma''$, $\tilde{\tilde{G}} \subset \tilde{G}$.

An upper estimate for the ratio $\nu(T^{-r} G_{m,\,s}) : \nu(G_{m-r,\,s_r})$. We use the estimates of §5, 2°, which were obtained for the study of the properties of quadrilaterals. First we find an estimate for $\Phi$. Let $x_{ru}$, $x_{rl}$, $x_{ll}$ be the upper right, lower right and lower left vertices of $G$. It is clear that

$$\max \frac{\cos\varphi(x')}{\cos\varphi(x'')} = \frac{\cos\varphi(x_{rl})}{\cos\varphi(x_{ll})} \cdot \frac{\cos\varphi(x_{ru})}{\cos\varphi(x_{rl})}\,.$$

We estimate the first ratio. From the definition of a decreasing curve and the fact that $\tilde{\tilde{G}} \subset \tilde{G}$, that is, $\sqrt{p_{rt}} \geqslant \frac{\mathrm{const}\ \delta_0}{(m-r)^4}$, it follows easily that

$$\left|\frac{\cos\varphi(x_{ru})}{\cos\varphi(x_{rl})} - 1\right| = \left|\frac{\cos\varphi(x_{ru}) - \cos\varphi(x_{rl})}{\cos\varphi(x_{rl})}\right| \leqslant$$
$$\leqslant \frac{\mathrm{const}\ p(\gamma_{rt}(\tilde{G}))}{\cos^2\varphi(x_{ru})} \leqslant \frac{\mathrm{const}\ (m-r)^{-115}\cdot\delta_0\sqrt{p_{rt}}}{p_{rt} - \delta_0 m(-r)^{-115}\sqrt{p_{rt}}} \leqslant \frac{\mathrm{const}}{(m-r)^{111}}\,.$$

To estimate $\cos\varphi(x_{rl}) : \cos\varphi(x_{ll})$ we include $\gamma_1(\tilde{G})$ in an increasing curve ending at $\frac{3\pi}{2}$ and consider the coordinate $p$ in relation to this curve. The resulting values of $p$ are marked with a prime. If $p'_{rt}$ is the $p'$-coordinate of $x_{rl}$, $\Delta p' = p'(\gamma_1(\tilde{G}'))$, then $p'_{rt} : p_{rt} \leqslant$ const (see (21)), $\frac{\Delta p' \lambda^{-(m-r)/2}}{\sqrt{p'_{rt}}} \leqslant \mathrm{const}\ \delta_0$, that is, $\Delta p' \leqslant \mathrm{const}\ \sqrt{p'_{rt}}\,\lambda^{(m-r)/2}\delta_0$. As above,

$$\left|\frac{\cos\varphi(x_{rl})}{\cos\varphi(x_{ll})} - 1\right| \leqslant \frac{\mathrm{const}\ \Delta p'}{\cos^2\varphi(x_{rl})} \leqslant \frac{\mathrm{const}\Delta p'}{p'_{rt}} \leqslant \frac{\mathrm{const}\ \delta_0 \lambda^{(m-r)/2}}{\sqrt{\Delta p'_{rt}}} \leqslant \mathrm{const}\left(\frac{m-r}{2}\right)^4 \lambda^{(m-r)/2}.$$

Finally we obtain

$$\max_{x',\,x'' \in \tilde{G}} \frac{\cos\varphi(x')}{\cos\varphi(x'')} \leqslant 1 + \frac{\mathrm{const}}{(m-r)^{111}}\,, \tag{29}$$

that is, $\Phi \leqslant \frac{\mathrm{const}}{(m-r)^{111}}$.

Now we estimate the value of $D$. This obviously reduces to an estimation of

$$\exp\{c_i\} = \max_{x \in \gamma_1} \frac{\cos\varphi(T^{-i}x)}{\cos\varphi(T^{-i}y)}\,.$$

Choose for $G$ the quadrilateral constructed in I.1. and consider a smooth partition $\xi_0$ of it. We denote by $\xi_{m-r}$ the image of the partition in $G_{m-r,\, s_r}$, as in §5. We estimate the ratio $\cos\varphi(T^{-s}x_1): \cos\varphi(T^{-s}x_2)$ for any pair $x_1,\ x_2 \in \tilde{G}$ belonging to the same $C_{\xi_{m-r}}$. Suppose that at least one of the points $T^{-s}x_1,\ T^{-s}x_2$ lies in the upper half of some regular component $M_{1i}$. If this is not so, then all the subsequent arguments are replaced by symmetrical ones.

We include $C_{\xi_{m-r-s}}$, containing $T^{-s}x_1,\ T^{-s}x_2$, in a decreasing curve ending at $\varphi = \frac{3\pi}{2}$ and assume that $p$ is consistent with the curve. If $p_1^{(s)} = p(T^{-s}x_1),\ p_2^{(s)} = p(T^{-s}x_2),\ p_2^{(s)} \geqslant p_1^{(s)},\ \Delta p^{(s)} = p_2^{(s)} - p_1^{(s)}$, then

$$\frac{\text{const}\,\Delta p^{(s)}}{\sqrt{p_2^{(s)}}} \leqslant \Delta p^{(s-1)} \quad \text{and}$$

$$\left|\frac{\cos\varphi(T^{-s}x_1)}{\cos\varphi(T^{-s}x_2)} - 1\right| \leqslant \frac{\text{const}\,\Delta p^{(s)}}{\cos^2\varphi(T^{-s}x_1)} \leqslant \frac{\text{const}\,\Delta p^{(s)}}{p_2^{(s)} - \Delta p^{(s)}} \leqslant \frac{\text{const}\,\Delta p^{(s-1)}\sqrt{p_2^{(s)}}}{p_2^{(s)} - \sqrt{p_2^{(s)}}\,\Delta p^{(s-1)}} \leqslant$$

$$\leqslant \frac{\text{const}\,\Delta p^{(s-1)}}{\sqrt{p_2^{(s)}} - \Delta p^{(s-1)}} \leqslant \frac{\text{const}\,\lambda^{s-1}(m-r-s)^{-115}\delta_0}{\sqrt{p_2^{(s)}} - \text{const}\,\lambda^{s-1}(m-r-s)^{-115}\delta_0} .$$

We use the fact that $T^{-s}x_1,\ T^{-s}x_2$ lie outside $V_{m-r-s,\,4}$. This gives $\sqrt{p_2^{(s)}} \geqslant \text{const}\ (m-r-s)^{-4}\delta_0$. Finally we obtain

$$\left|\frac{\cos\varphi(T^{-s}x_1)}{\cos\varphi(T^{-s}x_2)} - 1\right| \leqslant \text{const}\,\lambda^{s-1}(m-r-s)^{-111},$$

that is, $c_s = \text{const}\ \lambda^{s-1}(m-r-s)^{-111}$. Substituting this in the expression for $D$ and assuming that the inductive hypothesis is not satisfied, we may write

$$\frac{\nu(T^{-r}G_{m,\,s})}{\nu(G_{m-r,\,s_r})} \leqslant \frac{\nu(\tilde{\tilde{G}})}{\nu(\tilde{G})} \leqslant$$

$$\leqslant \text{const}\,K_0 \cdot \prod_{i=1}^{m-r}\left(1 + \text{const}\,q_1^i + \text{const}\,\lambda^i(m-r-i)^{-111}\right)\frac{p(\gamma_{rt}(\tilde{\tilde{G}})) + 2p_g(\tilde{G})}{p(\gamma_{rt}(\tilde{G})) - 2p_g(\tilde{G})} \leqslant$$

$$\leqslant \text{const}\,K_0 \cdot \frac{p(\gamma_{rt}(\tilde{\tilde{G}}))}{p(\gamma_{rt}(\tilde{G}))} \leqslant \text{const}\,K_0 m^{-35}.$$

Note that $K_0$ depends only on $x_0$.

A lower estimate of the ratio $\nu(T^{-r}G_{m,\,s}): \nu(G_{m-r,\,s_r})$.

We show that by the construction we have

$$\frac{\nu(T^{-r}G_{m,\,s})}{\nu(G_{m-r,\,s_r})} \geqslant \frac{\text{const}}{(m-r)^2} .$$

The contradiction with the previous estimates proves the validity of the inductive hypothesis.

By construction (see $c_4$)), if $T^{-1}G_{m,\,s} \subset O_{m-1,\,s_1,\,l_1}$ then

$$\nu(T^{-r}O_{m-1,\,s_1,\,l_1}) \geqslant \frac{1}{(m-r)^2}\,\nu(G'_{m-r-1,\,s_{r+1}}).$$

We show that $\nu(T^{-1}G_{m,s}) \geqslant \text{const}\ \nu(O_{m-1,s_1,l_1})$. Hence it follows that

$$\frac{\nu(T^{-r}G_{m,s})}{\nu(G_{m-r,s_r})} = \frac{\nu(T^{-1}G_{m,s})}{\nu(T^{-1}G_{m-r,s_r})} \geqslant \frac{\text{const}\,\nu(O_{m-1,s_1,l_1})}{\nu(G'_{m-r-1,s_{r+1}})} \cdot \frac{\nu(G'_{m-r-1,s_{r+1}})}{\nu(T^{-1}G_{m-r,s_r})} \geqslant$$
$$\geqslant \frac{\text{const}\,\nu(T^{-r}O_{m-1,s_1,l_1})}{\nu(G'_{m-r-1,s_{r+1}})} \geqslant \frac{\text{const}}{(m-r)^2}$$

It remains to show that

$$\nu(T^{-1}G_{m,s}) \geqslant \text{const}\,\nu(O_{m-1,s_1,l_1}).$$

As before, we assume that $\gamma_{rt}(G'_{m,s})$ is included in a decreasing curve ending, for definiteness, in $\varphi = \frac{3\pi}{2}$, and that the origin on the $p$-axis is consistent with this curve. $O_{m-1,s_1,l_1} - T^{-1}G_{m,s}$ consists of two quadrilaterals $G_u$, $G_l$ joined to the upper and lower ends of $\gamma_{rt}(O_{m-1,s_1,l_1})$, respectively. It is easy to see that $p(\gamma_{rt}(G_u)) \leqslant \text{const}\ \lambda^{m-1}\sqrt{p_{rt}(G_u)}\cdot\delta_0$, where $p_{rt}(G_u)$ is the coordinate of the upper end of $\gamma_{rt}(O_{m-1,s_1,l_1})$. We construct a quadrilateral $G''$, $G_u \subset G'' \subset O_{m-1,s_1,l_1}$ for which $p(\gamma_{rt}(G'')) = \frac{\sqrt{p_{rt}}\,\delta_0}{m^{250}}$. This can be done because $O_{m-1,s_1,l_1}$ is an essential component and $p(\gamma_{rt}(O_{m-1,s_1,l_1})) \geqslant \frac{\delta_0}{m^{222}}$ by the inductive hypothesis.

The same inequality as for the upper estimate of $\nu(T^{-r}G_{m,s}):\nu(G_{m-r,s_r})$ gives

$$\frac{\nu(G_u)}{\nu(O_{m-1,s_1,l_1})} \leqslant \frac{\nu(G_u)}{\nu(G'')} \leqslant \text{const}\,m^{250}\lambda^{m-1}.$$

The ratio $\nu(G_l):\nu(O_{m-1,s_1,l_1})$ is estimated similarly. Thus,

$$\frac{\nu(T^{-1}G_{m,s})}{\nu(O_{m-1,s_1,l_1})} \geqslant 1 - \text{const}\,m^{250}\lambda^{m-1}.$$

The inductive hypothesis is now completely proved.

PART III. AN ESTIMATE OF $\nu(\bigcap_{m\geqslant k_0} T^{-m}\Pi_m)$. We actually estimate $\nu(G - \bigcap_{m\geqslant k_0} T^{-m}\Pi_m)$.

III.1. We estimate the measure of the $R$-quadrilaterals $G_{m,s}$ for which $G_{m,s} \subset V_{m,\frac{3}{2}}$. Note that $\nu(V_{m,\frac{3}{2}}) \leqslant \frac{\delta_0^2}{m^3}$. For in the original variables $(r,\varphi)$ the domain $V_{m,\frac{3}{2}}$ has the form $\left\{(r,\varphi): |\cos\varphi| \leqslant \frac{\delta_0}{m^{3/2}}\right\}$. Therefore

$$\nu(V_m) = \text{const}\int\limits_{\frac{3\pi}{2}-\frac{\delta_0}{m^{3/2}}}^{\frac{3\pi}{2}} |\cos\varphi|\,d\varphi \leqslant \frac{\text{const}\,\delta_0^2}{m^3},$$

and the sum of the measures of the $R$-quadrilaterals that were left out at the beginning does not exceed $\frac{\mathrm{const}\,\delta_0^2}{k_0^2}$ and can be made smaller than $\frac{\alpha}{4}\nu(G)$, provided that $k_0$ is sufficiently large, since $\nu(G) \geqslant \mathrm{const}\ \delta_0^2$.

III.2. For a complete essential component or an incomplete essential component $O_{m-1,\,s_1,\,l_1}$ in cases c$_1$), c$_2$), c$_3$), from which the construction of $\Pi_m$ was derived, it follows from the estimates at the end of §5 that

$$\nu(O_{m-1,\,s_1,\,l_1}) - \nu(O_{m-1,\,s_1,\,l_1} \cap T^{-1}\Pi_m) \leqslant \mathrm{const}\, m^{250}\lambda_2^m \cdot \nu(O_{m-1,\,s_1,\,l_1}).$$

Therefore for the sum over such components

$$\sum_{m,\,s,\,l} [\nu(O_{m-1,\,s_1,\,l_1}) - \nu(O_{m-1,\,s_1,\,l_1} \cap T^{-1}\Pi_m)] \leqslant$$

$$\leqslant \mathrm{const} \sum_{m \geqslant k_0} m^{250}\lambda_2^m \sum_{s,\,l} \nu(O_{m-1,\,s,\,l}) \leqslant \mathrm{const} \sum_{m\geqslant k_0} m^{250}\lambda_2^m \nu(G) \leqslant \frac{\alpha}{4}\nu(G),$$

if $m \geqslant k_0$ and $k_0$ is sufficiently large.

III.3. The intersection $O_{m,\,s,\,l} \cap T^{-1}\Pi_{m+1} = \varnothing$, if $O_{m,\,s,\,l}$ is in case c$_4$) and $\nu(T^{-r}O_{m,\,s,\,l} \mid G_{m-r,\,s_r}) \leqslant \frac{1}{(m-r)^2}$ (see I.2.). In this situation, by construction, $T^{-1}O_{m,\,s,\,l}, \ldots, T^{-r+1}O_{m,\,s,\,l}$ are contained in components in case c$_4$), and for each $G_{m-r,\,s_r} \subset \Pi_m$ there is not more than one $O_{m,\,s,\,l}$ that is left out. Therefore the sum over those components is

$$\sum_{m,\,s,\,l} \nu(O_{m,\,s,\,l}) \leqslant \sum_{m-r\geqslant k_0} \frac{1}{(m-r)^2} \sum_{G_{m-r,\,s_r}} \nu(G_{m-r,\,s_r}) \leqslant \nu(G)\,\frac{\alpha}{4},$$

if $k_0$ is sufficiently large.

III.4. The last estimate is that of the number of inessential components and components of the second type. We show that by our choice of parameters the number of inessential components is bounded by a fixed constant.

The number of curves $\Sigma_i$ is, in general, infinite. The accumulation points of these curves are the I points $(p_1, \varphi_1), \ldots, (p_I, \varphi_I)$ that define periodic trajectories all of whose boundary reflections reduce to tangency and where, at the tangency points, $\partial Q$ lies on one side of the trajectory. Fix spherical neighbourhoods $U_1', \ldots, U_I'$ of these points such that $U_i' \cap U_j' = \varnothing$ for $i \neq j$ and let $U_1, \ldots, U_I$ be concentric neighbourhoods of half the radius. From the properties of the $\Sigma_j$ in the $U_i$ it follows (see §1) that the number of $\Sigma_j$ intersecting the complement of $\bigcup_i U_i$ is finite. Consequently, for each $G'_{m,\,s}$ the number of inessential components and components of the second type for which these $\Sigma_j$ are in their boundaries is finite and does not exceed some constant $P$.

On the $(p, \varphi)$-plane the curves $\Sigma_j$ in $U_i (i = 1, \ldots)$ (see §1 and Fig.3.) have the following form: there exists one curve $\Sigma_{i_0}$ and other curves $\Sigma_{i_1}, \Sigma_{i_2}, \ldots$ one of whose ends is in $\varphi = \frac{3\pi}{2}$ $\left(\text{or } \varphi = \frac{\pi}{2}\right)$, and the others are in $\Sigma_{i_0}$, and, consequently, are double points. We assume that these curves are enumerated so that $\mathrm{const} \leqslant \frac{t}{|\tau(\Sigma_{i_t})|} \leqslant \mathrm{const}$, where $\tau(\Sigma_{i_t}) = \min_{x \in \Sigma_{i_t}} \tau(x)$. Each $\Sigma_{i_t}$ ($t = 1, \ldots$) is given by a function $\varphi_t(r)$ defined on a segment $\Delta r_t \geqslant \frac{\mathrm{const}}{t^2}$. It is not difficult to see that the distance between two successive curves is not less than $\frac{\mathrm{const}}{t^4}$. In the variables $(p, \varphi)$ this is equivalent to saying that $\Sigma_{i_t}$ is given by an increasing function $\varphi_t(p)$ defined on a segment $\Delta p_t \geqslant \frac{\mathrm{const}}{t^4}$, and so, for any increasing curve $\gamma$ whose ends lie on $\Sigma_{i_t}$ and $\Sigma_{i_{t+1}}$,

$$p(\gamma) \geqslant \frac{\mathrm{const}}{t^8}.$$

Returning to our situation consider an $R$-quadrilateral $G'_{m,s}$, $G'_{m,s} \cap U_l \neq \varnothing$. Since $G'_{m,s} \cap V_{m,4} = \varnothing$, it follows that $G'_{m,s}$ intersects only the $\Sigma_{i_t}$ for which $t \leqslant \mathrm{const}\, m^2$. Consequently in $\gamma_{rt}(G'_{m,s})$, if we denote $z_t = z(p_t, \varphi_t) = \gamma_{rt}(G'_{m,s}) \cap \Sigma_{i_t}$, then for any two $z_t$, $z_{t_1}$ we have $|p_t - p_{t_1}| \geqslant \frac{\mathrm{const}}{m^{16}}$. In other words, components for which both ends of $\gamma_{rt}(O_{m,s,l})$ lie in $\bigcup_t \Sigma_{i_t}$ turn out to be essential. The number of remaining components does not exceed 2. Thus, the total number of inessential components and components of the second type does not exceed $P + 2 = \mathrm{const}$. For inessential components and components of the second type we have by analogy to §5.

$$\frac{\nu(O_{m,s,l})}{\nu(G_{m,s})} \leqslant \frac{\mathrm{const}}{m^{10}}.$$

Thus, if $\bigcup'_{m,s,l} O_{m,s,l}$ is the union of the inessential components and the components of the second type, then

$$\nu\Big(\bigcup'_{m,s,l} O_{m,s,l}\Big) \leqslant \sum_{m,s} \nu(G_{m,s}) \cdot \frac{(P+2)\,\mathrm{const}}{m^{10}} \leqslant$$

$$\leqslant (P+2)\,\mathrm{const} \sum_{m \geqslant k_0} \frac{1}{m^{10}} \cdot \sum_s \nu(G_{m,s}) \leqslant \frac{\alpha}{4}\nu(G),$$

if $k_0$ is sufficiently large.

III.5. RESULT. The area of the quadrilateral $G$ is greater than $\mathrm{const}\ \delta_0^2$. In our construction we reject quadrilaterals $G_{m,s}$ that intersect $V_{m,4}$ but not the complement of $V_{m,3/2}$, and we replace the remaining quadrilaterals by $G'_{m,s}$. From III.1. it follows that the area lost does not exceed $\alpha/4\ \nu(G)$. We then lose area under the passage from essential components to quadrilaterals contained in them. The total loss is estimated in III.2. and does not exceed $\alpha/4\ \nu(G)$. Finally, by the rejection of inessential components and components of the second type we lose an area not exceeding $\alpha/4\ \nu(G)$ (see III.4.) and by the rejection of essential components of relatively small measure we lose an area not exceeding $\alpha/4\ \nu(G)$ (see III.3.). Consequently the total area that we reject does not exceed $\alpha\nu(G)$. The proof of the theorem is now complete.

Replacing $T$ by $T^{-1}$ and decreasing curves by increasing curves we obtain Theorem 6.1′. in precisely the same way.

THEOREM 6.1′. *Let $x_0$ be such that $T^{-i}x_0$ is never a singular point of the boundary, that is,* $\cos\varphi(T^{-i}x_0)\neq 0$ *for* $i\geqslant 0$. *Then for any* $\alpha(0<\alpha\leqslant 1)$ *and* $C<\infty$ *there is a neighbourhood* $U_\varepsilon$ *of* $x_0$ *with the following property: for any increasing curve* $\gamma\subset U$, $p(\gamma)=\delta_0$ *we can construct a quadrilateral* $G$ *for which* $\gamma=\gamma_u(G)$, *and if* $G'=\{x\colon x\in G$ *and there is a regular segment of the l.c.t.f.* $\gamma^{(c)}(x)$ *through* $x$ *which intersects the upper and lower sides of* $G$, $p(\gamma^{(c)}(x))\geqslant C\delta_0\}$, *then* $\nu(G')\geqslant(1-\alpha)\nu(G)$. *We can also construct a quadrilateral with similar property for which* $\gamma$ *is the lower side.*

## §7. An estimate of the number of ergodic components. Final results

THEOREM 7.1. *Let* $x_0\in M_1$ *be such that* $\cos\varphi(T^ix_0)=0$ *for not more than one value of* $i(-\infty<i<\infty)$. *Then there exists a neighbourhood of* $x_0$ *that is contained* mod 0 *in one ergodic component of* $T$.

The proof of this theorem is also a generalization of a method of Hopf (see [4], [11], [18]). Let $f_1, f_2, \ldots$ be a countable set of continuous functions on $M_1$ that is dense in $L^2_\nu$. Put

$$\bar f_i(x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{s=0}^{n}f_i(T^sx),$$

where $\lim\limits_{n}\frac{1}{n+1}\sum\limits_{s=0}^{n}f_i(T^sx)=\bar f_i^+(x)$ exists and is equal to $\lim\frac{1}{n+1}\sum\limits_{s=0}^{n}f_i(T^{-s}x)=\bar f_i^-(x)$, and to 0 otherwise. We use the following simple fact whose verification is left to the reader: if in some neighbourhood $U$ of $x_0$ there is a subset $\tilde U\subset U$, $\nu(\tilde U)=\nu(U)$ satisfying the relations: for any $x', x''\in\tilde U$

$$\bar f_i(x')=\bar f_i(x'')\qquad(i=1,2,\ldots),$$

then $\tilde U$ is contained mod 0 in one ergodic component.

Let $U$ be a neighbourhood of $x_0$, $U'=\{x\colon x\in U$ and $\bar f_i^+(x)=\bar f_i^-(x)$ for all $i=1,2,\ldots\}$ and let $U''\subset U$ consist of those $x$ such that for the partitions $\zeta^+$, $\zeta^-$ into regular segments of the expanding and contracting foliations, respectively (see §4), the local fibres $C_{\zeta^+}(x)$, $C_{\zeta^-}(x)$ are non-empty and $\nu\{U'\mid C_{\zeta^+}(x)\}=\nu\{U'\mid C_{\zeta^-}(x)\}=1$. It is clear that $\nu(U)=\nu(U')=\nu(U'')$.

We show that if $U$ is a sufficiently small neighbourhood of $x_0$, then for any two points $x', x''\in U'\cap U''=\tilde U$ there is a finite chain of local fibres $\gamma_1,\gamma_2,\ldots,\gamma_t$ such that 1) $x'\in\gamma_1$, $x''\in\gamma_t$, 2) $\gamma_s\cap\gamma_{s-1}\subset U'$ and is non-empty for $s=2,\ldots,t$, and 3) if $\gamma_s$ is an l.e.t.f., then $\gamma_{s+1}$ is an l.c.t.f., and conversely. Let us derive the assertion of the theorem from this. Assume, for definiteness, that $\gamma_1$ and $\gamma_t$ are l.e.t.f. and put $y_s = \gamma_s \cap \gamma_{s+1}$ $(s = 1, 2, \ldots, t-1)$. From the properties of the chain $\gamma_1, \ldots, \gamma_t$ and the properties of transversal fibres it easily follows that for any $i$

$$\bar{f}_i(x') = \bar{f}_i^+(x') = \bar{f}_i^+(y_1) = \bar{f}_i^-(y_1) = \bar{f}_i^-(y_2) = \bar{f}_i^+(y_2) =$$
$$= \bar{f}_i^+(y_3) = \bar{f}_i^-(y_3) = \ldots = \bar{f}_i^+(y_{t-1}) = \bar{f}_i^+(x'') = \bar{f}_i(x''),$$

that is, $\bar{f}_i(x') = \bar{f}_i(x'')$. So the theorem follows from the general proposition stated above and the existence of a chain with the properties 1) - 3).

The construction of the chain $\gamma_1, \ldots, \gamma_t$. First let $x_0$ be such that the trajectory defined by it is never tangent to the boundary.

Let $x_0 = (p_0, \varphi_0)$, $x' = (p', \varphi')$, $x'' = (p'', \varphi'')$. Consider the rectangular neighbourhood $U = \{x: |p - p_0| \leqslant \rho_0, |\varphi - \varphi_0| \leqslant \rho_1\}$, $\rho_0, \rho_1$ are positive numbers. For some $\alpha$, $C$ $(0 < \alpha < 1;\ C < \infty)$ whose values will be given later choose $U$ so that Theorems 6.1. and 6.1$'$ apply to it.

Since $x' \in U''$, there is a regular segment $\gamma_1^{(c)}$ of the contracting foliation that passes through $x'$. This segment is given by a decreasing function $\varphi_0(p)$ defined in the interval $|p - p'| < \delta'$, for some $\delta' > 0$. We construct a strip $\Pi' = \{x: |\varphi - \varphi'| < \delta_0'\}$ where $\delta_0'$ is chosen so that every increasing curve $\gamma'$, $p(\gamma') \geqslant \delta'$ beginning in the lower half $\{\varphi - \varphi' < \delta_0'\}$ intersects the upper boundary $\{\varphi = \varphi' + \delta_0'\}$, and every decreasing curve $\gamma''$, $p(\gamma'') \geqslant \delta'$, beginning in the upper part $\{\varphi - \varphi' < \delta_0'\}$, intersects the lower boundary $\{\varphi = \varphi' - \delta_0'\}$. In particular, $\gamma_1^{(c)}$ intersects the upper and lower boundaries of $\Pi'$.

The choice of $C$. Take a strip $\Pi_1'$ of less than half the width, that is, $\Pi_1' = \left\{x: |\varphi - \varphi'| < \frac{1}{2}\delta_0\right\}$. Choose $C$ so that each increasing curve $\gamma \subset \Pi'$ whose point with least value $\varphi_{min}$ lies below $\Pi_1'$, that is, $\varphi' - \delta' < \varphi_{min} < \varphi' - \frac{1}{2}\delta'$ and $p(\gamma) \geqslant C\delta'$, intersects the upper boundary of $\Pi'$, that is, the line $\varphi = \varphi' + \delta'$, and an analogous condition is satisfied for decreasing curves beginning in the upper part of $\Pi'$, that is, such that $\varphi' + \frac{1}{2}\delta' < \varphi_{max} < \varphi' + \delta'$.

The choice of $\alpha (0 < \alpha \lesssim 1)$. If $\alpha$ is sufficiently near to 1, then for each decreasing curve $\gamma$ described in the last paragraph, and for the quadrilateral $G$ constructed by Theorem 6.1. whose left side is $\gamma$ we have $\nu\{x: x \in G'$ and the left end of the l.e.t.f. $C_{\zeta^+}(x)$ lies in the lower strip of the difference $\Pi' - \Pi_1'\} > 0$.

An analogous assertion holds for increasing curves.

Having the curve $\gamma_1^{(c)}$ we take its restriction to the strip $\Pi'$. Without fear of confusion we also denote the restriction by $\gamma_1^{(c)}$. Theorem 6.1. gives us a quadrilateral $G$, and by our conditions we may take an l.e.t.f. $\gamma_2^{(e)}$ inside it whose left end lies in the lower strip of $\Pi' - \Pi_1'$ and belongs to $U' \cap U''$ and $\gamma_2^{(e)} \subset U''$. Take the restriction of $\gamma_2^{(e)}$ in $\Pi'$ and construct, by Theorem 6.1$'$, a quadrilateral and take inside it an analogous l.c.t.f. beginning in the upper strip of $\Pi' - \Pi_1'$ and ending in the lower side $\varphi = \varphi' - \delta_0'$, etc. Continuing this process in the same way to the left of $x'$ we obtain a chain $\gamma_1', \gamma_2', \dots, \gamma_{t_1}'$ such that if $\gamma_s'$ is an l.e.t.f., then $\gamma_{s+1}'$ is an l.c.t.f., $\gamma_s' \subset U''$, $\gamma_s' \cap \gamma_{s+1}' \subset \Pi' \cap U'$, $\gamma_1'$ intersects the left boundary of $U$, $\gamma_{t_1}'$ intersects the right boundary of $U$. Obviously this chain can be constructed so that $y_s = \gamma_s' \cap \gamma_{s+1}'$ is for all $s$, a condensation point of $U' \cap \gamma_s'$.

Similarly we can construct for $x''$ a vertical strip $\Pi'' = \{x: |p - p''| \leq \delta''\}$ and a chain $\gamma_1'', \dots, \gamma_{t_2}''$, where $\gamma_{s+1}''$ is an l.e.t.f. if $\gamma_s''$ is an l.c.t.f., $\gamma_1''$ intersects the upper boundary of $U$, $\gamma_{t_2}''$ intersects the lower boundary of $U$, $\gamma_s'' \cap \gamma_{s+1}'' \in \Pi'' \cap U''$ and is a condensation point for $U' \cap \gamma_s$, $x'' \in \gamma_{j_0}''$.

Obviously the chains $\gamma_1', \dots, \gamma_{t_1}'$ and $\gamma_1'', \dots, \gamma_{t_2}''$ intersect in at least one point.

Let $\gamma_{r_1}' \cap \gamma_{r_2}'' \neq \varnothing$ and $y_{r_1-1}' = \gamma_{r_1}' \cap \gamma_{r_1-1}'$, $y_{r_1}' = \gamma_{r_1}' \cap \gamma_{r_1+1}'$, $y_{r_2-1}'' = \gamma_{r_2}'' \cap \gamma_{r_2-1}''$, $y_{r_2}'' = \gamma_{r_2+1}'' \cap \gamma_{r_2}''$. Construct a quadrilateral $G_1$ one of whose sides is a neighbourhood of $y_{r_1-1}'$ in $\gamma_{r_1+1}'$, and the opposite side is a neighbourhood of $y_{r_1}'$ in $\gamma_{r_1+1}'$, the other pair of opposite sides being formed from segments of transversal fibres of the same monotonicity as $\gamma_{r_1}'$. Construct an analogous quadrilateral $G_2$ for the other chain. If the neighbourhoods of the points $y_{r_1-1}'$, $y_{r_1}'$, $y_{r_2-1}''$, $y_{r_2}''$ are sufficiently small, then each local regular fibre in $G_1$ intersects each local regular fibre in $G_2$. Clearly $\nu(G_1 \cap G_2) > 0$. In addition, by the absolute continuity of the foliations (see §4 and [4], [11]), the local regular fibres whose ends belong to $U'$ form a set of full measure in both $G_1$ and $G_2$. Therefore we can find fibres $\tilde{\gamma}_{r_1} \subset G_1$, $\tilde{\tilde{\gamma}}_{r_2} \subseteq G_2$ whose ends and intersection $\tilde{\gamma}_{r_1} \cap \tilde{\tilde{\gamma}}_{r_2}$ belong to $U'$. Then $\gamma_1', \dots, \tilde{\gamma}_{r_1}, \tilde{\tilde{\gamma}}_{r_2}, \gamma_{r_2+1}'', \dots, \gamma_1''$ is the required chain.

Now we consider the case when $T^i x_0$ is tangent to the boundary. For definiteness we assume that $i \leq 0$. If $i = 0$ then, as will be shown later, Theorem 7.1 follows for $Tx_0$, that is, the existence of a neighbourhood of $Tx_0$ contained mod 0 in one ergodic component. In this neighbourhood there is a discontinuity curve of $T^{-1}$ dividing the neighbourhood into two semi-neighbourhoods, one of which passes under $T^{-1}$ into a semi-neighbourhood of $x_0$ ($\cos \varphi(x_0) = 0$), and thus the theorem is true in this case. Therefore we may take $i < 0$. Put $y_0 = T^{i-1} x_0$. Then apply Theorem 6.1 to $x_0$ and Theorem 6.1$'$ to $y_0$. We shall give the constants $\alpha$ and $C$ in the conditions of the theorems later. In a sufficiently small neighbourhood of $x_0$ we construct a curve $\Sigma(x_0)$ consisting of those $x$ such that $T^i x$ is tangent to the boundary. Since $i < 0$, the curve $\Sigma(x_0)$ is decreasing and $x_0 \in \Sigma(x_0)$. In the same way, in a sufficiently small neighbourhood of $y_0$ there is a curve $\Sigma(y_0)$ consisting of those $y$ for which $T^{-1} y$ is tangent to the boundary. Take also $y' = T^{i-1} x'$, $y'' = T^{i-1} x''$. As before, the latter constructions lead to a sufficiently small horizontal strip $\Pi'$ containing $x'$ and a strip $\Pi_1'$ of less than half the width.

We construct a decreasing curve $\gamma_1'$, which is an l.c.t.f. of $x'$ and intersects the side of $\Pi'$, $p(\gamma_1^{(c)}) = \delta_0'$. We may assume $\gamma_1' \cap \Sigma(x_0) = \phi$. By Theorem 6.1 we construct a quadrilateral $G_1'$ whose left side is $\gamma_1'$ and find a point $z_1' \in G_1'$ such that the l.e.t.f. $\gamma_2'$ of $z_1'$ intersects $\gamma_1'$, $\gamma_2' \in U''$, $p(\gamma_2') \geq \delta_0'$. Consider $T^{i-1} \gamma_2' = \tilde{\gamma}_2'$. If $\tilde{\gamma}_2'$ does not intersect $\Sigma(y_0)$, then we can construct a quadrilateral $G_2'$ whose upper side is $\tilde{\gamma}_2'$, choose $z_2'$ in it and pass through it an l.c.t.f. $\tilde{\gamma}_3'$. The constant $C$ in Theorem 6.1$'$ is chosen so that if $p(\tilde{\gamma}_3') \geqslant C\delta_0$, then $p(T^{-i+1}\gamma_3') \geqslant p(\gamma_3') \geqslant \delta_0'$. If $\tilde{\gamma}_2'$ intersects $\Sigma(y_0)$, or equivalently if $\gamma_2' = T^{-i+1}\tilde{\gamma}_2'$ intersects $\Sigma(x_0)$, then instead of the whole curve $\gamma_2'$ we take the curve $\gamma_{20}'$ from the point of intersection $\gamma_2' \cap \gamma_1'$ to the point of intersection $\gamma_2' \cap \Sigma(x_0)$, $p(\gamma_{20}') = \delta_{00}'$ and consider $T^{i-1}\gamma_{20}' = \tilde{\gamma}_{20}'$. We now construct a quadrilateral $G_2'$ whose upper side is $\tilde{\gamma}_{20}'$ and find $z_2'$ in it such that the l.c.t.f. $\gamma_3'$ through it intersects $\Sigma(y_0)$ in an interior point. Then $T^{-i+1}\tilde{\gamma}_3'$ intersects $\Sigma(x_0)$ and has a break, that is, a discontinuity of the derivative, at the point of intersection. We conduct the subsequent construction with the part of $T^{-i+1}\tilde{\gamma}_3'$ that lies on the opposite side of $\Sigma(x_0)$ to $x'$. In this way we obtain a chain intersecting the left and right sides of $U$. We then construct an analogous chain for $x''$. These chains intersect and the remaining argument is as before. The theorem is now proved.

REMARK. Similarly we can prove Theorem 7.1 for points where $\cos\varphi(T^i x_0) = 0$ for finitely many $i$.

It is easy to see that the set of points for which $\cos\varphi(T^i x) = 0$ for not more than one $i$ is the complement of a countable number of points and hence is linearly connected in each connected component of the boundary. Hence we obtain as corollaries:

COROLLARY 7.1. *Each connected component $M_{1i}$ consists* mod 0 *of one ergodic component of $T$.*

COROLLARY 7.2. *$T$ is ergodic.*

COROLLARY 7.3. *The system $\{S_t\}$ is ergodic and consequently is a $K$-system* (see §4).

## §8. The system of two discs on a torus

Let $E$ be the two-dimensional torus. Assume that inside $E$ there are two discs of identical mass and radius $\rho/2$ which move uniformly and linearly inside $E$ and collide according to the law of elastic collision.

The position of each disc is described by the coordinates of its centre $q_1^{(i)}$, $q_2^{(i)}$ $(i = 1, 2)$. Since the distance between the centres is not less than $\rho$, as coordinates in the space $Q$ of our system we may take a four-dimensional torus from which we remove the interior of the cylinder

$$(q_1^{(1)} - q_1^{(2)})^2 + (q_2^{(1)} - q_2^{(2)})^2 = \rho^2. \tag{30}$$

Thus, $Q$ is a closed manifold with a boundary, the latter being the three-dimensional cylinder (30).

The laws of elastic collision mean that this is a system of billiards type in $Q$: a moving point on reaching the cylinder $\partial Q$ is reflected so that the tangential component of the velocity is preserved, but the normal changes sign. Without loss of generality we assume the energy fixed so that the velocity is equal to 1.

The phase space $M$ of our system is the restriction to $Q$ of the tangent bundle of the four-dimensional torus, $\dim M = 7$. We denote by $\{S_t\}$ the one-parameter group of shifts along the trajectories, the invariant phase volume is

$d\mu = dq\cdot d\omega$, where $d\omega$ is the invariant volume in the three-dimensional sphere $S^{(3)}(q)$ which is the fibre over $q \in Q$. The projection of $M$ onto $Q$ is denoted by $\pi$.

On $M$ there acts a two-parameter group of translations, which commute with $\{S_t\}$. It is generated by the shifts

$$\{q_1^{(1)}, q_2^{(1)}, q_1^{(2)}, q_2^{(2)}\} \to \{q_1^{(1)}+r_1, q_2^{(1)}+r_2, q_1^{(2)}+r_1, q_2^{(2)}+r_2\}$$

for any $r_1$, $r_2$. A corollary of this, is the existence (which follows from Noether's theorem) of two first integrals for $\{S_t\}$: the components of the total momentum vector $\vec{I} = \{mv_1, mv_2\}$, where $v_1 = v_1^{(1)} + v_1^{(2)} = \dot{q}_1^{(1)} + \dot{q}_1^{(2)}$, $v_2 = v_2^{(1)} + v_2^{(2)} = \dot{q}_2^{(1)} + \dot{q}_2^{(2)}$.

We fix the value of the momentum $\vec{I} = \vec{I}_0$. The submanifold $M_{\vec{I}_0}$ of the phase space so chosen is a foliation over $Q$ whose fibres are unit circles $S^1(q)$, $\dim M_{\vec{I}_0} = 5$. Let $\tilde{Q}$ be the two-dimensional torus with the coordinates $(\tilde{q}_1, \tilde{q}_2)$, $\tilde{\tilde{Q}}$ the two-dimensional torus from which the circle $\tilde{\tilde{q}}_1^2 + \tilde{\tilde{q}}_2^2 < \rho^2$ is removed. Consider the mapping $\varphi$: $Q \to \tilde{Q} \times \tilde{\tilde{Q}}$ defined by

$$\varphi(q_1^{(1)}, q_2^{(1)}, q_1^{(2)}, q_2^{(2)}) = (q_1^{(1)}+q_1^{(2)}, q_2^{(1)}+q_2^{(2)}, q_1^{(1)}-q_1^{(2)}, q_2^{(1)}-q_2^{(2)}) = \\ = \{\tilde{q}_1^{(1)}, \tilde{q}_2^{(1)}, \tilde{\tilde{q}}_1^{(2)}, \tilde{\tilde{q}}_2^{(2)}\}.$$

The inverse image of each point under $\varphi$ consists of four points. Consequently $Q$ is a fourfold covering of $\tilde{Q} \times \tilde{\tilde{Q}}$. The natural mapping $\tilde{Q} \times \tilde{\tilde{Q}} \to \tilde{Q}$ generates a mapping $Q \to \tilde{Q}$. The inverse image of each point $q \in \tilde{Q}$ for this mapping consists of four regions each of which is isometric to $\tilde{\tilde{Q}}$. This mapping is denoted $\tilde{\pi}$.

Thus, $Q$ is a foliation whose base is the two-dimensional torus, and whose fibre is a disconnected sum of four manifolds, each isometric to $\tilde{\tilde{Q}}$. Corresponding to this decomposition of $Q$ there is a decomposition of $M$ for which $M_{\vec{I}_0}$ is a foliation whose base is the collection of tangent vectors to $\tilde{Q}$ in the direction $\vec{I}_0$, and whose fibre is the collection of tangent vectors to $\tilde{\tilde{Q}}$ of length $\sqrt{1-|I_0|^2}$. In accordance with this the flow $\{S_t\}$ in $M_{\vec{I}_0}$ decomposes, in the sense of ergodic theory (see [1]), into a skew product, where on the base $\tilde{Q}$ there is a conditionally periodic motion with velocity vector $\vec{I}_0$, and in the fibre there is a motion of billiards type to which the theory of the preceding sections applies. In particular, it follows that at almost every $x \in M_{\vec{I}_0}$ there is a one-dimensional l.e.t.f. and l.c.t.f. lying in the three-dimensional fibre $N_3(x)$ corresponding to the foliation of $M_{\vec{I}_0}$. In $N_3(x)$ these foliations are absolutely continuous and from the theory of the preceding sections and [4], [11] follows.

THEOREM 8.1. *For fixed $I_0$ the system $\{S_t\}$ is ergodic in $M_{\vec{I}_0}$, if the components of $\vec{I}_0$ are incommensurable. There exists a partition $\zeta$ whose elements are l.c.t.f. of $\{S_t\}$ and*

1) $S_t\zeta > \zeta$ *for* $t > 0$;

2) $\bigvee S_t\zeta = \varepsilon$;

3) $\bigwedge S_t\zeta = \alpha$, *where* $\alpha$ *is the partition into inverse images of points under* $\tilde{\pi}$;

4) $\{S_t\}$ *has discrete spectrum in the subspace of functions constant* mod 0 *on the elements of* $\alpha$;

5) $\{S_t\}$ *has countable Lebesgue spectrum in the orthogonal complement of this subspace*;

6) *the entropy of* $\{S_t\}$ *is* $\int \kappa^{(e)}(x)\, d\mu(x)$, *where* $\kappa^{(e)}(x)$ *is the curvature of the expanding fibre passing through* $x$.

## §9. Reductions and generalizations

Apart from the case presented in §8, the theory of §§1-7 applies with inessential changes to other situations. Consider, for example, the following problem. Let $Q$ be a square in the plane with a finite number of pairwise disjoint convex regions removed, and let $\{S_t\}$ be the dynamical system generated by the motion with unit velocity of a point inside $Q$, reflecting from the boundary under the law "angle of incidence equals angle of reflection". This system preserves the same measure as the billiards system of §1. We show that it is a $K$-system and a factor of the billiards system of §1.

Let $\hat{Q}$ be a two-dimensional torus realized as a square with sides of length 2 and with pairwise identification of the sides.

Consider the mapping $\hat{\chi}$: $\hat{Q} \to E_2$, where $E_2$ is the unit square, given by the formula

$$\hat{\chi}(q_1, q_2) = (q_1', q_2'),$$

where $q_i' = q_i$ if $0 \leqslant q_i \leqslant 1$, and $q_i' = 2 - q_i$ if $1 < q_i < 2$ $(i = 1, 2)$. We denote the inverse image of $Q$ under this map by $\tilde{Q}$. It is clear that $\tilde{Q}$ is a torus with a number of convex domains, four times the number in $Q$, removed. Let $\{\tilde{S}_t\}$ be billiards in $\tilde{Q}$. It is a $K$-system by §§1-7.

It is easy to see that $\hat{\chi}$ can be continued in a natural way to a mapping of tangent bundles. This mapping $\chi$ in the unit tangent bundle maps the phase space $\tilde{M}$ of $\{\tilde{S}_t\}$ onto the phase space $M$ of $\{S_t\}$. The mapping $\chi$ also commutes with the shifts: $\chi\tilde{S}_t = S_t\chi$. This means that $\{S_t\}$ is a factor system of $\{\tilde{S}_t\}$.

The inverse image of each point under $\chi$ consists of four points. Local transversal fibres of $\{\tilde{S}_t\}$ go over to local transversal fibres of $\{S_t\}$. Hence $\{S_t\}$ is a $K$-system.

We now dwell on the many-dimensional generalization of the situation. Let $Q$ be a $n$-dimensional torus with a finite number of strictly convex domains removed. Billiards in $Q$ is defined as for $n = 2$. Just as for $n = 2$ we define an equipment of unit normal vectors of a submanifold of codimension 1. Each equipment is a section of the tangent bundle. Given an equipment we can introduce the operator of the second quadratic form for the submanifold. A submanifold of codimension 1 with a given equipment is called convex (concave) if the operator of the second quadratic form is positive (negative) definite at each point. By the methods of §2 and 3 we may construct in this case local transversal $(n-1)$-dimensional fibres, which are convex and concave equipments of submanifolds. Complete transversal fibres have singularities of the same type as in the two-dimensional case. Using transversal fibres we can construct as in §4, an increasing partition. The investigation of absolute continuity and the number of ergodic components needs small changes.

By means of the reduction at the beginning of this section we can consider billiards in unit cubes with a finite number of convex domains removed.

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