MARKOV PARTITIONS AND C-DIFFEOMORPHISMS

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1. Pertinent Information on C-Diffeomorphisms and Formulation of the Results

Among all diffeomorphisms, C-diffeomorphisms [sometimes also called U-diffeomorphisms*] are distinguished by the maximum possible instability properties. Specifically, if T is a C-diffeomorphism of class C^r of some compact Riemannian manifold M of class C^∞, $x \in M$, then for any point x' near x the distance between $T^n x$ and $T^n x'$ increases with n at an exponential rate (either as $n \to +\infty$ or as $n \to -\infty$).

It is presumed that the reader is familiar with the conventional definition of the C-diffeomorphism [3, 4, 5]. The following fundamental property of C-diffeomorphisms is used in the present article:

For every point x the set of points x' for which $d(T^n x, T^n x') \to 0$ as $n \to \infty$ (d is the metric on the manifold) forms a k-dimensional submanifold $\Gamma^{(c)}(x)$ of class C^{r-1}; the set of points x' for which $d(T^n x, T^n x') \to 0$ as $n \to -\infty$ forms an l-dimension submanifold $\Gamma^{(e)}(x)$ of class C^{r-1}; here:

a) the numbers k and l do not depend on x, and $n = k + l$; at every point x the distance between the tangent spaces to $\Gamma^{(c)}(x)$ and $\Gamma^{(e)}(x)$ is greater than a fixed positive constant;

b) the set of submanifolds $\Gamma^{(c)}(x)$ forms a continuous k-dimensional foliation $\mathfrak{S}^{(k)}$, and the set of submanifolds $\Gamma^{(e)}(x)$ forms a continuous l-dimensional foliation $\mathfrak{S}^{(l)}$; each of these foliations is invariant: $T\mathfrak{S}^{(k)} = \mathfrak{S}^{(k)}$, $T\mathfrak{S}^{(l)} = \mathfrak{S}^{(l)}$;

c) there exist positive constants $a > 0$, $\lambda_c < 1$, $\lambda_e > 1$, such that if $d_c(d_e)$ is an induced metric on the layer $\Gamma^{(c)}$ ($\Gamma^{(e)}$), then

$$d_c(T^n x, T^n x') \leqslant a\lambda_c^n d_c(x, x'), \quad n \geqslant 0,$$

$$d_e(T^n x, T^n x') \leqslant a\lambda_e^{-n} d_e(x, x'), \quad n \leqslant 0.$$

For this reason, the foliation $\mathfrak{S}^{(k)}$ is called contractile, and all objects relating to it carry the index "c"; analogously, the foliation $\mathfrak{S}^{(l)}$ is called expansible, and all relevant objects carry the index "e."

Every layer $\Gamma^{(c)}$ and $\Gamma^{(e)}$ is homeomorphic to a Euclidean space.

Definition 1.1. A subset U of finite diameter of a layer $\Gamma^{(c)}$ or $\Gamma^{(e)}$ is called admissible if the set of its interior points is not empty and the Riemann volume of its boundary ∂U is equal to zero (in the sense of an induced metric on the layer).

Let O be a spherical neighborhood in M. We define a contractile local layer (CLL) in O as an admissible subset of the connected component of the intersection of $\Gamma^{(c)}$ with O. Expansible local layers (ELL) in O are introduced analogously. If the radius of O is sufficiently small, every CLL D_c intersects with every ELL D_e at most at one point. Henceforth, when speaking of the smallness of O, we refer to those dimensions of O for which this condition is indeed fulfilled.

Let D'_c and D''_c be two CLL. We call D'_c and D''_c canonically isomorphic if there exists a small neighborhood O, $D'_c \in O$, $D''_c \in O$, and a number $\rho > 0$, such that for every point $x \in D'_c$ a sphere $D_e(x; \rho)$ of radius ρ with center at a point x on an expansible layer intersects D''_c, and $\bigcup_{x \in D'_c} D_e(x; \rho) \subset O$. Here $D''_c \cap D_e(x; \rho) = y \in D''_c$, and the correspondence $\pi: x \to y$ is a one-to-one mapping of D'_c onto D''_c. We call this mapping π a canonical isomorphism.

More generally, let D'_c and D''_c be two arbitrary CLL. We assume that $D'_c = \bigcup_{i=1}^{k} D'_{c,i}$, $D''_c = \bigcup_{i=1}^{k} D''_{c,i}$ and that there exist chains of CLL $D'_{c,i,0} = D'_{c,i}$, $D'_{c,i,1}, \ldots, D'_{c,i,s} = D''_{c,i}$, such that $D'_{c,i,j}$ and $D'_{c,i,j+1}$ lie in one

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small spherical neighborhood O_{ij} and are canonically isomorphic therein (s does not depend on i, and $D'_{c,i}$, $D''_{c,i}$ are pairwise nonintersecting for different i). Then D'_c and D''_c are called canonically isomorphic.

Canonical isomorphism between ELL is defined analogously.

If D'_c and D''_c are canonically isomorphic, then $T^nD'_c$ and $T^nD''_c$ are also canonically isomorphic for every n.

A canonical isomorphism is continuous, because the foliations $\mathfrak{S}^{(k)}$ and $\mathfrak{S}^{(l)}$ are continuous. But more important is the fact that a canonical isomorphism is absolutely continuous, i.e., it takes the Riemann volume on one local layer into a measure equivalent (in the sense of absolute continuity) to the Riemann volume on another local layer. This statement is essentially proved in [4]. The arguments presented therein do not rely on the existence of a smooth invariant measure for T.

The property of absolute continuity of a canonical isomorphism demonstrates the uniqueness of the class of admissible sets that we have introduced; if U is admissible its canonical isomorphism image is also admissible.

Let $U \subset O$, and let O be sufficiently small. We pick a CLL $D_c \in O$ for which $\partial D_c \subset \partial O$. We call the intersection $D_c \cap U$ a CLL in U. An ELL in U is defined analogously.

Definition 1.2. A subset $U \subset O$ is called a parallelogram if all its CLL (ELL) are admissible and canonically isomorphic with one another.

Parallelograms U exist. It suffices for every point x to pick accessible CLL $D_c(x)$ and ELL $D_e(x)$ of sufficiently small dimensions and to set $U = \bigcup_{z \in D_c(x)} D_e(z) = \bigcup_{z \in D_e(x)} D_c(z)$, where $D_e(z)\,[D_c(z)]$ is an ELL [CLL] canonically isomorphic to $D_e(x)$ $[D_c(x)]$ and passing through $z \in D_c(x)\,[z \in D_e(x)]$. Every parallelogram U can be so represented as long as x is any point belonging to it and $D_c(x)$, $D_e(x)$ are local layers in U passing through it.

For a parallelogram U its boundary $\Gamma(U) = \bigcup_{z \in \partial D_c(x)} D_e(z) \cup \bigcup_{z \in \partial D_e(x)} D_c(z) = \Gamma_p(U) \cup \Gamma_c(U)$.. We define a partition of M into parallelograms $U_1, \ldots, U_r$ as a system of parallelograms such that $M = \bigcup_{i=1}^{r} \bar{U}_i$, $U_i \cap U_j \subset \Gamma(U_i) \cap \Gamma(U_j)$. If α is a partition into parallelograms, then $\Gamma_e(\alpha) = \bigcup_{i=1}^{r} \Gamma_e(U_i)$, $\Gamma_c(\alpha) = \bigcup_{i=1}^{r} \Gamma_c(U_i)$.

In the present article we describe a new method in the Markov partition.

Definition 1.3. A partition α into parallelograms is called a Markov partition if for some m > 0

$$\Gamma_e(T^{-m}\alpha) \subset \Gamma_e(\alpha), \quad \Gamma_c(T^m\alpha) \subset \Gamma_c(\alpha).$$

In a recent paper Adler and Weiss [2] give a very simple example of a Markov partition, when M is a two-dimensional torus and T is its algebraic automorphism. In this case (see Fig. 1), the partition consists entirely of two parallelograms, and m = 1. The sides of these parallelograms are formed of segments lying on the proper lines of our automorphism that pass through O.

THEOREM. Every C-diffeomorphism has a Markov partition. Moreover, for any $\varepsilon > 0$ it is possible to construct a Markov partition such that the diameter of every parallelogram is not greater than ε.

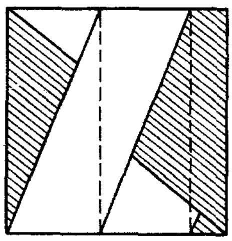

Fig. 1

This theorem is proved in the second part of the present paper. In the first part we deduce some results pertaining to C-diffeomorphisms, using only the extence of a Markov partition into parallelograms of sufficiently small dimensions.

If α is a Markov partition, then the partition $\alpha_1 = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-m+1}\alpha$, whose elements have the form $V = U_{i_0} \cap T^{-1}U_{i_1} \cap \ldots \cap T^{-m+1}U_{i_{m-1}}$, is also Markovian, with m = 1.

Let α_1 be a Markov partition with m = 1. One of its advantages is that any intersection $V_{i_0} \cap TV_{i_1} \cap \ldots \cap T^nV_{i_n} \neq \varnothing$ when and only when the pairwise intersections $V_{i_s} \cap TV_{i_{s+1}}$ (see Section 3), x = 0, 1, ..., n − 1, are not empty.

We introduce the matrix of intersections $\Pi = \|\pi_{ij}\|$, where $\pi_{ij} = 1$, if $V_i \cap TV_j \neq \emptyset$, $\pi_{ij} = 0$ otherwise. We examine the space Ω_Π of all infinite sequences $\omega = \{\ldots i_{-n} \ldots i_0 \ldots i_n \ldots\}$ for which $\pi_{i_s i_{s+1}} = 1$ for all s. We assign a weak topology to Ω_Π. The mapping $\Omega_\Pi \xrightarrow{\varphi} M$, where $\varphi(\omega) = \bigcap_{-\infty}^{\infty} T^n \overline{V}_{i_n}$, is a continuous mapping of Ω_Π onto all M. On a certain set $M_1 \subset M$ of the second category this is a one-to-one mapping, and $\varphi^{-1} | M_1$ is also continuous.

The fundamental class of C-diffeomorphisms is distinguished by the following condition: For some k the matrix Π^k consists of positive elements. C-diffeomorphisms meeting this condition are called transitive. In order for T to be transitive, it is necessary and sufficient that every layer $\Gamma^{(c)}$ and every layer $\Gamma^{(e)}$ be everywhere dense (see Section 3).

In the present article we investigate various invariant measures for transitive C-diffeomorphisms. We consider first the existence of invariant measures associated with smoothness.

Markov partitions make it possible to represent every C-diffeomorphism as a Markov chain, which has, of course, a continuous state space. Specifically, the state of our Markov chain is provided by the semiinfinite intersection $\bigcap_{-\infty}^{0} T^n \overline{V}_{i_n}$. Inasmuch as α_1 is Markovian, it follows that if this intersection is not empty, it represents a CLL D_c in V_{i_0}. It turns out (Section 2) that it is possible on every such D_c to specify $\tilde{\mu}(\cdot \mid D_c)$, uniformly equivalent to a normed Riemann volume on D_c, and a defining "transition probability," from D_c to D'_c is equal to $\tilde{\mu}(TV_{i_1} \mid D_c)$, and 0 in all other cases. For the transition probabilities thus defined the well-known Chapman-Kolmogorov equation from the theory of Markov processes is valid.

In Section 4, we study the Markov chain constructed in Section 2 for transitive C-diffeomorphisms. It turns out that it satisfies a very good regularity condition of the Doeblin type familiar in the theory of Markov chains (see [6]). This enables us to construct an invariant measure for our Markov chain. A second look at the C-diffeomorphism T yields the following theorem:

THEOREM 1. For every transitive C-diffeomorphism T there exists a measure $\tilde{\mu}_c$ positive on every open set and having the following properties:

1) $\tilde{\mu}_c$ is invariant relative to T, and T as an automorphism of a space with measure is a K-automorphism.

2) For every measurable partition ξ of a space M, the elements of which are CLL C_ξ, the conditional measure $\tilde{\mu}_c(\cdot \mid C_\xi)$ induced by $\tilde{\mu}_c$ on almost every C_ξ is equivalent to a normed Riemann volume on C_ξ.

The properties 1) and 2) of the measure $\tilde{\mu}_c$ are uniquely defined.

Replacing the contractile foliation by an expansible foliation, we construct a measure $\tilde{\mu}_e$ having analogous properties. It follows from Theorem 1 that if T has an invariant measure associated with smoothness, it is congruent with both $\tilde{\mu}_c$ and $\tilde{\mu}_e$. Clearly, it is possible to find examples in which the measures $\tilde{\mu}_c$ and $\tilde{\mu}_e$ are distinct and, therefore, singular with respect to the class of smooth measures.

The algebraic automorphisms of compact nilmanifolds and their small perturbations serve as currently known examples of transitive C-diffeomorphisms. The basic problem in the theory of transitive C-diffeomorphisms is to prove that every such C-diffeomorphism is topologically associated with an algebraic counterpart. In the fifth and last section we prove a number of theorems bearing on this fundamental problem.

THEOREM 5.1. On every layer $\Gamma^{(c)}$ of a contractile foliation it is possible to specify a σ-finite measure μ_c having the following properties:

a_1) There exists an algebraic number $h_c > 1$, such that for every $A \subset \Gamma^{(c)}$

$$\mu_c(T^{-1}A) = h\mu_c(A).$$

a_2) If $A \subset \Gamma^{(c)}$ is canonically isomorphic to $\tilde{A} \subset \tilde{\Gamma}^{(c)}$, then $\mu_c(A) = \mu_c(\tilde{A})$ [it is assumed here that $\mu_c(A)$ is a measure on $\Gamma^{(c)}$ and that $\mu_c(\tilde{A})$ is a measure on $\tilde{\Gamma}^{(c)}$].

Replacing the contractile foliation with an expansible foliation, we obtain a measure μ_e on the layers of the expansible foliation and a corresponding constant h_e. It turns out that $h_c = h_e = h$.

The properties a_1 and a_2 demonstrate that the measures μ_c and μ_e have certain homogeneity properties as natural measures induced by the Haar measure for algebraic automorphisms.

The constant h obtained in Theorem 5.1 is related to various properties of the diffeomorphism. For example, the following is true:

THEOREM 5.3. The topological entropy of a diffeomorphism T is equal to ln h.

Theorem 5.1 makes it possible to generate a topological classification of the C-diffeomorphisms of a two-dimensional torus.

THEOREM 5.2. Let M be a two-dimensional torus. Every C-diffeomorphism of M is topologically conjugate with an algebraic automorphism.

In general, the transformation formulated in Theorem 5.2 is singular, i.e., it takes a Lebesgue measure into a singular measure. We state one necessary and sufficient condition in order for it to be absolutely continuous. Specifically, let h'_μ and h''_μ be metric entropies of a diffeomorphism T when the $\widetilde{\mu}_c$ and $\widetilde{\mu}_e$ constructed in Theorem 1 are adopted as the invariant measure.

THEOREM 5.5. If $h = h'_\mu = h''_\mu$, then a homeomorphism taking T into an algebraic C-diffeomorphism is absolutely continuous.

In fact, it can even be shown that it is a diffeomorphism of the class C^r, but we will not concern ourselves with this.

We bring attention, finally, to Theorem 5.4, which bears on algebraic automorphisms of an n-dimensional torus.

THEOREM 5.4. Let M be an n-dimensional torus, T its algebraic automorphism, a C-diffeomorphism. Then T is metrically (i.e., in the measure-theoretic sense) conjugate with a Markov chain having a finite number of states. This Markov chain has the property of maximum entropy among all Markov chains for which the transition probabilities are positive when and only when they are positive for our chain.

Apropos chains of this type, see Parry [7].

The author is grateful to G. A. Margulis for the elimination of a number of errors in the original version of the article. We mention also G. A. Margulis's alternative approcah to the introduction of the measures μ_c and μ_e constructed in Theorem 5.1.

2. Construction of an Increasing Partition and Systems of Conditional Measures for C-Differomorphisms

As shown in [8], for every measure-preserving ergodic diffeomorphism T associated with smoothness and having a transversal foliation Z, there exists a measurable partition ζ with the following three proper properties:

1) $T\zeta > \zeta$.
2) $\prod_k T^k\zeta = \varepsilon \bmod 0$, where ε is a partition into single points.
3) $\bigcap_k T^k\zeta = \nu_Z$, where ν_Z is a measurable hull of a partition of Z into complete layers of the transversal foliation Z.

Making use of the Markov partition, we show that an analogous partition exists for every C-diffeomorphism, without assuming the existence of an invariant measure.

Thus, let T be a C-diffeomorphism, α an arbitrary partition into parallelograms $U_1, U_2, \ldots$. We assume for now that they are closed and that $U_i \cap U_j = \partial U_i \cap \partial U_j$.

LEMMA 2.1. There exists a number $\varepsilon > 0$, depending only on T, such that if the diameter of every LL in U_i is not greater than ε, then the partition α is a generating partition. This means that for any two points x and y, $x \neq y$, there exists a k, such that $T^k x$ and $T^k y$ belong to different parallelograms U_{i_1}, U_{i_2}, $U_{i_1} \cap U_{i_2} = \varnothing$.

Proof. We pick a number d > 0 having the following characteristics: If two points z_1 and z_2 lie on one layer of an expansible (or contractile) foliation and the distance between them in the metric of the layer is contained between d and d max $(\lambda_e, \lambda_c^{-1})$, then the points z_1 and z_2 belong to different parallelograms U_i, and the distance between in the metric M is larger than some $d_1 > 0$. The numbers d and d_1 depend only on T if the diameters of the LL in U_i are sufficiently small.

Let there be given two arbitrary points x and y belonging to the same U_i. We first examine the case when x and y do not lie on the same CLL in U_i. We find a point z, such that x and z lie on one ELL in U_i, while y and z lie on one CLL in U_i (if x and y lie on the same ELL, we let z = y). We pick a k > 0, such that the distance in the metric of the layer between $T^k x$ and $T^k z$ is between d and $d\lambda_e$. Then, by our choice of the number d, the distance between $T^k x$ and $T^k z$ is greater than d_1. On the other hand, the distance between T^k_z and T^k_y is not greater than the distance between them in the metric of the contractile layer and, hence, is not greater than the maximum diameter of the CLL in U_i. If this diameter is smaller than $d_1/4$, then $d(T^k x, T^k y) \geq d(T^k x, T^k z) - d(T^k z, T^k y) \geq d_1 - \frac{d_1}{4} = \frac{3}{4} d_i$. Consequently, $T^k x$ and $T^k y$ lie in different parallelograms U_i. The case when x and y lie on the same ELL is treated analogously, replacing the expansible layer by a contractile layer. This proves the lemma.

It follows from Lemma 2.1 that the infinite intersection $\bigcap_{-\infty}^{\infty} T^n U_{i_n}$ consists of at most one point. Inasmuch as the parallelograms U_i intersect, certain points may be represented in this form nonuniquely. It is readily seen that the set of nonuniqueness points has the form $E = \bigcup_{-\infty}^{\infty} T^n (\bigcup_i \partial U_i)$.

Now let α be a Markov partition satisfying the conditions of Lemma 2.1, and let m be the number involved in its definition. We set $\alpha_1 = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-m+1}\alpha$. The partition α_1 is also a partition into parallelograms. We designate the latter $V_1, V_2, \ldots$. Since $\Gamma_e(T^{-m}\alpha) \subset \Gamma_e(\alpha)$, $\Gamma_e(\alpha_1) = \Gamma_e(\alpha) \cup \Gamma_e(T^{-1}\alpha) \cup \cdots \cup \Gamma_e(T^{-m+1}\alpha)$, then $\Gamma_e(T^{-1}\alpha_1) \subset \Gamma_e(\alpha_1)$, $\Gamma_e(T^{-2}\alpha_1) \subset \Gamma_e(T^{-1}\alpha_1) \subset \Gamma_e(\alpha_1)$ etc., $\Gamma_e(T^{-n}\alpha_1) \subset \Gamma_e(\alpha_1)$ for any n > 0.

We now bring into the discussion the partition $\alpha^- = \prod_{n=0}^{\infty} T^{-n}\alpha$, the elements of which have the form $\bigcap_{n=0}^{\infty} T^{-n}U_{i_r}$. It is clear that $T\alpha^- = T\alpha \cdot \alpha^- \geqslant \alpha^-$. Lemma 2.1 implies that on the uniqueness set $\prod_{k=0}^{\infty} T^k\alpha^- = \prod_{-\infty}^{\infty} T^k\alpha = \varepsilon$. It turns out that α^- has a very simple form in the case of a Markov partition.

LEMMA 2.2. If $x \in V_i$, then an element $C_{\alpha^-}(x)$ of the partition α^-, containing the point x, represents a CLL $D_c(x)$ in V_i.

Proof. We denote by ξ a measurable partition, which on every V_i represents a partition into closed CLL. Plainly, $\alpha_1 \leq \xi$. If C_ξ is an arbitrary element of the partition ξ, then $T^{-n}C_\xi$ is representable as a sum of integer-valued elements C'_ξ of the same partition ξ, for otherwise there would exist a C'_ξ for which $C'_\xi \cap T^{-n}C_\xi \neq \varnothing$, $C'_\xi \setminus T^{-n}C_\xi \neq \varnothing$. But then part of $\partial(T^{-n}C_\xi)$ would lie strictly inside C'_ξ, and this is impossible, inasmuch as $\partial(T^{-n}C_\xi) \in \Gamma_p(T^{-n}\alpha_1) \subset \Gamma_p(\alpha_1)$. Consequently,

$$T^{-n}\xi \leqslant \xi,\ T^{-n}\alpha_1 \leqslant T^{-n}\xi \leqslant \xi \text{ и } \alpha_1^- = \prod_{k=0}^{\infty} T^{-k}\alpha_1 = \prod_{k=0}^{\infty} T^{-k}\alpha \leqslant \xi.$$

The truth of the equality $\prod_{k=0}^{\infty} T^{-k}\alpha = \xi$ is established most easily by referring to the proof of Lemma 2.1. It was shown there that if two points x and y belong to different elements of the partition ξ, then there exists a k > 0, such that $T^k x \in V_{i_1}$, $T^k y \in V_{i_2}$ for $V_{i_1} \cap V_{i_2} = \varnothing$. But then x and y lie in different elements of the partition $T^{-k}\alpha_1$. Consequently, $\alpha_1^- \geq \xi$, i.e., $\alpha_1^- = \alpha^- = \xi$, and the lemma is thus proved.

We have therefore constructed a partition α^- having the first two properties assigned to the partition ξ at the beginning of this section.

We now show that it is possible to define a natural "good" normed measure on every C_{α^-}.

LEMMA 2.3. On every C_{α^-} it is possible to define a normed measure $\tilde{\mu}(\cdot \mid C_{\alpha^-})$ having the following properties:

α_1) The measure $\tilde{\mu}$ is equivalent to the measure generated on a CLL by the Riemann volume; moreover, there exist constants C_1 and C_2, depending only on T, such that

$$C_1 \frac{\sigma_c(A \cap C_{\alpha^-})}{\sigma_c(C_{\alpha^-})} \leqslant \tilde{\mu}(A \mid C_{\alpha^-}) \leqslant C_2 \frac{\sigma_c(A \cap C_{\alpha^-})}{\sigma_c(C_{\alpha^-})}.$$

Here σ_c denotes the Riemann volume on the CLL; the density $\tilde{f}$ of the measure with respect to normed σ_c satisfies the Lipschitz condition with a fixed constant.

α_2) The measures $\tilde{\mu}$ are compatible in the following sense:* We consider $T^{-1}C_{\alpha^-} = C_{T^{-1}\alpha^-}$ and denote by $\tilde{\mu}(\cdot \mid C_{T^{-1}\alpha^-})$ the measure generated on $T^{-1}C_{\alpha^-} = C_{T^{-1}\alpha^-}$ by the natural transference of the measure $\tilde{\mu}(\cdot \mid C_{\alpha^-})$ to $C_{T^{-1}\alpha^-}$, i.e., $\tilde{\mu}(A \mid T^{-1}C_{\alpha^-}) = \tilde{\mu}(TA \mid C_{\alpha^-})$; then for every $A \subset C'_{\alpha^-} \subset C_{T^{-1}\alpha^-}$

$$\tilde{\mu}(A \mid C_{T^{-1}\alpha^-}) = \tilde{\mu}(A \mid C'_{\alpha^-})\, \tilde{\mu}(C'_{\alpha^-} \mid C_{T^{-1}\alpha^-}).$$

The properties α_1) and α_2) of the measure $\tilde{\mu}$ are uniquely defined.

Proof. Taking $C_{\alpha^-}(x)$, we construct $T^n C_{\alpha^-}(x) = C_{T^n\alpha^-}(T^n x)$. Clearly, T^n specifies a one-to-one mapping of $C_{\alpha^-}(x)$ onto $C_{T^n\alpha^-}(T^n x)$. On every $C_{T^n\alpha^-}(T^n x)$ we consider the normed measure generated by the Riemann volume, along with its image on $C_{\alpha^-}(x)$ under the action of T^{-n}. We designate this image μ_n.

For every n the measure μ_n is equivalent to the Riemann volume on $C_{\alpha^-}(x)$ because T^n is a continuous mapping. We denote the density of the measure μ_n with respect to the normed volume $\sigma_c \mid C_{\alpha^-}$ by f_n. It is seen at once that

$$\frac{f_n(y_1)}{f_n(y_2)} = \frac{\Delta_c(y_1) \ldots \Delta_c(T^{n-1}y_1)}{\Delta_c(y_2) \ldots \Delta_c(T^{n-1}y_2)}, \tag{1}$$

where $\Delta_c(y)$ is the Jacobian transformation of the Reimann volume on a CLL passing through y into the Riemann volume on a CLL passing through Ty. Of course, $\Delta_c(y) < 1$. By virtue of the smoothness of T and the properties of a contractile layer, for certain constants $C^{(3)} < \infty$ and $0 < \rho_1 < 1$

$$\left| \frac{\Delta_c(T^k y_1)}{\Delta_c(T^k y_2)} - 1 \right| \leqslant C^{(3)} \rho_c(T^k y_1, T^k y_2) \leqslant C^{(3)} \Lambda_c \lambda_c^k,$$

where Λ_c is the maximum diameter of the CLL C_{α^-}. It follows from this that $f_n(y_1)/f_n(y_2) \leqslant \prod_{k=0}^{\infty} (1 + C^{(3)} \Lambda_c \lambda_c^k) = C^{(4)}$. If we pick the y_1 for which $f_n(y_1) = 1$ $((\sigma_c(C_{\alpha^-}))^{-1} \int_{C_{\alpha^-}} f_n(x)\, d\sigma_{C_{\alpha^-}}(x) = 1$, we finally obtain

$$1/C^{(4)} \leqslant f_n(y) \leqslant C^{(4)}. \tag{2}$$

We now show that the functions f_n converge uniformly to C_{α^-}. We do this by analyzing the image of a normed measure on $C_{T^m\alpha^-}(T^m x)$ under the action of $T^{-(n-m)}$. This is a normed measure on $C_{T^n\alpha^-}(T^n x)$. Arguing as before, we find that the ratio of the values of the density of this measure at two points y_1, $y_2 \in C_{T^m\alpha^-}(T^m x)$ is equal to

$$\frac{\Delta_c(y_2) \ldots \Delta_c(T^{n-m-1}y_2)}{\Delta_c(y_1) \ldots \Delta_c(T^{n-m-1}y_1)}.$$

* The relation introduced below is the analog of the Chapman-Kolmogorov equation in the theory of Markov processes.

The latter ratio does not differ from unity by more than $C^{(5)}\Lambda_c\lambda_c^m$; $C^{(5)}$ is another constant. Consequently, there occurs on $C_{T^m\alpha^-}(T^m x)$ a measure whose density (with respect to the normed Riemann volume) does not differ from unity by more than $C^{(6)}\Lambda_c\lambda_c^m$. Under the action of T^m the density of this measure and unity, i.e., also a density with respect to the normed volume, are multiplied by the same number. Hence, we obtain

$$\left|\frac{f_n(x)}{f_m(x)}-1\right|\leqslant C^{(6)}\Lambda_c\lambda_c^m.$$

Consequently, the functions $f_n(x)$ converge uniformly, and our statement is proved.

We denote the limit of $f_n(x)$ by $\widetilde{f}_{C_{\alpha^-}}(x)=\widetilde{f}(x)$ and denote the measure $\widetilde{\mu}(A\,|\,C_{\alpha^-})=\frac{1}{\sigma_c(C_{\alpha^-})}\int\limits_{A\cap C_{\alpha^-}}\widetilde{f}(x)\,d\sigma_c(x)$ induced by that limit by $\widetilde{\mu}(\cdot\,|\,C_{\alpha^-})$.

It follows from (1) and (2) that $C_1\leq\widetilde{f}\leq C_2$, and $\widetilde{f}$ satisfies the Lipschitz condition with a fixed constant, inasmuch as this is true for $f_n(x)$. Moreover,

$$|\widetilde{f}(x)-f_n(x)|\leqslant C^{(6)}\Lambda_c\lambda_c^n. \tag{3}$$

Thus α_1) is proved.

We turn now to the property α_2). We have

$$\widetilde{\mu}(A\,|\,C_{T^{-1}\alpha^-}(x))=\widetilde{\mu}(TA\,|\,C_{\alpha^-}(Tx))=\lim_{n\to\infty}\mu_n'(TA\,|\,C_{\alpha^-}(Tx))=\lim_{n\to\infty}[\mu_n'(TA\,|\,C_{T\alpha^-}(Tx))\,\mu_n(C_{T\alpha^-}(Tx)\,|\,C_{\alpha^-}(Ty))].$$

Here μ_n' is the nominal measure induced on $C_{T\alpha^-}$ by the measure μ_n on C_{α^-}. But it quickly follows from the definition of μ_n that

$$\mu_n'(TA\,|\,C_{T\alpha^-}(Tx))=\mu_{n-1}(A\,|\,C_{\alpha^-}(x))=\mu_{n-1}(A\,|\,C_{\alpha^-}),$$

and passage to the limit as $n\to\infty$ gives the required relation.

It now remains for us to demonstrate the uniqueness of the measure $\widetilde{\mu}$. Making use of the fact that f satisfies the Lipschitz condition, we have

$$\widetilde{\mu}(A\,|\,C_{\alpha^-}(x))=\widetilde{\mu}(T^nA\,|\,T^nC_{\alpha^-}(x))=\widetilde{\mu}(T^nA\,|\,C_{T^n\alpha^-}(T^nx))$$

$$=\int\limits_{T^nA\cap C_{T^n\alpha^-}(T^nx)} f_{C_{\alpha^-}(T^nx)}(y)\,d\sigma_c(y)\cdot\left(\int\limits_{C_{T^n\alpha^-}} f_{C_{\alpha^-}(T^nx)}(y)\,d\sigma_c(y)\right)^{-1}$$

$$=\int\limits_{T^nA\cap C_{T^n\alpha^-}(T^nx)}\frac{f_{C_{\alpha^-}(T^nx)}(y)\,d\sigma_c(y)}{f_{C_{\alpha^-}(T^nx)}(x)}\cdot\left(\int\limits_{C_{T^n\alpha^-}}\frac{f_{C_{\alpha^-}(T^nx)}(y)\,d\sigma_c(y)}{f_{C_{\alpha^-}(T^nx)}(T^nx)}\right)^{-1}$$

and

$$|\widetilde{\mu}(A\,|\,C_{\alpha^-}(x))-\mu_n(A\,|\,C_{\alpha^-}(x))|\leqslant 2\,\mathrm{const}\,\mathrm{diam}\,C_{T^n\alpha^-}(T^nx)\to 0\quad\text{as}\quad n\to\infty.$$

If there existed a measure $\widetilde{\widetilde{\mu}}$ having the same properties, then for it also

$$|\widetilde{\widetilde{\mu}}(A\,|\,C_{\alpha^-}(x))-\mu_n(A\,|\,C_{\alpha^-}(x))|\leqslant 2\,\mathrm{const}\,\mathrm{diam}\,(C_{T^n\alpha^-}(T^nx)),$$

whence we obtain $\widetilde{\mu}(A\,|\,C_{\alpha^-}(x))=\widetilde{\widetilde{\mu}}(A\,|\,C_{\alpha^-}(x))$. The lemma is thus proved.

It is not a difficult matter to refine Lemma 2.3, showing that the density $\widetilde{f}(x)$ has on every CLL a smoothness class smaller by unity than the layer of our foliation.

In this section we also wish to discuss the dependence of the density $\tilde{f}_{C_{\alpha^-}}(x)$ on C_{α^-}. We examine the partition $\alpha^0_{-n} = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-n}\alpha = \alpha_1 \cdot T^{-1}\alpha_1 \cdot \ldots \cdot T^{-n+m}\alpha_1$ for $n > m$. This is a partition into smaller parallelograms than the parallelograms V_i representing the elements of the partition α_1. Since α is a Markov partition, every element $\Delta^0_{-n} \subset V_i$ of the partition α^0_{-n} consists of CLL in V_i. Consequently, if $C_c^{(i)}$ and $C_e^{(i)}$ are, respectively, the CLL and ELL defining the V_i, then every Δ^0_{-n} is defined by a subset of $C_e^{(i)}$ whose diameter does not exceed $\Lambda_e \lambda_e^{-n}$, where Λ_e is the maximum diameter of the Ell in all the U_i. Hence, it follows in particular that different CLL $C'_{\alpha^-} \in \Delta^0_{-n}$ and $C''_{\alpha^-} \in \Delta^0_{-n}$ are canonically isomorphic.

LEMMA 2.4. Let Δ^0_{-n} be an element of the partition α^0_{-n}, and let $C'_{\alpha^-}, C''_{\alpha^-} \in \Delta^0_{-n}$. We denote by $\pi: C'_{\alpha^-} \to C''_{\alpha^-}$ the canonical isomorphism between C'_{α^-} and C''_{α^-}. Then there exist constants C_3, $\varkappa > 0$, such that

$$\left| \frac{\tilde{\mu}(A \mid C'_{\alpha^-})}{\tilde{\mu}(\pi(A) \mid C''_{\alpha^-})} - 1 \right| \leqslant C_3 \max(\Lambda_c, \Lambda_e)(\max(\lambda_c, \lambda_e^{-1}))^{n\varkappa/2}.$$

Proof. Consider $T^{\left[\frac{n}{2}\right]} C'_{\alpha^-}$ and $T^{\left[\frac{n}{2}\right]} C''_{\alpha^-}$. It follows from Lemma 2.3 and from (3) that for any $q > 0$

$$\left| \frac{\tilde{\mu}(A \mid C'_{\alpha^-})}{\mu_q(A \mid C'_{\alpha^-})} - 1 \right| \leqslant C^{(6)} \Lambda_c \lambda_c^q, \quad \left| \frac{\tilde{\mu}(\pi(A) \mid C''_{\alpha^-})}{\mu_q(\pi(A) \mid C''_{\alpha^-})} - 1 \right| \leqslant C^{(6)} \Lambda_c \lambda_c^q.$$

Consequently,

$$\frac{\mu_q(A \mid C'_{\alpha^-})(1 - C^{(6)}\Lambda_c\lambda_c^q)}{\mu_q(\pi(A) \mid C''_{\alpha^-})(1 + C^{(6)}\Lambda_c\lambda_c^q)} \leqslant \frac{\tilde{\mu}(A \mid C'_{\alpha^-})}{\tilde{\mu}(\pi(A) \mid C''_{\alpha^-})} \leqslant \frac{\mu_q(A \mid C'_{\alpha^-})(1 + C^{(6)}\Lambda_c\lambda_c^q)}{\mu_q(\pi(A) \mid C''_{\alpha^-})(1 - C^{(6)}\Lambda_c\lambda_c^q)}.$$

By the definition of μ_q (see Lemma 2.3),

$$\mu_q(A \mid C'_{\alpha^-}) = \frac{\sigma_c(T^q(A \cap C'_{\alpha^-}))}{\sigma_c(T^q C'_{\alpha^-})}, \ \mu_q(\pi(A) \mid C''_{\alpha^-}) = \frac{\sigma_c(T^q(\pi(A) \cap C''_{\alpha^-}))}{\sigma_c(T^q C''_{\alpha^-})}.$$

We set $q = \left[\frac{n}{2}\right]$ and compare $\sigma_c(T^q C'_{\alpha^-})$ and $\sigma(T^q C''_{\alpha^-})$; $\sigma(T^q(A \cap C'_{\alpha^-}))$, and $\sigma(T^q(\pi(A) \cap C''_{\alpha^-}))$. Clearly, $T^q C'_{\alpha^-}$ and $T^q C''_{\alpha^-}$, as well as $T^q(A \cap C'_{\alpha^-})$ and $T^q(\pi(A) \cap C''_{\alpha^-})$, are canonically isomorphic. Therefore,

$$\sigma_c(T^q C'_{\alpha^-}) = \int_{T^q C''_{\alpha^-}} I(x)\, d\sigma_{C''_{\alpha^-}}(x), \quad \sigma_c(T^q(A \cap C'_{\alpha^-})) = \int_{T^q(\pi(A) \cap C''_{\alpha^-})} I(x)\, d\sigma_{C''_{\alpha^-}}(x),$$

where the expression for $I(x)$ is given in [4] [Eq. (5.3)]. It follows from this expression and from [10] that if $d_e(x, \pi(x)) \leq d$, then $|I(x) - I| \leq C^{(7)} d^{\varkappa}$ for certain constants $C^{(7)}$, $\varkappa > 0$. Inasmuch as the point $\pi(x)$ lies in the same element of the partition α^0_q for $x \in T^q C'_{\alpha^-}$, we can let $\Lambda_e \lambda_e^{-n/2}$ represent d. We can regard the dimensions of the original parallelograms U_i as so small that $C^{(6)}\Lambda_c < 1/2$, $C^{(7)}\Lambda_e < 1/2$. Then, assembling the ensuing inequalities, we have for a certain constant C_3, depending on $C^{(6)}$ and $C^{(7)}$,

$$\frac{\tilde{\mu}(A \mid C'_{\alpha^-})}{\tilde{\mu}(\pi(A) \mid C''_{\alpha^-})} \leqslant \frac{\mu_q(A \mid C'_{\alpha^-})}{\mu_q(\pi(A) \mid C''_{\alpha^-})} \cdot \frac{1 + C^{(6)}\Lambda_c\lambda_c^{n/2}}{1 - C^{(6)}\Lambda_c\lambda_c^{n/2}}$$

$$\leqslant \frac{(1 + C^{(6)}\Lambda_c\lambda_c^{n/2})(1 + C^{(7)}\Lambda_e\lambda_e^{-n\varkappa/2})}{(1 - C^{(6)}\Lambda_c\lambda_c^{n/2})(1 - C^{(7)}\Lambda_e\lambda_e^{-n\varkappa/2})} \leqslant 1 + C_3 \max(\Lambda_c, \lambda_e)[\max(\lambda_c, \lambda_e^{-1})]^{n\varkappa/2}.$$

A lower estimate is deduced analogously. This proves the lemma.

3. Symbolic Dynamics for C-Diffeomorphisms

In the preceding section we analyzed a system of closed parallelograms U_i of sufficiently small diameter $U_i \cap U_j = \partial U_i \cap \partial U_j$. We now consider the open kernels of these parallelograms forming the actual partition of the set $M_1 = M \quad E$, where $E = \bigcup_n T^n (\bigcup_i \partial U_i)$. We denote these open kernels by the same letters U_i. Lemma 2.1 states that every point $x \in M_1$ is uniquely representable in the form $x = \bigcap_{-\infty}^{\infty} T^n \bar{U}_{i_n}$ $= \bigcap_{-\infty}^{\infty} T^n U_{i_n}$, which is equivalent to $T^{-n} x \in \bar{U}_{i_n}, -\infty < n < \infty$. The set M_1 is an everywhere dense set of the type G_δ.

If we now go from the Markov partition α to the Markov partition $\alpha_1 = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-m+1}\alpha$ into parallelograms $V_1, V_2, \ldots, V_r$, then analogously $x = \bigcap_{-\infty}^{\infty} T^n \bar{V}_{i_n} = \bigcap_{-\infty}^{\infty} T^n V_{i_n}$ for $x \in M_1$.

Definition 3.1. An intersection matrix $\Pi = \|\pi_{ij}\|$ of the partition α_1 is defined as a matrix for which $\pi_{ij} = 1$ if $V_i \cap TV_j \neq \varnothing$, otherwise $\pi_{ij} = 0$.

THEOREM 3.1. Consider the space Ω_Π of infinite sequences $\omega = \{\ldots i_{-n} \ldots i_0 \ldots i_n \ldots\}$, such that $\pi_{i_s i_{s+1}} = 1$ for all s, and introduce a weak topology in Ω_Π. The mapping $\Omega_\Pi \xrightarrow{\varphi} M$, operating according to the formula $\varphi(\omega) = \bigcap_n T^n \bar{V}_{i_n}$, is a one-to-one mapping of Ω_Π onto all M. On the set M_1, it is a one-to-one mapping, and $\varphi^{-1}|M_1$ is continuous.

Proof. We refer to the subset Ω of those ω for which $i_{n_s} = l_{n_s}$, $s = 1, \ldots, k$, as a cylinder $\{l_{n_1} \ldots l_{n_k}\}$. We now show that $\{l_{n_1} \ldots l_{n_s}\} \in \Omega$ is not empty when and only when $T^{n_1} V_{l_1} \cap \ldots \cap T^{n_k} V_{l_k} \neq \varnothing$.

It is sufficient for the proof of this assertion to investigate cylinders of the form $\{l_0, l_1, \ldots l_n\}$. For n = 1 the assertion follows from the definition of Ω and Π. Assume that it has been proved for some n. Inasmuch as α_1 is Markovian, the intersections $V_{l_0} \cap TV_{l_1}$ consist of integer-valued ELL in V_{l_0}. Moreover, if D'_e and D''_e are arbitrary ELL in V_{l_1}, the intersections $TD'_e \cap V_{l_0}$ and $TD''_e \cap V_{l_0}$ are both empty or both nonempty at the same time. In fact, if $A = \{D'_e : TD'_e \cap V_{l_0} \neq \varnothing\}$ is not congruent with all of V_{l_1}, then $A \cap T^{-1} V_{l_0} = V_{l_1} \cap T^{-1} V_{l_0}$ is a parallelogram, part of whose boundary Γ_c lies inside V_{l_1}, contrary to the definition of the Markov partition. Furthermore,

$$V_{l_0} \cap TV_{l_1} \cap \ldots \cap T^{n+1} V_{l_{n+1}} = V_{l_0} \cap T(V_{l_1} \cap \ldots \cap T^n V_{l_{n+1}}).$$

For the intersection in the parentheses our assertion is proved by the induction hypothesis. Moreover, it is clear from the foregoing that this intersection consists of ELL lying inside V_{l_1}. But, as we have already witnessed, the intersection $V_{l_0} \cap TV_{l_1} \neq \varnothing$ means that for every ELL $D'_e \subset V_{l_1}$ the intersection $V_{l_0} \cap TD'_e \neq \varnothing$. Consequently, $V_{l_0} \cap \ldots \cap T^{n+1} V_{l_{n+1}} \neq \varnothing$, and our assertion is proved.

The assertion just proved implies that for every point $\omega \in \Omega_\Pi$ the intersection $\bigcap T^n \bar{V}_{i_n}$ actually defines a certain point $x \in M$, which is unique by virtue of Lemma 2.1. If $x \in E$, then the representation of x in the form $\bigcap_n T^n \bar{V}_{i_n}$ can be nonunique. On the set M_1 it is clearly unique. The continuity of $\varphi^{-1}| M_1$ follows from the fact that the diameter of every parallelogram $T^{-n} V_{i_n} \cap \ldots \cap T^n V_{i_n}$ tends to zero as $n \to \infty$. The theorem is thus proved.

Definition 3.2. A C-diffeomorphism T is called transitive if there exists a k_0 for which all elements of the matrix Π^{k_0} are positive definite.

It follows from the foregoing that the transitivity condition is equivalent to the assertion that $U_i \cap T^{k_0} U_j \neq \varnothing$ for any i and j.

We now expound necessary and sufficient conditions for the transitivity of T.

THEOREM 3.2. A C-diffeomorphism T is transitive when and only when every layer of every foliation thereof is everywhere dense.

Proof. We call a foliation almost periodic if for any $\varepsilon > 0$ there exists and $R(\varepsilon)$, such that every sphere of radius $R(\varepsilon)$ on any layer of our foliation forms an ε-net in M. We show first that T is transitive when and only when both foliations are almost periodic.

We denote by $U_{-\delta}(\alpha)$ the set of points separated from the boundary $\Gamma(\alpha)$ by a distance no smaller than δ. Let δ be so small that $U_{-\delta}(\alpha) \cap U_i \neq \varnothing$ for every i.

We assume now that both foliations are almost periodic. We pick an arbitrary point x and find an R, such that the sphere $D_c(x; R)$ of radius R on a contractile layer intersects all sets $U_{-\delta}(\alpha) \cap U_i$. Expanding $D_c(x; R)$ slightly, we find that for the expanded set $D'_c(x; R)$ its intersection with every U_i would be a sum of CLL $D_c(z)$ from U_i, i = 1, 2, ... Taking a sphere $D_e(x; \delta_1)$ of sufficiently small radius δ_1, we draw CLL canonically isomorphic to $D'_c(x; R)$ through all points $x \in D_e(x; \delta_1)$. For the ensuing parallelogram W the connected components of the intersection $W \cap U_i \neq \varnothing$ and represent parallelograms for which the CLL are CLL in U_i; the number of these parallelograms is equal to the number of CLL $D'_c(x; R) \cap U_i$. Let us consider $T^k W$. We assume k to be large enough that 1) $\operatorname{diam}_c(T^k D_c) < \delta/2$ for every CLL $D_c \in W$; 2) $T^k D_e(x; \delta_1) \cap (U_{-\delta}(\alpha) \cap U_j) \neq \varnothing$ for every j. Fulfillment of the latter condition is guaranteed by the almost-periodicity of the foliations. We now show that $T^k U_j \cap U_i \neq \varnothing$ for arbitrary i, j. We pick some parallelogram W_j representing the connected component of the intersection $W \cap U_j$. It is enough to show that $T^k W_j \cap U_i \neq \varnothing$. Inasmuch as the ELL D_e in W_j are canonically isomorphic to $D_e(x; \delta_1)$ and this isomorphism is established by means of the CLL D_c, it follows that $T^k D_e$ is canonically isomorphic to $T^k(D_e(x; \delta))$, and the distance between corresponding points does not exceed $\delta/2$. Hence, it follows that if $T^k D_e(x; \delta_1) \cap (U_{-\delta}(\alpha) \cap U_i) \neq \varnothing$, then $T^k D_e \cap U_i \neq \varnothing$ for every ELL $D_e \subset W_j$, i.e., $T^k W_j \cap U_i \neq \varnothing$. Consequently, T is transitive.

We assume now that T is transitive. We investigate the partition $T^{-t}\alpha \cdot \ldots \cdot T^t \alpha$. For sufficiently large t the diameter of each of its elements Δ^t_{-t} is not greater than $\varepsilon/2$. We pick a point $x \in M_1$. Then the CLL $C_{\alpha^-}(x) = \bigcap_{-\infty}^{0} T^n U_{i_n}$.

We examine $C_{T^{-t-k_0}\alpha^-}(x) = \bigcap_{-\infty}^{-t-k_0} T^n U_{i_n}$. Inasmuch as T is transitive, for any nonempty Δ^t_{-t} the intersection $C_{T^{-t-k_0}\alpha^-}(x) \cap \Delta^t_{-t} \neq \varnothing$. Consequently, $C_{T^{-t-k_0}\alpha^-}(x)$ forms an $\varepsilon/2$-net. Since $C_{T^{-t-k_0}\alpha^-}(x) = T^{-t-k_0}(C_{\alpha^-}(T^{t+k_0}x))$, the diameter of the CLL $C_{T^{-t-k_0}\alpha^-}(x)$ is not greater than $\Lambda_c \lambda_c^{-(t+k_0)}$ where Λ_0 is the maximum diameter of the CLL in all U_i. Consequently, the sphere $D_c(x; \Lambda_c \lambda_c^{-(t+k_0)})$ forms an $\varepsilon/2$-net.

If $x \notin M_1$, we investigate $T^{t+k_0} x \in \bar{U}_i$. We find an interior point $x_i \in M_1$ in $\bar{U}_i$. The CLL $D_c(T^{t+k_0}x)$ and CLL $D_c(x_i)$ in $\bar{U}_i$ are canonically isomorphic. Then $T^{-(t+k_0)} D_c(T^{t+k_0}x)$ and $T^{-(t+k_0)} D_c(x_i)$ are canonically isomorphic. If $t = t(\varepsilon)$ is sufficiently large, the distance between corresponding points in the metric of the expansible layer is not greater than $\varepsilon/2$. Since, $x_i \in M_1$, it follows that $T^{t+k_0} x_i \in M_1$, and, by the earlier proof, $C_{T^{-(t+k_0)}\alpha^-}(T^{-(t+k_0)} x_i)$ forms an $\varepsilon/2$-net. Consequently, $T^{-(t+k_0)} D_c(T^{t+k_0}x)$ is not greater than $\Lambda_c \lambda_c^{-(t+k_0)}$. This proves that the contractile foliation is almost periodic.

Expansible layers are investigated analogously.

We now show that the almost-periodicity of a foliation follows from the everywhere-density of each of its layers. Let $\varepsilon > 0$ be given. Then for every $x \in M$ there exists an $R(\varepsilon, x)$, such that the sphere of radius $R(\varepsilon, x)$ with center at the point x will form an ε-net. Picking $R'(\varepsilon, x) = 2R(\varepsilon, x)$, we find a neighborhood U_x of the point x, such that for every $y \in U_x$ the sphere of radius $R'(\varepsilon, x)$ with center at the point y will form an ε-net. This is admissible by the continuity of the foliation. The set of neighborhoods U_x forms a cover of M. Choosing a finite subcover from it, we find a single R for all x. This proves the theorem.

Throughout the remainder of this article we consider only transitive C-diffeomorphisms.

For $x \in M_1$ the element $C_{\alpha^-}(x)$ has the form $C_{\alpha^-}(x) = \bigcap_{-\infty}^{0} T^n U_{i_n}$. Any intersection $C_{\alpha^-}(x) \cap \bigcap_{k_0+1}^{\infty} T^n U_{i_n}$ is nonempty as long as $\bigcap_{k_0+1}^{\infty} T^n U_{i_n} \neq \varnothing$, and it consists of a finite number of points. Such an intersection represents an element of a measurable partition into $C_{\alpha^-}(x)$. The basis of this partition comprises all possible finite intersections $\bigcap_{k+1}^{l} T^n U_{i_n}$, $l \geqslant k_0 + 1$. We let $\mathfrak{S}_{k_0+1}^{\infty}$ represent the σ-algebra generated in M by these sets. We examine as a measure on $C_{\alpha^-}(x)$ the conditional measure $\sigma(\cdot \mid C_{\alpha^-}(x))$ generated by the Riemann volume σ in M. By virtue of the absolute continuity of the foliations (see [4]), it is equivalent to the measure σ_C generated by the Riemann volume on the CLL $C_{\alpha^-}(x)$. We inspect the values of this measure on $\mathfrak{S}_{k_0+1}^{\infty}$. It turns out that these measures are equivalent for different $C_{\alpha^-}(x)$. More precisely, there exist constants C_4 and C_5 (the indexing of the constants is based on technical considerations), such that for every $\Delta \in \mathfrak{S}_{k+1}^{\infty}$ and any x', $x'' \in M_1$

$$C_4 \leqslant \frac{\sigma(\Delta \mid C_{\alpha^-}(x'))}{\sigma(\Delta \mid C_{\alpha^-}(x''))} \leqslant C_5.$$

The proof of these inequalities by and large reiterates the proof given in [4] for the absolute continuity of a canonical isomorphism. We, therefore, present only the highlights. We observe that $\sigma(\cdot \mid C_{\alpha^-})$ is uniformly equivalent with respect to C_{α^-} to the Riemann volume (see [4], Lecture 5). For a volume element $d\sigma_C^{(n)}$ on the layer $T^{-n}C_{\alpha^-}(x)$ and its image $d\sigma_C^{(0)}$ on $C_{\alpha_-}(x')$ which contains the point x,

$$d\sigma_C^{(n)} = \frac{d\sigma_C^{(0)}}{\Delta_c(T^{n-1}x) \cdot \ldots \cdot \Delta_c(x)}.$$

We pick points $x'_1, \ldots, x'_{s_1}$, and $x_1'', \ldots, x''_{s_2}$ representing the intersections $C_{\alpha^-}(x') \cap \bigcap_{n+1}^{\infty} T^n U_{i_n}$ and $C_{\alpha^-}(x'') \cap \bigcap_{n+1}^{\infty} T^n U_{i_n}$, respectively. Proceeding as in [4], we verify that the ratio of the products Δ_C for different points x'_j, x''_j is uniformly bounded from above and below. Inasmuch as the layers $T^{-n}C_{\alpha^-}(x')$ and $T^{-n}C_{\alpha^-}(x'')$ are uniformly smooth with respect to n, it is possible to pick "identical" $d\sigma_C^{(n)}(x)$ and to deduce that the ratio of the corresponding differentials $d\sigma_C^{(0)}$ at different points x'_i and x''_j is uniformly bounded from above and below. The required assertion is a consequence of this and a general lemma from measure theory [4, 9].

4. Construction of an Invariant Measure (Proof of Theorem 1)

In Section 2 we constructed a partition α^- into CLL C_{α^-} and the measures $\tilde{\mu}(\cdot \mid C_{\alpha^-})$ on the elements of this partition. We note that α^- is not a partition in the fullest sense of the word; the intersection $\bar{C}'_{\alpha^-} \cap \bar{C}''_{\alpha^-} = \partial C'_{\alpha^-} \cap \partial C''_{\alpha^-}$ can also be nonempty. Of course, there is a real partition on the set $M_1 = M \setminus E\alpha^-$.

Below, when speaking of M as a space with measure, we are concerned with the measure generated by the normed Riemann volume σ, unless special mention is made to the contrary.

Let α be a Markov partition, $U_1, U_2, \ldots$ its elements. We now consider open U'_i, $\bar{U}'_i = U_i$. On the set M_1 the parallelograms U'_i form a real partition. Inasmuch as LL are admissible in U_i (see Section 1), for every $x \in M_1$ we have $\sigma_c(M_1 \mid \bar{C}_{T^{-n}\alpha^-}(x)) = \tilde{\mu}(M_1 \mid C_{T^{-n}\alpha^-}(x)) = 1$.

We designate $\alpha_{l_1}^{l_2} = T^{l_1}\alpha \cdot \ldots \cdot T^{l_2}\alpha$ for $l_1 \leq l_2$ (the possibility of l_1 and l_2 being equal to infinity is not excluded). It is clear that $\alpha_{l_1}^{l_2}$ is a partition of M_1 and that $T^l \alpha_{l_1}^{l_2} = \alpha_{l_1+l}^{l_2+l}$. We denote the elements of the partition $\alpha_{l_1}^{l_2}$ by $\Delta_{l_1}^{l_2}$.

Let A be an arbitrary element of the partition $\alpha_{l_1}^{l_2}$. We now wish to show that for every $x \in M_1$ there exists a $\lim_{n\to\infty} \tilde{\mu}(A \mid C_{T^{-n}\alpha^-}(x))$ independent of x. Since $\tilde{\mu}(A \mid C_{T^{-n}\alpha^-}(x)) = \tilde{\mu}(T^{-l_1}A \mid C_{T^{-n-l_1}\alpha^-}(T^{-l_1}x))$, it suffices to investigate those A which are elements of the partition α_l^0.

Let t be sufficiently large and fixed (the limitations on the values of t are indicated in the course of the proof). The collection of numbers $\tilde{\mu}(\Delta^{0}_{-t+1} | C_{T-n_{\alpha^-}}(x))$ represents the probability distribution of $\tilde{\mu}_n(x)$ on the elements of the partition α^{0}_{-t+1}. We prove the following lemma.

LEMMA 4.1. Let $\delta > 0$ be given. Then there exist $t_0(\delta)$ and $n_0(t, \delta) = n_0$, such that for any $t > t_0$; $n_1, n_2 > n_0(t, \delta)$; $x_1, x_2 \in M_1$

$$\mathrm{Var}\,(\tilde{\mu}_{n_1}(x_1) - \tilde{\mu}_{n_2}(x_2)) \leqslant \delta.$$

Proof. We have

$$\tilde{\mu}(\Delta^{0}_{-t+1} | C_{T-n_{\alpha^-}}(x))$$

$$= \sum_{\Delta^{-t}_{-2t-1}} \tilde{\mu}(\Delta^{-t}_{-2t-1} | C_{T-n_{\alpha^-}}(x)) \cdot \tilde{\mu}(\Delta^{0}_{-t+1} | \Delta^{-t}_{-2t-1} \cap C_{T-n_{\alpha^-}}(x)). \tag{4}$$

The collection of numbers $\tilde{\mu}(\Delta^{-t}_{-2t-1} | C_{T-n_{\alpha^-}}(x)) = \tilde{\mu}(T^t \Delta^{-t}_{-2t-1} | C_{T-n+t_{\alpha^-}}(T^t x))$ is nothing more than $\tilde{\mu}_{n-t}(T^t x)$. Furthermore, the collections of numbers $\tilde{\mu}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{-t}_{-2t-1} \cap C_{T-n_{\alpha^-}}(x)) = \tilde{\mu}(T^t \Delta^{0}_{-t+1} | T^t \tilde{\Delta}^{-t}_{-2t-1} \cap C_{T-n+t_{\alpha^-}}(T^t x))$ may be regarded as matrix elements $p_{n,x}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1})$, $\tilde{\Delta}^{0}_{-t+1} = T^t \tilde{\Delta}^{-t}_{-2t-1}$ of some stochastic matrix $P_{n,x}$. Using this matrix, we can rewrite (4) in the form

$$\tilde{\mu}_n(x) = \tilde{\mu}_{n-t}(T^t x) P_{n,x} = \tilde{\mu}_{n-2t}(T^{2t} x) P_{n-t, T^t x} \cdot P_{n,x} = \ldots = \tilde{\mu}_{n-ct}(T^{ct} x) P_{n-ct, T^{ct} x} \cdot \ldots \cdot P_{n,x}; \; c = \text{integer}. \tag{5}$$

Consequently, our problem reduces to the investigation of a product of stochastic matrices. We establish first that the matrices $P_{n,x}$ depend little on n and x. To this end, we define a constant matrix $\bar{P} = \| \bar{p}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1}) \|$, with which we compare the matrices $P_{n,x}$, and we show that for certain constants $C_6 < \infty$ and $\lambda < 1$

$$\left| \frac{p_{n,x}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1})}{\bar{p}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1})} - 1 \right| \leqslant C_6 \lambda^t \tag{6}$$

(these inequalities also imply that the matrix elements $P_{n,x}$ and $\bar{p}$ go to zero simultaneously). We first write

$$p_{n,x}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1}) = \tilde{\mu}(\Delta^{0}_{-t+1} | T^{-t} \tilde{\Delta}^{0}_{-t-1} \cap C_{T-n_{\alpha^-}}(x))$$

$$= \frac{\displaystyle\sum_{C_{T-t_{\alpha^-}} \subset C_{T-n_{\alpha^-}} \cap T^{-t} \tilde{\Delta}^{0}_{-t+1}} \tilde{\mu}(\Delta^{0}_{-t+1} | C_{T-t_{\alpha^-}}) \tilde{\mu}(C_{T-t_{\alpha^-}} | C_{T-n_{\alpha^-}}(x))}{\displaystyle\sum_{C_{T-t_{\alpha^-}} \subset C_{T-n_{\alpha^-}} \cap T^{-t} \tilde{\Delta}^{0}_{-t+1}} \tilde{\mu}(C_{T-t_{\alpha^-}} | C_{T-n_{\alpha^-}}(x))}.$$

In each Δ^{0}_{-t+1} we pick an arbitrary element $C_{\alpha^-} \in \Delta^{0}_{-t+1}$ and designate it $C_{\alpha^-}(\Delta^{0}_{-t+1})$. We set

$$\bar{p}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1}) = \tilde{\mu}(T^t \Delta^{0}_{-t+1} | C_{\alpha^-}(\tilde{\Delta}^{0}_{-t+1})).$$

This defines the stochastic matrix $\bar{P}$. If $y \in C_{T-t_{\alpha^-}}$, then $\tilde{\mu}(\Delta^{0}_{-t+1} | C_{T-t_{\alpha^-}}(y)) = \tilde{\mu}(T^t \Delta^{0}_{-t+1} | C_{\alpha^-}(T^t y))$. On the basis of Lemma 2.4

$$\frac{\tilde{\mu}(T^t \Delta^{0}_{-t+1} | C_{\alpha^-}(T^t y))}{\tilde{\mu}(T^t \Delta^{0}_{-t+1} | C_{\alpha^-}(\tilde{\Delta}^{0}_{-t+1}))} = \frac{\tilde{\mu}(T^t \Delta^{0}_{-t+1} | C_{\alpha^-}(T^t y))}{\bar{p}(\Delta^{0}_{-t+1} | \tilde{\Delta}^{0}_{-t+1})}$$

does not differ from unity by more than $C_6 \lambda^t$, where $C_6 = C_3 \max(\Lambda_c, \Lambda_p)$, $\lambda = [\max(\lambda_c, \lambda_p^{-1})]^{\varkappa/2}$. This proves (6).

We now establish that the matrices $P_{n,x}$ have one special property: We let $P^{(2)}_{n,x} = P_{n-t,T^t x} \cdot P_{n,x} = \overline{P}^2 = \| \overline{p}^{(2)}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1}) \|$, representing the square of the matrix $\overline{P}$, and we show that there exist constants C_7, C'_7, independent of t, such at for any Δ^0_{-t+1} and $\widetilde{\Delta}^0_{-t+1}$

$$C_7 \leqslant \frac{p^{(2)}_{n,x}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})}{\sigma(T^{2t}\Delta^0_{-t+1})} \leqslant C'_7, \quad C_7 \leqslant \frac{\overline{p}^{(2)}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})}{\sigma(T^{2t}\Delta^0_{-t+1})} \leqslant C'_7. \tag{7}$$

We point out first of all that if t is sufficiently large, then all $p^{(2)}_{n,x}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})$ and, hence, $\overline{p}^{(2)}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})$ are positive. Thus, $p^{(2)}_{n,x} = \widetilde{\mu}(T^{2t}\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1} \cap C_{T^{-n}\alpha^-}(x)) = \widetilde{\mu}(T^{2t+n}\Delta^0_{-t+1} | T^n\widetilde{\Delta}^0_{-t+1} \cap C_{\alpha^-}(T^n x))$. Let $T^n x \in U_i$. Now, if $\Delta^0_{-t+1} \neq \emptyset$, $\widetilde{\Delta}^0_{-t+1} \neq \emptyset$ and t is sufficiently large, then, as shown in Section 3, in the proof of Theorem 3.1, $U_i \cap T^n\widetilde{\Delta}^0_{-t+1} \cap T^{2t+n}\Delta^0_{-t+1} \neq \emptyset$. It was also shown there that the intersections $C_{\alpha^-}(y) \cap T^n\widetilde{\Delta}^0_{-t+1} \cap T^{2t+n}\Delta^0_{-t+1}$ are canonically isomorphic for all $y \in U_i \cap M_1$. Consequently, $C_{\alpha^-}(y) \cap T^n\widetilde{\Delta}^0_{-t+1} \cap T^{2t+n}\Delta^0_{-t+1} \neq \emptyset$ and $p^{(2)}_{n,x} > 0$.

In view of (6), it is sufficient in (7) to prove the latter inequality. Let

$$\overline{\overline{P}} = \| \overline{\overline{p}}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1}) \| = \| \widetilde{\mu}(T^{2t}\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1}) \|$$

and

$$\sigma(T^{2t}\Delta^0_{-t+1}) = \sum_{\widetilde{\widetilde{\Delta}}^0_{-t+1}} \sigma(T^{2t}\Delta^0_{-t+1} | \widetilde{\widetilde{\Delta}}^0_{-t+1}) \, \sigma(\widetilde{\widetilde{\Delta}}^0_{-t+1}).$$

For all $\widetilde{\widetilde{\Delta}}^0_{-t+1}$ we have $\sigma(T^{2t}\Delta^0_{-t+1} | \widetilde{\widetilde{\Delta}}^0_{-t+1}) > 0$. Furthermore,

$$\sigma(T^{2t}\Delta^0_{-t+1} | \widetilde{\widetilde{\Delta}}^0_{-t+1}) = \frac{1}{\sigma(\widetilde{\widetilde{\Delta}}^0_{-t+1})} \int_{\widetilde{\widetilde{\Delta}}^0_{-t+1}} \sigma(T^{2t}\Delta^0_{-t+1} | C_{\alpha^-}(x)) \, d\sigma.$$

We now observe that, inasmuch as $t > k_0$ (see the end of Section 3), for any C'_{α^-} and C''_{α^-}

$$C_4 \leqslant \frac{\sigma(T^{2t}\Delta^0_{-t+1} | C'_{\alpha^-})}{\sigma(T^{2t}\Delta^0_{-t+1} | C''_{\alpha^-})} \leqslant C_5.$$

In particular, setting $C''_{\alpha^-} = C_{\alpha^-}(\widetilde{\Delta}^0_{-t+1})$, we have

$$\sigma(T^{2t}\Delta^0_{-t+1}) \geqslant C_4 \sigma(T^{2t}\Delta^0_{-t+1} | C_{\alpha^-}(\widetilde{\Delta}^0_{-t+1})),$$

and, by virtue of Lemma 2.3 and the absolute continuity of the canonical isomorphism (Section 1),

$$\widetilde{\mu}(T^{2t}\Delta^0_{-t+1} | C_{\alpha^-}(\widetilde{\Delta}^0_{-t+1})) \leqslant C_2 \sigma_c(T^{2t}\Delta^0_{-t+1} | C_{\alpha^-}(\widetilde{\Delta}^0_{-t+1})).$$

Consequently,

$$\frac{\widetilde{\mu}(T^{2t}\Delta^0_{-t+1} | C_{\alpha^-}(\widetilde{\Delta}^0_{-t+1}))}{\sigma(T^{2t}\Delta^0_{-t+1})} \leqslant C_2 C_4^{-1}.$$

The lower inequality is derived analogously. We now compare $\overline{\overline{p}}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})$ and $\overline{p}^{(2)}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1})$. we have

$$\overline{p}^{(2)}(\Delta^0_{-t+1} | \widetilde{\Delta}^0_{-t+1}) = \sum_{\widetilde{\widetilde{\Delta}}^0_{-t+1}} \overline{p}(\widetilde{\widetilde{\Delta}}^0_{-t+1} | \widetilde{\Delta}^0_{-t+1}) \, \overline{p}(\Delta^0_{-t+1} | \widetilde{\widetilde{\Delta}}^0_{-t+1}).$$

On the other hand,

$$\bar{\bar{p}}(\Delta^0_{-t+1}\,|\,\tilde{\Delta}^0_{-t+1}) = \tilde{\mu}\,(T^{2t}\Delta^0_{-t+1}\,|\,C_{\alpha^-}(\tilde{\Delta}^0_{-t+1})) = \sum_{\tilde{\tilde{\Delta}}^0_{-t+1}} \tilde{\mu}\,(T^{2t}\Delta^0_{-t+1}\,|\,T^t\tilde{\tilde{\Delta}}^0_{-t+1}\cap C_{\alpha^-}(\tilde{\Delta}^0_{-t+1}))\,\tilde{\mu}\,(T^t\tilde{\tilde{\Delta}}^0_{-t+1}\,|\,C_{\alpha^-}(\tilde{\Delta}^0_{-t+1}))$$

Clearly, $T^t\tilde{\tilde{\Delta}}^0_{-t+1}\cap C_{\alpha^-}(\tilde{\Delta}^0_{-t+1}) = T^t C'_{\alpha^-}(\tilde{\tilde{\Delta}}^0_{-t+1})$, where $C'_{\alpha^-}(\tilde{\tilde{\Delta}}^0_{-t+1})$ is an element of the partition α^- belonging to $\tilde{\tilde{\Delta}}^0_{-t+1}$. Therefore,

$$\tilde{\mu}\,(T^{2t}\Delta^0_{-t+1}\,|\,T^t\tilde{\tilde{\Delta}}^0_{-t+1}\cap C_{\alpha^-}(\tilde{\Delta}^0_{-t+1})) = \tilde{\mu}\,(T^t\Delta^0_{-t+1}\,|\,C'_{\alpha^-}(\tilde{\tilde{\Delta}}^0_{-t+1})).$$

But, by virtue of Lemma 2.4,

$$\left|\frac{\tilde{\mu}\,(\Delta^0_{-t+1}\,|\,C'_{\alpha^-}(\tilde{\tilde{\Delta}}^0_{-t+1}))}{\tilde{\mu}\,(\Delta^0_{-t+1}\,|\,C_{\alpha^-}(\tilde{\Delta}^0_{-t+1}))} - 1\right| \leqslant C_6\lambda^t.$$

Consequently, $\left|\dfrac{\bar{p}^{(2)}(\Delta^0_{-t+1}\,|\,\tilde{\Delta}^0_{-t+1})}{\bar{\bar{p}}(\Delta^0_{-t+1}\,|\,\tilde{\Delta}^0_{-t+1})} - 1\right| \leqslant C_6\lambda^t$, and (7) is proved.

We state the next proposition in the form of a separate lemma. It represents a modification of the ergodic theorem for Markov chains under conditions of the Doeblin type (see [6]). The author is grateful to R. A. Minlos for a substantial simplification in the original version of this lemma.

LEMMA 4.2. Consider the class of probability distributions μ on the elements Δ^0_{-t+1} of the partition α^0_{-t+1}. There exists a constant C_9, depending only on T, such that for any two probability distributions μ' and μ''

$$\mathrm{Var}\,(\mu' P^{(2)}_{n,x} - \mu'' P^{(2)}_{n,x}) \leqslant (1 - C_9)\,\mathrm{Var}\,(\mu' - \mu''),\quad \mathrm{Var}\,(\mu'\bar{P}^2 - \mu''\bar{P}^2) \leqslant (1 - C_9)\,\mathrm{Var}\,(\mu' - \mu'').$$

Proof. We prove only the latter inequality. The first has an analogous proof. We designate $\bar{\mu}' = \mu'\bar{P}^2$, $\bar{\mu}'' = \mu''\bar{P}^2$ and introduce the sets $B^{(1)} = \{\Delta^0_{-t+1} : \bar{\mu}'(\Delta^0_{-t+1}) \geqslant \bar{\mu}''(\Delta^0_{-t+1})\}$, $B^{(2)} = \{\Delta^0_{-t+1} : \bar{\mu}'(\Delta^0_{-t+1}) \leqslant \bar{\mu}''(\Delta^0_{-t+1})\}$. It is clear that $B^{(1)}\cup B^{(2)} = M$. Therefore, either $\sigma\,(T^{2t}B^{(1)}) \geqslant 1/2$, or $\sigma\,(T^{2t}B^{(2)}) \geqslant 1/2$. Let the former condition prevail for definiteness. Then, letting $\sum^+$ denote summation over positive terms, we write

$$\mathrm{Var}\,(\bar{\mu}'' - \bar{\mu}') = 2\sum^+_{\Delta^0_{-t+1}} (\bar{\mu}''(\Delta^0_{-t+1}) - \bar{\mu}'(\Delta^0_{-t+1})) = 2\sum^+_{\Delta^0_{-t+1}}\sum_{\tilde{\Delta}^0_{-t+1}} \bar{p}^{(2)}(\Delta^0_{-t+1}\,|\,\tilde{\Delta}^0_{-t+1})\,(\mu''(\tilde{\Delta}^0_{-t+1}) - \mu'(\tilde{\Delta}^0_{-t+1}))$$

$$\leqslant 2\sum^+_{\tilde{\Delta}^0_{-t+1}} (\mu'(\tilde{\Delta}^0_{-t+1}) - \mu''(\Delta^0_{-t+1}))\,(1 - \bar{p}^{(2)}(B^{(1)}\,|\,\tilde{\Delta}^0_{-t+1}))$$

By virtue of (7), we have $\bar{p}\,(B^{(1)}\,|\,\tilde{\Delta}^0_{-t+1}) \geqslant C_7\sigma\,(T^{2t}B^{(1)}) \geqslant 0.5\,C_7$. Hence, we obtain

$$\mathrm{Var}\,(\bar{\mu}' - \bar{\mu}'') \leqslant (1 - 0.5\,C_7)\,\mathrm{Var}\,(\mu' - \mu'').$$

This is then the required inequality, as soon as we set $C_9 = 0.5C_7$. The lemma is proved.

We conclude now with the proof of Lemma 4.1. Let μ_0 be an eigenvector of the matrix $\bar{P}$. We show that the neighborhood $O_{\delta/2} = \{\mu : \mathrm{Var}\,(\mu - \mu_0) < \delta/2\}$ is invariant with respect to all stochastic matrices $p^{(2)}_{n,x}$ if t is sufficiently large. Making use of (6), we write

$$\frac{p^{(2)}_{n,x}(\Delta^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})}{\bar{p}^{(2)}(\Delta^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})} = \frac{\sum\limits_{\widetilde{\Delta}^0_{-t+1}} p_{n-t,T^t_x}(\widetilde{\widetilde{\Delta}}^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})\, p_{n,x}(\Delta^0_{-t+1}|\widetilde{\widetilde{\Delta}}^0_{-t+1})}{\sum\limits_{\widetilde{\widetilde{\Delta}}^0_{-t+1}} \bar{p}(\widetilde{\widetilde{\Delta}}^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})\, \bar{p}(\Delta^0_{-t+1}|\widetilde{\widetilde{\Delta}}^0_{-t+1})} \leqslant (1+C_6\lambda^t)^2 \leqslant 1+3C_6\lambda^t$$

for sufficiently large t and, analogously,

$$\frac{p^{(2)}_{n,x}(\Delta^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})}{\bar{p}^{(2)}(\Delta^0_{-t+1}|\widetilde{\Delta}^0_{-t+1})} \geqslant 1 - 3C_6\lambda^t .$$

Hence, it follows that $\mathrm{Var}(\mu P^{(2)}_{n,x} - \mu\bar{P}^2) \leqslant 3C_6\lambda^t$ for any μ. From Lemma 4.2, we now obtain

$$v = \mathrm{Var}(\mu P^{(2)}_{n,x} - \mu_0) = \mathrm{Var}(\mu\bar{P}^2 - \mu_0) + 3C_6\lambda^t \leqslant (1-C_9)\,\mathrm{Var}(\mu - \mu_0) + 3C_6\lambda^t .$$

Let t be so large that $0.5C_9\delta > 3C_6\lambda^t$. Then

$$v \leqslant (1-C_9)\frac{\delta}{2} + 3C_6\lambda^t = \frac{\delta}{2} - \frac{1}{2}C_9\delta + 3C_6\lambda^t < \frac{\delta}{2},$$

i.e., $\mu P^{(2)}_{n,x} \in O_{\delta/2}$, and our assertion is proved.

We saw [see (5)] that for $p < t$, $n = ct + p$, with c an integer,

$$\widetilde{\mu}_{n,x} = \widetilde{\mu}_{p,T^{(c-1)t}x} \cdot P_{t+p,T^{n-t}x} \cdot P_{t+2p,T^{n-2t}x} \cdot \ldots \cdot P_{n,x}.$$

It is readily inferred from Lemma 4.2 that there exists an n_0, independent of $x \in M_1$, such that $\widetilde{\mu}_{n,x} \in O_\delta$. Consequently, Lemma 4.1 is proved.

It is a direct consequence of Lemma 4.1 that for all sets A representing elements of some partition $\alpha^{l_2}_{l_1}$, there exists an x-independent limit

$$\lim_{n\to\infty} \widetilde{\mu}(A|C_{T^{-n}\alpha^-}(x)) = \widetilde{\mu}_x(A) = \widetilde{\mu}(A),$$

positive for every A.

We now show that the formulated limits $\widetilde{\mu}(A)$ are invariant with respect to T. We have from the property α_2) of Lemma 2.3

$$\widetilde{\mu}(A) = \widetilde{\mu}_x(A) = \widetilde{\mu}_{Tx}(TA) = \widetilde{\mu}(TA).$$

Thus, the limits $\widetilde{\mu}(A)$ are invariant with respect to T.

It is logical to adopt the limits $\widetilde{\mu}(A)$ as the values of the invariant measure we seek. We now investigate the problem: In what sense do the $\widetilde{\mu}(A)$ actually generate a measure defined on the σ-algebra of all Borel subsets of M? It follows at once from the definition that $\widetilde{\mu}$ is finite-additive. We look once again at the space Ω_Π that we constructed in Section 3. The relation $\widetilde{\mu}(\{l_{i_1}, \ldots, l_{i_n}\}) = \widetilde{\mu}(T^{i_1}U_{l_{i_1}} \cap \ldots \cap T^{i_n}U_{l_{i_n}})$ specifies a compatible system of finite-dimensional distributions on cyclindrical subsets of the space Ω_Π. According to the well-known Kolmogorov theorem, $\widetilde{\mu}$ can be continued to a denumerable-additive measure defined on the σ-algebra of all Borel subsets of Ω. We now recall that the mapping $\varphi : \Omega_\Pi \to M$ is one-to-one on M_1 and continuous. Therefore, $\varphi(A)$ for any Borel A is a Borel subset of M_1 and, hence, of M. Letting $\bar{\mu}(\varphi(A)) = \widetilde{\mu}(A)$, $\bar{\mu}(M \setminus M_1) = 0$, we obtain a certain Borel measure $\bar{\mu}$ in the space M. The measure $\bar{\mu}$, however, can prove to be irregular. We will show, nevertheless, that this is not true in our case. For this it is sufficient to establish that for the set $E = M \setminus M_1$ there exists a sequence of open sets $O_1 \supset O_2 \supset O_3 \supset \ldots$ such that $\bigcap O_k = E$ and $\lim_{i\to\infty} \bar{\mu}(O_i) = 0$.

We begin by noting that the conditional measure induced by $\bar{\mu}$ on an element $C_{\alpha^-} \overset{i\to\infty}{\cap} M_1$ of a partition α^- of the space M_1 coincides with $\tilde{\mu}(\cdot\,|C_{\alpha^-})$. Thus, we let $C_{\alpha^-}(x) = \bigcap_n \Delta^0_{-n}$. For almost every x, with respect to the measure $\bar{\mu}$, and $y \in M_1$, we have from Lemma 2.4

$$\bar{\mu}(\Delta_1^l | C_{\alpha^-}(x)) = \lim_{n\to\infty} \bar{\mu}(\Delta_1^l | \Delta^0_{-n}) = \lim_{n\to\infty} \frac{\bar{\mu}(\Delta_1^l \cap \Delta^0_{-n})}{\bar{\mu}(\Delta^0_{-n})} = \lim_{n\to\infty}\lim_{p\to\infty} \frac{\tilde{\mu}(\Delta_1^l \cap \Delta^0_{-n} | C_{T^{-p}\alpha^-}(y))}{\tilde{\mu}(\Delta^0_{-n} | C_{T^{-p}\alpha^-}(y))} =$$

$$= \lim_{n\to\infty}\lim_{p\to\infty} \frac{\sum\limits_{C_{\alpha^-}\in C_{T^{-p}\alpha^-}\cap\Delta^0_{-n}} \tilde{\mu}(\Delta_1^l | C_{\alpha^-})\,\tilde{\mu}(C_{\alpha^-} | C_{T^{-p}\alpha^-}(y))}{\sum\limits_{C_{\alpha^-}\in C_{T^{-p}\alpha^-}\cap\Delta^0_{-n}} \tilde{\mu}(C_{\alpha^-} | C_{T^{-p}\alpha^-}(y))} \leqslant \tilde{\mu}(\Delta_1^l | C_{\alpha^-}(x)) \cdot \lim_{n\to\infty}(1 + C_6\lambda^n) \leqslant \tilde{\mu}(\Delta_1^l | C_{\alpha^-}(x))$$

with an analogous estimate on the other side. Consequently, $\bar{\mu}(\Delta_1^l | C_{\alpha^-}(x))$ is congruent with the measure constructed in Lemma 2.3 almost everywhere with respect to $\bar{\mu}$, hence, it is equivalent to $\sigma_c | C_{\alpha^-}$.

We now formulate the required system of sets O_i for $E_e \setminus E_c = \bigcup_{-\infty}^{\infty} T^n\left(\bigcup_i(\Gamma_e(U_i)\setminus\Gamma_c(U_i))\right)$. It is sufficient to formulate it for one $\Gamma_e(\bar{U}_i)\setminus\Gamma_c(U_i)$, all other sets being treated analogously. Inasmuch as all U_i consist of admissible local layers, for every CLL D_c we have $\sigma(\Gamma_e(U_i)|D_c) = 0$. We pick an $x \in U_i$ and a CLL $D_c(x)$ representing a sphere of radius δ (in the metric of the layer) sufficiently large that some neighborhood $\Gamma_e(U_i) \cap D_c(x_i)$ is contained in $D_c(x)$. By the regularity of the measure $\sigma(\cdot\,|D_c)$ there exists a diminishing sequence of open sets $O_j^{(c)} \subset D_c$, such that $\sigma(O_j^{(c)}) \to 0$ and $\bigcap_j O_j^{(c)} \supset \Gamma_e(U_i) \cap D_c$. But then the open sets $O_j = \bigcup_{C_{\alpha^-}} \pi(O_j^{(c)}|C_{\alpha^-})$, where the C_{α^-} are such that the closure $\bar{C}_{\alpha^-}\cap(\Gamma_e(U_i)\setminus\Gamma_c(U_i))$ and the canonical isomorphism image of the sets $O_j^{(c)}$, viz. $\pi(O_j^{(c)}|C_{\alpha^-})$, cover $\Gamma_e(U_i)\setminus\Gamma_c(U_i)$. We now have

$$\bar{\mu}(O_j) = \int \bar{\mu}(O_j|C_{\alpha^-})\,d\bar{\mu} = \int \bar{\mu}(\pi(O_j^{(c)})|C_{\alpha^-})d\bar{\mu} \leqslant \text{const}\,\sigma(O_j^{(c)}) \to 0.$$

Consequently, $\bar{\mu}(\bigcap_j O_j) \to 0$ for $j \to \infty$.

We direct our attention now to $\Gamma_c(U_i)$. We saw (Section 1) that $\Gamma_c(U_i) = \bigcup_{z\in\partial D_p^{(i)}} \bar{D}_c(z)$, where $D_e^{(i)}$ is any ELL in U_i. Moreover, the closed set $\Gamma_c(U_i)$ intersects with other $\Gamma_c(U_{i'})(\partial U_{i'} \cap \partial U_i \neq \varnothing)$. We find a number k_1, such that for every ELL D_e in U_j, $j = 1, \ldots,$ the intersection $T^{-k_1}D_e \cap U_{j'}$ consists of more than one connected component (for any j and j'). We examine $T^{-k_1}U_i$. Clearly, $T^{k_1}U_i \cap U_j \neq \varnothing$. We set O'_1 equal to the sum of those connected components of the intersections $T^{k_1}U_{i'} \cap U_j$ whose closure contain $T^{k_1}\Gamma_c(U_i)$ and $O_1 = T^{-k_1}O'_1$. We investigate the structure of the set O_1. We may regard all $U_{i'}$ as contained in some fixed spherical neighborhood W. We choose a complete ELL D_e in W. The intersection $D_e \cap O_1$ consists of a finite number of open sets (one for each $U_{i'}$). Every point $x \in D_e \cap \Gamma_c(U_i)$ will have some neighborhood consisting of points belonging to O_1 and of points belonging to $\Gamma_c(U_i)$. Therefore, if we add the set $\Gamma_c(U_i)$ to O_1, we obtain an open set O_1^* containing $\Gamma_c(U_i)$. It is clear that $C_{\alpha^-}(x) \cap O_1^* = C_{\alpha^-}(x) \cap O_1$ for $x \in M_1$. Now that we have the set O_1^*, we consider every connected component $O_{jj'}$ of the intersection $T^{k_1}O_1^* \cap U_j$ and separate the part whose boundary intersects with $T^{2k_1}\Gamma_c(U_j)$. We take the union of all $T^{-2k_1}O_{jj'}$ and add to it $\Gamma_c(U_i)$. As before, we obtain an open set O_2^*, such that $\Gamma_c(U_i) \subset O_2^* \subset O_1^*$. We construct $O_3^* \subset O_2^*$, $O_4^* \subset O_3^*$, etc., analogously.

Our task now is to show that $\bar{\mu}(O_n^*) \to 0$ as $n \to \infty$. We establish that for some constant $\bar{\lambda} < 1$ the inequality $\bar{\mu}(O_{n+1}^*) \leqslant \bar{\lambda}\bar{\mu}(O_n^*)$ holds. Let us consider the connected component P of the intersection $T^{k_1 n}O_n^* \cap \dot{U}_i$. Here P comprises canonically isomorphic ELL. The intersection $T^{k_1(n+1)}O_{n+1}^* \cap T^{k_1}P$ is generated as follows. Pick an ELL $D_e \subset P$; then $T^{k_1}D_e$ is the sum of ELL lying in all possible U_j. We pick those $D'_e \subset T^{k_1}D_e$ whose boundary intersects with $T^{(n+1)k_1}\Gamma_c(U_i)$. Clearly, in this case the boundary of D'_e must at least partially intersect with the boundary $\partial(T^{k_1}D_e)$. By the choice of k_1, however, among the ELL constituting $T^{k_1}P$ there are necessarily D'_e that lie wholly within $T^{k_1}D_e$. Consequently, $T^{k_1(n+1)}O^*_{n+1} \cap T^{k_1}P$ is a proper subset of $T^{k_1}P$, and it suffices for us to show that the inequality $\bar{\mu}(T^{k_1(n+1)}O^*_{n+1} \cap T^{k_1}P) \leqslant \bar{\lambda}\bar{\mu}(T^{k_1}P)$. is valid for some $\bar{\lambda} < 1$. We note that $T^{k_1}P$ is a parallelogram; its ELL have a diameter limited to a constant K independent of n. If we examine different CLL D_c belonging to $T^{k_1}P$, then $K_1 \leqslant \sigma_c(D'_c)/\sigma_c(D''_c) \leqslant K_2$ for certain constants K_1 and K_2. Hence, it follows that $K_3 \leqslant \bar{\mu}(Q')/\bar{\mu}(Q'') \leqslant K_4$ for the connected components Q of the intersection $T^{k_1}P \cap U_j$ for suitable constants K_3 and K_4. Inasmuch as scarcely all the components belong to $T^{k_1(n+1)}O^*_{n+1}$, , we have the following, denoting by $\sum'$ summation over those connected components which belong to $T^{k_1(n+1)}O^*_{n+1}$:

$$\frac{\bar{\mu}(T^{k_1}P \cap T^{k_1(n+1)}O^*_{n+1})}{\bar{\mu}(T^{k_1}P)} = \frac{\sum'\bar{\mu}(Q)}{\sum''\bar{\mu}(Q)} \leqslant 1 - \frac{K_3}{K_4 N},$$

where N is the total number of connected components. Clearly, N does not depend on n, and our assertion is proved. Thus, the measure $\bar{\mu}$ is regular.

By its construction the measure $\bar{\mu}$ is invariant with respect to T. It induces a conditional measure $\tilde{\mu}$ on almost every C_{α^-}. Since we have $\lim_{n\to\infty} \bar{\mu}(A|C_{T^{-n}\alpha^-}(x)) = \bar{\mu}(A)$ for every $x \in M_1$, the transformation T with invariant measure $\bar{\mu}$ is a K-automorphism.

We prove the uniqueness of $\bar{\mu}$. Let there be another measure $\bar{\bar{\mu}}$ having the same properties. We pick a Markov partition α and the measure induced by it on C_{α^-}. It will satisfy the properties α_1) and α_2) of Lemma 2.3, hence, it will coincide with the measure $\tilde{\mu}$ constructed in that connection. But then $\bar{\mu} = \bar{\bar{\mu}}$, because the values $\bar{\mu}$ are defined according to $\tilde{\mu}$.

The property 2 needed in Theorem 1 is a direct consequence of the fact that it is fulfilled for the partitions $T^n\alpha^-$. This completes the proof of Theorem 1.

5. Several Theorems on the Construction of C-Automorphisms

Let T, as before, be a transitive C-diffeomorphism.

THEOREM 5.1. On every layer $\Gamma^{(c)}$ of a contractile foliation it is possible to specify a σ-finite measure μ_c having the following properties:

a_1) There exists a constant $h_c > 1$, depending only on T, such that for every set $A \subset \Gamma^{(c)}$

$$\mu_c(T^{-1}A) = h_c\mu_c(A).$$

a_2) If $A' \subset \Gamma^{(c)}$ is canonically isomorphic to $A'' \subset \tilde{\Gamma}^{(c)}$, then $\mu_c(A') = \mu_c(A'')$.

Similarly, replacing T by T^{-1}, it is possible to specify a measure μ_e on the layers of an expansible foliation and a corresponding constant h_e. It turns out that $h_c = h_e$.

Proof. Consider a Markov partition α, and let $\alpha_1 = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-m+1}\alpha$, where m is the number involved in the definition of the Markov partition. Let $V_1, V_2, \ldots, V_r$ be parallellograms representing the elements of the partition α_1. It follows from a_1) that the quantity μ_c (C_{a^-}) should take on a constant value for all $C_{a^-} \in V_i$. Therefore $\mu_c(C_{a^-}) = \mu_c^{(i)}$. Let us try to find the numbers $\mu_c^{(i)}$ and the number h_c. It follows from a_2) that $\mu_c^{(i)}$ must satisfy the following equation:

$$h_c\mu_c^{(i)} = \sum_j \mu_c^{(j)}\pi_{ji}, \qquad i = 1, \ldots, r, \tag{8}$$

or, in vector form

$$h_c\boldsymbol{\mu}_c = \boldsymbol{\mu}_c\Pi. \tag{8'}$$

We see from this that the vector $\mu_c\{\mu_c^{(1)}, \dots \mu_c^{(r)}\}$ is an eigenvector of a matrix Π having nonnegative components, and h_c is the corresponding eigenvalue. If Π^{k_0} has positive-definite elements, then this eigenvector is unique correct to a multiplier, and the number h_c is unique as well. We note, incidentally, that h_c is an algebraic number.

We assign definite values to the numbers $\mu_c^{(i)}$. Now we let $C_{T\alpha^-} \subset C_{\alpha^-}$ be an element of the partition $T\alpha^-$. We define $\mu_c(C_{T\alpha^-})$ so as to meet a_1) and a_2). To do this, we need to set $\mu_c(C_{T\alpha^-}) = h_c^{-1}\mu_c(T^{-1}C_{T\alpha^-}) = h_c^{-1}\mu_c(C'_{\alpha^-})$. We assume, in general, that the $\mu_c(C_{T^n\alpha^-})$ have already been determined. We set $\mu_c(C_{T^{n+1}\alpha^-}) = h_c^{-1}\mu_c(T^{-1}C_{T^{n+1}\alpha^-}) = h_c^{-1}\mu_c(C'_{T^n\alpha^-})$.

We show that such a definition is proper. We observe that $\sum_{C_{T\alpha^-}\subset C^0_{\alpha^-}} \mu_c(C_{T\alpha^-}) = h_c^{-1}\sum\mu_c(T^{-1}C_{T\alpha^-}) = \mu_c(C^0_{\alpha^-})$, since $\bigcup T^{-1}C_{T\alpha^-} = T^{-1}C^0_{\alpha^-}$, and for $C^0_\alpha-$ we have the system (8), (8').

We assume that the following relations hold for all $n \le n_0$:

$$\mu_c(C^0_{T^{n-1}\alpha^-}) = \sum_{C_{T^n\alpha^-}\subset C^0_{T^{n-1}\alpha^-}} \mu_c(C_{T^n\alpha^-}), \quad \mu_c(C_{T^n\alpha^-}) = h_c^{-1}\mu_c(T^{-1}C_{T^n\alpha^-}).$$

Then from the definition for $n + n_0 + 1$, we have

$$\sum_{C_{T^{n+1}\alpha^-}\subset C^0_{T^n\alpha^-}} \mu_c(C_{T^{n+1}\alpha^-}) = h_c^{-1}\sum\mu_c(T^{-1}C_{T^{n+1}\alpha^-})$$

But $T^{-1}(\bigcup C_{T^n\alpha^-}) = T^{-1}C^0_{T^n\alpha^-} = C^0_{T^{n-1}\alpha^-}$, hence the latter sum on the right is equal to $\mu_c(C^0_{T^{n-1}\alpha^-})$, and by the induction hypothesis $\mu_c(C^0_{T^{n-1}\alpha^-}) = h_c\mu_c(C^0_{T^n\alpha^-})$. We obtain

$$\sum_{C_{T^{n+1}\alpha^-}\subset C^0_{T^n\alpha^-}} \mu_c(C_{T^{n+1}\alpha^-}) = \mu_c(C^0_{T^n\alpha^-}).$$

Consequently, μ_c is a finite-additive function on the system of sets $C_{T^n\alpha^-}\subset C^0_{\alpha^-}$. As in the preceding section, it has been established that μ_c generates a regular Borel measure on every $C^0_\alpha-$. Continuing it additively, we obtain a measure μ_c on every layer of the contractile foliation.

Inasmuch as $\mu_c| C^0_\alpha-$ complies with the condition a_1), μ_c will then meet the same condition on every layer. All that remains is to verify a_2). If A' and π(A') lie inside the same V_i, the equation $\mu_c(A') = \mu_c(\pi(A'))$ follows from the construction of μ_c. Now let $A' \in V_i$, $\pi(A') = A'' \in \bar V_i \setminus V_i$. Then

$$A'' = \bigcup_j (\bar V_i \cap \bar V_j \cap A'') = \bigcup_j A''_j.$$

We set $A'_j = \pi^{-1}(A''_j)$. We bring into the discussion any CLL $A'''_j \subset V_j$ canonically isomorphic to A''_j and thus to A'_j. We then show that $\mu_c(A'_j) = \mu_c(A'''_j) = \mu_c(A''_j)$. Since every canonical isomorphism decomposes into a chain of mappings of the type $A'_j \xrightarrow{\pi} A'''_j$, this will ultimately lead to the property a_2).

First let A'_j be an element of the partition $T^n\alpha^-$ for some $n \ge 0$. Now $T^{-n}A'_j = C_{\alpha^-} \in V_{i_1}$. If $T^{-n}A'''_j \in V_{i_1}$, then A'''_j is also an element of the partition $T^n\alpha^-$, and $\mu_c(T^{-n}A'_j) = \mu_c(T^{-n}A'''_j)$ by definition. From a_1), however, we have $\mu_c(A'_j) = \mu_c(A'''_j)$.

We investigate the case when $T^{-n}A'_j$ and $T^{-n}A'''_j$ do not lie in the same parallelogram V_{i_1}. Let $A' = \bigcup C'_{T^{n_1}\alpha^-}$, $n_1 > n$, $T^{-n_1}A'_j = \bigcup T^{-n_1}C'_{T^{n_1}\alpha^-} = \bigcup C'_{\alpha^-}$. Now, if $C''' = \pi(C'_{T^{n_1}\alpha^-}) \subset A'''_j$, then $\pi(T^{-n_1}C'_{T^{n_1}\alpha^-}) = \pi(C'_{\alpha^-}) = T^{-n}C'''$. If $C'_\alpha-$ and $T^{-n}C'''$ lie in the same parallelogram V_l, then we are entitled to use our earlier argumentation, deducing that C''' is an element of the partition $T^{n_1}\alpha^-$ and $\mu_c(C''') = \mu_c(C'_{T^{n_1}\alpha^-})$. Conquently, the points $x \in A'_j$, for which $T^{-n}C_{T^n\alpha^-}(x)$ and $T^{-n}(\pi(C_{T^n\alpha^-}(x)))$ belong to different parallelograms, for all n fall outside the scope of our investigation. Here $d_e(T^{-n}x, T^{-n}\pi(x)) \to 0$ as $n \to \infty$, $\bar{C}_{\alpha+}(T^{-n}x) \cap \bar{C}_{\alpha+}(T^{-n}\pi(x)) \neq \varnothing$ and enters into $\Gamma_c(\alpha_1)$. It is easy to show (in fact, the corresponding arguments are exactly analogous to the proof of regularity of the measure $\tilde{\mu}_c$ in Section 4) that for the set B of such x we have $\mu_c(B) = 0$, $\mu_c(\pi(B)) = 0$. It turns out, therefore, that $A'_j = B \cup \bigcup \bar{C}'_{T^n\alpha^-}$, where the summation is taken over the open elements of the partition $T^n_{\alpha^-}$ that do not intersect B. Inasmuch as $\mu_c(B) = \mu_c(\pi(B)) = 0$ and $\mu_c(C'_{T^n\alpha^-}) = \mu_c(\bar{C}'_{T^n\alpha^-}) = \mu_c(\pi(C'_{T^n\alpha^-})) = \mu_c(C'''_{T^n\alpha^-})$, it follows that $\mu_c(A'_j) = \mu_c(A'''_j)$. Thus, a_2) has been proved.

The expansible foliation is investigated analogously. All that remains is to show that $h_c = h_e$. We note that in the case of an expansible foliation the matrix Π^* for the system of equations of the type (8) will be conjugate to the matrix Π corresponding to the contractile foliation. The degree of the matrix Π^* will also consist of positive elements. The equation $h_c = h_e$ ensues from the fact that a positive eigenvalue for such matrices corresponds with a positive eigenvalue for the conjugate matrix. The theorem is thus proved.

THEOREM 5.2. Let T be a C-diffeomorphism of a two-dimensional torus. Then T is topologically conjugate with an algebraic automorphism.

Proof. In the two-dimensional case a C-diffeomorphism is always transitive. In addition, it readily follows from the Lefschetz formula in the two-dimensional case that T has at least one stationary point. Let x_0 be one of them. We now use our foliations to construct a certain covering of the torus M by a plane $R^2 = (\mu^1, \mu^2)$.

We orient our foliations in some way. Let $\Gamma_0^{(c)}$ and $\Gamma_0^{(e)}$ be oriented complete layers passing through x_0. We pick an arbitrary point (μ^1, μ^2) of the plane R^2. We find on $\Gamma_0^{(c)}$ a point x_c, such that $\mu_c([x_0, x_c]) = |\mu^1|$, and if $\mu^1 \geq 0$, then x_c is the end point of the interval $[x_0, x_c]$, but if $\mu^1 \leq 0$, the opposite is true. We draw a layer of an expansible foliation through the point x_c and find a point x on that layer, such that $\mu_e([x_c, x]) = |\mu^2|$ with an analogous condition on the sign of μ^2. The correspondence $(\mu^1, \mu^2) \to x$ is a certain mapping of the plane R^2 onto the torus M. We want to show that this mapping is a covering.

We introduce a metric on the torus. Let x' and x'' be nearby points. Then there exists a parallelogram, such that x' and x'' are two opposite vertices of it, while the opposite sides are segments of the same foliation. Then μ_c and μ_e assume the same value on the corresponding sides; let us call these values d_1 and d_2, and set $d(x', x'') = d_1^2 + d_2^2$.

Theorem 5.1 tells us that the mapping we have constructed is an isometric covering of the plane R^2 with metric $\mu_1^2 + \mu_2^2$ onto a torus with metric d. Hence, it follows that the images of every point form a lattice on the plane, and our torus is derived by factorization of the plane on the lattice corresponding to the point x_0.

We investigate the following linear transformation of the plane: $(\mu^1, \mu^2) \to (h_c\mu^1, h_e^{-1}\mu^2)$. It is seen at once that this transformation takes the image of every point of the torus into the same image of another point, and the corresponding transformation of the torus in the original C-diffeomorphism. This proves the theorem.

In Theorem 5.1 a certain numerical characteristic $h = h_c = h_e$ of the C-diffeomorphism T is obtained. It follows from its definition that h is an algebraic number. We now show that it coincides with certain other characteristics of T.

THEOREM 5.3. The topological entropy of the diffeomorphism T is equal to ln h.

Proof. The basic definitions and facts pertaining to topological entropy may be found in [1].

We adopt a Markov partition α and let $\alpha_1 = \alpha \cdot T^{-1}\alpha \cdot \ldots \cdot T^{-m+1}\alpha$. Let $V_1, \ldots, V_r$ be the elements of the partition α_1. For every V_i we consider the V_j for which $\bar{V}_i \cap \bar{V}_j \neq \varnothing$, and put $\bar{V}'_i = \bar{V}_i \cup \bigcup \bar{V}_j$, where the latter summation is taken over such V_j. Let V'_i be an open kernel of the closed set $\bar{V}'_i$. It is readily seen that $V'_i \supset \bar{V}_i$, so that the sets $\bar{V}'_i$ form an open covering of M. We pick the partition α fine enough that the diameters of the CLL and ELL in V'_i are not greater than the $\varepsilon > 0$ indicated in Lemma 2.1.

Then the proof of 2.1 carries over without any modification, and the open covering α'_1 by the sets V'_i is a generator in the sense indicated in the statement of Lemma 2.1. It follows from this and from the results of [1] that the topological entropy can only be calculated with respect to the covering α'_1.

We now investigate the finite intersections $(\Delta')_0^n = V'_{i_0} \cap TV'_{i_1} \cap \ldots \cap T^n V'_{i_n}$, representing elements of the covering $(\alpha'_1)_0^n = \alpha'_1 \cap T\alpha'_1 \cap \ldots \cap T^n\alpha'_1$, and, in addition to these, the elements $\Delta_0^n = V_{i_0} \cap TV_{i_1} \cap \cdots \cap T^n V_{i_n}$ of the partition $(\alpha_1)_0^n = \alpha_1 \cdot T^{-1}\alpha_1 \cdot \ldots \cdot T^{-n}\alpha_1$. We show that there exists a constant K, independent of n, such that every element $(\Delta')_0^n$ contains at most K elements of the partition α_0^n. From this we arrive at the statement of theorem.

We analyze the subcovering of the covering $(\alpha')_0^n$ consisting of elements $(\Delta')_0^n = V'_{i_0} \cap TV'_{i_1} \cap \ldots \cap T^n V'_{i_n}$ such that $\Delta_0^n = V_{i_0} \cap TV_{i_1} \cap \ldots \cap T^n V_{i_n} \neq \varnothing$. Since distinct Δ_0^n intersect only on the boundary, this will indeed be a covering of M. The number of elements of this covering is equal to $N(\alpha_0^n)$, i.e., the number of non-empty elements of the partition α_0^n. On the other hand, if β is a subcovering of $(\alpha')_0^n$ with the minimum power $N(\beta)$, then $KN(\beta) \geq N(\alpha_0{}^n)$. Finally, we have $K^{-1}N(\alpha_0^n \leq N(\beta) \leq N(\alpha_0^n)$. Consequently, the limiting behavior of $\frac{1}{n} \ln N(\beta)$ is the same as that of $\frac{1}{n} \ln N(\alpha_0^n)$.

Let $N(\alpha_0^n, V_i)$ be the number of elements Δ_0^n of the partition α_0^n, such that $T^{-n}\Delta_0^n \subset V_i$. The vector N_n with components $N(\alpha_0^n, V_i) = N_n(V_i)$ satisfies the set of equations $N_{n+1} = N_n\Pi$, where, as we recall, Π is the intersection matrix; $\pi_{ij} = 1$, if $V_i \cap TV_j \neq \varnothing$, otherwise $\pi_{ij} = 0$. But then, as we are well aware,

$$\lim_{n\to\infty} \frac{1}{n} \ln N(\alpha_0^n) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_i N_n(V_i) = \ln h.$$

Consequently, the topological entropy is equal to ln h.

We now have to find the constant K. We observe that if the dimensions of the parallelograms V_i are sufficiently small, then every V'_i and every $T^{-1}V'_i$ can be included inside a sphere of radius ε. Let $O_\varepsilon(i)$ be a sphere of radius ε containing V'_j. We draw a complete CLL in $O_\varepsilon(i)$, and find out how many elements $C_{\alpha^-} \in V'_i$ of the partition α^- intersect with it. We designate the maximum number of these elements with respect to the entire complete CLL by K_1. We now observe that if we pick some complete CLL in $T^{-1}O_\varepsilon(j)$, and find for it the number of elements $C_{\alpha^-} \in V'_i$ contained therein, again finding the maximum, we then obtain the same constant K_1. It is seen at once that K_1 is not greater than the maximum number K_2 of parallelograms V_j lying in one V'_i. The following is easily proved by induction: Consider $V'_{i_0} \cap \ldots \cap T^n V'_{i_n}$; inside $V_j \subset V'_{i_0}$ this intersection is contained within the closure of the union of a collection of elements Δ_0^n whose number does not exceed K_1. Hence, it follows that $V'_{i_0} \cap \ldots \cap T^n V'_{i_n}$ is contained within the closure of the union of a collection of elements Δ_0^n whose total number does not exceed K_1K_2. Consequently, every element of the covering $(\alpha'_1)_0^n$ contains at most K_1K_2 elements Δ_0^n of the partition $(\alpha_1)_0^n$. The theorem is completely proved.

For any C-diffeomorphisms in the algebraic case the Markov constructions we formulated acquire a curious metric characteristic.

THEOREM 5.4. Let M be an n-dimensional torus, T its algebraic automorphis, a C-diffeomorphism. Then T is metrically (i.e., in the measure-theoretic sense) conjugate with a Markov chain having a finite number of states. This Markov chain has a special property, to be spelled out in the proof.

Proof. We investigate a Markov partition α_1 for T. It is a generator for T in the measure-theoretic sense. With it a torus M with Lebesgue measure is mapped mod 0 into a space of sequences ω. We show that the image of the Lebesgue measure for this mapping specifies a stationary Markov chain in Ω.

It is readily seen that on every C_{α^-} the conditional measure $\tilde{\mu}(\cdot \mid C_{\alpha^-})$ induced by the Lebesgue measure is a simply uniform measure. Within one V_i the canonical isomorphism between distinct CLL C_{α^-} takes one conditional measure into another owing to the algebraic quality of T.

But then $\tilde{\mu}(TV_j | C_{\alpha^-})$ is identical for all $C_{\alpha^-} \in V_i$, i.e., it depends only on V_i. We can therefore let $\tilde{\mu}(TV_j | C_{\alpha^-}) = \varkappa_{ij}$. The numbers $\varkappa_{ij}$ are then the transition probabilities of our Markov chain with states V_i.

If L_i is a k-dimensional area of any $C_{\alpha^-} \in V_i$, then $\varkappa_{ij} = L_j L_i^{-1} h^{-1}$, where h, $\{L_i\}$ are the eigenvalue and components of the eigenvector of the intersection matrix Π. As shown by Parry [7], a Markov chain having these transition probabilities is a unique stationary measure in the space Ω with maximum entropy. The theorem is proved.

We turn once again to the C-diffeomorphisms of a two-dimensional torus. The measure μ_C constructed in Theorem 5.1 can be singular with respect to the Riemann length on a layer of a contractile foliation. We recall that this layer is a smooth submanifold of class C^r. Consequently, a homeomorphism that takes T into an algebraic diffeomorphism can be, and frequently is, singular, i.e., nondifferentiable. We now state a necessary and sufficient condition for it to be differentiable.

Let h be the topological entropy of T. We adopt the invariant measures $\tilde{\mu}'$ and $\tilde{\mu}''$ that we constructed in the preceding section for an expansible and a contractile foliation, respectively. Then the transformation T, as a measure-preserving transformation, has an ordinary, i.e., metric, entropy for each measure. We designate them h'_m and h''_m.

THEOREM 5.5. If $h = h'_m = h''_m$, then a homeomorphism taking T into an algebraic C-diffeomorphism is absolutely continuous.

Proof. Let T_a be an algebraic C-diffeomorphism of a two-dimensional torus, and let U be a homeomorphism taking T_a into T. We adopt a Markov partition α_1 for T_a. Its image $\beta_1 = U\alpha_1$ is a Markov partition for T. Inasmuch as $h'_m(T) = h'_m(\beta;\ T) = h$, where $h'_m(\beta_1;\ T)$ is the stepwise entropy of the partition β_1, computed according to the measure $\tilde{\mu}$, it follows from a theorem of Parry [7] that $\tilde{\mu}$ induces in the space Ω a Markov chain with conditional probabilities $\varkappa_{ij}$, computed according to the intersection matrix Π, as in Theorem 5.4. But this measure induces in Ω a Lebesgue measure on a torus for the mapping therein of M with the diffeomorphism T_a. Hence, it follows that U takes a Lebesgue measure into the measure $\tilde{\mu}$.

We let μ_0 designate the image of the Lebesgue measure. Since $\mu_0 = \tilde{\mu}$, it follows from Theorem 5.1, therefore, that μ_0 induces a measure equivalent to length on the CLL. Replacing T by T^{-1}, we infer that μ_0 coincides with the invariant measure constructed for an expansible foliation and inducing a measure equivalent to length on the ELL. Consequently, μ_0 induces a length-equivalent measure on the CLL for any partition and induces a length-equivalent measure on the ELL for any partition. We show that μ_0 is thus a measure equivalent to the Lebesgue measure σ on M.

Let $A \subset V_i$ and $\sigma(A) = 0$. We investigate the partitions $\alpha^- | V_i$ and $\alpha^+ | V_i$. Inasmuch as $\sigma(A) = 0$, we have $\sigma(A_1) = 0$ for $A_1 = \{x : \mu(A | C_{\alpha^-}(x)) \neq 0\}$. The set A_1 comprises ELL. It follows from the absolute continuity of the foliations that the intersections $A_1 \cap C_{\alpha^-}$ are canonically isomorphic for distinct C_{α^-} and $\tilde{\mu}(A_1 | C_{\alpha^-}) = \sigma(A_1 | C_{\alpha^-}) = 0$ for every C_{α^-}. Consequently, $\tilde{\mu}(A_1) = \mu_0(A_1) = 0$. If $A_2 = V_i \setminus A_1$, then, by virtue of the relation $0 = \sigma(A \cap A_2) = \int_{V_i \cap A_2} \sigma(A | C_{\alpha^+}(x))\, d\sigma$, we deduce that $\sigma(A | C_{\alpha^+}(x)) = 0$ on every $C_{\alpha^+} \in A_1$. But since μ_0 induces a length-equivalent measure on almost every C_{α^+}, then $\mu_0(A | C_{\alpha^+}(x)) = 0$. Consequently, $\mu_0(A \cap A_2) = 0$, and, finally, $\mu_0(A) = 0$.

Thus, we have obtained $\mu_0 \ll \sigma$. There now exists, invariant with respect to T, a set of positive σ-measure on which the measure μ_0 is concentrated. This set should necessarily comprise mod 0 both complete layers of a contractile foliation and complete layers of an expansible foliation. By virtue of absolute continuity, the set must coincide with M correct to a factor of σ-measure zero.

Thus, μ_0 is equivalent to σ, and the theorem is proved.

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