CHAPTER 4

Markov Partitions

We have studied dynamics of TA-maps and symbolic dynamics in the previous chapters. Our main task in this chapter is to show that every TA-homeomorphism has Markov partitions, and by them the TA-homeomorphism is realized on a symbolic dynamics. This symbolic dynamics is applied often to the ergodic theory of TA-homeomorphisms.

§4.1 Markov partitions and subshifts

Markov partitions for Anosov diffeomorphisms were first constructed by Sinai [Si1]. After that, Bowen [Bo2] found Markov partitions for basic sets of Axiom A diffeomorphisms. Ruelle [Ru] constructed Markov partitions for homeomorphisms satisfying some conditions that means local product structures.

Let $f: X \to X$ be a homeomorphism of a compact metric space. Let $\Delta(\varepsilon) = \{(x,y) : d(x,y) \le \varepsilon\}$ for $\varepsilon > 0$. Then we say that f has *local product structure* if the following conditions (A) and (B) are satisfied :

(A) There are $\delta_0 > 0$ and a continuous map $[\ ,\ ] : \Delta(\delta_0) \to X$ such that for $x, y, z \in X$

$$[x,x] = x, \quad [[x,y],z] = [x,z], \quad [x,[y,z]] = [x,z]$$
$$f[x,y] = [f(x), f(y)]$$

when the two sides of these relations are defined.

(B) There exist $0 < \delta_1 < \delta_0/2$ and $0 < \rho < \delta_1$ such that for each $x \in X$, letting

$$V^u_{\delta_1}(x) = \{y \in W^u_{\delta_0}(x) : d(x,y) < \delta_1\},$$
$$V^s_{\delta_1}(x) = \{y \in W^s_{\delta_0}(x) : d(x,y) < \delta_1\},$$
$$N_x = [V^u_{\delta_1}(x), V^s_{\delta_1}(x)],$$

the conditions hold :

(a) N_x is an open set of X and $\operatorname{diam}(N_x) < \delta_0$,
(b) $[\ ,\ ] : V^u_{\delta_1}(x) \times V^s_{\delta_1}(x) \to N_x$ is a homeomorphism,
(c) $N_x \supset B_\rho(x)$ where $B_\rho(x) = \{y \in X : d(x,y) \le \rho\}$.

If $f: X \to X$ has POTP, for $\delta > 0$ small we have $W^s_\delta(x) \cap W^u_\delta(y) \neq \emptyset$ when x is very near to y. If, in addition, f is expansive, then we see that $W^s_\delta(x) \cap W^u_\delta(y)$ is a set consisting of single point and it is denoted as $[x,y]$.

Throughout this section let $f: X \to X$ be a TA-homeomorphism of a compact metric space.

Theorem 4.1.1. *The homeomorphism f has a local product structure.*

Proof. Let $e > 0$ be an expansive constant for f and fix $\varepsilon_0 = e/4$. Then there is $0 < \delta_0 < \varepsilon_0$ such that $W^s_{\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(y) = [x, y]$ for $x, y \in X$ with $d(x, y) \le \delta_0$.

First we show that $[\ ,\ ] : \Delta(\delta_0) \to X$ is continuous. Suppose a sequence $\{(x_n, y_n)\}$ of $\Delta(\delta_0)$ converges to $(x, y) \in \Delta(\delta_0)$. Put $z_n = [x_n, y_n]$. Since X is compact, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ that converges to $z \in X$. Since $z_{n_j} \in W^s_{\varepsilon_0}(x_{n_j})$, we have $d(f^i(x_{n_j}), f^i(z_{n_j})) \le \varepsilon_0$ for $i \ge 0$ and n_j, and so $d(f^i(x), f^i(z)) \le \varepsilon_0$ for $i \ge 0$. Thus, $z \in W^s_{\varepsilon_0}(x)$. Similarly, $z \in W^u_{\varepsilon_0}(y)$ and $z = [x, y]$. This shows that $\{z_n\}$ converges to $[x, y]$.

It is clear that $[x, x] = x$ for all $x \in X$. Since $[x, y] \in W^s_{\varepsilon_0}(x)$, we have $[[x, y], z] \in W^s_{2\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(z)$ and then $[[x, y], z] = [x, z]$ by expansivity. Similarly, $[x, [y, z]] = [x, z]$. It is easily checked that $f[x, y] = [f(x), f(y)]$ by uniform continuity.

To conclude the theorem we must prove (B). To do so define $g_1 : X \times \Delta(\delta_0) \to \mathbb{R}$ by

$$g_1(x, (y, z)) = d(x, [y, z])$$

for $x \in X$ and $(y, z) \in \Delta(\delta_0)$. Then g_1 is continuous and $g_1(x, (x, x)) = 0$. By uniform continuity of g_1 we can find $0 < \delta_1 < \delta_0/2$ such that $\operatorname{diam}(\{x, y, z\}) < 2\delta_1$ implies $d(x, [y, z]) < \delta_0/3$. If $(y, z) \in V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$, then $d(x, [y, z]) < \delta_0/3$ and thus $\operatorname{diam}(N_x) < \delta_0$.

To show openness of N_x let $w \in N_x$. Then there are $y \in V^u_{\delta_1}(x)$ and $z \in V^s_{\delta_1}(x)$ with $w = [y, z]$. Since $d(x, w) < \delta_0/3$, we can find maps

$$p_u : B_{\delta_0/3}(w) \to W^u_{\varepsilon_0}(x), \quad p_s : B_{\delta_0/3}(w) \to W^s_{\varepsilon_0}(x)$$

by $p_u(v) = [v, x]$ and $p_s(v) = [x, v]$ for $v \in B_{\delta_0/3}(w)$. They are clearly continuous. Since $w = [y, z]$, we have $p_u(w) = [y, x] = y$ and $p_s(w) = z$. Thus there is a neighborhood $U \subset B_{\delta_0/3}(w)$ of w in X such that $p_u(U) \subset V^u_{\delta_1}(x)$ and $p_s(U) \subset V^s_{\delta_1}(x)$. If $v \in U$, then $v \in N_x$ since $v = [[v, x], [x, v]]$ by expansivity. This implies that N_x is open in X. Therefore, (a) was proved.

(b) is easily checked as follows. Define a map $h : N_x \to V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$ by

$$h(w) = ([w, x], [x, w]), \quad w \in N_x.$$

Obviously, h is continuous and h is the inverse map of $[\ ,\ ]$; i.e. h is a homeomorphism.

To see (c) put

$$g_2(x, y) = \operatorname{diam}\{x, [y, x], [x, y]\}$$

for $(x, y) \in \Delta(\delta_0)$. Then g_2 is a continuous map and there is $0 < \rho < \delta_1$ such that $d(x, y) < \rho$ implies $g_2(x, y) < \delta_1$. This shows that $[y, x] \in V^u_{\delta_1}(x)$ and $[x, y] \in V^s_{\delta_1}(x)$ and therefore $y = [[y, x], [x, y]] \in N_x$. $\square$

Remark 4.1.2. Let $x \in X$. Then $[y,z] \in N_x$ for $y,z \in N_x$.

For $y,z \in N_x$ there exist $u_1,u_2 \in V^u_{\delta_1}(x)$ and $v_1,v_2 \in V^s_{\delta_1}(x)$ such that $y = [u_1,v_1]$ and $z = [u_2,v_2]$. Thus $[y,z] = [[u_1,v_1],[u_2,v_2]] = [u_1,v_2] \in N_x$.

For convention we write

$$D^u_{x,y} = V^u_{\delta_1}(x) \cap N_y, \quad D^s_{x,y} = V^s_{\delta_1}(x) \cap N_y$$

for $x,y \in X$.

Lemma 4.1.3. *For $x,y \in X$ with $d(x,y) < \rho$, $D^\sigma_{x,y}$ is an open neighborhood of x in $V^\sigma_{\delta_1}(x)$ $(\sigma = s,u)$, and the maps*

$$[\ \ ,y] : D^u_{x,y} \to D^u_{y,x}, \quad [y,\ \ ] : D^s_{x,y} \to D^s_{y,x}$$

are homeomorphisms.

Proof. Since N_y is open in X, $D^\sigma_{x,y}$ is open in $V^\sigma_{\delta_1}(x)$. If $d(x,y) \le \rho$, then $x \in B_\rho(y) \subset N_y$ and $x \in D^\sigma_{x,y}$. Thus $D^\sigma_{x,y}$ is an open neighborhood of x in $V^\sigma_{\delta_1}(x)$. Let $z \in D^u_{x,y}$. Since $z \in N_y$, we have $[z,y] \in V^u_{\delta_1}(y)$. Since $z \in V^u_{\delta_1}(x) \subset N_x$ and $y \in B_\rho(x) \subset N_x$, we have $[z,y] \in N_x$. Thus $[z,y] \in D^u_{y,x}$. Similarly we have $[z,x] \in D^u_{x,y}$ for $z \in D^u_{y,x}$. That $[D^u_{x,y},y] = D^u_{y,x}$ follows from the properties of $[\ \ ,\ \ ]$. By (b) the map $[\ \ ,y] : D^u_{x,y} \to D^u_{y,x}$ is a homeomorphism and $[\ \ ,x] : D^u_{y,x} \to D^u_{x,y}$ is its inverse map. The same result is true for $\sigma = s$. $\square$

Lemma 4.1.4. *For $x \in X$*

(a) *$fV^s_{\delta_1}(x) \cap V^s_{\delta_1}(f(x))$ is open in $V^s_{\delta_1}(f(x))$,*

(b) *$f^{-1}V^u_{\delta_1}(x) \cap V^u_{\delta_1}(f^{-1}(x))$ is open in $V^u_{\delta_1}(f^{-1}(x))$.*

Proof. Take $w \in f(N_x) \cap V^s_{\delta_1}(f(x))$. Then $f^{-1}(w) \in N_x$ and $f^{-1}(w) = [y,z]$ for some $y \in V^u_{\delta_1}(x)$ and $z \in V^s_{\delta_1}(x)$. Since $V^u_{\delta_1}(z) \subset W^u_{\delta_0}(z)$, we have $f^{-1}(w) \in W^u_{\delta_0}(z)$.

On the other hand, since $w \in V^s_{\delta_1}(f(x))$, we have $w \in V^s_{\delta_1}(f(x)) \subset W^s_{2\delta_0}(f(z))$. By expansivity we have $f^{-1}(w) = z$ and so $w = f(z) \in fV^s_{\delta_1}(x)$. Therefore

$$f(N_x) \cap V^s_{\delta_1}(f(x)) = fV^s_{\delta_1}(x) \cap V^s_{\delta_1}(f(x)).$$

Since N_x is open in X, we have (a) and similarly (b). $\square$

A subset R of X is called a *rectangle* if $\mathrm{diam}(R) \le \rho$ and $[x,y] \in R$ for $x,y \in R$.

Throughout this section let R denote a rectangle of X.

Remark 4.1.5. Since ρ is chosen such that $\rho < \delta_1$, we have $[x, y] = V^s_{\delta_1}(x) \cap V^u_{\delta_1}(y)$ for $x, y \in \mathrm{cl}(R)$. Thus $\mathrm{cl}(R)$ is a rectangle.

For convention we write

$$V^s(x, R) = V^s_{\delta_1}(x) \cap R, \quad V^u(x, R) = V^u_{\delta_1}(x) \cap R$$

for $x \in X$. Denote as int $V^\sigma(x, R)$ the interior of $V^\sigma(x, R)$ in $V^\sigma_{\delta_1}(x)$ ($\sigma = s, u$) and write

$$\partial V^s(x, R) = V^s(x, R) \setminus \text{ int } V^s(x, R), \quad \partial V^u(x, R) = V^u(x, R) \setminus \text{ int } V^u(x, R).$$

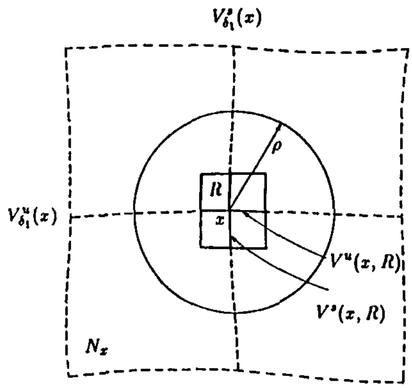

Figure 15

Lemma 4.1.6. *Let $x, y \in R$. Then*
(a) $R = [V^u(x, R), V^s(x, R)]$,
(b) $[\partial V^u(x, R), V^s(x, R)] = [\partial V^u(y, R), V^s(y, R)]$,
(c) $[V^u(x, R), \partial V^s(x, R)] = [V^u(y, R), \partial V^s(y, R)]$.

Proof. Since R is a rectangle, clearly $[V^u(x, R), V^s(x, R)] \subset R$. Let $z \in R$, then

$$[z, x] \in R \cap V^u_{\delta_1}(x) = V^u(x, R)$$

and $[x, z] \in V^s(x, R)$. Thus

$$z = [[z, x], [x, z]] \in [V^u(x, R), V^s(x, R)]$$

(a) was proved.

Since $\mathrm{diam}(R) \leq \rho$ and $y \in R$, we have $R \subset B_\rho(y) \subset N_y$ and so $V^u(x, R) \subset D^u_{x,y}$. Similarly, $V^u(y, R) \subset D^u_{y,x}$. Since $[V^u(y, R), y] = V^u(y, R)$, by Lemma 4.1.3 we have

$$[\partial V^u(y, R), x] = \partial V^u(x, R)$$

and thus

$$\begin{aligned}[\partial V^u(x,R), V^s(x,R)] &= [[\partial V^u(y,R),x],[x,V^s(y,R)]] \\ &= [\partial V^u(y,R), V^s(y,R)]\end{aligned}$$

(b) was proved. (c) is shown in the same way. □

We write

$$\partial^s R = [\partial V^u(x,R), V^s(x,R)], \quad \partial^u R = [V^u(x,R), \partial V^s(x,R)].$$

By Lemma 4.1.6, $\partial^s R$ and $\partial^u R$ do not depend on $x \in R$. Since R is a rectangle, $\partial^s R \subset R$ and $\partial^u R \subset R$. Denote as $\mathrm{int}(R)$ the interior of R in X and write

$$\partial R = R \setminus \mathrm{int}(R).$$

Lemma 4.1.7.
(a) $\mathrm{int}(R) = [\mathrm{int}V^u(x,R), \mathrm{int}V^s(x,R)]$ *for* $x \in R$,
(b) $\partial R = \partial^s R \cup \partial^u R$.

Proof. Since $R \subset N_x$ and N_x is open in X, the interior of R in N_x coincides with $\mathrm{int}(R)$. By Lemma 4.1.3, (a) follows from (B) (b). Thus

$$\begin{aligned}\partial R &= R \setminus \mathrm{int}(R) \\ &= [V^u(x,R), V^s(x,R)] \setminus [\mathrm{int}V^u(x,R), \mathrm{int}V^s(x,R)] \\ &= \partial^s R \cup \partial^u R. \square\end{aligned}$$

Remark 4.1.8. $\mathrm{int}(R)$ is a rectangle. This follows from Lemma 4.1.7 (a).

Lemma 4.1.9. *Let* $x \in R$. *Then* $\mathrm{int}V^\sigma(x,R) = V^\sigma(x, \mathrm{int}(R))$ *for* $\sigma = s, u$.

Proof. Since $\mathrm{int}(R)$ is open in X, $V^s(x,\mathrm{int}(R))$ is open in $V^s_{\delta_1}(x)$. Thus $V^s(x,\mathrm{int}(R)) \subset \mathrm{int}V^s(x,R)$. Let $z \in \mathrm{int}V^s(x,R)$, then $z = [x,z]$ and by Lemma 4.1.7 (a), $z \in \mathrm{int}(R)$. Therefore, $z \in V^s(x,\mathrm{int}(R))$. The same result is true for $\sigma = u$. □

Lemma 4.1.10. *Let* $x \in R$. *Then* $\mathrm{cl}(V^\sigma(x,R)) = V^\sigma(x,\mathrm{cl}(R))$ *for* $\sigma = s, u$.

Proof. Since $\mathrm{cl}(R)$ is a rectangle, we have

$$[x, \mathrm{cl}(R)] \subset \mathrm{cl}(R) \cap V^s_{\delta_1}(x) = V^s(x, \mathrm{cl}(R)).$$

Let $z \in V^s(x,\mathrm{cl}(R))$, then $z \in V^s_{\delta_1}(x)$ and $z = [x,z] \in [x,\mathrm{cl}(R)]$. Thus $V^s(x,\mathrm{cl}(R)) = [x,\mathrm{cl}(R)]$. Since $[x,\mathrm{cl}(R)]$ is closed in X, we have $\mathrm{cl}(V^s(x,R)) \subset V^s(x,\mathrm{cl}(R))$. Similarly, $\mathrm{cl}(V^u(x,R)) \subset V^u(x,\mathrm{cl}(R))$. By Lemma 4.1.6 (a),

$$R = [V^u(x,R), V^s(x,R)]$$

and thus

$$R \subset [\mathrm{cl}(V^u(x,R)), \mathrm{cl}(V^s(x,R))]$$

which is closed in X. Thus

$$\begin{aligned}\mathrm{cl}(R) &\subset [\mathrm{cl}(V^u(x,R)), \mathrm{cl}(V^s(x,R))] \\ &\subset [V^u(x,\mathrm{cl}(R)), V^s(x,\mathrm{cl}(R))] = \mathrm{cl}(R).\end{aligned}$$

Therefore,

$$\begin{aligned}V^s(x,\mathrm{cl}(R)) &= [x,\mathrm{cl}(R)] = [x,[\mathrm{cl}(V^u(x,R)), \mathrm{cl}(V^s(x,R))]] \\ &= [x, \mathrm{cl}(V^s(x,R))] = \mathrm{cl}(V^s(x,R)).\square\end{aligned}$$

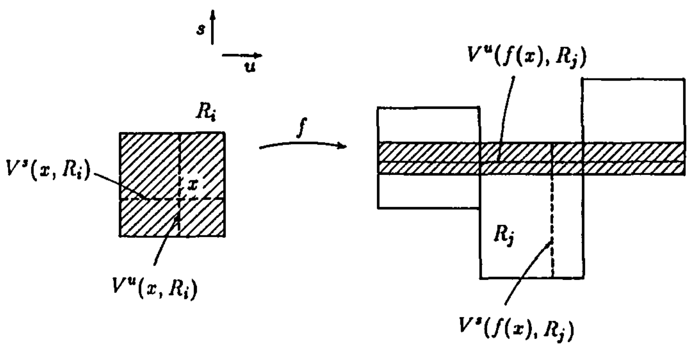

Figure 16

A rectangle R is said to be *proper* if $R = \mathrm{cl}(\mathrm{int}(R))$. A *Markov partition* for a homeomorphism $f: X \to X$ of a compact metric space is a finite cover $\{R_1, \cdots, R_m\}$ of X such that

(a) each R_i is a proper rectangle,
(b) $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$,
(c) let $x \in \mathrm{int}(R_i) \cap f^{-1}\mathrm{int}(R_j)$, then

$$fV^s(x,R_i) \subset V^s(f(x),R_j) \quad \text{and} \quad fV^u(x,R_i) \supset V^u(f(x),R_j).$$

§4.2 Construction of Markov partitions

With the preparation of §4.1 we construct a Markov partition for a TA-homeomorphism $f: X \to X$ of a compact metric space by using the method in Bowen [Bo1].

We say that a subset A of X is γ-*dense* if for every $x \in X$ there exists $y \in A$ such that $d(x, y) < \gamma$.

Choose $0 < \beta < \min\{\rho/2, e\}$ such that $d(x, y) < \beta$ implies $\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} < \delta_1$. Since f has POTP, let $0 < \alpha < \beta/2$ be a number such that any α-pseudo orbit is $\beta/2$ traced by some point of X. We choose $0 < \gamma < \alpha/2$ such that

$$d(x, y) < \gamma \Longrightarrow d(f(x), f(y)) < \alpha/2.$$

Let $P = \{P_1, \cdots, P_r\}$ be a γ-dense finite set of X and define

$$\Sigma(P) = \{(q_j) \in P^{\mathbf{Z}} : d(f(q_j), q_{j+1}) < \alpha,\ j \in \mathbb{Z}\}.$$

Given $q \in \Sigma(P)$ there is a unique $\theta(q) \in X$ which $\beta/2$-traces q. This implies that for any $x \in X$ there is $q \in \Sigma(P)$ with $x = \theta(q)$. Thus $\theta : \Sigma(P) \to X$ is surjective and the diagram

$$\begin{array}{ccc} \Sigma(P) & \xrightarrow{\sigma} & \Sigma(P) \\ \theta\downarrow & & \downarrow\theta \\ X & \xrightarrow[f]{} & X \end{array} \qquad \text{commutes .}$$

Here σ denotes the shift map defined as usual.

Now define

$$T_s = \{\theta(q) : q \in \Sigma(P), q_0 = P_s\}, \quad 1 \le s \le r.$$

Then $\operatorname{diam}(T_s) \le \beta$ and $T = \{T_1, \cdots, T_r\}$ is a cover of X. Fix $1 \le s \le r$. If $x, y \in T_s$, then there exist $q, q' \in \Sigma(P)$ such that

$$x = \theta(q), \ \ y = \theta(q') \ \ \text{and} \quad q_0 = q_0' = P_s.$$

Put $q_j^* = q_j$ for $j \ge 0$, $q_j^* = q_j'$ for $j \le 0$ and write $q^* = (q_j^*) \in \Sigma(P)$. Then we have

$$\begin{aligned} d(f^j(\theta(q^*)), f^j(\theta(q))) < \beta \quad &\text{for} \quad j \ge 0, \\ d(f^j(\theta(q^*)), f^j(\theta(q'))) < \beta \quad &\text{for} \quad j \le 0. \end{aligned}$$

Since $\beta < e$ and

$$[x, y] = [\theta(q), \theta(q')] \in W^s_\beta(\theta(q)) \cap W^u_\beta(\theta(q')),$$

we have $[x, y] = \theta(q^*)$, which is contained in T_s. Therefore, T_s is a rectangle.

We next prove that $\theta : \Sigma(P) \to X$ is continuous. If this is false, then there is $\lambda > 0$ so that for every N we can find $q^N, \tilde{q}^N \in \Sigma(P)$ with $q_j^N = \tilde{q}_j^N$, $|j| \le N$, such that

$$d(\theta(q^N), \theta(\tilde{q}^N)) > \lambda.$$

Let $x_N = \theta(q^N)$ and $y_N = \theta(\tilde{q}^N)$. Then

$$d(f^j(x_N), f^j(y_N)) < 2\beta, \quad |j| \le N.$$

By taking a subsequence we have $x_N \to x$ and $y_N \to y$ as $N \to \infty$. Then $d(f^j(x), f^j(y)) \le 2\beta$ for all $j \in \mathbb{Z}$ and so $x = y$. On the other hand, since $d(x,y) \ge \lambda$, we have a contradiction.

Continuity of θ ensures that T_s is closed in X. Let $x \in X$ and define

$$T(x) = \{T_j \in T : x \in T_j\},$$
$$T^*(x) = \{T_k \in T : T_k \cap T_j \neq \emptyset \text{ for some } T_j \in T(x)\}.$$

The set $Z = X \setminus \bigcap_{j=1}^{r} \partial T_j$, where each ∂T_j is the boundary in X, is open in X. Here we define

$$Z^* = \{x \in X : V_{\delta_1}^s(x) \cap \partial^s T_k = \emptyset, V_{\delta_1}^u(x) \cap \partial^u T_k = \emptyset \text{ for all } T_k \in T^*(x)\}.$$

Lemma 4.2.1. *Z^* is dense in X.*

Proof. For $x \in X$ we define

$$\partial_x^s = \bigcup\{\partial^s T_k : T_k \in T^*(x)\}, \quad \partial_x^u = \bigcup\{\partial^u T_k : T_k \in T^*(x)\}.$$

Since $\operatorname{diam}(T_k) \le \beta$, we have

$$\bigcup\{T_k : T_k \in T^*(x)\} \subset B_\rho(x) \subset N_x$$

and thus

$$[\partial_x^s, x] = \bigcup_{T_k \in T^*(x)} [\partial^s T_k, x] = \bigcup_{T_k \in T^*(x)} [\partial V^u(y_k, T_k), x] \subset V_{\delta_1}^u(x)$$

where $y_k \in T_k$. Since $V^u(y_k, T_k) \subset D_{y_k,x}^u$, by Lemma 4.1.3 we have $[\partial V^u(y_k, T_k), x] \subset D_{x,y_k}^u$. But $[\partial V^u(y_k, T_k), x]$ is nowhere dense in D_{x,y_k}^u and so is $V_{\delta_1}^u(x)$. Thus $[\partial_x^s, x]$ is nowhere dense in $V_{\delta_1}^u(x)$.

Let $x \in Z$, then there exists an open neighborhood $U_x \subset N_x$ of x in X such that $T(x) = T(y)$ for any $y \in U_x$, from which $\partial_y^\sigma = \partial_x^\sigma$ for $\sigma = s, u$. Let us define

$$U_x' = U_x \bigcap [V_{\delta_1}^u(x) \setminus [\partial_x^s, x], V_{\delta_1}^s(x) \setminus [x, \partial_x^u]].$$

By (B)(b) we have that U'_x is dense in U_x. For $y \in U'_x$ there exist $y_1 \in V^u_{\delta_1}(x) \setminus [\partial^s_x, x]$ and $y_2 \in V^s_{\delta_1}(x) \setminus [x, \partial^u_x]$ such that $y = [y_1, y_2]$. Thus $V^s_{\delta_1}(y) \cap \partial^s_x = \emptyset$. Indeed, if $z \in V^s_{\delta_1}(y) \cap \partial^s_x$, then

$$y_1 = [y, x] = [[y, z], x] = [z, x] \in [\partial^s_x, x],$$

thus contradicting.

Similarly, $V^u_{\delta_1}(y) \cap \partial^u_x = \emptyset$. Therefore, $y \in Z^*$ and so $U'_x \subset Z^* \neq \emptyset$ for $x \in Z$. Since U_x is an open neighborhood of x, Z^* is dense in Z. Since Z is dense in X, Z^* is dense in X. □

For $T_j, T_k \in T$ with $T_j \cap T_k \neq \emptyset$ we define

$$\begin{aligned}
T^1_{j,k} &= \{x \in T_j : V^u_{\delta_1}(x) \cap T_k \neq \emptyset,\ V^s_{\delta_1}(x) \cap T_k \neq \emptyset\},\\
T^2_{j,k} &= \{x \in T_j : V^u_{\delta_1}(x) \cap T_k \neq \emptyset,\ V^s_{\delta_1}(x) \cap T_k = \emptyset\},\\
T^3_{j,k} &= \{x \in T_j : V^u_{\delta_1}(x) \cap T_k = \emptyset,\ V^s_{\delta_1}(x) \cap T_k \neq \emptyset\},\\
T^4_{j,k} &= \{x \in T_j : V^u_{\delta_1}(x) \cap T_k = \emptyset,\ V^s_{\delta_1}(x) \cap T_k = \emptyset\}.
\end{aligned}$$

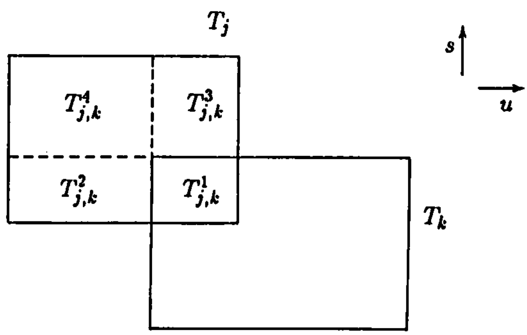

Figure 17

Remark 4.2.2. If $T_j \cap T_k \neq \emptyset$ then $T^1_{j,k} = T_j \cap T_k$ and $T_j = \bigcup_{n=1}^4 T^n_{j,k}$.

Remark 4.2.3. Each $T^n_{j,k}$ is a rectangle. Indeed, it is clear that $T^1_{j,k}$ is a rectangle. To see that $T^2_{j,k}$ is a rectangle, let $x, y \in T^2_{j,k}$. Then $z \in V^u_{\delta_1}(y) \cap T_k \neq \emptyset$ and $V^u_{\delta_1}(x) \cap T_k = \emptyset$. Since $z = [z, y] = [z, [x, y]]$ and $d(z, [x, y]) \leq 2\beta < \delta_1$, we have $z \in V^u_{\delta_1}([x, y]) \cap T_k \neq \emptyset$. If $z' \in V^s_{\delta_1}([x, y]) \cap T_k \neq \emptyset$, then $z' = [[x, y], z'] = [x, z']$. Since $d(z', x) \leq 2\beta < \delta_1$, we have $z' \in V^s_{\delta_1}(x) \cap T_k \neq \emptyset$, a contradiction. Thus $V^s_{\delta_1}([x, y]) \cap T_k = \emptyset$ and $[x, y] \in T^2_{j,k}$. This implies that $T^2_{j,k}$ is a rectangle. It is checked by the similar way that $T^3_{j,k}$ and $T^4_{j,k}$ are rectangles.

Lemma 4.2.4. *For $T_j, T_k \in T$ with $T_j \cap T_k \neq \emptyset$*

$$\begin{aligned}&\mathrm{int}(T^n_{j,k}) = \\ &\{x \in T^n_{j,k} : V^s_{\delta_1}(x) \cap (\partial^s T_j \cup \partial^s T_k) = \emptyset,\ V^u_{\delta_1}(x) \cap (\partial^u T_j \cup \partial^u T_k) = \emptyset\}\end{aligned}$$

for $1 \leq n \leq 4$.

Proof. Take $x \in \mathrm{int}(T^n_{j,k})$ and $z \in V^s_{\delta_1}(x) \cap \partial^s T_j \neq \emptyset$. By the definition of $\partial^s T_j$ there exist $z_1 \in \partial V^u(x, T_j)$ and $z_2 \in V^s(x, T_j)$ such that $z = [z_1, z_2]$. Since

$$x = [z_2, z_1] \in \partial V^u(x, T_j) \cap V^s(x, T_j),$$

we have

$$x = [[z_1, z_2], [z_2, z_1]] = z_1 \in \partial^s T_j.$$

This contradicts $x \in \mathrm{int}(T_j)$ by Lemma 4.1.7 (b).

If $z \in V^s_{\delta_1}(x) \cap \partial^s T_k$, then $z = [z_1, z_2]$ for some $z_1 \in \partial V^u(z, T_k)$ and some $z_2 \in V^s(z, T_k)$. Thus

$$z = [[z_1, z_2], [z_2, z_1]] = z_1$$

and so $z \in \partial V^u(z, T_k)$. Since $d(x, z) \leq \rho$, by Lemma 4.1.3 there is a homeomorphism

$$[\ , z] : D^u_{x,z} \to D^u_{z,x}$$

and then $[x, z] = z$ since $z \in V^s_{\delta_1}(x)$. Note that $D^u_{x,z} \cap T^n_{j,k}$ is a neighborhood of x in $D^u_{x,z}$, and that $V^u(z, T_k)$ has no interior points in T_k. Then we can find the following

$$v \in D^u_{x,z} \cap T^n_{j,k} \quad \text{with } [v, z] \notin V^u(z, T_k).$$

For this v we have $V^s_{\delta_1}(v) \cap T_k = \emptyset$. Indeed, if $w \in V^s_{\delta_1}(v) \cap T_k \neq \emptyset$, then $[v, z] = [[w, v], z] \in V^u(z, T_k)$. But $[v, z] \notin V^u(z, T_k)$, thus contradicting.

Therefore, $V^s_{\delta_1}(x) \cap (\partial^s T_j \cap \partial^s T_k) = \emptyset$ when $x \in \mathrm{int}(T^n_{j,k})$. By the similar way we can prove that

$$x \in \mathrm{int}(T^n_{j,k}) \Longrightarrow V^u_{\delta_1}(x) \cap (\partial^u T_j \cap \partial^u T_k) = \emptyset.$$

Suppose that

$$x \in \partial^s T^n_{j,k} \quad \text{and} \quad V^s_{\delta_1}(x) \cap (\partial^s T_j \cup \partial^s T_k) = \emptyset.$$

Then $x \in \partial V^u(x, T^n_{j,k})$. If $x \in \partial V^u(x, T_j)$, then

$$x \in [\partial V^u(x, T_j), V^s(x, T_j)] = \partial^s T_j$$

and so $V^s_{\delta_1}(x) \cap \partial^s T_j \neq \emptyset$, thus contradicting. Therefore, $x \in \mathrm{int}(V^u(x, T_j))$.

Since $d(x, y_k) \leq \rho$ for $y_k \in T_k$, by Lemma 4.1.3 there is a homeomorphism $[\ \ , y_k] : D^u_{x,y_k} \to D^u_{y_k,x}$. Since $V^s_{\delta_1}(x) \cap \partial^s T_k = \emptyset$ and $\partial V^u(y_k, T_k) \subset \partial^s T_k$, we have $V^s_{\delta_1}(x) \cap \partial V^u(y_k, T_k) = \emptyset$ and thus $[x, y_k] \notin \partial V^u(y_k, T_k)$. Since T_k is closed, $V^u(y_k, T_k)$ is closed in X by Lemma 4.1.10. Since $V^u(y_k, T_k) \subset D^u_{y_k,x}$, $\partial V^u(y_k, T_k)$ is the boundary of $V^u(y_k, T_k)$ in $D^u_{y_k,x}$. Thus there is a neighborhood $U^u_x \subset V^u(x, T_j)$ of x in D^u_{x,y_k} such that

$$[U^u_x, y_k] \subset V^u(y_k, T_k) \quad \text{or} \quad [U^u_x, y_k] \subset D^u_{y_k,x} \setminus V^u(y_k, T_k),$$

namely $V^s_{\delta_1}(v) \cap T_k \neq \emptyset$ for all $v \in U^u_x$ or $V^s_{\delta_1}(v) \cap T_k = \emptyset$ for all $v \in U^u_x$.

On the other hand, since $U^u_x \subset V^u(x, T_j)$, we have that either $V^u_{\delta_1}(v) \cap T_k \neq \emptyset$ for all $v \in U^u_x$, or $V^u_{\delta_1}(v) \cap T_k = \emptyset$ for all $v \in U^u_x$. Therefore, $U^u_x \subset T^n_{j,k}$ which contradicts $x \in \partial V^u(x, T^n_{j,k})$. It was proved that for $x \in T^n_{j,k}$

$$V^s_{\delta_1}(x) \cap (\partial^s T_j \cup \partial^s T_k) = \emptyset \Longrightarrow x \notin \partial^s T^n_{j,k}.$$

Similarly we can prove that $V^u_{\delta_1}(x) \cap (\partial^u T_j \cup \partial^u T_k) = \emptyset$ implies $x \notin \partial^u T^n_{j,k}$. Therefore the conclusion is obtained by Lemma 4.1.7 (b) □

Remark 4.2.5. Let $x \in Z^*$. If $T_j \cap T_k \neq \emptyset$ for $T_j \in T(x)$ and $T_k \in T$, by Lemma 4.2.4 we have $x \in \mathrm{int}(T^n_{j,k})$ for some n.

For $x \in Z^*$ define

$$R(x) = \bigcap \{\mathrm{int}(T^n_{j,k}) : T_j \cap T_k \neq \emptyset \text{ for } T_j \in T(x), T_k \in T \text{ and } x \in \mathrm{int}(T^n_{j,k})\}.$$

Remark 4.2.6. $R(x)$ is an open rectangle and for $y \in R(x)$, $R(x) = R(y)$ and $T(x) = T(y)$.

The former is clear. If $T_k \in T(y)$ and $T_j \in T(x)$, then $y \in T_j \cap T_k \neq \emptyset$ and so $y \in R(x) \subset T^1_{j,k}$ and $T_k \in T(x)$. Therefore, $T(y) \subset T(x)$. It is clear that $T(x) \subset T(y)$. Thus $T(x) = T(y)$. By definition, $R(x) = R(y)$ for $y \in R(x)$.

Since T is finite, so is $\{R(x) : x \in Z^*\}$. Thus there exist $x_1, \cdots, x_m \in Z^*$ such that

$$Z^* = R(x_1) \cup \cdots \cup R(x_m)$$

is a disjoint union.

Lemma 4.2.7. *Let $x \in R(x_i) \cap f^{-1}R(x_j)$ for $i \neq j$. Then*

(a) $$fV^s(x, R(x_i)) \subset V^s(f(x), R(x_j)),$$

(b) $$fV^u(x, R(x_i)) \supset V^u(f(x), R(x_j)).$$

Proof. Fix $x \in R(x_i) \cap f^{-1}R(x_j)$ and let $v \in X$. Then $v = \theta(q)$ for some $q \in \Sigma(P)$. Suppose $q_0 = P_s$ and $q_1 = P_t$ for $P_s, P_t \in P = \{P_1, \cdots, P_r\}$.

If $w \in V^s(v, T_s)$, then there is $q' \in \Sigma(P)$ such that $w = \theta(q')$ and $q'_0 = P_s$. Thus

$$w = [v, w] = [\theta(q), \theta(q')] = \theta(qq')$$

so that

$$f(w) = f \circ \theta(qq') = \theta \circ \sigma(qq') \in T_t.$$

Here $qq' = (\cdots, q'_{-2}, q'_{-1}, q_0, q_1, q_2, \cdots)$. Since $w = [v, w] \in V^s_{\delta_1}(v)$ and $d(v, w) < \beta$, we have $d(f(v), f(w)) < \delta_1$ and so

$$f(w) \in V^s_{\delta_1}(f(v)) \cap T_t = V^s(f(v), T_t).$$

Since w is arbitrary, we have

$$(1) \qquad fV^s(v, T_s) \subset V^s(f(v), T_t)$$

and in the similar way

$$(2) \qquad fV^u(v, T_s) \supset V^u(f(v), T_t).$$

Let $y \in V^s(x, R(x_i))$. Then $y \in V^s_{\delta_1}(x)$ and $R(x) = R(y) = R(x_i)$. We first prove that $T(f(x)) = T(f(y))$. If $f(x) \in T_j$ and $f(x) = \theta \circ \sigma(q)$ where $q_1 = P_j$ and $q_0 = P_s$, then $x = \theta(q) \in T_s$. By (1) we have

$$f(y) \in fV^s(x, T_s) \subset V^s(f(x), T_j)$$

and thus $f(y) \in T_j$. Similarly, $f(x) \in T_j$ when $f(y) \in T_j$.

Next we prove that if $T_j \in T(f(x)) = T(f(y))$ and $T_j \cap T_k \neq \emptyset$ for $T_k \in T$ then $f(x), f(y) \in T^n_{j,k}$. Since $f(y) \in V^s(f(x), T_j)$, $f(x)$ and $f(y)$ belong to $T^1_{j,k} \cup T^3_{j,k}$ or $T^2_{j,k} \cup T^4_{j,k}$. Suppose $V^u_{\delta_1}(f(y)) \cap T_k = \emptyset$ and $V^u_{\delta_1}(f(x)) \cap T_k \neq \emptyset$. Note that this assumption is equivalent to

$$(3) \qquad V^u(f(y), T_j) \cap T_k = \emptyset, \quad V^u(f(x), T_j) \cap T_k \neq \emptyset.$$

Take $f(z) \in V^u(f(x), T_j) \cap T_k$. Let $f(x) = \theta \circ \sigma(q)$ for $q \in \Sigma(P)$ with $q_1 = P_j$ and $q_0 = P_s$. By (2) we have

$$f(z) \in V^u(f(x), T_j) \subset fV^u(x, T_s).$$

Let $f(z) = \theta \circ \sigma(q')$ for $q' \in \Sigma(P)$ with $q'_1 = P_k$ and $q'_0 = P_t$. Then $z \in T_t$ and $z \in T_s \cap T_t \neq \emptyset$. Since $x \in T_s$, we have $T_s \in T(x) = T(y)$. Since $z \in V^u(x, T_s) \cap T_t$ and x, y belong to the same set $T^n_{s,t}$, it follows that $V^u(y, T_s) \cap T_t \neq \emptyset$. Thus there is $z' \in V^u(y, T_s) \cap T_t$ and then

$$z'' = [z, y] = [z, z'] \in V^s(z, T_t)$$

and by (1)

$$f(z'') \in V^s(f(z), T_k).$$

Since $f(z), f(y) \in T_j$, we have

$$f(z'') = [f(z), f(y)] \in V^u(f(y), T_j)$$

which contradicts (3).

Now we denote

$$R(f(x))' = \bigcap \{T^n_{j,k} : T_j \cap T_k \neq \emptyset \text{ for } T_j \in T(f(x)), T_k \in T \text{ and } f(x) \in T^n_{j,k}\}.$$

Then, $\mathrm{int}(R(f(x))') = R(f(x)) = R(x_j)$ since $f(x) \in R(x_j)$. Since $\mathrm{diam}(R(x_j)) \leq \beta$, we have $fV^s(x, R(x_j)) \subset V^s_{\delta_1}(f(x))$ and thus

$$fV^s(x, R(x_j)) \subset V^s(f(x), R(f(x))').$$

Lemma 4.1.4 (a) ensures that $fV^s_{\delta_1}(x) \cap V^s_{\delta_1}(f(x))$ is open in $V^s_{\delta_1}(f(x))$ and $R(x_j)$ is also open in X. Thus $fV^s(x, R(x_i))$ is open in $V^s_{\delta_1}(f(x))$. Therefore, by Lemma 4.1.9

$$fV^s(x, R(x_i)) \subset \mathrm{int}(V^s(f(x), R(x_j))) = V^s(f(x), R(x_j)).$$

Similarly we can prove that $V^u(f(x), R(x_j)) \subset fV^u(x, R(x_i))$. □

With the above preparations we have the following theorem.

Theorem 4.2.8. *Let $f: X \to X$ be a homeomorphism of a compact metric space. If f is a TA-homeomorphism, then there exists in X Markov partitions with arbitrarily small diameter.*

Proof. Let $R_j = \mathrm{cl}(R(x_j))$ for $1 \leq j \leq m$. Then $\mathcal{R} = \{R_1, \cdots, R_m\}$ is a Markov partition of X. Indeed, $\mathcal{R}$ is a cover of X, and each R_j is a proper rectangle with $\mathrm{diam}(R_j) \leq \beta$ and $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$. The remainder of the proof is to check that if $x \in \mathrm{int}(R_i) \cap f^{-1}\mathrm{int}(R_j)$ then $fV^s(x, R_i) \subset V^s(f(x), R_j)$ and $fV^u(x, R_i) \supset V^u(f(x), R_j)$.

By Lemma 4.2.7

$$fV^s(x, R(x_i)) \subset V^s(f(x), R(x_j))$$

and by Lemma 4.1.10

$$fV^s(x, R_i) \subset V^s(f(x), R_j)$$

and then for $y \in R_i \cap f^{-1}(R_j)$

$$\begin{aligned} fV^s(y, R_i) &= f[y, V^s(x, R_i)] \\ &= [f(y), fV^s(x, R_i)] \subset [f(y), V^s(f(x), R_j)] \\ &= V^s(f(y), R_j). \end{aligned}$$

The proof of one half has been given. The proof of the other half is similar and is therefore omitted. □

Remark 4.2.9. Let $f: X \to X$ be a homeomorphism of a compact metric space and let $\Omega(f)$ be the nonwandering set of f in X. If f is a TA-homeomorphism, then $f_{|\Omega(f)}$ has POTP by Theorem 3.1.8 and it is expansive. Thus $\Omega(f)$ has Markov partitions by Theorem 4.2.8. $\Omega(f)$ is expressed as the finite disjoint union $\Omega(f) = \bigcup \Omega_j$ of basic sets Ω_j which are open and closed in $\Omega(f)$ (see Theorems 3.1.2 and 3.1.11). Thus $f_{|\Omega_j} : \Omega_j \to \Omega_j$ is a TA-homeomorphism. Therefore, each Ω_j has Markov partitions.

§4.3 Symbolic dynamics

Let $f: X \to X$ be a TA-homeomorphism of a compact metric space.

Let $\mathcal{R} = \{R_1, \cdots, R_m\}$ be a Markov partition of a basic set Ω_s and define the transition matrix $A = A(\mathcal{R})$ by

$$A_{ij} = \begin{cases} 1 & \text{if } \operatorname{int}(R_i) \cap f^{-1}(\operatorname{int}(R_j)) \neq \emptyset \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 4.3.1. *Suppose $x \in R_i \cap f^{-1}(R_j)$ and $A_{ij} = 1$. Then $fV^s(x, R_i) \subset V^s(f(x), R_j)$ and $fV^u(x, R_i) \supset V^u(f(x), R_j)$.*

Proof. This is the same as the last part of the proof of Theorem 4.2.8. □

Lemma 4.3.2. *Let $D \subset V^s_{\delta_1}(x) \cap \Omega_s$ and $C \subset V^u_{\delta_1}(x) \cap \Omega_s$. Then the rectangle $[C, D]$ is proper if and only if $D = \operatorname{cl}(\operatorname{int}(D))$ and $C = \operatorname{cl}(\operatorname{int}(C))$ as subsets of $V^s_{\delta_1}(x) \cap \Omega_s$ and $V^u_{\delta_1}(x) \cap \Omega_s$ respectively.*

Proof. This follows from (B) (b). □

Let R, S be two rectangles in Ω_s. Then S is called a *u-subrectangle* of R if
(a) $S \neq \emptyset, S \subset R$ and S is proper,
(b) $V^u(y, S) = V^u(y, R)$ for $y \in S$.

Lemma 4.3.3. *Suppose S is a u-subrectangle of R_i and $A_{ij} = 1$. Then $f(S) \cap R_j$ is a u-subrectangle of R_j.*

Proof. Take $x \in R_i \cap f^{-1}(R_j)$ and put $D = V^s(x, R_i) \cap S$. Since S is a u-subrectangle, we have

$$S = \bigcup \{V^u(y, R_i) : y \in D\} = [V^u(x, R_i), D].$$

Since S is proper and nonempty, by Lemma 4.3.2 we have $\emptyset \neq D = \operatorname{cl}(\operatorname{int}(D))$ and

$$f(S) \cap R_j = \bigcup \{fV^u(y, R_i) \cap R_j : y \in D\}.$$

Since $A_{ij} = 1$, we have that $f(y) \in R_j$ for $y \in D$ and $y \in R_i \cap f^{-1}(R_j)$. Thus

$$fV^u(y, R_i) \cap R_j = V^u(f(y), R_j)$$

by Lemma 4.3.1 and so

$$f(S) \cap R_j = \bigcup \{V^u(y, R_j) : y \in f(D)\} = [V^u(f(x), R_j), f(D)].$$

Since $R_j = [V^u(f(x), R_j), V^s(f(x), R_j)]$ by Lemma 4.1.6 and R_j is proper, $V^u(f(x), R_j)$ is also proper. Since f maps $V^s(x, R_i)$ homeomorphically onto a neighborhood of $V^s(f(x), R_j)$, we have $f(D) = \mathrm{cl}(\mathrm{int}(f(D)))$ and so $f(S) \cap R_j$ is proper by Lemma 4.3.2. Since $f(D) \neq \emptyset$, we have $f(D) \cap R_j \neq \emptyset$. If $y'' \in f(S) \cap R_j$, then $y'' \in V^u(y', R_j)$ for some $y' \in f(D)$ and thus

$$V^u(y'', R_j) = V^u(y', R_j) \subset f(S) \cap R_j.$$

Therefore

$$V^u(y'', R_j) = V^u(y'', f(S) \cap R_j)$$

for $y'' \in f(S) \cap R_j$. This implies that $f(S) \cap R_j$ is a u-subrectangle of R_j. □

Theorem 4.3.4. *Let Σ_A be the compact subset of $Y_m^{\mathbb{Z}}$ defined by*

$$\Sigma_A = \{x = (x_i) : A_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}$$

and $\sigma : \Sigma_A \to \Sigma_A$ be the shift map defined by $\sigma((x_i)) = (x_{i+1})$ as usual. For each $a \in \Sigma_A$, the set $\bigcap \{f^{-j}(R_{a_j}) : j \in \mathbb{Z}\}$ consists of a single point which is denoted by $\pi(a)$. The map $\pi : \Sigma_A \to \Omega_s$ is a continuous surjection such that the diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Omega_s & \xrightarrow[f]{} & \Omega_s \end{array} \qquad \textit{commutes ,}$$

and π is injective on the Baire set $Y = \Omega_s \setminus \bigcup \{f^j(\partial \mathcal{R}) : j \in \mathbb{Z}\}$.

Proof. If $A_{a_i a_{i+1}} = 1$ for $1 \le i \le n-1$, by using Lemma 4.3.3 inductively we have that

$$\bigcap_{j=1}^{n} f^{n-j}(R_{a_j}) = R_{a_n} \bigcap f(\bigcap_{j=1}^{n-1}(R_{a_j}))$$

is a u-subrectangle of R_{a_n}. Thus

$$K_n(a) = \bigcap \{f^{-j}(R_{a_j}) : -n \le j \le n\}$$

is nonempty. Since $K_n(a) \supset K_{n+1}(a) \supset \cdots$, we have

$$K(a) = \bigcap_{-\infty}^{\infty} f^{-j}(R_{a_j}) = \bigcap_{1}^{\infty} K_n(a) \neq \emptyset.$$

If $x, y \in K(a)$, then $f^j(x), f^j(y) \in R_{a_j}$ are close for all $j \in \mathbb{Z}$ and so $x = y$ by expansivity. Thus $K(a)$ is a single point. Here we define $\pi(a) = K(a)$.

Since $K(\sigma(a)) = \bigcap f^{-j}(R_{a_{j+1}}) = f(\bigcap f^{-j}(R_{a_j})) = f(K(a))$, we have $\pi \circ \sigma = f \circ \pi$. Continuity of π is easily checked. Since $\partial\mathcal{R}$ is nowhere dense, Y is a Baire set.

For $x \in Y$ take $a_j \in Y_m$ with $f^j(x) \in R_{a_j}$ for all $j \in \mathbb{Z}$. Then $f^j(x) \in \text{int}(R_{a_j})$ and so $A_{a_j a_{j+1}} = 1$. Thus $a = (a_j) \in \Sigma_A$ and $x = \pi(a)$. If $x = \pi(b)$, then $f^j(x) \in R_{b_j}$ and $b_j = a_j$ since $f^j(x) \notin \partial\mathcal{R}$. Thus π is injective on Y. Since $\pi(\Sigma_A)$ is a compact subset of Ω_s containing Y, we have $\pi(\Sigma_A) = \Omega_s$. □

Theorem 4.3.5. *The shift map $\sigma : \Sigma_A \to \Sigma_A$ is topologically transitive. If $f_{|\Omega_s}$ is topologically mixing, so is $\sigma : \Sigma_A \to \Sigma_A$.*

Proof. Let U and V be nonempty open subsets of Σ_A. Then there exist $a, b \in \Sigma_A$ and $N > 0$ such that

$$U \supset U_1 = \{x \in \Sigma_A : x_i = a_i, |i| \leq N\},$$
$$V \supset V_1 = \{x \in \Sigma_A : x_i = b_i, |i| \leq N\},$$

and

$$\emptyset \neq \text{int}(K_N(a)) = \bigcap_{-N}^{N} f^{-j}(\text{int}(R_{a_j})) = U_2,$$
$$\emptyset \neq \text{int}(K_N(b)) = \bigcap_{-N}^{N} f^{-j}(\text{int}(R_{b_j})) = V_2.$$

If $\pi(x) \in U_2$, then $f^j \circ \pi(x) \in R_{x_j}$ and $f^j \circ \pi(x) \in \text{int}(R_{a_j})$, which implies $x_j = a_j$ for $|j| \leq N$. Thus, $\pi^{-1}(U_2) \subset U_1$. Similarly, $\pi^{-1}(V_2) \subset V_1$. Since $f_{|\Omega_s}$ is topologically transitive, we have $f^n(U_2) \cap V_2 \neq \emptyset$ for some n. Then

$$\begin{aligned}\emptyset \neq \pi^{-1}(f^n(U_2) \cap V_2) &= \pi^{-1}(f^n(U_2)) \cap \pi^{-1}(V_2) \\ &\subset \sigma^n(U) \cap V.\end{aligned}$$

By the same argument we can prove that σ is topologically mixing if $f_{|\Omega_s}$ is. □

Theorem 4.3.6 (Bowen [Bo3]). *Under the notations and assumptions of Theorem 4.3.4, there exists an integer d such that $\pi : \Sigma_A \to \Omega_s$ is at most d-to-one map, i.e.* $\text{card}(\pi^{-1}(x)) \leq d$ *for all $x \in \Omega_s$.*

Remark 4.3.7. $\Sigma_A \ni x$ is a periodic point if and only if $\pi(x)$ is.

Since $f \circ \pi = \pi \circ \sigma$, it is clear that if x is a periodic point then $\pi(x)$ is. If $f^n(y) = y$ where $y = \pi(a)$, then $\pi^{-1}(y) \supset \{a, \sigma^n(a), \sigma^{2n}(a), \cdots\}$. Since $\pi^{-1}(y)$ is finite by Theorem 4.3.6, a is a periodic point of σ.

For the proof of Theorem 4.3.6 we need some notations and lemmas.

As above let $\mathcal{R} = \{R_1, \cdots, R_m\}$ be a Markov partition for $f|_{\Omega_s}$. For $R_i, R_j \in \mathcal{R}$ we define

$$t(R_i, R_j) = \begin{cases} 1 & \text{if } f(\text{int}(Ri)) \cap \text{int}(R_j) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Sigma = \{(R_{n_i})_{-\infty}^{\infty} : R_{n_i} \in \mathcal{R} \text{ and } t(R_{n_i}, R_{n_{i+1}}) = 1, i \in \mathbb{Z}\}.$$

Then Σ is a compact metric space under the topology defined as usual. The shift $\sigma : \Sigma \to \Sigma$ defined by $\sigma((R_{n_i}))_i = R_{n_{i+1}}$, $i \in \mathbb{Z}$, is a homeomorphism. Obviously (Σ, σ) is topologically conjugate to (Σ_A, σ). Thus we identify (Σ_A, σ) with (Σ, σ) under the conjugacy.

Define $\text{Set}(x) = \{R \in \mathcal{R} : x \in R\}$ for $x \in \Omega_s$.

Lemma 4.3.8.
(a) $\text{Set}(x) = \{R \in \mathcal{R} : R = x_0$ *for some* $(x_i) \in \pi^{-1}(x)\}$,
(b) $\text{Set}([x, y]) \supset \text{Set}(x) \cap \text{Set}(y)$.

Proof. (a): If $x = \pi((x_i))$, then $x_0 = R \in \text{Set}(x)$. That there is such a (x_i) for any member of $\text{Set}(x)$ was proved in Theorem 4.3.4.

(b): If $x, y \in R$, then $[x, y] \in R$ since R is a rectangle. □

Lemma 4.3.9. *Let z be near enough x so that $[x, z]$ is defined.*
(a) *If $z \notin \partial^u \mathcal{R}$, for $\delta > 0$ there exists*

$$z' \in V_\delta^s(z) \setminus \bigcup_{-\infty}^{\infty} f^n(\partial^u \mathcal{R})$$

such that $\text{Set}(z') = \text{Set}(z)$ *and* $[x, z'] \notin \partial^u \mathcal{R}$.
(b) *If $z \notin \partial^s \mathcal{R}$, for $\delta > 0$ there exists*

$$z' \in V_\delta^u(z) \setminus \bigcup_{-\infty}^{\infty} f^n(\partial^s \mathcal{R})$$

such that $\text{Set}(z') = \text{Set}(z)$ *and* $[z', x] \notin \partial^s \mathcal{R}$.

Proof. Remember that $\partial^u \mathcal{R} = \cup\{\partial^u R : R \in \mathcal{R}\}$ where

$$\partial^u R = [V^u(x, R), \partial V^s(x, R)],$$
$$\partial V^s(x, R) = V^s(x, R) \setminus \text{int}(V^s(x, R)).$$

Since $\partial V^s(x, R)$ is closed and nowhere dense in $V_\delta^s(x)$, we see that if $z \notin \partial^u \mathcal{R}$ then

$$V = \bigcap_{R \in \text{Set}(z)} \text{int}(V^s(z, R)) \setminus \bigcup_{R \notin \text{Set}(z)} R$$

is an open neighborhood of z in $V^s_\delta(z)$. Obviously, $\mathrm{Set}(z') = \mathrm{Set}(z)$ for all $z' \in V$ and $V \cap \partial^u\mathcal{R} = \emptyset$. Since $\partial^u\mathcal{R} \cap V^s_\delta(x)$ is contained in a finite union of nowhere dense set of the form $\partial V^s(y, R)$, it is clear that $\partial^u\mathcal{R} \cap V^s_\delta(x)$ is nowhere dense in $V^s_\delta(x)$. Since each $f^{-n}(V^s_\delta(z))$ can be covered by finitely many $V^s_\delta(x)$, we have that

$$f^n(\partial^u\mathcal{R}) \cap V^s_\delta(z) \subset f^n(\cup_x V^s_\delta(x) \cap \partial^u\mathcal{R})$$

is nowhere dense in $V^s_\delta(z)$. By Baire's theorem, $\cup_n f^n(\partial^u\mathcal{R}) \cap V^s_\delta(z)$ is nowhere dense in $V^s_\delta(z)$. By Lemma 4.1.3, $[x, \ \ ] : V^s_\delta(z) \cap N_x \to V^s_\delta(x) \cap N_z$ is a homeomorphism. Therefore, $[x, z'] \notin \partial^u\mathcal{R}$ for some $z' \in V^s_\delta(z) \setminus \cup_n f^n(\partial^u\mathcal{R})$.
□

Lemma 4.3.10. *Let $R_1, R_2, R_3 \in \mathcal{R}$. Suppose that*

$$R_1 \neq R_2, \quad x \in R_1 \cap R_2 \cap f^{-1}(R_3),$$
$$A_{13} = 1, \quad A_{23} = 1.$$

Then $x \in \partial^u\mathcal{R}$.

Proof. Lemma 4.3.1 gives

$$V = V^u(x, R_1) \cap V^u(x, R_2) \supset f^{-1}V^u(f(x), R_3).$$

Since $V^u(f(x), R_3)$ contains an open subset contained in every neighborhood of $f(x)$ in $V^u_{\delta_1}(f(x))$, V contains an open subset contained in every neighborhood of x in $V^u_{\delta_1}(x)$. As the proof of Lemma 4.3.9 we can find $x' \in V$ such that $x' \notin \partial^s\mathcal{R}$. Since $R_1 \neq R_2$, we have $V \subset R_1 \cap R_2 \subset \partial R_1$. Since $\partial R_1 = \partial^u R_1 \cup \partial^s R_1$, we have $x' \in \partial^u R_1$ and then $V^u(x', R_1) \subset \partial^u R_1 \subset \partial^u\mathcal{R}$. Therefore, $x \in \partial^u\mathcal{R}$.
□

Lemma 4.3.11. (a) *Suppose $z \notin \partial^u\mathcal{R}$. For $R_i \in \mathrm{Set}(f(z))$ there is a unique $R_j = T_z(R_i) \in \mathrm{Set}(z)$ such that $A_{ij} = 1$. The map $T_z : \mathrm{Set}(f(z)) \to \mathrm{Set}(z)$ is surjective. If $y \in V^s_{\delta_1}(z) \setminus \partial^u\mathcal{R}$ and $\mathrm{Set}(y) = \mathrm{Set}(z)$, then $\mathrm{Set}(f(y)) = \mathrm{Set}(f(z))$ and $T_y = T_z$.*

(b) *Suppose $z \notin \partial^s\mathcal{R}$. For $R_i \in \mathrm{Set}(f^{-1}(z))$ there is a unique $R_j = T'_z(R_i) \in \mathrm{Set}(z)$ such that $A_{ij} = 1$. The map $T'_z : \mathrm{Set}(f^{-1}(z)) \to \mathrm{Set}(z)$ is surjective. If $y \in V^u_{\delta_1}(z) \setminus \partial^s\mathcal{R}$ and $\mathrm{Set}(y) = \mathrm{Set}(z)$, then $\mathrm{Set}(f^{-1}(y)) = \mathrm{Set}(f^{-1}(z))$ and $T'_y = T'_z$.*

Proof. By Lemma 4.3.8, for $R_i \in \mathrm{Set}(f(z))$ there is $(w_i) \in \pi^{-1}(f(z))$ with $w_0 \in R_i$ and so $\sigma^{-1}((w_i)) \in \pi^{-1}(z)$. Thus $z \in R_{w_{-1}}$. Since $(w_i) \in \Sigma_A$, we have $A_{w_{-1}w_0} = A_{w_{-1}i} = 1$. By Lemma 4.3.10, w_{-1} is a unique since $z \notin \partial^u\mathcal{R}$. Thus T_z is well defined.

If $R_i \in \mathrm{Set}(z)$, then there is $(z_i) \in \pi^{-1}(z)$ with $z_0 \in R_i$. Then $R_{z_1} \in \mathrm{Set}(f(z))$ and $A_{z_0 z_1} = 1$. Thus $T_z(R_{z_1}) = R_{z_0}$ and T_z is surjective.

Take $y \in V^s_{\delta_1}(z) \setminus \partial^u\mathcal{R}$ with $\mathrm{Set}(y) = \mathrm{Set}(z)$. Then $y \in V^s(z, R_i)$ for $R_i \in \mathrm{Set}(z)$. If $R_j \in \mathrm{Set}(f(z))$ and $R_i = T_z(R_j)$, then Lemma 4.3.1 shows that

$$f(y) \in fV^s(z, R_i) \subset V^s(f(z), R_j) \subset R_j.$$

Thus $\mathrm{Set}(f(z)) \subset \mathrm{Set}(f(y))$. Symmetrically, $\mathrm{Set}(f(y)) \subset \mathrm{Set}(f(z))$. Thus $\mathrm{Set}(f(y)) = \mathrm{Set}(f(z))$. Since the definition of T_z depends only on the sets $\mathrm{Set}(z)$ and $\mathrm{Set}(f(z))$ (not z itself), we can easily check that $T_y = T_z$. (b) is proved in the same way. □

Let $\delta_1 > 0$ be as in (B). For $x \in \Omega_s$ define

$$J_s(x) = \{\mathcal{D} \subset \mathcal{R} : \delta_1 > \forall\delta, \exists z \in V^s_\delta(x) \setminus \partial^u\mathcal{R} \text{ with } \mathrm{Set}(z) = \mathcal{D}\},$$
$$J_u(x) = \{\mathcal{D} \subset \mathcal{R} : \delta_1 > \forall\delta, \exists z \in V^u_\delta(x) \setminus \partial^s\mathcal{R} \text{ with } \mathrm{Set}(z) = \mathcal{D}\},$$

Note that we can choose $\delta = \delta(x)$ so small that

$$\mathrm{Set}(z) \in J_s(x) \text{ if } z \in V^s_\delta(x) \setminus \partial^u\mathcal{R},$$
$$\mathrm{Set}(z) \in J_u(x) \text{ if } z \in V^u_\delta(x) \setminus \partial^s\mathcal{R}.$$

Let $N > 0$ and K_N be the set of all n-tuples $(a_1, a_2, \cdots, a_n)$ of integers with $1 \le n \le N$, $1 \le a_i \le N$ and $a_1 \ge a_2 \ge \cdots \ge a_n$. Define a partial ordering $\le$ on K_N by $(a_1, \cdots, a_n) \ge (b_1, \cdots, b_m)$ if either $n > m$ or $N = m$ and $b_i \ge a_i$ for all $1 \le i \le m$ (note that this ordering is not the natural condition $a_i \ge b_i$). We write $(a_1, \cdots, a_n) > (b_1, \cdots, b_m)$ if $(a_1, \cdots, a_n) \ge (b_1, \cdots, b_m)$, but $(a_1, \cdots, a_n) \ne (b_1, \cdots, b_m)$.

Let $\sharp E$ denote the cardinality of E and $\mathcal{P}(B)$ denote the family of subsets of B. Fix $N \ge 2^{2\sharp\mathcal{R}}$. For $L \in \mathcal{P}(\mathcal{P}(\mathcal{R}))$ we define $L^* \in K_N$ to be $(a_1, \cdots, a_n)$ where $n = \sharp L$ and $a_1, \cdots, a_n$ denote the cardinalities of the n-sets in L.

Lemma 4.3.12. *$J_s(f(x))^* \le J_s(x)^*$ and $J_u(f^{-1}(x))^* \le J_u(x)^*$.*

Proof. Define the map $R^s_x : J_s(x) \to J_s(f(x))$ by $R^s_x(\mathcal{D}) = \mathrm{Set}(f(z))$ for $\mathcal{D} \in J_s(x)$. Lemma 4.3.11 ensures that R^s_x is well defined. We show that R^s_x is surjective.

If $\mathcal{E} \in J_s(f(x))$, by Lemma 4.3.9, for arbitrarily small $\delta > 0$ we can find $w \in V^s_\delta(f(x)) \setminus \partial^u\mathcal{R}$ with $\mathrm{Set}(w) = \mathcal{E}$ and $w \notin f(\partial^u\mathcal{R})$. Then $\mathcal{E} = R^s_x(\mathrm{Set}(f^{-1}(w)))$.

Let $J_s(x)^* = (a_1, \cdots a_n)$ where $a_i = \sharp\mathcal{D}_i$ and $J_s(x) = \{\mathcal{D}_1, \cdots, \mathcal{D}_n\}$, and let $J_s(f(x))^* = (b_1, \cdots, b_m)$ where $b_i = \sharp E_i$ and $J_s(f(x)) = \{\mathcal{E}_1, \cdots, \mathcal{E}_m\}$. Then R^s_x induces a surjection $g: [1, n] \to [1, m]$ by $R^s_x(\mathcal{D}_i) = \mathcal{E}_{g(i)}$. Here we can consider the two cases :

$$n > m \text{ and } (a_1, \cdots, a_n) > (b_1, \cdots, b_m),$$
$$n = m \text{ and } g \text{ is bijective} .$$

Considering the second case, for a certain z we have the surjection $T_z : \mathcal{E}_{g(i)} \to \mathcal{D}_i$ and thus $b_{g(i)} \ge a_i$. For any i there is $j \in [1, i]$ with $g(j) \ge i$. Otherwise

$g([1,i]) \subset [1, i-1]$ which contradicts that g is one-to-one. Then $b_i \geq b_{g(j)} \geq a_j \geq a_i$. Here we use the fact that all a_i and b_i are arranged in descending order. The second statement is proved in the same way. $\square$

We say that $x \in \Omega_s$ is an *s-branch* (*u-branch*) point if there exist $(x_i), (y_i) \in \pi^{-1}(x)$ with $x_0 = y_0$ but $x_1 \neq y_1$ (but $x_{-1} \neq y_{-1}$).

Lemma 4.3.13. *If $x \in \Omega_s$ is an s-branch point, then $J_s(x)^* > J_s(f(x))^*$. If $x \in \Omega_s$ is a u-branch point, then $J_u(x)^* > J_u(f^{-1}(x))^*$.*

Proof. Let x be an s-branch point. It is enough to show that one of T_z with $z \in V^s_{\delta(x)}(x) \setminus \partial^u\mathcal{R}$ is not bijective. Indeed, if $(a_1, \cdots, a_n) = (b_1, \cdots, b_n)$, then g is a bijective and so

$$\sum_{i=1}^{n} b_i = \sum_{j=1}^{n} a_i \leq \sum_{i=1}^{n} b_{g(i)} = \sum_{i=1}^{n} b_i.$$

We here use that $a_i \leq b_{g(i)}$. To have the equality we need $a_i = b_{g(i)}$ for all i, i.e. T_z is always a bijection. Since x is an s-branch point, we can find $R_i \in \mathrm{Set}(x)$ and $R_j, R_k \in \mathrm{Set}(f(x))$ with $R_j \neq R_k$ and $A_{ij} = A_{ik} = 1$. By Lemma 4.3.9 we can choose

$$z \in \{V^s(x, R_i) \cap V^s_{\delta(x)}(x)\} \setminus \partial^u\mathcal{R},$$

and since

$$fV^s(x, R_i) \subset V^s(f(x), R_j) \cap V^s(f(x), R_k)$$

(by Lemma 4.3.1), we have $R_j, R_k \in \mathrm{Set}(f(z))$ and $T_z(R_j) = T_z(R_k) = R_i$. Therefore T_z is not bijective. $\square$

Proof of Theorem 4.3.6. Let $c = \sharp\mathcal{R}$ and e be the length of the longest chain in the partially ordered set $(K_N, \leq)$. Let $0 \leq n_1 < n_2 < \cdots < n_k$ be all the nonnegative integers such that $f^{n_i}(x)$ is an s-branch point. By Lemmas 4.3.12 and 4.3.13

$$J_s(f^{n_1}(x))^* > J_s(f^{n_2}(x))^* > \cdots > J_s(f^{n_k}(x))^*,$$

which shows $k \leq e$.

Let A_n be the set of all sequences $(R_{k_0}, \cdots, R_{k_n})$ of elements of $\mathcal{R}$ such that there is $(x_i) \in \pi^{-1}(x)$ with $x_i \in R_{k_i}$ for all $0 \leq i \leq n$. By the definition of s-branch points, $\sharp A_{n+1} = \sharp A_n$ unless $f^n(x)$ is an s-branch point. Thus $\sharp A_{n+1} \leq c\sharp A_n$ for every n. Since $\sharp A_0 \leq c$, we have

$$\sharp A_n \leq c^{k+1} \leq c^{e+1}$$

for all $n \geq 0$. Thus there exist at most c^{e+1} possibilities for $(x_i)_0^\infty$ with $(x_i) \in \pi^{-1}(x)$.

Similarly there exist at most c^{e+1} possibilities for $(x_i)_{-\infty}^0$ with $(x_i) \in \pi^{-1}(x)$. Therefore, $\sharp\pi^{-1}(x) \leq c^{2(e+1)}$. $\square$