CHAPTER 1

Some Simple Examples

The theory of smooth dynamical systems can be thought of as an outcrop of a more general theory known as *topological dynamics.* Topological dynamics deals with continuous actions of any topological group G on a topological space X. Smooth dynamical systems are smooth actions of the group $\mathbf{R}$ or $\mathbf{Z}$ on a differentiable manifold X. Naturally, adding extra structure in this way revolutionizes the subject, just as it does when one makes the transition from point-set topology to differential calculus. However, in our two opening chapters we are, for the most part, laying basic foundations in which differentiability does not play a significant role. Thus these chapters have a flavour of topological dynamics. We begin by illustrating a few fundamental definitions with some simple examples. We sidestep a detailed discussion of *group action* in the text, since this would slow us down at the outset. We do, however, return to this point in the appendix to this chapter.

I. FLOWS AND HOMEOMORPHISMS

Let G denote either the additive topological group $\mathbf{R}$ of real numbers or the additive (discrete) topological group $\mathbf{Z}$ of integers. A *dynamical system* on a topological space X is a continuous map $\phi: G \times X \to X$ such that, for all $x \in X$ and, for all $s, t \in G$,

(1.1)
$$\phi(s+t, x) = \phi(s, \phi(t, x))$$
$$\phi(0, x) = x.$$

(1.2) Examples. For any X, the *trivial* dynamical system is defined by $\phi(t, x) = x$. For $X = \mathbf{R}$, $\phi(t, x) = e^t x$ defines a C^ω (analytic) dynamical system on X.

The space X is called the *phase space* of ϕ. If X is a differentiable manifold and ϕ is a C^r map, for $r \geq 0$, then we call ϕ a C^r dynamical system. *Throughout the book the adjective "smooth" means "C^r for some $r \geq 1$".*

Let ϕ be a dynamical system on X. Given $t \in G$, we define the partial map $\phi^t : X \to X$ by $\phi^t(x) = \phi(t, x)$. If $G = \mathbf{R}$, we sometimes call ϕ^t the *time t map* of ϕ. Similarly, given $x \in X$, we define the partial map $\phi_x : G \to X$ by $\phi_x(t) = \phi(t, x)$. Note that if ϕ is C^r, then so are ϕ^t and ϕ_x. Equation (1.1) may be written as

(1.3)
$$\phi^{s+t} = \phi^s \phi^t$$
$$\phi^0 = id.$$

Throughout the book juxtaposition of maps denotes composition and id denotes the relevant identity map, here the identity on X.

(1.4) Proposition. *For all $t \in G$, ϕ^t is a homeomorphism. If ϕ is C^r then ϕ is a C^r diffeomorphism.*

Proof. By equations (1.3) $\phi^t \phi^{-t} = \phi^{-t} \phi^t = \phi^0 = id$. That is, ϕ^t is invertible, and its inverse is ϕ^{-t}, which is C^r when ϕ is C^r. □

For brevity we sometimes denote $\phi(t, x)$ by $t \cdot x$ when the context makes it obvious which dynamical system is under discussion. With this convention, equations (1.1) and (1.3) become

(1.5)
$$(s + t) \cdot x = s \cdot (t \cdot x)$$
$$0 \cdot x = x.$$

If $G = \mathbf{R}$ then the dynamical system ϕ is called a *flow* on X, or a *one-parameter group of homeomorphisms* of X. If $G = \mathbf{Z}$, then ϕ is completely determined by the homeomorphism ϕ^1, and it is usual to talk in terms of the homeomorphism rather than the dynamical system ϕ (sometimes called a *discrete dynamical system* or *discrete flow*) that it generates.

II. ORBITS

Let ϕ be a dynamical system on X. We define a relation $\sim$ on X by putting $x \sim y$ if and only if there exists $t \in G$ such that $\phi^t(x) = y$.

(1.6) Proposition. *The relation* $\sim$ *is an equivalence relation* □

The equivalence classes of $\sim$ are called *orbits* of ϕ, or of the homeomorphism ϕ^1 in the case $G = \mathbf{Z}$. For each $x \in X$, the equivalence class containing x is called the *orbit through* x. It is the image of the partial map $\phi_x : G \to X$. We denote it $G.x$ when it is clear to which dynamical system we refer. Proposition 1.6 implies that two orbits either coincide or are disjoint. We denote the quotient space $X/\sim$ by X/ϕ, and call it the *orbit space* of ϕ. The quotient map, which takes x to its equivalence class, is denoted $\gamma_\phi : X \to X/\phi$, or just $\gamma : X \to X/\phi$ when no ambiguity can occur. As usual, we give X/ϕ the finest topology with respect to which γ is continuous (that is, a subset U of X/ϕ is open in X/ϕ if and only if $\gamma^{-1}(U)$ is open in X).

III. EXAMPLES OF DYNAMICAL SYSTEMS

(1.7) Every orbit of the trivial dynamical system $t.x = x$ is a singleton $\{x\}$. (From now on we shall usually denote both the point and the subset ambiguously by x.)

(1.8) If $G = \mathbf{R}$ the non-trivial flow $t.x = e^t x$ of Example 1.2 has three orbits, namely the origin and the positive and negative half lines. In Figure 1.8 the arrows on the orbits indicate the *orientations* induced on them by the flow. That is to say, they give the direction in which $t.x$ moves as t increases.

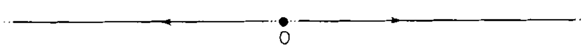

FIGURE 1.8

(1.9) For all $t, x \in \mathbf{R}$, put $t.x = x + t$. This flow has only one orbit, $\mathbf{R}$ itself.

FIGURE 1.9

(1.10) For all $t, x \in \mathbf{R}$, put $t.x = (x^{1/3} + t)^3$. Again this flow has $\mathbf{R}$ as its only orbit.

The preceding three examples are flows on the space $\mathbf{R}$ of real numbers with its standard topology. We now give a rather pathological example on $\mathbf{R}^\flat$, which is the set $\mathbf{R}$ with the indiscrete topology, in which the only open sets are $\mathbf{R}$ itself and the empty set.

(1.11) Regard $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of rational numbers, and, using the axiom of choice, extend 1 to a basis $\mathscr{B}$ of $\mathbf{R}$. Let $T:\mathbf{R}\to\mathbf{R}$ be the unique $\mathbf{Q}$-linear map such that $T(1)=1$ and $T(t)=0$ for all other $t\in\mathscr{B}$. Put $t\,.\,x=x+T(t)$ for all $t\in\mathbf{R}$ and all $x\in\mathbf{R}^\flat$. Then the orbit through x of this flow on $\mathbf{R}^\flat$ is the set $\{x+q\colon q\in\mathbf{Q}\}$.

Let S^1 denote the circle $\mathbf{R}/\mathbf{Z}$ (see Example 1.3 of Appendix A) and let $[x]\in S^1$ denote the equivalence class of $x\in\mathbf{R}$.

(1.12) *Rotation.* Define a diffeomorphism $f\colon S^1\to S^1$ by $f([x]=[x+\theta]$, for some fixed $[\theta]\in S^1$. Then each orbit of f consists of s points if θ is rational, $\theta=r/s$ with r, s coprime integers and $s>0$. If, on the other hand, θ is irrational, then every orbit of f is dense in S^1, for the orbit through $[0]$ is the infinite cyclic subgroup of S^1 generated by $[\theta]$.

(1.13) *Rotation flows.* For any $\theta\in\mathbf{R}$, put $t\,.\,[x]=[x+\theta t]$. If $\theta=0$ we have the trivial flow on S^1. Otherwise we have the single orbit S^1. If we embed S^1 in the plane by the standard embedding $[x]\mapsto(\cos 2\pi x, \sin 2\pi x)$, then the rotation is anti-clockwise if θ is positive, and clockwise if θ is negative. We call θ the *speed* of the flow.

The next four examples describe flows on the plane $\mathbf{R}^2$. It is sometimes convenient to identify $\mathbf{R}^2$ with the complex line $\mathbf{C}$, since the two are indistinguishable as topological spaces.

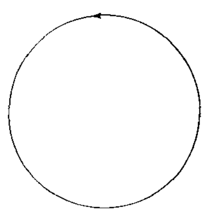

FIGURE 1.13 $(\theta>0)$

(1.14) For all $t\in\mathbf{R}$ and for all $(x, y)\in\mathbf{R}^2$, put $t\,.\,(x, y)=(x\,e^t, y\,e^t)$. The origin is the only point orbit, and all other orbits are open rays issuing from the origin.

(1.15) If the formula giving the previous example is changed slightly to $t\,.(x, y)=(x\,e^t, y\,e^{-t})$, the *phase portrait* (i.e. the partition of the phase space into orbits) is radically altered, since the new flow has only two orbits beginning at the origin. The picture (without arrows) is familiar to anyone who has sketched contours of a real valued function on $\mathbf{R}^2$. It is associated with a *saddle-point* of the function. We shall investigate later the connection between contours and flows (see Example 3.3).

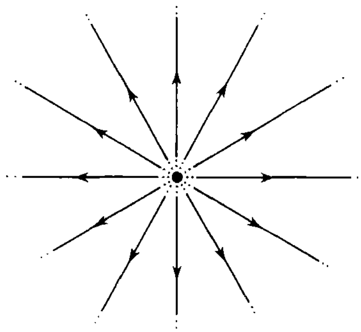

FIGURE 1.14

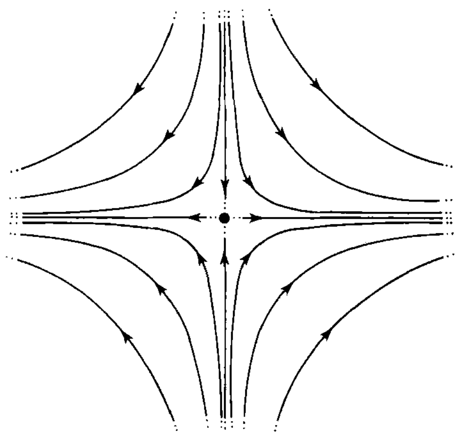

FIGURE 1.15

(1.16) For all $t \in \mathbf{R}$ and $z = x + iy \in \mathbf{C}$, put $t \cdot z = z\, e^{it}$. The origin is a point orbit, and the other orbits are all circles with centre the origin.

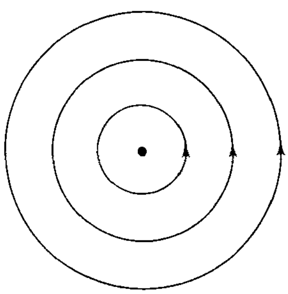

FIGURE 1.16

(1.17) For all $t \in \mathbf{R}$ and $z \in \mathbf{C}$, put $t \,.\, z = z\, e^{(i-1)t}$. The origin is a point orbit, and all other orbits spiral in towards it.

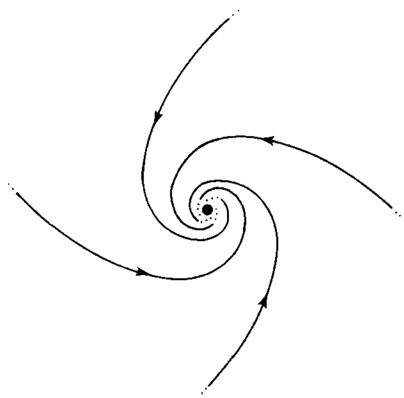

FIGURE 1.17

(1.18) Exercise. Sketch the phase portraits of the flows on $\mathbf{R}^2$ given by

(i) $t \,.\, (x, y) = (x, tx + y)$,

(ii) $t \,.\, (x, y) = (x\, e^{-t}, (tx + y)\, e^{-t})$.

(1.19) Exercise. Describe the orbit spaces of the dynamical systems (1.7)–(1.17). Which are Hausdorff topological spaces?

IV. CONSTRUCTING SYSTEMS

There are various ways of constructing new dynamical systems from given ones.

(1.20) *Product.* Let $\phi: G \times X \to X$ and $\psi: G \times Y \to Y$ be dynamical systems. The *product* of the two systems is the dynamical system on $X \times Y$ defined, for all $t \in G$ and $(x, y) \in X \times Y$, by

$$t \,.\, (x, y) = (t \,.\, x, t \,.\, y).$$

We shall usually denote the product system by $\phi \times \psi$, although this is strictly speaking an abuse of notation (the cartesian product of the maps ϕ and ψ has domain $G \times X \times G \times Y$, not $G \times X \times Y$, and takes (s, x, t, y) to $(s \,.\, x, t \,.\, y)$). Note that the product of two discrete dynamical systems is the system corresponding to the cartesian product of their generating homeomorphisms.

One can regard this construction either as a way of building up complicated examples from simple ones, or, perhaps more importantly, as a way of decomposing complicated examples into simple ones. For instance Example 1.14 is the product of two copies of Example 1.8, and Example 1.15 is the product of Example 1.8 and the flow $t.x = xe^{-t}$ on $\mathbf{R}$ (see Figure 1.20).

FIGURE 1.20

(1.21) Example. Let ϕ be the flow of Example 1.8 and let ψ be the rotation flow of Example 1.13 (for some $\theta > 0$, say). Then $\phi \times \psi$ is a flow on the circular cylinder $\mathbf{R} \times S^1$. The circle $\{0\} \times S^1$ is an orbit, and all other orbits spiral away from it.

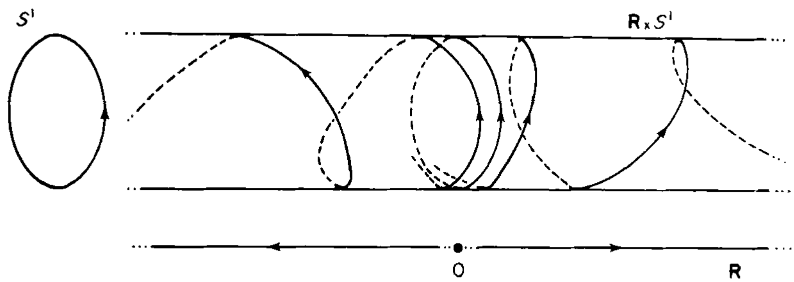

FIGURE 1.21

(1.22) *Embedding.* Let $\phi: \mathbf{R} \times X \to X$ be any flow. For any $t \in \mathbf{R}$, $\phi^t: X \to X$ is a homeomorphism which generates a discrete dynamical system $\psi: \mathbf{Z} \times X \to X$. We say that ϕ^t (or ψ) is *embedded* in ϕ. For example, any rotation of S^1 (Example 1.12) is embedded in any non-trivial rotation flow on S^1 (Example 1.13).

It is not true that every homeomorphism of every topological space X can be embedded in a flow on X. For example, when X is a pair of points (with the discrete topology) there is only one flow on X, namely the trivial flow, so only the identity homeomorphism of X is embeddable (in a flow on X). However there is another homeomorphism of X, namely the one which interchanges the two points. More generally, any embeddable homeomorphism of X is isotopic to the identity, since ϕ itself supplies an isotopy from ϕ^t to ϕ^0. Thus, for example, the homeomorphism $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = -x$ is not embeddable.

Summing up, every flow on X yields a homeomorphism of X (in fact, many homeomorphisms), but the reverse is not usually true. This is rather unfortunate, since we wish to develop the theories of flows and homeomorphisms side by side, and we need some way of associating with a given homeomorphism f a flow with similar properties. We can do this provided we allow the flow to be on a larger space Y. Thus we have the space X embedded in the space Y and the homeomorphism of X embedded in a flow ϕ on Y. The standard, and most economical, way of constructing the space Y and the flow ϕ is known as *suspension* (warning: this is different from the construction of the same name in algebraic topology).

(1.23) *Suspension.* Let $f: X \to X$ be a homeomorphism (generating a discrete dynamical system ψ). Let $\sim$ be the equivalence relation defined on $\mathbf{R} \times X$ by $(u, x) \sim (v, y)$ if and only if $u = v + m$ for some $m \in \mathbf{Z}$ and $y = f^m(x)$. Then there is a flow $\phi: \mathbf{R} \times Y \to Y$ on $Y = (\mathbf{R} \times X)/\sim$ defined by $\phi(t, [u, x]) = [u + t, x]$ where $[u, x]$ denotes the equivalence class of $(u, x) \in \mathbf{R} \times X$. The flow ϕ is called the *suspension* of the homeomorphism f (or of ψ). For any $u \in R$, the restriction of ϕ^1 to any *cross section* $[u, X]$ with the obvious identification $[u, X] = X$ coincides with f.

For example, if $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = -x$, then the suspension is a flow on the open Möbius band, and all its orbits are topologically circles (see Figure 1.23).

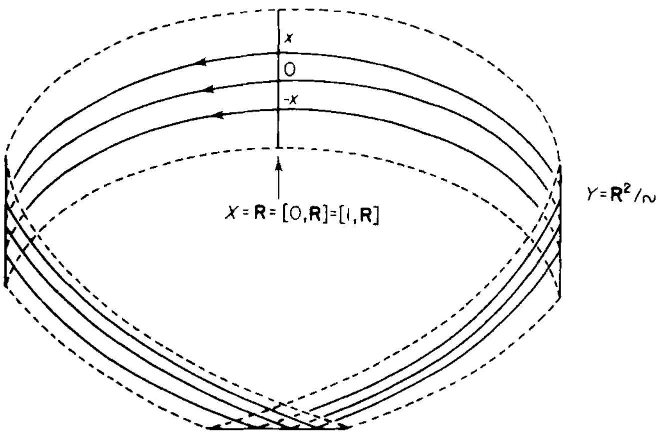

FIGURE 1.23

(1.24) Example. A *rational flow* ϕ on the torus $T^2 = S^1 \times S^1$ is given by suspending a rational rotation of S^1 (Example 1.12). Each orbit of ϕ is topologically a circle. The two diagrams in Figure 1.24 illustrate one orbit of ϕ for a rotation $\theta = \frac{2}{3}$. The first shows the torus cut open. To glue it together again, we need to identify the top and bottom edge AB and then the two

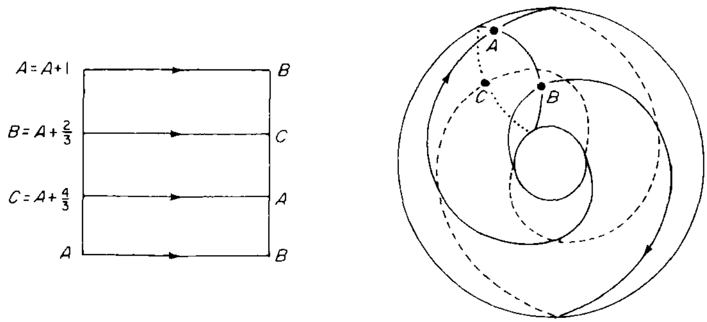

FIGURE 1.24

ends of the cylinder as indicated by the lettering. We then get the second picture, which portrays T^2 embedded as an "anchor ring" or "tyre inner tube" in $\mathbf{R}^3$. Each orbit of ϕ winds twice round "the air inside the tube" and three times round "the axis of the wheel".

(1.25) Example. An *irrational flow* ϕ on T^2 is defined by suspending an irrational rotation of S^1. In this case, ϕ has neither point orbits nor circle orbits. Every orbit of ϕ is the image of $\mathbf{R}$ under a continuous injection. However, none of these embeddings is a topological embedding; in fact, every orbit of ϕ is dense in T^2.

Rational and irrational flows on T^2 may be given another definition as the product of two rotation flows on S^1 (Example 1.13). One gets a rational flow when the ratio of the speeds of the two factors is rational, and an irrational flow when it is irrational. We discuss the relation between systems given by the two different constructions in Chapter 2 (Example 2.9). We also show there that all rational flows are essentially the same, but that there are infinitely many different types of irrational flow. (This is initially rather startling; looking at the pictures one might almost expect the reverse to be the case.) For the moment we just draw the important conclusion that *the phase portrait of a product flow is not uniquely determined by the phase portraits of its factors,* since the phase portraits of rational and irrational flows are completely different topologically, but come from factors with identical phase portraits.

(1.26) *Induced systems.* Let $\phi: G \times X \to X$ be a dynamical system on X, let $\alpha: G \to G$ be a continuous automorphism of the additive group G, and let $h: X \to Y$ be a homeomorphism. Then $\psi = h\phi(\alpha \times h)^{-1}$ is a dynamical system on Y. We call ψ the dynamical system *induced from ϕ by the pair* (α, h) (or *by h*, if $\alpha = id$). For example, if $h = id: X \to X$ and $\alpha = -id: \mathbf{R} \to \mathbf{R}$, we obtain ϕ^-, the *reverse flow* of ϕ, given by $\phi^-(t, x) = \phi(-t, x)$. Intuitively in ϕ^- points move along the orbits of ϕ with the same speed but in the opposite direction.

Notice that the continuous automorphisms of $\mathbf{R}$ are exactly the linear maps $t \mapsto ct$ where c is a non zero real constant. One talks of *speeding up* (or *slowing down*) the flow *by the factor c* when one changes to the flow induced by (α, id). The continuous automorphisms of $\mathbf{Z}$ are just id and $-id$, so in the discrete case the homeomorphism induced from f is just hfh^{-1} or its inverse.

(1.27) Example. Let $h: \mathbf{R}\times S^1 \to \mathbf{R}^2\backslash\{0\}$ be the homeomorphism defined for all $x \in \mathbf{R}$ and all $z \in S^1$, by $h(x, z) = e^x z$, and let ζ be the flow on $R\times S^1$ defined in Example 1.21. Then h induces a flow on $\mathbf{R}^2\backslash\{0\}$ which we may extend to a flow on $\mathbf{R}^2$ by making the origin a point orbit. The phase portrait of this flow is illustrated in Figure 1.27.

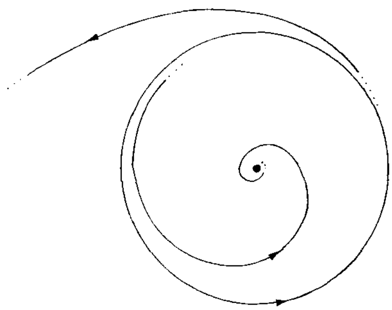

FIGURE 1.27

The construction of induced systems is not in itself a particularly interesting process, since one usually regards the induced systems as being equivalent to the original ones. However the following generalization produces genuinely new systems.

(1.28) *Quotient systems.* Let $\phi: G\times X \to X$ be a dynamical system on X and let $\sim$ be an equivalence relation on X such that, for all $t \in G$ and all $x, y \in X$, $\phi(t, x) \sim \phi(t, y)$ if (and hence only if) $x \sim y$. Then ϕ induces a dynamical system ψ, called the *quotient* system, on the quotient space $X/\sim$ by $\psi(t, [x]) = [\phi(t, x)]$, where $t \in \mathbf{R}$ and $[x]$ is the equivalence class of $x \in X$.

(1.29) Example. If f and g are *commuting homeomorphisms* of X (i.e. $fg = gf$) then f takes orbits of g onto orbits of g and hence induces a homeomorphism of the orbit space of g. This is an example of a quotient system, where ϕ is the discrete dynamical system generated by f and $\sim$ is the equivalence relation giving orbits of g as equivalence classes. Similarly if ϕ and ψ are *commuting flows* on X (i.e. $\phi^s\psi^t = \psi^t\phi^s$ for all $s, t \in \mathbf{R}$) then ϕ

induces a quotient flow on the orbit space X/ψ. Notice that such a pair of commuting flows on X is equivalent to an action of $\mathbf{R}^2$ on X (define $(s, t) . x$ to be $\phi^s(\psi^t(x))$).

(1.30) Example. Define an equivalence relation $\sim$ on $\mathbf{R}^n$ by $x \sim y$ if and only if $x - y \in \mathbf{Z}^n$. The quotient $\mathbf{R}^n/\sim$ is the n-dimensional torus T^n, the cartesian product of n copies of the circle S^1. Let f be a linear automorphism of $\mathbf{R}^n$ whose matrix A (with respect to the standard basis of $\mathbf{R}^n$) is in $GL_n(\mathbf{Z})$. That is to say, A has integer entries and $\det A = \pm 1$ (or equivalently, A^{-1} also has integer entries). Then f maps $\mathbf{Z}^n$ onto itself, and hence f and f^{-1} preserve the relation $\sim$. Thus f induces a homeomorphism (in fact, a diffeomorphism) of T^n. Suppose further that A has no eigenvalues on the unit circle in the Argand diagram $\mathbf{C}$. Then we call the induced homeomorphism of T^h a *hyperbolic toral automorphism*. Thus, for example, we use the term if

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{but not if } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Considering for a moment the particular case

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

we observe that the phase portrait of f is qualitatively the same as that of the time 1 map, ϕ^1 say, of the flow in Example 1.15. Thus the saddle point picture in Figure 1.15 is still a good illustration for it, with the reservation that the hyperbolae are merely unions of certain orbits, and do not have any special significance. The straight lines, on the other hand, are very significant; they are the loci of points whose iterates under positive or negative iterates of f tend to the fixed point 0. The important difference between f and ϕ^1 is that for f these lines have irrational slope, and so the corresponding curves on T^2 for the induced toral automorphism wrap themselves densely around T^2. This rather complicated behaviour is typical of hyperbolic toral automorphisms.

Another remarkable fact about any hyperbolic toral automorphism $g : T^n \to T^n$ is that both its periodic and non-periodic point sets are dense in T^n. Of course, a point $x \in T^n$ is *periodic* if $g^r(x) = x$ for some $r > 0$.

(1.31) Theorem. *The periodic set of the hyperbolic toral automorphism $g : T^n \to T^n$ is precisely $\mathbf{Q}^n/\sim$, where $\mathbf{Q}$ is the rational numbers.*

Proof. First suppose that $x \in \mathbf{Q}^n$, and let m be the L.C.M. of the denominators of the coordinates of x. Since A, the matrix of f, has integer entries, the coordinates of $f^r(x)$ $(r > 0)$ are integer combinations of those of x, and hence their denominators have L.C.M. dividing m. There are only finitely many points $[y]$ of T^n corresponding to points $y \in \mathbf{Q}^n$ whose coordinates satisfy the above condition. Thus there are only finitely many possible values for $[f^r(x)] = g^r([x])$, and hence $g^r([x]) = g^s([x])$ for some $r > s > 0$. Thus $g^{r-s}([x]) = [x]$, and $[x]$ is a periodic point of g.

Conversely, suppose that $x \in \mathbf{R}^n$ is such that $[x] \in T^n$ is a periodic point of g. Then $g^r([x]) = [x]$ for some $r > 0$, and hence $(f^r - id)(x) = y$, say, is in $\mathbf{Z}^n$. Now $f^r - id$ is a linear automorphism (since 1 is not an eigenvalue of f^r) with integer entries. Thus $(f^r - id)^{-1} = (\det(f^r - id))^{-1}$ adj $(f^r - id)$ has rational entries, and hence $x = (f^r - id)^{-1}(y)$ is in $\mathbf{Q}^n$. □

Hyperbolic toral automorphisms are the simplest examples of Anosov diffeomorphisms (see Chapter 7) on compact manifolds. They first aroused interest as a counterexample (due to R. Thom) to the conjecture that structurally stable diffeomorphisms (in the sense of Chapter 7) have finite periodic sets.

(1.32) Exercise. Verify that induced and quotient systems are indeed dynamical systems.

(1.33) Exercise. Let ϕ and ψ be flows on $\mathbf{R}^2$ defined by $\phi(t, z) = z\,e^t$ and $\psi(t, z) = z\,e^{it}$. Describe (i) the flow that ψ induces on the orbit space of ϕ, and (ii) the flow that ϕ induces on the orbit space of ψ.

V. PROPERTIES OF ORBITS

A cursory examination of the preceding examples reveals at least five topologically different types of orbit in the case of flows, and infinitely many in the case of homeomorphisms. We now make some remarks which are directly concerned with sorting out orbits into topological types.

Let ϕ be a dynamical system on a topological space X. For each $x \in X$, the subset $G_x = \{t \in G : \phi(t, x) = x\}$ is a subgroup of G called the *stabilizer* (or *isotropy subgroup*) *of* x, or *of* ϕ *at* x.

(1.34) Proposition. *If X is a T_1 space, then for all $x \in X$, G_x is a closed subgroup of G.*

Proof. The subset $\{x\}$ is closed, and $G_x = \phi_x^{-1}(\{x\})$. □

We must place some sort of separation condition on X if we require closed stabilizers. Example 1.11 shows that if X has a very coarse topology then $\mathbf{R}_x$ may be a rather unpleasant non-closed subgroup of $\mathbf{R}$. However one may get by with weaker conditions than in the above proposition.

(1.35) Exercise. Prove that Proposition 1.34 still holds with T_1 replaced by T_0.

The connection between the stabilizer of a point and the topology of its orbit is made as follows:

(1.36) Proposition. *For all $x \in X$, there is a continuous bijection*

$$\beta : G/G_x \to G \cdot x.$$

Proof. Let $\pi: G \to G/G_x$ be the quotient map. Define β by $\beta(\pi(t)) = t \cdot x$. To show that β is well defined, let $\pi(s) = \pi(t)$. Thus $s - t \in G_x$, whence $(s-t) \cdot x = x$ and so $s \cdot x = t \cdot x$.

Suppose now that $\beta(\pi(s)) = \beta(\pi(t))$, i.e. that $s \cdot x = t \cdot x$. Thus $s - t \in G_x$, and so $\pi(s) = \pi(t)$. This proves that β is injective. Since β is trivially surjective, we have shown that β is bijective.

Finally, let $A = U \cap (G \cdot x)$ be an open subset of $G \cdot x$, where U is open in X. Then $W = \phi_x^{-1}(U)$ is open in G. But $W = \pi^{-1}\beta^{-1}(U)$. Hence $\beta^{-1}(U)$ is open in G/G_x, by definition of the topology on G/G_x. □

(1.37) Corollary. *If X is Hausdorff and G/G_x is compact, then the orbit $G \cdot x$ is homeomorphic to G/G_x* □

To capitalize on this corollary, we must analyse the possible closed (by Proposition 1.34) subgroups of G.

In the case $G = \mathbf{Z}$, with ϕ generated by the homeomorphism f, G_x is either (i) the zero subgroup or (ii) isomorphic to $\mathbf{Z}$ itself and generated by some positive integer n. Case (ii) gives a compact quotient space, a set of n distinct points. Correspondingly the orbit is a set of n distinct points $\{x, f(x), \ldots, f^{n-1}(x)\}$, and x satisfies $f^n(x) = x$. We say that x is *periodic* of *period* n and write per $x = n$. If $n = 1$ we call x a *fixed point.*

One may easily prove that any closed subgroup H of $\mathbf{R}$ is either (i) the zero subgroup, (ii) $\mathbf{R}$ itself, or (iii) an infinite cyclic subgroup generated by some positive number t_0. (*Hint*: if $H \neq \{0\}$, $\inf\{t \in H: t > 0\} \in H$.) Thus if $G = \mathbf{R}$ and X is Hausdorff, we have correspondingly the three possibilities that G/G_x is (i) homeomorphic to $\mathbf{R}$ itself (ii) a single point or (iii) homeomorphic to the circle. In case (ii) the orbit $G \cdot x$ is the single point x, which is called a *fixed point* of ϕ. In case (iii) $G \cdot x$ is homeomorphic to the circle S^1, and $\phi_x: \mathbf{R} \to X$ is periodic of period t_0. One calls $G \cdot x$ a *periodic orbit, circle orbit, cycle* or *closed orbit.* The last term is open to the criticism that point orbits are also closed in X, and orbits in case (i) may very well be closed in X as well (see Example 1.9 above. We shall show in Corollary 2.36 below that this never happens if X is compact). However, it is probably the commonest of the four terms, so we shall employ it whenever it does not create actual confusion. We call t_0 the *period* of the orbit $G \cdot x$, and write per $x = t_0$.

(1.38) Exercise. Let X be Hausdorff. Prove that the *fixed point set* Fix ϕ of a dynamical system ϕ on X is closed in X. Find, among the examples of this chapter, counterexamples to the following statements:

(i) The *periodic point set* Per f of a homeomorphism $f: X \to X$ is closed in X.

(ii) The set of all periodic points with a given period n_0 of a homeomorphism $f: X \to X$ is closed in X.

(iii) the set of all points on periodic orbits with a given period t_0 of a flow ϕ on X is closed in X.

(1.39) Exercise. Show that if p is a fixed point of a flow ϕ then, for any neighbourhood U of p there exists a neighbourhood V of p such that $\phi(t, x) \in U$ for all $x \in V$ and all $t \in [0, 1]$. Deduce that $\phi(t, x) \to p$ as $t \to \infty$ if and only if $\phi(n, x) \to p$ as $n \to \infty$, where $t \in \mathbf{R}$ and $n \in \mathbf{Z}$. Prove, similarly, that if $\mathbf{R}.p$ is a closed orbit of a flow ϕ on a metric space then the distance $d(\phi(t, x), \phi(t, p)) \to 0$ as $t \to \infty$ in $\mathbf{R}$ if and only if $\phi(n\tau, x) \to \phi(n\tau, p) = p$ as $n \to \infty$ in $\mathbf{Z}$, where τ is the period of $\mathbf{R}.p$.

(1.40) Exercise. We have shown that every orbit of a flow on a Hausdorff space either is a point, is homeomorphic to the circle or is the image of $\mathbf{R}$ under a continuous bijection. Prove that these three possibilities are mutually exclusive.

In case (i) the continuous bijection β of Proposition 1.36 is (identified with) the map $\phi_x: \mathbf{R} \to \mathbf{R}.x$. It may or may not be the case that this map is a homeomorphism. It is in most of the examples given above, but it is not for irrational flow on the torus (Example 1.25). It turns out that the orbit $\mathbf{R}.x$ is homeomorphic to $\mathbf{R}$ (or, equivalently, ϕ_x is a homeomorphism) if and only if it is locally compact. For this result, see Theorem 2.51 below, or construct a proof directly, in which case you will probably need the Baire category theorem (see § 5.3 of Chapter 9 of Bourbaki [1]).

There is, we should finally say, one rather obvious topological property that is shared by orbits of flows on all topological spaces, Hausdorff or not.

(1.41) Proposition. *Every orbit of every flow is connected.*

Proof. For any flow ϕ on any space X and for all $x \in X$, ϕ_x is continuous, and so preserves the connectedness of $\mathbf{R}$. □

Appendix 1

I. GROUP ACTIONS

Let X be a topological space. The set Homeo X of all homeomorphisms of X (onto itself) acquires the structure of a group when one defines the group product of two such homeomorphisms f and g to be the composed map fg. The identity of Homeo X is the identity map $id: X \to X$ (sending each $x \in X$ to itself), and the group inverse of f is the inverse map f^{-1} (defined, for all $x, y \in X$, by $f^{-1}(y) = x$ if and only if $f(x) = y$). One would like to turn Homeo X into a *topological group* (that is, to give it a topology with respect to which the multiplication map $(f, g) \to fg$ and the inversion map $f \to f^{-1}$ are continuous). One would then define an *action* of a topological group G on X to be a continuous homomorphism $\alpha: G \to \text{Homeo}\, X$. The triple (G, X, α) is known as a *topological transformation group. Topological dynamics* is the study of topological transformation groups.

The reader may already have encountered the *compact-open* (C.O.) *topology* as an example of a function space topology. A sub-basis for this topology for Homeo X is given by all subsets of the form $\{f \in \text{Homeo}\, X : f(K) \subset U\}$, where K, U range respectively over all compact and all open subsets of X. Thus every open subset in the C.O. topology is a union of finite intersections of subsets of the given form. Homeo X is not always a topological group with respect to the C.O. topology. Group multiplication may fail to be continuous when X is not locally compact, and so may inversion even when X is locally compact. However X *is* a topological group with respect to the C.O. topology if X is either compact Hausdorff or locally connected, locally compact Hausdorff. Moreover the former condition suggests that by compactifying X we may obtain a suitable topology (called the Arens G-topology) on Homeo X for any locally compact Hausdorff X. The interested reader is referred to § 3 of Chapter 9 of Bourbaki [1] together with its associated exercises.

The connection between group actions and dynamical systems is fairly obvious. Let G be a topological group and let X be a topological space such that Homeo X is a topological group with respect to the C.O. topology. Any map $\alpha: G \to \text{Homeo } X$ determines a map $\phi: G \times X \to X$ given by $\phi(g, x) = \alpha(g)(x)$. The map α is a homomorphism if and only if, for all $x \in X$ and all $g, h \in G$,

(A.1) $$\phi(g*h, x) = \phi(g, \phi(h, x)),$$

and (redundantly)

(A.2) $$\phi(e, x) = x,$$

where $*$ is the group operation of G and e is the identity element of G. When $G = \mathbf{R}$ or $\mathbf{Z}$ we have the equations (1.1) in the definition of "dynamical system". Conversely, if $\phi \times X \to X$ is a continuous map satisfying (1) and (2) then the argument of Proposition 1.4 shows that ϕ determines a map $\alpha: G \to \text{Homeo } X$ by $\alpha(g) = \phi^g$, and α is a homomorphism, as above. To complete the picture we have to relate continuity of α and ϕ. It is straightforward to show that if ϕ is continuous then α is continuous. One checks that the inverse image of a sub-basic open set is open in G; this requires no extra conditions on G and X. Showing, conversely, that if α is continuous then ϕ is continuous requires a little more care. The obvious argument needs X to be locally compact (i.e. for all $x \in X$, every neighbourhood of x contains a compact neighbourhood of x). However this is not usually a serious restriction, since we are already assuming some such condition in order to make Homeo X a topological group.

Later in the book we restrict our attention to smooth manifolds X and smooth dynamical systems ϕ. Corresponding to such systems we would naturally expect a smooth version of the above theory of group actions. The analogue of a topological group in the smooth theory is a *Lie group*, which is a C^∞ manifold with a group structure such that composition and inversion are C^∞ maps. The group need not be abelian. A (*left*) *Lie transformation group* or (*left*) *action* of a Lie group G on a C^∞ manifold X is usually defined to be a C^∞ map $\phi: G \times X \to X$ satisfying the multiplicative version of (1.1)

$$\phi(gh, x) = \phi(g, \phi(h, x)), \qquad \phi(e, x) = x.$$

See, for example, Warner [1] or Brickell and Clark [1]. Another approach would be to define an action as a C^∞ homomorphism of G into $\text{Diff}^\infty X$, the space of all C^∞ diffeomorphisms of X, but it is not straightforward to give $\text{Diff}^\infty X$ a Lie group structure (see Leslie [1]).

There is a good deal of general theory of Lie group actions, especially for compact G. We shall not investigate this theory; we are quite happy to work with the particular groups $\mathbf{R}$ and $\mathbf{Z}$.