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Sinai Billiards Under Small External Forces

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Abstract. Consider a particle moving freely on the torus and colliding elastically with some fixed convex bodies. This model is called a periodic Lorentz gas, or a Sinai billiard. It is a Hamiltonian system with a smooth invariant measure, whose ergodic and statistical properties have been well investigated. Now let the particle be subjected to a small external force. This new system is not likely to have a smooth invariant measure. Then a Sinai-Ruelle-Bowen (SRB) measure describes the evolution of typical phase trajectories. We find general sufficient conditions on the external force under which the SRB measure for the collision map exists, is unique, and enjoys good ergodic and statistical properties, including Bernoulliness and an exponential decay of correlations.

1 Introduction

Let $\mathcal{B}_1, \ldots, \mathcal{B}_s$ be open convex domains on the unit 2-dimensional torus $\mathbb{T}^2$. Assume that $\bar{\mathcal{B}}_i \cap \bar{\mathcal{B}}_j = \emptyset$ for $i \neq j$, and for each i the boundary $\partial\mathcal{B}_i$ is a C^3 smooth closed curve with non-vanishing curvature.

Consider a particle of unit mass moving in $Q := \mathbb{T}^2 \setminus \cup_i \mathcal{B}_i$ according to equations

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = \mathbf{F} \tag{1.1}$$

where $\mathbf{q} = (x, y)$ is the position vector, $\mathbf{p} = (u, v)$ is the momentum (or velocity) vector, and $\mathbf{F}(x, y, u, v) = (F_1, F_2)$ is a stationary force (the force is independent of time). Upon reaching the boundary $\partial Q = \cup_i \partial\mathcal{B}_i$, the particle reflects elastically, according to the usual rule

$$\mathbf{p}^+ = \mathbf{p}^- - 2\,(\mathbf{n}(\mathbf{q}) \cdot \mathbf{p}^-)\,\mathbf{n}(\mathbf{q}) \; . \tag{1.2}$$

Here $\mathbf{q} \in \partial Q$ is the point of reflection, $\mathbf{n}(\mathbf{q})$ is the unit normal vector to ∂Q pointing inside Q , and $\mathbf{p}^-$, $\mathbf{p}^+$ are the incoming and outgoing velocity vectors, respectively.

The case $\mathbf{F} = 0$ corresponds to the ordinary billiard dynamics on the table Q. It preserves the kinetic energy $K = \frac{1}{2}||\mathbf{p}||^2$, so that one can fix it, usually by setting $||\mathbf{p}|| = 1$. Then the phase space of the system is a compact three-dimensional manifold $\mathcal{M}_0 := Q \times S^1$, with identification of incoming and outgoing velocity vectors, i.e. $\mathbf{p}^-$ and $\mathbf{p}^+$ in (1.2), at every point of reflection. The dynamics Φ_0^t on $\mathcal{M}_0$ preserves the Liouville measure, which is simply a uniform measure on $\mathcal{M}_0$.

In the study of billiards, one usually considers the following two-dimensional cross-section of $\mathcal{M}_0$:

$$M_0 := \{(\mathbf{q}, \mathbf{p}) \in \mathcal{M}_0 : \mathbf{q} \in \partial Q, \ (\mathbf{p} \cdot \mathbf{n}(\mathbf{q})) \geq 0\} \tag{1.3}$$

which consists of all outgoing velocity vectors at reflection points. Then the first return map $T_0 : M_0 \to M_0$ is well defined, it is called the collision map or billiard map. The cross-section M_0 can be parameterized by (r, φ), where r is the arclength parameter along ∂Q and $\varphi \in [-\pi/2, \pi/2]$ is the angle between $\mathbf{p}$ and $\mathbf{n}(\mathbf{q})$. In these coordinates, $M_0 = \partial Q \times [-\pi/2, \pi/2]$. The map T_0 preserves a finite smooth measure on M_0, induced by the Liouville measure on $\mathcal{M}_0$. It is given by

$$d\nu_0 = \text{const} \cdot \cos\varphi \, dr \, d\varphi \ .$$

Since each obstacle $\mathcal{B}_i$ is convex, it acts as a scatterer, so that parallel bundles of trajectories diverge upon reflection, see Fig. 1. Billiard with this property are said to be dispersing, or Sinai billiards. The map T_0 and the flow Φ_0^t for dispersing billiards are proved to be hyperbolic (i.e., they have one positive and one negative Lyapunov exponents), ergodic, mixing, K-mixing and Bernoulli [Si, GO]. The map T_0 enjoys strong statistical properties: exponential decay of correlations and satisfies a central limit theorem and weak invariance principle [BSC2, Y1, Ch2].

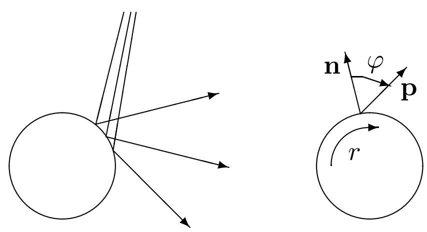

Figure 1. Scattering effect and the coordinates r, φ.

Let us now assume that the configuration of scatterers has *finite horizon* meaning that the the free motion of the billiard particle in Q is uniformly bounded (by a constant $L > 0$). In other words, any straight line of length L on the torus intersects one of the $\mathcal{B}_i$. Under this condition, in addition to all the cited properties, the flow Φ_0^t satisfies a central limit theorem and weak invariance principle [BSC2], hence the billiard particle satisfies a diffusion equation [BSC2]. It is very likely that the flow Φ_0^t enjoys exponential decay of correlations as well, but this is not proven yet. The assumption of finite horizon seems to be necessary for the above properties, because in billiards without horizon the moving particle exhibits superdiffusive (ballistic) behavior [Bl], and the correlations seem to decay very slowly, as const$\cdot t^{-1}$, see, e.g., [FM1, FM2].

Very little is known in the general case $\mathbf{F} \neq 0$, though. Clearly, a large force can change the dynamics dramatically, so that the properties of the dynamics will be determined by the character of $\mathbf{F}$ in (1.1) more than by the scattering effect of collisions with obstacles. Hence, the dynamics can be of quite generic nature. It is presently understood, due to the KAM theory, that generic mechanical (even Hamiltonian) systems are not completely hyperbolic or ergodic – typically, chaotic regions in the phase space coexist with elliptic islands of stability. So, we have to restrict ourselves to small forces that will not overcome the scattering effect of collisions with obstacles. Thus we will keep the dynamics close enough to the original billiard. Then we can hope that many properties of the system with force will be "inherited" from the original billiard model. It is now clear that the assumption on finite horizon will be necessary – without it the effect of even a small force $\mathbf{F}$ may accumulate to a dangerous level during long runs between collisions.

The system (1.1)-(1.2) with a force $\mathbf{F} \neq 0$ may be hyperbolic but is likely to admit no smooth invariant measure. Then the evolution of typical phase trajectories is governed by the so called Sinai-Ruelle-Bowen (SRB) measures. Those measures are characterized by smooth conditional distributions on unstable manifolds. The SRB measures are the only physically observable measures, they are called non-equilibrium steady states in the language of statistical mechanics. We refer the reader to [GC, Ru, Y2] for more discussion on SRB measures and their role in hyperbolic dynamics and physics.

There is a remarkable example of the system (1.1)-(1.2) well studied in the literature. Let $\mathbf{F}$ be a small constant electric force, possibly combined with a small magnetic force, with a Gaussian thermostat added, see (2.8) below. An SRB measure was constructed and its strong ergodic and statistical properties were mathematically proved [CELS1, CELS2] for this particular model. Certain transport laws of physics were then rigorously derived, including Ohm's law, Einstein relation, Green-Kubo formula, etc. Similar models are now getting more and more popular in physics.

The purpose of this paper is to find general classes of forces $\mathbf{F}$ for which the system (1.1)-(1.2) has an SRB measure with good ergodic and statistical properties. In fact, we try to consider the forces as general as possible, assuming only what seems to be necessary.

First, we will assume that the force $\mathbf{F}$ is small, as we said above. The only other major assumption we need is an additional integral of motion. If none exists, then the phase space of the system is a four-dimensional non-compact manifold $Q \times \mathbb{R}^2$. Then we would face two almost hopeless problems. First, we can only ensure that two nonzero Lyapunov exponents exist – they are inherited from the original billiard, but the other two may be zero or arbitrary small. This makes the system, essentially, only partially hyperbolic with little chance for any good ergodic or statistical properties. To make things worse, the non-compactness of the phase space makes the existence of physically interesting invariant measures very unlikely. In fact, without a proper temperature control (thermostating), the system will usually heat up ($||\mathbf{p}|| \to \infty$) or cool down ($||\mathbf{p}|| \to 0$), which effectively rules out interesting invariant measures.

Hence, we will assume that the dynamics preserves a smooth function $\mathcal{E}(\mathbf{q}, \mathbf{p})$, an integral of motion, and its level surface is a compact 3-D manifold. We now turn to exact assumptions on the force $\mathbf{F}$ in our model.

2 The model and main results

Here we state our assumptions on the force $\mathbf{F}$.

Assumption A (additional integral). A smooth function $\mathcal{E}(\mathbf{q}, \mathbf{p})$ is preserved by the dynamics Φ^t defined by (1.1)-(1.2). Its level surface, $\mathcal{M} := \{\mathcal{E}(\mathbf{q}, \mathbf{p}) = \text{const}\}$ is a compact 3-D manifold. Two extra assumptions are made for convenience:
(A1) $||\mathbf{p}|| \neq 0$ on $\mathcal{M}$,
(A2) for each $\mathbf{q} \in Q$ and $\mathbf{p} \in S^1$ the ray $\{(\mathbf{q}, s\mathbf{p}),\ s > 0\}$ intersects the manifold $\mathcal{M}$ in exactly one point.

Under the assumptions (A1)-(A2), $\mathcal{M}$ can be parameterized by (x, y, θ), where $(x, y) = \mathbf{q} \in Q$ and $0 \leq \theta < 2\pi$ is a cyclic coordinate, the angle between $\mathbf{p}$ and the positive x-axis. The dynamics (1.1)-(1.2) restricted to $\mathcal{M}$ is a flow that we denote by Φ^t. In the coordinates (x, y, θ) the equations of motion (1.1) can be rewritten as

$$\dot{x} = p\cos\theta, \qquad \dot{y} = p\sin\theta, \qquad \dot{\theta} = ph \tag{2.1}$$

where

$$p = ||\mathbf{p}|| > 0 \quad \text{and} \quad h = (-F_1 \sin\theta + F_2 \cos\theta)/p^2 \ .$$

It is also useful to note that

$$\dot{p} = F_1 \cos\theta + F_2 \sin\theta \tag{2.2}$$

Both $h = h(x, y, \theta)$ and $p = p(x, y, \theta)$ are assumed to be C^2 smooth functions on $\mathcal{M}$. Due to our assumption (A1), we have

$$0 < p_{\min} \leq p \leq p_{\max} < \infty \ . \tag{2.3}$$

Note that at time of reflection, the angle θ changes discontinuously, say from θ^- to θ^+. The law (1.2) then imposes the restriction

$$p(x, y, \theta^-) = p(x, y, \theta^+) \tag{2.4}$$

on the function p at every point $(x, y) \in \partial Q$.

For a function f on $\mathcal{M}$, let f_x, f_y, f_θ denote the partial derivatives of f. Denote by $||f||_{C^2}$ the maximum of f and its first and second partial derivatives over $\mathcal{M}$. Now put

$$B_0 = \max\{p_{\min}^{-1}, ||p||_{C^2}, ||h||_{C^2}\} \ . \tag{2.5}$$

Assumption B (smallness of the force). We assume that the force F and its first derivatives are small enough. This means that

$$\delta_0 = \max\{|h|, |h_x|, |h_y|, |h_\theta|\}$$

is sufficiently small. More precisely, we require that for any given $B_* > 0$ there should be a small $\delta_* = \delta_*(Q, B_*)$ such that all our results will hold whenever $B_0 < B_*$ and $\delta_0 < \delta_*$.

Remark. The geometric curvature of the trajectories of the particle on the torus is $\dot{\theta}/p = h$. By Assumption B, it is small, and so the trajectories are nearly straight lines. This has an important implication: no trajectory can collide with one body $\mathcal{B}_i$ more than once during a short interval of time. Hence, the distance between collisions is uniformly bounded below by a positive constant $L_{\min} > 0$. The time between collisions is uniformly bounded below as well, by $t_{\min} = L_{\min}/p_{\max}$.

Lastly, we state our assumption on finite horizon:

Assumption C (finite horizon). There is an $L > 0$ so that every straight line of length L on the torus $\mathbb{T}^2$ crosses at least one obstacle $\mathcal{B}_i$.

Remark. Under Assumptions B and C, every trajectory of length $L_{\max}$ for the system (1.1)-(1.2), for some $L_{\max} > L$, must hit a scatterer $\mathcal{B}_i$. So, the collision-free path of the particle is uniformly bounded by $L_{\max}$. The time between collisions is uniformly bounded as well, by $t_{\max} = L_{\max}/p_{\min}$.

Consider the two-dimensional cross-section of the manifold $\mathcal{M}$:

$$M := \{(\mathbf{q}, \mathbf{p}) \in \mathcal{M} : \mathbf{q} \in \partial Q, (\mathbf{p} \cdot \mathbf{n}(\mathbf{q})) \geq 0\} \tag{2.6}$$

which, as M_0 in (1.3), consists of all outgoing velocity vectors at reflection points. Then the first return map $T : M \to M$ is then well defined, we also call it collision map.

The cross-section M can be parameterized by (r, φ), where r is the arclength parameter along ∂Q and $\varphi \in [-\pi/2, \pi/2]$ is the angle between $\mathbf{p}$ and $\mathbf{n}(\mathbf{q})$. In these coordinates, $M = \partial Q \times [-\pi/2, \pi/2]$, the same as M_0 in (1.3). We choose the orientation of the coordinates r and φ as shown on Fig. 1 (where r and φ increase in the direction of arrows). Also, denote by $K(r) > 0$ the curvature of the curve ∂Q at the point with coordinate r.

There are two particularly interesting types of forces satisfying our assumptions A and B:

Type 1 forces (potential forces). Consider an isotropic force $\mathbf{F} = \mathbf{F}(\mathbf{q})$ (independent of $\mathbf{p}$) such that $\mathbf{F} = -\nabla U$, where $U = U(\mathbf{q})$ is a (small) potential function. Note that $U(x, y)$ must be a smooth function on the torus, so it is necessarily a periodic function in x and y. These forces preserve the total energy $T = \frac{1}{2}||\mathbf{p}||^2 + U(\mathbf{q})$. In this case we can set $T = 1/2$, so that $||\mathbf{p}||^2 = 1 - 2U(\mathbf{q}) \approx 1$, assuming $U(\mathbf{q})$ be small.

Remark. Type 1 forces preserve the Lebesgue measure $dx\, dy\, d\theta$ on the manifold $\mathcal{M}$, since the divergence of the vector field (2.1) vanishes. This follows from the equation $\mathbf{F} = -\nabla U$ by direct calculations. Therefore, the collision map $T : M \to M$ also preserves a smooth measure ν.

Sinai billiards under type 1 forces have been studied in numerous papers, and, in most cases, ergodicity and Bernoulli property were established.

Type 2 forces (isokinetic forces). Consider forces satisfying $(\mathbf{F} \cdot \mathbf{p}) = 0$. They preserve the kinetic energy, i.e. $K = \frac{1}{2}||\mathbf{p}||^2$ =const. In this case we can set $||p|| = 1$, as in billiards. Note that the equations in (2.1) hold with $p = 1$ and $|h| = ||\mathbf{F}||$ (the sign of h is determined by the direction of $\mathbf{F}$).

Example of type 2 forces: thermostating. Let $\mathbf{F}$ be an arbitrary force. One can modify the equations (1.1) so that the kinetic energy will be preserved:

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = \mathbf{F} - \alpha\mathbf{p} \quad \text{where} \quad \alpha = (\mathbf{F} \cdot \mathbf{p})/(\mathbf{p} \cdot \mathbf{p}) \tag{2.7}$$

It is easy to verify that $||\mathbf{p}||$ =const. The added term $\alpha\mathbf{p}$ is called a Gaussian thermostat, it satisfies the Gaussian principle of least constraint. Also, α is called the Gaussian friction coefficient.

Example : electric and magnetic fields. A well studied example of a force of type 2 is the following, see [CELS1]

$$\mathbf{F}(\mathbf{q}, \mathbf{p}) = \mathbf{E} + [\mathbf{B} \times \mathbf{p}] - \alpha\mathbf{p} \ . \tag{2.8}$$

Here $\mathbf{E}$ is a small constant electric field, $\mathbf{B}$ is a small constant magnetic field (a vector in $\mathbb{R}^3$ perpendicular to the billiard table Q), and $\alpha\mathbf{p}$ is the Gaussian thermostat $\alpha = (\mathbf{E} \cdot \mathbf{p})/(\mathbf{p} \cdot \mathbf{p})$.

If $\mathbf{E} = 0$ (thus $\alpha = 0$), then the system preserves the Lebesgue measure $dx\,dy\,d\theta$ on $\mathcal{M}$, just as does the pure billiard dynamics Φ_0^t. If $\mathbf{E} \neq 0$, then the system has no absolutely continuous invariant measure, but has a unique SRB measure with good ergodic and statistical properties. We refer the reader to [CELS1] for a detailed study of the system (2.8). Recently, M. Wojtkowski found explicit conditions on the field $\mathbf{E}$ under which the the system is hyperbolic [W2, W3].

Now we state the main results of this paper.

Theorem 2.1 *Under Assumptions A, B, and C, the map $T : M \to M$ is a uniformly hyperbolic map with singularities. It admits a unique SRB measure ν, which is positive on open sets, K-mixing and Bernoulli.*

The next theorem concerns the statistical properties of the map $T : M \to M$. Let $\mathcal{H}_\eta$ be the class of Hölder continuous functions on M with exponent $\eta > 0$:

$$\mathcal{H}_\eta = \{f : M \to \mathbb{R} \,|\, \exists C > 0 : |f(X) - f(Y)| \le C\,[\mathrm{dist}(X, Y)]^\eta, \ \forall X, Y \in M\}$$

We say that (T, ν) has exponential decay of correlations for Hölder continuous functions if for all $\eta > 0$ there is $\lambda = \lambda(\eta) \in (0, 1)$ such that for all $f, g \in \mathcal{H}_\eta$ and some $C = C(f, g) > 0$ we have

$$\left| \int_M (f \circ T^n) g \, d\nu - \int_M f \, d\nu \int_M g \, d\nu \right| \le C\lambda^{|n|} \tag{2.9}$$

for all $n \in \mathbb{Z}$. We say that (T, ν) satisfies the central limit theorem for Hölder continuous functions if for all $\eta > 0, f \in \mathcal{H}_\eta$ with $\int f \, d\nu = 0$, there is $\sigma_f \geq 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \stackrel{\text{distr}}{\longrightarrow} N(0, \sigma_f^2) \tag{2.10}$$

which means the convergence in distribution to the normal law $N(0, \sigma_f^2)$. Furthermore, $\sigma_f = 0$ iff f is cohomologous to zero, i.e. $f = g \circ T - g$ for some $g \in L^2(\nu)$

Theorem 2.2 *The measure ν enjoys exponential decay of correlations and satisfies the central limit theorem. The decay of correlations is uniform in the force $\mathbf{F}$, i.e. the constants λ and C in (2.9) are independent of $\mathbf{F}$.*

We only remark that Theorem 2.1 easily implies that the flow $\Phi^t : \mathcal{M} \to \mathcal{M}$ is fully hyperbolic and has a unique SRB measure μ that is ergodic and positive on open sets. In a forthcoming paper, we will prove that the flow Φ^t is actually mixing and Bernoulli and satisfies the central limit theorem.

3 Hyperbolicity of Φ^t and T

Our first goal is to prove that the flow Φ^t on $\mathcal{M}$ is hyperbolic, i.e. it has one positive and one negative Lyapunov exponents. The hyperbolicity is usually obtained by constructing a family of invariant cones in the tangent space [W1]. For Sinai billiards, invariant cones have a clear geometrical interpretation. Unstable cones correspond to divergent bundles of trajectories, and stable cones - to convergent bundles of trajectories. Any divergent bundle of trajectories remains divergent upon reflections off convex obstacles, as in Fig. 1, this easily implies the invariance of the unstable cones. Similarly, any convergent bundle of trajectories remains convergent in the past (as they flow backwards).

We will prove that in our dynamics a sufficiently divergent bundle of trajectories remains divergent in the future. Obviously, we need to consider runs between collisions carefully and make sure that the divergence is not lost there. We use some new techniques to do that.

Let $P = (x, y, \theta) \in \mathcal{M}$ be an arbitrary point and $P = (dx, dy, d\theta) \in \mathcal{T}_P\mathcal{M}$ a tangent vector at P. Pick a smooth curve $P_s = (x_s, y_s, \theta_s) \subset \mathcal{M}$ tangent to the vector dP at the point P, i.e. assume that $P_0 = P$ and $P_0' = dP$. In the calculations below, we denote differentiation with respect to the auxiliary parameter s by primes and that with respect to time t by dots. In particular, $\dot{P} = (\dot{x}, \dot{y}, \dot{\theta}) = (p\cos\theta, p\sin\theta, ph)$ is the velocity vector of the flow Φ^t. It is not to be confused with the velocity vector $(\dot{x}, \dot{y}) = (p\cos\theta, p\sin\theta)$ of the moving particle on the torus, the latter will be referred to as the *particle velocity.*

Now consider $P_{st} = (x_{st}, y_{st}, \theta_{st}) := \Phi^t P_s$. The points P_{st} make a two-dimensional surface $\mathcal{S}$ in $\mathcal{M}$. It is standard that

$$D\Phi^t(dP) = P_{0t}' = \frac{d}{ds} P_{st}|_{s=0} \ .$$

In subsequent formulas, all the calculations will be done at the point P_{0t}, where $s = 0$, and for brevity we will often drop the subscript $0t$. Note that the vectors $P' = (x', y', \theta')$ and

$$\dot{P} = (\dot{x}, \dot{y}, \dot{\theta}) = (p\cos\theta, p\sin\theta, ph) \tag{3.1}$$

are both tangent vectors to $\mathcal{S}$ at the point P $(= P_{0t})$.

We introduce two quantities

$$v = x'\cos\theta + y'\sin\theta \quad \text{and} \quad w = -x'\sin\theta + y'\cos\theta\ . \tag{3.2}$$

It is easy to see that v is the component of the vector (x', y') parallel to the particle velocity $(\dot{x}, \dot{y})$, and w is the perpendicular component of (x', y'). Solving (3.2) for x', y' gives

$$x' = v\cos\theta - w\sin\theta \quad \text{and} \quad y' = v\sin\theta + w\cos\theta\ . \tag{3.3}$$

Now let

$$\alpha = v/w \quad \text{and} \quad \kappa = (\theta' - vh)/w\ . \tag{3.4}$$

So, α is the cotangent of the angle between the vector (x', y') and the particle velocity $(\dot{x}, \dot{y})$. To describe κ geometrically, consider the one parameter family of trajectories $\{(x_{st}, y_{st})\}$ on the torus, where s is the parameter of the family and t is the internal parameter along each trajectory. Then κ is the curvature of the orthogonal cross-section of this family. Furthermore, $\kappa > 0$ corresponds to divergent families, $\kappa < 0$ to convergent families, and $\kappa = 0$ to parallel families. Also, note that $|w|$ is the width of that family in the direction perpendicular to the particle velocity, per unit increment of the parameter s.

Now, consider two vectors

$$U = (\cos\theta, \sin\theta, h) \quad \text{and} \quad R = (-\sin\theta, \cos\theta, \kappa)\ .$$

Both are tangent vectors to the surface $\mathcal{S}$, as it follows from the equations

$$\dot{P} = pU \quad \text{and} \quad P' = vU + wR\ .$$

The vector U is obtained by taking a unit vector $(\cos\theta, \sin\theta)$ in the direction of the particle velocity $(\dot{x}, \dot{y})$ and lifting it to a tangent vector to the surface $\mathcal{S}$. Similarly, the vector R is obtained by taking a unit vector $(-\sin\theta, \cos\theta)$ in the perpendicular direction and lifting it to a tangent vector to the surface $\mathcal{S}$.

Denote by p_U and p_R the 'scaled' directional derivatives of the function p along the vectors U, R, respectively, defined by

$$p_U = p_x\cos\theta + p_y\sin\theta + p_\theta h, \qquad p_R = -p_x\sin\theta + p_y\cos\theta + p_\theta\kappa\ .$$

Similarly,

$$h_U = h_x\cos\theta + h_y\sin\theta + h_\theta h, \qquad h_R = -h_x\sin\theta + h_y\cos\theta + h_\theta\kappa\ .$$

It is then straightforward that

$$p' = p_U v + p_R w \qquad \text{and} \qquad h' = h_U v + h_R w \tag{3.5}$$

upon direct differentiation and substitution of (3.3) and using

$$\theta' = \kappa w + hv \ . \tag{3.6}$$

It is also easy to see that

$$\dot{p} = p_U p \qquad \text{and} \qquad \dot{h} = h_U p \ .$$

Lemma 3.1 *The evolution of the quantities κ, w, α is given by the equations*

$$\dot{\kappa}/p = -\kappa^2 - h^2 + h_R \tag{3.7}$$

$$\dot{w} = p\kappa w \tag{3.8}$$

and

$$\dot{\alpha} = -p\kappa\alpha + p_U\alpha + p_R + ph \ . \tag{3.9}$$

Proof. First, note that $dx'/dt = (\dot{x})' = p'\cos\theta - p\sin\theta \cdot \theta'$ and similarly $dy'/dt = p'\sin\theta + p\cos\theta \cdot \theta'$. Also, $d\theta'/dt = (\dot{\theta})' = p'h + ph'$. Hence

$$\dot{v} = p' + phw \qquad \text{and} \qquad \dot{w} = p\theta' - phv = p\kappa w \ .$$

Then direct differentiation of the equations (3.4) and substitution of (3.5) completes the proof. □

Let τ be the length parameter along the trajectory (x_{0t}, y_{0t}) on the torus, i.e. $d\tau/dt = p$. Then the equations (3.7)-(3.9) can be rewritten as

$$d\kappa/d\tau = -\kappa^2 - h^2 + h_R \tag{3.10}$$

$$dw/d\tau = \kappa w \tag{3.11}$$

and

$$d\alpha/d\tau = -\kappa\alpha + h + p_U\alpha/p + p_R/p \tag{3.12}$$

Now consider a reflection at some $\partial\mathcal{B}_i \subset \partial Q$ experienced by the family P_{st}. For every s, the trajectory P_{st} reflects in $\partial\mathcal{B}_i$ at some moment of time $t = t_s$. The outgoing velocity vectors of these trajectories (taken immediately after the reflection) make a curve γ in M, we call it the *trace* of the family P_{st} (on M). Let γ satisfy an equation $\varphi = \varphi(r)$ in the coordinates r, φ introduced after (2.6). Note that the curve γ is also parameterized by s, because each point corresponds to a trajectory of the family P_{st}.

The quantities $\theta, \alpha, \kappa, w, v, p, h$ may change discontinuously at the reflection. Denote by $\theta^-, \alpha^-, \kappa^-$, etc., their values before the reflection and by $\theta^+, \alpha^+, \kappa^+$, etc., their values after the reflection. Actually, we have $p^+ = p^-$ by (2.4).

Lemma 3.2 *The derivative $t' = dt_s/ds$ satisfies*

$$t' = \mp(w^{\pm} \tan\varphi \pm v^{\pm})/p^{\pm} \ . \tag{3.13}$$

The derivative dr/ds on γ satisfies

$$dr/ds = \mp w^{\pm}/\cos\varphi \ . \tag{3.14}$$

The derivative of the function $\varphi = \varphi(r)$ satisfies

$$d\varphi/dr = \mp K(r) + \kappa^{\pm}\cos\varphi \mp h^{\pm}\sin\varphi \ . \tag{3.15}$$

Recall that $K(r) > 0$ is the curvature of the boundary $\partial\mathcal{B}_i \subset \partial Q$ at the point $r \in \partial Q$ on the torus $\mathbb{T}^2$.

Proof. Let the boundary $\partial\mathcal{B}_i$ satisfy an equation $G(x, y) = 0$, where the function G is chosen so that its gradient vector (G_x, G_y) is a normal vector to $\partial\mathcal{B}_i$ pointing inside Q (outside $\mathcal{B}_i$). Then t_s satisfies the equation $G(x_{st_s}, y_{st_s}) = 0$. Differentiating with respect to s before and after the reflection gives, respectively,

$$[(x')^{\pm} + (\dot{x})^{\pm}t']G_x + [(y')^{\pm} + (\dot{y})^{\pm}t']G_y = 0 \ . \tag{3.16}$$

A simple geometric analysis shows that, in the orientation of φ specified after (2.6), we have

$$G_y/G_x = \tan(\theta^{+} + \varphi) = \tan(\theta^{-} - \varphi) \ . \tag{3.17}$$

Solving (3.16) for t' then gives

$$t' = -\frac{(x')^{\pm} + (y')^{\pm}\tan(\theta^{\pm} \pm \varphi)}{(\dot{x})^{\pm} + (\dot{y})^{\pm}\tan(\theta^{\pm} \pm \varphi)} \ .$$

Substitution of (3.1) and (3.3) yields (3.13).

Next, we have $|dr/ds| = \sqrt{(x' + \dot{x}t')^2 + (y' + \dot{y}t')^2}$, where $x', \dot{x}, y', \dot{y}$ are all taken either before the reflection or after it. Using (3.1), (3.3) and (3.13) and taking into account our orientation of the coordinate r (to determine the sign of dr/ds) yields (3.14).

Another simple geometric inspection shows that $K(r) = -d(\tan^{-1}(G_x/G_y))/dr$ in our orientation of the coordinate r. Therefore, using (3.17) gives

$$d\varphi/dr = -K(r) - d\theta^{+}/dr = K(r) + d\theta^{-}/dr \ .$$

Lastly,

$$\frac{d\theta^{\pm}}{dr} = \frac{d\theta^{\pm}}{ds}\Big/\frac{dr}{ds} = \mp\Big[(\theta')^{\pm} + (\dot{\theta})^{\pm}t'\Big]\Big/[w^{\pm}/\cos\varphi] \ .$$

Now substituting (3.1), (3.6), and (3.13) proves (3.15). Lemma is proved. □

Lemma 3.3 *At each reflection, we have $v^+ = v^-$, i.e. v remains unchanged. Also, $w^+ = -w^-$ and hence $\alpha^+ = -\alpha^-$. The variable κ changes by the rule*

$$\kappa^+ = \kappa^- + \Delta\kappa \tag{3.18}$$

where

$$\Delta\kappa = \frac{2K(r) + (h^+ + h^-)\sin\varphi}{\cos\varphi} \ . \tag{3.19}$$

Generally, there is no relation between h^+ and h^-.

All this directly follows from the previous lemma. □

Note that by setting $h \equiv 0$ in (3.19) we recover the well known equation $\Delta\kappa = 2K(r)/\cos\varphi$ for billiards derived by Sinai, see e.g. [Si, BSC1].

Observe that $\Delta\kappa$ does not depend on the family of trajectories P_{st} (i.e., on α, κ, w). It only depends on the point $(r, \varphi) \in M$. Hence it is a (smooth) function on the cross-section M, we call it $\Theta(r, \varphi)$, i.e.

$$\Theta(r,\varphi) = \frac{2K(r) + (h^+ + h^-)\sin\varphi}{\cos\varphi} \ .$$

Note that this function has a positive lower bound,

$$\Theta(r,\varphi) \geq \Theta_{\min} = 2\min_r K(r) - 2\delta_0 > 0$$

but it is not bounded above (as, indeed, $\cos\varphi$ may be arbitrary close to zero during almost ‘grazing’ reflections).

We now consider a family of trajectories that are divergent before some reflection, i.e. assume $\kappa^- > 0$. Then $\kappa^+ > \Theta_{\min}$ by (3.18), so the curvature of the family is big enough after the reflection. Denote by L the length of the trajectory on the torus between the current and the next reflection points, and parameterize this trajectory segment by the length parameter τ, $0 \leq \tau \leq L$. Recall that the free path between consecutive reflections is uniformly bounded, hence $L_{\min} \leq L \leq L_{\max}$.

Lemma 3.4 *Let $\kappa^- > 0$. Then*

$$\frac{1}{(\kappa^+)^{-1} + \tau} - \delta_1 \leq \kappa_\tau \leq \frac{1}{(\kappa^+)^{-1} + \tau} + \delta_1 \tag{3.20}$$

for all $0 < \tau < L$. Here δ_1 is a small constant that depends only on δ_0 in Assumption B and on $\Theta_{\min}$.

Proof. The equation (3.10) and Assumption B imply

$$-\kappa^2 - \delta_0\kappa - 2\delta_0 \leq d\kappa/d\tau \leq -\kappa^2 + \delta_0\kappa + 2\delta_0$$

assuming that $\kappa > 0$. Hence,

$$-(\kappa + \delta')^2 \leq d\kappa/d\tau \leq -(\kappa - \delta')^2 + (\delta'')^2 \tag{3.21}$$

where $\delta' = (2\delta_0)^{1/2}$ and $\delta'' = (4\delta_0)^{1/2}$. The smallness of δ', δ'' and the initial bound $\kappa_0 = \kappa^+ \geq \Theta_{\min} > 0$ allows direct integration of (3.21) resulting in

$$\frac{1}{(\kappa^+ + \delta')^{-1} + \tau} - \delta' \leq \kappa_\tau \leq \delta'' \frac{Ae^{2\delta''\tau} + 1}{Ae^{2\delta''\tau} - 1} + \delta'$$

where

$$A = \frac{\kappa^+ - \delta' + \delta''}{\kappa^+ - \delta' - \delta''}$$

(this, in particular, justifies the assumption $\kappa > 0$). Since δ', δ'' are small, one can now easily obtain (3.20) with $\delta_1 = \delta' + 2\delta''$. □

Convention (on δ's). Throughout the paper, we denote by δ_i various small constants that depend on the domain Q and δ_0 in Assumption B so that all $\delta_i \to 0$ as $\delta_0 \to 0$. Hence, all those constants are effectively assumed to be small enough.

Denote

$$\kappa_{\min} := \frac{1}{\Theta_{\min}^{-1} + L_{\max}} - \delta_1 \quad \text{and} \quad \kappa_{\max}^- := \frac{1}{L_{\min}} + \delta_1 \ .$$

Corollary 3.5 *If a family of trajectories is divergent before a reflection at time t_0, i.e., $\kappa_{t_0-0} > 0$, then for all $t > t_0$ we have $\kappa_t > \kappa_{\min} > 0$, i.e. the curvature of the family stays bounded away from zero. In addition, at each reflection that occurs after the time t_0 we have $\kappa^- \leq \kappa_{\max}^-$, i.e. the curvature of the family falling upon ∂Q is uniformly bounded above.*

We call a family of trajectories P_{st} *strongly divergent* on a time interval (t_1, t_2) if $\kappa_t \geq \kappa_{\min}$ for all $t_1 < t < t_2$ (and then, of course, for all $t > t_1$). We emphasize the following:

"Invariance principle". Any strongly divergent family of trajectories remains strongly divergent in the future under the flow Φ^t. We note that later on some additional restrictions on the class of strongly convergent families will be assumed (see, e.g., the convention on α's below), but this invariance principle will hold.

Remark. We do not assume that the derivatives p_x, p_y, p_θ are small, they are just bounded as the function p is smooth on a compact manifold $\mathcal{M}$. It is important, though, that the function $p_U = \dot{p}/p = d(\ln p)/dt$ has uniformly bounded integrals along any orbit segment of the flow:

$$\left| \int_{t_1}^{t_2} p_U \, dt \right| \leq \text{const} = \ln(p_{\max}/p_{\min}) < \infty$$

for any $t_1 < t_2$.

Lemma 3.6 *There are constants $\alpha_{\max}$ and $\bar{\alpha}_{\max}$ such that for any strongly divergent family of trajectories on the interval (t_0, ∞) we have $|\alpha_t| \leq \alpha_{\max}$ eventually, for all $t > t_1$ (where t_1 depends on α_{t_0}). Moreover, if $|\alpha_{t_0}| < \alpha_{\max}$, then $|\alpha_t| \leq \bar{\alpha}_{\max}$ for all $t > t_0$.*

Proof. At every reflections, α simply changes sign, i.e. $|\alpha^+| = |\alpha^-|$. Due to (3.12), we have

$$d\alpha/d\tau = -\kappa(\alpha - p_\theta/p) + (p_U/p)\alpha + (-p_x \sin\theta + p_y \cos\theta)/p + h \ . \tag{3.22}$$

Note that the terms p_θ/p, p_U/p, and $(-p_x \sin\theta + p_y \cos\theta)/p + h$ are uniformly bounded. Since $\kappa \geq \kappa_{\min} > 0$, the first term in (3.22) drives α back whenever it gets too large. The influence of the second term, $(p_U/p)\alpha$, is uniformly bounded by the previous remark. □

Convention (on α's). In all that follows, we will only consider strongly divergent families that satisfy $|\alpha_t| \leq \alpha_{\max}$ for all relevant t. We also assume that the "invariance principle" holds, as we may in view of Lemma 3.6.

Lemma 3.7 *For any strongly divergent family of trajectories on an interval* (t_0, ∞)*, its width* $|w|$ *grows exponentially in time:*

$$|w_t| = |w_{t_0}| \exp\left(\int_{t_0}^t p_u \kappa_u \, du\right) \geq |w_{t_0}| \, e^{c(t-t_0)}$$

where $c = p_{\min}\kappa_{\min} > 0$.

We also need the invariance and exponential growth for sufficiently convergent families of trajectories as they flow backwards in time. The following trick will do the job.

Time reversal principle. There is a convenient way to study backward dynamics Φ^t as $t \to -\infty$. Consider the involution map $\mathcal{I} : (x, y, \theta) \mapsto (x, y, \theta + \pi)$ on $\mathcal{M}$. The flow

$$\Phi_-^t := \mathcal{I} \circ \Phi^{-t} \circ \mathcal{I}$$

is governed by the equations

$$\dot{x} = p_- \cos\theta, \quad \dot{y} = p_- \sin\theta, \quad \dot{\theta} = p_- h_- \tag{3.23}$$

where $p_-(x, y, \theta) = p(x, y, \theta + \pi)$ and $h_-(x, y, \theta) = -h(x, y, \theta + \pi)$. So, equations (3.23) are similar to (2.1). The new flow Φ_-^t satisfies Assumption A, quite obviously, and Assumption B, because the function h_- and its partial derivatives are the negatives of those of h. Thus, all the properties of the flow Φ^t also hold for Φ_-^t.

It is clear that convergent families of trajectory and their backward evolution correspond to divergent families of the flow Φ_-^t and their forward evolution. Hence, all the properties we proved and assumptions we made for divergent families have their counterparts for convergent families. We will say that a family is *strongly convergent* on an interval (t_t, t_2) if $\kappa_t \leq -\kappa_{\max,-}$ where $\kappa_{\max,-} > 0$ is the constant defined just as $\kappa_{\max}$, but for the flow Φ_-. Our convention on α's and the "invariance principle" (under the backward dynamics) apply to strongly convergent families, and they grow (in terms of the width w) exponentially in time as $t \to -\infty$.

Remark. In a particular case where

$$p(x, y, \theta) = p(x, y, \theta + \pi) \quad \text{and} \quad h(x, y, \theta) = -h(x, y, \theta + \pi)$$

the flows Φ^t and Φ^t_- coincide. Then we say that the flow Φ^t is time reversible. Time reversibility is quite common in many models of direct physical origin. For example, potential forces (type 1) are always time reversible. The model (2.8) is time reversible, though, only if $\mathbf{B} \neq 0$. Generally, time reversibility does not follow from Assumptions A and B.

We now arrive at the first major theorem.

Theorem 3.8 (Hyperbolicity) *The flow $\{\Phi^t\}$ on $\mathcal{M}$ is hyperbolic with respect to any invariant measure, i.e. it has one positive and one negative Lyapunov exponent almost everywhere. The unstable tangent vector $dP^u = (dx^u, dy^u, d\theta^u)$ at a point $P \in \mathcal{M}$ corresponds to a family of trajectories that is strongly divergent at all times ($-\infty < t < \infty$). The stable tangent vector $dP^s = (dx^s, dy^s, d\theta^s)$ corresponds to a strongly convergent family of trajectories at all times. The angle between the particle velocity $(\dot{x}, \dot{y})$ and the vector (dx^u, dy^u) is uniformly bounded away from zero, and the same holds for the angle between $(\dot{x}, \dot{y})$ and (dx^s, dy^s).*

Having established hyperbolicity for the flow Φ^t, we can project unstable and stable vectors on the cross section M, and hence the following

Corollary 3.9 *The first return map $T : M \to M$ induced by the flow Φ^t is hyperbolic, too, with respect to any invariant measure.*

In the rest of this section, we prove that T is, essentially, a *uniformly* hyperbolic map.

Let $X = (r, \varphi) \in M$ and $V = (dr, d\varphi) \in \mathcal{T}_X M$. We call V an unstable vector if it is a tangent vector to the trace $\varphi = \varphi(r)$ of a strongly unstable family P_{st}. Similarly, V is a stable vector if it is tangent to the trace of a strongly stable family.

Lemma 3.10 (Uniform hyperbolicity - 1) *There is a constant $B_1 > 1$ such that for every nonzero unstable vector $V = (dr, d\varphi)$ we have $B_1^{-1} \leq d\varphi/dr \leq B_1$. Similarly, for any stable vector $V \neq 0$ we have $-B_1 \leq d\varphi/dr \leq -B_1^{-1}$. As a result, the angles between stable and unstable vectors are bounded away from zero.*

Proof. The lemma follows from (3.15), the bound $0 < \kappa^- \leq \kappa^-_{\max}$ in Corollary 3.5, and a similar bound $-\kappa^-_{\max} \leq \kappa^+ < 0$ for strongly convergent families. $\square$

Convention (on B's), We denote by $B_i > 0$ constants that only depend on the domain Q and the bounds on the function $p(x, y, \theta)$ and its derivatives. Such constants are called *global constants*.

Remark. All our claims about unstable vectors here have their obvious counterparts for stable vectors, as in the above lemma. For brevity, we will only state the claims for unstable vectors.

Lemma 3.11 (Uniform hyperbolicity - 2) *Let V and $\tilde{V}$ be two unstable vectors at a point $X \in M$ and T^n continuous at X. Then the angle between the unstable vectors $DT^n(V)$ and $DT^n(\tilde{V})$ at the point T^nX is less than $C\lambda^n$, where $C > 0$ and $\lambda \in (0,1)$ are global constants.*

In other words, the cones made by unstable vectors shrink uniformly and exponentially fast under DT^n as $n \to \infty$.

Proof. Let V and $\tilde{V}$ be tangent vectors to the traces $\varphi = \varphi(r)$ and $\varphi = \tilde{\varphi}(r)$ of two strongly unstable families P_{st} and $\tilde{P}_{st}$. According to (3.15), $|d\varphi/dr - d\tilde{\varphi}/dr| \leq |\kappa^- - \tilde{\kappa}^-| \cos\varphi$, hence it is enough to prove that $|\kappa_n^- - \tilde{\kappa}_n^-| \leq C\lambda^n$, where κ_n^- and $\tilde{\kappa}_n^-$ are taken at the point T^nX before the reflection. Note that $\Delta := \kappa - \tilde{\kappa}$ satisfies $d\Delta/d\tau = -(\kappa + \tilde{\kappa} - h_\theta)\Delta$ according to (3.10), and does not change at reflections due to Lemma 3.3. Hence, $|\Delta_\tau| \leq |\Delta_0| e^{-a\tau}$ where $a = 2\kappa_{\min} - \delta_0 > 0$. □

Now denote by $V_1 = (dr_1, d\varphi_1) = DT(V)$ the image of vector V under DT. It is a tangent vector at $X_1 = TX$. If V is an unstable vector, then so is V_1. Let V and V_1 be tangent vectors to the traces left on M by a strongly divergent family P_{st} at the points X and X_1, respectively. Denote by L the length of the trajectory segment on the torus between the points X and X_1, and parameterize that segment by the length parameter τ, $0 \leq \tau \leq L$. Denote by w^+, κ^+, etc. the quantities introduced in Sect. 3 taken for the family P_{st} immediately after the reflection at the point X, and by w_1^-, κ_1^-, etc. the corresponding quantities before the reflection at the point X_1.

Lemma 3.12 *For any unstable vector V*

$$e^{-\delta_2}[1 + \kappa^+ L] \leq \frac{|w_1^-|}{|w^+|} \leq e^{\delta_2}[1 + \kappa^+ L] \tag{3.24}$$

with some small $\delta_2 > 0$.

Proof. Combining (3.11) and (3.20) and integrating with respect to τ from 0 to L yields (3.24) with $\delta_2 := \delta_1 L_{\max}$. □

Note that integrating from 0 to any $\tau < L$ in the above proof gives

$$e^{-\delta_2}[1 + \kappa^+ \tau] \leq \frac{|w_\tau|}{|w^+|} \leq e^{\delta_2}[1 + \kappa^+ \tau]\ . \tag{3.25}$$

In the theory of dispersing billiards, a convenient norm of stable and unstable vectors is often used, it is called the p-norm: $|V|_p = \cos\varphi |dr|$, and respectively $|V_1|_p = \cos\varphi_1 |dr_1|$. This is not really a norm in $\mathcal{T}_X M$, since $|W|_p = 0$ for some $W \neq 0$, but at least $|V|_p > 0$ for every stable and unstable vector $V \neq 0$ due to Lemma 3.10. Now (3.24) can be rewritten as

$$e^{-\delta_2}[1 + \kappa^+ L] \leq \frac{|V_1|_p}{|V|_p} \leq e^{\delta_2}[1 + \kappa^+ L] \tag{3.26}$$

because $|V_1|_p / |V|_p = |w_1^-| / |w^+|$, as it follows by applying (3.14) to V and V_1.

Note that in the pure billiard dynamics $\delta_2 = 0$, and we recover a standard formula $|V_1|_p/|V|_p = 1 + \kappa^+ L$, see [Si].

The inequality (3.26) shows that the p-norm of unstable vectors grows monotonically and exponentially in time (= the number of collisions), i.e. for all $n \geq 1$

$$|DT^n(V)|_p/|V|_p \geq \Lambda^n \tag{3.27}$$

where $\Lambda > 1$ is a global constant, say

$$\Lambda = 1 + \kappa_{\min} L_{\min}/2 \ . \tag{3.28}$$

The p-metric plays the role of the so called adapted metric of Axiom A systems. It also follows from (3.19) and (3.26) that the expansion of V under DT is mainly determined by $\cos\varphi$:

$$\frac{B_2^{-1}}{\cos\varphi} \leq \frac{|V_1|_p}{|V|_p} \leq \frac{B_2}{\cos\varphi} \tag{3.29}$$

for some constant $B_2 > 0$.

We will primarily work with the Euclidean metric $|V| = \sqrt{(dr)^2 + (d\varphi)^2}$. It is clear that for stable and unstable vectors $V \neq 0$, which satisfy Lemma 3.10, we have

$$1 \leq \frac{|V|\cos\varphi}{|V|_p} \leq B_3 \tag{3.30}$$

for some constant $B_3 > 0$. Then (3.29) and (3.30) imply

$$\frac{B_4^{-1}}{\cos\varphi_1} \leq \frac{|DT(V)|}{|V|} \leq \frac{B_4}{\cos\varphi_1} \tag{3.31}$$

for some constant $B_4 > 0$.

Lemma 3.13 (Uniform hyperbolicity - 3) *For any unstable vector V where DT^n is defined*

$$|DT^n(V)|/|V| \geq B_5 \Lambda^n \tag{3.32}$$

for global constants $\Lambda > 1$ and $B_5 > 0$.

Proof. Indeed, due to (3.27), (3.29) and (3.30)

$$|DT^n(V)| \geq |DT^n(V)|_p \geq \Lambda^{n-1}|DT(V)|_p \geq \Lambda^{n-1}\frac{B_2^{-1}|V|_p}{\cos\varphi} \geq \Lambda^{n-1}B_2^{-1}B_3^{-1}|V| \ .$$

4 The properties of the billiard map $T : M \to M$

Here we study stable and unstable curves, and singularity curves, for the billiard map T on the cross-section M. Certain technical properties of those curves are necessary for the construction and further study of SRB measures. In the theory of dynamical systems, these properties are called *curvature bounds, distortion bounds, absolute continuity, alignment* etc. The proofs of these properties are, unfortunately, quite involved. To make things worse, the proofs are not always available even in the pure billiard case – some of these facts are just known as folklore, whose proofs have never been published. For the sake of completeness, we provide here full proofs of all these facts.

Definition. A smooth curve $\gamma \subset M$ given by $\varphi = \varphi(r)$ is called an *unstable curve* (or a *stable curve*) if it is the trace of a strongly divergent (resp., strongly convergent) family of trajectories.

Our "invariance principle" for strongly divergent families implies that the class of unstable curves is invariant under T^n, $n \geq 1$, and the class of stable curves is invariant under T^{-n}, $n \geq 1$. We will refer to this as the "invariance principle" for unstable curves.

Lemma 4.1 (Curvature bounds) *There are constants $B_{\max}$ and $\bar{B}_{\max}$ such that for any C^2 smooth unstable curve γ its images $T^n\gamma$ satisfy $|d^2\varphi/dr^2| \leq B_{\max}$ eventually, for all $n \geq n_\gamma$. Moreover, if γ itself satisfies $|d^2\varphi/dr^2| \leq B_{\max}$, then all its images $T^n\gamma$, $n \geq 1$, satisfy $|d^2\varphi/dr^2| \leq \bar{B}_{\max}$.*

We note that a similar property for pure billiard dynamics is known [Y1, Ch2], but hardly a complete proof was ever published. Our proof certainly covers the pure billiard case.

Proof. Let $\varphi = \varphi(r)$ be an unstable curve, the trace of a strongly divergent family P_{st}. Differentiating (3.15) gives

$$\frac{d^2\varphi}{dr^2} = \frac{dK(r)}{dr} + \frac{d\kappa^-}{dr}\cos\varphi - \kappa^- \sin\varphi\frac{d\varphi}{dr} + \frac{dh^-}{dr}\sin\varphi + h^-\cos\varphi\frac{d\varphi}{dr} \ .$$

Since ∂Q is C^3 smooth, the term dK/dr is bounded. The term κ^- is bounded by Corollary 3.5, and $d\varphi/dr$ is bounded by Lemma 3.10. Now, using (3.5) and (3.13) gives $dh^-/ds = h_R^- w^- + h_U^- w^- \tan\varphi$. Hence, due to (3.14), $dh^-/dr = (dh^-/ds)/(dr/ds) = h_R^- \cos\varphi + h_U^- \sin\varphi$, so $|dh^-/dr| \leq (4 + \kappa_{\max}^-)\delta_0$.

It then remains to estimate the term $d\kappa^-/dr$. First, according to (3.14)

$$\frac{d\kappa^-}{dr} = \frac{d\kappa^-}{ds}\Big/\frac{dr}{ds} = [(\kappa')^- + (\dot{\kappa})^- t'] \cdot \frac{\cos\varphi}{w^-} \ .$$

Substituting (3.7) and (3.13) gives

$$d\kappa^-/dr = (\kappa'/w)^- \cos\varphi - [(\kappa^-)^2 + (h^-)^2 - h_R^-] \cdot (\sin\varphi - \alpha^- \cos\varphi)$$

Here all the terms are bounded except, possibly, the term $(\kappa'/w)^-$. So, it is enough to prove that the quantity $\Xi := \kappa'/w$ is bounded by a global constant before every reflection. Direct differentiation and using (3.7) yields

$$d\kappa'/dt = d\dot{\kappa}/ds = -2p\kappa\kappa' - p_\theta w\kappa^3 - D_1 w\kappa^2 + ph_\theta\kappa'$$

where D_1 is an expression involving first and second order derivatives of the functions p and h. All those derivatives are bounded, since these functions are C^2 smooth on a compact manifold $\mathcal{M}$, hence $|D_1|$ is bounded by a global constant. Now, by using (3.8),

$$\begin{aligned} d\Xi/dt &= (d\kappa'/dt)/w - \kappa'\dot{w}/w^2 \\ &= -3p\kappa\Xi + ph_\theta\Xi - p_\theta\kappa^3 - D_1\kappa^2 \ . \end{aligned} \tag{4.1}$$

Now consider a reflection experienced by the family P_{st} and denote by Ξ^- and Ξ^+ the values of Ξ before and after the reflection. Differentiating (3.18)-(3.19) gives

$$\frac{d\kappa^+}{dr} = \frac{d\kappa^-}{dr} + \frac{2K(r)\sin\varphi}{\cos^2\varphi}\cdot\frac{d\varphi}{dr} + \frac{2K'}{\cos\varphi} + \frac{H_1}{\cos^2\varphi} - h_\theta^+\kappa^+\sin\varphi \ . \tag{4.2}$$

Here we denote $K' = dK/dr$, which is bounded on ∂Q, and H_1 is a small quantity, see below.

Convention (on D's and H's), We will denote by D_i variable quantities whose absolute values are bounded above by global constants, i.e. $|D_i| \le B_i$ for some global constant B_i. We will also denote by H_i variable quantities whose absolute values are bounded by some small constants depending on δ_0 in Assumption B, i.e. $|H_i| \le \delta_i^*$ where $\delta_i^* \to 0$ as $\delta_0 \to 0$, i.e. δ_i^* satisfy our convention on δ's.

Note that $d\kappa^+/dr = (d\kappa^+/ds)/(dr/ds) = [(\kappa')^+ + (\dot{\kappa})^+ t']/(-w^+/\cos\varphi)$ and, similarly, $d\kappa^-/dr = [(\kappa')^- + (\dot{\kappa})^- t']/(w^-/\cos\varphi)$, where we used (3.14). Substituting these into (4.2) and using (3.7), (3.15), (3.18), and (3.13) yields

$$\Xi^+ = -\Xi^- + \Delta\Xi \ , \tag{4.3}$$

where

$$\Delta\Xi = -\frac{6K^2(r)\sin\varphi}{\cos^3\varphi} + \frac{D_2}{\cos^2\varphi} + \frac{H_2}{\cos^3\varphi} \ . \tag{4.4}$$

Here D_2 is an expression involving $K', \kappa^-, \alpha^\pm$ and other bounded quantities.

Eqs. (4.1) and (4.3)-(4.4) completely describe the evolution of the quantity Ξ in time. Since (4.1) is a linear differential equation, we can decompose $\Xi = \Xi_1 + \Xi_2$ so that

$$d\Xi_1/dt = -3p\kappa\Xi_1 + ph_\theta\Xi_1 \quad \text{and} \quad d\Xi_2/dt = -3p\kappa\Xi_2 + ph_\theta\Xi_2 - p_\theta\kappa^3 - D_1\kappa^2 \tag{4.5}$$

and at every reflection

$$\Xi_1^+ = -\Xi_1^- \quad \text{and} \quad \Xi_2^+ = -\Xi_2^- + \Delta\Xi \ . \tag{4.6}$$

Initially, at a time t_0+0 when the family P_{st} just leaves the curve γ (its trace on M), we set $\Xi_1(t_0+0)=\Xi(t_0+0)$ and $\Xi_2(t_0+0)=0$.

Now, since $|\Xi_1|$ does not change during reflections, the first equation (4.5) implies that

$$|\Xi_1(t)| \le |\Xi_1(t_0)| \cdot e^{-a(t-t_0)} \qquad \text{for } t > t_0 \tag{4.7}$$

where $a = 3p_{\min}\kappa_{\min} - \delta_0 > 0$. Hence, the component Ξ_1 converges to zero exponentially fast.

Claim. There is a global constant $B_6 > 0$ such that $|\Xi_2(t-0)| \le B_6$ for every moment of reflection $t > t_0$.

We prove the claim inductively. Suppose $|\Xi_2(t_1-0)| \le B_6$ before a reflection at some time $t_1 > t_0$. During the interval from t_1 to the next reflection, t_2, we decompose $\Xi_2 = \Xi_{21} + \Xi_{22}$ as in (4.5), so that

$$d\Xi_{21}/dt = -3p\kappa\Xi_{21} + ph_\theta\Xi_{21} \quad \text{and} \quad d\Xi_{22}/dt = -3p\kappa\Xi_{22} + ph_\theta\Xi_{22} - p_\theta\kappa^3 - D_1\kappa^2 \tag{4.8}$$

and initially set $\Xi_{21}(t_1+0) = -\Xi_2(t_1-0)$ and $\Xi_{22}(t_1+0) = \Delta\Xi$, where $\Delta\Xi$ is given by (4.4) and taken at the reflection at t_1.

Similarly to (4.7), we now have

$$|\Xi_{21}(t_2-0)| \le |\Xi_2(t_1-0)| \cdot e^{-a(t_2-t_1)} \ . \tag{4.9}$$

The equation (4.4) shows that $\Xi_{22}(t_1+0) = \Delta\Xi$ is of order $O(1/\cos^3\varphi) = O(\kappa^3(t_1+0))$. It is then convenient to "link" Ξ_{22} with κ^3 and consider the ratio $g(t) := \Xi_{22}(t)/\kappa^3(t)$. First, by (3.19) and (4.4)

$$g(t_1+0) = \Xi_{22}(t_1+0)/\kappa^3(t_1+0) \le B'$$

with some global constant B'. Then, (4.8) and (3.7) imply

$$dg/dt = -p_\theta + H_3 g + D_3/\kappa \ .$$

Hence, $|g|$ stays bounded by a global constant between the two reflections, i.e. $|g(t)| \le B''$ for all $t_1 < t < t_2$. Therefore, $|\Xi_{22}(t_2-0)| \le B''(\kappa^-_{\max})^3$ and

$$|\Xi_2(t_2-0)| \le B_6 e^{-at_{\min}} + B''(\kappa^-_{\max})^3 \ .$$

Hence, an appropriate choice of B_6 ensures that $|\Xi_2(t_2-0)| \le B_6$. The claim is proved.

Lemma 4.1 now easily follows. □

Convention. In all that follows we will only consider unstable curves that satisfy $|d^2\varphi/dr^2| \le B_{\max}$. Hence, all our unstable curves will have uniformly bounded geometric curvature. The same goes, of course, to stable curves. We also assume that the "invariance principle" for unstable curves holds, as we may in view of Lemma 4.1.

This convention is equivalent to the requirement that for any strongly divergent family, immediately before any reflection,

$$\Xi^- = (\kappa')^-/w^- \le B_7 \tag{4.10}$$

for some global constant B_7.

Also note that the proof of the claim in the proof of Lemma 4.1 implies that, under the above convention, all strongly divergent families satisfy

$$|\Xi|/\kappa^3 = |\kappa'|/(\kappa^3|w|) \le B_8 \tag{4.11}$$

for some global constant B_8. This will be used later.

We now turn to distortion bounds, but first, a remark is in order. Let γ be an unstable curve on which T^n is continuous for some $n \ge 1$. We know that T^n expands γ exponentially fast in n, due to (3.32). We now need to compare the expansion rates at different points of γ and ensure that those rates vary slowly over γ (this property is referred to as 'bounded distortions'). However, at almost grazing reflections, when $\cos\varphi \approx 0$, the expansion of unstable curves is highly nonuniform, and so distortions are unbounded. To fix the situation, we consider the so called homogeneous unstable curves.

We partition M into countably many rectangular domains I_k, for $k = 0$ and $|k| \ge k_0$, where $k_0 > 1$ is a large constant to be specified later. For every $k \ge k_0$ we put

$$I_k = \{(r,\varphi) :\, \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\}$$

and

$$I_{-k} = \{(r,\varphi) :\, -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\}$$

and lastly

$$I_0 = \{(r,\varphi) :\, -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}\ .$$

The domains I_k are called homogeneity strips, they are also used in the study of pure billiard systems [BSC2, Y1, Ch2].

We say that an unstable curve $\gamma \subset M$ is *homogeneous* if it is entirely contained in one homogeneity strip I_k. Note that if γ is a homogeneous unstable curve, then for every point $X = (r,\varphi) \in \gamma$ we have

$$\cos\varphi \ge B_9^{-1}|\gamma|^{2/3} \tag{4.12}$$

where $B_9 > 0$ is a global constant. Here and on $|\gamma|$ denotes the length of γ in the Euclidean metric $(dl)^2 = (dr)^2 + (d\varphi)^2$.

Let γ be an unstable curve, $X \in \gamma$ and T^n continuous at X. Denote by $J_{\gamma,n}(X)$ the expansion factor of the curve γ under T^n at the point X, i.e. $J_{\gamma,n}(X) := |DT^nV|/|V|$ for any tangent vector V to γ at X.

Lemma 4.2 (Distortion bounds) *Let γ be an unstable curve on which T^n is continuous. Assume that $\gamma_i := T^i\gamma$ is a homogeneous unstable curve for each $0 \le i \le n$. Then for all $X, Y \in \gamma$*

$$|\ln J_{\gamma,n}(X) - \ln J_{\gamma,n}(Y)| \le B_{10}|\gamma_n|^b$$

for some global constants $B_{10} > 0$ and $b > 0$ (in fact, $b = 1/3$).

We note that the corresponding property for pure billiard dynamics is known [Ch2], but only a proof of a somewhat weaker statement was published [BSC2]. Our proof covers the pure billiard case, too.

Proof. Note that $J_{\gamma,n}(X) = \prod_{i=0}^{n-1} J_{\gamma_i,1}(T^iX)$. Hence, it is enough to prove the lemma for $n = 1$, because $|\gamma_i|$ grows exponentially in i due to (3.32). So we put $n = 1$.

Let P_{st} be a strongly divergent family whose trace on M is the curve γ. We will use the notation adopted before Lemma 3.12. Consider $J_{\gamma,1}(X)$ as a function of $X_1 = (r_1, \varphi_1) = TX \in \gamma_1$, and parameterize γ_1 by r_1. It is enough to prove that

$$\left|\frac{d\ln J_{\gamma,1}}{dr_1}\right| \le \frac{B}{|\gamma_1|^{2/3}} \tag{4.13}$$

for some global constant $B > 0$. Then Lemma 4.2 (with $n = 1$) would follow by integration over γ_1. The bound (4.13), in turn, follows from

$$\left|\frac{d\ln J_{\gamma,1}}{dr_1}\right| \le \frac{B\cos\varphi_1}{\cos\varphi} + \frac{B}{\cos\varphi_1} \tag{4.14}$$

with a global constant $B > 0$, by applying (4.12) to both γ and γ_1, and because $|\gamma| \ge B_4^{-1}|\gamma_1|\cos\varphi_1$, which follows from (3.31).

We now prove (4.14). We have $|V| = |dr|\sqrt{1 + (d\varphi/dr)^2} = |ds|\,|w^+|(\cos\varphi)^{-1}$ $\sqrt{1 + (d\varphi/dr)^2}$, and similarly for $|V_1|$, hence

$$J_{\gamma,1}(X) = \frac{|V_1|}{|V|} = \frac{|w_1^-|}{|w^+|} \cdot \frac{\cos\varphi}{\cos\varphi_1} \cdot \frac{\sqrt{1 + (d\varphi_1/dr_1)^2}}{\sqrt{1 + (d\varphi/dr)^2}} = J' \cdot J'' \cdot J'''$$

where J', J'', J''' simply denote the first, second and third factors in this expression. We bound them separately. First,

$$\left|\frac{d\ln J'''}{dr_1}\right| \le \frac{|d\varphi_1/dr_1| \cdot |d^2\varphi_1/dr_1^2|}{1 + (d\varphi_1/dr_1)^2} + \frac{|d\varphi/dr| \cdot |d^2\varphi/dr^2|}{1 + (d\varphi/dr)^2} \cdot \left|\frac{dr}{dr_1}\right| .$$

Note that $|dr/dr_1| \le B_4' \cos\varphi_1$ for some global constant $B_4' > 0$ due to (3.31). Hence, $|d\ln J'''/dr_1|$ is uniformly bounded due to Lemmas 3.10 and 4.1.

Next,

$$\left|\frac{d\ln J''}{dr_1}\right| \le \left|\frac{d\varphi_1/dr_1}{\cos\varphi_1}\right| + \left|\frac{d\varphi/dr}{\cos\varphi}\right| \cdot \left|\frac{dr}{dr_1}\right| \le \frac{B_1}{\cos\varphi_1} + \frac{B_1 B_4' \cos\varphi_1}{\cos\varphi}$$

as required by (4.14).

It remains to consider $\ln J'(X) = \int_{t_0}^{t_1} \kappa p\, dt$, cf. (3.8), where t_0 and t_1 denote the moments of reflection at X and X_1, respectively. First, $d\ln J'/dr_1 = (d\ln J'/ds)/(dr_1/ds)$, and

$$\begin{aligned} d\ln J'/ds &= -\kappa^+ p^+ dt_0/ds + \kappa_1^- p_1^- dt_1/ds + \int_{t_0}^{t_1} (\kappa p' + \kappa' p)\, dt \\ &= \kappa^+(w^+ \tan\varphi + v^+) + \kappa_1^-(w_1^- \tan\varphi_1 - v_1^-) + \int_{t_0}^{t_1} (\kappa p' + \kappa' p)\, dt \ . \end{aligned}$$

Note that $|w^+/w_1^-| \le 2(\kappa^+ L_{\min})^{-1}$ by (3.24) and for all $w = w(t)$, $t_0 < t < t_1$, we also have by (3.25)

$$|w/w_1^-| \le |w/w^+| \cdot |w^+/w_1^-| \le 4L_{\min}^{-1}[(\kappa^+)^{-1} + \tau] \ . \tag{4.15}$$

Combining the above formulas and (3.14) yields

$$\begin{aligned} |d\ln J'/dr_1| \le \quad & 2 \quad L_{\min}^{-1}|\tan\varphi + \alpha^+|\cos\varphi_1 + \kappa_{\max}^-(|\sin\varphi_1| + |\alpha_1^-|\cos\varphi_1) \\ & + \quad \cos\varphi_1 \int_{t_0}^{t_1} |\kappa p' + \kappa' p|/|w_1^-|\, dt \ . \end{aligned}$$

The first two terms are clearly properly bounded, as required by (4.14). The integral term can be estimated by (3.5) and (4.11), so it does not exceed

$$\cos\varphi_1 \int_{t_0}^{t_1} \kappa\, |p_U\alpha + p_R| \cdot |w/w_1^-| + B_8\kappa^3 p\, |w/w_1^-|\, dt \ .$$

Using (4.15) and an obvious $|p_U\alpha + p_R| \le \mathrm{const}\cdot(1+\kappa)$ shows that the last expression does not exceed

$$\cos\varphi_1 \int_{t_0}^{t_1} B'\kappa^3 p |w/w_1^-|\, dt \le \cos\varphi_1 \int_{t_0}^{t_1} B''[(\kappa^+)^{-1} + \tau]^{-2} p\, dt$$

where B' and B'' are some global constants. A direct integration shows that the last expression is bounded by

$$\cos\varphi_1 \int_0^L B''[(\kappa^+)^{-1} + \tau]^{-2}\, d\tau \le \mathrm{const} \cdot \cos\varphi_1 \kappa^+ \ .$$

Since $\kappa^+ \le \mathrm{const}\cdot(1 + 1/\cos\varphi)$, the last expression is properly bounded as required by (4.14). This completes the proof of (4.14). Lemma 4.2 is proved. □

Before we turn to the absolute continuity, one useful observation should be made.

Volume compression. The volume $dV = dx\,dy\,d\theta$ in the phase space $\mathcal{M}$ is not necessarily invariant under Φ^t. Its rate of change is given by the divergence of the flow Φ^t :

$$\frac{d}{dt}(\ln dV_t) = p_x \cos\theta + p_y \sin\theta + p_\theta h + ph_\theta = p_U + ph_\theta \ . \tag{4.16}$$

Under our assumptions, ph_θ is small. The function p_U has uniformly bounded integrals along orbits, see Remark before Lemma 3.6. Therefore, for all $X \in \mathcal{M}$ and $t > 0$

$$B_{11}^{-1} e^{-\delta_4 t} < |d\Phi^t(X)| < B_{11} e^{\delta_4 t} \tag{4.17}$$

for some small constant $\delta_4 > 0$ and a global constant B_{11}. Also, let P_{st} be a strongly convergent or divergent family on a time interval (t_1, t_2) that does not experience singularities (grazing reflections) for $t_1 < t < t_2$. Then for any $X, Y \in P_{st_1}$ we have

$$B_{12}^{-1} < |d\Phi^{t_2 - t_1}(X)| / |D\Phi^{t_2 - t_1}(Y)| < B_{12} \tag{4.18}$$

with a global constant B_{12}. Indeed, the trajectories $\Phi^t X$ and $\Phi^t Y$ exponentially converge to each other due to the uniform hyperbolicity of Φ^t, and the smoothness of the functions in (4.16) then proves (4.18).

Similar inequalities hold for the map T and the element $d\nu_0$ of the smooth invariant measure ν_0 of the billiard map T_0. Recall that $d\nu_0 = \text{const} \cdot \cos\varphi\, dr\, d\varphi$. The elements dV and $d\nu_0$ are related by

$$dV = dx\,dy\,d\theta = p\cos\varphi\, dr\, d\varphi\, dt = \text{const} \cdot p\, d\nu_0\, dt \tag{4.19}$$

in the immediate vicinity of the cross-section M. Note that the corresponding relation for pure billiard systems (with $p = 1$) holds everywhere in the phase space $\mathcal{M}_0$, cf. [Si].

Denote by $|DT^n|_0$ the Jacobian of T^n with respect to the measure ν_0. Now (4.17) and (4.19) imply

$$B_{11}^{-1} e^{-\delta_4 n} < |DT^n|_0 < B_{11} e^{\delta_4 n} \ . \tag{4.20}$$

Also, let $\gamma \subset M$ be a stable or unstable curve on which T^n is continuous, and $T^n\gamma$ also stable (resp., unstable) curve. Then for any $X, Y \in \gamma$ (4.18) and (4.19) imply

$$B_{12}^{-1} < |DT^n(X)|_0 / |DT^n(Y)|_0 < B_{12} \ . \tag{4.21}$$

We use the same notation δ_4, B_{11}, B_{12} here, even though the values of these constants in (4.17)-(4.18) and (4.20)-(4.21) may be different.

Lemma 4.3 (Absolute continuity) *Let ξ be a stable curve, $X, Y \in \xi$, and γ_1, γ_2 two unstable curves crossing ξ at X and Y, respectively. Assume that T^n is continuous on ξ and $T^i\xi$ is a homogeneous stable curve for each $0 \le i \le n$. Then*

$$|\ln J_{\gamma_1, n}(X) - \ln J_{\gamma_2, n}(Y)| \le B_{13} \tag{4.22}$$

where B_{13} is a global constant.

Proof. For any $Z \in \xi$, let $J_{\xi,n}(Z)$ be the contraction factor of ξ under T^n at the point Z, i.e. $J_{\xi,n}(Z) = |DT^n(V)|/|V|$ for any tangent vector V to ξ at Z. By Lemma 4.2 (applied to ξ) we have

$$|\ln J_{\xi,n}(X) - \ln J_{\xi,n}(Y)| \le B_9 |\xi|^b \le B' \tag{4.23}$$

for a global constant B'.

Let $|DT^n(Z)|_e$ denote the Jacobian of T^n at $Z \in M$ with respect to the Lebesgue measure $dr\, d\varphi$ on M, i.e. $|DT^n(Z)|_e = |DT^n(Z)|_0 \cos\varphi(Z)/\cos\varphi(T^n Z)$. Since both ξ and $T^n\xi$ are homogeneous curves, (4.21) implies

$$B^{-1} < |DT^n(X)|_e/|DT^n(Y)|_e < B \tag{4.24}$$

for a global constant B. Now (4.23) and (4.24), along with Lemma 3.10, prove (4.22). □

Next, we describe the singularities of the map T. Let $\mathcal{S}_0 = \partial Q \times \{\varphi = \pm\pi/2\}$ be the natural boundary of M. Put $\mathcal{S}_n = T^n \mathcal{S}_0$ for all $n \in \mathbb{Z}$, and $\mathcal{S}_{m,n} = \cup_{i=m}^n \mathcal{S}_i$ for $-\infty \le m \le n \le \infty$. On the sets $\mathcal{S}_{-n,-1}$ and $\mathcal{S}_{1,n}$ the maps T^n and T^{-n}, respectively, are discontinuous.

We will also need the set

$$\mathcal{D}_0 = \cup_{k \ge k_0} \{\varphi = \pm(\pi/2 - k^{-2})\}$$

the union of countably many parallel lines in M separating the homogeneity strips. Put $\mathcal{D}_n = T^n \mathcal{D}_0$ for all $n \in \mathbb{Z}$, and $\mathcal{D}_{m,n} = \cup_{i=m}^n \mathcal{D}_i$ for $-\infty \le m \le n \le \infty$.

Lemma 4.4 (Alignment) *For each $n \ge 1$ the set $\mathcal{S}_n$ is a finite union of C^2 unstable curves. The set $\mathcal{S}_{-n}$ is finite union of stable curves. Similarly, the set $\mathcal{D}_n$ is a countable union of unstable curves and $\mathcal{D}_{-n}$ is a countable union of stable curves. The curvature of all these curves in M is bounded by a global constant.*

Proof. One only need to prove this for $n = 1$, due to the invariance of unstable (stable) curves under T (resp., T^{-1}). Since the curves of $\mathcal{D}_0$ converge to $\mathcal{S}_0$, then their images (components of $\mathcal{D}_1$) converge to $\mathcal{S}_1$, so it is enough to prove the lemma for $\mathcal{D}_1$. Consider a curve γ in $\mathcal{D}_0$ given by $\varphi = \varphi_0 = \pm(\pi/2 - k^{-2})$ with some $|k| \ge k_0$. It is the trace of a family P_{st} that can be naturally parameterized by $s = r$, and we set $t = 0$ on that curve. Note that x', y' is a unit tangent vector to ∂Q. It is then easy to compute $v^+ = \sin\varphi_0$, $w^+ = -\cos\varphi_0$, $(\theta')^+ = -K(r)$, and $\kappa^+ = (K(r) + h^+ \sin\varphi_0)/\cos\varphi_0$. Hence, $\kappa^+ \ge \Theta_{\min}$ and so the family P_{st} is strongly divergent for $t > 0$.

We now prove the boundedness of curvature. The above natural parameterization $r = s$ does not satisfy our convention on α's when k is large. But for any point $X = (r, \varphi) \in \gamma$ we can reparameterize the outgoing family P_{st}, $t > 0$, with a new parameter s so that $v^+ = 0$ and $w = 1$ at X. In this parameterization, as one can compute directly,

$$(\kappa')^+ = -\frac{dK(r)/dr + \sin\varphi_0 dh^+/dr}{\cos^2\varphi_0} + \frac{p\sin\varphi_0[(\kappa^+)^2 + (h^+)^2 - h_R^+]^2}{\cos^3\varphi_0}\ .$$

Now we see that $(\kappa')^+ = O(\cos^{-3}\varphi_0) = O((\kappa^+)^3)$, so we are in the position of the proof of the claim in the proof of Lemma 4.1. Just like then, we get a uniform bound $\kappa' \le B_6$ before the next reflection occurs. This proves that the curvature of the curve $T\gamma$ is bounded by a global constant. $\square$

Corollary 4.5 *Unstable curves are uniformly transversal to the boundary $\partial M = \mathcal{S}_0$ and to the components of the singularity set $\mathcal{S}_{-n}$, $n \ge 1$ (and to those of $\mathcal{D}_{-n}$, $n \ge 1$). Stable curves are uniformly transversal to the boundary $\partial M = \mathcal{S}_0$ and to the components of the singularity set $\mathcal{S}_n$, $n \ge 1$ (and to those of $\mathcal{D}_n$, $n \ge 1$).*

The following continuation property is standard [BSC2, Ch2]:

Remark (Continuation property). Each endpoint, X, of every smooth curve $\gamma \subset \mathcal{S}_{-n,0}$, $n \ge 1$, lies either on $\mathcal{S}_0 = \partial M$ or on another smooth curve $\gamma' \subset \mathcal{S}_{-n,0}$ that itself does not terminate at X. Hence, each curve $\gamma \in \mathcal{S}_{-n,0}$ can be continued monotonically up to $\mathcal{S}_0 = \partial M$ by other curves in $\mathcal{S}_{-n,0}$.

5 Growth of unstable curves

Here we discuss iterations of unstable curves under the action of T. We prove a version of the so called "growth lemma", a key element in the modern studies of ergodic and statistical properties of hyperbolic dynamical systems.

Let $\gamma \subset M$ be an unstable curve of small length ε and $m \ge 1$. The map T^m is defined on $\gamma \setminus \mathcal{S}_{-m,0}$. By the "invariance principle" for unstable curves, the set $\gamma_m := T^m(\gamma \setminus \mathcal{S}_{-m,0})$ is a union of some unstable curves. Denote by $K_m(\gamma)$ the number of those curves (connected components of γ_m). By Lemma 3.13 (uniform hyperbolicity) the total length of γ_m is $\ge B_5\Lambda^m\varepsilon$. However, the effect of growth of γ_m with m may be effectively eliminated if $B_5\Lambda^m \ll K_m(\gamma)$. In that case applying T^m to γ may produce nothing but a bunch of curves that are even shorter that γ. If that happens for all m, the very existence of SRB measures would be doubtful, if not hopeless. Fortunately, $K_m(\gamma)$ only grows linearly with m, provided ε is small enough. We prove this below.

First, note that $K_m(\gamma)-1$ is the number of points of intersection $\gamma \cap \mathcal{S}_{-m,-1}$. A point $X \in M$ where $k \ge 2$ smooth curves of the set $\mathcal{S}_{-m,0}$ meet is called a *multiple singularity point*, and k is its *multiplicity*. Denote by K_m the maximal multiplicity of all $X \in M$ for a given m.

Lemma 5.1 *For each $m \ge 1$ there is an $\varepsilon_m > 0$ such that for any unstable curve $\gamma \subset M$ of length $\varepsilon < \varepsilon_m$ we have $K_m(\gamma) \le K_m$.*

The lemma easily follows from the properties of unstable curves and the singularity set $\mathcal{S}_{-m,0}$ proved in the previous section.

Lemma 5.2 (Multiplicity bound) *There is a global constant $C_0 > 0$ such that $K_m \le C_0 m$ for all $m \ge 1$.*

We note that a linear bound on K_m was first observed by Bunimovich for pure billiard dynamics, see [BSC2]. It is now understood that it is the continuity of the flow Φ^t that implies the linear bound on K_m. We give a proof of this fact different from the original one in [BSC2].

Proof. If K_m curves of $\mathcal{S}_{-m,0}$ meet at X, then a neighborhood $U(X)$ of X is divided by those curves into some L_m parts (sectors), and clearly $K_m \leq L_m$. We now will show that $L_m \leq C_0 m$ for some $C_0 > 0$.

On each of the L_m parts of $U(X)$ the map T^m is continuous and can be extended by continuity to the point X. Thus, $T^m X$ can be defined in L_m different ways. To see exactly how that happens, first note that the real time trajectory $\Phi^t X$ is well and uniquely defined for all $t > 0$. This trajectory may be tangent to ∂Q at one or more points. We call such points tangent (grazing) reflections. Now, the $L_m > 1$ different versions of T^m at X are possible precisely when the trajectory $\Phi^t X$ has tangential reflections: each of those reflections can be counted as either a "hit" (making an iteration of T) or a "miss" (skipping it in the construction of T).

Note that the real time elapsed until the mth iteration of T (in any of its versions) is less than $m\tau_{\max}$. Hence, there can be no more that $C_1 m$ reflections (both tangential and regular ones) involved in the construction of T^m at X, where $C_1 = \tau_{\max}/\tau_{\min}$ is a global constant. Let $\tilde{m} \leq C_1 m$ be the number of tangential reflections among the first $C_1 m$ reflections on the trajectory $\Phi^t X$. It seems that, with a choice of hit or miss at every tangential reflection, we would have up to $2^{\tilde{m}}$ versions of T^m at X. That would be too many for us. Fortunately, relatively few sequences of hits and misses materialize, as we show next.

Note that there can be no more than C_1 tangential reflections in a row. Consider a string of p consecutive tangential reflections on the trajectory $\Phi^t X$, $t > 0$, with $1 \leq p \leq C_1$. Let $Y' = \Phi^{t'} X \in M$ be the last regular reflection point on the trajectory $\Phi^t X$ before the above string. If there are previous tangential reflections on $\Phi^t X$, $0 < t < t'$, then the neighborhood $U(Y') \subset M$ is already divided into some L' parts (sectors) according to the hit/miss sequences arisen in those reflections. The boundaries of those L' sectors of $U(Y')$ are curves in $\mathcal{S}_{1,n'}$ for some $n' > 0$, so they are increasing curves in the r, φ coordinates (by Lemma 4.4). Now, there are at most 2^p possible hit/miss sequences on the string of p tangential reflections that we have right after the point Y'. Accordingly, $U(Y')$ is divided into $\leq 2^p$ parts (sectors) along some curves in $\mathcal{S}_{-p,-1}$, which are decreasing curves (by Lemma 4.4). So, we have two partitions of $U(Y')$: one into L' sectors by increasing curves, and the other into $\leq 2^p$ sectors by decreasing curves. These two partitions combined divide $U(Y')$ into no more than $L' + 2^p$ parts, as it is clear from Fig. 2. So, each string of p consecutive tangential reflections adds $\leq 2^p$ (i.e., $\leq 2^{C_1}$) parts to the partition of $U(X)$ by $\mathcal{S}_{-m,-1}$. Hence, $L_m \leq 2^{C_1} C_1 m$. $\square$

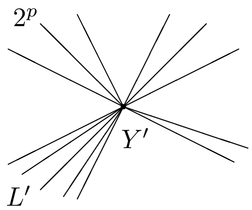

Figure 2. The partition of the neighborhood $U(Y')$.

Lemmas 5.1-5.2 effectively guarantee the growth of sufficiently short unstable curves under T^m. Precisely, if m is large enough, and the unstable curve γ is short enough, then the expansion factor $B_5\Lambda^m$ of γ under T^m is larger than the "cutting factor" $K_m(\gamma) \le C_0 m$.

We can now proceed exactly as in [Ch2]. A scheme developed there for the pure billiard dynamics perfectly works for us here, it can be repeated almost word by word. We refer the reader to [Ch2] and only describe certain major steps in the scheme necessary for our further analysis.

We start by cutting M along the boundaries of the homogeneity strips I_k thus making $M = \cup_k I_k$ a disconnected countable union of strips I_k. This makes the map T discontinuous on the set $\Gamma = \mathcal{S}_{-1} \cup \mathcal{D}_{-1}$. Note that after cutting M into these strips, any connected unstable curve $\gamma \subset M$ will be automatically homogeneous.

Then we fix a higher iteration $T_1 = T^m$ of the map T, with m picked so that $C_0 m < B_5\Lambda^m - 1$. The map T_1 uniformly expands unstable vectors: $|DT_1(V)| \ge \Lambda_1|V|$ with $\Lambda_1 := B_5\Lambda^m > 1$ for all unstable vectors V by Lemma 3.13. The map T_1 has singularity set $\Gamma_1 = \Gamma \cup T^{-1}\Gamma \cup \cdots \cup T^{-m+1}\Gamma = \mathcal{S}_{-m,-1} \cup \mathcal{D}_{-m,-1}$. Note also that $\Lambda_1 > K_m + 1$ by Lemma 5.2, so that T_1 expands sufficiently short unstable curves faster than the singularity set $\mathcal{S}_{-m,-1}$ breaks them into pieces.

For any smooth curve $\gamma \subset M$ we denote by ρ_γ the metric on γ induced by the Euclidean metric on M and by m_γ the Lebesgue measure on γ generated by ρ_γ. Note that $m_\gamma(\gamma) = |\gamma|$ is the length of the curve γ.

An important remark is now in order. Let γ be a homogeneous unstable curve, $n \ge 1$, and $\xi \subset T_1^n\gamma$ any connected (and hence homogeneous and unstable) curve. Consider the measure $m_\xi^{(n)} := T_{1,*}^n m_\gamma|_\xi$, i.e. the image of the Lebesgue measure m_γ under $T_1^n = T^{mn}$ conditioned on ξ. It is a probability measure on ξ absolutely continuous with respect to the Lebesgue measure m_ξ, and its density

$f_\xi^{(n)} = dm_\xi^{(n)}/dm_\xi$ satisfies

$$\frac{f_\xi^{(n)}(X)}{f_\xi^{(n)}(Y)} = \frac{J_{\gamma,mn}(T^{-mn}Y)}{J_{\gamma,mn}(T^{-mn}X)} \qquad \text{for all } X, Y \in \xi \ . \tag{5.1}$$

Lemma 4.2 (distortion bounds) implies that

$$|\ln f_\xi^{(n)}(X) - \ln f_\xi^{(n)}(Y)| \le B_{10}|\xi|^{1/3} \ . \tag{5.2}$$

Key Remark. By making $|\xi|$ smaller, we can make the the density $f_\xi^{(n)}$ almost constant on ξ, uniformly in ξ, γ and n. In what follows we only work with unstable curves of small length, less than some $\rho_0 > 0$. We will assume that ρ_0 is small enough, hence all the measures $m_\xi^{(n)}$ on curves $\xi \subset T_1^n\gamma$ will be almost uniform.

For $n \ge 1$ denote by $\Gamma_1^{(n)} = \Gamma_1 \cup T_1^{-1}\Gamma_1 \cup \cdots \cup T_1^{n-1}\Gamma_1$ the singularity set for T_1^n. For any $\delta > 0$ let $\mathcal{U}_\delta$ denote the δ-neighborhood of the set $\Gamma_1 \cup \partial M$.

Let $\rho_0 > 0$, $n \ge 0$, and $\gamma \subset M$ an unstable curve (which is automatically homogeneous). Let $\xi \subset \gamma$ be a disjoint union of open subintervals of γ, and for every $X \in \xi$ denote by $\xi(X)$ the subinterval of ξ containing the point X. We call ξ a (ρ_0, n)-subset (of γ) if for every $X \in \xi$ the set $T_1^n\xi(X)$ is a single homogeneous unstable curve of length $\le \rho_0$ (in particular, ξ does not intersect the set $\Gamma_1^{(n)}$). Define a function $r_{\xi,n}$ on ξ by

$$r_{\xi,n}(X) = \rho_{T_1^n\xi(x)}(T_1^nX, \partial T_1^n\xi(X)) \tag{5.3}$$

which is simply the distance from T_1^nX to the nearest endpoint of the curve $T_1^n\xi(X)$ (measured along this curve). In particular, note that $r_{\gamma,0}(X) = \rho_\gamma(X, \partial\gamma)$. We will use shorthand $m_\gamma(r_{\xi,n} < \varepsilon)$ for $m_\gamma(X \in \xi : \ r_{\xi,n}(X) < \varepsilon)$

Proposition 5.3 ("Growth lemma") *There is a global constant $\alpha_0 \in (0,1)$ and positive global constants $\beta_0, \beta_1, \beta_2, \kappa, \sigma, \zeta$ with the following property. For any sufficiently small $\rho_0, \delta > 0$ and any homogeneous unstable curve $\gamma \subset M$ of length $\le \rho_0$, there is an open $(\rho_0, 0)$-subset $\xi_\delta^0 \subset \gamma \cap \mathcal{U}_\delta$ and an open $(\rho_0, 1)$-subset $\xi_\delta^1 \subset \gamma \setminus \mathcal{U}_\delta$ (one of these subsets may be empty) such that $m_\gamma(\gamma \setminus (\xi_\delta^0 \cup \xi_\delta^1)) = 0$ and for all $\varepsilon > 0$ we have*

$$m_\gamma(r_{\xi_\delta^1,1} < \varepsilon) \le \alpha_0\Lambda_1 \cdot m_\gamma(r_{\gamma,0} < \varepsilon/\Lambda_1) + \varepsilon\beta_0\rho_0^{-1}m_\gamma(\gamma) \ , \tag{5.4}$$

$$m_\gamma(r_{\xi_\delta^0,0} < \varepsilon) \le \beta_1\delta^{-\kappa}\, m_\gamma(r_{\gamma,0} < \varepsilon) \tag{5.5}$$

and

$$m_\gamma(\xi_\delta^0) = m_\gamma(\gamma \cap \mathcal{U}_\delta) \le \beta_2\, m_\gamma(r_{\gamma,0} < \zeta\delta^\sigma) \ . \tag{5.6}$$

A general meaning of the above inequalities is the following: (5.4) ensures that the curves in the set $T_1\xi_\delta^1$ are, on the average, long enough; (5.6) asserts that the total measure of the set ξ_δ^0 is small enough; and (5.5) guarantees that the connected components of ξ_δ^0 are not too tiny (hence, they will grow under T_1^n fast enough).

The proof of this proposition repeats word by word the proof of an identical proposition for the pure billiard case. That proof was given in [Ch2] (see the proof of the estimates (2.6)–(2.8) in Section 7 there). It was based on certain facts about billiards which were all listed in [Ch2]. Here we have proved the corresponding facts for our model in Sections 3 and 4. We even tried to use similar notation for the convenience of the reader. Thus here we can refer to [Ch2] for the proof of the above proposition.

Corollary 5.4 *For any sufficiently small $\rho_0 > 0$ and any homogeneous unstable curve $\gamma \subset M$ of length $\le \rho_0$ there is an open $(\rho_0, 1)$-subset $\xi^1 \subset \gamma$ such that $m_\gamma(\gamma \setminus \xi^1) = 0$ and for all $\varepsilon > 0$ we have*

$$m_\gamma(r_{\xi^1,1} < \varepsilon) \le \alpha_0\Lambda_1 \cdot m_\gamma(r_{\gamma,0} < \varepsilon/\Lambda_1) + \varepsilon\beta_0\rho_0^{-1} m_\gamma(\gamma) \ . \tag{5.7}$$

Also, for any $n \ge 2$ there is an open (ρ_0, n)-subset $\xi^n \subset \gamma$ such that $m_\gamma(\gamma \setminus \xi^n) = 0$ and for all $\varepsilon > 0$ we have

$$\begin{aligned} m_\gamma(r_{\xi^n,n} < \varepsilon) &\le (\alpha_1\Lambda_1)^n \cdot m_\gamma(r_{\gamma,0} < \varepsilon/\Lambda_1^n) + \varepsilon\beta_3\rho_0^{-1}(1 + \alpha_1 + \cdots + \alpha_1^{n-1}) m_\gamma(\gamma) \\ &\le \alpha_1^n\varepsilon + \varepsilon\beta_3\rho_0^{-1}(1-\alpha_1)^{-1} m_\gamma(\gamma) \ . \end{aligned} \tag{5.8}$$

Lastly, for all sufficiently small $\delta > 0$ we have

$$\begin{aligned} m_\gamma(\gamma \cap T_1^{-n}\mathcal{U}_\delta) &\le \beta_4\, m_\gamma(r_{\xi^n,n} < \zeta\delta^\sigma) \\ &\le \beta_4\alpha_1^n\zeta\delta^\sigma + \beta_4\zeta\delta^\sigma\beta_3\rho_0^{-1}(1-\alpha_1)^{-1} m_\gamma(\gamma) \ . \end{aligned} \tag{5.9}$$

Here $\alpha_1 \in (\alpha_0, 1)$ and $\beta_3 > \beta_0$, $\beta_4 > \beta_2$ are some global constants.

Proof. The bound (5.7) follows from (5.4) by taking the limit $\delta \to 0$. The bound (5.8) follows from (5.7) by induction on n, this induction argument was explained in detail on pp. 432–433 in [Ch1]. The first inequality in (5.9) is obtained by applying (5.6) to every connected curve in $T_1^n\xi^n$, where ξ^n is the set involved in (5.8). The second inequality in (5.9) then follows directly from the bound (5.8).

We note that the necessity to slightly increase the constants $\alpha_0, \beta_0, \beta_2$ (to $\alpha_1, \beta_3, \beta_4$ respectively) results from the slight non-uniformity of the measure $m_\xi^{(n)}$ with respect to the Lebesgue measure m_ξ on every connected component ξ of the set $T_1^n\xi^n$. In view of our Key Remark, we can make $\rho_0 > 0$ small enough, so that the increase of α_0 will be small, hence α_1 will be still less than one, because the requirement $\alpha_1 < 1$ is crucial. □

Now we fix a $\rho_0 > 0$ satisfying Proposition 5.3. We also fix a small $q \in (0, 1)$ and let $\rho_1 = \rho_0 q(1 - \alpha_1)/4\beta_3$. For any homogeneous unstable curve $\gamma \subset M$ of length $\leq \rho_0$ and $n \geq 1$ let $\xi^n \subset \gamma$ be the set involved in (5.8). Denote

$$\xi^n(\rho_1) = \{X \in \xi^n : |T_1^n \xi^n(X)| \geq \rho_1\} \ .$$

In other words, $T_1^n \xi^n(\rho_1)$ will be the union of long enough (longer than ρ_1) components of $T_1^n \xi^n$. A direct calculation based on (5.8) yields :

Corollary 5.5 *For all* $n \geq n(\gamma) := \ln m_\gamma(\gamma)/\ln \alpha_1 + \ln(q/\rho_1)/\ln \alpha_1$ *we have*

$$m_\gamma(\xi^n(\rho_1)) \geq (1-q)\, m_\gamma(\gamma) \ .$$

This means that in the set $T_1^n \gamma$, sufficiently long components (longer than ρ_1) will be prevalent after $n(\gamma)$ iterations of T_1. Note that ρ_0, ρ_1, q are global constants (independent of the force $\mathbf{F}$).

We complete this section with the construction of stable and unstable manifolds.

An unstable curve $\gamma \subset M$ is called an unstable fiber (or unstable manifold) if for all $n \geq 1$ the map T^{-n} is defined on γ and $T^{-n}\gamma$ is also an unstable curve. Likewise, γ is a stable fiber if $T^n\gamma$ is a stable curve for all $n \geq 0$.

Note that for an unstable fiber γ we have $\mathrm{diam}(T^{-n}\gamma) \to 0$ as $n \to \infty$. Similarly, for a stable fiber γ we have $\mathrm{diam}(T^n\gamma) \to 0$ as $n \to \infty$.

The above notion corresponds to a standard definition of stable and unstable manifolds for hyperbolic dynamical systems. It is not very helpful in the case of billiards, because of the lack of proper distortion bounds. Such bounds are only available on homogeneous stable and unstable curves, as we have seen in Section 4. Hence, we adopt the following:

Definition. An unstable curve $\gamma \subset M$ is called an unstable homogeneous fiber, or *h-fiber*, if for all $n \geq 0$ the curve $T^{-n}\gamma$ is a homogeneous unstable curve. Similarly, $\gamma \subset M$ is a stable h-fiber if for all $n \geq 0$ the curve $T^n\gamma$ is a homogeneous stable curve.

Clearly, stable and unstable h-fibers are automatically ordinary stable and unstable fibers. But generally, h-fibers are shorter than ordinary fibers. In other words, an ordinary fiber can be a union (finite or countable) of h-fibers.

We now prove that h-fibers exist and are abundant in M. The hyperbolicity of the flow Φ^t or the map T does not automatically provide the existence of h-fibers, though, because both the flow and the map have singularities.

For $\varepsilon > 0$, denote by $\mathcal{U}_\varepsilon^-$ the ε-neighborhood of $\mathcal{S}_0 \cup \mathcal{S}_{-1} \cup \mathcal{D}_0$, and by $\mathcal{U}_\varepsilon^+$ the ε-neighborhood of $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{D}_0$. Let

$$M_\varepsilon^\pm = \{X \in M : T^{\pm n}X \notin \mathcal{U}^\pm_{\varepsilon\Lambda^{-n}} \quad \text{for all} \quad n \geq 1\}$$

(here and on Λ is the global constant defined by (3.28)). The following is standard [Pe, Y1, Ch1]:

Fact. For every point $X \in M_\varepsilon^-$, an unstable h-fiber $\gamma^u(X)$ exists and stretches by at least $c_0\varepsilon$ in both directions from X (where $c_0 > 0$ is a global constant). Similarly, for every point $X \in M_\varepsilon^+$, a stable h-fiber $\gamma^s(X)$ exists and stretches by at least $c_0\varepsilon$ in both directions from X.

In the notation of the previous section, we have $r_{\gamma^u(X),0}(X) \geq c_0\varepsilon$ for every $X \in M_\varepsilon^-$, and $r_{\gamma^s(X),0}(X) \geq c_0\varepsilon$ for every $X \in M_\varepsilon^+$.

Proposition 5.6 *For ν_0-almost every point $X \in M$ there are stable and unstable h-fibers $\gamma^u(X)$ and $\gamma^s(X)$ through X. Moreover,*

$$\nu_0(X : r_{\gamma^u(X),0}(X) \leq \varepsilon) \leq C\varepsilon \quad \text{and} \quad \nu_0(X : r_{\gamma^s(X),0}(X) \leq \varepsilon) \leq C\varepsilon$$

for some global constant $C > 0$. In particular, the union of h-fibers shorter than ε has ν_0-measure less than const $\cdot\, \varepsilon$.

Proof. Since the set $\mathcal{S}_0 \cup \mathcal{S}_{\pm 1}$ is a finite union of smooth compact curves, the ν_0 measure of its ε-neighborhood is less than const$\cdot\varepsilon$. A similar fact for the set $\mathcal{D}_0$ can be verified by direct inspection. Then $\nu_0(\mathcal{U}_\varepsilon^-) \leq B'\varepsilon$ for some global constant B'. Due to (4.20), for all $n \geq 1$ we have $\nu_0(T^n\mathcal{U}^-_{\varepsilon\Lambda^{-n}}) \leq B'B_{11}\varepsilon(e^{-\delta_4}\Lambda)^{-n}$. Therefore, $\nu_0(M_\varepsilon^-) \geq 1 - B\varepsilon$ for some global constant B. A similar bound holds for M_ε^+. Now the proposition follows from the above fact. □

We record a few standard facts about h-fibers, which follow from the properties proved in Sections 3-4, in the same way as in the pure billiard case [BSC2]:
(1) if a sequence of h-fibers γ_n^u, $n \geq 1$, converges to a curve γ in the C^0 metric, then γ is an h-fiber.
(2) For every point $x \in M_\varepsilon^-$ the h-fiber $\gamma^u(X)$ is unique, i.e. h-fibers do not cross each other or branch out. The same holds for every $X \in M_\varepsilon^+$ and $\gamma^s(X)$.

6 A Sinai-Ruelle-Bowen measure for the map T

For any unstable h-fiber $\gamma \subset M$, a unique probability measure ν_γ, absolutely continuous with respect to the Lebesgue measure m_γ with density $f_\gamma = d\nu_\gamma/dm_\gamma$, is defined by the following condition:

$$\frac{f_\gamma(X)}{f_\gamma(Y)} = \lim_{n\to\infty} \frac{J_{T^{-n}\gamma,n}(T^{-n}Y)}{J_{T^{-n}\gamma,n}(T^{-n}X)} \qquad \text{for all } X, Y \in \gamma \tag{6.1}$$

(compare this to (5.1)). The existence of the the limit (6.1) is guaranteed by Lemma 4.2 (distortion bounds). We call ν_γ the u-SRB measure on γ. Observe that u-SRB measures are conditionally invariant under T, i.e. for any subsegment $\gamma_1 \subset T\gamma$, the measure $T_*\nu_\gamma|\gamma_1$ (the image of ν_γ under T conditioned on γ_1) coincides with ν_{γ_1}.

Note that the density f_γ is a pointwise limit of the densities $f_\gamma^{(n)}$ introduced in the previous section, as $n \to \infty$. The bound (5.2) implies a similar bound for

f_γ. So, according to our Key Remark, all the u-SRB densities are almost constant on unstable h-fibers of length $\leq \rho_0$.

Definition. A T-invariant ergodic probability measure ν on M is called a Sinai-Ruelle-Bowen (SRB) measure if its conditional distributions on unstable h-fibers are absolutely continuous. In that case the conditional measure $\nu|_\gamma$ is the u-SRB measures ν_γ on every unstable h-fiber.

The significance of SRB measures lies in the following facts. For any SRB measure ν there is a set $B \subset M$ of positive Lebesgue measure (called sometimes the basin of attraction) such that for every $X \in B$ and any continuous function $f : M \to \mathbb{R}$

$$\frac{f(X) + f(TX) + \cdots + f(T^{n-1}X)}{n} \to \int_M f(X)\, d\nu$$

as $n \to \infty$. Thus, the measure ν describes the distribution of trajectories of points $X \in B$, which are physically observable (detectable) since $\nu_0(B) > 0$. Hence, SRB measures are physically observable.

The first goal of this section is to prove the existence and finiteness of SRB measures. We first prove a similar claim for the map $T_1 = T^m$ introduced in the previous section.

In [Pe], Pesin found sufficient conditions for the existence of SRB measures for a wide class of hyperbolic maps with singularities (he called them generalized hyperbolic attractors), which included the class we study here. We restate Pesin's existence theorem in our notation. Denote by m the Lebesgue measure on M.

Theorem 6.1 (see [Pe]) *The map T_1 admits at least one and at most countably many SRB measures, provided the following two conditions hold. First, there are constants $C_1 > 0, q_1 > 0$ such that for all $\varepsilon > 0, n \geq 1$*

$$m(T_1^{-n}\mathcal{U}_\varepsilon) \leq C_1 \varepsilon^{q_1} \ . \tag{6.2}$$

Second, there is an unstable h-fiber $\gamma \subset M$ and constants $C_2 > 0, q_2 > 0$ such that for all $\varepsilon > 0, n \geq 1$

$$m_\gamma(\gamma \cap T_1^{-n}\mathcal{U}_\varepsilon) \leq C_2 \varepsilon^{q_2} \ . \tag{6.3}$$

Each SRB measure is K-mixing and Bernoulli, up to a finite cycle.

Recall that $\mathcal{U}_\varepsilon$ stands for the ε-neighborhood of the set $\Gamma_1 \cup \partial M$.

Later Sataev [Sa] showed that the number of SRB measures is finite under two additional conditions: there are constants $C_3 > 0, q_3 > 0$ such that for every homogeneous unstable curve $\gamma \subset M$ there are $n_\gamma \geq 1$ and $C_\gamma > 0$ such that for all $\varepsilon > 0$

$$m_\gamma(\gamma \cap T_1^{-n}\mathcal{U}_\varepsilon) \leq C_\gamma \varepsilon^{q_3}\, m_\gamma(\gamma) \quad \text{for all} \ \ n > 0 \tag{6.4}$$

and

$$m_\gamma(\gamma \cap T_1^{-n}\mathcal{U}_\varepsilon) \leq C_3 \varepsilon^{q_3}\, m_\gamma(\gamma) \quad \text{for all} \ \ n > n_\gamma \ . \tag{6.5}$$

We now verify Pesin's and Sataev's conditions.

Proposition 6.2 *The map T_1 satisfies (6.2)-(6.5). Hence, T_1 admits at least one and at most finitely many SRB measures. Every SRB measure is K-mixing and Bernoulli, up to a finite cycle.*

Proof. We foliate M by smooth unstable curves whose collection we denote by $\Gamma^* = \{\gamma\}$. We require that the length of each $\gamma \in \Gamma^*$ be ρ_0 (except for the corners of M and narrow strips I_k, where the curves are necessarily shorter). Let m^*_γ be the conditional measure on each $\gamma \in \Gamma^*$ induced by the Lebesgue measure m on M, and m^* the factor measure on Γ^*. If the foliation is smooth enough and ρ_0 small enough, then every m^*_γ will have almost uniform density with respect to the Lebesgue measure m_γ. In fact, the curves γ can be chosen as parallel line segments, then the measures m^*_γ will be exactly uniform. Now the condition (6.2) easily follows from the bound (5.9) by integration over Γ^* with respect to the factor measure m^*, which is a straightforward calculation. The conditions (6.3) and (6.4) are direct consequences of (5.9). Lastly, the inequality (6.5) follows from (5.9) whenever $\alpha_1^n < m_\gamma(\gamma)$, i.e. $n > \ln m_\gamma(\gamma)/\ln \alpha_1$. □

Proposition 6.3 *The map T admits at least one and at most finitely many SRB measures. Every SRB measure is K-mixing and Bernoulli, up to a finite cycle.*

Proof. Let ν be an SRB measure for the map $T_1 = T^m$. Then the measure $(\nu + T_*\nu + \cdots + T_*^{m-1}\nu)/m$ will be an SRB measure for the map T, hence the existence part. Now, let ν be an SRB measure for T. If it is ergodic for T_1, then it is an SRB measure for T_1. Otherwise ν has at most m ergodic components (with respect to T_1), each of which is an SRB measure for T_1. This proves Proposition 6.3. □

The following proposition gives Theorem 2.2 modulo Theorem 2.1, whose proof is yet to be completed.

Proposition 6.4 *Each SRB measure ν of the map $T_{\mathbf{F}}$ enjoys the exponential decay of correlations (2.9) and satisfies the central limit theorem (2.10). The correlation bound (2.9) is uniform in* $\mathbf{F}$.

Proof. This follows from a general theorem proved in [Ch2]. That theorem is stated for generic hyperbolic maps satisfying certain assumptions. All the assumptions have been already verified in Sections 3-5. The uniformity of the correlation bound follows from the fact that all the constants in the crucial estimates in Sections 3-5 (most notably, in the "growth lemma" 5.3) are global, i.e. independent of $\mathbf{F}$. Thus we obtain Proposition 6.4. □

The uniqueness of an SRB measure for T requires a more elaborate argument. We recall that the space M in the coordinates (r, φ) does not depend on the force $\mathbf{F}$ in (1.1). So we consider all the maps $T = T_{\mathbf{F}}$ as defined on the same space M. For $\mathbf{F} = 0$, we get the billiard map T_0. Recall that the space M is cut into countably many strips, I_k, hence all stable and unstable curves will be automatically homogeneous.

The classes of stable and unstable curves depend on $\mathbf{F}$, but only slightly, as it follows from our definitions in Sections 3 and 4. For simplicity, we intersect these classes over all relevant $\mathbf{F}$'s. Hence, from now on, stable and unstable curves mean such curves for all relevant maps $T = T_{\mathbf{F}}$. On the contrary, stable and unstable h-fibers depend on $\mathbf{F}$ strongly (not only their directions, but even more their sizes), so we will denote them by $\gamma_{\mathbf{F}}^{s,u}(X)$, respectively, for $X \in M$.

For any $\rho > 0$, consider the class $\mathcal{C}_u(\rho)$ of unstable curves $\gamma \subset M$ of length $\geq \rho$. Denote by $\overline{\mathcal{C}_u(\rho)}$ its closure in the Hausdorff metric. Recall that the Hausdorff metric defines the distance between two compact subsets $A, B \subset M$ by

$$\mathrm{dist}(A,B) = \max\{\max_{X\in A} \mathrm{dist}(X,B), \max_{Y\in B} \mathrm{dist}(Y,A)\}$$

(it is just the C^0 metric if restricted to continuous curves in M). Recall that our unstable curves are at least C^2, their tangent vectors satisfy the uniform bound in Lemma 3.10 and their curvature is uniformly bounded by Lemma 4.1. Therefore, all the curves in the class $\overline{\mathcal{C}_u(\rho)}$ are of length $\geq \rho$, at least C^1 (but not necessarily C^2), and their tangent vectors satisfy the same bound in Lemma 3.10. We will call curves $\gamma \subset \cup_{\rho>0}\overline{\mathcal{C}_u(\rho)}$ *generalized unstable curves*. Similarly, generalized stable curves are defined, and we denote their class respectively by $\cup_{\rho>0}\overline{\mathcal{C}_s(\rho)}$.

We call a rhombus $R \subset M$ a domain bounded by two unstable curves and two stable curves (called the sides of R). We say that a generalized unstable curve γ *straddles* R if $\gamma \subset R$ and the endpoints of γ lie on the (opposite) stable sides of R. We say that γ *properly crosses* R if γ intersects the middle half of each stable side of R and the points of intersection divide γ into three parts of which the smallest one is $\gamma \cap R$. Similar notion are defined for generalized stable curves. For a rhombus R, let $R^*_{\mathbf{F}}$ be the set of points $X \in R$ such that both $\gamma^u_{\mathbf{F}}(X)$ and $\gamma^s_{\mathbf{F}}(X)$ properly cross R.

Lemma 6.5 *There is a rhombus $R \subset M$ such that $\nu_0(R^*_0) > 0$.*

This easily follows from Proposition 5.6. □

Note that we do not claim that $\nu_0(R^*_{\mathbf{F}}) > 0$ for all $\mathbf{F}$, or even for any $\mathbf{F} \neq 0$. This will follow from our further results, see Corollary 6.10, etc.

We fix a rhombus R that satisfies the above lemma. Since it does not depend on the force $\mathbf{F}$, it is a "global" object, just like our constants B_i in the previous sections.

For any generalized unstable curve γ and $n \geq 1$ let $\gamma_{\mathbf{F}}(n)$ denote the union of intervals $\xi \subset \gamma$ such that $T^n_{\mathbf{F}}\xi$ is one generalized unstable curve that straddles R. Also, let $\gamma^p_{\mathbf{F}}(n)$ denote the union of intervals $\xi \subset \gamma$ such that $T^n_{\mathbf{F}}\xi$ is one generalized unstable curve that properly crosses R.

Lemma 6.6 *There are global constants $\tilde{n} = \tilde{n}(\rho_1, R) \geq 1$ and $\tilde{\beta}_1 = \tilde{\beta}_1(\rho_1, R) > 0$ such that for every generalized unstable curve $\gamma \subset M$ of length $\geq \rho_1$ and all $n \geq \tilde{n}$*

$$m_\gamma(\gamma^p_0(n)) \geq \tilde{\beta}_1\, m_\gamma(\gamma)\ .$$

In other words, for all $n \geq \tilde{n}$ the image $T_0^n\gamma$ will contain a certain positive fraction (characterized by $\tilde{\beta}_1$) of curves that properly cross the rhombus R.

This lemma is proved in [BSC2] (see Theorem 3.13 there) under the additional assumption that γ is an h-fiber. However, the past images $T_0^{-k}\gamma$, $k > 0$, are not involved in that theorem or its proof, and, clearly, there is no difference between unstable h-fibers and generalized unstable curves as far as their forward iterations are concerned. Thus, Theorem 3.13 in [BSC2] extends to generalized unstable curves.

Lemma 6.7 *For any force* $\mathbf{F}$ *satisfying Assumptions A and B and every generalized unstable curve* $\gamma \subset M$ *of length* $\geq \rho_1$ *and all* $n \in [\tilde{n}, \tilde{n} + m]$

$$m_\gamma(\gamma_{\mathbf{F}}(n)) \geq \tilde{\beta}_2\, m_\gamma(\gamma)$$

where $\beta_2 > 0$ *is a global constant, and* m *is again the fixed power of* T*, i.e. such that* $T_1 = T^m$*.*

In other words, for every $n = \tilde{n}, \ldots, \tilde{n} + m$ the image $T_{\mathbf{F}}^n\gamma$ will contain a certain positive fraction of curves that straddle the rhombus R.

Proof. Let γ be a generalized unstable curve and $n \in [\tilde{n}, \tilde{n}+m]$. Consider the curves $\xi \in T_0^n\gamma$ that properly cross R (such curves exist by Lemma 6.6). Let $\mathbf{F}$ be small enough, so that the map $T_{\mathbf{F}}$ is a small enough perturbation of T_0, in particular the singularity sets of these maps are close enough to each other. Let also $\gamma' \subset M$ be a generalized unstable curve sufficiently close to γ in the Hausdorff metric. We claim that if $\mathbf{F} \approx 0$ and $\gamma' \approx \gamma$, then to every curve $\xi \in T_0^n\gamma$ that properly crosses R there corresponds a curve $\xi' \in T_{\mathbf{F}}^n\gamma'$ that is close to ξ and has almost the same length. We emphasize that we first fix γ and n, and then assume that $\mathbf{F} \approx 0$ and $\gamma' \approx \gamma$, for the given γ and n. Note that ξ' will be one curve (not broken by singularities), because of the continuation property from the end of Section 4. One can easily see that, by that property, if any long enough generalized unstable curve γ intersects the singularity set for T_0, then any generalized unstable curve γ' close enough to γ (in the Hausdorff metric) intersects the singularity set for $T_{\mathbf{F}}$ with $\mathbf{F} \approx 0$, and vice versa. This justifies our claim. Now, since ξ' is close to ξ, and ξ properly crosses R, then ξ' crosses both stable sides of R, and so the curve $\xi' \cap R$ straddles R.

Thus, given γ and n, there is an open neighborhood $\mathcal{V}(\gamma, n)$ of the curve γ in the class $\overline{\mathcal{C}_u(\rho_1)}$ equipped with the Hausdorff metric and a $\delta_0(\gamma, n) > 0$ such that any curve $\gamma' \subset \mathcal{V}(\gamma, n)$ satisfies

$$m_{\gamma'}(\gamma'_{\mathbf{F}}(n)) \geq \tilde{\beta}_2\, m_{\gamma'}(\gamma') \tag{6.6}$$

for some global constant $\tilde{\beta}_2 > 0$ and all $\mathbf{F}$'s that satisfy Assumptions A and B with $\delta_0 < \delta_0(\gamma, n)$. The finite intersection

$$\mathcal{V}(\gamma) := \cap_{n=\tilde{n}}^{\tilde{n}+m} \mathcal{V}(\gamma, n)$$

is also an open neighborhood of the curve γ in the class $\overline{\mathcal{C}_u(\rho_1)}$. Any curve $\gamma' \subset \mathcal{V}(\gamma)$ satisfies the inequality (6.6) for all $n \in [\tilde{n}, \tilde{n}+m]$ and with all $\mathbf{F}$'s that satisfy Assumptions A and B with

$$\delta_0 < \delta_0(\gamma) := \min_{\tilde{n} \le n \le \tilde{n}+m} \delta_0(\gamma, n) \ .$$

Since the class $\overline{\mathcal{C}_u(\rho_1)}$ is obviously compact in the Hausdorff metric, there is a finite cover of $\overline{\mathcal{C}_u(\rho_1)}$ by some $\mathcal{V}(\gamma_j)$, $1 \le j \le J$. This proves the lemma for all forces satisfying Assumptions A and B with

$$\delta_0 < \delta_* := \min_{1 \le j \le J} \delta_0(\gamma_j) \ .$$

□

Remark. Note that in the proof of Lemma 6.7 we have put a new restriction $\delta_0 < \delta_*$ on δ_0 that enters Assumption B. This restriction is probably much more severe than any of the restrictions on δ_0 we needed before. Therefore, the uniqueness of the SRB measure probably holds for much smaller forces $\mathbf{F}$ than the hyperbolicity of $T_{\mathbf{F}}$ and the existence and finiteness of SRB measures do. We therefore expect that in physical models where $\mathbf{F}$ changes from $\mathbf{F} = 0$ continuously (such as by increasing the strength of an electrical field [CELS1]), one first observes a unique non-smooth SRB measure, then a finite collection of SRB measures, and then non-SRB stationary states. Such experiments were done, for example, in [DM]. This discussion is related to the physically important issue of the range of applicability of the linear response theory – see van Kampen's objections [K] and some counterarguments in [CELS1].

Lemma 6.7 and Corollary 5.5 easily imply the following two corollaries.

Corollary 6.8 *There are global constants $\tilde{n}_1 \ge 1$ and $\tilde{\beta}_3 > 0$ such that for any force $\mathbf{F}$ satisfying Assumptions A and B with $\delta_0 < \delta_*$ and every generalized unstable curve $\gamma \subset M$ of length $\ge \rho_1$ and all $n \ge \tilde{n}_1$*

$$m_\gamma(\gamma_{\mathbf{F}}(n)) \ge \tilde{\beta}_3 \, m_\gamma(\gamma) \ .$$

The main difference from Lemma 6.7 is that now *all* $n \ge \tilde{n}_1$ are covered, rather than $n \in [\tilde{n}, \tilde{n}+m]$.

Corollary 6.9 *There are global constants $\tilde{n}_2 \ge 1$ and $\tilde{\beta}_4 > 0$ such that for any force $\mathbf{F}$ satisfying Assumptions A and B with $\delta_0 < \delta_*$ and every generalized unstable curve $\gamma \subset M$ of length $|\gamma| = \varepsilon > 0$ and all*

$$n \ge \tilde{n}(\varepsilon) := \ln \varepsilon / \ln \alpha_1 + \tilde{n}_2 \tag{6.7}$$

we have

$$m_\gamma(\gamma_{\mathbf{F}}(n)) \ge \tilde{\beta}_4 \, m_\gamma(\gamma) \ . \tag{6.8}$$

Let $R^u_{\mathbf{F}}$ be the set of points $X \in R$ such that the unstable h-fiber $\gamma^u(X) \cap R$ straddles R.

Corollary 6.10 *There is a global constant $\tilde{\beta}_R > 0$ such that for any $T_{\mathbf{F}}$ with $\delta_0 < \delta_*$ and any SRB measure ν of $T_{\mathbf{F}}$ we have $\nu(R^u_{\mathbf{F}}) > \tilde{\beta}_R$. Furthermore, let ν be not mixing, so that by Proposition 6.3, $T_{\mathbf{F}}$ permutes a finite number of subsets $X_1, \dots, X_k \subset M$ on each of which $T^k_{\mathbf{F}}$ is mixing. In this case we have $\nu(R^u_{\mathbf{F}} \cap X_i) > 0$ for every $i = 1, \dots, k$.*

Proposition 6.11 *For any force $\mathbf{F}$ satisfying Assumptions A and B with $\delta_0 < \delta_*$ the SRB measure of the map $T_{\mathbf{F}}$ is unique and mixing.*

We first adopt a definition.

Definition. Let $\gamma^s_{\mathbf{F}}$ be a stable h-fiber. For $\varepsilon > 0$, let $\Gamma_\varepsilon(\gamma^s_{\mathbf{F}})$ denote the union of all stable h-fibers in M that are ε-close to $\gamma^s_{\mathbf{F}}$ in the Hausdorff metric. We call $\gamma^s_{\mathbf{F}}$ a *density h-fiber* if for every $\varepsilon > 0$ the set $\Gamma_\varepsilon(\gamma^s_{\mathbf{F}})$ has positive Lebesgue measure in M. Note that in this case for any generalized unstable curve $\xi \subset M$ that crosses $\gamma^s_{\mathbf{F}}$, the set $\xi \cap \Gamma_\varepsilon(\gamma^s_{\mathbf{F}})$ has positive m_ξ measure, by the absolute continuity Lemma 4.3. Similarly, we introduce unstable density h-fibers.

Lemma 6.12 *For each map $T_{\mathbf{F}}$ there are density h-fibers. In fact, their union has full Lebesgue measure. If $\gamma^s_{\mathbf{F}}$ is a density h-fiber, then all the connected components of $T^{-n}\gamma^s_{\mathbf{F}}$ are density h-fibers, too, for every $n \geq 1$.*

Proof. The first two claims follows from Proposition 5.6. To prove the last one, we see $n = 1$ and note that $T^{-1}_{\mathbf{F}}$ is piecewise smooth and its singularities are unstable curves with the continuation property. Then we use induction on n. □

Proof of Proposition 6.11. Let $\gamma^s_{\mathbf{F}}$ be a density h-fiber. By the above Lemma and Corollary 6.9 (actually applied to stable curves), there exist density h-fibers in $T^{-n}\gamma^s_{\mathbf{F}}$ for some $n \geq 1$ that straddle the rhombus R. This, along with Corollary 6.10, proves Proposition 6.11. □

Proposition 6.13 *For any force $\mathbf{F}$ satisfying Assumptions A and B with $\delta_0 < \delta_*$ the SRB measure ν of the map $T_{\mathbf{F}}$ is positive on open sets. Moreover, for every small round disk $D \subset M$ we have*

$$\nu(D) \geq c_1[\nu_0(D)]^{1+\delta_5}$$

for some global constant $c_1 > 0$ and small constant $\delta_5 > 0$ depending on δ_0 (i.e., $\delta_5 \to 0$ as $\delta_0 \to 0$).

Proof. Since the disk D is connected, it belongs in one homogeneity strip I_k, and so the quantity $\cos\varphi$ does not vary too much over D, i.e. the measure ν_0 is almost proportional to the Lebesgue measure m on D. We can find a rhombus $R_D \subset D$ whose opposite sides are parallel straight lines and which is big enough so that, say,

$\nu_0(R_D) \geq \nu_0(D)/10$. Now, we foliate the rhombus R_D by parallel stable segments γ that straddle R_D and are parallel to the stable sides of R_D. Hence, all γ's in our foliation have the same length, ε. Note that $\varepsilon \geq c_2\sqrt{\nu_0(D)}$ with a global constant $c_2 > 0$. For any γ in our foliation of R_D and $n \geq 1$ let $\gamma_{\mathbf{F}}(-n)$ denote the union of intervals $\xi \subset \gamma$ such that $T_{\mathbf{F}}^{-n}\xi$ is one stable curve that straddles the rhombus R (fixed earlier). Corollary 6.9 (actually, its dual statement for stable curves) implies that $m_\gamma(\gamma_{\mathbf{F}}(-n)) > \tilde{\beta}_4\, m_\gamma(\gamma)$ for all $n \geq \tilde{n} := \tilde{n}(\varepsilon)$. Consider the set

$$R_D(-n) := \cup_\gamma \gamma_{\mathbf{F}}(-n)$$

where the union is taken over all γ in our foliation of R_D. Our previous estimates imply that $m(R_D(-n)) \geq \tilde{\beta}_4\, m(R_D)$ and hence $\nu_0(R_D(-n)) \geq \tilde{\beta}_5\, \nu_0(R_D)$ for all $n \geq \tilde{n}$, and with the global constant $\tilde{\beta}_5 = \tilde{\beta}_5/2$.

Now the volume compression bounds (4.20) imply

$$\nu_0(T_{\mathbf{F}}^{-n} R_D(-n)) \geq B_{11}^{-1} e^{-\delta_4 n} \tilde{\beta}_5\, \nu_0(R_D)$$

for all $n \geq \tilde{n}$. We put $n = \tilde{n} = \tilde{n}(\varepsilon)$ given by (6.7) and obtain

$$\nu_0(T_{\mathbf{F}}^{-\tilde{n}} R_D(-\tilde{n})) \geq c'\varepsilon^{\delta_6}\nu_0(R_D) \geq c''\,[\nu_0(R_D)]^{1+\delta_6/2}$$

with some positive global constants c', c'' and a small constant $\delta_6 = \delta_4/\ln\alpha_1$. We put $\delta_5 = \delta_6/2$.

Next observe that the set $T_{\mathbf{F}}^{-\tilde{n}} R_D(-\tilde{n})$ is a union of stable curves that straddle our fixed rhombus R. Lemma 4.3 (absolute continuity) then implies that for any unstable curve ξ that straddles R we have

$$m_\xi(\xi \cap T_{\mathbf{F}}^{-\tilde{n}} R_D(-\tilde{n})) \geq c'''\,[\nu_0(R_D)]^{1+\delta_5} m_\xi(\xi)$$

with a global constant $c''' > 0$. This bound combined with Corollary 6.10 yields

$$\nu(T_{\mathbf{F}}^{-\tilde{n}} R_D(-\tilde{n})) \geq \tilde{c}\,[\nu_0(R_D)]^{1+\delta_5}$$

for some global constant $\tilde{c} > 0$ and the SRB measure ν of the map $T_{\mathbf{F}}$. The $T_{\mathbf{F}}$-invariance of ν completes the proof of Proposition 6.13. $\square$

Theorems 2.1 and 2.2 are now proved.

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