APPENDIX C

The Contraction Mapping Theorem

Let X and Y be non-empty metric spaces, with distance function denoted by d. Let κ be any positive number. A map $f: X \to Y$ is *Lipschitz* (*with constant* κ) if, for all x and $x' \in X$,

$$d(f(x), f(x')) \leqslant \kappa d(x, x').$$

The chords of the graph of f have slope $\leqslant \kappa$ (see Figure C.1). Clearly any

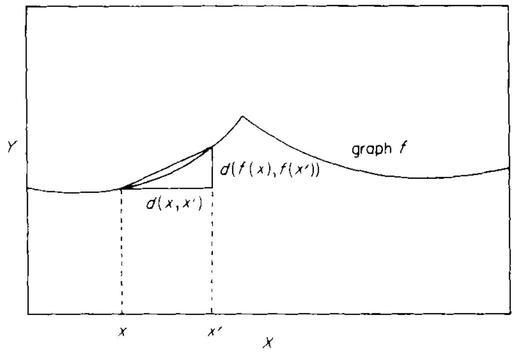

FIGURE C.1

Lipschitz map is continuous (in fact, uniformly continuous). An invertible Lipschitz map with Lipschitz inverse is sometimes called a *Lipeomorphism*. A map f is *locally Lipschitz* if every $x \in X$ has a neighbourhood on which f is Lipschitz.

(C.2) Proposition. *Let X and Y be subsets of Banach spaces and let $f: X \to Y$ be a map. If f is C^1 with $|Df(x)|$ bounded by κ and if X is convex, then f is Lipschitz with constant κ. In particular, any C^1 map is locally Lipschitz.*

Conversely, if f is Lipschitz with constant κ and f is differentiable at x then $|Df(x)| \leq \kappa$.

Proof. These are immediate consequences of the mean value theorem (see § 4 of Chapter 5 of Lang [2]) and the definition of differentiability. □

(C.3) Exercise. Which of the following maps are Lipschitz?

(i) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin^2 x$,

(ii) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^{1/3}$,

(iii) $f: \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x^2 + y^2$,

(iv) $f: \mathbf{E} \to \mathbf{R}$ defined by $f(x) = |x|$, for any norm $|\ |$ on a vector space $\mathbf{E}$.

We say that f is a (*metric*) contraction if it is Lipschitz with constant $\kappa < 1$. If f is invertible and f^{-1} is a contraction we call f an *expansion*. When $X \cap Y$ is non-empty, a *fixed point* of f is any $x \in X \cap Y$ such that $f(x) = x$. One of the simplest and yet most widely used of all fixed point theorems is due to Banach and Cacciopoli. The idea is as follows. Suppose that $\chi: X \to X$ is a contraction, with Lipschitz constant $\kappa < 1$. Let $x_0 \in X$, and choose a number r with $r(1-\kappa) > d(x_0, \chi(x_0))$. If $B_r(x)$ denotes the closed ball with centre x and radius r in X, then $B_{\kappa r}(\chi(x_0))$ is contained in $B_r(x_0)$ (see Figure C.4). Since χ

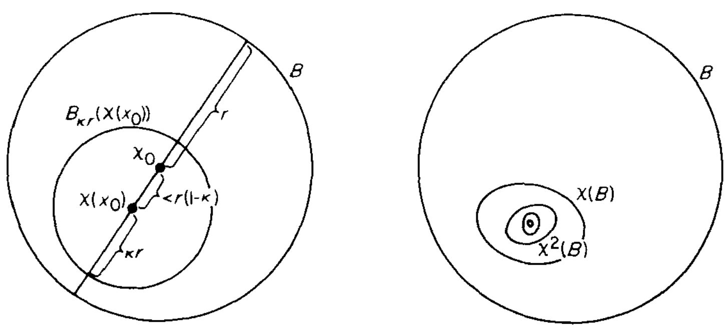

FIGURE C.4

maps $B = B_r(x_0)$ into $B_{\kappa r}(\chi(x_0))$, the iterates $\chi^n(B)$ for $n = 0, 1, 2, \ldots$ form a nested sequence, as in Figure C.4. Since χ decreases diameters by a factor κ, it is intuitively obvious that there is single point at the core of the sequence, and this must be a fixed point of f.

It is almost as quick to give a proper proof:

(C.5) Theorem. (*Contraction mapping theorem*) *A contraction $\chi: X \to Y$ has at most one fixed point. If $X = Y$ and X is complete then χ has a fixed point.*

Proof. Let x and x' be fixed points of χ. Then

$$d(x, x') = d(\chi(x), \chi(x')) \leqslant \kappa d(x, x')$$

where $\kappa < 1$ is the Lipschitz constant of χ. Thus $d(x, x') = 0$, and so $x = x'$.

Suppose that $X = Y$ is complete. Let $x \in X$. Consider the sequence $(\chi^n(x))_{n\geqslant 0}$. For all integers $n \geqslant m \geqslant 0$

$$\begin{aligned} d(\chi^n(x), \chi^m(x)) &\leqslant \kappa^m d(\chi^{n-m}(x), x) \\ &\leqslant \kappa^m \sum_{r=0}^{n-m-1} d(\chi^{r+1}(x), \chi^r(x)) \\ &\leqslant \kappa^m \sum_{r=0}^{n-m-1} \kappa^r d(\chi(x), x) \\ &\leqslant \kappa^m (1-\kappa)^{-1} d(\chi(x), x), \end{aligned}$$

which tends to 0 as $m \to \infty$. Thus the sequence is a Cauchy sequence. Since X is complete, the sequence converges to some limit, l say, in x. By the continuity of χ,

$$\chi(l) = \chi\Big(\lim_{n\to\infty} \chi^n(x)\Big) = \lim_{n\to\infty} \chi(\chi^n(x)) = l. \qquad \square$$

We often find in applications that there is a variable parameter present, and that we need to know how the fixed point depends on this parameter. Let us be more precise. A map $\chi: X \times Y \to Z$ is *uniformly Lipschitz on the first factor* if for some constant $\kappa > 0$ and all $y \in Y$ the map $\chi_y: X \to Z$ taking x to $\chi(x, y)$ is Lipschitz with constant κ. Similarly for the second factor. Clearly χ is Lipschitz if and only if it is uniformly Lipschitz on both factors. It is a *uniform contraction* on either factor if it is uniformly Lipschitz on that factor with constant <1. When χ is a uniform contraction on the second factor, say, and $Y = Z$ is complete, each map χ^x has a unique fixed point, which we denote by $g(x)$. This defines the *fixed point map* $g: X \to Y$ of χ. Such a map, satisfying for all $x \in X$

(C.6) $$\chi(x, g(x)) = g(x)$$

may, of course, exist even when the above Lipschitz conditions do not hold. We now investigate the extent to which properties of χ influence properties of g.

(C.7) Theorem. *Let $\chi: X \times Y \to Z$ be a uniform contraction on the second factor, and let $g: X \to Y$ satisfy* (C.6). *If χ is continuous then g is continuous. If χ is Lipschitz then g is Lipschitz. If, further, X is a subset of a Banach space* **E**, *Y and Z are subsets of a Banach space* **F** *and χ is C^r then g is C^r* $(r \geqslant 1)$. *If, further, $D\chi$ is C^{r-1}-bounded then Dg is C^{r-1}-bounded.*

Proof. We denote the distance from p to p' by $|p-p'|$. Let $\kappa > 1$ be a Lipschitz constant for χ on the second factor. Then, for all x and $x' \in X$,

$$\begin{aligned}
|g(x)-g(x')| &= |\chi(x, g(x))-\chi(x', g(x'))| \\
&\leqslant |\chi(x, g(x))-\chi(x', g(x))| \\
&\quad + |\chi(x', g(x))-\chi(x', g(x'))| \\
&\leqslant |\chi(x, g(x))-\chi(x', g(x))| + \kappa|g(x)-g(x')|,
\end{aligned}$$

and so

$$|g(x)-g(x')| \leqslant (1-\kappa)^{-1}|\chi(x, g(x))-\chi(x', g(x))|.$$

Thus g is continuous when χ is continuous, and Lipschitz when χ is Lipschitz.

Now suppose $X \subset \mathbf{E}$, $Y \cup Z \subset \mathbf{F}$ and that χ is Lipschitz and C^r $(r \geqslant 1)$. Then for all $(x, y) \in X \times Y$, $|D_2\chi(x, y)| \leqslant \kappa$, and thus $id - D_2\chi(x, y)$ is a linear homeomorphism of $\mathbf{F}$. We first show that g is differentiable at $x \in X$, with

(C.8) $$Dg(x) = T(x)D_1\chi(x, g(x))$$

where $T(x) = (id - D_2\chi(x, g(x)))^{-1}$. For all sufficiently small $\xi \in \mathbf{E}$,

$$\begin{aligned}
&|g(x+\xi)-g(x)-T(x)D_1\chi(x, g(x))(\xi)| \\
&\quad \leqslant |T(x)||g(x+\xi)-g(x)-D_2\chi(x, g(x))(g(x+\xi)-g(x)) \\
&\qquad -D_1\chi(x, g(x))(\xi)| \\
&\quad = |T(x)||\chi(x+\xi, g(x+\xi))-\chi(x, g(x)) \\
&\qquad -D\chi(x, g(x))((x+\xi, g(x+\xi))-(x, g(x)))|.
\end{aligned}$$

By the differentiability of χ, this expression is $o(|(\xi, g(x+\xi)-g(x))|)$ as $|(\xi, g(x+\xi)-g(x))| \to 0$, whence $o(|\xi|)$ as $|\xi| \to 0$ (since g is Lipschitz). This gives differentiability of g.

The proof that g is C^r is by induction on $r(\geqslant 0)$. The case $r=0$ is trivial, since g is Lipschitz. The inductive step is clear, since (C.8) expresses Dg as a composite

(C.9) $$X \xrightarrow{(id,\, g)} X \times Y \xrightarrow{(D_1\chi,\, D_2\chi)} L(\mathbf{E}, \mathbf{F}) \times B \xrightarrow{id \times \rho}$$
$$\xrightarrow{id \times \rho} L(\mathbf{E}, \mathbf{F}) \times L(\mathbf{F}, \mathbf{F}) \xrightarrow{\text{comp}} L(\mathbf{E}, \mathbf{F}),$$

where B is the ball with centre 0 and radius κ in $L(\mathbf{F}, \mathbf{F})$, and $\rho : B \to L(\mathbf{F}, \mathbf{F})$ is the C^∞-bounded uniformly C^∞ map sending T to $(id - T)^{-1}$. Note that comp is here continuous bilinear.

The last part comes, similarly, by induction using Lemma B.7. □

Notice that the proof of continuity of g works in principle when X is merely a topological space. Note also that continuity of χ is implied by continuity of the maps χ_y for $y \in Y$.

We now take the theory one stage further. Our attitude is that results in the text such as Theorem 3.45 (relating a change in a vector field to the corresponding change in its integral curves) should be immediate applications of theorems in this section. To achieve this, we introduce a further parameter, taking values in a topological space A. The spaces X, Y and Z are as in Theorem C.7. We are now, however, given a map $\chi: A \times X \times Y \to Z$ such that, for each $a \in A$, $\chi^a: X \times Y \to Z$ is a uniform contraction on the second factor with constant $\kappa < 1$. We also have, for each $a \in A$, a fixed point map $g^a: X \to Y$ satisfying $g^a(x) = \chi^a(x, g^a(x))$ for all $x \in X$.

(C.10) Theorem. *Let $a_0 \in A$. Suppose that, for all $a \in A$, χ^a is C^r $(r \geqslant 0)$ and, if $r > 0$, Lipschitz. Suppose also that $D\chi^a$ is C^{r-1}-bounded and that $\chi^a - \chi^{a_0}$ is C^0-bounded. Then $g^a - g^{a_0}$ is C^r-bounded. If, further, $D\chi^{a_0}$ is uniformly C^{r-1} and the map $\alpha: A \to C^r(X \times Y, Z)$ taking a to $\chi^a - \chi^{a_0}$ is continuous at a_0, then the map $\beta: A \to C^r(X, Y)$ taking a to $g^a - g^{a_0}$ is continuous at a_0.*

Proof. By Theorem C.7 Dg^a is C^{r-1}-bounded for all $a \in A$. Also, for all $a \in A$ and $x \in X$,

$$\begin{aligned}|g^a(x) - g^{a_0}(x)| &= |\chi^a(x, g^a(x)) - \chi^{a_0}(x, g^{a_0}(x))| \\ &\leqslant |\chi^a(x, g^a(x)) - \chi^{a_0}(x, g^a(x))| \\ &\quad + |\chi^{a_0}(x, g^a(x)) - \chi^{a_0}(x, g^{a_0}(x))| \\ &\leqslant |\chi^a - \chi^{a_0}|_0 + \kappa |g^a(x) - g^{a_0}(x)|,\end{aligned}$$

and

$$|g^a(x) - g^{a_0}(x)| \leqslant (1-\kappa)^{-1} |\chi^a - \chi^{a_0}|_0.$$

This completes the proof that $g^a - g^{a_0}$ is C^r-bounded. It also gives continuity of β at a_0, when α is continuous at a_0, in the $r = 0$ case. We complete the proof by induction. Suppose that $\beta: A \to C^k(X, Y)$ is continuous at a_0. To perform the inductive step, we show that the map $\gamma: A \to C^k(X, L(\mathbf{E}, \mathbf{F}))$ taking a to Dg^a is continuous at a_0.

First note that, by hypothesis, the map from A to $C^k(X, X \times Y)$ taking a to $(0, g^a - g^{a_0})$ is continuous at a_0. So is the map $(a \mapsto D\chi^a)$ from A to $C^k(X \times Y, L(\mathbf{E} \times \mathbf{F}, \mathbf{F}))$. Now (id, g^{a_0}) has a C^{k-1}-bounded derivative, and $D\chi^{a_0}$ is uniformly C^k. We may apply Theorem B.18 and the $s = 0$ argument from Theorem B.15 to show that the composition map from $C^k(X, X \times Y) \times C^k(X \times Y, L(\mathbf{E} \times \mathbf{F}, \mathbf{F}))$ to $C^k(X, L(\mathbf{E} \times \mathbf{F}, \mathbf{F}))$ taking (θ, ϕ) to $(\phi(\theta + (id, g^{a_0}))$ is continuous at $((0, 0), D\chi^{a_0})$. Thus the map λ from A to

$C^k(X, L(\mathbf{E}\times\mathbf{F}, \mathbf{F}))$ taking a to $D\chi^a(id, g^a)$ is continuous at a_0. We identify $C^k(X, L(\mathbf{E}\times\mathbf{F}, \mathbf{F}))$ with $C^k(X, L(\mathbf{E}, \mathbf{F}))\times C^k(X, L(\mathbf{F}, \mathbf{F}))$ by the canonical isomorphism. The second component of λ takes values in $C^k(X, B)$, where B is as in the proof of Theorem C.7. We now describe a decomposition of the map γ. One first applies λ. Then one operates on the second factor by ρ_*, where ρ is as in the proof of Theorem C.7. Finally one takes the compositional product of the two factors (continuous bilinear, by Lemma B.5). Since λ is continuous at a_0, and the maps that follow are continuous, γ is continuous at a_0. $\square$

(C.11) Exercise. (*Lipschitz inverse mapping theorem*) Let B be the closed ball with centre 0 and radius b (possibly $b=\infty$) in a Banach space $\mathbf{E}$. Let $T:\mathbf{E}\to\mathbf{E}$ be a (topological) linear automorphism, and let $\eta:B\to\mathbf{E}$ be Lipschitz with constant $\kappa<|T^{-1}|^{-1}$ and such that $\eta(0)=0$. Let C be the closed ball with centre 0 and radius $b(|T^{-1}|^{-1}-\kappa)$ in $\mathbf{E}$. Prove that, for all $y\in C$, there is a unique $x\in B$ such that $(T+\eta)(x)=y$. (*Hint*: rewrite this as $x=T^{-1}(y-\eta(x))$.) Hence, if $D=\operatorname{int} C$ and we write $x=g(y)$, then $g(D)$ is an open neighbourhood of 0 in B and the map $g:D\to g(D)$ is inverse to the restriction $T+\eta:g(D)\to D$. Prove that g is Lipschitz, and C^r $(r\geq 1)$ when η is C^r. Deduce the following local form:

> If f is a C^r $(r\geq 1)$ map of some open subset of $\mathbf{E}$ into $\mathbf{E}$ and if $Df(x_0)$ is an automorphism then there exist open neighbourhoods U of x_0 and V of $f(x_0)$ such that the restriction $f: U\to V$ is a C^r diffeomorphism.

(C.12) Exercise. (*Immersive mapping theorem*) Prove that if $f: X\to Y$ is a C^r map of manifolds $(r\geq 1)$ and f is immersive at x_0 then f restricts to a C^r embedding of some neighbourhood of x_0. (*Hint*: Assume that X and Y are open in Banach spaces $\mathbf{E}$ and $\mathbf{F}$, $x_0=f(x_0)=0$, $\mathbf{F}=\mathbf{E}\times\mathbf{G}$ and $Df(0)=(id, 0)$. Apply the inverse mapping theorem to the map $\phi:X\times\mathbf{G}\to\mathbf{F}$ defined by $\phi(x, z)=f(x)+(0, z)$.)

(C.13) Exercise. (*Submersive mapping theorem*) Prove that if $f:X\to Y$ is a C^r map of manifolds $(r\geq 1)$ and f is submersive at x_0 then some neighbourhood of x_0 in $f^{-1}(f(x_0))$ is a C^r submanifold of X modelled on $\ker Df(x_0)$. (*Hint*: Assume that X and Y are open in Banach spaces $\mathbf{E}$ and $\mathbf{F}$, $x_0=f(x_0)=0$, $\mathbf{E}=\mathbf{F}\times\ker Df(0)$ and $Df(0)$ is projection to the first factor. Apply the inverse mapping theorem to the map $\phi:X\to\mathbf{E}$ defined by $\phi(x)=\phi(x_1, x_2)=(f(x), x_2)$.)

(C.14) Exercise. (*Implicit mapping theorem*) The implicit mapping theorem is, basically, concerned with solving the equation

(C.15) $$T(y)+\eta(x, y)=0$$

for y in terms of x, where T is an automorphism of a Banach space $\mathbf{F}$, x takes values in a topological space X and η is Lipschitz on the second factor. The theorem is usually presented in a local form, where we are given a single solution $y = b$ when $x = a$, and have to show the existence of a unique continuous map $x \mapsto g(x)$ defined on some neighbourhood of a in X such that $g(a) = b$ and

$$\textbf{(C.16)} \qquad T(g(x)) + \eta(x, g(x)) = 0$$

for all x in the neighbourhood. We can always modify η so that $a = b = 0$.

Let B be the closed ball in F with centre 0 and radius b (possibly $b = \infty$). Suppose that $x = 0$, $y = 0$ satisfies (C.15) and let $\eta: X \times B \to \mathbf{F}$ be uniformly Lipschitz on the second factor with constant $\kappa < |T^{-1}|^{-1}$. Suppose that, for all $x \in X$, $|\eta(x, 0)| \leq |T^{-1}|^{-1} - \kappa$. Prove that there is a unique map $g: X \to B$ satisfying (C.16) for all $x \in X$. (*Hint*: Rewrite (C.15) as $y = -T^{-1}\eta(x, y) = 0$.) Prove that g is continuous if η is continuous. Prove that if X is open in a Banach space $\mathbf{E}$ then g is Lipschitz if η is Lipschitz, and C^r $(r \geq 1)$ if η is C^r. Deduce the following local form:

Let X and Y be open subsets of Banach spaces $\mathbf{E}$ and $\mathbf{F}$ respectively, and let $f: X \times Y \to \mathbf{F}$ be C^r $(r \geq 1)$ with $f(a, b) = 0$ and $D_2 f(a, b)$ an automorphism, for some $(a, b) \in X \times Y$. Prove that there exist neighbourhoods U of a in X and V of b in Y such that there is a unique map $g: U \to V$ satisfying $f(x, g(x)) = 0$ for all $x \in U$. Moreover the map g is C^r.