A Measure Associated with Axiom-A Attractors
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Source: *American Journal of Mathematics*, Vol. 98, No. 3 (Autumn, 1976), pp. 619-654
Published by: The Johns Hopkins University Press
Stable URL: https://www.jstor.org/stable/2373810
Accessed: 26-03-2020 18:26 UTC

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A MEASURE ASSOCIATED WITH AXIOM-A ATTRACTORS.

By DAVID RUELLE.*

Abstract. The future orbits of a diffeomorphism near an Axiom-A attractor are investigated. It is found that their asymptotic behavior is in general described by a fixed probability measure μ carried by the attractor. The measure μ has an exponential cluster property, and satisfies a variational principle.

0. Introduction. Let M be a manifold, f a diffeomorphism of M and $x \in M$. One may hope that, in cases of some generality, the asymptotic behaviour of $f^k x$ for $k \to \infty$ is described by a probability measure μ_x in the sense that

$$\operatorname*{vague\,lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{f^k x} = \mu_x \tag{0.1}$$

where δ_y denotes the unit mass at y. In other words

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^k x) = \mu_x(\varphi)$$

for every real continuous function on M.

In the present paper we show that (0.1) holds for almost all x in a neighbourhood of an *attractor* satisfying *Axiom-A* of Smale [22], and that $\mu_x = \mu$ does not depend on x. The measure μ is characterized by a variational principle: it makes maximum the expression

$$h(\mu, f) + \mu(\log \lambda)$$

where h is the measure-theoretical *entropy* (see for instance [4]) and λ is an *expansion coefficient* defined as follows. Choose a Riemann metric on M and let σ be the measure defined by the induced metric on the unstable manifolds [22] of points of Ω, then λ is given by a Radon-Nikodym derivative:

$$\lambda(fx) = \frac{d(f\sigma)}{d\sigma}(fx) \tag{0.2}$$

Manuscript received November 15, 1973.

*Part of this work was done while the author was visiting the Mathematics Department of the University of California at Berkeley.

American Journal of Mathematics, Vol. 98, No. 3, pp. 619–654

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We shall need the concepts of *Markov partition*, discussed below in §1, and of *weak Bernouilli partition*. The finite partition $\mathcal{Q} = (A_i)_{i \in I}$ is weak Bernouilli for the dynamical system (Ω, μ, f) if, for each $\epsilon > 0$, there exists n such that the partition $Q = (Q_1, \dots, Q_q) = \bigvee_{k=n+m+1}^{n+2m} f^k \mathcal{Q}$ is ϵ-independent of $P = (P_1, \dots, P_p) = \bigvee_{k=1}^{m} f^k \mathcal{Q}$. This means that

$$\sum_{j=1}^{q} |\mu(P_i \cap Q_j)/\mu(P_i) - \mu(Q_j)| < \epsilon \tag{0.3}$$

(except perhaps for some P_i the union of which form a set of μ-measure $\leqslant \epsilon$). According to the Friedman-Ornstein theorem [10] the existence of a weak Bernouilli partition implies that the dynamical system (Ω, μ, f) is isomorphic to a Bernouilli shift.

We state our results now for an Axiom-A attractor Ω on a compact manifold, and assume for simplicity that the (full) unstable manifolds [22] are dense in Ω, or equivalently that Ω is connected. A somewhat more general case will actually be treated below (see Theorem 1.5).

THEOREM. *Let f be a C^2 diffeomorphism of a compact manifold M, and Ω a connected Axiom-A attractor. Let $U = \{x \in M : f^n x \to \Omega$ when $n \to \infty\}$. There exists then a f-invariant probability measure μ on Ω with the following properties.*

Let the measure $\nu \geqslant 0$ have continuous density with respect to some Riemann volume on M, and assume $\operatorname{supp} \nu \subset U$, *then*

$$\operatorname*{vague\,lim}_{k \to \infty} f^k \nu = \|\nu\| \cdot \mu \tag{0.4}$$

There exists $\tilde{U} \subset U$ such that $\tilde{U}$ has measure zero with respect to the Riemann volume and, if $x \in U \setminus \tilde{U}$,

$$\operatorname*{vague\,lim}_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{f^k x} = \mu \tag{0.5}$$

$\tilde{U}$ is necessarily dense in U unless Ω is a point.

μ is the only f-invariant probability measure on Ω which makes maximum the expression

$$h(\mu, f) + \mu(\log \lambda) \tag{0.6}$$

where h is the entropy and λ is defined by (0.2), *using any smooth Riemann metric on M. The maximum of* (0.6) *is* 0.

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If $\mathcal{Q}$ is a Markov partition of Ω, it is a weak Bernouilli partition for the dynamical system (Ω,μ,f), and this system is isomorphic to a Bernouilli shift.*

There exist $C,k>0$ such that if the functions φ',φ'' are C^1 in a neighborhood of Ω, the following exponential cluster property holds

$$|\mu\big((\varphi'\circ f^{-m'})\cdot(\varphi''\circ f^{-m''})\big)-\mu(\varphi')\cdot\mu(\varphi'')|\leqslant C\|\varphi'\|_1\cdot\|\varphi''\|_1 e^{-k|m'-m''|} \tag{0.7}$$

where $\|\varphi\|_1$ is the C^1-norm of φ.

In the case of Anosov diffeomorphisms, Sinai [20], [21] proves (0.4) and (essentially) (0.7), and Azencott [3] shows that (Ω,μ,f) is isomorphic to a Bernouilli shift. Sinai [21] shows that the conditional measure induced by μ on unstable manifolds is equivalent to the measure defined by the Riemann metric, and that μ is characterized among f-invariant measures by this property. Sinai [21] also uses λ to characterize the measure μ in a manner related to but different from the maximizing of (0.6) indicated above.

To prove the above theorem we shall to a large extent follow the lines of thought of Sinai, and use both the theory of differentiable dynamical systems and techniques of statistical mechanics. For the sake of completeness, we shall go rapidly through a lot of material which is essentially not new, but which does not appear to be present in readily quotable form in the literature.

It is expected that the present work on diffeomorphisms can be extended to vector fields. Consider a differential equation

$$\frac{d\xi}{dt}=X(\xi)$$

and let ξ_x be the solution with initial condition $\xi_x(0)=x$. The conjecture is that on an Axiom-A attractor Ω there exists a probability measure μ such that, for almost all x near Ω,

$$\lim_{T\to\infty}\frac{1}{T}\int_T^0\varphi(\xi_x(t))\,dt=\mu(\varphi) \tag{0.8}$$

for continuous φ. The measure μ would have properties similar to those given in the theorem above. In particular, an exponential cluster property similar to (0.7) would hold. Such a result would be significant for the understanding of turbulence in hydrodynamics because, following the ideas of [17], it would explain the occurrence of a "continuous spectrum" after the onset of turbulence.

*This could be deduced from (0.6) using Bowen's results [6], [7].

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Note added in proof. For the study of vector fields (Axiom-A flows), see R. Bowen and D. Ruelle, "The ergodic theory of Axiom-A flows," *Inventiones Math.*, **29** (1975), 181-202.

1. Generalities and statement of results. We consider a situation slightly more general than that described in the Introduction.

Let M be a finite dimensional Riemann manifold, $U \subset M$ an open set and $f: U \rightarrow M$ a C^r embedding ($r \geqslant 1$). Let $\Lambda \subset U$ be invariant, i.e. $f\Lambda = \Lambda$. The set Λ is *hyperbolic* if the tangent bundle of M restricted to Λ has a continuous splitting

$$T_\Lambda M = E^+ \oplus E^- \tag{1.1}$$

invariant under Df and if there exist $C > 0$ and $\theta < 1$ such that

$$\|(Df^n)|E^+\| \leqslant C\theta^n, \quad \|(Df^{-n})|E^-\| \leqslant C\theta^n \tag{1.2}$$

for all positive integers n. A Riemann metric on M is *adapted* to Λ if one can take $C = 1$ in (1.2).

THEOREM 1.1.* *Let Λ be a compact hyperbolic set for $f: U \rightarrow M$, then M has a smooth Riemann metric adapted to Λ. Let d be the distance for such a metric.*

One can choose $\delta > 0$ and $\bar{\theta} < 1$ such that if

$$W_x^\pm = \{ y \in M : d(f^{\pm n}y, f^{\pm n}x) < \delta \quad \text{for all} \quad n \geqslant 0\} \tag{1.3}$$

the following properties hold for all $x \in \Lambda$.

(a) *For all $n \geqslant 0$, and $y, z \in W_x^\pm$,*

$$d(f^{\pm n}y, f^{\pm n}z) \leqslant \bar{\theta}^n d(y, z) \tag{1.4}$$

(b) *There is a neighbourhood A of x in Λ and a continuous map*

$$\psi : A \rightarrow C^r(B_x, M)$$

where B_x is the unit ball of $E_x^\pm$, $\psi(y)$ is an embedding, $\psi(y)(0) = y$, and

$$W_y^\pm = \{ z \in \psi(y)(B_x) : d(z, y) < \delta \}$$

for all $y \in A$.

*See Smale [22], Hirsch and Pugh [12], and references quoted there.

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(c) *$W_x^{\pm}$ is tangent to $E_x^{\pm}$ at x.*
(d) *If $y \in \Lambda$, $W_x^{\pm} \cap W_y^{\pm}$ is an open subset of $W_x^{\pm}$.*

The W_x^+ are called *stable manifolds*, the W_x^- are called *unstable manifolds for f.*

Notice that, under the assumptions of the theorem, f restricted to Λ is *expansive* with *expansive constant* γ for any $\gamma < \delta$. This means that if $x, y \in \Lambda$ and $d(f^n x, f^n y) \leqslant \gamma$ for all n, then $x = y$. [If $d(f^n x, f^n y) \leqslant \gamma < \delta$ for all n, (1.3) gives $f^n y \in W_{f^n x}^-$ for all n, and (1.4) gives

$$d(x,y) \leqslant \bar{\theta}^n d(f^n x, f^n y) \leqslant \bar{\theta}^n \delta \to 0$$

when $n \to +\infty$].

If $x \in \Lambda$, let σ, σ_1 be the measures defined by the Riemann metric respectively on W_x^- and W_{fx}^-, then the *expansion coefficient* λ is defined by the Radon-Nicodym derivative

$$\lambda(fx) = \frac{d(f\sigma)}{d\sigma_1}(fx)$$

Clearly $0 < \lambda < \theta^u$ where $u = \dim E_x^-$.

We shall say that the compact invariant set Λ is *attracting* for $f: U \mapsto M$ if

$$\cap_{n \geqslant 0} f^n U = \Lambda.$$

We can then choose U such that $fU \subset U$ [let V be an open neighbourhood of Λ with compact closure $\bar{V}$ such that $\bar{V} \subset U. f\bar{V} \subset U$; for some N we have then $\cap_1^{N+1} f^n \bar{V} \subset V$ and hence $fU' \subset U'$ if $U' = \cap_0^N f^n V$].

PROPOSITION 1.2. *Let Λ be an attracting compact hyperbolic set for $f: U \to U$ and choose δ sufficiently small.*

(a) *For all $x \in \Lambda$, $W_x^- \subset \Lambda$*
(b) *If $x \in \Lambda$ and $y, z \in W_x^-$ then $W_y^+ \cap W_z^+ = \emptyset$ when $y \neq z$*
(c) *If $x \in \Lambda$, $\cup \{W_y^+ : y \in W_x^-\}$ is an open neighbourhood of x in M.*
(d) *There exists $\epsilon > 0$ such that if $x, y \in \Lambda$ and $d(x,y) \leqslant \epsilon$, $W_x^+ \cap W_y^-$ consists of exactly one point $[x,y]$ and the map $[.,.]: \{(x,y) \in \Lambda \times \Lambda : d(x,y) \leqslant \epsilon\} \to \Lambda$ is continuous.*

We may assume that the δ-neighbourhood of Λ is contained in U, then if $y \in W_x^-$, we have $f^{-n} y \in U$ for all $n \geqslant 0$ and therefore $y \in \cap_{n \geqslant 0} f^n U = \Lambda$, proving (a). If $y, z \in W_x^-$, and $W_y^+ \cap W_z^+ \neq \emptyset$, we have

$$z \in \tilde{W}_y^+ = \{z \in M : d(f^n z, f^n y) < 2\delta \quad \text{for all} \quad n \geqslant 0\}$$

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But if 2δ is sufficiently small, Theorem 1.1 (b) and (c) imply that $\tilde{W}_y^+ \cap \tilde{W}_x^- = \{y\}$ and therefore $z = y$, proving (b). Finally (c)* and (d) are also easy consequences of Theorem 1.1 (b) and (c).

The above properties imply the existence of a Markov partition on Λ, according to Sinai [18], [19], and Bowen [5]†. We start with some definitions.

Let $x \in \Lambda, C \subset W_x^- \cap \Lambda, D \subset W_x^+ \cap \Lambda$ and suppose that C, D are the closure of their interior respectively as subsets of $W_x^- \cap \Lambda$ and $W_x^+ \cap \Lambda$. If C, D are not empty, the set $A = [C, D]$ is called a *rectangle*. Proposition 1.2 (*d*) implies that $[.,.]: C \times D \to A$ is a homeomorphism. A finite cover $\mathcal{A} = \{A_1, \ldots, A_p\}$ of Λ by rectangles $A_i = [C_i, D_i]$ is a rectangle partition if $A_i \cap A_j \subset \partial A_i \cap \partial A_j$ for $i \neq j$‡. $\mathcal{A}$ is a *Markov partition* if, in addition

$$f[C_i, x] \supset [C_j, f(x)], \quad f[x, D_i] \subset [f(x), D_j]$$

whenever $x \in \operatorname{int} A_i \cap f^{-1} \operatorname{int} A_j$.

THEOREM 1.3. *Let Λ be an attracting compact hyperbolic set for $f: U \mapsto U$. Then, f restricted to Λ has a Markov partition $\mathcal{A} = \{A_1, \ldots, A_P\}$ where $A_i = [C_i, D_i], C_i \subset W_{x_i}^-, D_i \subset W_{x_i}^+, x_i \in \Lambda$, $i = 1, \ldots, P$.*

We define

$$t(A_i, A_j) = \begin{cases} 1 \text{ if } f(\operatorname{int} A_i) \cap \operatorname{int} A_j \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

and let

$$\prod = \{ (A_{i_n}) \in \mathcal{A}^{\mathbf{Z}} : t(A_{i_n}, A_{i_{n+1}}) = 1 \qquad \text{for all} \qquad n \in \mathbf{Z} \}$$

Putting on $\mathcal{A}$ the discrete topology, and on $\mathcal{A}^{\mathbf{Z}}$ the product topology, $\mathcal{A}^{\mathbf{Z}}$ is compact metrizable, and so is $\prod$ as a closed subset of $\mathcal{A}^{\mathbf{Z}}$. The shift τ defined by

$$\tau((A_{i_n})_{n \in \mathbf{Z}}) = (A_{i_{n+1}})_{n \in \mathbf{Z}}$$

is a homeomorphism of $\prod$.

*To check (c), Lemma (4.1) of [11] may be used.

†More precisely Theorem 1.1 (a) and Proposition 1.2 (d) yield "Fact 1" of [5], from which the existence of a Markov partition follows.

‡The boundary $\partial A = A \backslash \operatorname{int} A$ of a rectangle $A = [C, D]$ is of the form $\partial A = \partial^+ A \cup \partial^- A$ where $\partial^+ A = [\partial C, D]$, $\partial^- A = [C, \partial D]$ and ∂C, ∂D are the boundaries of C and D considered as subsets of $W_x^- \cap \Lambda$ and $W_x^+ \cap \Lambda$ respectively.

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PROPOSITION 1.4*. *If* $\xi=(A_{i_n})\in\coprod$, *then* $\cap_{n\in Z}f^{-n}A_{i_n}$ *consists of a single point* $\pi(\xi)$. The map π is a continuous from $\coprod$ onto Λ and

$$f\circ\pi=\pi\circ\tau$$

We state now the main results of this paper.

THEOREM 1.5. *Let* Λ *be an attracting compact hyperbolic set for* $f: U\mapsto U$ *where* f *is* $C^r, r\geqslant 2$. *There is a finite family* (μ_α) *of* f*-invariant probability measures with disjoint supports* $\Omega_\alpha\subset\Lambda$ *such that the following properties hold.*

(a) Ω_α *is an Axiom-A attractor, and the family* (Ω_α) *contains all Axiom-A attractors in* U. *One can write*

$$\mu_\alpha=N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha}\mu_{\alpha\beta}$$

where the probability measures $\mu_{\alpha 1},\ldots,\mu_{\alpha N_\alpha}$ *are cyclically permuted by* f, *and their supports* $\Omega_{\alpha 1},\ldots,\Omega_{\alpha N_\alpha}$ *are the connected components of* Ω_α.

(b) *Let the measure* ν *have support in* U *and continuous density with respect to the Riemann volume. If* N *is the least common multiple of the* N_α, *then*

$$\operatorname*{vague\,lim}_{k\to\infty} f^{kN}\nu$$

exists and is a linear combination of the $\mu_{\alpha\beta}$.

(c) *There is a set* $\tilde U\subset U$ *such that* $\tilde U$ *has measure zero with respect to the Riemann volume and that, if* $x\in U\setminus\tilde U$,

$$\operatorname*{vague\,lim}_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_{f^kx}=\mu_{\alpha(x)}$$

(d) μ_α *is the only* f*-invariant probability measure on* Ω_α *which makes maximum the expression* $h(\mu,f)+\mu(\log\lambda)$. *This maximum is* 0.

(e) *Let* $\mathcal{R}_{\alpha\beta}$ *be a Markov partition of* $\Omega_{\alpha\beta}$ *with respect to* $f^N\alpha$. *Then* $\mathcal{R}_{\alpha\beta}$ *is a weak Bernouilli partition for the dynamical system* $(\Omega_{\alpha\beta},\mu_{\alpha\beta},f^N\alpha)$, *and this system is isomorphic to a Bernouilli shift.*

(f) *There exists* $C,k>0$ *such that if the functions* φ',φ'' *are* C^1 *in a*

*See for instance Bowen [5].

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neighbourhood of $\Omega_{\alpha\beta}$, *then*

$$|\mu_{\alpha\beta}((\varphi'\circ f^{-m'N_\alpha})\cdot(\varphi''\circ f^{-m''N\alpha}))-\mu_{\alpha\beta}(\varphi')\,\mu_{\alpha\beta}(\varphi'')|$$
$$\leqslant C\|\varphi'\|_1\cdot\|\varphi''\|_1 e^{-k|m'-m''|}$$

where $\|\varphi\|_1$ *denotes the* C^1 *norm of* φ.

Remark 1.6. In view of (a) we may in the above theorem replace (Λ,f) by (Ω_α,f) or $(\Omega_{\alpha\beta},f^{N_\alpha})$. Notice in particular the following facts

(a′) If f is topologically transitive on Λ and the periodic points are dense in Λ, then there is only one Ω_α, which is equal to Λ.

(b′) Let the measure $\nu\geqslant 0$ have support in a sufficiently small neighbourhood of $\Omega_{\alpha\beta}$, and continuous density with respect to the Riemann volume, then

$$\operatorname*{vague\,lim}_{k\to\infty} f^{kN_\alpha}\nu=\|\nu\|\cdot\mu_{\alpha\beta}$$

(c′) There is an open neighbourhood U_α of Ω_α, and $\tilde U_\alpha\subset U_\alpha$ with $\tilde U_\alpha$ of measure zero with respect to the Riemann volume such that, if $x\in U_\alpha\setminus\tilde U_\alpha$,

$$\operatorname*{vague\,lim}_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_{f^kx}=\mu_\alpha.$$

It will be shown that $\tilde U_\alpha$ is necessarily dense in U_α unless Ω_α is a finite orbit of f (see 3.9).

2. Preliminaries to the Proof of Theorem 1.5. In the following lemmas we assume that Λ is an attracting compact hyperbolic set for $f:U\mapsto U$, where f is $C^r, r\geqslant 2$. We use the general terminology of Section 1.

LEMMA 2.1. *Define* $0_x=\cup\{W_y^+:y\in W_x^-\}$ *and* $p:0_x\to W_x^-$ *be the continuous map such that* $z\in W_{pz}^+$.

(a) *If* ν *is any measure with support in* 0_x,

$$\operatorname*{vague\,lim}_{n\to+\infty} f^n(\nu-p\nu)=0 \tag{2.1}$$

(b) *If* ν *has continuous density with respect to the Riemann volume on* M, $p\nu$ *has continuous density with respect to the measure* σ *defined by the Riemann metric on* W_x^-.

(c) *Let* $C\subset W_x^-\cap\Lambda, D\subset W_x^+\cap\Lambda$, *and* $A=[C,D]$ *be a rectangle. Given* $y\in D$, *define* $C'=[C,y]$ *and let* σ,σ' *be the measures on* C,C' *defined by the*

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Riemann metric. If $\nu = h'\sigma'$, *and* h' *is continuous on* C', *we have*

$$p\nu = h\sigma \tag{2.2}$$

where

$$h(pz) = h'(z)F(pz),\ F(pz) = \prod_{k=1}^{\infty} \frac{\lambda(f^k z)}{\lambda(f^k pz)} \tag{2.3}$$

Part (a) follows from tbe fact that

$$d(f^n pz, f^n z) \leqslant \delta.\theta^n \to 0 \text{ as } n \to \infty.$$

The proof of (b) and (c) is given in Appendix A.

Lemma 2.2. *We use the notation and conditions of Lemma* 2.1 (*c*), *and write*

$$T = \sup_{z\in\Lambda} \| T_z f \|,\ \gamma = \frac{2|\log\theta|}{\log T + |\log\theta|} \tag{2.4}$$

(a) *The mapping* $p^{-1}: C \mapsto C'$ *is Hölder continuous with Hölder exponent* γ.

(b) *The function* F *is Hölder continuous with Hölder exponent* $\gamma/2$.

The proof of this lemma is given in Appendix A.

Lemma 2.3. *We denote by* $\mathcal{Q} = \{A_1, \ldots, A_P\}$ *a Markov partition of* f *restricted to* $\Lambda: A_i = [C_i, D_i], C_i \subset W_{x_i}^-, D_i \subset W_{x_i}^+, x_i \in \operatorname{int} A_i, i = 1, \ldots, P$. *Let also* χ_i *be the characteristic function of* A_i *in* Λ, σ_i *the measure defined by the Riemann metric on* C_i, *and* $p_i: A_i \mapsto C_i$ *the map* $p_i(z) = [z, x_i]$.

Let $\mathcal{C}(C_i)$ *be the space of real continuous functions on* C_i *and* $\mathcal{C} = \oplus_{i=1}^P \mathcal{C}(C_i)$. *We define a linear map* $\varphi \to \mathcal{L}\varphi$ *of* $\mathcal{C}$ *into itself by*

$$\sum_j (\mathcal{L}\varphi)_j \cdot \sigma_j = \sum_j p_j\left[\chi_j \cdot \left(f \Sigma_{i: t(A_i, A_j) = 1} \varphi_i \cdot \sigma_i\right)\right] \tag{2.5}$$

(a) *We have*

$$(\mathcal{L}\varphi)_j(z_j) = \sum_i F_{ij}(z_j) \cdot \left[\varphi_i \circ f^{-1} \circ p_{ij}^{-1}(z_j)\right] \tag{2.6}$$

where p_{ji} *is the restriction of* p_j *to* $\chi_j f C_i$ *and*

$$F_{ij}(z_j) = \lambda\left(p_{ji}^{-1} z_j\right) \prod_{k=1}^{\infty} \frac{\lambda\left(f^k p_{ji}^{-1} z_j\right)}{\lambda\left(f^k z_j\right)} \tag{2.7}$$

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if $t(A_i,A_j)=1$, $F_{ij}=0$ *if* $t(A_i,A_j)=0$.

Let now

$$\prod{}^{+}=\left\{(A_{i_n})\in\mathcal{Q}^{\mathbf{P}}:t(A_{i_n},A_{i_{n+1}})=1 \text{ for all } n\in\mathbf{P}\right\} \tag{2.8}$$

where $\mathbf{P}=\{n\in\mathbf{Z}:n>0\}$. *If* $\xi=(A_{i_n})\in\prod^{+}$, *then* $C_{i_1}\cap\bigcap_{n=0}^{\infty}f^{-n}A_{i_{n+1}}$ *consists of a single point* $\pi_+(\xi)$ *of* C_{i_1}.

When $t(A_i,A_{i_1})=1$, *we write* $(A_i,\xi)=(A_i,A_{i_1},A_{i_2},\ldots)\in\prod^{+}$. *We introduce* $F_i\in\mathcal{C}(\prod_{+})$ *by*

$$F_i(\xi)=F_{ii_1}\circ\pi_+(\xi) \qquad \text{if} \qquad \xi=(A_{i_n})$$

In particular $F_i(\xi)=0$ *if* $t(A_i,A_{i_1})=0$. *Finally, a linear map* $L:\mathcal{C}(\prod^{+})\to\mathcal{C}(\prod^{+})$ *is defined by*

$$(L\psi)(\xi)=\sum_i F_i(\xi)\psi(A_i,\xi)$$

(b) $L(\varphi\circ\pi_+)=(\mathcal{L}\varphi)\circ\pi_+$ *when* $\varphi\in\mathcal{C}$

(c) *Restricted to* $\{\xi=(A_{i_n})\in\prod^{+}:t(A_i,A_{i_1})=1\}$, *the function* F_i *is of the form*

$$\exp\sum_{n=0}^{\infty}\Phi(A_i,A_{i_1},\ldots,A_{i_n})$$

where

$$\sup_{i,i_1,\ldots,i_n}|\Phi(A_i,A_{i_1},\ldots,A_{i_n})|\leqslant K\theta'^n$$

and $K>0, 0<\theta'<1$.

(a) follows directly from Lemma 2.1(c).

If $\xi=(A_{i_n})\in\prod^{+}$, then $\cap_{n=0}^{\infty}f^{-n}A_{i_{n+1}}$ is of the form $[y,D_{i_1}]$ and the intersection of this set with C_{i_1} consists of a single point $\pi_+(\xi)$.

We note the following facts

$$\left\{p_{i_1i}^{-1}\circ\pi_+(\xi)\right\}=fC_i\cap A_{i_1}\cap\bigcap_{n=1}^{\infty}f^{-n}A_{i_{n+1}}$$

$$\left\{f^{-1}\circ p_{i_1i}^{-1}\circ\pi_+(\xi)\right\}=C_i\cap f^{-1}A_{i_1}\cap\bigcap_{n=1}^{\infty}f^{-n-1}A_{i_{n+1}}=\{\pi_+(A_i,\xi)\}.$$

From this (b) follows:

$$\begin{aligned}(\mathcal{L}\varphi)_{i_1}(\pi_+\xi)&=\sum_i F_{ii_1}(\pi_+\xi).\left[\varphi_i\circ f^{-1}\circ p_{i_1i}^{-1}\circ\pi_+(\xi)\right]\\&=\sum_i\left[F_{ii_1}\circ\pi_+(\xi)\right].\left[\varphi_i\circ\pi_+(A_i,\xi)\right]\\&=(L(\varphi\circ\pi_+))(\xi)\end{aligned}$$

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To prove (c), we notice first that, by Lemma 2.2, $\log F_{ii_1}$ is Hölder continuous with exponent $\gamma/2$ on C_{i_1}. We may assume diam $C_i < \delta$. Therefore if $\xi = (A_{i_k})$, $\xi' = (A_{i'_k})$ and $i_k = i'_k$ for $1 \leqslant k \leqslant n$, we have $d(f^k\pi_+\xi, f^k\pi_+\xi') < \delta$ for all $k \in \mathbf{Z}$ with $k < n$. Hence

$$d(\pi_+\xi, \pi_+\xi') < \delta.\theta^{n-1}$$

hence

$$|\log F_i(\xi)| < K$$

$$|\log F_i(\xi) - \log F_i(\xi')| < K\theta'^{n+1}$$

for some $K > 0$, and $\theta' = \theta^{\gamma/2}$. One can thus choose the $\Phi(A_i, A_{i_1}, \dots, A_{i_k})$ recursively on k such that

$$\left|\log F_i(\xi) - \sum_{k=0}^{n} \Phi(A_i, A_{i_1}, \dots, A_{i_k})\right| < K\theta'^{n+1}$$

$$\sup_{i, i_1, \dots, i_n} |\Phi(A_i, A_{i_1}, \dots, A_{i_n})| < K\theta'^{n}.$$

LEMMA 2.4. *We use the notation and assumptions of Lemma* 2.3.

(a) $\sigma_i(\partial C_i) = 0$ *and, if* $\mathcal{L}^*$ *is the adjoint of* $\mathcal{L}$,

$$\mathcal{L}^* \oplus_{i=1}^{P} \sigma_i = \oplus_{i=1}^{P} \sigma_i$$

(b) *If the probability measure* ν *has continuous density with respect to the Riemann volume on* M *and support in a sufficiently small neighbourhood of* Λ, *there exist continuous functions* $h_i \geqslant 0$ *on* C_i *such that* $\sum_i h_i \cdot \sigma_i$ *is a probability measure and*

$$\operatorname*{vague\,lim}_{n \to +\infty} f^n(\nu - \Sigma h_i \cdot \sigma_i) = 0$$

(c) *If* $n > 0$, *then*

$$\sum_j (\mathcal{L}^n \varphi)_j \cdot \sigma_j = \sum_j p_j\left[\chi_j \cdot \left(f^n \sum_i \varphi_i \cdot \sigma_i\right)\right]$$

(d) *There is a unique measure* $\omega \geqslant 0$ *on* $\coprod^+$ *such that its image by* π_+ *is* $\oplus_i \sigma_i$. *Furthermore* $L^*\omega = \omega$ *where* L^* *is the adjoint of* L.

Let σ_i be the product of σ_i by the characteristic function of $\operatorname{int} C_i$. Write $\sigma^+ = \oplus_i \sigma_i, \sigma^- = \oplus_i \sigma_i^-$. If $\varphi \geqslant 0$ we have (see (2.5))

$$(\mathcal{L}^*\sigma^+)(\varphi) = \sum_j \int_{C_j} [(\mathcal{L}\varphi)_j(z_j)].\sigma_j(dz_j) \geqslant \sigma^+(\varphi)$$

$$(\mathcal{L}^*\sigma^-)(\varphi) = \sum_j \int_{\operatorname{int} C_j} [(\mathcal{L}\varphi)_j(z_j)].\sigma_j(dz_j) \leqslant \sigma^-(\varphi).$$

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Thus

$$\mathfrak{L}^*\sigma^- \leqslant \sigma^- \leqslant \sigma^+ \leqslant \mathfrak{L}^*\sigma^+$$

and, since $\mathfrak{L}^*$ is positively preserving

$$0 \leqslant \mathfrak{L}^{*n}\sigma^- \leqslant \ldots \leqslant \mathfrak{L}^*\sigma^- \leqslant \sigma^- \leqslant \sigma^+ \leqslant \mathfrak{L}^*\sigma^+ \leqslant \ldots \leqslant \mathfrak{L}^{*n}\sigma^+.$$

Using the Hahn-Banach theorem construct now measures $\omega^{\pm}$ on $\coprod^+$ such that $0 \leqslant \omega^- \leqslant \omega^+$ and $\pi^+\omega^{\pm} = \sigma^{\pm}$. From Lemma 2.3 (b) we see that for $n \geqslant 0$

$$\pi_+(L^{*n}\omega^{\pm}) = \mathfrak{L}^{*n}\sigma^{\pm}.$$

We use now the machinery of Appendix B. From Lemma B. 3(a) we find that for sufficiently large n

$$\omega^+(L^n 1) \leqslant \sum_{i=1}^{p} \omega^+\left(\coprod_i^+\right).D. \inf_{\xi \in \coprod_i^+} (L^n 1)(\xi).$$

Choose $D' > 0$ such that for all i

$$\omega^+\left(\coprod_i^+\right) = \sigma_i(C_i) \leqslant D'\sigma_i(\operatorname{int} C_i) = D'\omega^-\left(\coprod_i^+\right).$$

Then

$$\begin{aligned} \omega^+(L^n 1) &\leqslant DD'\omega^-(L^n 1) = DD'\sigma^-(\mathfrak{L}^n 1) \\ &= DD'(\mathfrak{L}^{*n}\sigma^-)(1) \leqslant DD'\sigma^-(1). \end{aligned}$$

Therefore $\|L^{*n}\omega^+)\| = \omega^+(L^n 1)$ is bounded. Furthermore

$$(L^{*n}\omega^+)\left(\coprod_i^+\right) = \|\mathfrak{L}^{*n}\sigma_i\| \geqslant \|\sigma_i\|.$$

If ω is a vague limit of the measures $n^{-1}\Sigma_{k=1}^n L^{*n}\omega^+$ we have $L^*\omega = \omega$ and $\omega\left(\coprod_i^+\right) > 0$ for each i. With this choice of ω the conditions of Proposition B.1 are satisfied. Applying Proposition B.1 (f) with $S = \pi_+^{-1}\Sigma_{i=1}^P \partial C_i$ we obtain $\omega(\pi_+^{-1}\Sigma_{i=1}^P \partial C_i) = 0$ ($\tau g^{-1}S \subset g^{-1}S$ because $\mathcal{C}$ is a Markov partition). Since $\omega^+ \leqslant \omega$, and $\pi^+\omega^+ = \sigma^+$, we find $\sigma_i(\partial C_i) = 0$. This implies $\mathfrak{L}^*\sigma^+ = \sigma^+$, concluding the proof of (a).

It follows from (a) that a positive measure ω on $\coprod^+$ such that $\pi^+\omega = \oplus_i \sigma_i$ satisfies

$$\omega\left(\coprod_{i_1 \ldots i_n}^+\right) = \sigma_{i_1}\left(C_{i_1} \cap \bigcap_{k=1}^{n} f^{n-1} A_{i_n}\right)$$

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where

$$\coprod_{i_1\dots i_n}^{+}=\left\{\xi=(\xi_n)_{n\in\mathbf{P}}\in\coprod^{+}:\xi_1=A_{i_1},\dots,\xi_n=A_{i_n}\right\}.$$

This determines ω uniquely. Since $\pi_+(L^*\omega)=\mathfrak{L}^*(\pi_+\omega)=\oplus_i\sigma_i=\pi_+\omega$, we have $L^*\omega=\omega$, proving (d).

To prove (b) notice first that, since the D_i have dense interior, near each $x\in\Lambda$ there is $x'\in\Lambda$ such that $W_{x'}^-$ has an empty intersection with each $\partial^-A_i=[C_i,\partial D_i]$. We choose a finite number of such points $x_1',\dots,x_{L'}'$, so that the open sets $0'_k=\cup\{W_y^+:y\in W_{x'_k}^-\}$ cover a neighbourhood of Λ in U. We assume that the support of ν is contained in this neighbourhood. Using a partition of unity we write $\nu=\sum_k\nu_k$ where the support of ν_k is contained in $0_k'$ and $\nu_k\geqslant 0$ has continuous density with respect to the Riemann volume. By Lemma 2.1 (b), $P_k'\nu_k=g_k'.\sigma_k'$ where $p_k':0_k'\to W_{x'_k}^-$ is defined by $z\in W_{p_k'z}^+$, $g_k'\geqslant 0$ is a continuous function with compact support on $W_{x'_k}^-$, and σ_k' is the measure defined by the Riemann metric on $W_{x'_k}^-$. By Lemma 2.1 (c) we may write

$$\sum_k p_i(\chi_i g'_k\sigma'_k)=h_i\sigma_i$$

so that part (b) of the present lemma holds.

We prove (c) by induction on n:

$$\begin{aligned}\sum_j(\mathfrak{L}^n\varphi)_j.\sigma_j&=\sum_j p_j\left[\chi_j.\left(f\sum_i(\mathfrak{L}^{n-1}\varphi)_i.\sigma_i\right)\right]\\&=\sum_j p_j\left[\chi_j.\left(f\sum_l p_l\left[\chi_l.\left(f^{n-1}\sum_i\varphi_i.\sigma_i\right)\right]\right)\right]\\&=\sum_j p_j\left[\chi_j.\left(f\left[\left(\sum_l\chi_l\right).f^{n-1}\sum_i\varphi_i.\sigma_i\right]\right)\right]\\&=\sum_j p_j\left[\chi_j.\left(f^h\sum_i\varphi_i.\sigma_i\right)\right]\end{aligned}$$

because $p_jfp_l=p_jf$ (definition of a Markov partition).

3. Proof of Theorem 1.5. Let $g:\coprod\mapsto\coprod^+$ be given by $g(A_{i_n})_{n\in\mathbf{Z}}=(A_{i_n})_{n\in\mathbf{P}}$ and write $\coprod_i=\left\{(A_{i_n})\in\coprod:i_1=i\right\}$, $\coprod_i^+=\left\{(A_{i_n})\in\coprod^+:i_1=i\right\}$. It

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follows from Section 2 that the following diagrams are commutative

$$\begin{tikzcd}
\prod \arrow[rr, "\pi"] \arrow[dd, "g"'] & & \Lambda \arrow[dd] & \\
& \prod_i \arrow[ul, hook] \arrow[rr] \arrow[dd] & & A_i \arrow[ul, hook] \arrow[dd, "p_i"] \\
\prod^+ \arrow[rr, "\pi_+"] & & \Sigma C_i & \\
& \prod_i^+ \arrow[ul, hook] \arrow[rr] & & C_i \arrow[ul, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod \arrow[r, "\pi"] \arrow[d, "\pi"'] & \Lambda \arrow[d, "f"] \\
\prod \arrow[r, "\pi"] & \Lambda
\end{tikzcd}$$

with the restriction that $\Lambda \mapsto \Sigma C_i$ is uniquely defined only on $\cup_i$ int A_i. We also have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\prod^+) \arrow[d, "L"'] & \oplus_i \mathcal{C}(C_i) \arrow[l, "\circ \pi_+"'] \arrow[d, "\mathcal{L}"] \\
\mathcal{C}(\prod^+) & \oplus_i \mathcal{C}(C_i) \arrow[l, "\circ \pi_+"']
\end{tikzcd}$$

The left vertical line of this diagram can be studied by methods of statistical mechanics. The relevant results are given in Appendix B. We have noticed in Section 2 that the conditions of Proposition B.1 are satisfied in the present situation; this will enable us to prove Theorem 1.5.

If $\rho_{[i]}$ is defined by Proposition B.1 (c), the following measures are f-invariant:

$$\bar{\mu}_{[i]} = N_{[i]}^{-1} \sum_{\beta=1}^{N_{[i]}} \pi\tau^{\beta}\rho_{[i]}.$$

We call μ_α the *distinct* $\bar{\mu}_{[i]}$ which occur. It will follow from Lemma 3.2 (d) that $\operatorname{supp}\bar{\mu}_{[i]}$ and $\operatorname{supp}\bar{\mu}_{[j]}$ either coincide or are disjoint and from Remark 3.4 (a) that $\bar{\mu}_{[i]} = \bar{\mu}_{[j]}$ if $\operatorname{supp}\bar{\mu}_{[i]} = \operatorname{supp}\bar{\mu}_{[j]}$. Therefore the measures μ_α have disjoint supports as announced.

3.1. *Proof of* (a). Having defined $\mu_\alpha = \bar{\mu}_{[i]}$, let $\Omega_{\alpha\beta} = \operatorname{supp}\ \pi\tau^{\beta}\rho_{[i]}$ for $\beta = 1, \ldots, N_\alpha$ where N_α is the smallest integer such that $\Omega_{\alpha N_\alpha} = \operatorname{supp}\pi\rho_{[i]}$. Since $\operatorname{supp}\pi\tau^{\beta}\rho_{[i]}$ and $\operatorname{supp}\pi\tau^{\gamma}\rho_{[i]}$ are either disjoint or identical by Lemma 3.2 (d) below, $\Omega_{\alpha 1}, \ldots, \Omega_{\alpha N_\alpha}$ are disjoint and cyclically permuted by f. Clearly one can write $\mu_\alpha = N_\alpha^{-1}\Sigma_{\beta=1}^{N_\alpha} \mu_{\alpha\beta}$ where $\operatorname{supp}\mu_{\alpha\beta} = \Omega_{\alpha\beta}$. Then $\mu_{\alpha 1}, \ldots, \mu_{\alpha N_\alpha}$ are probability measures cyclically permuted by f. The $\Omega_{\alpha\beta}$ are Axiom-A attractors with

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respect to f^{N_α} (Lemma 3.2 (b)) and therefore Ω_α is an Axiom A attractor for f. The $\Omega_{\alpha\beta}$ are also connected (Lemma 3.2 (c)). To complete the proof of (a) we have to show that there are no Axiom A attractors in U except those of the family (Ω_α). This is done in Remark 3.4 (c) below.

LEMMA 3.2. *(a) The support of $\rho_{[i]}$ is $\bigcap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}$. The action of $\tau^{N_{[i]}}$ on supp $\rho_{[i]}$ is topologically transitive and periodic points are dense.*

(b) *The support of $\pi\rho_{[i]}$ is $\cap_{n\geqslant 0} f^{nN_{[i]}} \cup_{j\in[i]}A_j$, and supp $\pi\rho_{[i]}$ is an Axiom A attractor for $f^{N[i]}$.*

(c) supp $\pi\rho_{[i]}$ *is connected.*

(d) *For any two maximal classes $[i]$, $[j]$, the sets supp $\pi\rho_{[i]}$, supp $\pi\rho_{[j]}$ are either disjoint or identical.*

(a) By definition $\operatorname{supp}\rho_{[i]}\subset\prod_{[i]}$. Thus by $\tau^{N_{[i]}}$-invariance $\operatorname{supp}\rho_{[i]}\subset\bigcap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}$. Let $E=\{\xi\in\prod:\xi_k=i_k \text{ for } 1\leqslant k\leqslant l+1\}$ where $t_{i_1i_2}=\cdots=t_{i_li_{l+1}}=1$, and $E^+=gE$. We have $\omega(E^+)>0$ because $\omega=L^{*l}\omega$ and $\omega(\prod_j^+)>0$ for all j. Suppose $i_1\in[i]$, i.e. $E^+\subset\prod_{[i]}^+$. Since $\psi_{[i]}$ does not vanish on $\prod_{[i]}^+$, $\rho_{[i]}(E)=(\psi_{[i]}\cdot\omega)(E^+)>0$. Every neighbourhood of a point of $\bigcap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}$ contains a set of the form $\tau^{nN_{[i]}}E$, and therefore $\bigcap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}\subset\operatorname{supp}\rho_{[i]}$.

Taking for l a multiple of $N_{[i]}$, it is easily seen that E contains points of period $l+M$. This implies the density of $\tau^{N_{[i]}}$-periodic points in $\operatorname{supp}\rho_{[i]}$. Similarly, it is easy to prove the existence of a dense $\tau^{N_{[i]}}$-orbit (topological transitivity).

(b) We have

$$\begin{aligned}\operatorname{supp}\pi\rho_{[i]}&=\pi\operatorname{supp}\rho_{[i]}=\pi\bigcap_{n\geqslant 0}\tau^{nN_{[i]}}\prod\nolimits_{[i]}\\&=\bigcap_{n\geqslant 0}f^{nN_{[i]}}\pi\prod\nolimits_{[i]}=\bigcap_{n\geqslant 0}f^{nN_{[i]}}\cup_{j\in[i]}A_i.\end{aligned}$$

In particular, $\operatorname{supp}\pi\rho_{[i]}$ is a union of sets $[C_j,y]$ where $y\in A_j, j\in[i]$. Let $x\in\operatorname{supp}\pi\rho_{[i]}, x\in U_{j\in[i]}$ [int C_j,D_j], and $f^{pN_{[i]}}x=x$ for some p. Such points are dense in $\operatorname{supp}\pi\rho_{[i]}$. Since $[C_j,x]\subset\operatorname{supp}\pi\rho_{[i]}$ for some $j\in[i]$, $\operatorname{supp}\pi\rho_{[i]}$ contains a small piece of W_x^- around x, and since $f^{npN_{[i]}}x=x$ for all $n\geqslant 1$ it follows that $W_x^-\subset\operatorname{supp}\pi\rho_{[i]}$. It is now possible to choose a finite family (x_k) of points x as above, such that the sets $0_{x_k}=\cup\{W_y^+:y\in W_{x_k}^-\}$ cover $\operatorname{supp}\pi\rho_{[i]}$. Since $W_{x_k}^-\subset\operatorname{supp}\pi\rho_{[i]}$,

$$\cap_{n\geqslant 0}f^{nN_{[i]}}\left[\cup_k 0_{x_k}\right]=\operatorname{supp}\pi\rho_{[i]}$$

showing that $\operatorname{supp}\pi\rho_{[i]}$ is an attracting set for $f^{N_{[i]}}$. The action of $f^{N_{[i]}}$ on the invariant compact set $\operatorname{supp}\pi\rho_{[i]}$ also satisfies hyperbolicity (because Λ is hyperbolic), topological transitivity and density of periodic points by part (a) of the

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present lemma. Thus $\operatorname{supp} \pi\rho_{[i]}$ is an Axiom A attractor for $f^{N_{[i]}}$.

(c) Let x be as in the proof of (b) and $W = \cup_{n \geqslant 0} f^{npN_{[i]}} W_x^-$, then $W \subset \operatorname{supp} \pi\rho_{[i]}$ and one sees readily, using (a), that $\pi^{-1}W$ is dense in $\operatorname{supp} \rho_{[i]}$. Thus W is dense in $\operatorname{supp} \pi\rho_{[i]}$, and $\operatorname{supp} \pi\rho_{[i]}$ is connected because W is connected.

(d) Since $\operatorname{supp} \pi\rho_{[i]}$, $\operatorname{supp} \pi\rho_{[j]}$ are both Axiom A attractors for f^N, they are either disjoint or identical.

3.3. *Proof of (b).* We first study vague $\lim f^{kM}\nu$. In view of Lemma 2.4(b), it is equivalent to study vague $\lim f^{kM} \Sigma h_i \sigma_i$. Using Lemma 2.1(a) and Lemma 2.4(c) we see that this is the same as vague $\lim f^{kM} \Sigma_j (\mathcal{L}^{lM} h)_j \sigma_j$. We have by Lemma 2.4(d)

$$\begin{aligned} \Sigma_j (\mathcal{L}^{lM} h)_j \sigma_j &= \pi_+ \left\{ \left[(\mathcal{L}^{lM} h) \circ \pi_+ \right] \cdot \omega \right\} \\ &= \pi_+ \left\{ \left[L^{lM} (h \circ \pi_+) \right] \cdot \omega \right\}. \end{aligned}$$

For large $l, L^{lM}(h \circ \pi_+)$ is uniformly close to $P(h \circ \pi_+) = \Sigma_{[i]} c_{[i]} \psi_{[i]}$ therefore, using also (B.4),

$$\begin{aligned} \operatorname*{vague\,lim}_{k \to \infty} f^{kM}\nu &= \operatorname*{vague\,lim}_{k \to \infty} f^{kM} \pi_+ \left[(\Sigma_{[i]} c_{[i]} \psi_{[i]}) \omega \right] \\ &= \Sigma_{[i]} c_{[i]} \operatorname*{vague\,lim}_{k \to \infty} f^{kM} \pi_+ g \rho_{[i]} \end{aligned} \tag{3.1}$$

Notice that by appropriate choice of ν, all real values of the $c_{[i]}$ can be achieved.

Let $x_j = \pi(\xi^{(j)})$ where the x_j are the points associated with the Markov partition $\mathcal{Q}$. If $\xi = (\xi_n)_{n \in \mathbf{Z}} \in \prod_j$, write

$$\gamma\xi = \left(\cdots, \xi_{-1}^{(j)}, \xi_0^{(j)}, A_j, \xi_2, \xi_3, \cdots \right).$$

This defines a continuous map $\gamma : \prod \to \prod$, and

$$\pi_+ g\xi = \pi\gamma\xi.$$

Therefore

$$f^{kM} \pi_+ g\rho_{[i]} = f^{kM} \pi\gamma\rho_{[i]} = \pi\tau^{kM} \gamma \tau^{-kM} \rho_{[i]}$$

which tends to $\pi\rho_{[i]}$ when $k \to \infty$. Therefore

$$\operatorname*{vague\,lim}_{k \to \infty} f^{kM}\nu = \sum_{[i]} c_{[i]} \pi \rho_{[i]} \tag{3.2}$$

The conclusion of the proof is given as part (b) of the remark below.

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Remark 3.4. We now fill in some gaps in the above arguments. By (a) the set $\Omega_{\alpha\beta} = \operatorname{supp} \pi\rho_{[i]}$ is an Axiom A attractor for f^{N_α}. Let us introduce a Markov partition $\mathcal{Q}_{\alpha\beta}$ of $\Omega_{\alpha\beta}$ with respect to f^{N_α} and consider instead of $(\Lambda, f, \mathcal{Q})$ the system $(\Omega_{\alpha\beta}, f^{N_\alpha}, \mathcal{Q}_{\alpha\beta})$. Using the connectedness of $\Omega_{\alpha\beta}$ we find that for this new system there is only one maximal class, to which there corresponds a unique measure $\mu_{\alpha\beta}$. Therefore if $\operatorname{supp} \nu$ is contained in a sufficiently small neighbourhood of $\Omega_{\alpha\beta}$, (3.2) yields now

$$\operatorname*{vague\,lim}_{k\to\infty} f^{kM'}\nu = c\mu_{\alpha\beta} \tag{3.3}$$

(a) Comparison of (3.3) and (3.2) shows that $\pi\rho_{[i]} = \mu_{\alpha\beta}$.
(b) Inserting this in (3.2) we find

$$\operatorname*{vague\,lim}_{k\to\infty} f^{kM}\nu = \Sigma c_{\alpha\beta}\mu_{\alpha\beta} \tag{3.4}$$

(c) If Ω were an Axiom A attractor distinct from the Ω_α, it would be disjoint from them, and this would contradict (3.4).

3.5. *Proof of* (c). In view of (a) and (b) it suffices to prove that there is an open neighbourhood U of $\Omega_{\alpha\beta}$ and $\tilde{U} \subset U$ with $\tilde{U}$ of measure zero with respect to the Riemann volume on M such that, if $x \in U \setminus \tilde{U}$

$$\operatorname*{vague\,lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{f^{kn_\alpha}x} = \mu_{\alpha\beta} \tag{3.5}$$

If $\mathcal{Q}_{\alpha\beta} = (A_i')$ is a Markov partition of $(\Omega_{\alpha\beta}, f^{N_\alpha})$, where $A_j' = [C_j', D_j']$, the sets $U_j = \cup\{W_y^+ : y \in C_j'\}$ cover a neighbourhood of $\Omega_{\alpha\beta}$. Suppose $\tilde{C}_j \subset C_j'$ and $\tilde{C}_j$ has measure zero with respect to the measure defined by the Riemann metric on C_j'. Let $\tilde{U}_j = \cup\{W_j^+ : y \in \tilde{C}_j\}$. Then, by Lemma 2.1 (b), $\tilde{U}_j$ has measure zero with respect to the Riemann volume on M. It suffices therefore to prove (3.5) for $x \in U_j \setminus \tilde{U}_j$, or equivalently for $x \in C_j \setminus \tilde{C}_j$ (because the left-hand side of (3.5) is the same for $x \in W_y^-$ and for $x = y$).

Let $[j]$ be the unique maximal class corresponding to the Markov partition $\mathcal{Q}_{\alpha\beta}$ of $(\Omega_{\alpha\beta}, f^{N_\alpha})$ (see Remark 3.4) and $\rho_{[j]}'$ the corresponding measure, such that $\pi'\rho_{[j]}' = \mu_{\alpha\beta}$ [we let $\pi', g', \cdots$ correspond to π, $g, \cdots$ when $(\Lambda, f, \mathcal{Q})$ are replaced by $(\Omega_{\alpha\beta}, f^{N_\alpha}, \mathcal{Q}_{\alpha\beta})$]. The measure defined by the Riemann metric on $\Sigma C_j'$ is $\pi'_+\omega'$, and is absolutely continuous with respect to $\pi'_+ g'\rho_{[j]}' = \pi'_+(\psi_{[j]}' \cdot \omega')$ because $\psi_{[j]}'$ is bounded away from zero. Therefore it suffices to prove (3.5) for $x = \pi'_+ g'\xi'$ for $\rho_{[j]}'$-almost all ξ', or to prove

$$\operatorname*{vague\,lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{\tau'^k \xi'} = \rho'_{[j]}$$

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for $\rho'_{[j]}$ -almost all ξ'. But this follows from the ergodic theorem and the τ'-ergodicity of $\rho'_{[j]}$ (see Proposition B.1(d)).

3.6. *Proof of* (d). For the proof of (d) we can take $\Omega_\alpha = \Lambda$ and therefore $\coprod = \cup_{m=1}^{N_{[i]}} \tau^m \coprod_{[i]}$. Given a probability measure μ on Ω_α there is a probability measure ρ on $\coprod$ such that $\mu = \pi\rho$ [by Hahn-Banach]. Assuming that μ is f-invariant, one can choose ρ to be τ-invariant [by Markov-Kakutani]. We have then

$$h(\mu, f) \leqslant h(\rho, \tau) \tag{3.6}$$

[consider the partition $(\pi^{-1}\tilde{A}_j)$ associated with each partition $(\tilde{A}_j)$ of Ω_α].

The function H on $\coprod$ is defined (see Proposition B.1 (e) and Lemma 2.3) by

$$\begin{aligned}\exp H(\xi) &= \lambda\left(p_{i_2 i_1}^{-1}\pi_+ g\tau\xi\right) \prod_{k=1}^{\infty} \frac{\lambda\left(f_{i_2 i_1}^{k-1}\pi_+ g\tau\xi\right)}{\lambda\left(f^k\pi_+ g\tau\xi\right)} \\ &= \lambda\left(f p_{i_1} f^{-1}\pi\tau\xi\right) \prod_{k=1}^{\infty} \frac{\lambda\left(f^{k+1} p_{i_1} f^{-1}\pi\tau\xi\right)}{\lambda\left(f^k p_{i_2}\pi\tau\xi\right)}\end{aligned}$$

if $\xi = (A_{i_n})_{n\in\mathbf{Z}}$.

The expression $\rho(H)$ is thus the limit when $n\to\infty$ of the value of ρ at the function

$$\xi \to \log\lambda\left(f p_{i_1}\pi\xi\right) + \sum_{k=1}^{n} \left[\log\lambda\left(f^{k+1} p_{i_1}\pi\xi\right) - \log\lambda\left(f^k p_{i_2}\pi\tau\xi\right)\right]$$

or, using the τ-invariance of ρ, at the function

$$\begin{aligned}&\xi \to \log\lambda\left(f p_{i_1}\pi\xi\right) + \sum_{k=1}^{n} \left[\log\lambda\left(f^{k+1} p_{i_1 1}\pi\xi\right) - \log\lambda\left(f^k p_{i_2}\pi\xi\right)\right] \\ &\quad = \log\lambda\left(f^{n+1} p_{i_1}\pi\xi\right)\end{aligned}$$

or at

$$\xi \to \log\lambda\left(f^{n+1} p_{i_{-n}} f^{-n-1}\pi\xi\right)$$

which tends to $\log(\lambda\circ\pi)$ as $n\to\infty$. Therefore

$$\mu(\log\lambda) = \rho(\log(\lambda\circ\pi)) = \rho(H) \tag{3.7}$$

By (3.6) and (3.7) we have thus

$$h(\mu, f) + \mu(\log\lambda) \leqslant h(\rho, \tau) + \rho(H)$$

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and, by Proposition B.1 (e), the maximum of the right-hand side is reached exactly when $\rho = \bar{\rho}_{[i]} = N_{[i]}^{-1}\Sigma_{m=1}^{N_{[i]}}\tau^m\rho_{[i]}$ and is 0. To prove that the maximum of the left-hand side is reached for $\mu = \mu_\alpha$, and is 0, it suffices to show that

$$h(\mu_\alpha, f) + \mu_\alpha(\log\lambda) = h(\bar{\rho}_{[i]}, \tau) + \bar{\rho}_{[i]}(H)$$

or, in view of (3.7), that

$$h(\mu_\alpha, f) = h(\bar{\rho}_{[i]}, \tau).$$

This will result from the fact that the systems $(\Omega_\alpha, \mu_\alpha, f)$ and $\left(\prod, \rho_{[i]}, \tau\right)$ are isomorphic, as we now prove.

We first show that $\mu_{\alpha\beta}(\partial A_i) = 0$. Writing $\partial^+ = \cup_i[\partial C_i, D_i]$, $\partial^- = \cup_i[C_i, \partial D_i]$, we have $f\partial^+ \subset \partial^+, f^{-1}\partial^- \subset \partial^-$. Therefore using the f-invariance of μ_α,

$$\mu_\alpha(\partial^\pm) = \mu_\alpha\left(\bigcap_{n \geqslant 0} f^{\pm n}\partial^\pm\right)$$

Since $\rho_{[i]}$ is $\tau^{N_{[i]}}$-ergodic by Proposition B.1 (d), the $\tau^{N_{[i]}}$-invariant sets $\pi^{-1}\bigcap_{n \geqslant 0} f^{\pm n}\partial^\pm$ have $\rho_{[i]}$-measure 0 or 1 for each $[i]$. The choice 0 is imposed by the fact that $\operatorname{supp}\mu_{\alpha\beta} = \Omega_{\alpha\beta}$ is not contained in the closed set $\bigcap_{n \geqslant 0} f^{\pm n}\partial^\pm$ for any β. The set $S = \cup_{n \in \mathbf{Z}} f^n \cup_i \partial A_i$ has thus μ_α-measure 0, and since π^{-1} is one-to-one on $\Omega_{\alpha\beta} \setminus S$ we see that the systems $(\Omega_\alpha, \mu_\alpha, f)$ and $\left(\prod, \bar{\rho}_{[i]}, \tau\right)$ are isomorphic.

3.7. *Proof of* (e). Let us replace $\Omega_\alpha, f, \mathfrak{A}$ by $\Omega_{\alpha\beta}, f^{N_\alpha}, \mathfrak{A}_{\alpha\beta}$ is the isomorphism just described, and use Proposition B.1(d): (e) results immediately.

3.8. *Proof of* (f). Let $(\prod, \tau)$ be constructed with respect to the system $(\Omega_{\alpha\beta}, f^{N_\alpha})$ rather than (Λ, f). If $\xi', \xi'' \in \prod$ and $\xi_i' = \xi_i''$ for $|i| \leqslant n$, then $d(\pi\xi', \pi\xi'') < \delta \cdot \bar{\theta}^n$ by (1.4). Therefore if φ is C^1 on a neighbourhood of $\Omega_{\alpha\beta}$ we have

$$|(\varphi \circ \pi)(\xi)| \leqslant \|\varphi\|_0$$

$$|(\varphi \circ \pi)(\xi') - (\varphi \circ \pi)(\xi'')| \leqslant \|\varphi\|_1 \delta\bar{\theta}^n.$$

One can then choose $\psi_k(\xi_{-k}, \ldots, \xi_k)$ recursively on k so that

$$\left|(\varphi \circ \pi)(\xi) - \sum_{n=0}^{k} \psi_k(\xi_{-k}, \ldots, \xi_k)\right| \leqslant C\bar{\theta}^{n+1}$$

$$\sup_\xi |\psi_{(k)}(\xi_{-k}, \ldots, \xi_k)| \leqslant C\bar{\theta}^n$$

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where $C = \|\varphi\|_0 + \frac{\delta}{\theta}\|\varphi\|_1$. From this and Proposition B.1(d), one readily obtains a proof of (f).

3.9. *Proof of Remark* 1.6(c′). The set $\cup\{W_y^+ : y \in \Omega_\alpha$ and y periodic$\}$ is dense in a neighbourhood of Ω_α. If x belongs to this set

$$\operatorname*{vague\,lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{f^k x}$$

is a measure with finite support, which is necessarily different from μ_α if Ω_α is not a finite set.

Appendix A*. Let V be a submanifold of U and $y \in V$. Denote by σ, σ_1 the measures on V and fV defined by the Riemann metric, then $f\sigma$ is absolutely continuous with respect to σ_1 and

$$\frac{d(f\sigma)}{d\sigma_1}(fy) = \Delta(T_{fy}fV)$$

where $T_y V$ denotes the tangent space to V at y and Δ is a C^{r-1} function on a bundle with basis U and fiber over y consisting of the linear subspaces of $T_y M$.

Before proving Lemma 2.1, we study a related problem.

LEMMA A.1 *Let* 0_x *and* p *be as in Lemma* 2.1, *and write* $W_x^- = W$. *Assume that* W' *is a* C^r *submanifold of* 0_x *such that* p *restricted to* W' *is a homeomorphism onto* V *and that the following condition is satisfied:* $(T)\, T_{f^n z} f^n W'$ *and* $T_{f^n pz} f^n W$ *become close exponentially fast as* $n \to +\infty$, *uniformly with respect to* $z \in W'$.

Denote by σ, σ' *the measures on* W *and* W' *defined by the Riemann metric. Then if* g' *is continuous with compact support on* W',

$$p(g'\sigma') = g\sigma \tag{A.1}$$

where

$$g(pz) = g'(z)F(pz) \tag{A.2}$$

$$F(pz) = \prod_{k=1}^{\infty} \frac{\Delta\left(T_{f^k z} f^k W'\right)}{\Delta\left(T_{f^k pz} f^k W\right)} \tag{A.3}$$

*This appendix extends to Axiom-A attractors known results on the absolute continuity of foliations for Anosov diffeomorphisms. See Anosov [1], Anosov and Sinai [2], Pugh and Shub [14].

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To prove (A.1) it suffices to produce a sequence of maps $p_n: W' \to W$ such that $\sup_z d(p_n z, pz) \to 0$ and $p_n(g'\sigma') \to g\sigma$ (in norm) when $n \to +\infty$. In fact p_n need only be defined on a part W_n' of W' such that $\sigma'(W' \setminus W_n') \to 0$, $\sigma(W \setminus p_n W_n') \to 0$ when $n \to +\infty$.

We shall in fact construct maps α_n from $V_n' \subset f^n W'$ to $V_n \subset f^n W$ with the following properties

(a) $\lim_{n \to +\infty} (f^n \sigma')(f^n W' \setminus V_n') = 0$

(b) $\lim_{n \to +\infty} (f^n \sigma)(f^n W \setminus V_n) = 0$

(c) $\sup_{z_n \in V_n'} d^*(\alpha_n z_n, f^n \circ p \circ f^{-n} z_n) \leqslant C_1 \theta^n$

where $C_1 > 0$ *and* d^* *is the distance for the Riemann metric on* V_n.

(d) *If* σ_n, σ_n' *denote the restriction of* σ, σ' *to* $f^{-n}V_n', f^{-n}V_n$ *respectively, then*

$$\lim_{n \to +\infty} \left[(f^{-n} \circ \alpha_n \circ f^n)(g'\sigma_n') - g\sigma_n \right] = 0 \qquad \text{(in norm)}$$

(a), (b), (c), (d) imply that $p_n = f^{-n} \circ \alpha_n \circ f^n$ and $W_n' = f^{-n} V_n'$ have the desired properties enumerated above. In particular the expanding character of f^n restricted to W (see (1.4)) and (c) imply that $d(p_n z, pz) \to 0$. More generally and precisely

$$d\left(f^{n-k} p_n z, f^{n-k} pz\right) \leqslant C_1 \theta^{n+k} \tag{A.4}$$

We come now to the construction of α_n, V_n, V_n'. We first cover a neighbourhood of Λ in U with a finite number of charts Γ_i so that in each chart the stable manifolds W_y^+ stay "roughly parallel" to some coordinate plane $\prod_i$, and let 2ϵ be a Lebesgue number for this covering. For sufficiently large n we will define finite families (V_{nk}), (V_{nk}') of disjoint open subsets of $f^n W$ and $f^n W'$ respectively, and put

$$V_n = \cup_k V_{nk}, \quad V_n' = \cup_k V_{nk}'.$$

For each k there will be a chart $\Gamma_{i(k)}$ covering both V_{nk} and V_{nk}', and a diffeomorphism $\alpha_{nk}: V_{nk}' \to V_{nk}$ consisting in projection parallel to the coordinate plane $\prod_{i(k)}$; α_n will then be defined so that its restriction to V_{nk}' is α_{nk}.

The V_{nk}, V_{nk}' may be constructed as follows. Consider a maximal family of nonoverlapping spheres* with radius $\epsilon/4$ on $f^n W$. The spheres with the same centers and radius $\epsilon/2$ cover $f^n W$. We can refine this covering to a partition $(\tilde{V}_{nk})$, choose a chart $\Gamma_{i(k)}$ covering each $\tilde{V}_{nk}$, and finally shrink $\tilde{V}_{nk}$ to an open

*With respect to the distance given by the Riemann metric on $f^n W$.

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set V_{nk} so that the $\alpha_{nk}^{-1} V_{nk} = V'_{nk}$ are disjoint, and (a), (b) hold. The imprecisely stated condition that the stable manifolds W_y^+ should be roughly parallel to $\prod_i$ is to ensure the existence of the maps α_{nk}, and of a constant $C > 0$ independent of n such that

$$d(\alpha_n(z_n), z_n) \leqslant Cd(f^n \circ p \circ f^{-n} z_n, z_n) \tag{A.5}$$

for all $z_n \in E_n$. (c) follows readily from (A.5) and the definition of stable manifolds, which yields

$$d(f^n \circ p \circ f^{-n} z_n, z_n) \leqslant \delta \cdot \theta^n$$

There remains to check (d), i.e.,

$$\lim_{n \to +\infty} [\, p_n(g'\sigma'_n) - g\sigma_n] = 0$$

where $p_n = f^{-n} \circ \alpha_n \circ f^n$. By (A.4) we know already that $d(p_n z, pz) \to 0$ uniformly, hence $g' \circ p_n^{-1}$ tends to $g' \circ p^{-1}$ uniformly when $n \to +\infty$. Therefore, using (A.2), it suffices to show that

$$\lim_{n \to +\infty} [\, p_n \sigma'_n - F\sigma_n] = 0.$$

Denote by ρ_n, ρ'_n the measures defined on V_n, V'_n by the Riemann metric. From (c) follows that $T_{f^n p f^{-n} z_n} V_n$ and $T_{\alpha_n z_n} V_n$ become close uniformly in z_n when $n \to +\infty$. In view of assumption (T) of the lemma, $T_{z_n} V'_n$ and $T_{\alpha_n z_n} V_n$ also become close uniformly in z_n when $n \to +\infty$. Since α_n is locally a projection this implies that

$$\alpha_n \rho'_n = \eta_n \rho_n$$

where the positive function η_n tends to 1 uniformly when $n \to +\infty$. A simple calculation then shows that

$$p_n \sigma'_n = (f^{-n} \circ \alpha_n \circ f^n) \sigma'_n = F_n \sigma_n$$

where

$$F_n(p_n z) = \frac{\Delta(T_{fz} fW') \cdots \Delta(T_{f^n z} f^n W')}{\Delta(T_{fp_n z} fW) \cdots \Delta(T_{f^n p_n z} f^n W)} \eta_n(f^n p_n z)$$

In view of (A.4), the uniform bounds on second derivatives given by Theorem

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1.1 (b), and the fact that Δ is C^1, we have

$$\left|\frac{\Delta\left(T_{f^{n-k}p_n z}f^{n-k}W\right)}{\Delta\left(T_{f^{n-k}pz}f^{n-k}W\right)}-1\right|\leqslant C_2\theta^{n+k}$$

for some $C_2>0$. From this it is clear that $F_n\to F$ uniformly when $n\to+\infty$, concluding the proof of the lemma.

A.2. *Proof of the Lemma* 2.1 (b) *and* (c). To prove (b) we notice first that by considering $f^n\nu$ instead of ν (for sufficiently large n) and using a partition of unity, we can assume that the support of ν is close to W. It is then possible to cover this support by a smooth family of disjoint manifolds $W'C'$-close to W and satisfying therefore the conditions of Lemma A.1. We then verify (b) by writing ν as an integral of measures carried by the W', and applying Lemma A.1.

To prove (c) notice that h' can be extended to a continuous function with support in a neighbourhood of C in W_y^-. It suffices then to apply Lemma A.1; (A.2) and (A.3) yield (2.3).

A.3. *Proof of Lemma* 2.2. Let d^* denote the distance for the Riemann metric on f^kC' or f^kC. If $u,v\in C'$, we have

$$d^*(f^kpu,f^kpv)\leqslant T^kd^*(pu,pv).$$

Let n be the largest integer such that

$$(T/\theta)^n\leqslant\delta/d^*(pu,pv) \tag{A.6}$$

Then

$$d^*(f^{n+1}u,f^{n+1}pu)\leqslant\delta\cdot\theta^{n+1}\leqslant T^{n+1}d^*(pu,pv)$$
$$d^*(f^{n+1}v,f^{n+1}pv)\leqslant\delta\cdot\theta^{n+1}\leqslant T^{n+1}d^*(pu,pv)$$

and if $d^*(pu,pv)$ is sufficiently small this implies

$$d^*(f^{n+1}u,f^{n+1}v)\leqslant 4T^{n+1}d^*(pu,pv) \tag{A.7}$$

hence

$$d^*(u,v)\leqslant 4(T\theta)^{n+1}d^*(pu,pv).$$

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Using now (A.6) we have

$$d^*(u,v) \leqslant 4T\theta d^*(pu,pv)\exp\left[\log(T\theta)\frac{\log(\delta/d^*(pu,pv))}{\log(T/\theta)}\right]$$
$$=4T\theta d^*(pu,pv)\left[\delta/d^*(pu,pv)\right]^{1-\gamma}$$
$$=4T\theta\delta^{1-\gamma}\left[d^*(pu,pv)\right]^{\gamma}.$$

This proves part (a) of the lemma.

Using (A.7), (A.6) we obtain also

$$\sum_{k=1}^{n+1}\left[d^*(f^kpu,f^kpv)+d^*\left(f^ku,f^kv\right)\right]$$
$$\leqslant d^*(pu,pv)\sum_{k=1}^{n+1}\left[T^k+4T^{n+1}\theta^{n+1-k}\right.$$
$$=d^*(pu,pv)\left[\frac{T^{n+2}-T}{T-1}+4T^{n+1}\frac{1-\theta^{n+1}}{1-\theta}\right]$$
$$\leqslant\left(\frac{T^2}{T-1}+\frac{4T}{1-\theta}\right)T^nd^*(pu,pv)$$
$$\leqslant\left(\frac{T^2}{T-1}+\frac{4T}{1-\theta}\right)d^*(pu,pv)\exp\left[\log T\cdot\frac{\log(\delta/d^*(pu,pv))}{\log(T/\theta)}\right]$$
$$=\left(\frac{T^2}{T-1}+\frac{4T}{1-\theta}\right)d^*(pu,pv)\left[\delta/d^*(pu,pv)\right]^{1-\gamma/2}$$
$$=\left(\frac{T^2}{T-1}+\frac{4T}{1-\theta}\right)\delta^{1-\gamma/2}\left[d^*(pu,pv)\right]^{\gamma/2} \tag{A.8}$$

We have

$$\lambda(f^ku)=\Delta\left(T_{f^ku}f^kC'\right),\quad \lambda(f^kpu)=\Delta\left(T_{f^kpu}f^kC\right).$$

In view of the uniform bounds on second derivatives given by Theorem 1.1 (b), and the fact that Δ is C^1, we find

$$|\log\lambda(f^kpu)-\log\lambda(f^kpv)|+|\log\lambda(f^ku)-\log\lambda(f^kv)|$$
$$\leqslant C_3\left[d^*(f^kpu,f^kpv)+d^*(f^ku,f^kv)\right]$$

so that by (A.8)

$$\sum_{k=1}^{n+1}|\log\frac{\lambda(f^ku)}{\lambda(f^kpu)}-\log\frac{\lambda(f^kv)}{\lambda(f^kpv)}|\leqslant C_4\left[d^*(pu,pv)\right]^{\gamma/2} \tag{A.9}$$

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On the other hand as $k \to \infty$, $T_{f^k u} f^k C'$ and $T_{f^k pu} f^k C$ become close exponentially fast, with a bound $C_5 \theta^k$. Therefore

$$\sum_{k=n+2}^{\infty} \left|\log \frac{\lambda(f^k u)}{\lambda(f^k pu)} - \log \frac{\lambda(f^k v)}{\lambda(f^k pv)}\right|$$

$$\leqslant C_6 \sum_{k=n+2}^{\infty} \theta^k = \frac{C_6 \theta}{1-\theta} \theta^{n+1} \leqslant \frac{C_6 \theta}{1-\theta} \exp\left[\log \theta \cdot \frac{\log(\delta / d^*(pu,pv))}{\log(T/\theta)}\right]$$

$$= \frac{C_6 \theta}{1-\theta} \left[\delta / d^*(pu,pv)\right]^{-\gamma/2} = C_7 \left[d^*(pu,pv)\right]^{\gamma/2} \tag{A.10}$$

From (A.9) and (A.10) we obtain

$$|\log F(pu) - \log F(pv)| \leqslant C_8 \left[d^*(pu,pv)\right]^{\gamma/2}$$

proving part (b) of the lemma.

Appendix B*. Proposition B.1. *Let $\mathcal{Q}$ be a finite set with the discrete topology and, for each $i, j \in \mathcal{Q}$, let $t_{ij} \in \{0,1\}$ be given. Let also $\Phi(i_0, i_1, \ldots, i_n) \in \mathbf{R}$ be defined for each $n \geqslant 0$, $i_0, i_1, \ldots, i_n \in \mathcal{Q}$, and satisfy*

$$|\Phi(i_0, i_1, \ldots, i_n)| \leqslant K \theta^n$$

where $K > 0$, $0 < \theta < 1$.

We give $\mathcal{Q}^{\mathbf{P}}$ and $\mathcal{Q}^{\mathbf{Z}}$ the product topology and consider the following compact subsets:

$$\textstyle\prod^+ = \left\{\xi = (\xi_n)_{n \in \mathbf{P}} : t_{\xi_n \xi_{n+1}} = 1 \text{ for all } n \in \mathbf{P}\right\}$$

$$\textstyle\prod = \left\{\xi = (\xi_n)_{n \in \mathbf{Z}} : t_{\xi_n \xi_{n+1}} = 1 \text{ for all } n \in \mathbf{Z}\right\}$$

If $i \in \mathcal{Q}$, $\xi \in \prod^+$ we write

$$F_i(\xi) = t_{i\xi_1} \cdot \exp \sum_{n=1}^{\infty} \Phi(i, \xi_1, \ldots, \xi_n) \tag{B.1}$$

and we introduce a linear map $L : \mathcal{C}(\prod^+) \mapsto \mathcal{C}(\prod^+)$ by

$$(L\psi)(\xi) = \sum_{i \in \mathcal{Q}} F_i(\xi) \psi(i, \xi) \tag{B.2}$$

*This appendix largely follows the ideas of Ruelle [15]. For the Bernouilli property we follow G. Gallavotti, Ising model and Bernouilli schemes in one dimension [Commun. math. Phys. **32**, 183–190 (1973)] and F. Ledrappier, Mesures d'equilibre sur un réseau [Commun. math. Phys. **33**, pp. 119–129 (1973).]

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where (i,ξ) is defined when $t_{i\xi_1}=1$, and equal to $(i,\xi_1,\cdots)\in\coprod^+$.

Finally we write $\coprod_i^+=\{\xi\in\coprod^+:\xi_1=i\}$, $\coprod_i=\{\xi\in\coprod:\xi_1=i\}$ and we assume the existence of a measure $\omega\geqslant 0$ on $\coprod^+$ such that $\omega(\coprod_i^+)>0$ for all $i\in\mathcal{Q}$, and $L^*\omega=\omega$, where L^* is the adjoint of L.

(a) *If t^n denotes the* n-th *power of the matrix $t=(t_{ij})$, we can choose $M\in\mathbf{P}$ such that $(t^{2M})_{ij}>0$ if and only if $(t^M)_{ij}>0$. We define an equivalence relation on $\mathcal{Q}$ such that $i\sim j$ means $i=0$, or $(t^M)_{ij}>0$ and $(t^M)_{ji}>0$. If $[i]$, $[j]$ are the equivalence classes of i,j and $(t^N)_{ij}>0$ we write $[i]\prec[j]$. The relation $\prec$ is an order, and $\sim$, $\prec$ do not depend on the choice of M above. We shall write $\coprod_{[i]}^+=\cup_{j\in[i]}\coprod_i^+$, $\coprod_{[i]}=\cup_{j\in[i]}\coprod_i$.*

(b) *There is a positivity preserving linear map $P:\mathcal{C}(\coprod^+)\mapsto\mathcal{C}(\coprod^+)$ such that*

$$\lim_{n\to\infty}L^{nM}\psi=P\psi \tag{B.3}$$

uniformly. The set $P\{\psi:\psi\geqslant 0$ and $\omega(\psi)=1\}$ is a simplex. Its vertices $\psi_{[i]}$ are indexed by the maximal classes $[i]$ for the order $\prec$. The support of $\psi_{[i]}$ is $\coprod_{[i]}^+$ and $\psi_{[i]}$ is bounded away from zero on $\coprod_{[i]}^+$. If the support of ψ is contained in $\coprod_j$ then $P\psi$ is a linear combination of the $\psi_{[i]}$ with $[i]$ maximal and $[j]\prec[i]$. The functions $\psi_{[i]}$ are permuted by L. We denote by $N_{[i]}$ the smallest integer >0 such that $L^{N_{[i]}}[i]\psi_{[i]}=\psi_{[i]}$.

(c) *Define $g:\coprod\to\coprod_+$ by*

$$g(\xi_n)_{n\in\mathbf{Z}}=(\xi_n)_{n\in\mathbf{P}}$$

Then for each maximal class $[i]$ there is a unique probability measure $\rho_{[i]}$ on $\coprod$, invariant under the shift $(\xi_n)_{n\in\mathbf{Z}}\mapsto(\xi_{n+M})_{n\in\mathbf{Z}}$ on $\coprod$, and such that

$$g\rho_{[i]}=\psi_{[i]}\cdot\omega \tag{B.4}$$

The measures $\rho_{[i]}$ are permuted by the shift $\tau:(\xi_n)_{n\in\mathbf{Z}}\mapsto(\xi_{n+1})_{n\in\mathbf{Z}}$ and $\rho_{[i]}$ has period $N_{[i]}$.

(d) *Let Q be the partition of $\coprod$ consisting of the sets $\coprod_j$. We assume that $[i]$ is maximal and let $Q_{[i]}$ be $\vee_{k=0}^{N_{[i]}-1}\tau^{-k}Q$ restricted to $\coprod_{[i]}$. Then $Q_{[i]}$ is a weak Bernouilli partition of $(\coprod_{[i]},\rho_{[i]},\tau^{N_{[i]}})$ and this dynamical system is isomorphic to a Bernouilli shift. Furthermore there are $a>0$, $b>0$, such that if $\varphi,\varphi'\in\mathcal{C}(\coprod_{[i]})$ where $\varphi(\xi)$ depends only on the ξ_l with $l\leqslant 0$ and $\varphi'(\xi)$ depends only on the ξ_l with $l\geqslant n$, then*

$$|\rho_{[i]}(\varphi\cdot\varphi')-\rho_{[i]}(\varphi)\cdot\rho_{[i]}(\varphi')|\leqslant ae^{-bn}\rho_{[i]}(|\varphi|)\rho_{[i]}(|\varphi'|) \tag{B.5}$$

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(e) *The measure* $\rho_{[i]}$ *is the unique probability measure on* $\prod_{[i]}$, *invariant under* $\tau^{N_{[i]}}$, *which makes maximum the expression*

$$h\left(\rho,\tau^{N_{[i]}}\right)+\rho(G) \tag{B.6}$$

where h denotes the entropy and

$$G(\xi)=\sum_{m=1}^{N[i]}\sum_{n=m}^{\infty}\Phi(\xi_m,\ldots,\xi_n)$$

The measure $N_{[i]}^{-1}\sum_{m=1}^{N_{[i]}}\tau^m\rho_{[i]}$ *is the unique probability measure on* $\cup_{m=1}^{N_{[i]}}\tau^m\prod_{[i]}$, *invariant under* τ, *which makes maximum the expression*

$$h(\rho,\tau)+\rho(H) \tag{B.7}$$

where

$$H(\xi)=\sum_{n=1}^{\infty}\Phi(\xi_1,\ldots,\xi_n).$$

The maxima of (B.6) *and* (B.7) *are* 0.

(f) *Let* S *be a closed subset of* $\prod^+$ *such that* $\tau g^{-1}S\subset g^{-1}S$ *and* $S\cap\prod_{[i]}^+\neq\prod_{[i]}^+$ *for each maximal* $[i]$. *Then* $\omega(S)=0$.

B.2. *Proof of* (a). There are finitely many ways to choose the elements of a finite dimensional matrix zero or non-zero. Therefore we can choose M such that $(t^{2M})_{ij}>0$ if and only if $(t^M)_{ij}>0$. By induction, when $M'\in\mathbf{P}$, $(t^{MM'})_{ij}>0$ if and only if $(t^M)_{ij}>0$. From this it is clear that the relations $\sim$, $\prec$ do not depend on the choice of M. Furthermore if $(t^M)_{ij}>0$ and $(t^M)_{jk}>0$, then $(t^M)_{ik}>0$. Thus $\prec$ is an order.

LEMMA B.3. *There exists* $D>0$ *and, given* $\delta>0$, *there exists* $k_0\in\mathbf{P}$ *such that if* $0\leqslant\psi\in\mathcal{C}(\prod^+)$ *and* $\psi(\xi)$ *depends only on* $\xi_1,\ldots,\xi_m$, *the following properties hold.*

(a) *Let* $n\geqslant m+M-1$. *If* $\xi',\xi''\in\prod^+$ *and* $\xi_1'=\xi_1''$ *or* $[\xi_1']\prec[\xi_1'']$, *then*

$$(L^n\psi)(\xi')\leqslant D(L^n\psi)(\xi'') \tag{B.8}$$

(b) *Let* $n\geqslant m$, *and* $k\geqslant k_0$, *then*

$$L^n\psi=\sum_{l=1}^{\infty}\psi_l,\quad 0\leqslant\psi_l\leqslant\delta^{l-1}\psi_1 \tag{B.9}$$

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where $\psi_l(\xi)$ *depends only on* $\xi_1,\ldots,\xi_{lk}$.

We may write $\psi(\xi)=\tilde{\psi}(\xi_1,\ldots,\xi_m)$. Then

$$(L^n\psi)(\xi)=\sum_{i_1,\ldots,i_n}\tilde{\psi}(i_1,\ldots,i_m)t_{i_1i_2}\cdots t_{i_n\xi_1}\exp H_{i_1\cdots i_n}(\xi) \tag{B.10}$$

where

$$\begin{aligned} H_{i_1\cdots i_n}(\xi) &= \sum_{p=1}^{n}\left[\sum_{q=p}^{n}\Phi(i_p,\ldots,i_q)+\sum_{r=1}^{\infty}\Phi(i_p,\ldots,i_n,\xi_1,\ldots,\xi_r)\right] \\ &= \sum_{i\leqslant p\leqslant q\leqslant n-M}\Phi(i_p,\ldots,i_q)+\sum_{n-M<p\leqslant q\leqslant n}\Phi(i_p,\ldots,i_q)+R \end{aligned} \tag{B.11}$$

with

$$|R|\leqslant 2K\sum_{l=1}^{\infty} l\theta^l=K_1.$$

(a). We have thus

$$\begin{aligned} &e^{-K_1}(L^n\psi)(\xi) \\ &\quad\leqslant \sum_{i_1\cdots i_{n-M+1}}\left[\tilde{\psi}(i_1,\ldots,i_m)t_{i_1i_2}\cdots t_{i_{n-M}i_{n-M+1}}\exp\sum_{1\leqslant p\leqslant q\leqslant n-M}\Phi(i_p,\ldots,i_q)\right] \\ &\quad\quad\times T(i_{n-M+1},\xi_1) \\ &\quad\leqslant e^{K_1}(L^n\psi)(\xi) \end{aligned}$$

where

$$T(j_1,\xi_1)=\sum_{j_2\cdots j_M}t_{j_1j_2}\cdots t_{j_M\xi_1}\exp\sum_{1\leqslant p\leqslant q\leqslant M}\Phi(j_p,\ldots,j_q)$$

has an upper bound K_2 and, if $[j_1]\prec[\xi_1]$, a lower bound K_3^{-1}. If $\xi_1'=\xi_1''$ or $[\xi_1']\prec[\xi_1'']$ then $[i_{n-M+1}]\prec[\xi_1']$ implies $[i_{n-M+1}]\prec[\xi_1'']$, thus

$$e^{-K_1}(L^n\psi)(\xi')\leqslant K_2K_3e^{K_1}(L^n\psi)(\xi'')$$

and (B.8) is proved with $D=K_2K_3e^{2K_1}$.

(b). Taking $k\in\mathbf{P}$, let

$$S_l=\sum_{p=1}^{n}\sum_{r=(l-1)k+1}^{lk}\left[\Phi(i_p,\ldots,i_n,\xi_1,\ldots,\xi_r)+3K\theta^{n-p+r}\right].$$

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Then

$$0 \leqslant S_1 < 4K\frac{\theta}{(1-\theta)^2}, \quad 0 \leqslant S_l \leqslant 2\theta^{k(l-1)}S_1$$

and

$$\begin{aligned}\exp \sum_{p=1}^{n} \sum_{r=1}^{\infty} \Phi(i_p, \ldots, i_n, \xi_1, \ldots, \xi_r) \\ &= \exp\left[-3K\sum_{p=1}^{n}\sum_{r=1}^{\infty}\theta^{n-p+r}\right]\prod_{l=1}^{\infty} e^{S_l} \\ &= \exp\left[-3K\frac{1-\theta^n}{1-\theta}\cdot\frac{\theta}{1-\theta}\right]\cdot\left\{e^{S_1} + \sum_{l=2}^{\infty} e^{S_1+\cdots+S_{l-1}}(e^{S_l}-1)\right\} \\ &= \sum_{l=1}^{\infty} E_l \end{aligned} \tag{B.12}$$

For sufficiently large k (chosen independently of n, p, $i_1, \ldots, i_n, \xi$) we have

$$0 \leqslant E_l \leqslant \delta^{l-1}E_1$$

Introducing (B.12) into (B.11), (B.10) proves (B.9).

LEMMA B.4. *Suppose that the class* $[i]$ *is maximal for the order* $\prec$. *If* $\psi \in \mathcal{C}(\prod^+)$ *vanishes on* $\prod^+/\prod^+_{[i]}$ *so does* $L^M\psi$. *A map* $L_{[i]}: \mathcal{C}(\prod^+_{[i]}) \mapsto \mathcal{C}(\prod^+_{[i]})$ *is thus naturally defined by the restriction of* L^M *to* $\mathcal{C}(\prod^+_{[i]})$ *identified to the subspace of* $\mathcal{C}(\prod^+)$ *consisting of those* ψ *which vanish on the complement of* $\prod^+_{[i]}$. *Let* $\omega_{[i]}$ *be the restriction of* ω *to* $\prod^+_{[i]}$.

(a) *There exists* $\psi_{[i]} \in \mathcal{C}(\prod^+_{[i]})$ *such that* $L_{[i]}\psi_{[i]} = \psi_{[i]}$, $\omega_{[i]}(\psi_{[i]}) = 1$, *and* $\psi_{[i]}(\xi') \leqslant D\psi_{[i]}(\xi'')$ *for all* $\xi', \xi'' \in \prod^+_{[i]}$.

(b) *There exist* $\eta \in (0, \frac{1}{3})$ *and* $p \geqslant 1$ *such that if* $0 \leqslant \psi \in \mathcal{C}(\prod^+_{[i]})$ *and* ψ *depends only on* $\xi_1, \ldots, \xi_{(m-1)M}$, *then*

$$L^m_{[i]}\psi = C^{(0)}_\psi \psi_{[i]} + \sum_{l=1}^{\infty} \psi^{(0)}_l, \quad C^{(0)}_\psi \geqslant 0$$

where $\psi^{(0)}_l \geqslant 0$, $\psi^{(0)}_l(\xi)$ *depends only on* $\xi_1, \ldots, \xi_{(lp-1)M}$ *and*

$$\omega_{[i]}(\psi^{(0)}_1) \leqslant (1-2\eta)\omega_{[i]}(\psi)$$

$$\psi^{(0)}_l(\xi) \leqslant \eta^l \omega_{[i]}(\psi)\cdot\psi_{[i]}(\xi) \qquad \text{for } l \geqslant 2$$

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(c) *Under the conditions of* (b) *we have*

$$L_{[i]}^{m+rp}\psi = C_\psi^{(r)}\psi_{[i]} + \sum_{l=1}^{\infty}\psi_l^{(r)}, \qquad C_\psi^{(r)} \geqslant 0$$

where each $\psi_l^{(r)}$ *is a sum* $\sum_{k\geqslant 0} L_{[i]}^{kp}\psi_{l,k}^{(r)}$ *such that* $\psi_{l,k}^{(r)} \geqslant 0$ *and* $\psi_{l,k}^{(r)}(\xi)$ *depends only on* $\xi_1,\ldots,\xi_{(lp+kp-1)M}$. *Furthermore*

$$\omega_{[i]}(\psi_1^{(r)}) \leqslant (1-2\eta)(1-\eta)^r\omega_{[i]}(\psi)$$

and, for $l \geqslant 2$,

$$\psi_l^{(r)}(\xi) \leqslant \eta^l(1-\eta)^r\omega_{[i]}(\psi)\cdot\psi_{[i]}(\xi)$$

Let $0 \leqslant \psi \in \mathcal{C}(\prod_{[i]}^+)$. Applying Lemma B.3 (with $k=(p-1)M$) we find that if ψ depends only on $\xi_1,\ldots,\xi_{(m-1)M}$, then

$$L_{[i]}^m\psi(\xi') \leqslant DL_{[i]}^m\psi(\xi'') \tag{a$'$}$$

for $\xi',\xi'' \in \prod_{[i]}^+$.

$$L_{[i]}^m\psi = \sum_{l=1}^{\infty}\psi_l, \quad 0 \leqslant \psi_l \leqslant \delta^{l-1}\psi_1 \tag{b$'$}$$

where $\psi_l(\xi)$ depends only on $\xi_1,\ldots,\xi_{l(p-1)M}$.

If ψ_0 is the constant $[\omega_{[i]}(1)]^{-1}$, the sequence $\dfrac{1}{n}\sum_{m=1}^{n} \mathrm{L}_{[i]}^m\psi_0$ has, because of (b′), a uniformly convergent subsequence and, using (a′), we see that it satisfies (a) of the present lemma. Furthermore

$$\psi_{[i]} = \sum_{l=1}^{\infty}\psi_{[i]l}, \quad 0 \leqslant \psi_{[i]l} \leqslant \delta^{l-1}\psi_{[i]1} \tag{B.13}$$

Notice the inequalities

$$\psi_1 \geqslant (1-\delta)L_{[i]}^m\psi \geqslant \frac{1-\delta}{D}\cdot\frac{\omega_{[i]}(\psi)}{\omega_{[i]}(1)}$$

$$\psi_{[i]l} \leqslant \delta^{l-1}\psi_{[i]1} \leqslant \delta^{l-1}\psi_{[i]} \leqslant \delta^{l-1}\frac{D}{\omega_{[i]}(1)}\,.$$

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We write

$$\psi = C\psi_{[i]} + \left[\psi_1 - C\left(\frac{\delta D}{(1-\delta)\omega_{[i]}(1)} + \psi_{[i]1} \right) \right] + \sum_{l=2}^{\infty} \left[\psi_l + C\left(\frac{\delta^{l-1} D}{\omega_{[i]}(1)} - \psi_{[i]l} \right) \right]$$

$$= C\psi_{[i]} + \psi_1^{(0)} + \sum_{l=2}^{\infty} \psi_l^{(0)}.$$

If $C = \left(\frac{1-\delta}{D} \right)^2 \omega_{[i]}(\psi)$ we have

$$\psi_1^{(0)} \geqslant \frac{1-\delta}{D} \cdot \frac{\omega_{[i]}(\psi)}{\omega_{[i]}(1)} - \frac{C}{\omega_{[i]}(1)} \left(\frac{\delta D}{1-\delta} + D \right) = 0$$

and also $\psi_l^{(0)} \geqslant 0$ for $l \geqslant 2$. Furthermore, since $\psi_{[i]1} \geqslant (1-\delta)\psi_{[i]}$,

$$\psi_1^{(0)} \leqslant \psi_1 - C\psi_{[i]1} \leqslant \psi - C(1-\delta)\psi_{[i]}$$

$$\omega_{[i]}\left(\psi_1^{(0)}\right) \leqslant \left(1 - \frac{(1-\delta)^3}{D^2}\right)\omega_{[i]}(\psi)$$

and, for $l \geqslant 2$,

$$\psi_l^{(0)} \leqslant \psi_l + \frac{C}{\omega_{[i]}(1)} \delta^{l-1} D \leqslant \delta^{l-1} \left[L_{[i]}^m \psi + \frac{CD}{\omega_{[i]}(1)} \right]$$

$$\leqslant \delta^{l-1} \left[D \frac{\omega_{[i]}(\psi)}{\omega_{[i]}(1)} + \frac{CD}{\omega_{[i]}(1)} \right] \leqslant \delta^{l-1} \left[D^2 + (1-\delta)^2 \right] \omega_{[i]}(\psi) \cdot \psi_{[i]}.$$

From these inequalities it follows that (b) of the present lemma holds if δ is chosen sufficiently small.

To prove (c) we proceed by induction on r, writing

$$\psi_l^{(r)} = L_{[i]}^p \psi_{l+1}^{(r-1)} + \left(\psi_1^{(r-1)}\right)_l^{(0)}$$

Since $0 < \eta < \frac{1}{3}$ we have

$$\omega_{[i]}(\psi_1^{(r)}) = \omega_{[i]}(\psi_2^{(r-1)}) + \omega_{[i]}\left(\left(\psi_1^{(r-1)}\right)_1^{(0)}\right)$$

$$\leqslant \left[\eta^2 (1-\eta)^{r-1} + (1-2\eta)^2 (1-\eta)^{r-1} \right] \omega_{[i]}(\psi)$$

$$\leqslant (1-2\eta)(1-\eta)^r \omega_{[i]}(\psi)$$

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For $l \geqslant 2$,

$$\psi_l^{(r)} \leqslant \eta^{l+1}(1-\eta)^{r-1}\omega_{[i]}(\psi)\cdot\psi_{[i]} + \eta^l\omega_{[i]}(\psi_1^{(r-1)})\cdot\psi_{[i]}$$
$$\leqslant \eta^l(1-\eta)^r\omega_{[i]}(\psi)\cdot\psi_{[i]}.$$

B.5. *Proof of* (b). In view of (B.8), $L^n 1$ is bounded on $\coprod_{[i]}^+$ by $D\omega(\coprod^+)/\omega(\coprod_{[i]}^+)$ uniformly in n, $[i]$. Since L is positivity preserving, it suffices to prove that $L^{nM}\psi$ tends uniformly to a limit under the assumption that $\psi \geqslant 0$ and $\psi(\xi)$ depends only on $\xi_1, \ldots, \xi_m$.

We know that $L^{nM}\psi$ has the same total mass with respect to ω as ψ (because $L^*\omega = \omega$). In fact it follows readily from the definitions that L^{nM} redistributes the mass carried by $\coprod_j^+$ to the sets $\coprod_{[i]}^+$ such that $[j] \prec [i]$. Therefore, if $[i]$ is a minimal class among those such that $(\psi\cdot\omega)\coprod_{[i]}^+ > 0$, then $(L^{nM}\psi\cdot\omega)\coprod_{[i]}^+$ decreases to a limit when $n\to\infty$, and this limit is zero unless $[i]$ is maximal (use (B.8) and the definition of L^{nM}). Iterating this argument we find that unless $[i]$ is maximal $(L^{nM}\psi\cdot\omega)\coprod_{[i]}^+$ will tend to zero. By (B.8), $L^{nM}\psi$ tends thus uniformly to zero on those $\coprod_{[i]}^+$ such that $[i]$ is not maximal.

By Lemma B.4 (c), when $[i]$ is maximal, the restriction of $L^{(m+rp+1)M}\psi$ to $\coprod_{[i]}$ tends uniformly to $C_{[i]}\psi_{[i]}$, with $C_{[i]} \geqslant 0$, when $r\to\infty$. It is then easy to check the first part of (b).

The fact that L permutes the $\psi_{[i]}$ follows from the geometric characterization of the $\psi_{[i]}$ as vertices of the simplex $P\{\psi : \psi \geqslant 0 \text{ and } \omega(\psi) = 1\}$, which is linearly mapped onto itself by L.

Remark. B.6 Since $L^{N_{[i]}}$ is positivity preserving and $L^{N_{[i]}}\psi_{[i]}$, a map $L'_{[i]} : \mathcal{C}(\coprod_{[i]}^+) \to \mathcal{C}(\coprod_{[i]}^+)$ is naturally defined by the restriction of $L^{N_{[i]}}$ to $\mathcal{C}(\coprod_{[i]}^+)$ identified to the subspace of $\mathcal{C}(\coprod^+)$ consisting of those ψ which vanish on the complement of $\coprod_{[i]}^+$.

If $0 \leqslant \psi \in \mathcal{C}(\coprod_{[i]}^+)$ and $\psi(\xi)$ depends only on $\xi_1, \ldots, \xi_{mN_{[i]}}$, then, for some a_0, $b_0 > 0$,

$$|(L'_{[i]})^{m+n}\psi - [\omega(\psi)]\cdot\psi_{[i]}| \leqslant [a_0\omega(\psi)]e^{-b_0 n} \tag{B.14}$$

To see this it suffices to write (for n large enough)

$$(L'_{[i]})^{m+n} = (L'_{[i]})^q L_{[i]}^l$$

with $0 \leqslant q < M$, and to apply Lemma B.4 (c).

B.7. *Proof of* (c). The functions of the form $\psi\circ g\circ\tau^{kM}$ where $\psi \in \mathcal{C}(\coprod^+)$ and $k \in \mathbf{Z}$ are dense in $\mathcal{C}(\coprod)$ by the Stone-Weierstrass theorem, and

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if $\rho_{[i]}$ satisfies the required conditions we must have

$$\rho_{[i]}(\psi \circ g \circ \tau^{kM}) = \rho_{[i]}(\psi \circ g) = (g\rho_{[i]})(\psi) = \omega(\psi_{[i]} \cdot \psi) \tag{B.15}$$

Conversely, using (B.15) as definition, it is easily seen that $\mu_{[i]}$ has the desired properties provided we show that the definition $(*)$ is unique. This amounts to checking that, if $l > 0$,

$$\omega\big(\psi_{[i]} \cdot (\psi \circ g \circ \tau^{lM} \circ g^{-1})\big) = \omega(\psi_{[i]} \cdot \psi)$$

We have indeed

$$\begin{aligned}
\omega\big(\psi_{[i]} \cdot (\psi \circ g \circ \tau^{lM} \circ g^{-1})\big) &= (L*^{lM}\omega)\big(\psi_{[i]} \cdot (\psi \circ g \circ \tau^{lM} \circ g^{-1})\big) \\
&= \omega\Big(L^{lM}\big(\psi_{[i]} \cdot (\psi \circ g \circ \tau^{lM} \circ g^{-1})\big)\Big) \\
&= \omega\big((L^{lM}\psi_{[i]}) \cdot \psi\big) = \omega(\psi_{[i]} \cdot \psi)
\end{aligned}$$

By a similar calculation we find

$$\begin{aligned}
(\tau\rho_{[i]})(\psi \circ g) = \rho_{[i]}(\psi \circ g \circ \tau) &= \omega\big(\psi_{[i]} \cdot (\psi \circ g \circ \tau \circ g^{-1})\big) \\
&= \omega((L\psi_{[i]}) \cdot \psi)
\end{aligned} \tag{B.16}$$

Since the $\psi_{[i]}$ are permuted by L, the $\mu_{[i]}$ are permuted by τ in the same manner.

B.8. *Proof of* (d). Let $\varphi = \psi \circ g$ and $\varphi' = \psi' \circ g \circ \tau^{(m+n)N_{[i]}}$ where $\psi(\xi)$, $\psi'(\xi)$ depend only on $\xi_1, \dots, \xi_{mN_{[i]}}$. Then, by (B.15),

$$\begin{aligned}
\rho_{[i]}(\varphi \cdot \varphi') &= \omega\big(\psi_{[i]} \cdot \psi \cdot (\psi' \circ g \circ \tau^{(m+n)N}[\,i\,] \circ g^{-1})\big) \\
&= \omega\Big(L^{(m+n)N_{[i]}}\big(\psi_{[i]} \cdot \psi \cdot (\psi' \circ g \circ \tau^{(m+n)N_{[i]}} \circ g^{-1})\big)\Big) \\
&= \omega\big(\psi' \cdot L^{(m+n)N_{[i]}}(\psi_{[i]} \cdot \psi)\big).
\end{aligned}$$

Using (B.14) and the fact (B.13) that $\psi_{[i]}(\xi)$ can be approximated by a function of $\xi_1, \dots, \xi_{lN_{[i]}}$ exponentially fast in n, we obtain

$$|L^{(m+n)N_{[i]}}(\psi_{[i]} \cdot \psi) - [\,\omega(\psi_{[i]} \cdot \psi)\,]\psi_{[i]}| \leqslant a_1 \omega(|\psi|) e^{-b_1 n}$$

hence

$$|\rho_{[i]}(\varphi \cdot \varphi') - \omega(\psi_{[i]} \cdot \psi)\omega(\psi_{[i]} \cdot \psi')| \leqslant a_1 e^{-b_1 n} \omega(|\psi|)\omega(|\psi'|)$$

or

$$|\rho_{[i]}(\varphi\varphi') - \rho_{[i]}(\varphi)\rho_{[i]}(\varphi')| \leqslant a_2 e^{-b_1 n} \rho_{[i]}(|\varphi|)\rho_{[i]}(|\varphi'|).$$

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From this (B.5) readily follows. Let now (φ_α) and (φ'_β) be the families of characteristic functions corresponding to the partitions $\bigvee_{k=0}^{m-1}\tau^{-kN_{[i]}}Q_{[i]}$ and $\bigvee_{k=n+m-1}^{n+2m-1}\tau^{-kN_{[i]}}Q_{[i]}$, then

$$\sum_\alpha\left|\frac{\mu_{[i]}(\varphi_\alpha\varphi'_\beta)}{\mu_{[i]}(\varphi'_\beta)}-\mu_{[i]}(\varphi_\alpha)\right|<a_2e^{-b_1n}$$

For sufficiently large n this is arbitrarily small, showing that $Q_{[i]}$ is weak Bernouilli, therefore $(\prod,\mu_{[i]},\tau^{N_{[i]}})$ is isomorphic to a Bernouilli shift.

B.9. *Proof of* (e). Let $\omega_{[i]}$ be the restriction of $\omega/\omega(\prod_{[i]}^+)$ to $\prod_{[i]}^+$, and $L'_{[i]}$ be as in Remark B.6. Then one checks easily that

$$\omega_{[i]}=L'^*_{[i]}\omega_{[i]}=\operatorname*{vague\,lim}_{m\to\infty}(L'^*_{[i]})^m\omega_{[i]}$$

$$g\rho_{[i]}=\operatorname*{vague\,lim}_{m\to\infty}(g\tau^{mN_{[i]}}g^{-1})\omega_{[i]}.$$

From this it results that $\rho_{[i]}$ is a *Gibbs state* (see Dobrushin [8], [9]) for a suitable statistical mechanical system), more precisely a *one-dimensional classical lattice system*. The underlying *lattice* consists here of blocks of length $N_{[i]}$ of $\mathbf{Z}$, the configuration space is $\cap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}$, the *interaction* is obtained from Φ by restriction to $\cap_{n\geqslant 0}\tau^{nN_{[i]}}\prod_{[i]}$ and then passing to the division of $\mathbf{Z}$ into blocks of length $N_{[i]}$. There is only one Gibbs state in the present situation: this can be seen for instance from (B.5) and a slight extension of the uniqueness criterion in Dobrushin [8] (see Ruelle [16].)

Translation invariant Gibbs states are characterized by a variational principle (Lanford-Ruelle [13]). Reference [13] applies only to the case where $t_{ij}=1$ for all i,j, but can be extended to the present situation. A complete proof is given in Ruelle [16]. This yields (B.6), and (B.7) follows immediately. The maximum of (B.6) is the pressure, and is 0 because $L^*\omega=\omega$.

Notice that $\operatorname{supp} g\rho_{[i]}=\prod_{[i]}^+\cap\operatorname{supp}\omega=\operatorname{supp}\omega_{[i]}$ is $\prod_{[i]}^+$ because $\omega_{[i]}=L'^*_{[i]}\omega_{[i]}$ and $\omega_{[i]}(\prod_{[i]}^+)>0$ for all $j\in[i]$. This will be used in the proof of (f).

B.10. *Proof of* (f). Let χ_s be the characteristic function of S. Then

$$\omega(S)\leqslant\omega(\chi_s\circ g\circ\tau^n\circ g^{-1})=\omega\big(L^n(\chi_s\circ g\circ\tau^n\circ g^{-1})\big)=\omega(\chi_s\cdot L^n1).$$

In view of (b), (c), it suffices to prove that the following expression vanishes for all maximal $[i]$:

$$\omega(\chi_s\cdot\psi_{[i]})=(g\rho_{[i]})(S).$$

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Since $\operatorname{supp} g\rho_{[i]} = \coprod_{[i]}^{+}$ (see end of Section B.9) and $S \cap \coprod_{[i]}^{+} \neq \coprod_{[i]}^{+}$, $(g\rho_{[i]})(S) < 1$, hence

$$1 > (g\rho_{[i]})(S) = \rho_{[i]}(g^{-1}S) = \rho_{[i]}\left(\bigcap_{m \geq 0} \tau^{mN_{[i]}} g^{-1}S \right) \tag{B.17}$$

Since $\rho_{[i]}$ is $\tau^{N_{[i]}}$-ergodic by (d), the right-hand side of (B.17) is 0 or 1, and therefore vanishes.

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