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HIT-AND-RUN FROM A CORNER*

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Abstract. We show that the hit-and-run random walk mixes rapidly starting from any interior point of a convex body. This is the first random walk known to have this property. In contrast, the ball walk can take exponentially many steps from some starting points. The proof extends to sampling an exponential density over a convex body.

Key words. sampling, random walks, isoperimetric inequalities

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1. Introduction. Consider a random walk in $\mathbb{R}^n$. It starts somewhere and at each step moves to a randomly chosen "neighboring" point (which could be the current point). With a suitable choice of the "neighbor" transition, the steady state distribution of such a walk can be the uniform distribution over a convex body or, indeed, any reasonable distribution in $\mathbb{R}^n$. For example, to sample uniformly from a convex body K, the *ball walk* at a point x chooses a point y uniformly in a ball of fixed radius centered at x and then goes to y if y is in K; else, the step is wasted and it stays at x.

In the last decade and a half, there has been much progress in analyzing these walks [1, 2, 4, 6, 8, 9, 11, 12]. In [8] it was shown that the ball walk mixes in $O^*(n^3)$ steps from a *warm* start after appropriate preprocessing. (A warm start means that the starting point is chosen from a distribution that already is not too far from the target in the sense that its density at any point is at most twice the density of the target distribution. The O^* notation suppresses logarithmic factors and dependence on other parameters like error bounds.) While this result is sufficient to get polynomial-time algorithms for important applications, it is rather cumbersome to generate a warm start and increases the complexity substantially. Kannan and Lovász [6] have shown that the ball walk mixes in $O^*(n^3)$ time from any starting point if wasted steps are not counted. However, the ball walk can take an exponential number of (mostly wasted) steps to mix from some starting points, e.g., a point close to the apex of a rotational cone. (This is because most of the volume of the ball around the start is outside the cone.) Moreover, even starting from a fairly deep point (i.e., the distance to the boundary is much larger than the ball radius), the mixing time can be exponential.[1] The only known way to avoid this problem is to invoke a warm start; it has been an open question as to whether there is a random walk that mixes rapidly starting from, say, the center of gravity of the convex body.

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[1]Random walks on a discrete subset of $\mathbb{R}^n$ (e.g., the lattice walk) avoid this local conductance problem but have other complications that make their convergence less efficient, although still polynomial. Also, one is sampling from a discrete subset, which might be acceptable for applications but is a bit unsatisfactory.

Is there a random walk that mixes rapidly starting from a(ny) single point? *Hit-and-run*, introduced by Smith [15], is defined as follows:

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a uniform random point along the chord $\ell \cap K$.

It was proved in [15] that the stationary distribution of the hit-and-run walk is the uniform distribution π_K over K. In [10], it was shown that hit-and-run mixes in $O^*(n^3)$ steps from a warm start after appropriate preprocessing; i.e., it is no worse than the ball walk. In this paper, we show that it actually mixes rapidly from *any* interior starting point.

To be more precise, the mixing time *can* be big if we start from a very tight corner. But our bound will be logarithmic in the distance; thus, if, e.g., the convex body is described by a system of linear inequalities with rational coefficients, and the starting point is given by rational coordinates, then the mixing time will be polynomial in the input data.

To derive this mixing result, we prove the following theorem that still assumes a bound on the density of the starting distribution.

THEOREM 1.1. *Let K be a convex body that contains a ball of radius r and is contained in a ball of radius R. Let σ be a starting distribution and let σ^m be the distribution of the current point after m steps of hit-and-run in K. Let $\varepsilon > 0$, and suppose that the density function $d\sigma/d\pi_K$ is bounded by M except on a set S with $\sigma(S) \le \varepsilon/2$. Then for*

$$m > 10^{10}\frac{n^2R^2}{r^2}\ln\frac{M}{\varepsilon},$$

the total variation distance of σ^m and π_K is less than ε.

The condition on the starting density captures the case when the L_2 distance of σ and π is bounded (as shown in section 5). The theorem improves on existing bounds by reducing the dependence on M and ε from polynomial (which is unavoidable for the ball walk) to logarithmic, while maintaining the optimal dependence on r, R, and n. To bound the convergence to stationarity when starting from a specific point at distance d from the boundary, we do one step and then (if this is not too short) we apply Theorem 1.1 with the starting distribution obtained this way.

COROLLARY 1.2. *Under the conditions of Theorem 1.1, suppose that the starting distribution σ is concentrated on a single point in K at distance d from the boundary. Then for*

$$m > 10^{11}\frac{n^3R^2}{r^2}\ln\frac{R}{d\varepsilon},$$

the total variation distance of σ^m and π_K is less than ε.

At the heart of this theorem is a bound of $\Omega(r/nR)$ on the conductance of every subset (see Theorem 4.2). (For the ball walk, the conductance of small sets can be arbitrarily small; therefore the need for a warm start.) As we discuss in section 5, the condition that K is contained in a ball of radius R can be replaced by the weaker condition that its second moment is at most R^2, i.e., $\mathsf{E}_K(|x - z_K|^2) \le R^2$, where z_K is the centroid of K. The mixing time goes up by a factor of $O(\ln^2(M/\varepsilon))$. For a body in near-isotropic position, $R/r = O(\sqrt{n})$ and so the number of steps required is $O(n^3 \ln^3(M/\varepsilon))$. It follows that hit-and-run mixes in $O(n^4 \ln^3(n/d))$ steps starting from a point at distance d from the boundary. Such a guarantee is not possible for the ball walk.

Our main tool is a new isoperimetric inequality (section 2). To formulate an isoperimetric inequality, one considers a partition of a convex body K into three sets S_1, S_2, S_3 such that S_1 and S_2 are "far" from each other and the inequality bounds the minimum possible volume of S_3 relative to the volumes of S_1 and S_2. All previous inequalities have viewed the distance between S_1 and S_2 as the *minimum* distance between points in S_1 and points in S_2. For example, if $d(S_1, S_2)$ is the minimum Euclidean distance between S_1 and S_2, then

$$\mathrm{vol}(S_3) \geq \frac{2d(S_1, S_2)}{D} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\},$$

where D is the diameter of K [3, 7]. One can get a similar inequality using the cross-ratio distance (see section 2) instead of the Euclidean distance [10]:

$$\mathrm{vol}(S_3) \geq d_K(S_1, S_2) \frac{\mathrm{vol}(S_1)\mathrm{vol}(S_2)}{\mathrm{vol}(K)}.$$

In this paper, by means of a weight function $h(x)$ on K that measures the distance between S_1 and S_2 as a certain average distance, we obtain a more general inequality that can be much stronger. We formulate it for general logconcave functions in Theorem 2.1. For a convex body, it says that

$$\mathrm{vol}(S_3) \geq \mathsf{E}_K(h(x)) \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}.$$

The weight $h(x)$ at a point x is restricted only by the cross-ratio distance between pairs u, v from S_1, S_2, respectively, for which $x \in [u, v]$. In general, the weight $h(x)$ can be much higher than the minimum cross-ratio distance between S_1 and S_2.

Hit-and-run can be extended to sampling general densities f in $\mathbb{R}^n$ as follows:

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a random point y along the line ℓ chosen from the distribution induced by f on ℓ.

The stationary distribution of this walk is π_f, the probability measure with density f. It has been shown that it is efficient for any logconcave density from a warm start [13]. (Similar results are also known for the ball walk with a Metropolis filter [13].) It is natural to ask if hit-and-run is rapidly mixing from any starting point even for arbitrary logconcave functions. There are some technical problems with extending the results of this paper to arbitrary logconcave functions; but we make some progress in this direction by showing that this is indeed the case for an exponential density over a convex body. This class of density functions is interesting for other reasons as well—these are the functions used in "simulated annealing" and in the fastest volume algorithm [14]. We prove the following theorem in section 6. The condition on the starting density captures the case of bounded L_2-norm; the proof uses the same isoperimetric inequality (see Theorem 2.1).

THEOREM 1.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body and let f be a density supported on K which is proportional to $e^{-a^T x}$ for some vector $a \in \mathbb{R}^n$. Assume that the level set of f of probability $1/8$ contains a ball of radius r and that $\mathsf{E}_f(|x - z_f|^2) \leq R^2$, where z_f is the centroid of f. Let σ be a starting distribution and let σ^m be the distribution of the current point after m steps of hit-and-run applied to f. Let $\varepsilon > 0$, and suppose that the density function $d\sigma/d\pi_f$ is bounded by M except on a set S with $\sigma(S) \leq \varepsilon/2$. Then for*

$$m > 10^{30} \frac{n^2 R^2}{r^2} \ln^5 \frac{MnR}{r\varepsilon},$$

the total variation distance of σ^m and π_f is less than ε.

1.1. Overview of analysis. We wish to bound the rate of convergence of the Markov chain underlying hit-and-run to the uniform distribution π_K on the convex body K. For this we use the notion of *conductance*, which is defined as follows: For any measurable subset $S \subseteq K$ and $x \in K$, we denote by $P_x(S)$ the probability that a step from x goes to S. If $0 < \pi_K(S) < 1$, then the conductance $\phi(S)$ is defined as

$$\phi(S) = \frac{\int_{x\in S} P_x(K \setminus S)\, d\pi_K}{\min\{\pi_K(S), \pi_K(K \setminus S)\}}.$$

The minimum value over all subsets S is the conductance, ϕ, of the Markov chain. Lovász and Simonovits [12], extending a result of Jerrum and Sinclair [5], have shown that the mixing rate (roughly, the number of steps needed to halve the distance to the stationary distribution) is bounded by $O(1/\phi^2)$ (and is at least $1/\phi$).

The main part of our proof shows that the conductance of the hit-and-run Markov chain is $\Omega(r/nR)$. All previous attempts to bound the conductance of geometric random walks could prove only that the conductance of "large" subsets is large, namely, that the conductance bound for a subset S was proportional to $\pi_K(S)$. For this reason, one had to limit the probability that we start in one of bad small sets, which leads to the use of a warm start. As mentioned earlier, the example of starting at a point x near the apex of a rotational cone shows that the ball walk can in fact take exponentially many steps from some starting points: Most points of a ball around x are outside the cone, and hence most steps from x are wasted.

Hit-and-run, on the contrary, exhibits a sizable (inverse polynomial) drift toward the base of the cone. Thus, although the initial steps are tiny, they quickly get larger and the current point moves away from the apex. By bounding the conductance, we show that this phenomenon is general; i.e., hit-and-run mixes rapidly starting from any interior point of a convex body. To prove this, we use the new isoperimetric inequality. Besides the inequality, a key observation in the proof is that the "median" step length from points in K is a concave function.

2. A weighted isoperimetric inequality. To analyze the walk, we use a non-Euclidean notion of distance [10]. Let u, v be two distinct points in K, let $\ell(u, v)$ denote the line through u and v, and let p, q be the endpoints of the segment $\ell(u, v) \cap K$, so that the points appear in the order p, u, v, q along $\ell(u, v)$. Then,

$$d_K(u, v) = \frac{|u - v||p - q|}{|p - u||v - q|}.$$

For two subsets S_1, S_2 of K, we define

$$d_K(S_1, S_2) = \min_{u\in S_1, v\in S_2} d_K(u, v).$$

Theorem 2.5 from [13] asserts the following: If f is a logconcave function on a convex set K, $\varepsilon > 0$ and $S_1 \cup S_2 \cup S_3$ is a partition of K into three measurable sets such that for any $u \in S_1$ and $v \in S_2$ we have $d_K(u, v) \geq \varepsilon$, then

$$\int_K f(x)\, dx \int_{S_3} f(x)\, dx \geq \varepsilon \int_{S_1} f(x)\, dx \int_{S_2} f(x)\, dx. \tag{1}$$

We prove the following related result.

THEOREM 2.1. *Let K be a convex body in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}_+$ be a logconcave function and let $h : K \to \mathbb{R}_+$ be an arbitrary function. Let S_1, S_2, S_3 be any partition* *of K into measurable sets. Suppose that for any pair of points $u \in S_1$ and $v \in S_2$ and any point x on the chord of K through u and v,*

$$h(x) \leq \frac{1}{3}\min(1, d_K(u,v)).$$

Then

$$\frac{\int_{S_3} f(x)\,dx}{\min\left\{\int_{S_1} f(x)\,dx, \int_{S_2} f(x)\,dx\right\}} \geq \frac{\int_K h(x)f(x)\,dx}{\int_K f(x)\,dx}.$$

Remark. For a logconcave density function f and corresponding distribution π_f, the conclusion of the theorem can be stated as

$$\pi_f(S_3) \geq \mathsf{E}_f(h(x))\min\{\pi_f(S_1), \pi_f(S_2)\}.$$

Proof. We can assume that $\int_{S_1} f(x)\,dx \leq \int_{S_2} f(x)\,dx$. Suppose that the conclusion is false. Then there exists an $A \leq 1/2$ such that

$$\int_K f(x)\,dx = \frac{1}{A}\int_{S_1} f(x)\,dx$$

and

$$\int_{S_3} f(x)\,dx < A\int_K h(x)f(x)\,dx.$$

Now we invoke the localization lemma, specifically the version given in Corollary 2.4 of [7]. This implies that there exist two points $a, b \in K$ and a linear function $\ell:\ [0,1] \to \mathbb{R}_+$ with the following properties. Set

$$F(t) = \ell(t)^{n-1} f(ta + (1-t)b),\ \ G(t) = h(ta + (1-t)b),$$

and

$$J_i = \{t \in [0,1] : ta + (1-t)b \in S_i\} \quad (i = 1, 2, 3);$$

then

$$\int_0^1 F(t)\,dt = \frac{1}{A}\int_{J_1} F(t)\,dt,$$
$$\int_{J_3} F(t)\,dt < A\int_0^1 G(t)F(t)\,dt,$$

and hence

$$\int_0^1 F(t)\,dt \int_{J_3} F(t)\,dt < \int_{J_1} F(t)\,dt \int_0^1 G(t)F(t)\,dt.$$

For $u, v \in K$, let M_{uv} denote the maximum of $h(x)$ over the chord through u and v; then

$$\int_0^1 G(t)F(t)\,dt \leq M_{ab}\int_0^1 F(t)\,dt,$$

and so

$$\int_{J_3} F(t)\,dt < M_{ab} \int_{J_1} F(t)\,dt. \tag{2}$$

We also have

$$\int_{J_1} F(t)\,dt = A \int_0^1 F(t)\,dt \le \frac{1}{2} \int_0^1 F(t)\,dt. \tag{3}$$

Let $u \in J_1$ and $v \in J_2$, and (say) $u < v$. Then by hypothesis,

$$\frac{v-u}{u(1-v)} \ge d_K(ua + (1-u)b, va + (1-v)b) \ge 3M_{ab},$$

and hence by the one-dimensional case of (1), we have

$$\int_0^1 F(t)\,dt \int_{J_3} F(t)\,dt \ge 3M_{ab} \int_{J_1} F(t)\,dt \int_{J_2} F(t)\,dt.$$

Comparing this with (2), we get

$$\int_0^1 F(t)\,dt > 3 \int_{J_2} F(t)\,dt.$$

Using this and (3), it follows that

$$\begin{aligned} \int_{J_3} F(t)\,dt &= \int_0^1 F(t)\,dt - \int_{J_1} F(t)\,dt - \int_{J_2} F(t)\,dt \\ &> \left(1 - \frac{1}{2} - \frac{1}{3}\right) \int_0^1 F(t)\,dt \\ &= \frac{1}{6} \int_0^1 F(t)\,dt. \end{aligned}$$

But then (2) and (3) imply that $M_{ab} > 1/3$, a contradiction. $\square$

3. Bounding the step size. For $x \in K$, let y be a random step from x. Following [10], we define $F(x)$ as

$$\mathsf{P}\left(|x-y| \le F(x)\right) = \frac{1}{8}. \tag{4}$$

Roughly speaking, this is the "median" step length from x. The goal of this section is to bound this function from below by a concave function.

For $x \in K$, let

$$\lambda(x,t) = \frac{\operatorname{vol}(K \cap (x+tB))}{\operatorname{vol}(tB)}$$

denote the fraction of a ball of radius t around x that intersects K. For a fixed $\gamma \ge 0$, define $s: K \to \mathbb{R}_+$ as

$$s(x) = \sup\{t \in \mathbb{R}_+ : \lambda(x,t) \ge \gamma\}.$$

The value $s(x)$ is a measure of how close x is to the boundary of K. Its somewhat complicated definition guarantees some useful properties.

LEMMA 3.1. *For any $\gamma > 0$, $s(x)$ is a concave function.*

Proof. Let $x_1, x_2 \in K$ with $s(x_1) = r_1$ and $s(x_2) = r_2$. Let $A_i = K \cap (x_1 + r_i B)$ $(i = 1, 2)$. Let $x = (x_1 + x_2)/2$ and consider $A = (A_1 + A_2)/2$. By convexity, $A \subseteq K$. Further, any point $y \in A$ can be written as

$$y = \frac{1}{2}(x_1 + z_1 \; + \; x_2 + z_2) = x + \frac{z_1 + z_2}{2}$$

for some z_1, z_2 such that $|z_1| \le r_1$ and $|z_2| \le r_2$. Thus,

$$A \subseteq K \cap \left(x + \frac{r_1 + r_2}{2} B \right).$$

Next, by the Brunn–Minkowski inequality,

$$\begin{aligned} \mathrm{vol}(A)^{\frac{1}{n}} &\ge \frac{1}{2}\left(\mathrm{vol}(A_1)^{\frac{1}{n}} + \mathrm{vol}(A_2)^{\frac{1}{n}} \right) \\ &\ge \frac{1}{2}\gamma^{\frac{1}{n}} \mathrm{vol}(B)^{\frac{1}{n}} (r_1 + r_2) \\ &= \gamma^{\frac{1}{n}} \mathrm{vol}\left(\frac{r_1 + r_2}{2} B \right)^{\frac{1}{n}}. \end{aligned}$$

It follows that

$$\mathrm{vol}\left(K \cap \left(x + \frac{r_1 + r_2}{2} B \right) \right) \ge \mathrm{vol}(A) \ge \gamma \mathrm{vol}\left(\frac{r_1 + r_2}{2} B \right)$$

and thus $s(x) \ge (r_1 + r_2)/2$. $\square$

LEMMA 3.2. *If $\gamma \ge 63/64$, then for all $x \in K$,*

$$F(x) \ge \frac{s(x)}{32}.$$

Proof. Set $s = s(x)$. Let p denote the fraction of the surface of the ball $x + (s/2)B$ that is not in K. Then

$$\mathrm{vol}((x + sB) \setminus K) \ge p\mathrm{vol}(sB) - \mathrm{vol}((s/2)B).$$

By the definition of s,

$$\mathrm{vol}((x + sB) \setminus K) \le (1 - \gamma)\mathrm{vol}(sB),$$

and hence

$$p \le 1 - \gamma + 2^{-n} \le \frac{1}{32}.$$

Take a random line ℓ through x; then with probability at least $1 - 2p$, $\ell \cap (x + (s/2)B) \subseteq K$. If this happens, then for the point y chosen uniformly from $\ell \cap K$, we have

$$\mathsf{P}\left(|y - x| \le \frac{s}{32} \;\mid\; \ell \right) \le \frac{1}{16},$$

and so

$$\mathsf{P}\left(|y-x| \le \frac{s}{32}\right) \le \frac{1}{16} + \frac{15}{16}\cdot\frac{1}{16} < \frac{1}{8}.$$

This proves the lemma. ∎

Let us quote Corollary 4.6 in [8] as the following lemma.

LEMMA 3.3. *Suppose K contains a ball of radius r. Then,*

$$\int_K \int_{y\in x+tB\setminus K} dy\, dx \le \frac{t\sqrt{n}}{2r}\mathrm{vol}(K)\mathrm{vol}(tB).$$

This lemma can be used to bound the average value of $s(x)$ from below, as follows.

LEMMA 3.4. *Suppose K contains a unit ball. Then,*

$$\int_K s(x)\, dx \ge \frac{1-\gamma}{\sqrt{n}}\mathrm{vol}(K).$$

Proof. From Lemma 3.3,

$$\int_K \lambda(x,t)\, dx \ge \left(1 - \frac{t\sqrt{n}}{2}\right)\mathrm{vol}(K).$$

On the other hand,

$$\int_K \lambda(x,t)\, dx \le \gamma\mathrm{vol}(K) + (1-\gamma)\mathrm{vol}(\{x \in K:\ \lambda(x,t)\ge\gamma\})$$

and so

$$\mathrm{vol}(\{x\in K:\ \lambda(x,t)\ge\gamma\}) \ge \left(1 - \frac{t\sqrt{n}}{2(1-\gamma)}\right)\mathrm{vol}(K).$$

Using this,

$$\begin{aligned}\int_K s(x)\, dx &= \int_0^\infty \mathrm{vol}(\{x\in K:\ \lambda(x,t)\ge\gamma\})\, dt\\ &\ge \mathrm{vol}(K)\int_0^\infty \left(1 - \frac{t\sqrt{n}}{2(1-\gamma)}\right)^+ dt\\ &= \frac{1-\gamma}{\sqrt{n}}\mathrm{vol}(K).\end{aligned}$$ ∎

4. A scale-free bound on the conductance. For a point $u \in K$, let P_u be the distribution obtained by taking one hit-and-run step from u. Then (as shown in [10]),

$$P_u(A) = \frac{2}{\mathrm{vol}_{n-1}(\partial B)}\int_A \frac{dx}{\ell(u,x)|x-u|^{n-1}}, \tag{5}$$

where $\ell(u,x)$ is the length of the chord through u and x.

Let $d_{tv}(P,Q)$ denote the total variation distance between distributions P and Q. The following lemma from [10] connects the geometric distance of two points to the variation distance of the distributions obtained by taking one hit-and-run step.

LEMMA 4.1 (see [10]). *Let $u, v \in K$. Suppose that*

$$d_K(u,v) < \frac{1}{8} \text{ and } |u-v| < \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}.$$

Then

$$d_{tv}(P_u, P_v) < 1 - \frac{1}{500}.$$

The main theorem of this section is the following.

THEOREM 4.2. *Let K be a convex body in $\mathbb{R}^n$ of diameter D, containing a unit ball. Then the conductance of hit-and-run in K is at least $\frac{1}{2^{24}nD}$.*

Proof. Let $K = S_1 \cup S_2$ be a partition into measurable sets. We will prove that

$$\int_{S_1} P_x(S_2)\,dx \ge \frac{1}{2^{24}nD} \min\{\text{vol}(S_1), \text{vol}(S_2)\}. \tag{6}$$

We can read the left-hand side as follows: We select a random point X from the uniform distribution and make one step to get Y. What is the probability that $X \in S_1$ and $Y \in S_2$? It is well known that this quantity remains the same if S_1 and S_2 are interchanged.

Consider the points that are deep inside these sets, i.e., unlikely to jump out of the set:

$$S_1' = \left\{x \in S_1 : P_x(S_2) < \frac{1}{1000}\right\}$$

and

$$S_2' = \left\{x \in S_2 : P_x(S_1) < \frac{1}{1000}\right\}.$$

Let S_3' be the rest, i.e., $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $\text{vol}(S_1') < \text{vol}(S_1)/2$. Then

$$\int_{S_1} P_x(S_2)\,dx \ge \frac{1}{1000}\text{vol}(S_1 \setminus S_1') \ge \frac{1}{2000}\text{vol}(S_1),$$

which proves (6).

So we can assume that $\text{vol}(S_1') \ge \text{vol}(S_1)/2$ and, similarly, that $\text{vol}(S_2') \ge \text{vol}(S_2)/2$. For any $u \in S_1'$ and $v \in S_2'$,

$$d_{tv}(P_u, P_v) \ge 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{500}.$$

Thus, by Lemma 4.1, either

$$d_K(u,v) \ge \frac{1}{8} \tag{7}$$

or

$$|u-v| \ge \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}. \tag{8}$$

We want to apply Theorem 2.1 to the partition S_1', S_2', S_3' and the function $h(x) = s(x)/(48D\sqrt{n})$, where $s(x)$ is as defined in section 3 with $\gamma = 63/64$. To verify the condition, let $u \in S_1'$, $v \in S_2'$, and x be any point on the chord pq through u and v (where p is the endpoint closer to u than v). Clearly $h(x) \le 1/3$. If (7) holds, then $h(x) \le d_K(u,v)/3$ is trivial. Then suppose that (8) holds. Let, e.g., x be between u and q. Then, using the concavity of s (Lemma 3.1), we have

$$\begin{aligned} s(x) \le \frac{|x-p|}{|u-p|} s(u) &\le 32 \frac{|q-p|}{|u-p|} F(u) \quad \text{(using Lemma 3.2)} \\ &\le 16 \frac{|q-p|}{|u-p|} \sqrt{n} |u-v| \quad \text{(using (8) above)} \\ &= 16 d_K(u,v) \sqrt{n} |q-v| \\ &\le 16 d_K(u,v) D \sqrt{n}, \end{aligned}$$

and hence $h(x) \le d_K(u,v)/3$ follows again. Thus, Theorem 2.1 applies with f being the uniform density and we get

$$\begin{aligned} \frac{\mathrm{vol}(S_3')}{\min\{\mathrm{vol}(S_1'), \mathrm{vol}(S_2')\}} &\ge \frac{1}{48D\sqrt{n}} \cdot \frac{1}{\mathrm{vol}(K)} \int_K s(x)\, dx \\ &> \frac{1}{4000nD}. \end{aligned}$$

Here we have used Lemma 3.4 with $\gamma = 63/64$. Therefore,

$$\begin{aligned} \int_{S_1} P_x(S_2)\, dx &\ge \frac{1}{2} \cdot \frac{1}{1000} \mathrm{vol}(S_3') \\ &\ge \frac{1}{2^{23} nD} \min\{\mathrm{vol}(S_1'), \mathrm{vol}(S_2')\} \\ &\ge \frac{1}{2^{24} nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}, \end{aligned}$$

which again proves (6). $\square$

5. Proof of the mixing bound. First note that hit-and-run is invariant under a scaling of space (i.e., there is a $1-1$ mapping between the random walk in K and cK) and thus the conductance bound of $\Omega(r/nR)$ follows by considering K/r. Next, suppose we start with an M-warm distribution σ; i.e., for any subset S of K, $\sigma(S) \le M\pi_K(S)$. Then using Corollary 1.5 of [12], the distribution σ^m obtained after m steps satisfies

$$d_{tv}(\sigma^m, \pi_K) \le \sqrt{M} \left(1 - \frac{\phi^2}{2}\right)^m$$

and thus after $m > Cn^2 \frac{R^2}{r^2} \ln \frac{M}{\varepsilon}$ steps (C is a constant), the total variation distance of σ^m and π_K is less than ε.

If we know only that $\sigma \le M\pi_K$ except for the subsets of a set S with $\sigma(S) < \varepsilon/2$, then we think of a random point of K as being generated with probability $1 - \varepsilon/2$ from a distribution σ' that is $(2M)$-warm with respect to π_K and with probability $\varepsilon/2$ from some other distribution. After m steps, we have

$$d_{tv}(\sigma^m, \pi_K) \le \frac{\varepsilon}{2} + \left(1 - \frac{\varepsilon}{2}\right) \sqrt{\frac{2M}{\varepsilon}} \left(1 - \frac{\phi^2}{2}\right)^m,$$

which implies Theorem 1.1.

In some applications, the L_2-norm of σ w.r.t. π is bounded; i.e., suppose that

$$\int_K \left(\frac{d\sigma}{d\pi}\right)^2 d\pi \leq M.$$

This will also be sufficient for mixing. The set

$$S = \left\{x : \frac{d\sigma}{d\pi} > \frac{2M}{\varepsilon}\right\}$$

has measure $\pi(S)$ of at most $\varepsilon/2$. So we can apply the mixing theorem with $2M/\varepsilon$ in place of M.

As mentioned in the introduction, Theorem 1.1 can be strengthened to require only that $\mathsf{E}_K(||x - z_K||^2) \leq R^2$ with a small increase in the mixing time. It is well known that the volume of K outside a ball of radius $R\ln(2/\delta)$ is at most a $\delta/2$ fraction. Thus the conductance of any subset of measure x is at least

$$\phi(x) = \frac{cr}{nR\ln(2/x)}$$

for some constant c. Then the average conductance theorem of [6] implies that after $m > C(n^2R^2/r^2)\ln^3(M/\varepsilon)$ steps (where C is a constant), we get that $d_{tv}(\sigma^m, \pi_K) \leq \varepsilon$.

Finally, Corollary 1.2 follows by bounding M for the distribution obtained after one step of hit-and-run.

6. Exponential density over a convex body. Here we extend the main theorem to sample an exponential density over a convex body. We will use the following notation. Let f be a density function in $\mathbb{R}^n$. For any line ℓ in $\mathbb{R}^n$, let $\mu_{\ell,f}$ be the measure induced by f on ℓ, i.e.,

$$\mu_{\ell,f}(S) = \int_{p+tu\in S} f(p + tu)dt,$$

where p is any point on ℓ and u is a unit vector parallel to ℓ. We abbreviate $\mu_{\ell,f}$ by μ_ℓ if f is understood, and also $\mu_\ell(\ell)$ by μ_ℓ. The probability measure $\pi_\ell(S) = \mu_\ell(S)/\mu_\ell$ is the *restriction* of f to ℓ.

For two points $u, v \in \mathbb{R}^n$, let $\ell(u, v)$ denote the line through them. Let $[u, v]$ denote the segment connecting u and v, and let $\ell^+(u, v)$ denote the semiline in ℓ starting at u and not containing v. Furthermore, let

$$\begin{aligned} f^+(u, v) &= \mu_{\ell,f}(\ell^+(u, v)), \\ f^-(u, v) &= \mu_{\ell,f}(\ell^+(v, u)), \\ f(u, v) &= \mu_{\ell,f}([u, v]). \end{aligned}$$

For any $T > 0$, let $L(T) = \{x : f(x) \geq T\}$ be the level set of function value T. It will be convenient to let π_n denote the volume of the unit ball in $\mathbb{R}^n$.

6.1. Distance. The following "distance" was used in [13]:

$$d_f(u, v) = \frac{f(u, v)f(\ell(u, v))}{f^-(u, v)f^+(u, v)}.$$

(This quantity is not really a distance, since it does not satisfy the triangle inequality. To get a proper distance function, one could consider $\ln(1 + d_f(u, v))$; but it will be more convenient to work with d_f.)

Note that when f is the uniform distribution over a convex set K, then $d_f(u, v) = d_K(u, v)$. The next lemma describes how the two are related in general.

LEMMA 6.1. *Let f be a logconcave density function in $\mathbb{R}^n$ whose support is a convex body K. Let $G = \max_K f(x)/\min_K f(x)$.*

1. $d_f(u, v) \geq d_K(u, v)$.
2.
$$d_K(u, v) \geq \frac{\min\{3, d_f(u, v)\}}{6(1 + \ln G)}.$$

The first inequality is Lemma 5.9 in [13], and the second inequality is a direct implication of Lemma 5.11 in [13].

6.2. Step size. Let f be a density function whose support is a convex body K. We define three parameters that all measure the local smoothness of f. First, for a fixed β and γ, we define

$$\lambda(x, t) = \frac{\mathrm{vol}((x + tB) \cap L(\beta f(x)))}{\mathrm{vol}(tB)} \quad \text{and} \quad s(x) = \sup\{t \in \mathbb{R}_+ : \lambda(x, t) \geq \gamma\}.$$

Second, we define $F(x)$ by

$$\mathsf{P}\left(|x - y| \leq F(x)\right) = \frac{1}{8},$$

where y is a random step from x. Third, we define $\alpha(x)$ (as in [13]) as the smallest $s \geq 3$ for which a hit-and-run step y from x satisfies

$$\mathsf{P}(f(y) \geq s f(x)) \leq \frac{1}{16}.$$

We will shortly fix $\beta = 3/4$ and $\gamma = 63/64$. Note that $\lambda(x, t)$, $s(x)$, and $F(x)$ as defined here are generalizations of the definitions in section 3 (where $f(x)$ was the uniform density over K).

The following lemma was proved in [13].

LEMMA 6.2 (see [13, Lemma 6.10]).

$$\pi_f(u : \alpha(u) \geq t) \leq \frac{16}{t}.$$

Our next lemma extends a crucial property of $s(x)$ to exponential functions (it does not hold for general logconcave functions).

LEMMA 6.3. *Suppose $f(x)$ is proportional to $e^{-a^T x}$ in a convex body K and zero outside. Then for any fixed $\beta, \gamma > 0$, the function $s(x)$ is concave.*

Proof. Let $x_1, x_2 \in K$ with $s(x_1) = r_1$ and $s(x_2) = r_2$. Define

$$A_1 = \{y \in x_1 + r_1 B : f(y) \geq \beta f(x_1)\} \quad \text{and} \quad A_2 = \{y \in x_2 + r_2 B : f(y) \geq \beta f(x_2)\}.$$

Now let $x = (x_1 + x_2)/2$ and consider $A = (A_1 + A_2)/2$. Any point $y \in A$ can be written as

$$y = x + \frac{z_1 + z_2}{2}$$

for some z_1, z_2 such that $z_1 \in r_1 B$ and $z_2 \in r_2 B$. Thus

$$A \subseteq x + \frac{r_1 + r_2}{2} B.$$

Also, since $f(x)$ is proportional to $e^{-a^T x}$, we have $f((x+y)/2) = \sqrt{f(x)f(y)}$ and so for any $y \in A$,

$$f(y) = f\left(\frac{y_1 + y_2}{2}\right) = \sqrt{f(y_1)f(y_2)},$$

where $y_1 \in A_1$ and $y_2 \in A_2$. By the definition of these subsets, $f(y_1) \geq \beta f(x_1)$ and $f(y_2) \geq \beta f(x_2)$. Thus

$$f(y) \geq \beta\sqrt{f(x_1)f(x_2)} = \beta f\left(\frac{x_1 + x_2}{2}\right) = \beta f(x)$$

and so

$$A \subseteq \left\{ y \in x + \frac{r_1 + r_2}{2} B : f(y) \geq \beta f(x) \right\}.$$

Finally, by the Brunn–Minkowski inequality,

$$\begin{aligned} \mathrm{vol}(A)^{\frac{1}{n}} &\geq \frac{1}{2}\left(\mathrm{vol}(A_1)^{\frac{1}{n}} + \mathrm{vol}(A_2)^{\frac{1}{n}}\right) \\ &\geq \frac{1}{2}(\gamma \pi_n)^{\frac{1}{n}}(r_1 + r_2) \\ &= \gamma^{\frac{1}{n}} \mathrm{vol}\left(\frac{r_1 + r_2}{2} B\right)^{\frac{1}{n}}. \end{aligned}$$

It follows that $s(x) \geq (r_1 + r_2)/2$. ∎

Next, we bound the expected value of $s(x)$.

LEMMA 6.4. *Let f be any logconcave density such that the level set of f of measure* $1/8$ *contains a ball of radius r. Then with $\beta = 3/4$ and $\gamma = 63/64$,*

$$\mathsf{E}_f(s(x)) \geq \frac{r}{2^{10}\sqrt{n}}.$$

Proof. Let L_0 be the level set

$$L_0 = \{x : f(x) \geq f_0\},$$

such that the measure of L_0 is $1/8$. For $i = 1, 2, \ldots$, consider the level sets

$$L_i = \left\{ x : f(x) \geq \left(\frac{3}{4}\right)^i f_0 \right\}.$$

Note that since f is logconcave, each L_i is a convex body. We will first bound $\mathsf{E}_f(1 - \lambda(x,t))$ as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) \int_{y\in x+tB: f(y)<3f(x)/4} \frac{dy}{\mathrm{vol}(tB)}\,dx \\
&\le \frac{1}{8} + \sum_{i>0} \frac{f_0}{(4/3)^{i-1}} \int_{x\in L_i\setminus L_{i-1}} \int_{y\in x+tB\setminus L_i} \frac{dy}{\mathrm{vol}(tB)}\,dx \\
&\le \frac{1}{8} + \sum_{i} \frac{f_0}{(4/3)^{i-1}} \int_{x\in L_i} \int_{y\in x+tB\setminus L_i} \frac{dy}{\mathrm{vol}(tB)}\,dx \\
&\le \frac{1}{8} + \frac{t\sqrt{n}}{2r} \sum_{i} \frac{f_0}{(4/3)^{i-1}} \mathrm{vol}(L_i).
\end{aligned}$$

In the last step, we applied Lemma 3.3 to the convex set L_i which contains a ball of radius r by assumption. Now for any $x \in L_i \setminus L_{i-1}$,

$$\frac{f_0}{(4/3)^i} \le f(x) < \frac{f_0}{(4/3)^{i-1}}.$$

Using this,

$$\begin{aligned}
\sum_i \frac{f_0}{(4/3)^{i-1}} \mathrm{vol}(L_i) &\le \sum_i \frac{4f_0}{(4/3)^{i-1}} \mathrm{vol}(L_i \setminus L_{i-1}) \\
&\le \frac{16}{3} \int_{\mathbb{R}^n} f(x)\,dx < 6.
\end{aligned}$$

Thus,

$$\mathsf{E}_f(1-\lambda(x,t)) \le \frac{1}{8} + \frac{3t\sqrt{n}}{r}.$$

Next, since $\lambda(x,t)$ can be at most 1, we get

$$\int_{x:\lambda(x,t)\ge 3/4} f(x)\,dx \ge \frac{1}{2} - \frac{12t\sqrt{n}}{r}.$$

We will use the following claim to complete the proof: If $\lambda(x,t) \ge 3/4$, then for $c > 1$,

$$\lambda(x, t/c) \ge 1 - e^{-(\frac{c}{4}-1)^2/2}.$$

To see the claim, note that since $\lambda(x,t) \ge 3/4$, there must be a ball of radius $t/2\sqrt{n}$ inside K centered at x. The claim then follows by applying Lemma 4.4 in [13].

Setting $c = 16$ above, we get $\lambda(x, t/16) \ge 1 - e^{-9/2} > 63/64$. Using this,

$$\begin{aligned}
\int_{\mathbb{R}^n} s(x)f(x)\,dx &\ge \frac{1}{16} \int_{t=0}^{\infty} \int_{x:\lambda(x,t)\ge 3/4} f(x)\,dx\,dt \\
&\ge \frac{1}{16} \int_{t=0}^{\infty} \left(\frac{1}{2} - \frac{12t\sqrt{n}}{r}\right)^+ dt \\
&\ge \frac{r}{2^{10}\sqrt{n}}. \qquad \square
\end{aligned}$$

We can also relate the maximum value of $s(x)$ to the diameter D.

LEMMA 6.5. *Let* $G = \frac{\max_K f(x)}{\min_K f(x)}$, *where* f *is proportional to* $e^{-a^T x}$ *with support* K. *Suppose* K *has diameter* D. *Then,*

$$\max_K s(x) \le \min\left\{\frac{2\sqrt{n}D}{\ln G}, D\right\}.$$

Proof. Let $t = 1/|a|$. Then along the direction of a, the function value drops by $1/e$ each time we move distance t. Hence,

$$t \le \frac{D}{\ln G}.$$

On the other hand, for any point x, we claim that

$$s(x) \le 2t\sqrt{n}.$$

To see this, consider the nearest point y along the line through x in the direction of a with $f(y) \le f(x)/2$. This point satisfies $|x - y| \le t$. Now the portion of the ball $x + s(x)B$ in the half-space $\{z : a^T z \ge a^T y\}$ must have volume at most $1/4$ of the volume of $s(x)B$ by the definition of $s(x)$ (in a ball of radius $2t\sqrt{n}$, a half-space at distance t from the center cuts off at least $1/4$ of the volume of the ball). This implies the inequality. The lemma follows. ∎

The next lemma is about the step size along a given line.

LEMMA 6.6. *Let* f *be logconcave and* ℓ *be any line through a point* x. *Let* p, q *be intersection points of* ℓ *with the boundary of* $L(F/8)$, *where* F *is the maximum value of* f *along* ℓ, *and let* $s = \max\{|x - p|/32, |x - q|/32\}$. *Choose a random point* y *on* ℓ *from the distribution* π_ℓ. *Then*

$$\mathsf{P}(|x - y| > s) > \frac{3}{4}.$$

Proof. We will use the following observation. For any logconcave function g that is nonincreasing on an interval $[a, b]$

$$\int_{[a,b]} g(x)\,dx \ge |a - b|\frac{g(a) - g(b)}{\ln g(a) - \ln g(b)}.$$

The proof is by noting that the exponential function with value $g(a)$ at a and $g(b)$ at b is a lower bound on any such function.

In our case, suppose f attains its maximum at a point $z \in [p, q]$. Then, applying the observation separately to the intervals $[p, z]$ and $[z, q]$, we get

$$\int_{[p,q]} f(x)\,dx \ge \frac{7F}{8\ln 8}|p - q|.$$

Also, by Lemma 3.5(a) in [13] (whose proof uses a similar reduction to the exponential function), $\mathsf{P}(y \in [p, q]) \ge 7/8$. We now consider two cases. If $x \in [p, q]$, then $s \le |p - q|/32$ and thus

$$\mathsf{P}(|x - y| \le s) \le \frac{2sF}{\int_{[p,q]} f(x)\,dx} \le \frac{\ln 8}{14} < \frac{1}{4}.$$

Suppose $x \notin [p, q]$. Let u be the unit vector along $p - q$. Then,

$$|[x - su, x + su] \cap [p, q]| \le \frac{|p - q|}{32},$$

and thus

$$\mathsf{P}(|x-y| \le s) \le \frac{1}{8} + \frac{F|p-q|/32}{7F|p-q|/8\ln 8} < \frac{1}{4}. \qquad \square$$

Finally, $s(x)$ gives a lower bound on $F(x)$ as in section 3.

LEMMA 6.7. *If* $\gamma \ge 63/64$ *and* $\beta \ge 3/4$, *then*

$$F(x) \ge \frac{s(x)}{64}.$$

Proof. We need to prove the following: If $x \in \mathbb{R}^n$ and $s > 0$ satisfies

$$\mathrm{vol}\big((x+sB) \cap \{f \le \beta f(x)\}\big) \le (1-\gamma)\mathrm{vol}(x+sB),$$

then for a hit-and-run step y from x,

$$\mathsf{P}\left(|x-y| \le \frac{s}{64}\right) \le \frac{1}{8}. \tag{9}$$

Let p denote the fraction of the surface of $x+(s/2)B$ in the set $\{f \le \beta f(x)\}$. Clearly

$$\begin{aligned}\mathrm{vol}\big((x+sB) \cap \{f \le \beta f(x)\}\big) &\ge p\mathrm{vol}(x+sB) - \mathrm{vol}(x+(s/2)B)\\ &= (p-2^{-n})\mathrm{vol}(x+sB),\end{aligned}$$

and by our hypothesis on s,

$$p \le 1-\gamma+2^{-n} \le \frac{1}{32}.$$

Thus if we choose a random line through x, with probability at least 15/16 it will intersect the surface of $x+(s/2)B$ in points z_1, z_2 with $f(z_i) \ge \beta f(x)$.

Suppose that we have chosen such a line, and let u be a unit vector parallel to this line. Then we have

$$f(x+tu) \ge \beta f(x) \qquad \left(-\frac{s}{2} \le t \le \frac{s}{2}\right)$$

and also (by logconcavity)

$$f(x+tu) \le \beta^{-2|t|/s} f(x) \qquad (-\infty < t < \infty).$$

We have

$$\mathsf{P}\left(|x-y| \le s/64\right) = \int_{-s/64}^{s/64} f(x+tu)\,dt \Big/ \int_{-\infty}^{\infty} f(x+tu)\,dt.$$

Here

$$\int_{-\infty}^{\infty} f(x+tu)\,dt \ge \int_{-s/2}^{s/2} f(x+tu)\,dt \ge s\beta f(x),$$

while

$$\int_{-s/64}^{s/64} f(x+tu)\,dt \le \frac{s}{32}\beta^{-1/32} f(x),$$

and thus

$$\mathsf{P}\left(|x-y| \le \frac{s}{64}\right) \le \frac{1}{32}\beta^{-33/32} < \frac{1}{16}.$$

Thus the probability that $|x-y| \le s/64$ is bounded by $2p + 1/16 \le 1/8$. This proves the lemma. $\square$

6.3. Conductance. For a point $u \in K$, let P_u be the distribution obtained by taking one hit-and-run step from u. Let $\mu_f(u,x)$ be the integral of f along the line through u and x. Then,

$$P_u(A) = \frac{2}{n\pi_n} \int_A \frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}}. \tag{10}$$

The next lemma is analogous to Lemma 4.1. It holds for any logconcave density f, although we know how to use it only for the exponential density. Its proof is closely related to that of Lemma 7.2 in [13].

LEMMA 6.8. *Let $u, v \in K$. Suppose that*

$$d_f(u,v) < \frac{1}{128 \ln(3+\alpha(u))} \quad \text{and} \quad |u-v| < \frac{1}{4\sqrt{n}} \max\{F(u), F(v)\}.$$

Then

$$d_{tv}(P_u, P_v) < 1 - \frac{1}{500}.$$

Proof. We will show that there exists a set $A \subseteq K$ such that $P_u(A) \geq \frac{1}{2}$ and for any subset $A' \subset A$,

$$P_v(A') \geq \frac{1}{200} P_u(A').$$

To this end, we define certain "bad" lines through u. Let σ be the uniform probability measure on lines through u.

Let B_1 be the set of lines that are not almost orthogonal to $u-v$, in the sense that for any point $x \neq u$ on the line,

$$|(x-u)^T(u-v)| > \frac{2}{\sqrt{n}} |x-u||u-v|.$$

The measure of this subset can be bounded as $\sigma(B_1) \leq 1/8$.

Next, let B_2 be the set of all lines through u which contain a point y with $f(y) > 2\alpha(u)f(u)$ (see section 6.2 for the definition of α). By Lemma 3.5(a) in [13], if we select a line from B_2, then with probability at least $1/2$, a random step along this line takes us to a point x with $f(x) \geq \alpha(u)f(u)$. From the definition of $\alpha(u)$, this can happen with probability at most $1/16$, which implies that $\sigma(B_2) \leq 1/8$.

Let A be the set of points x in K which are not on any of the lines in $B_1 \cup B_2$, and which are far from u in the sense of Lemma 6.6:

$$|x-u| \geq \frac{1}{32} \max\{|u-p|, |u-q|\}.$$

Applying Lemma 6.6 to each such line, we get

$$P_u(A) \geq \left(1 - \frac{1}{8} - \frac{1}{8}\right) \frac{3}{4} > \frac{1}{2}.$$

We will show that for any subset $A' \subseteq A$,

$$P_v(A') \geq \frac{1}{200} P_u(A')$$

using the next two claims.

Claim 1. For every $x \in A$,

$$|x - v| \le \left(1 + \frac{1}{n}\right) |x - u|.$$

Claim 2. For every $x \in A$,

$$\mu_f(v, x) < 64 \frac{|x - v|}{|x - u|} \mu_f(u, x).$$

Claim 1 is easy to prove (cf. [10]), and the proof of Claim 2 is identical to that given in [13]. Thus, for any $A' \subset A$,

$$\begin{aligned} P_v(A') &= \frac{2}{n\pi_n} \int_{A'} \frac{f(x)\,dx}{\mu_f(v, x)|x - v|^{n-1}} \\ &\ge \frac{2}{64n\pi_n} \int_{A'} \frac{|x - u| f(x)\,dx}{\mu_f(u, x)|x - v|^{n}} \\ &\ge \frac{2}{64en\pi_n} \int_{A'} \frac{f(x)\,dx}{\mu_f(u, x)|x - u|^{n-1}} \\ &\ge \frac{1}{64e} P_u(A'). \end{aligned}$$

The lemma follows. □

We are now ready to state and prove the main theorem.

THEOREM 6.9. *Let f be a density in $\mathbb{R}^n$ proportional to $e^{-a^T x}$ whose support is a convex body K of diameter D. Assume that any level set of measure $1/8$ contains a ball of radius r. Then for any subset S, with $\pi_f(S) = p \le 1/2$, the conductance of hit-and-run satisfies*

$$\phi(S) \ge \frac{r}{10^{13} n D \ln(\frac{nD}{rp})}.$$

Proof. The proof has the same structure as that of Theorem 4.2.

Let $K = S_1 \cup S_2$ be a partition into measurable sets, where $S_1 = S$ and $p = \pi_f(S_1) \le \pi_f(S_2)$. We will prove that

$$\int_{S_1} P_x(S_2)\,dx \ge \frac{r}{10^{13} n D \ln \frac{nD}{rp}} \pi_f(S_1). \tag{11}$$

Consider the points that are deep inside these sets:

$$S_1' = \left\{ x \in S_1 : P_x(S_2) < \frac{1}{1000} \right\} \quad \text{and} \quad S_2' = \left\{ x \in S_2 : P_x(S_1) < \frac{1}{1000} \right\}.$$

Let S_3' be the rest, i.e., $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $\pi_f(S_1') < \pi_f(S_1)/2$. Then

$$\int_{S_1} P_x(S_2)\,dx \ge \frac{1}{1000} \pi_f(S_1 \setminus S_1') \ge \frac{1}{2000} \pi_f(S_1),$$

which proves (11).

So we can assume that $\pi_f(S_1') \geq \pi_f(S_1)/2$ and, similarly, that $\pi_f(S_2') \geq \pi_f(S_2)/2$.

Next, define the exceptional subset W as the set of points u for which $\alpha(u)$ is very large.

$$W = \left\{ u \in S : \alpha(u) \geq \frac{2^{27} nD}{rp} \right\}.$$

By Lemma 6.2,

$$\pi_f(W) \leq \frac{rp}{2^{23} nD}.$$

Next, for any $u \in S_1' \setminus W$ and $v \in S_2' \setminus W$,

$$d_{tv}(P_u, P_v) \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{500}.$$

Thus, by Lemma 6.8, either

$$d_f(u,v) \geq \frac{1}{128 \ln(3 + \alpha(u))} \geq \frac{1}{2^{12} \ln \frac{nD}{rp}}$$

or

$$|u - v| \geq \frac{1}{4\sqrt{n}} \max\{F(u), F(v)\}.$$

But by Lemmas 6.1 and 6.7, this implies that either

$$d_K(u,v) \geq \frac{1}{2^{15} \ln \frac{nD}{rp} (1 + \ln G)} \tag{12}$$

or

$$|u - v| \geq \frac{1}{2^8 \sqrt{n}} \max\{s(u), s(v)\} \tag{13}$$

holds. Now, by Lemma 6.5, condition (12) implies that

$$d_K(u,v) \geq \frac{1}{2^{17} \ln \frac{nD}{rp}} \frac{\max s(x)}{\sqrt{n} D}. \tag{14}$$

Next, we define

$$h(x) = \frac{s(x)}{2^{19} D \sqrt{n} \ln \frac{nD}{rp}}$$

and apply Theorem 2.1 to the partition $S_1' \setminus W$, $S_2' \setminus W$ and the rest. If (14) holds, then clearly $h(x) \leq d_K(u,v)/3$. Otherwise, (13) holds. Let x be a point on the chord pq of K, say between u and q. Then, using the concavity of s (Lemma 6.3),

$$\begin{aligned} s(x) &\leq \frac{|x-p|}{|u-p|} s(u) \leq 2^8 \frac{|q-p|}{|u-p|} \sqrt{n} |u - v| \\ &\leq 2^8 d_K(u,v) D \sqrt{n} \end{aligned}$$

and hence $h(x) \le d_K(u,v)/3$ again. Thus,

$$\begin{aligned}\pi_f(S_3') &\ge \mathsf{E}_f(h)\pi_f(S_1' \setminus W)\pi_f(S_2' \setminus W) - \pi_f(W) \\ &\ge \frac{r}{2^{30} nD \ln \frac{nD}{ra}} \pi_f(S_1).\end{aligned}$$

Here we have used Lemma 6.4 and the bound on $\pi_f(W)$. Therefore,

$$\begin{aligned}\int_{S_1} P_x(S_2)\, dx &\ge \frac{1}{2} \cdot \frac{1}{1000} \pi_f(S_3') \\ &\ge \frac{r}{10^{13} nD \ln \frac{nD}{rp}} \pi_f(S_1),\end{aligned}$$

which again proves (11). ∎

6.4. Mixing time. Since f satisfies $\mathsf{E}_f(|x - z_f|^2) \le R^2$, we consider the restriction of f to the ball of radius $R \ln(4e/a)$ around z_f and then by Lemma 5.17 in [13], the measure of f outside this ball is at most $a/4$. In the proof of the conductance bound, we can consider the restriction of f to this set. In the bound on the conductance for a set of measure a, the diameter D is effectively replaced by $R \ln(4e/a)$.

The bound on the mixing time then follows by applying Theorem 6.9 along with either Corollary 1.6 in [12] or the average conductance theorem of [6]. For the latter, we have that for any subset of measure x, the conductance is at least

$$\phi(x) \ge \frac{cr}{nR \ln(nR/rx) \ln(4e/x)} \ge \frac{cr}{nR \ln^2(nR/rx)}.$$

Then the theorem of [6] implies that after $m > C(n^2R^2/r^2)\ln^5(MnR/r\varepsilon)$ steps, we have $d_{tv}(\sigma^m, \pi_f) < \varepsilon$.

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