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CHARACTERISTIC LYAPUNOV EXPONENTS AND SMOOTH ERGODIC THEORY

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PART I

§1. Introduction

1.1. The smooth ergodic theory studies the ergodic properties of smooth dynamical systems on smooth compact Riemannian manifolds, preserving a given normalized measure, denoted by ν, which is compatible with the smoothness (that is, equivalent to the Riemannian volume). By a dynamical system of class C^r on a manifold M we understand either a C^r-diffeomorphism $f\colon M \to M$ (and also the system $\{f^n\}$, $n \in \mathbf{Z}$ with discrete time, or cascade, generated by it), or a one-parameter group f^t of diffeomorphisms of M that is differentiable with respect to $t \in \mathbf{R}$ (a dynamical system with continuous time, or a flow), which is given by a C^r-vector field X on M, such that $X(x) = \left.\frac{df^t(x)}{dt}\right|_{t=0}$, $x \in M$. We are interested in ergodic properties such as the existence of an ergodic component of positive measure (in particular, ergodicity), the positiveness of entropy, and, even stronger, the K-property and Bernoullian property (the exact definitions are given in §2).

1.2. We consider the space of all dynamical systems of class C^r preserving a given measure ν and equipped with the C^r-topology. This space is denoted by $\mathrm{Diff}^r(M, \nu)$ in the case of diffeomorphisms and by $\Gamma^r(TM, \nu)$ in the case of flows. A subset of this space is said to be massive if it contains an everywhere dense G_δ-subset. A property of dynamical systems is said to be typical if it is satisfied for the elements of some massive subset in $\mathrm{Diff}^r(M, \nu)$ (or in $\Gamma^r(TM, \nu)$). The concept of being typical is very useful when dealing with the problem of describing the ergodic properties of dynamical systems. Above all we note that there is no hope of solving this problem for all dynamical systems without exception, since the effects arising from such a fact are extremely complex. It is natural that we should attempt to find typical ergodic properties of dynamical systems.

A similar approach, linked with the concept of being typical in the space of all dynamical systems of class C^r ($\mathrm{Diff}^r(M)$ or $\Gamma^r(TM)$) was systematically developed by Smale [44]. The up-to-date state of this problem is reflected in Shub's paper [43]. In contrast to the topological case where hypotheses on the typicality of many "nice" properties did not get corroborated, in the case of smooth dynamical systems with an invariant measure hypotheses on the typicality of positive entropy and of the existence of an ergodic component of positive measure have not been disproved.

In this paper we formulate certain conditions to be imposed on a dynamical system, which represent a very weak (in fact, as weak as possible) variant of the so-called "hyperbolicity conditions". These conditions are sufficient to deduce essential information about ergodic properties. At the same time it is likely that the diffeomorphisms and flows satisfying them form a massive set in $\mathrm{Diff}^r(M, \nu)$ or in $\Gamma^r(TM, \nu)$. In any case, such dynamical systems exist on a large class of manifolds, apart from trivial exceptions in small dimensions (1 or 2 for a flow, 1 for a diffeomorphism); see §1.9. Moreover, a number of systems of "classical" character which have for a long time attracted attention to a certain extent satisfy our conditions (see §1.3 and §1.9). For some of these examples the information on ergodic properties that can be obtained on the basis of the general theory to be developed here (see §1.6 and §1.7), had not been known earlier.

1.3. Before we give the exact definition of the dynamical systems studied in this paper, we discuss some variants of the "hyperbolicity conditions" that differ, so to speak, by a degree of "rigidity" of the conditions to be imposed.

Historically, the first idea about the hyperbolic behaviour of trajectories was one that can be described by saying that near any fixed trajectory $\{f^t(x)\}$ the behaviour of its neighbouring trajectories resembles the behaviour of the trajectories in the neighbourhood of a saddle point. In other words, hyperbolic behaviour is described in infinitesimal terms and is specified by means of certain conditions (the so-called "*U*" conditions) on the differential of the dynamical system. These conditions mean (see [1], [3]) that the tangent space at every point $x \in M$ can be decomposed in an invariant way into two subspaces E_{1x} and E_{2x} (in the case of a flow the direction of motion must be added to them), so that the maps $df_x^t | E_{1x}$ and $df_x^t | E_{2x}$ are, respectively, a contraction and an expansion with coefficients that are uniform with respect to the phase space.

It has been noted long ago that local instability of trajectories – one of the methods of expressing this are the hyperbolicity conditions – is closely linked with the statistical properties of a dynamical system. The fact is that on a compact manifold a "typical" hyperbolic trajectory is distributed in the phase space in a very complex and irregular way, so that such trajectories must get "mixed up". This offers ground for hoping that a dynamical system will be ergodic and have positive entropy.

The role of the instability of trajectories was already noted in the thirties by Hedlund (see the survey [38]) and E. Hopf [31], who studied the ergodic properties of geodesic flows on certain compact Riemannian manifolds with negative curvature. The investigation of other (geometric and topological) properties of these flows had been started even earlier by Hadamard [37] and was significantly developed in papers of a number of authors, especially Morse (see Hedlund's survey [38] and §3 of Anosov's paper [1]). A condition of hyperbolicity was explicitly formulated and systematically used as a basic assumption for the first time by Anosov [1]. The dynamical systems he considered received the name of *U*-systems (*U*-diffeomorphisms and *U*-flows). At present the class of *U*-systems is fairly well studied (see [1]–[3], [18], [32], [33], [39]). We note that the "*U*" conditions are a very strong (in fact, as strong as possible) variant of the "hyperbolicity conditions". This situation accounts for the fact that, on the one hand, the *U*-systems have a good many ergodic properties (for example, *U*-diffeomorphisms are *K*-automorphisms, see [1], and, as has become clear recently, they are isomorphic to the Bernoulli automorphisms, see [41] and §8 of this paper), and on the other hand, they are not typical in the sense defined above.[1] (Moreover, for purely topological reasons the *U*-systems exist by no means on every manifold; see [33]).

[1] However, they form an open set in the space of all smooth dynamical systems (see [1]).

1.4. As already mentioned in this paper we study dynamical systems that satisfy very weak "hyperbolicity conditions". In our case hyperbolicity is partial and non-uniform (besides, by no means all trajectories are "hyperbolic", but only those generating a set of full or at least of positive measure). "Partiality" means that the two subspaces E_{1x} and E_{2x} (which occur in the "U" conditions) do not generate the whole tangent space at $x \in M$, and "non-uniformity" means that the inequalities expressing the variation $\| df_x^t v \|$ ($t \in \mathbf{Z}$ or $t \in \mathbf{R}$) with increment t for vectors $v \in E_{1x}$ or $v \in E_{2x}$ are not uniform with respect to x. An intermediate variant of the "hyperbolicity conditions", when hyperbolicity is partial but uniform, leads to the case of the so-called partially hyperbolic dynamical systems (see [5]).

DEFINITION. A dynamical system f^t on a manifold M is said to be non-uniform and partially hyperbolic if there are an invariant set $\Lambda \subset M$, $\nu(\Lambda) > 0$, families of subspaces E_{1x}, $E_{2x} \subset T_xM$, depending measurably on $x \in \Lambda$, and measurable functions $\lambda(x)$, $\mu(x)$, $C(x)$, $K(x)$, $\varepsilon(x)$, $x \in \Lambda$ such that for any $t \in \mathbf{Z}$ (or $t \in \mathbf{R}$) and $x \in \Lambda$

$$(1.1)\quad \begin{cases} 0<\lambda(x)<\mu(x), & \mu(x)-\lambda(x)>\varepsilon(x), \quad 1-\lambda(x)>\varepsilon(x)>0, \\ \mu(f^t(x))=\mu(x), & \lambda(f^t(x))=\lambda(x), \quad \varepsilon(f^t(x))=\varepsilon(x); \end{cases}$$

$$(1.2)\quad T_xM=E_{1x}\oplus E_{2x}, \quad df^tE_{ix}=E_{if^t(x)} \quad (i=1,\ 2);$$

$$(1.3)\quad \begin{cases} C(f^t(x))\leqslant C(x)\,e^{\varepsilon(x)|t|}, \quad K(f^t(x))\,e^{-\varepsilon(x)|t|}, \\ C(x)>0, \quad K(x)>0; \end{cases}$$

$$(1.4)\quad \begin{cases} \| df^tv\|_{f^t(x)}\leqslant C(x)\,\lambda^t(x)\,\|v\|_x & \text{for} \quad v\in E_{1x}, \quad t>0, \\ \| df^tv\|_{f^t(x)}\geqslant C^{-1}(x)\,\mu^t(x)\,\|v\|_x & \text{for} \quad v\in E_{2x}, \quad t>0; \end{cases}$$

and the angle $\gamma(x)$ between E_{1x} and E_{2x} admits the estimate

$$(1.5)\quad \gamma(x)\geqslant K(x);$$

(here $\|\cdot\|_x$ denotes the norm induced on the tangent space T_xM by the Riemannian metric of M).

The conditions (1.3) signify that the estimates (1.4) do not "worsen too much" along the trajectory $\{f^t(x)\}$. Actually, the functions $C(x)$ and $K(x)$ oscillate along a "typical" trajectory and the magnitude of the oscillations is estimated by (1.3). In non-uniform partially hyperbolic dynamical systems, in contrast to U-systems, the subspaces $E_{1f^t(x)}$ and $E_{2f^t(x)}$, generally speaking, approach each other from time to time with a small and, as (1.5) shows, exponential speed. We note that, in essence, the conditions (1.3) and (1.5) are not additional (and, as it may appear at a first glance, artificial) restrictions: as will become clear below, they are automatically satisfied on a set of trajectories of full measure.

1.5. Another approach to the definition of our "hyperbolicity conditions" is connected with the characteristic Lyapunov exponents. (The definition of the characteristic exponents $\chi^+(x, v)$, $x \in M$, $v \in T_xM$ is in §3).

Let $x \in M$. We consider a trajectory $\{f^t(x)\}$ and the family of maps

$$df^t_x\colon T_xM \to T_{f^t(x)}M. \tag{1.6}$$

We are interested in the case when this family is regular in the sense of Lyapunov. The exact definition of this concept will be given in §3. Here we note the following. By the multiplicative ergodic theorem of Oseledets [19] and Millionshchikov [16] (see also §3, Theorem 3.5), for almost every (with respect to the measure ν) point x the family of maps (1.6) is regular in the sense of Lyapunov. If we neglect sets of measure zero (which is natural within the framework of ergodic theory), then the conditions (1.1)–(1.5) of non-uniform partial hyperbolicity are equivalent to the following:

(1.7) the set $\Lambda = \{x \in M$: there is a vector $v \in T_xM$ for which $\chi^+(x, v) < 0\}$ has positive measure.

(It is not difficult to show that this set is measurable and invariant with respect to f.) The fact that the condition above follows from the conditions (1.1)–(1.5) of non-uniform partial hyperbolicity is almost evident. The converse is proved in [22] (see Theorem 1.1).

1.6. We now indicate two results which describe ergodic properties of dynamical systems satisfying (1.7). The first result consists in the fact that these dynamical systems have positive entropy. It follows from the formulae obtained in §5 for the calculation of the entropy of an arbitrary dynamical system of class C^2 with respect to a measure ν, namely: the entropy is equal to the integral of the sum of the positive characteristic Lyapunov exponents. This proposition has been known as a conjecture for more than ten years; it was discussed in 1965 in the school on ergodic theory in Khumsan. At that time rougher upper bounds were known [15], [13], and a little later Margulis obtained an exact upper estimate. Although Margulis's result is well known to the specialists and is even mentioned in the literature (see [17]), it is still unpublished. A proof of an exact lower estimate based on the method of [21] and developed in [22], was obtained by the author in collaboration with Katok. We emphasize one interesting (not to say odd) fact: Margulis's result is true for dynamical systems of class C^1, whereas the lower estimate is obtained for systems of class C^2 (it can be proved for class $C^{1+\varepsilon}$). A proposition equivalent to this formula was proved for U-systems by Sinai [27]. A rougher lower bound for the entropy of partially hyperbolic systems is due to Brin [6].

We emphasize that the entropy being positive and the condition (1.7) are equivalent. From this it follows that the π-partition (see §2) of dynamical systems satisfying this condition is non-trivial. A complete description of the π-partition is given in §6.

Now we can formulate more precisely the problem of which we talked at the very beginning: is the set of dynamical systems satisfying (1.7) and, consequently, having positive entropy, massive? Even stronger is the assertion that dynamical systems having non-zero Lyapunov exponents almost everywhere are typical. In the two-dimensional case and also for Hamiltonian dynamical systems on manifolds of any dimension with sufficient smoothness this assertion is not true by virtue of the well-known Kolmogorov–Arnol'd–Moser theorem (see [14], [4], [40]). About possible effects in the case of less smoothness we mention Takens's paper (a Russian translation is in [33]) and also the recent result of Newhouse on the density in the C^1-topology on all two-dimensional manifolds except the torus of the measure preserving diffeomorphisms with periodic points of general elliptic type (see [47]). In the general case of dimension greater than 2 none of the results available at present gives reasons to doubt the validity of the assertion in question.

1.7. The remaining results obtained in our paper refer to dynamical systems satisfying the condition, which is stronger than (1.7), of "complete" non-uniform hyperbolicity, namely (we restrict ourselves to diffeomorphisms):

(1.8) the set $\Lambda = \{x \in M: \chi^+(x, v) \neq 0$ for any $v \in T_xM\}$ has positive measure.

Our basic results are contained in Theorems 7.2, 7.9, and 8.1. In Theorem 7.2 we establish the existence of an ergodic component of positive measure, lying in Λ; more precisely:

there are sets $\Lambda_n (n = 0, 1, 2, \ldots)$ such that

a) $\bigcup_{n \geq 0} \Lambda_n = \Lambda$, $\Lambda_{n1} \cap \Lambda_{n2} = \varnothing$ if $n_1 \neq n_2$;

b) $\nu(\Lambda_0) = 0$, $\nu(\Lambda_n) > 0$ for $n > 0$;

c) $f(\Lambda_n) = \Lambda_n$, for $n \geq 0$;

d) the automorphism $f | \Lambda_n$, $n > 0$, is ergodic.

Of course, in practice it is important to know the number of ergodic components of positive measure. Theorem 7.8 gives sufficient conditions for the diffeomorphism $f | \Lambda$ to be ergodic.

Theorems 7.9 and 8.1 establish the K-property and the Bernoullian property, namely:

there are a sequence of numbers $n_i (i = 1, 2, \ldots)$ and measurable sets $\Lambda_i^j (j = 1, \ldots, n_i)$ such that

a) $\bigcup_{j=1}^{n_i} \Lambda_i^j = \Lambda_i$, $\Lambda_i^{j_1} \cap \Lambda_i^{j_2} = \varnothing$ if $j_1 \neq j_2$;

b) $f(\Lambda_i^j) = \Lambda_i^{j+1} (j = 1, \ldots, n_i - 1)$, $f(\Lambda_i^{n_i}) = \Lambda_i^1$;

c) the automorphism $f^{n_i} | \Lambda_i^1$ is isomorphic to a Bernoulli automorphism.

1.8. In §9 the results obtained for diffeomorphisms are carried over to flows. Here the theorems on ergodicity are modified in the obvious way.

The situation is different with the K-property, even in the case of U-systems. As Anosov has proved [1], an alternative holds: either f^t is a K-flow (if the linear operator U_{f^t} in $L^2(M, \nu)$ generated by it does not have eigenfunctions other than constants), or f^t is obtained by a suspension of a U-flow (if U_{f^t} has a non-constant eigenfunction). In our case a certain analogue of this theorem holds.

1.9. An example of a flow with non-zero exponents was constructed by the author [20]. In the case of discrete time similar examples were constructed by Anosov and Blokhin (unpublished). In these examples the characteristic exponents are non-zero almost everywhere (but not everywhere). Furthermore, in these examples the partitions into ergodic components have the form described above. Katok and Grines have recently constructed examples of diffeomorphisms with exponents that are non-zero almost everywhere, on any two-dimensional manifold and on an n-dimensional disk. Furthermore, on a three-dimensional manifold a diffeomorphism can be constructed (by means of a suspension) for which all the exponents except one are non-zero almost everywhere. Moreover, in the case of two-dimensional manifolds these diffeomorphisms satisfy the conditions of Theorems 7.5, 7.6, and of Corollary 7.2, therefore, they are isomorphic to Bernoulli automorphisms.

A particular place among our examples belongs to the geodesic flows on compact Riemannian manifolds, because along with examples of an algebraic nature they have for a long time served as a field of application of the methods of the theory of dynamical systems of hyperbolic structure. For instance, geodesic flows on compact Riemannian manifolds with negative curvature are U-flows (see [1]). The next step consists in the study of geodesic flows on Riemannian manifolds having non-positive curvature or "small" sections of positive curvature (more precisely, manifolds equipped with Riemannian metrics without focal points). The first result in this direction for flows on surfaces of genus greater than 1 was proved by Kramli [12] (he established the existence of an ergodic component of positive measure). In § 10 we prove a result that generalizes Kramli's theorem: we show that flows on surfaces of genus greater than 1 are ergodic and even have the Bernoullian property. The proof of this proposition is based firstly on the results obtained in §9; secondly on Eberlein's results describing the structure of the equation in variations for a geodesic flow and the topological properties of these flows [36]; thirdly, on the construction of horospheres for a broad class of Riemannian manifolds (see § 10).

We mention a deep informal connection between our results and those of Sinai and Bunimovich on metric properties of the so-called billiard dynamical systems (see [7]–[9], [28]), which are a model for a number of problems in statistical physics. Although these systems do not fit formally into our scheme because they are discontinuous, in their constructions Sinai and Bunimovich actually use the absence of zero exponents on some "nice" set at all points of which the dynamical system is continuous. The methods of Sinai and Bunimovich are closely connected with the use of the specific structure of the phase space in billiard systems, but the general scheme of the proof of ergodicity and of the K-property, and also the nature of the difficulties to be overcome, are similar to ours.

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1.10. Throughout the paper we use the following notation:

$\langle , \rangle$ and $\|\cdot\|$ are the standard scalar product and the corresponding norm in the Euclidean space $\mathbf{R}^n$.

$\mathbf{Z}^+$ and $\mathbf{R}^+$ are the set of non-negative integers and the set of non-negative real numbers.

M is a smooth compact Riemannian manifold (without loss of generality the manifold may be considered C^∞-smooth).

$\langle , \rangle_x$ and $\|\cdot\|_x$ are the scalar product and the corresponding norm in the tangent space T_xM, defined by a (C^∞-) smooth Riemannian metric on the tangent bundle TM (sometimes the subscript x will be dropped).

ρ and d are the distance induced on M and TM, respectively, by the Riemannian metric.

ν is a measure equivalent to the measure induced by the Riemannian metric.

$B(x, r)$ is the open ball with centre at x and of radius r on M.

f: $M \to M$ is a ν-measure preserving C^r-diffeomorphism of M, $r \geqslant 2$.

f^t: $M \to M$ is a flow on M defined by a ν-measure preserving C^r-vector field X, $r \geqslant 1$.

§2. Prerequisites from ergodic theory

We assume that the reader is familiar with the basic concepts of general measure theory and of ergodic theory to the extent of §1–§4 of Halmos's book [30] and §1–§3 of Rokhlin's paper [25]. In this section we briefly touch on concepts connected with measurable partitions, entropy, ergodicity, the K-property, and we also give some new results due to Ornstein on Bernoullian systems. For a more detailed exposition we recommend Rokhlin's papers [25] and [26] and that of Ornstein and Weiss [41].

2.1. The measurable spaces $(M, \mathfrak{A}, \nu)$ with measure to be considered by us (M is a set, $\mathfrak{A}$ some σ-algebra of its subsets, called measurable, ν a finite or even normalized measure; the measurable spaces with measure will henceforth be denoted only by the first letter M) are Lebesgue spaces, that is, they are isomorphic mod 0 to the interval [0, 1] together with the σ-algebra of its Borel subsets and Lebesgue–Stieltjes measure. (We recall that an isomorphism mod 0 of two measurable spaces M and N is a bijective map χ: $M' \to N'$, $M' \subset M$, $N' \subset N$, $M = M'$(mod 0), $N = N'$(mod 0), for which the image and the inverse image of any measurable set in M' or N', respectively, is measurable and has the same measure.)

Partitions of a space M will be denoted by ξ, η, ζ etc., and the elements of a partition ξ by C_ξ (or by $C_\xi(x)$ - the element of the partition that contains the point $x \in M$). If ξ is a finite partition, then we also write $\xi = \{C_1, \ldots, C_n\}$, where $C_i \in \mathfrak{A}$. A partition ξ is said to be measurable (see [25], §1.7) if it has a system of conditional measures, that is, a system of measures $\nu(\cdot|C_\xi)$ (and of corresponding σ-algebras $\mathfrak{A}(C_\xi)$) such that:

1) for almost every $C_\xi \in M/\xi$ the measurable space $(C_\xi, \mathfrak{A}(C_\xi), \nu(\cdot|C_\xi))$ is a Lebesgue space;[1]

2) for any measurable set $A \subset M$ and almost every $C_\xi \in M/\xi$ the set $A \cap C_\xi$ is measurable in C_ξ; the function $\nu(A \cap C_\xi)$ is measurable on M/ξ and

$$\nu(A) = \int_{M/\xi} \nu(A \cap C_\xi | C_\xi)\, d\nu_\xi.$$

Let ξ and η be measurable partitions. The notation $\xi \leqslant \eta$ means that η is a refinement of ξ, that is, every element of η is contained in some element of ξ. The relation $\xi \leqslant \eta$ is a partial ordering. We write $\xi \leqslant \eta$ and $\xi = \eta$ also when the relations hold only mod 0.

If $\{\xi_\alpha\}$ is a system of measurable partitions, then there exists the product $\bigvee_\alpha \xi_\alpha$, which is defined to be the measurable partition ξ with the two properties (see [26]): $\xi_\alpha \leqslant \xi$ for any α, and if $\xi_\alpha \leqslant \xi'$ for any α, then $\xi \leqslant \xi'$. (The product of finitely or countably many measurable partitions $\xi_n = \{C_1^{(n)}, \ldots, C_{m_n}^{(n)}\}$ consists of elements of the form $\bigcap_n C_{i_n}^{(n)}$.)

If $\{\xi_\alpha\}$ is a system of measurable partitions, then there exists the intersection $\bigwedge_\alpha \xi_\alpha$, which is defined to be the measurable partition ξ with the two properties (see [26]): $\xi_\alpha \leqslant \xi$ for any α, and if $\xi_\alpha \geqslant \xi'$ for any α, then $\xi \geqslant \xi'$.

We denote by ε_0 the partition of M into its individual points, and by ν the trivial partition whose only element is M. Two measurable partitions ξ and η are said to be independent if $\nu(C_\xi \cap C_\eta) = \nu(C_\xi) \cdot \nu(C_\eta)$ for any

[1] If ξ is a measurable partition, then there is a natural way of turning the factor set M/ξ into a measurable Lebesgue space with measure, which we denote by ν_ξ (see [25], §1.5).

elements C_ξ and C_η.

Let f: $M \to M$ be an automorphism and ξ a measurable partition. The partition consisting of the sets $\{f(C_\xi)\}$ is measurable and is denoted by $f\xi$. A measurable partition ξ is said to be invariant with respect to f if $f\xi = \xi$, and increasing if $f\xi \geqslant \xi$. We put

$$\xi^- = \bigvee_{k=0}^{\infty} f^{-k}\xi, \quad \xi_f = \bigvee_{k=-\infty}^{\infty} f^k\xi, \quad \nu(\xi) = \bigwedge_{k=0}^{\infty} f^{-k}\xi.$$

A partition is said to be generating if $\xi^- = \varepsilon_0$; bilateral generating if $\xi_f = \varepsilon_0$; and exhaustive if it is increasing and $\bigvee_{k=0}^{\infty} f^k\xi = \varepsilon_0$.

2.2. Let ξ be a measurable partition of a Lebesgue space M and $C_1, C_2, \ldots$ the elements of ξ having positive measure. We put[1]

$$H(\xi) = \begin{cases} -\sum\limits_k \nu(C_k) \log \nu(C_k) & \text{if } \nu(M \setminus \bigcup\limits_k C_k) = 0, \\ +\infty, & \text{if } \nu(M \setminus \bigcup\limits_k C_k) > 0. \end{cases}$$

For any measurable partition η a partition ξ induces a measurable partition on almost every element C_η. Its entropy, calculated with respect to the conditional measure $\nu(\cdot \mid C_\eta)$, is denoted by $H(\xi \mid C_\eta)$ and its mean $\int_M H(\xi \mid C_\eta)\, d\nu$ by $H(\xi \mid \eta)$ and is called the conditional entropy of ξ with respect to η. The entropy of an automorphism f is the quantity

$$h(f) = \sup_\xi H(\xi \mid f^{-1}\xi^-),$$

where the supremum is taken over all measurable partitions. It can be shown that the supremum in this formula may be taken over the set of finite measurable partitions (see [25], §9.1). We put

$$h(f, \xi) = H(\xi \mid f^{-1}\xi^-) = H(f\xi \mid \xi^-).$$

The following relations hold (see [25], §§5.5, 5.10, 8.2, 9.3, 9.5):

1. For any measurable partitions $\xi, \eta, \zeta, \xi', \xi \leqslant \xi'$

$$H(\xi \mid \eta) \leqslant H(\xi' \mid \eta), \quad H(\xi \mid \zeta \vee \eta) \leqslant H(\xi \mid \eta).$$

2. $h(f, \xi \vee \eta) \leqslant h(f, \xi) + h(f, \eta)$.
3. $h(f^n) = |n|\, h(f)$ for any $n \in \mathbf{Z}$.
4. If ξ_n is an increasing sequence of finite measurable partitions such that $\xi_n \to \varepsilon_0$, then $h(f, \xi_n) \nearrow h(f)$.

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[1] Usually it is assumed that logarithms are taken to the base 2. For us, however, it is more convenient to deal with natural logarithms.

For an equivalent definition and other properties of the entropy, see [25], §§7, 8, 9.

2.3. An automorphism f is said to be ergodic if any measurable invariant function is constant mod 0. An automorphism f is called a K-automorphism if there is an increasing exhaustive partition for which $\nu(\xi) = \nu$. K-automorphisms are ergodic and have positive entropy.

Among the invariant partitions with zero entropy there is a greatest partition $\pi(f)$, the so-called Pinsker partition for f (see [25], §11.5). If ξ is an exhaustive partition, then $\nu(\xi) \geqslant \pi(f)$ (see [25], §12.1). Hence it follows that $\pi(f) = \nu$ if f is a K-automorphism. The converse is also true: if $\pi(f) = \nu$, then f is a K-automorphism (see [25], §12.5).

2.4. Let $f:(M, \nu) \to (N, \mu)$ be a measurable injective map. The measure μ is carried by f into a measure μ^* on M: if $A \subset M$ and $f(A) \subset N$ are measurable, then $\mu^*(A) = \mu(f(A))$. If μ^* is absolutely continuous with respect to ν so that $d\mu^*(x) = \rho(x)d\nu(x)$ (where $\rho(x)$ is a positive measurable function and $x \in M$), then f is said to be absolutely continuous, and $\rho(x)$ the Jacobian of f.

2.5. Let ξ be a finite measurable ordered partition. We say that a certain property is true for ε-almost every element of ξ if the measure of the union of those elements for which this property does not hold is less than ε. Let $\{\xi_i\}_1^n$ and $\{\eta_i\}_1^n$ ($i = 1, \ldots, n$) be two sequences of partitions of spaces (X, ν) and (Y, μ), respectively, where $\xi_i = \{A_1^{(i)}, \ldots, A_m^{(i)}\}$, $\eta_i = \{B_1^{(i)}, \ldots, B_m^{(i)}\}$. Then the notation $\{\xi_i\}_1^n \sim \{\eta_i\}_1^n$ means that for any k_i, $1 \leqslant k_i \leqslant m$, $1 \leqslant i \leqslant n$

$$\nu\left(\bigcap_1^n A_{k_i}^{(i)}\right) = \mu\left(\bigcap_1^n B_{K_i}^{(i)}\right).$$

We say that

$$\bar{d}(\{\xi_i\}_1^n, \ \{\eta_i\}_1^n) \leqslant \varepsilon$$

if there are partitions $\bar{\xi}_i = \{\bar{A}_1^{(i)}, \ldots, \bar{A}_m^{(i)}\}$, $\bar{\eta}_i = \{\bar{B}_1^{(i)}, \ldots, \bar{B}_m^{(i)}\}$ of the space $(Z, \varkappa)$ such that $\{\xi_i\}_1^n \sim \{\bar{\xi}_i\}_1^n$, $\{\eta_i\}_1^n \sim \{\bar{\eta}_i\}_1^n$ and

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varkappa(\bar{A}_j^{(i)} \Delta \bar{B}_j^{(i)}) \leqslant \varepsilon.$$

A partition ξ is said to be very weak Bernoullian for an automorphism f (ξ is a VWB-partition) if for every $\varepsilon > 0$ there is a $N_0 = N_0(\varepsilon)$ such that for any $N' > N \geqslant N_0$, $n \geqslant 0$, and ε-almost every element $A \in \bigvee_N^{N'} f^k \xi$

$$\bar{d}(\{f^{-i}\xi\}_1^n, \ \{f^{-i}\xi \mid A\}_1^n) \leqslant \varepsilon$$

(the partition $\xi \mid A$ is considered with respect to normalized measure).

An automorphism f of a measurable space M is said to be Bernoullian if it is isomorphic to the standard Bernoullian scheme (see [25], §3.6). An equivalent definition is: there is a bilateral generating partition ξ such that $f^i\xi$ and $f^j\xi$ are independent for any i and j, $i \neq j$. Bernoullian automorphisms are K-automorphisms.

THEOREM 2.1 (see [41], Theorems A and B). *Let $\xi_1 \leqslant \xi_2 \leqslant \ldots$ be an increasing sequence of* VWB-*partitions of a space (X, ν) for an automorphism f, such that $\xi_n \to \varepsilon_0$. Then f is Bernoullian.*

We say that a map θ: $X \to Y$ of the measure spaces (X, ν) and (Y, μ) is ε-measure preserving if there is a set $E \subset X$, $\nu(E) \leqslant \varepsilon$, such that for every measurable set $A \subset X \setminus E$

$$|\mu(\theta(A))(\nu(A))^{-1} - 1| \leqslant \varepsilon.$$

REMARK 2.1. Later (§8) when we study measure preserving or ε-measure preserving maps of various measure spaces, we assume that the measures in these spaces are normalized. In particular, when a subset A of a space (X, ν) with normalized measure is mapped onto the whole space, then in speaking of the measure on A we have in mind not ν but $\frac{1}{\nu(A)}\nu$.

Let $\{\xi_i\}_1^n$ be a sequence of partitions of a measure space (X, ν) and let $x \in X$. The $\{\xi_i\}_1^n$-name of the point x is the numerical sequence $l_i = l_i(x)$ defined by the condition $x \in A_{l_i}^{(i)}$, $\xi_i = \{A_1^{(i)}, \ldots, A_{k_i}^{(i)}\}$. The function $l_i(x)$ is called the nominal function of the sequence of partitions $\{\xi_i\}_1^n$. We define a function e: $\mathbf{Z}^+ \to \mathbf{Z}^+$ by $e(0) = 0$, $e(n) = 1$ for $n > 0$.

THEOREM 2.2 (see [41], Lemma 1.3). *Let $\{\xi_i\}_1^n$ and $\{\eta_i\}_1^n$ be two sequences of partitions of spaces (X, ν) and (Y, μ), respectively, with nominal functions $l_i(x)$ and $m_i(y)$. If there exists an ε-measure preserving map θ: $X \to Y$ for which $\frac{1}{n}\sum_{i=1}^{n} e(l_i(x) - m_i(\theta(x))) \leqslant \varepsilon$ for all $x \in X$ except, perhaps, some set E of measure $\nu(E) \leqslant \varepsilon$, then $\bar{d}(\{\xi_i\}_1^n, \{\eta_i\}_1^n) \leqslant 16\varepsilon$.*

The following statement about K-automorphisms will be needed later.

THEOREM 2.3 (see [41], Lemma 2.2). *Let f be a K-automorphism of a Lebesgue space (M, ν), ξ a finite partition of (M, ν), $\delta > 0$, and $B \subset M$ measurable. There is a $N_0 > 0$ such that for any $N' > N \geqslant N_0$ and δ-almost every element $A \in \bigvee_{N}^{N'} f^k\xi$*

$$\left|\frac{\nu(A \cap B)}{\nu(A)} - \nu(B)\right| \leqslant \delta. \tag{2.1}$$

§3. Basic properties of the characteristic exponents of dynamical systems

3.1. In this section we present a version of the theory of characteristic Lyapunov exponents in a form convenient for our purpose. Since we are going to study the differentials of smooth dynamical systems on compact manifolds, the definitions and results from the theory of Lyapunov exponents to be quoted below are not given in their most general form. The account we follow was proposed by Anosov and differs somewhat from the usual one (see, for example, [10], Ch. 1).

DEFINITION 3.1. A characteristic exponent is defined to be a measurable function χ: $TM \to \mathbf{R}$ satisfying the following conditions for any $x \in M$, v_1, v_2, $v \in TM$:

1. $-\infty < \chi(x, v) < \infty$, $v \neq 0$, $\chi(x, 0) = -\infty$;
2. $\chi(x, \alpha v) = \chi(x, v)$, $\alpha \in \mathbf{R}$, $\alpha \neq 0$;
3. $\chi(x, v_1) + \chi(x, v_2) \leqslant \max\{\chi(x, v_1), \chi(x, v_2)\}$.

The number $\chi(x, v)$ is called the characteristic exponent of the vector $v \in T_xM$ at x.

It can be shown that for every $x \in M$ the restriction of χ to the subspace T_xM takes at most n values other than $-\infty$. We denote these values, arranged in increasing order, by

$$\chi_1(x) < \chi_2(x) < \ldots < \chi_{s(x)}(x), \qquad s(x) \leqslant n. \tag{3.1}$$

Let us put

$$L_i(x) = \{v \in T_xM \colon \chi(x, v) \leqslant \chi_i(x)\}. \tag{3.2}$$

The subspaces $L_i(x)$ form a filtration of T_xM, that is,

$$0 = L_0(x) \subset L_1(x) \subset \ldots \subset L_{s(x)}(x) = T_xM. \tag{3.3}$$

We put $\dim L_i(x) = k_i(x)$, $k_0(x) = 0$. The integral-valued functions $s(x)$, $k_1(x)$, $\ldots$, $k_{s(x)}(x)$ and the families of subspaces $L_i(x)$ $(i = 1, \ldots, s(x))$ depend measurably on x. Conversely, let $s(x) \leqslant n$ be an integral-valued function, let $\chi_1(x)$, $\ldots$, $\chi_{s(x)}(x)$ be measurable functions satisfying (3.1), and let (3.3) be a filtration, depending measurably on x, where $\dim L_i(x) = k_i(x)$ $(i = 1, \ldots, s(x))$. Then the function $\chi(x, v)$ defined by

$$\chi(x, v) = \chi_i(x), \qquad v \in L_i(x) \setminus L_{i-1}(x) \qquad (i = 1, \ldots, s(x)) \tag{3.4}$$

is measurable and determines a characteristic exponent in TM.

We consider the cotangent bundle T^*M. For a point $x \in M$ and a subspace $L \subset T_xM$ we denote by $L^\perp$ the annihilator of L. Now (3.3) induces a filtration on T_x^*M of the form

$$T_x^*M = L_0^\perp(x) \supset L_1^\perp(x) \supset \ldots \supset L_{s(x)}^\perp(x) = 0.$$

This filtration, together with the functions $s^*(x) = s(x)$, $\chi_i^*(x) = \chi_i(x)$ determines a characteristic exponent in T^*M, which we denote by χ^*. It can also be described in the following way.

Let $|\cdot|$ be the trivial normalization of the field $\mathbf{R}$ of real numbers, that is, $|0| = 0$, $|\alpha| = 1$, $\alpha \in \mathbf{R} \setminus \{0\}$. Then for every $x \in M$ the function

$|v|_x = e^{\chi(x,v)}$ defines a non-Archimedean normalization of the linear space T_xM over the trivial normalization of **R**. (A normalization is called non-Archimedean if it satisfies condition 3 of Definition 3.1.) Conversely, any non-Archimedean normalization $|\cdot|_x$ in T_xM over the trivial normalization of **R**, depending measurably on x, determines a characteristic exponent in TM by the formula $\chi(x, v) = \log |v|_x$. We introduce a non-Archimedean normalization in T_x^*M by putting

$$|\varphi|_x^* = \sup_{v \in T_xM \setminus \{0\}} \frac{|\varphi(v)|}{|v|_x}.$$

If $\varphi \in L_{i-1}^{\perp}(x) \setminus L_i^{\perp}(x)$, then it can be easily verified that

$$|\varphi|_x^* = e^{-\chi_i(x)}. \tag{3.5}$$

Thus, the exponent defined by $|\cdot|_x^*$ coincides with χ^*.

3.2. Let f^t be a dynamical system on a manifold M, that is, a diffeomorphism or a flow ($t \in \mathbf{Z}$ in the first case and $t \in \mathbf{R}$ in the second) preserving the measure ν.

We consider the function

$$\chi^+(x, v) = \varlimsup_{t \to +\infty} \frac{1}{t} \log \|df^t v\|. \tag{3.6}$$

It can be shown (see [19]) that χ^+ is measurable and satisfies the conditions 1, 2, and 3 of Definition 3.1. Hence it defines a certain exponent, the so-called characteristic Lyapunov exponent of f^t. If φ is a measurable function on M, then its characteristic Lyapunov exponent at x is defined as

$$\chi^+(\varphi(x)) = \varlimsup_{t \to +\infty} \frac{1}{t} \log |\varphi(f^t(x))|. \tag{3.7}$$

The exponent $\chi^-(\varphi(x))$ for $t \to -\infty$ is similarly defined. We say that a vector v (a function φ) has an exact exponent at x if in (3.6) (or (3.7)) the upper limit can be replaced by the limit.

The exponents $\chi^+(x, v)$ and $\chi^+(\varphi(x))$ are invariant under f^t. Thus, for any $x \in M$ and any t

$$\chi_i(x) = \chi_i(f^t(x)), \qquad k_i(x) = k_i(f^t(x)),$$
$$s(x) = s(f^t(x)), \qquad L_i(f^t(x)) = df^t L_i(x).$$

Here $\chi_i(x)$, $k_i(x)$, $s(x)$, and $L_i(x)$ are the values and the filtration connected with them for the exponent $\chi^+(x, v)$.

The dynamical system f^t generates a dynamical system $d'f^t$ on the cotangent bundle (the so-called codifferential): $d'f^t = ((df^t)^*)^{-1}$.

If on the right-hand side of (3.6) df^t is replaced by $d'f^t$, we obtain an exponent $\chi^{+'}$ on the cotangent bundle, which is called the adjoint exponent.

Let $x \in M$ and let the vector $v \in T_xM$ and the functional $\varphi \in T_x^*M$ be such that $\varphi(v) = 1$. Then for any t

(3.8) $$d'f^t\varphi(df^tv) = \varphi(v) = 1.$$

Therefore, $\| d'f^t\varphi \| \geqslant \| df^tv \|^{-1}$. Hence, the exponents χ^+ and $\chi^{+'}$ satisfy the so-called adjointness condition

(3.9) $$\chi^{+'}(x, \varphi) + \chi^+(x, v) \geqslant 0.$$

By (3.5), for a given $\varphi \in T_x^*M$ there is a vector $v \in T_xM$ such that $\varphi(v) = 1$, and

(3.10) $$\chi^*(x, \varphi) = -\chi^+(x, v).$$

Therefore, by the adjointness condition for any $\varphi \in T_x^*M$

$$\chi^{+'}(x, \varphi) \geqslant \chi^*(x, \varphi).$$

The measurable function

$$\gamma(x) = \max_{\varphi \in T_x^*M} (\chi^{+'}(x, \varphi) - \chi^*(x, \varphi))$$

is called the irregularity coefficient of χ^+. The exponent χ^+ is said to be regular at x if $\gamma(x) = 0$, and a point at which this condition is satisfied is called forward regular. If x is forward regular, then so is $f^t(x)$ for any t.

3.3. We give another equivalent definition of forward regularity. Let $x \in M$ and consider the filtration (3.3) at x. A normalized basis $e(x) = \{e_i(x)\} \in T_xM$ is called normal if the first $k_1(x)$ vectors lie in $L_1(x)$, the next $k_2(x) - k_1(x)$ vectors in $L_2(x)\backslash L_1(x)$, and so on. If $e(x) = \{e_i(x)\}$ is a normal basis at x, then the vectors $e_i^t(x) = \dfrac{df^te_i(x)}{\|df^te_i(x)\|}$ form a normal basis $e^t(x)$ at $f^t(x)$.

We consider the dual basis $e'(x) = \{e_i'(x)\}$ on T_x^*M. The adjointness condition (3.9) means that $\chi^+(x, e_i'(x)) + \chi_x^{+'}(x, e_i(x)) \geqslant 0$. The defect of this pair of bases is defined to be the function

$$\gamma(x, e(x), e'(x)) = \max_i \{\chi^+(x, e_i(x)) + \chi^{+'}(x, e_i'(x))\}.$$

THEOREM 3.1.

$$\gamma(x) = \min \gamma(x, e(x), e'(x)),$$

where the minimum is taken over all pairs $(e(x), e'(x))$ of dual bases.

PROOF. It can be shown (see [10], Theorem 2.6.13) that the minimum in (3.11) is attained on any pair of dual bases under the condition that $e(x)$ is normal. We denote this minimum by $\bar{\gamma}(x)$. In what follows the dependance on x of the quantities under consideration is omitted in the notation. We select a normal basis e and its dual basis e'. For some i, $1 \leqslant i \leqslant \dim M$,

$$\bar{\gamma} = \chi^+(e_i) + \chi^{+'}(e_i'),$$

where $e_i'(e_i) = 1$. We consider the filtration (3.3) at x. Let $e_i' \in L_{j-1}^\perp \backslash L_j^\perp$;

then $e_i \in L_j \setminus L_{j-1}$. By (3.4) and (3.5) we have $\chi^+(e_i) = \chi_j = -\chi^*(e_i')$. Thus, $\bar{\gamma} = \chi^{+'}(e_i') - \chi^*(e_i') \leqslant \gamma$. On the other hand, for some $\varphi \in T_x^*M$

$$\gamma = \chi^{+'}(\varphi) - \chi^*(\varphi).$$

We choose a vector v satisfying (3.10). There is a normal basis e containing v. Then the dual basis e' contains the functional φ. Thus,

$$\gamma = \chi^{+'}(\varphi) + \chi^+(v) \leqslant \gamma(e, e') = \bar{\gamma}.$$

The theorem is now proved.

From it we obtain the following result.

THEOREM 3.2 (Perron [10]). *The exponents χ^+ and $\chi^{+'}$ are regular at x if and only if $\chi_i(x) = -\chi_i'(x)$, and the filtration connected with $\chi^{+'}$ consists of the subspaces $L_i^\perp(x)$. Any basis dual to a normal one is normal.*

We need the following properties of forward regular points.

THEOREM 3.3 (see [10], Theorem 22.1.2). *The following statements are equivalent*:

1. *The point x is forward regular, and $e(x) = \{e_i(x)\}$ is a normal basis.*
2. *The ordered basis $e(x)$ splits into blocks*

$$e(x) = \{e_{k_1(x)}(x), \ldots, e_{k_{s(x)}(x)}(x)\},$$

so that

$$\chi^+(x, e_i(x)) = \chi_j(x), \quad k_{j-1}(x) < i \leqslant k_j(x)$$

and the function $\Gamma_j(x)$, the volume of the parallelipiped spanned by the vectors $e_i(x)$, $k_{j-1}(x) < i \leqslant k_j(x)$, has the exact exponent

$$\chi^+(\Gamma_j(x)) = \sum_{k_{j-1}(x) < i \leqslant k_j(x)} \chi^+(x, e_i(x)) = (k_j(x) - k_{j-1}(x))\chi_j(x). \tag{3.12}$$

COROLLARY 3.1. *If x is forward regular, then for any $v \in T_xM$ the exponent $\chi^+(x, v)$ is exact.*

If on the right-hand side of (3.6) instead of the upper limit as $t \to +\infty$ we consider the upper limit as $t \to -\infty$, we obtain the exponent χ^- on TM.

A point x is said to be backward regular if it is forward regular for the exponent χ^-. A point x is said to be regular if it is both forward and backward regular.

If x is regular, then so is $f^t(x)$ for any t. Furthermore, the values of the characteristic exponents χ^+ and χ^- and also the filtrations connected with them are compatible with each other.

THEOREM 3.4 (see [19], Theorem 4, [10], §11, 22). *If x is regular, then there are subspaces $E_i(x)$ $(i = 1, \ldots, s(x))$ such that*

1. $L_i(x) = \bigoplus_{j=1}^{k_i(x)} E_j(x) \qquad (i = 1, \ldots, s(x)).$

2. $\lim\limits_{t\to\pm\infty} \frac{1}{|t|} \log \| df^t v \| = \pm \chi_j(x)$ *uniformly in* $v \in E_j(x)$.

3. $\chi^-(\Gamma_j(x)) = -\chi^+(\Gamma_j(x)) = (k_j(x) - k_{j-1}(x))\chi_j(x)$, *where* $\Gamma_j(x)$ *is the volume of the parallelipiped in* $E_j(x)$.

4. *If* $E_j(f^t(x))$ *is a subspace at* $f^t(x)$, *then* $df^t E_j(x) = E_j(f^t(x))$.

5. *If* σ *is a subset of the set of natural numbers from* 1 *to* $s(x)$, *if* $P_\sigma(x) = \bigoplus_{i\in\sigma} E_i(x)$, *and if* $\gamma_{\sigma_1,\sigma_2}(x)$ *is the angle between* $P_{\sigma_1}(x)$ *and* $P_{\sigma_2}(x)$, *then* $\chi^+(\gamma_{\sigma_1,\sigma_2}(x)) = \chi^-(\gamma_{\sigma_1,\sigma_2}(x)) = 0$.

REMARK. At a regular point we also have the decomposition $T_x^*M = \bigoplus_{j=1}^{s(x)} E_j'(x)$ with similar properties with respect to the exponent $\chi^{+'}$; moreover, if $e(x) = \{e_i(x)\}$ is a normal basis for which $e_i(x) \in E_j(x)$ for $k_{j-1}(x) < i \leqslant k_j(x)$ and if $e'(x) = \{e_i'(x)\}$ is the dual basis, then $e_i'(x) \in E_j'$ for $k_{j-1}(x) < i \leqslant k_j(x)$.

3.4. In what follows we use the fact that for a smooth dynamical system regularity is a typical property with respect to any invariant measure. This was proved by Oseledets (see [19]) in a more general formulation (for linear extensions of dynamical systems on a Lebesgue space) and in a similar form by Millionshchikov (see [16]); in [19] it is called the multiplicative ergodic theorem. For it plays the same role in the study of the above-mentioned linear extensions as Birkhoff's ergodic theorem does in the study of dynamical systems on a Lebesgue space. We quote the Oseledets–Millionshchikov theorem for dynamical systems on smooth manifolds.

THEOREM 3.5. *Let* f^t *be a dynamical system on a smooth compact manifold* M *that preserves Borel measure. Then with respect to this measure almost every point* $x \in M$ *is regular. The function* $s(x)$ *and the subspaces* $E_1(x), \ldots, E_{s(x)}(x)$ (see Theorem 3.4) *depend measurably on* x.

§4. Properties of local stable manifolds

4.1. In this section we formulate some results we need later on local stable manifolds, which generalize the corresponding statements on properties of local contracting fibres in the theory of U-systems. Detailed proofs are in our paper [22]. The exposition will necessarily be brief and quite formal. However, since these technical results are fundamental for all further developments, before dealing with them we given an informal account of the questions to be studied here and describe the steps we have to follow in order to pass from (1.7) to the study of metric properties of dynamical systems.

The first step consists in establishing the conditions of non-uniform partial hyperbolicity (1.1)–(1.5) on the set of regular points $\widetilde{\Lambda} \subset \Lambda$ (see §§ 4.2 and 4.3; the set Λ is defined in (1.7)). In what follows with the help of the inequalities (1.3) and (1.5) we succeed to some extent in reproducing the scheme of arguments developed by Anosov and Sinai in the theory of U-systems. More precisely, we wish to "lower" the subspaces E_{1x}, $x \in \tilde{\Lambda}$, f_n-invariantly on the manifold M. In other words, by considering a fixed trajectory $\{f^n(x_0)\}$ we want to find out whether our "conditions of hyperbolicity" of the family $\{df^n_{x_0}\}$ imply some "conditional stability" of the trajectory $\{f^n(x_0)\}$, that is, whether there is a smooth submanifold $V(x_0) \subset M$ such that the $\{f^n(x)\}$ with the initial value $x \in V(x_0)$ approach $\{f^n(x_0)\}$ asymptotically.

Better known is a similar question on the conditional stability of the trivial solution $x(t) \equiv 0$ of the system of differential equations $dx/dt = A(t)x + f(t, x)$, where x and f are vectors, $A(t)$ is a uniformly bounded and uniformly continuous matrix, and $d_x f(t, x)$ satisfies a Lipschitz condition with a sufficiently small constant, uniform with respect to t. It is known that when part of the characteristic Lyapunov exponents of the "first approximation" system $dx/dt = A(t)x$ is negative, this alone is not sufficient for the "conditional stability" of the perturbed system. An appropriate counterexample was given by Perron (see [10], §30.3). It turns out that regularity in the sense of Lyapunov of the system $dx/dt = A(t)x$ (or of the family $\{df^n_{x_0}\}$, respectively) is sufficient for the theorem on "conditional stability" to hold. For differential equations with an analytic right-hand side such a theorem was proved by Lyapunov (see [10], §16). For differential equations with smooth right-hand side in the case of absolute stability (when all characteristic Lyapunov exponents are negative) the corresponding theorem was proved by Malkin (his result is obtained in [10], §16 as a corollary to an even more general theorem, also concerning the case of absolute stability). Finally, in [22] the author has proved a theorem on "conditional stability" for families of maps $\{df^n_x\}$ of class $C^{1+\varepsilon}$ satisfying the condition of non-uniform partial hyperbolicity. This theorem is called to play the same role in our case as the famous Hadamard–Perron theorem (see [1], §3) in the general theory of U-systems: it helps in the construction of local stable manifolds $V(x)$ (see §4.4).

We mention some difficulties connected with the study of local stable manifolds. Firstly, they are defined only for almost all $x \in \Lambda$, that is, on some "perforated" subset of M. Secondly, the submanifolds $V(x)$ depend on x "discontinuously" (but in a certain sense "measurably").[1] In particular, the "measures" of these submanifolds are discontinuous functions. Moreover, the "measures" of the $V(f^n(x))$ for sufficiently large $n > 0$ may prove to be arbitrarily small. The surmounting of these is based on the following considerations. By means of (1.5) and (1.3) the "measures" of the

[1] We also note that there is a certain arbitrariness in the construction of local stable manifolds: for example, they depend on the choice of the Riemannian metric on M (for more details see §6).

$V(f^n(x))$, $n > 0$, can be estimated from below. Namely (see Theorem 4.1),

$$\text{"measure" of } V(f^n(x)) \geqslant (\text{"measure" of } V(x))e^{-\varepsilon(x)n}.$$

Thus, the "measures" of $V(f^n(x))$ decrease more slowly than their points approach each other (by means of (1.4) it can be shown that the rate of approach of these points is bounded above by $(\lambda(x) + \varepsilon(x))^n$) a fact on which the proofs of the majority of propositions of a technical nature are based. Next we consider the so-called families of local stable manifolds, that is, the collection of submanifolds $V(x)$, where x ranges over one of the (non-invariant!) subsets $\Lambda^l \subset \Lambda$ ($l = 1, 2, \ldots$), which are chosen in a special way so that the estimates (1.1)–(1.5) are uniform on Λ^l (and "worsen" with the increase of l; see §4.3).

Among the properties of the families of local stable manifolds one of the most important is that of absolute continuity. Roughly speaking, this means that the intersection of any set $A \subset M$ of measure zero with almost every local stable manifold $V(x)$ from the family is a set of measure zero (we have in mind the measure induced by the Riemannian metric on $V(x)$, regarded as a smooth submanifold in M), and conversely, every set with this property is of measure zero. Our definition of absolute continuity and the theorem on absolute continuity in §4.5 generalize the corresponding concepts and assertions in the theory of U-systems (see [1], §5; [3]) and of partially hyperbolic dynamical systems (see [5], §2). In fact, absolute continuity is the bridge on which we can pass from the differential properties of a dynamical system (that is, from the properties of the equation in variations) to its metric properties. In this context we mention that, as Sinai has shown [27], the presence in a dynamical system of an invariant contracting absolutely continuous foliation (which he calls a transversal foliation) ensures certain ergodic properties, for example, the positiveness of entropy (see also [6]).

4.2. Let f be a C^r-diffeomorphism on M, $r \geqslant 2$, preserving the measure ν and satisfying (1.7). We consider the set $\tilde{\Lambda}$ of regular points in Λ. From Theorem 3.5 it follows that $\nu(\tilde{\Lambda}) = \nu(\Lambda)$. Let $x \in \tilde{\Lambda}$ and consider the filtration (3.3) and the subspaces $E_j(x)$ ($j = 1, 2, \ldots, s(x)$) at x (see Theorem 3.4). We denote by $k(x)$ the largest natural number such that $\chi^+(x, v) < 0$ for every $v \in L_{k(x)}(x)$. Then $1 \leqslant k(x) < s(x)$ and $k(f(x)) = k(x)$. We set

$$\left\{\begin{array}{ll} E_{1x} = \bigoplus\limits_{j=1}^{k(x)} E_j(x), & E_{2x} = \bigoplus\limits_{j=k(x)+1}^{s(x)} E_j(x), \\ \lambda(x) = e^{\chi_{k(x)}(x)}, & \mu(x) = e^{\chi_{k(x)+1}(x)}. \end{array}\right. \tag{4.1}$$

By Theorem 3.5, the functions $\lambda(x)$, $\mu(x)$ and the subspaces E_{1x}, E_{2x} depend measurably on x and satisfy the conditions

$$\left\{\begin{array}{l} 0 < \lambda(x) < 1 \leqslant \mu(x), \\ \lambda(f(x)) = \lambda(x), \quad \mu(f(x)) = \mu(x). \end{array}\right. \tag{4.2}$$

$$(4.3) \qquad T_xM = E_{1x} \oplus E_{2x}, \qquad dfE_{ix} = E_{if(x)} \qquad (i = 1,\ 2).$$

PROPOSITION 4.1 (see [22], Theorem 1.1.1). *There exist measurable functions* $C(x, \varepsilon), K(x, \varepsilon), \varepsilon > 0,\ x \in \widetilde{\Lambda}$, *such that:*

1) *for any* $m \in \mathbf{Z}$

$$(4.4) \qquad \begin{cases} C(f^m(x), \varepsilon) \leqslant C(x, \varepsilon)\, e^{4\varepsilon|m|}, \\ K(f^m(x), \varepsilon) \geqslant K(x, \varepsilon)\, e^{-\varepsilon|m|}; \end{cases}$$

2) *for any* $n \in \mathbf{Z}^+$

$$(4.5) \qquad \begin{cases} \|df^n v\| \leqslant C(x, \varepsilon)\, \lambda^n(x)\, e^{\varepsilon n} \|v\|, & \\ & v \in E_{1x}, \\ \|df^{-n} v\| \geqslant C^{-1}(x, \varepsilon)\, \lambda^{-n}(x)\, e^{-\varepsilon n} \|v\|, & \\ \|df^n v\| \geqslant C^{-1}(x, \varepsilon)\, \mu^n(x)\, e^{-\varepsilon n} \|v\|, & \\ & v \in E_{2x}; \\ \|df^{-n} v\| \leqslant C(x, \varepsilon)\, \mu^{-n}(x)\, e^{\varepsilon n} \|v\|, & \end{cases}$$

3) *the angle* $\gamma(x)$ *between* E_{1x} *and* E_{2x} *admits the estimate* $\gamma(x) \leqslant K(x, \varepsilon)$.

4.3. For integers $s > r \geqslant 1$ we consider the sets $\widetilde{\Lambda}_{s,r} = \left\{x \in \widetilde{\Lambda}: \frac{r-1}{s} < \lambda(x) \leqslant \frac{r}{s} < \frac{r+2}{s} \leqslant \mu(x)\right.$, where s is the smallest number satisfying these inequalities for a certain $r\}$.

It is obvious that the $\widetilde{\Lambda}_{s,r}$ are measurable and f-invariant; also, $\bigcup_{s>r\geqslant 1} \widetilde{\Lambda}_{s,r} = \widetilde{\Lambda}$, and if $s_1 \neq s_2$ or $r_1 \neq r_2$, then $\widetilde{\Lambda}_{s_1,r_1} \cap \widetilde{\Lambda}_{s_2,r_2} = \emptyset$.

For $x \in \widetilde{\Lambda}_{s,r}$ and $N \in \mathbf{Z}^+$ we put

$$(4.6) \qquad \varepsilon(x) = \varepsilon_s = \frac{1}{100N} \log\left(1 + \frac{2}{s}\right).$$

The function $\varepsilon(x)$ is measurable and invariant. Of course, it also depends on the choice of N, but in our notation we do not indicate this dependence explicitly (see Remark 4.4). Setting $\varepsilon = \varepsilon(x)$, $C(x) = C(x, \varepsilon(x))$, $K(x) = K(x, \varepsilon(x))$ in Proposition 4.1, we find that on Λ the diffeomorphism f satisfies the conditions of non-uniform partial hyperbolicity (1.3)–(1.5).

For every integer $l \geqslant 1$ we put

$$\widetilde{\Lambda}^l_{s,r} = \{x \in \widetilde{\Lambda}_{s,r}: C(x, \varepsilon(x)) \leqslant l, \quad K^{-1}(x, \varepsilon(x)) \leqslant l\}.$$

The sets $\widetilde{\Lambda}^l_{s,r}$ are measurable, $\bigcup_{l\geqslant 1} \widetilde{\Lambda}^l_{s,r} = \widetilde{\Lambda}_{s,r}$, $\widetilde{\Lambda}^l_{s,r} \subset \widetilde{\Lambda}^{l+1}_{s,r}$, and for any $x \in \widetilde{\Lambda}^l_{s,r}$, $n \in \mathbf{Z}^+$, $m \in \mathbf{Z}$

$$
(4.7)\quad \begin{cases} \| df^n_{f^m(x)} v \| \leqslant l \left(\frac{r}{s}\right)^n e^{\varepsilon_s n + 4\varepsilon_s |m|} \| v \|, & \\ & v \in E_{1f^m(x)}, \\ \| df^{-n}_{f^m(x)} v \| \geqslant l^{-1} \left(\frac{r}{s}\right)^{-n} e^{-\varepsilon_s n - 4\varepsilon_s |m|} \| v \|, & \\ \| df^n_{f^m(x)} v \| \geqslant l^{-1} \left(\frac{r+2}{s}\right)^n e^{-\varepsilon_s n - 4\varepsilon_s |m|} \| v \|, & \\ & v \in E_{2f^m(x)}; \\ \| df^{-n}_{f^m(x)} v \| \leqslant l \left(\frac{r+2}{s}\right)^{-n} e^{\varepsilon_s n + 4\varepsilon_s |m|} \| v \|, & \end{cases}
$$

$$
(4.8)\qquad l^{-1} e^{-\varepsilon_s |m|} \leqslant \gamma(f^m(x)).
$$

We denote by $\Lambda^l_{s,r}$, $s > r \geqslant 1$, the set of points $x \in M$ such that
(4.9) there are subspaces E_{1x} and E_{2x} for which $T_x M = E_{1x} \oplus E_{2x}$;
(4.10) the estimates (4.7) hold for the vectors $v \in df^m E_{ix} (i = 1, 2)$;
(4.11) the angle $\gamma(f^m(x))$ between $df^m E_{1x}$ and $df^m E_{2x}$ satisfies (4.8).
PROPOSITION 4.2 (see [22], Theorem 1.3.1).

1. $\widetilde{\Lambda}^l_{s,r} \subset \Lambda^l_{s,r} \subset \Lambda$, $\Lambda^l_{s,r} \subset \Lambda^{l+1}_{s,r}$.

2. *The set* $\Lambda^l_{s,r}$ *is closed.*
3. E_{1x} *and* E_{2x} *depend continuously on* x *in* $\Lambda^l_{s,r}$.
4. *For any integer* q *and* $l \geqslant 1$ *there is an* $\alpha = \alpha(l, q, s) > 0$ *such that* $f^q(\Lambda^l_{s,r}) \subset \Lambda^\alpha_{s,r}$.
5. *The set* $\Lambda_{s,r} = \bigcup_{l \geqslant 1} \Lambda^l_{s,r}$ *is f-invariant. If* $r_1, s_1, r_2, s_2 \in \mathbf{Z}^+$, $s_1 > r_1 \geqslant 1$, $s_2 > r_2 \geqslant 1$ *are such that* $\frac{r_1}{s_1} < \frac{r_2}{s_2} < \frac{r_2+2}{s_2} < \frac{r_1+2}{s_1}$, *then* $\Lambda_{r_1,s_1} \subset \Lambda_{r_2,s_2}$.

We put

$$
\hat{\Lambda} = \bigcup_{s > r \geqslant 1} \Lambda_{s,r}, \qquad A_k = \{x \in \hat{\Lambda}\colon \dim E_{1x} = k\},
$$

$$
\Lambda_{k,s,r} = A_k \cap \Lambda_{s,r}, \quad \Lambda^l_{k,s,r} = A_k \cap \Lambda^l_{s,r},
$$

$$
\widetilde{\Lambda}_{k,s,r} = A_k \cap \widetilde{\Lambda}_{s,r}, \quad \widetilde{\Lambda}^l_{k,s,r} = A_k \cap \widetilde{\Lambda}^l_{s,r}.
$$

4.4. We consider the set $\Lambda_{s,r}$ for some s and r, $s > r \geqslant 1$ (which in this subsection we take as fixed). We choose an arbitrary number $\varkappa_{s,r}$ such that

$$
(4.12)\qquad \frac{r}{s} e^{3\varepsilon_s} < \varkappa_{s,r} < e^{-5\varepsilon_s}.
$$

Let $\delta(x)$ be a positive measurable function on $\Lambda_{s,r}$. We set

$$
B^i(\delta(x)) = \{u \in E_{ix} : \| u \|_x < \delta(x)\} \qquad (i = 1, 2),
$$

$$
B(\delta(x)) = B^1(\delta(x)) \times B^2(\delta(x)),
$$

$$
(4.13)\qquad U(x, \delta(x)) = \exp_x B(\delta(x)).
$$

THEOREM 4.1 (see [22], Theorem 2.2.1). *There exist measurable functions $\delta(x)$, $\delta'(x)$, $A(x)$, a family of maps $\varphi(x)$: $B^1(\delta(x)) \to B^2(\delta(x))$ of class C^{r-1}, depending measurably on $x \in \Lambda_{s,r}$, and a constant $L > 0$ such that*

1) *the set $V(x) = \{\exp_x(u, \varphi(x)u) : u \in B^1(\delta(x))\}$ is a submanifold in M of class C^{r-1};*

2) $x \in V(x)$;

3) $T_x V(x) = E_{1x}$;

4) *for $y \in V(x)$ and $n \in \mathbf{Z}^+$ we have $f^n(y) \in U(f^n(x), \delta'(f^n(x)))$,*

$$\rho(f^n(x), f^n(y)) \leqslant LA(x)\, (\varkappa_{s,r})^n \rho(x, y); \tag{4.14}$$

5) *Suppose that $y \in U(x, \delta(x))$ and let $C > 0$ be a constant such that $f^n(y) \in U(f^n(x), \delta'(f^n(x)))$, $\rho(f^n(x), f^n(y)) \leqslant C(\varkappa_{s,r})^n$ for any $n \in \mathbf{Z}^+$; then $y \in V(x)$;*

6) *for any $m \in \mathbf{Z}^+$*

$$\left\{ \begin{array}{ll} \delta'(f^m(x)) \geqslant \delta'(x)\, e^{-5\varepsilon(x)m}, & \delta'^{\,l}_{s,r} = \inf\limits_{x \in \Lambda^l_{s,r}} \delta'(x) > 0, \\ \delta(f^m(x)) \geqslant \delta(x)\, e^{-10\varepsilon(x)m}, & \delta^l_{s,r} = \inf\limits_{x \in \Lambda^l_{s,r}} \delta(x) > 0; \end{array} \right. \tag{4.15}$$

$$A(f^m(x)) \leqslant A(x)\, e^{5\varepsilon(x)m}, \quad A^l_{s,r} = \sup_{x \in \Lambda^l_{s,r}} A(x) < \infty \tag{4.16}$$

and $\delta'(x) > \delta(x)$;

7) $f(V(x)) \cap U(f(x), \delta(f(x))) \subseteq V(f(x))$;

8) *there is a measurable function $G(x)$, $x \in \Lambda_{s,r}$, such that $G(f(x)) = G(x)$ and $G_{s,r} = \sup\limits_{x \in \Lambda_{s,r}} G(x) < \infty$, and for any $y \in V(x)$*

$$d(T_y V(x), T_x V(x)) \leqslant G(x) A^2(x) \rho(x, y). \tag{4.17}$$

DEFINITION 4.1. $V(x)$ is called a local stable manifold passing through $x \in \Lambda_{s,r}$.

REMARK 4.1. By means of Theorem 4.1 we can construct a local stable manifold at every point of $\widetilde{\Lambda}_{s,r}$ for any $s > r \geqslant 1$ and, consequently (since these sets are disjoint), at every point of $\widetilde{\Lambda}$. In fact, this can be done at every point of $\hat{\Lambda}$.

REMARK 4.2. Theorem 4.1.5, expresses a certain property of "uniqueness", which can also be stated in the following form.

5'). *For any $\varepsilon > 0$ there are functions $\delta'_\varepsilon(x)$, $\delta_\varepsilon(x)$, $x \in \Lambda_{s,r}$ satisfying* (4.15) *and such that if $y \in U(x, \delta_\varepsilon(x))$ and*

$$f^n(y) \in U(f^n(x), \delta'_\varepsilon(f^n(x))), \quad \rho(f^n(x), f^n(y)) \leqslant Ce^{-\varepsilon n}$$

for any $n > 0$ *and a certain* $C > 0$, *then* $y \in V(x)$.

PROOF. We choose the number N in (4.6) so large that $6\varepsilon(x) \leqslant \varepsilon$ for any $x \in \Lambda_{s,r}$. We put $\varkappa_{s,r} = e^{-6\varepsilon_s}$. By means of Theorem 4.1, we construct from $\varepsilon(x)$ and $\varkappa_{s,r}$ the functions $\delta'(x) = \delta'_\varepsilon(x)$ and $\delta(x) = \delta_\varepsilon(x)$. Let y and C satisfy the conditions of 5′). Then

$$\rho(f^n(x),\ f^n(y)) \leqslant Ce^{-\varepsilon n} \leqslant Ce^{-6\varepsilon(x)n} = C(\varkappa_{s,r})^n.$$

Hence, by Theorem 4.1. 5), we find that $y \in V(x)$.

Some additional properties of local stable manifolds are described in the following propositions.

THEOREM 4.2 (see [22], Theorem 2.3.1). 1. *If* $x, y \in \Lambda_{s,r}$, $y \in U \times (x, \frac{1}{4}\delta(x))$, $y \notin V(x)$, *then* $V(x) \cap V(y) \cap U(x, \frac{1}{4}\delta(y)) = \varnothing$.

2. *If* $x \in \Lambda_{s,r}$, $y \in V(x) \cap \Lambda_{s,r}$, *then* $V(y) \cap U(x, \delta(x)) \subseteq V(x)$.

3. *If* $x \in \Lambda^l_{s,r}$, $x_i \in \Lambda^l_{s,r} (i = 1, 2, \ldots)$, *and* $x_i \to x$, *then* $V(x_i) \cap U(x, q) \to V(x) \cap U(x, q)$ *in the* C^1*-topology, where* $0 < q < \delta^l_{s,r}$.

THEOREM 4.3 (see [22], Proposition 2.3.1). 1. *Let* $x \in \widetilde{\Lambda}$, $y \in V(x)$. *Then* y *is forward regular and* $s(x) = s(y)$, $\chi_i(x) = \chi_i(y)$ $(i = 1, \ldots, s(x))$.

2. *Let* $x \in \widetilde{\Lambda}^l_{s,r}$, $y \in \widetilde{\Lambda} \cap V(x)$. *Then there is a* $K = K(l, s, r)$ *such that for any* $n \in \mathbf{Z}^+$

$$\|df^n v\|_{f^n(y)} \leqslant K\left(\frac{r}{s}\right)^n e^{5\varepsilon_s n}\|v\|_y,\quad v \in E_{1y} = T_yV(x), \tag{4.18}$$

$$\|df^n v\|_{f^n(y)} \geqslant K^{-1}\left(\frac{r+2}{s}\right)^n e^{-5\varepsilon_s n}\|v\|_y,\quad v \in E_{2y}. \tag{4.19}$$

3. *Let* $x \in \Lambda_{s,r}$, $y \in V(x)$. *Then* $\chi^+(y, v) < 0$ *for any* $v \in T_yV(x)$.

REMARK 4.3 (see [22], Remark 2.3.1). There is an $\alpha^l_{s,r}$ such that for any $x, y \in \Lambda^l_{k,s,r}$, $y \in U(x, \alpha^l_{s,r})$

$$V(y) \cap U\left(x, \frac{1}{8}\ \alpha^l_{s,r}\right) = \{\exp_x(u,\ \varphi_y(u)) : u \in B(\alpha^l_{s,r})\},$$

where $\varphi_y : B(\alpha^l_{s,r}) \to E_{2x}$ is a map of class C^{r-1}.

DEFINITION 4.2. The family of local stable manifolds $S^l_{k,s,r}(x)$, $x \in \Lambda^l_{k,s,r}$, is the collection of local stable manifolds passing through $y \in \Lambda^l_{k,s,r} \cap U(x, \frac{1}{8}\delta^l_{s,r})$.

4.5. Let x be a density point of $\Lambda^l_{k,s,r}$. We choose a number q, $0 < q < \frac{1}{8}\alpha^l_{s,r}$, and put

$$\hat{\Lambda}^l_{k,s,r}(x) = \bigcup_{y \in \Lambda^l_{k,s,r} \cap U(x,q)} V(y) \cap U(x,\ q). \tag{4.20}$$

In the neighbourhood $U(x, q)$ we consider an open smooth submanifold

W for which the set $\exp_x^{-1} W$ is the graph of a smooth map $\psi\colon U \to E_{1x}$ defined in some neighbourhood $U \subset E_{2x}$ by $t(\psi(u)) = u$, $u \in U$, where t is the projection on E_{2x} parallel to E_{1x}. We set

$$(4.21) \qquad |W| = \max_{u \in U} \|\psi(u)\|_x + \max_{u \in U} \|d\psi(u)\|_x.$$

There is a constant $\varepsilon^l_{k,s,r} > 0$ such that if $|W| \leqslant \varepsilon^l_{k,s,r}$, then W intersects every $V(y)$, $y \in \Lambda^l_{k,s,r} \cap U(x, q)$ in at most one point, and this intersection is transversal. A submanifold W satisfying these conditions is called transversal to the family $S^l_{k,s,r}(x)$.

Let W^1 and W^2 be two smooth submanifolds transversal to $S^l_{k,s,r}(x)$. There are open submanifolds $\widetilde{W}^1 \subset W^1$ and $\widetilde{W}^2 \subset W^2$ for which the succession map is defined:

$$p\colon \hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^1 \to \hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^2.$$

Namely, if $y = \widetilde{W}^1 \cap V(w)$, $w \in \Lambda^l_{k,s,r} \cap U(x, q)$, then

$$(4.22) \qquad p(y) = \widetilde{W}^2 \cap V(w).$$

DEFINITION 4.3. A family $S^l_{k,s,r}(x)$ is said to be absolutely continuous if any succession map constructed as above is absolutely continuous.

This definition generalizes that of absolute continuity given by Anosov in the case of U-systems (see [1], §5, §17, §19; this property is discussed there and its role in metric theory is indicated; see also the definition of absolute continuity in [5], §2).

THEOREM 4.4. *There are constants $q^l_{s,r}$, $J^l_{s,r}$ such that:*

1. *the family $S^l_{k,s,r}(x)$ is absolutely continuous in $U(x, q^l_{s,r})$;*
2. *if y is a density point of $\hat{\Lambda}^l_{k,s,r} \cap \widetilde{W}^1$, then the Jacobian $J(p)(y)$ satisfies the condition*

$$(4.23) \qquad |J(p)(y) - 1| \leqslant J^l_{s,r}(|W^1| + |W^2|).$$

4.6. Let x be a density point of $\Lambda^l_{k,s,r}$, W an open smooth submanifold transversal to $S^l_{k,s,r}(x)$, $x \in W$, and $A \subset W \cap \hat{\Lambda}^l_{k,s,r}$. We put

$$\hat{A} = \bigcup_{z \in \Lambda^l_{k,s,r},\ V(z) \cap A \neq \varnothing} (V(z) \cap U(x, q^l_{s,r})).$$

Let μ be the measure induced on W by the restriction to W of the Riemannian metric of M. We introduce a new measure $\hat{\mu}$ on W by putting for any measurable set A

$$\hat{\mu}(A) = \nu(A).$$

Let ν_z denote the measure induced on $V(z)$ by the Riemannian metric.

PROPOSITION 4.3.1. *The measure $\hat{\mu}$ is absolutely continuous with respect to μ.*

2. *The partition* ζ *of* $\hat{\Lambda}$ *into submanifolds* $V(z)$, $z \in \Lambda^l_{k,s,r}$, $V(z) \cap A \neq \emptyset$, *is measurable and the conditional measure* $\hat{\nu}_z$ *on an element of the partition is absolutely continuous with respect to* ν_z.

3. $\nu_z(V(z)) > 0$ *for almost every* $y \in A$, *where* $z \in \Lambda^l_{k,s,r}$ *and* $V(z) \cap A = y$.

In the subsequent propositions we use mostly not Theorem 4.4 itself, but the property of local stable manifolds, which follows from it and is established in Proposition 4.3. (The inequality (4.23) is used explicitly only in the proof of the Bernoullian property.) As Sinai has done (see [27]), this property can be used as a base for the definition of absolute continuity. In our case, however (as in the case of U-systems), it is convenient to establish this property by using succession maps (about this, see the remark in [1], p. 205). From Theorem 4.4 and Proposition 4.3 the following results are immediately obtained.

PROPOSITION 4.4. *Let* x *be a density point of* $\Lambda^l_{k,s,r}$, W *a smooth submanifold transversal to* $S^l_{k,s,r}(x)$, *and* $N \subset W$ *a set of Lebesgue measure zero in* W. *Then*

$$\nu\Big(\bigcup_{w\in\Lambda^l_{k,\,s,\,r},\ V(w)\cap W\in N} (V(w)\cap U(x,\, q^l_{s,\,r}))\Big)=0.$$

PROPOSITION 4.5. *Let* $\nu(\Lambda_{k,s,r}) > 0$. *There is a set* $N \subset M$ *of measure zero such that for any* $l \in \mathbf{Z}^+$ *and* $x \in \Lambda^l_{k,s,r} \setminus N$

$$\nu_x(V(x)\setminus\Lambda^l_{k,\,s,\,r})=0.$$

4.7. In this subsection we reformulate the preceding statements for diffeomorphisms satisfying (1.8).

Let f be a C^2-diffeomorphism of M preserving the measure ν and $\widetilde{\Lambda}$ the set of the regular points in Λ, $\nu(\widetilde{\Lambda}) = \nu(\Lambda)$ (see Theorem 3.5). We define the functions $\lambda(x)$, $\mu(x)$ and the subspaces E_{1x}, E_{2x} by (4.1). Then

$$\mu(x)>1. \tag{4.24}$$

Making use of Proposition 4.1 we construct measurable functions $C(x,\varepsilon)$ and $K(x,\varepsilon)$, $\varepsilon>0$, $x \in \widetilde{\Lambda}$. For an integer $s > 1$ we set

$\widetilde{\Lambda}_s = \{x \in \widetilde{\Lambda}\colon \lambda(x) \leqslant 1 - 1/s < (1 - 1/s)^{-1} \leqslant \mu(x)$, where s is the smallest number satisfying these inequalities$\}$.

It is clear that the $\widetilde{\Lambda}_s$ are measurable f-invariant sets; also that $\bigcup_{s>1} \widetilde{\Lambda}_s = \widetilde{\Lambda}$ and if $s_1 \neq s_2$ then $\Lambda_{s_1} \cap \Lambda_{s_2} = \emptyset$. We define the function $\varepsilon(x)$ for $x \in \Lambda_s$ by (4.6). For $l \geqslant 1$ we put

$$\widetilde{\Lambda}^l_s=\{x\in\widetilde{\Lambda}_s\colon\ C(x,\,\varepsilon(x))\leqslant l,\ \ K^{-1}(x,\,\varepsilon(x))\leqslant l\}.$$

The sets $\widetilde{\Lambda}^l_s$ are measurable, $\bigcup_{l\geqslant 1} \widetilde{\Lambda}^l_s = \widetilde{\Lambda}_s$, and $\widetilde{\Lambda}^l_s \subset \widetilde{\Lambda}^{l+1}_s$. Moreover, for any $x \in \widetilde{\Lambda}^l_s$, $m \in \mathbf{Z}$, $n \in \mathbf{Z}^+$ (4.7) and (4.8) hold if r/s is replaced by

$(s-1)/s$ and $(r+2)/s$ by $s/(s-1)$. We define the sets Λ_s^l by (4.9)–(4.11) (as before, in (4.7) and (4.8) r/s is to be replaced by $(s-1)/s$ and $(r+2)/s$ by $s/(s-1)$). The following is proved like Proposition 4.2.

PROPOSITION 4.6. 1. $\widetilde{\Lambda}_s^l \subset \Lambda_s^l \subset \Lambda$, $\Lambda_s^l \subset \Lambda_s^{l+1}$.

2. *The set* Λ_s^l *is closed.*

3. *The subspaces* E_{1x} *and* E_{2x} *depend continuously on* x *in* Λ_s^l.

4. *For any integer* q *and* $l \geqslant 1$ *there is an* $\alpha = \alpha(l, q, s) \in \mathbf{Z}^+$ *such that* $f^q(\Lambda_s^l) \subset \Lambda_s^\alpha$.

5. *The set* $\Lambda_s = \bigcup_{l\geqslant 1} \Lambda_s^l$ *is f-invariant, and* $\Lambda_{s_1} \subset \Lambda_{s_2}$ *for any* $1 < s_1 < s_2$.

We consider the diffeomorphism f^{-1}. By Theorem 3.4, it has non-zero characteristic exponents at any point of $\widetilde{\Lambda}$, therefore, the preceding arguments are applicable to it. Let the bar above symbols for functions, subspaces, and sets mean that they are constructed for f^{-1}. Then by Theorems 3.4 and 4.1, for $x \in \widetilde{\Lambda}$,

$$\bar{\lambda}(x) = \mu^{-1}(x), \quad \bar{\mu}(x) = \lambda^{-1}(x), \quad \bar{E}_{1x} = E_{2x}, \quad \bar{E}_{2x} = E_{1x},$$
$$\bar{\varepsilon}(x) = \varepsilon(x), \quad \bar{C}(x, \bar{\varepsilon}(x)) = C(x, \varepsilon(x)), \quad \bar{K}(x, \bar{\varepsilon}(x)) = K(x, \varepsilon(x)),$$
$$\bar{\Lambda}_s^l = \Lambda_s^l, \quad \bar{\Lambda}_s = \Lambda_s, \quad \bar{\widetilde{\Lambda}}_s^l = \widetilde{\Lambda}_s^l, \quad \bar{\widetilde{\Lambda}}_s = \widetilde{\Lambda}_s.$$

We put

$$\hat{\Lambda} = \bigcup_{s>1} \Lambda_s, \quad \Lambda_{k,s} = A_k \cap \Lambda_s, \quad \Lambda_{k,s}^l = A_k \cap \Lambda_s^l.$$

It is obvious that $\Lambda_{k,s}^l$ is measurable and that $\Lambda_{k,s}$ is measurable and invariant. From Theorem 3.5 it follows that $\hat{\Lambda} = \Lambda$ (mod 0).

By applying Theorem 4.1 to f and f^{-1} we can construct local stable manifolds, which we denote by $V^-(x)$ and $V^+(x)$, respectively. The manifold $V^+(x)$ is called local unstable manifold (with respect to f). These manifolds are defined in $U(x, \delta(x))$ (see (4.13)), and the measurable function $\delta(x)$ satisfies the third inequality (4.15) and the condition

$$\delta_s^l = \inf_{x\in\Lambda_s^l} \delta(x) > 0. \tag{4.25}$$

In addition, they have the properties in Theorems 4.1 and 4.2. We remark that the function $A(x)$ in Theorem 4.1 satisfies the first inequality (4.16) and the condition

$$A_s^l = \sup_{x\in\Lambda_s^l} A(x) < \infty. \tag{4.26}$$

From Theorems 4.1, 4.2 and Proposition 4.6 (see also Remark 4.3) we obtain the following result.

PROPOSITION 4.7. *There is a measurable function* $r(x, \varepsilon)$ *such that for any* $y, z \in \hat{\Lambda} \cap U(x, r(x, \varepsilon))$ *the submanifolds* $V^-(y)$ *and* $V^+(y)$ *intersect transversally at a unique point* $w \in U(x, \varepsilon)$. *Moreover,* (4.27) $$r_s^l(\varepsilon) = \inf_{x \in \Lambda_s^l} r(x, \varepsilon) > 0.$$

Let $x \in \Lambda_{k,s}^l$. We denote by $S_{k,s}^{-l}(x)$ and $S_{k,s}^{+l}(x)$ the families of local stable and local unstable manifolds at x, respectively. These families are absolutely continuous in some neighbourhood $U(x, q_s^l)$, and the Jacobian $J(p)$ of the succession map p (see (4.22)) defined by the submanifolds W^1 and W^2 transversal to one of these families, satisfies the condition (see (4.23)):

(4.28) $$| J(p)(y) - 1 | \leqslant J_s^l(| W^1 | + | W^2 |),$$

where J_s^l is a certain constant.

PART II

§5. The entropy of smooth dynamical systems

5.1. Let f be a C^2-diffeomorphism of M preserving the measure ν. We consider the collection of distinct values $\chi_i(x)$ ($i = 1, \ldots, s(x)$), arranged in increasing order, of the characteristic exponent χ^+ at $x \in M$, and let $q_i(x) = k_i(x) - k_{i-1}(x)$ be the multiplicity of the corresponding value (see §3). Let $k(x)$ be the number of negative values of χ^+ at x.

THEOREM 5.1. *The entropy $h(f)$ of the diffeomorphism f satisfies the equality*

(5.0) $$h(f) = -\int_M \sum_{i=1}^{k(x)} q_i(x)\chi_i(x)\, d\nu(x)$$

(*for $k(x) = 0$ the empty sum is taken to be zero*).

PROOF. As noted in Introduction, the fact that the entropy does not exceed the quantity on the right-hand side of (5.0) was proved by Margulis. In §5.2 we shall show that

$$h(f) \geqslant -\int_M \sum_{i=1}^{k(x)} q_i(x)\chi_i(x)\, d\nu(x).$$

The idea of the proof of this theorem is due to Katok. Here we use the method proposed by Sinai to prove the corresponding proposition for systems with transversal foliations (see [27]).

5.2. We consider the set Λ defined in (1.7). If $\nu(\Lambda) = 0$ then the lower estimate is obvious. We therefore assume that $\nu(\Lambda) > 0$ and consider the set $\widetilde{\Lambda}$ of regular points in Λ. For $x \in \widetilde{\Lambda}$ we denote by $J^n(x)$ the Jacobian of the restriction $df^n | E_{1x}$ and set $\lambda_i(x) = e^{\chi_i(x)}$, $g(x) = \prod_{i=1}^{k(x)} (\lambda_i(x))^{q_i(x)}$. We fix $\varepsilon > 0$ and put

$$\widetilde{\Lambda}_m = \{x \in \widetilde{\Lambda}:\ m\varepsilon < g(x) \leqslant (m+1)\varepsilon\},$$
$$\widetilde{\Lambda}_{m,k,s,r} = \widetilde{\Lambda}_m \cap \widetilde{\Lambda}_{k,s,r}.$$

We evaluate the entropy of the restriction $f|\widetilde{\Lambda}_{m,k,s,r}$ with respect to the conditional measure ν_1 ($\nu_1(A) = \frac{1}{\nu(\widetilde{\Lambda}_{m,k,s,r})}\nu(A)$, $A \subset \widetilde{\Lambda}_{m,k,s,r}$ being a measurable set). From Theorem 3.4 and the definition (3.7) we obtain the following result.

LEMMA 5.1. *There is a measurable function $L(x, \varepsilon)$, $x \in \widetilde{\Lambda}$, $\varepsilon > 0$, such that for any $n \in \mathbf{Z}^+$*

$$J^n(x) \leqslant L(x, \varepsilon)(g(x))^n e^{\varepsilon n}.$$

We put

$$\widetilde{\Lambda}^l_{m,k,s,r} = \{x \in \widetilde{\Lambda}_m \cap \widetilde{\Lambda}^l_{k,s,r}:\ L(x, \varepsilon) \leqslant l\}.$$

For any $\alpha_1 > 0$ and sufficiently large $l \in \mathbf{Z}^+$ we have

$$\nu_1(\widetilde{\Lambda}^l_{m,k,s,r}) \geqslant 1 - \alpha_1. \tag{5.1}$$

Let ξ be any finite measurable partition of M every element of which satisfies the conditions:

1) C_ξ is homeomorphic to a ball and has a piecewise smooth boundary;
2) $\operatorname{diam} C_\xi \leqslant \alpha^l_{s,r}$ (see Remark 4.3).

We put $\Lambda^l = \bigcup_{x \in \widetilde{\Lambda}^l_{m,k,s,r}} (V(x) \cap C_\xi(x))$. By Theorem 4.2, the set Λ^l is measurable. We divide every element $C_\xi(x)$ into sets $V(y) \cap C_\xi(x)$, $y \in C_\xi(x) \cap \widetilde{\Lambda}^l_{m,k,s,r}$. (This partition is well-defined by Remark 4.3 and Theorem 4.1.1.) Then we complete the resulting partition of Λ to a partition η of $\hat{\Lambda} = \bigcup_{-\infty < n < \infty} f^n(\Lambda^l)$, by adding the element $\hat{\Lambda}\setminus\Lambda^l$. Since $\widetilde{\Lambda}^l_{m,k,s,r} \subset \Lambda^l \subset \Lambda \subset \widetilde{\Lambda}_{m,k,s,r}$(mod 0), by (5.1)

$$\nu_1(\hat{\Lambda} \setminus \Lambda^l) \leqslant \alpha_1, \quad \nu_1(\widetilde{\Lambda}_{m,k,s,r} \setminus \hat{\Lambda}) \leqslant \alpha_1. \tag{5.2}$$

We consider the measurable partition η^- of $\hat{\Lambda}$ and denote by $\nu_{\bar{x}}$ the conditional measure on $C_{\eta^-}(x)$. We set $\hat{f} = f|\hat{\Lambda}$. By what we have said in §2.2, the entropy of $f|\widetilde{\Lambda}_{m,k,s,r}$ has the following lower estimate:

$$h(f|\widetilde{\Lambda}_{m,k,s,r}) \geqslant h(\hat{f}) = \frac{1}{n} h(\hat{f}^n) \geqslant \frac{1}{n} H(\hat{f}^n\eta|\eta^-),$$

where $n \in \mathbf{Z}^+$ is arbitrary. To estimate the last expression we find a lower bound for

$$H(\hat{f}^n\eta|C_{\eta^-}(x)) = -\int\limits_{C_{\eta^-}(x)} \log \frac{\nu_{\bar{x}}(C_{\eta^-}(x) \cap C_{f^n\eta}(y))}{\nu_{\bar{x}}(C_{\eta^-}(x))}\, d\nu_{\bar{x}}(y).$$

First we describe a typical element of η^-, and then we show how to evaluate the conditional measure on this element. To simplify the notation we do not indicate the dependance on m, k, s, r of the constants occurring in the subsequent lemmas. We write

$$\partial\xi = \bigcup_{y\in M} \partial C_\xi(y), \qquad B_\delta = \{x\colon \rho(x, \partial\xi) \leqslant \delta\}.$$

The following statement is easily seen to be true.

LEMMA 5.2. *There is a constant* $C_1 > 0$ *such that* $\nu_1(B_\delta) \leqslant C_1\delta$ *for any* $\delta > 0$.

Let $x \in \Lambda^l$, $r \leqslant \alpha^l_{s,r}$. We denote by $B_\eta(x, r)$ the ball with centre at x and radius r on $C_\eta(x)$.

LEMMA 5.3. *For any* $\alpha_2 > 0$ *there are a* $q(l)$ *and a set* $A^l \subset \Lambda^l$, $\nu_1(\Lambda^l \setminus A^l) \leqslant \alpha_2$, *such that* $C_{\eta^-}(x) \supset B_\eta(x, q(l)$ *for any* $x \in A^l$.

PROOF. We put

$$D_q = \{x \in \Lambda^l\colon \text{ there is a } y \in B_\eta(x, q) \setminus C_{\eta^-}(x)\}.$$

If $x \in D_q$, then we can find $n \in \mathbf{Z}^+$ and $y \in B_\eta(x, q)$ such that $y \notin C_{f^{-n}\eta}(x)$. Hence, $f^n(z) \in \partial\xi$ for any $z \in B_\eta(x, q)$. Consequently, by Theorem 4.1. 4) and 6), $f^n(x) \in B_{C_2(\varkappa_{s,r})^n q}$, where $C_2 = C_2(l)$ is a constant. From this and Lemma 5.2 it follows that $\nu_1(D_q) \leqslant \sum_{n=0}^{\infty} C_2(\varkappa_{s,r})^n q \leqslant C_3 q$, where $C_3 = C_3(l)$ is a constant. To prove the lemma it is enough to set $q(l) = \alpha_2 C_3^{-1}$, $A^l = \Lambda^l \setminus D_{q(l)}$.

We denote by ν_y the measure induced by the Riemannian metric on the local stable manifold $V(y)$, $y \in \widetilde{\Lambda}^l_{m,k,s,r}$.

LEMMA 5.4. *There is a* $C_4 = C_4(l)$ *such that for any* $x \in \Lambda^l$, $x \in V(y)$, $y \in \widetilde{\Lambda}^l_{m,k,s,r}$, $n \in \mathbf{Z}^+$

$$\nu_y(f^n(C_\eta(x))) \leqslant C_4 J^n(y).$$

The proof follows immediately from Lemma 3.2.5 of [22]. From Proposition 4.3 and Lemma 5.3 we find that for $x \in A^l$, $x \in V(y)$, $y \in \widetilde{\Lambda}^l_{m,k,s,r}$

$$C_5^{-1} \leqslant d\nu_{\bar{x}}/d\nu_y \leqslant C_5, \tag{5.3}$$

where $C_5 = C_5(l)$ is a constant.

From what we have said above and from Lemmas 5.1, 5.3, 5.4 and the inequality (5.3) it follows that for any $x \in A^l$ and $n \in \mathbf{Z}^+$

$$H(\hat{f}^n\eta \mid C_{\eta^-}(x)) \geqslant -\log\,[C_5^2 C_4 l\,((m+1)\,\varepsilon)^n e^{\varepsilon n} (V(B_\eta(x, q(l))))^{-1}] = I_n, \tag{5.4}$$

where $V(B_\eta(x, q(l)))$ is the Riemannian volume of $B_\eta(x, q(l))$. Since $V(B_\eta(x, q(l))) > C_6(q(l))^k$, where $C_6 > 0$ is a constant,

$$(5.5)\quad I_n \geqslant -\log\,[C_5^2 C_4 l\,(C_6\,(q\,(l))^k)^{-1}] -$$
$$-n\,(\,\log\,(\varepsilon\,(m+1))+\varepsilon) \geqslant -C_7\,(l) - n\ \ (\log\,g\,(x)+\varepsilon) - 2n/m$$

($C_7(l) > 0$ is a constant). Integrating (5.4) over the elements of η^- and using (5.5) and the fact that $\nu_1(\widetilde{\Lambda}_{m,k,s,r}\backslash A^l) \leqslant 2\alpha_1 + \alpha_2$ (see (5.2) and Lemma 5.3) we obtain

$$\frac{1}{n} H\,(\hat{f}^n \eta\,|\,\eta^-) \geqslant \frac{1}{n} I_n \nu_1\,(A^l) \geqslant$$
$$\geqslant \Big[-\int\limits_{\widetilde{\Lambda}_{m,k,s,r}} \sum_{i=1}^{k(x)} q_i\,(x)\,\chi_i\,(x)\,d\nu_1\,(x) - \frac{1}{n} C_7\,(l) - \varepsilon - 2m^{-1} \Big]\,(1 - 2\alpha_1 - \alpha_2) =$$
$$= -\frac{1}{\nu\,(\widetilde{\Lambda}_{m,k,s,r})} \int\limits_{\widetilde{\Lambda}_{m,k,s,r}} \sum_{i=1}^{k\,(x)} q_i\,(x)\,\chi_i\,(x)\,d\nu\,(x) - \beta,$$

where β can be chosen arbitrarily small if ε, α_1, and α_2 are chosen sufficiently small and n sufficiently large. Summing the resulting inequalities for various m, k, s, r (we recall that the sets $\widetilde{\Lambda}_{m,k,s,r}$ are pairwise disjoint), we obtain the required estimate.

REMARK. A similar result is proved in [48]. However, the proof given there is not complete: it consists of several separate stages, but it is not always clear how they are to be realized. This refers to the question of principle whether absolute continuity (or something of the kind) is used in proving the lower estimate for the entropy, and if so, then how exactly (and also how to prove it); it remains unclear whether the author had in mind arguments of the kind that appear natural in the light of this paper or other ones.

§6. "Measurable foliations". Description of the π-partition

6.1. In this section we construct a special f-invariant partition of Λ, which is similar to the partition into global contracting fibres for U-systems: almost every element of the partition is a mod 0 smooth immersed submanifold of M, contracting under the action of f^n. Following Anosov and Sinai [3], we call such a partition a "foliation of Λ", adding the term "measurable" to indicate that, in general, the continuous dependence of the fibres is not assumed (see [1], §4).[1] The construction of this partition is achieved by means of local stable manifolds, but instead of the usual glueing procedure, which is applied in the theory of U-systems, but is not suitable in our case, we use a different method.

For $x \in \widetilde{\Lambda}$ we set

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[1] An extract definition of a "measurable foliation" can be given (however, we do not need it): it is a partition ξ of Λ for which there is a sequence of sets $F_n \subset \Lambda$ such that $F_n \subset F_{n+1}$, $\bigcup_n F_n = \Lambda$ and the partition $\xi\,|\,F_n$ is a mod 0 continuous foliation (see Definition 7.1).

$$(6.1) \qquad W(x) = \bigcup_{n=-\infty}^{\infty} f^{-n}(V(f^n(x))).$$

For $x \in M \setminus \widetilde{\Lambda}$ we put $W(x) = \{x\}$. The following result is a corollary to Theorems 4.1–4.3.

THEOREM 6.1 (see [23], Theorem 3). *Let $x, y \in \widetilde{\Lambda}$. The following statements hold*:

1. $W(x) \cap W(y) = \emptyset$ *if* $y \notin W(x)$;
2. $W(x) = W(y)$ *if* $y \in W(x)$;
3. $W(x)$ *is a k-dimensional immersed submanifold of class* C^{r-1} *without boundary*;
4. $f^n(W(x)) = W(f^n(x))$, $n \in \mathbf{Z}$;
5. *if* $y \in W(x)$, *then* $\rho_{f^n(W(x))}(f^n(x), f^n(y)) \to 0$ *as* $n \to \infty$.

(Here $\rho_{f^n(W(x))}$ is the distance induced on $f^n(W(x))$ by the Riemannian metric);

6. *if* $\mu(x)$ *denotes the measure induced on* $W(x)$ *by the Riemannian metric, then* $\mu(x)(W(x) \setminus \widetilde{\Lambda}) = 0$ *for almost all* $x \in \widetilde{\Lambda}$.

Let x be a density point of $\Lambda^l_{k,s,r}$ and $A \subset \Lambda^l_{k,s,r} \cap U(x, \alpha^l_{s,r})$ a measurable set of positive measure (the number $\alpha^l_{s,r}$ is defined in Remark 4.3). For $y \in A$ we denote by $n_i(y)$ ($i = 1, 2, \dots$) the successive moments when the half-trajectory $\{f^n(y)\}$, $n \geqslant 0$, hits A, and by $D(y, q)$ the ball with centre at y and radius q on $V(y)$. There is a $q^l_{s,r} > 0$ such that $D(y, q^l_{s,r}) \subset V(y)$ for any $y \in A$. We take an arbitrary number q, $0 < q < \frac{1}{2} q^l_{s,r}$, and choose an open subset $U(y) \subset V(y)$ such that $D(y, q) \subset U(y)$ for any $y \in A$.

THEOREM 6.2 (see [23], Theorem 5). *For almost every* $y \in A$

$$W(y) = \bigcup_{i=1}^{\infty} f^{-n_i(y)}(U(f^{n_i(y)}(y))).$$

PROOF. There is a set N of measure zero such that for any $y \in A \setminus N$ the sequence $n_i(y)$ is infinite. Let $y \in A \setminus N$, $z \in W(y)$, and $z \notin \bigcup_{i=1}^{\infty} f^{-n_i(y)}(U(f^{n_i(y)}(y)))$. By Theorem 6.1.5, for sufficiently large $i > 0$ we have $\rho_{f^{n_i(y)}(W(x))}(f^{n_i(y)}(y), f^{n_i(y)}(z)) \leqslant \frac{1}{2} q$. Therefore $f^{n_i(y)}(z) \in D(f^{n_i(y)}(y), \frac{q}{2}) \subset U(f^{n_i(y)}(y))$. This contradiction proves the theorem.

There is a certain ambiguity in what should be understood by local stable manifolds. For example, in Theorem 4.1 instead of $\delta'(x)$ and $\delta(x)$ we can take the functions $\varepsilon\delta'(x)$ and $\varepsilon\delta(x)$, respectively, where $0 < \varepsilon \leqslant 1$, and with their help we can construct "new" local stable manifolds of "smaller dimension". Theorem 6.2 shows that this procedure (and others like it) does not affect (up to a set of measure zero) the definition of $W(x)$. It can be shown that, in general, these sets do not depend mod 0 on the method of constructing local stable manifolds (for more details, see [23]).

We denote by W the partition of M consisting of the sets $W(x)$. The following theorem is similar to the corresponding result of Sinai for systems with a transversal field (see [27], Theorem 5.2).

THEOREM 6.3 (see [23], Theorem 1). *There is a partition η of M with the following properties*:

1. *For almost every $x \in \widetilde{\Lambda}$ the element $C_\eta(x)$ is a* mod 0 *open subset of $W(x)$*;

2. $f\eta \geqslant \eta$;

3. $\bigvee\limits_{0}^{\infty} f^k\eta = \varepsilon$;

4. $\bigwedge\limits_{-\infty}^{0} f^k\eta = \nu(W)$;

5. $h(f, \eta) = h(f) = \int\limits_M \sum\limits_{i=1}^{k(x)} q_i(x)\chi_i(x)\,d\nu(x)$.

For completeness of presentation we reproduce here a sketch of a proof of this theorem, omitting details. It is sufficient to construct a partition η on each of the sets $\widetilde{\Lambda}_{k,s,r}$, because these sets are disjoint and f-invariant. We restrict ourselves to the case when the automorphism $f\,|\,\widetilde{\Lambda}_{k,s,r}$ is ergodic (Sinai in [27] considers precisely the case of an ergodic f). Let x be a density point of $\widetilde{\Lambda}^l_{k,s,r}$ for some l. We put

$$A(x) = \widetilde{\Lambda}^l_{k,s,r} \cap U(x, \alpha^l_{s,r}), \qquad P = \bigcup_{y \in A(x)} V(y).$$

It is easy to see that $\widetilde{\Lambda}_{k,s,r} = \bigcup\limits_{n=-\infty}^{\infty} f^n(P)$ (mod 0). We consider the manifold $\widetilde{W}$ that contains x and is transversal to the family $S^l_{k,s,r}(x)$. For $z \in A(x)$ we set $y(z) = V(z) \cap \widetilde{W}$ and denote by $V(y(z))$ the local stable manifold containing $y(z)$ (this point does not necessarily belong to $\widetilde{\Lambda}_{k,s,r}$). There is a $q^l_{s,r} > 0$ such that $D(y(z), q^l_{s,r}) \subset V(y(z))$. We put $R = \bigcup\limits_{x \in A(x)} D(y(z), \frac{1}{2}q^l_{s,r})$ and consider the partition $\widetilde{\xi}$ of R into the sets $D(y(z), \frac{1}{2}q^l_{s,r})$; we complete it by the element $\widetilde{\Lambda}_{k,s,r} \setminus R$ to obtain a partition ξ of $\widetilde{\Lambda}_{k,s,r}$. We put $\eta = \xi^-$ and claim that η has the required properties.

We consider the sets $K_n = \{z \in R\colon H(\xi | C_{f^n\xi}(f^n(z))) > 0\}$. From (4.14) and (4.16) it follows that for any $z \in R$ and $n > 0$

$$\operatorname{diam}(f^n(C_\xi(z))) \leqslant C(l)(\varkappa_{s,r})^n q^l_{s,r}, \tag{6.2}$$

where $C(l)$ is a constant. From what we have said above and Theorem 4.2 it follows that $w \in f^n(K_n)$ for sufficiently large n (namely, for n such that $C(l)(\varkappa_{s,r})^n \leqslant 1/2$) lies on some local stable manifold and its distance to the boundary of some element C_ξ on the corresponding manifold $V(y(z))$ is not greater than $C(l)\,(\varkappa_{s,r})^n\, q^l_{s,r}$. Therefore, by Proposition 4.3 there is a $C_1(l)$ such that for all $n > 0$

$$\hat{\nu}_y(f^n(K_n)\,|\,C_\xi) \leqslant C_1(l)\,C(l)\,q^l_{s,r}(\varkappa_{s,r})^n.$$

Denoting by ν_ξ the measure in $R\backslash\widetilde{\xi}$, we obtain by Proposition 4.3

$$\nu(K_n) = \nu(f^n(K_n)) = \int_R \hat{\nu}_y(f^n(K_n)\,|\,C_{\widetilde{\xi}}(y))\,d\nu_{\widetilde{\xi}}(y) \leqslant C_1(l)\,C(l)\,q^l_{s,r}(\varkappa_{s,r})^n\,\nu(R).$$

Therefore $\sum_n \nu(K_n) < \infty$. Consequently, almost every x belongs to finitely many sets K_n. Thus, for almost all the elements of ξ the number l of those moments $n > 0$ for which $f^n(C_\xi)$ intersects more than one element of ξ is finite. Hence, for almost every $z \in R$

$$H(\eta\,|\,C_\xi(z)) = H(f^{-1}\xi \vee \ldots \vee f^{-k}\xi\,|\,C_\xi(z)) \leqslant$$
$$\leqslant \sum_{i=1}^{l} H(f^{-i}\xi\,|\,C_\xi(z)) = \sum_{i=1}^{l} H(\xi\,|\,f^i(C_\xi(z))) < \infty.$$

This proves part 1. Part 2 is obvious. Part 3 follows from (6.2), which shows that as $n \to \infty$ the diameters of almost all the elements of $f^n\xi$ tend to zero. Part 4 follows from the method of constructing η and from Theorem 6.2. The proof of Part 5 is similar to that of Theorem 5.1.

From Theorem 6.3 and Theorems 12.1–12.4 in [25] we obtain the following result.

THEOREM 6.4 (see [23], Theorem 2). *The Pinsker partition $\pi(f)$ of a diffeomorphism f satisfies the equality $\pi(f) = \nu(W)$.*

§7. Ergodicity of a diffeomorphism with non-zero exponents on a set of positive measure. The K-property

7.1. Let f be a C^2-diffeomorphism of a manifold M, preserving the measure ν, for which the set Λ defined by (1.8) has positive measure. In this section we describe the partition into ergodic components for the diffeomorphism f on Λ.

Let x be a density point of $\Lambda^l_{k,s}$. We put

$$P^l_{k,s}(x, r) = \bigcup_{w \in U(x,r) \cap \Lambda^l_{k,s}} V^-(w) \cup V^+(w). \tag{7.1}$$

We call $P^l_{k,s}(x, r)$ the lattice of local stable manifolds at x.

THEOREM 7.1. *Let x be a density point of $\Lambda^l_{k,s}$. There is an*

$r = r(l, s)$ such that the diffeomorphism f on the set

$$Q(x) = \bigcup_{n \in \mathbf{Z}} f^n(P^l_{k,s}(x, r)) \tag{7.2}$$

is ergodic.

PROOF. First we prove a statement similar to Theorem 4.4 in [3].

LEMMA 7.1. *For any f-invariant function φ there is a set $N \subset M$ of measure zero such that $\varphi(z_1) = \varphi(z_2)$ for any $z_1, z_2 \in P^l_{k,s}(x, r)\setminus N$, $z_1, z_2 \in V^-(w)$ or $z_1, z_2 \in V^+(w), w \in \Lambda^l_{k,s} \cap U(x, r)$.*

PROOF. For any invariant function $\psi(x)$ we set

$$\begin{cases} \bar{\psi}^+(x) = \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \psi(f^k(x)), \\ \bar{\psi}^-(x) = \lim\limits_{n\to-\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \psi(f^k(x)), \\ \bar{\psi}(x) = \lim\limits_{n\to\infty} \dfrac{1}{2n+1} \sum\limits_{k=-n}^{n} \psi(f^k(x)). \end{cases} \tag{7.3}$$

From Birkhoff's ergodic theorem it follows that $\bar{\psi}^+(x)$, $\bar{\psi}^-(x)$, and $\bar{\psi}(x)$ are defined almost everywhere and that $\bar{\psi}^+(x) = \bar{\psi}^-(x) = \bar{\psi}(x) \pmod 0$. Thus, the limits (7.3) exist and are equal everywhere outside a set N of measure zero. If z_1 and z_2 satisfy the conditions of Lemma 7.1, then by Theorem 4.1.4, $\rho(f^n(z_1), f^n(z_2)) \to 0$ as $n \to \infty$ (or $\rho(f^{-n}(z_1), f^{-n}(z_2)) \to 0$ as $n \to \infty$). Since ψ is continuous, $\bar{\psi}(z_1) = \bar{\psi}^+(z_1) = \bar{\psi}^+(z_2) = \bar{\psi}(z_2)$ ($\bar{\psi}(z_1) = \bar{\psi}^-(z_1) = \bar{\psi}^-(z_2) = \bar{\psi}(z_2)$, respectively). To prove our lemma it is now sufficient to remark that since the continuous functions are dense in L_2, the functions $\bar{\psi}(x)$ are dense in the set of f-invariant functions.

Let φ be an f-invariant function and N the set constructed in Lemma 7.1. We put (see Proposition 4.7)

$$r = \min\left\{\frac{1}{8}\delta^l_s,\ q^l_s,\ r^l_s\left(\frac{1}{8}\alpha^l_s\right),\ r^l_s(q^l_s)\right\}. \tag{7.4}$$

By Proposition 4.3 we can find a point $w_0 \in \Lambda^l_{k,s} \cap U(x, r)$, $w_0 \notin N$, for which

$$\nu^-_{w_0}(V^-(w_0) \cap N) = 0, \qquad \nu^+_{w_0}(V^+(w_0) \cap N) = 0,$$

where $\nu^-_{w_0}$ and $\nu^+_{w_0}$ are the measures induced by the Riemannian metric on $V^-(w_0)$ and $V^+(w_0)$, respectively. We put $R^- = \cup\, V^-(w)$, $R^+ = \cup\, V^+(w)$, where the union is taken over all points $w \in \Lambda^l_{k,s} \cap U(x, \delta^l_s)$ for which $V^-(w) \cap V^+(w_0) \in N$ and, correspondingly, $V^+(w) \cap V^-(w_0) \in N$. From Proposition 4.4 it follows that $\nu(R^-) = 0$ and $\nu(R^+) = 0$. Let

$z_1, z_2 \in P^l_{k,s}(x, r)\setminus(R^- \cup R^+ \cup N)$. We claim that $\varphi(z_1) = \varphi(z_2)$. We define points $w_i \in \Lambda^l_{k,s} \cap U(x, r)$ so that $z_i \in V^+(w_i)$ or $z_i \in V^-(w_i)$ $(i = 1, 2)$. There are four possible cases, depending on the location of z_1 and z_2:

$$1)\ z_1 \in V^+(w_1),\quad z_2 \in V^+(w_2);\quad 2)\ z_1 \in V^-(w_1),\quad z_2 \in V^+(w_2);$$
$$3)\ z_1 \in V^+(w_1),\quad z_2 \in V^-(w_2);\quad 4)\ z_1 \in V^-(w_1),\quad z_2 \in V^-(w_2).$$

We consider only the first two cases; the other two can be treated similarly.

1) Let $y_i \in V^+(w_i) \cap V^-(w_0)$ $(i = 1, 2)$. (By Proposition 4.7 and (7.4) these submanifolds intersect each other.) From the definition of R^+ it is clear that $y_i \notin N$. Therefore, $\varphi(z_1) = \varphi(y_1) = \varphi(y_2) = \varphi(z_2)$.

2) Let $y_1 = V^-(w_1) \cap V^+(w_0)$, $y_2 = V^+(w_2) \cap V^-(w_0)$. It is easy to see that $y_1, y_2 \notin N$. Since $w_0 \notin N$,

$$\varphi(z_1) = \varphi(y_1) = \varphi(w_0) = \varphi(y_2) = \varphi(z_2).$$

This proves the theorem.

PROPOSITION 7.1. *Let x be a density point of $\Lambda^l_{k,s}$. Then $Q(x) \subset \Lambda_{k,s}(\mathrm{mod}\ 0)$.*

PROOF. Since

$$Q(x) \supset P^l_{k,s}(x, r) \supset \Lambda^l_{k,s} \cap U(x, r),$$

$\nu(\Lambda_{k,s} \cap Q(x)) > 0$. On the other hand, the automorphism $f \mid Q(x)$ is ergodic. Therefore, $Q(x) = \Lambda_{k,s} \cap Q(x)\,(\mathrm{mod}\ 0) \subset \Lambda_{k,s}$.

The following theorem describes the partition into ergodic components for the diffeomorphism $f \mid \Lambda$. Its proof follows from Theorem 7.1 and the fact that almost every point of Λ is a density point of $\Lambda^l_{k,s}$ for some l, k, s.

THEOREM 7.2. *There are sets $\Lambda_i \subset \Lambda (i = 0, 1, 2, \ldots)$ such that*

1) $\Lambda_i \cap \Lambda_j = \varnothing$, $i \neq j$, $\bigcup_{i=0}^{\infty} \Lambda_i = \Lambda$;

2) $\nu(\Lambda_0) = 0$, $\nu(\Lambda_i) > 0$ *for* $i > 0$;

3) $f(\Lambda_i) = \Lambda_i$;

4) *the automorphism $f \mid \Lambda_i$ is ergodic for $i > 0$.*

7.2. In this subsection we continue the study of properties of local stable manifolds (see Theorems 4.1–4.3 and Proposition 4.5).

THEOREM 7.3. *Let $\nu(\Lambda_{k,s}) > 0$. There is a set $N \subset M$ of measure zero such that for any $x \in \Lambda_{k,s}\setminus N$*

$$\nu^-_x(V^-(x) \setminus \Lambda_{k,s}) = 0, \quad \nu^+_x(V^+(x) \setminus \Lambda_{k,s}) = 0;$$

ν^-_x and ν^+_x are the measures induced on $V^-(x)$ and $V^+(x)$ by the Riemannian metric.

The proof is a modification of that of Proposition 4.5.

THEOREM 7.4. *Let $k, s \in \mathbf{Z}^+$ be fixed. There is a function $\psi\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ such that if $x_1, x_2 \in \widetilde{\Lambda}^l_{k,s}$ and $y = V^-(x_1) \cap V^+(x_2)$, then $y \in \Lambda^{\psi(l)}_{k,s}$ (the* *bar denotes closure).*

PROOF. Suppose first that $x_1, x_2 \in \widetilde{\Lambda}^l_{k,s}$. From Theorem 4.3.1 it follows that y is regular (because it is both forward and backward regular), and

$$E_{1y} = T_y V^-(x_1),\ E_{2y} = T_y V^+(x_2),\ \lambda(y) = \lambda(x_1),\ \mu(y) = \mu(x_2).$$

Hence

$$\dim E_{1y} = \dim E_{1x_1} = k, \quad \lambda(y) \leqslant (s-1)/s < 1 < (s+1)/s \leqslant \mu(y).$$

Thus, $y \in \widetilde{\Lambda}_{k,s}$. In particular, $\varepsilon(y) = \varepsilon(x) = \varepsilon_s$. Furthermore, by Theorem 4.3. 2) for the vectors $v \in E_{1y}$ and $v \in E_{2y}$ the estimates (4.18) and (4.19) hold. By Theorem 4.1.8) and Theorem 4.2 . 3) there is a constant K_1 such that $\inf K(y) \geqslant K_1$, where the supremum is taken over all points $y = V^-(x_1) \cap V^+(x_2)$, $x_1, x_2 \in \bar{\widetilde{\Lambda}}{}^l_{k,s}$. Thus, $\gamma(f^m(y)) \geqslant K_1^{-1}e^{-\varepsilon_s m}$. Let $\psi(l)$ be the smallest integer for which $\max\ \{K, K_1\} \leqslant \psi(l)$. From what has been said above it follows that $y \in \Lambda^{\psi(l)}_{k,s}$. Now let $x_1, x_2 \in \bar{\widetilde{\Lambda}}{}^l_{k,s}$, $y = V^-(x_1) \cap V^+(x_2)$. We choose sequences $x_1^n \to x_1$ and $x_2^n \to x_2$, where $x_1^n, x_2^n \in \widetilde{\Lambda}^l_{k,s}$, and set $y^n = V^-(x_1^n) \cap V^+(x_2^n)$. (For sufficiently large n this intersection is non-empty.) From what was said above it follows that $y^n \in \widetilde{\Lambda}^{\psi(l)}_{k,s}$. Therefore, by Proposition 4.6. 2), $y^n \to y \in \Lambda^{\psi(l)}_{k,s}$, and the theorem is proved.

7.3. In this subsection we formulate sufficient conditions for Λ to be open mod 0, and we discuss the question of the ergodicity of f on Λ.

DEFINITION 7.1. A continuous $(\delta(x), k)$-foliation ξ of a set $X \subset M$ is a partition of X (also denoted by ξ) having the following properties:

1. For every $x \in X$ there is a smooth immersed k-dimensional submanifold $V(x) \ni x$ such that $C_\xi(x) = V(x) \cap X$. Then $V(x) \cap B(x, \delta(x))$ is called the local fibre of ξ.

2. $V(x) \cap B(x, \delta(x))$ is the image of a smooth map $\varphi(x)\colon D \to M$, where D is the ball with centre at 0 and radius 1 in $\mathbf{R}^k$. The map $\varphi\colon X \cap B(x, \delta(x)) \to C^1(D, M)$ is continuous.

If $X = M$, then our definition of a continuous $(\delta(x), k)$-foliation coincides with that of a continuous foliation in [1].

A mod 0 continuous $(\delta(x), k)$-foliation of X is defined to be a partition of this set that is a continuous $(\delta(x), k)$-foliation of some set $X_1 \subset X$ such that $\nu(X_1) = \nu(X)$.

A mod 0 continuous $(\delta(x), k)$-foliation is said to be invariant if $f(C_\xi(x)) = C_{f\xi}(x)$ for almost every $x \in X$.

We put (see (6.1))

$$W^-(x) = \bigcup_{n=-\infty}^{\infty} f^{-n}(V^-(f^n(x))), \quad W^+(x) = \bigcup_{n=-\infty}^{\infty} f^{-n}(V^+(f^n(x))).$$

The sets $W^-(x)$ and $W^+(x)$ have the properties stated in Theorems 6.1 and 6.2 (when we consider $W^+(x)$, we must replace n by $-n$ in Theorems 6.1.5) and 6.2).

We consider $\Lambda_{k,s}$ and denote by W^- and W^+ the partition of this set into the sets $W^-(x)$ and $W^+(x)$, respectively.

THEOREM 7.5. *Suppose that W^- is a* mod 0 *continuous $(\delta(x), k)$-foliation of $\Lambda_{k,s}$. Then any ergodic component of positive measure lying in $\Lambda_{k,s}$ is a* mod 0 *open set.*

PROOF. Let Q be an f-invariant set of positive measure such that $Q \subset \Lambda_{k,s}$, and suppose that $f \mid Q$ is ergodic. By Theorem 7.1 there is an $x \in Q$ that is a density point of $\Lambda^l_{k,s}$ for some l, such that $Q = Q(x)$ (mod 0) (see (7.2)). By Theorem 7.3 we can assume that (with respect to the measure ν^+_x) almost every point $y \in V^+(x)$ lies in $\Lambda_{k,s}$. For $y \in \Lambda_{k,s}$ we denote by $B^-(y, q)$ the ball with centre at y and radius q on $W^-(y)$. We fix an integer $m > 0$ and put

$$R(q) = \bigcup_{y \in V^+(x)} B^-(y, q), \quad R^m(q) = \bigcup_{y \in V^+(x) \cap \Lambda^m_{k,s}} B^-(y, q),$$

$$R(q, Y) = \bigcup_{y \in Y} B^-(y, q),$$

where $Y \subset V^+(x)$ is a measurable set with respect to ν^+_x. Since W^- is a mod 0 continuous $(\delta(x), k)$-foliation, we can find a $q_0 > 0$ such that $B^-(y, q) \subset W^-(y)$ for any $y \in V^+(x) \cap \Lambda_{k,s}$. By Theorems 4.1 and 4.2 there is a q^m_s such that $B^-(y, q^m_s) \subset V^-(y)$ for any $y \in V^+(x) \cap \Lambda^m_{k,s}$. We fix q, $0 < q \leqslant q^m_s$. For $y \in R^m\left(\frac{1}{2}q\right)$ we denote by $n_i(y)$ $(i = 1, 2, \dots)$ the consecutive moments of the return of the half-trajectory $\{f^n(y)\}$, $n > 0$, in $R^m\left(\frac{1}{2}q\right)$, and by $z_i \in V^+(x) \cap \Lambda^m_{k,s}$ a sequence of points such that $f^{n_i(y)}(y) \in B^-\left(z_i, \frac{1}{2}q\right)$. The following is proved in the same way as Theorem 6.2.

LEMMA 7.2. *For almost every $y \in R^m\left(\frac{1}{2}q\right)$*

$$W^-(y) = \bigcup_{i>0} f^{-n_i(y)}(B^-(z_i, q)). \tag{7.5}$$

We denote by $\xi(q_0)$ and $\xi^m(q_0)$ the partition of $R(q_0)$ and $R^m(q_0)$ into the sets $B^-(y, q_0)$, respectively.

LEMMA 7.3. *The partition $\xi^m\left(\frac{1}{2}q_0\right)$ is measurable; the conditional measure on the elements of this partition is absolutely continuous with respect to ν^-_y; the measure in $R^m\left(\frac{1}{2}q_0\right) \big/ \xi^m\left(\frac{1}{2}q_0\right)$ (which by the* mod 0 *continuity of W^- is naturally identified with $V^+(x) \cap \Lambda^m_{k,s}$) is absolutely continuous with respect to ν^+_x restricted to $V^+(x) \cap \Lambda^m_{k,s}$.*

PROOF. We put $q = \min\left\{\frac{1}{100}q_0, q^m_s\right\}$. Making use of Lemma 7.2, we choose for almost every $w \in V^+(x) \cap \Lambda^m_{k,s}$ a point

$y(w) \in B^-\left(w, \frac{1}{2}q\right) \subset R^m\left(\frac{1}{2}q\right)$ for which (7.5) holds. (This can be achieved so that the map χ: $V^+(x) \cap \Lambda^m_{k,s} \to R^m\left(\frac{1}{2}q\right)$ that associates with a point $w \in V^+(x) \cap \Lambda^m_{k,s}$ the point $y(w)$ is measurable.) We represent the set $R^m\left(\frac{3}{4}q_0\right)$ as a union of sets R_n, $R^m\left(\frac{3}{4}q_0\right) = \bigcup_{n>0} R_n$, where

$$R_n = \bigcup_{y(w)} \bigcup_{n_i(y(w)) \leqslant n} \left(f^{-n_i(y(w))}(B^-(z_i, q)) \cap R^m\left(\frac{3}{4}q_0\right)\right).$$

For every $\varepsilon > 0$ there is a $N > 0$ and a set $Y \subset V^+(x) \cap \Lambda^m_{k,s}$ such that $\nu^+_x(V^+(x) \cap \Lambda^m_{k,s} \setminus Y) \leqslant \varepsilon$ and $R\left(\frac{1}{2}q_0, Y\right) \subset \bigcup_{n \leqslant N} R_n$. Since f is a diffeomorphism, it follows from Proposition 4.3 that $\xi^m\left(\frac{1}{2}q_0\right) | R_n$ satisfies the required conditions for any $n > 0$. The result now follows because ε is arbitrary.

Applying Lemma 7.3 in turn to $\xi^m\left(\frac{1}{2}q_0\right)$ $(m = 1, 2, \ldots)$ we find that $\xi\left(\frac{1}{2}q\right)$ satisfies the conclusions of Lemma 7.3. Since W^- is a mod 0 continuous $(\delta(x), k)$-foliation, from what was said above it follows that $R\left(\frac{1}{2}q_0\right)$ is open mod 0.By Theorem 7.1 $Q \supset R\left(\frac{1}{2}q_0\right)$, consequently, $Q = \bigcup_{-\infty<n<\infty} f^n\left(R\left(\frac{1}{2}q_0\right)\right)$ is a mod 0 open set, and the theorem is proved.

COROLLARY 7.1. *Under the conditions of Theorem* 7.5, $\Lambda_{k,s}$ *is a* mod 0 *open set* (this follows from Theorems 7.2 and 7.5).

REMARK 7.1. From our results we obtain a proposition first proved by other methods by Bowen and Ruelle in [34]: if f is a ν-preserving C^2-diffeomorphism of a connected manifold M, and Λ a hyperbolic set of positive measure (that is, an invariant set on which the conditions of uniform hyperbolicity are satisfied; see [32]), then $\Lambda = M$. For from what was said in §4.2 it follows that $\Lambda = \Lambda_{k,s}$ for some k, s, in particular, Λ is closed. It is also easy to see that W^- and W^+ are continuous on Λ, so that by Corollary 7.1 Λ is open mod 0. Hence, if $x \in \Lambda$ is a density point of Λ, then $B(x, r) \subset \Lambda$ for some r, where r can be chosen not to depend on x. From this and the fact that M is connected it follows easily that $\Lambda = M$, so that f is a U-diffeomorphism.

THEOREM 7.6. *Let W be a* mod 0 *continuous invariant* $(\delta(x), k)$-*foliation of $\Lambda_{k,s}$ satisfying the following conditions*:

1. $W(x) \supset V^-(x)$ *for any* $x \in \Lambda_{k,s}$.

2. *There are a $\delta_0 > 0$ and a measurable function $n(x)$ on $\Lambda_{k,s}$ such that for almost every $x \in \Lambda_{k,s}$ and any $n \geqslant n(x)$*

$$f^{-n}(V^-(x)) \subset B_W(f^{-n}(x),\ \delta_0),$$

where $B_W(x, \delta_0)$ *is the ball with centre at* x *and radius* δ_0 *on* $W(x)$.

Then W^- *is a* mod 0 *continuous* $(\delta(x), k)$*-foliation of* $\Lambda_{k,s}$ *for some function* $\delta(x)$.

PROOF. We consider a density point x of $\Lambda^l_{k,s}$ for some l. Applying Theorem 6.2 to the set $A = \Lambda^l_{k,s} \cap U(x, \frac{1}{8}\delta^l_s)$ and taking into account conditions 1 and 2 of the theorem, we find that $W^-(y) \supset B_W(y, \delta_0)$ for almost any $y \in A$. The statement to be proved now results from the mod 0 continuity of W and the condition $W^-(x) \subset W(x)$ for every $x \in \Lambda_{k,s}$.

THEOREM 7.7. *Suppose that* W *is a* mod 0 *continuous invariant* $(\delta(x), 1)$*-foliation of* $\Lambda_{1,s}$ *such that* $W(x) \supset V^-(x)$ *for any* $x \in \Lambda_{1,s}$. *Then* W^- *is a* mod 0 *continuous* $(\delta(x), 1)$*-foliation of* $\Lambda_{1,s}$. *Moreover,* $W^-(x) = W(x)$ *for almost every* $x \in \Lambda_{1,s}$.

PROOF. Let $x \in \Lambda_{1,s}\backslash N$ (the set N is constructed in Theorem 7.3). For $z \in V^-(x)$ we put $s(z) = \rho_{V^-(x)}(x, z)$. Next, let $\bar{y}(x) = \exp_x(\delta(x), \varphi_x(\delta(x))) \in V^-(x)$ (see Theorem 4.1.1). We choose a point $y(x)$ such that $s(y(x)) = \frac{1}{2}\ s(\bar{y}(x))$. The map $\chi\colon \widetilde{\Lambda} \to M$ that associates with $x \in \widetilde{\Lambda}$ the point $y(x)$ is measurable. We consider the positive measurable function $C(z, \varepsilon_s)$, $z \in V^-(x)$ (see §4.3). From what has been said above, it follows that there is a function $C(x) > 0$ such that

$$\nu_x^-(\{z \in V^-(x)\colon s(z) \leqslant s(y),\ C(z, \varepsilon_s) \leqslant C(x)\}) \geqslant \frac{1}{4}.$$

Therefore, by (4.5),

$$\rho_{W(f^{-n}(x))}(f^{-n}(x), f^{-n}(y)) = \int_0^{s(y)} \|df^{-n}(z)\|\, ds \geqslant \frac{1}{4}[C(x)]^{-1}\left(1 - \frac{1}{s}\right)^{-n}. \tag{7.6}$$

We define a number $n(x)$ such that $\frac{1}{4}\ [C(x)]^{-1}\left(1 - \frac{1}{s}\right)^{-n} \geqslant \delta(x)$ for any $n \geqslant n(x)$. From what we have said above it follows that $n(x)$ is measurable. Thus condition 2 of Theorem 7.6 is satisfied, consequently, W^- is a mod 0 continuous $(\delta_1(x), 1)$-foliation of $\Lambda_{1,s}$, $\delta_1(x) \leqslant \delta(x)$. Since $W(x) \supset V^-(x)$ for $x \in \Lambda_{1,s}$ and W is invariant, $W^-(x) \subset W(x)$ for $x \in \Lambda_{1,s}$. On the other hand, by (7.6) and Theorem 6.2 $W^-(x) \supset B_W(x, R)$ for any $R > 0$ and almost every $x \in \Lambda_{1,s}$. Therefore, $W^-(x) = W(x)$ for almost every $x \in \Lambda_{1,s}$. In particular, W^- is a mod 0 continuous $(\delta(x), 1)$-foliation of $\Lambda_{1,s}$, and the theorem is proved.

DEFINITION 7.2. A diffeomorphism f is said to be topologically transitive if for any two open sets A and B there is an n such that $f^n(A) \cap B = \emptyset$ (equivalently: there is an everywhere dense trajectory).

THEOREM 7.8. *Suppose that a diffeomorphism f is topologically transitive and satisfies the conditions of one of the Theorems 7.5, 7.6, or 7.7. Then $f \mid \Lambda_{k,s}$ is ergodic.*

PROOF. Let A, $B \subset \Lambda_{k,s}$ be two ergodic components of f of positive measure. Then $\nu(f^n(A) \cap B) = 0$ for any $n \in \mathbf{Z}^+$. Since, by Theorem 7.5, A and B are open mod 0 and f is topologically transitive, $\nu(f^n(A) \cap B) > 0$ for some $n \in \mathbf{Z}^+$. This contradiction proves the theorem.

7.4. Let Λ be the f-invariant set of positive measure defined in (1.8), and Λ_i the sets constructed in Theorem 7.2.

THEOREM 7.9. *For every $i = 1, 2, \ldots$ there is a decomposition*

$$\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_i^j, \quad n_i \in \mathbf{Z}^+, \quad \Lambda_i^{j_1} \cap \Lambda_i^{j_2} = \phi, \quad j_1 \neq j_2,$$

with the following properties:

1. $f(\Lambda_i^j) = \Lambda_i^{j+1}$ $(j = 1, 2, \ldots, n_i - 1)$, $f(\Lambda_i^{n_i}) = \Lambda_i^1$.
2. $f^{n_i} \mid \Lambda_i^1$ *is a K-automorphism.*

PROOF. We consider a set Λ_i, $i > 0$, which by Theorem 7.1 can be represented in the form $\Lambda_i = \bigcup_{n=-\infty}^{\infty} f^n(P_{k,s}^l(x, r))$ (mod 0), where x is a density point of $\Lambda_{k,s}^l$ for some l, k, s, r and $P_{k,s}^l(x, r)$ is the lattice of local manifolds at x (see (7.1)). Let $w \in \Lambda_{k,s}^l \cap U(x, r)$ and

$$y_j = y_j(w) = \begin{cases} V^-(w) \cap V^+(x) & (j = 1), \\ V^+(w) \cap V^-(x) & (j = 2). \end{cases}$$

We denote by $V_1(y_1)$ the local stable and by $V_2(y_2)$ the unstable manifold containing y_j, and by $B_j(y_j, q)$ the ball with centre at y_j and radius q on $V_j(y_j)$. There is a $q(l) > 0$ such that for any $y_j = y_j(w)$ $B_j(y_j, q(l)) \subset V_j(y_j)$. We put $R_j = \bigcup_{w \in U(x,r) \cap \Lambda_{k,s}^l} B_j\left(y_j(w), \frac{1}{2} q(l)\right)$ $(j = 1, 2)$ and consider the partition ξ_j of R_j into the sets $B_j\left(y_j(w), \frac{q(l)}{2}\right)$.

Let π be the Pinsker partition for $f \mid \Lambda_i$. From Theorem 6.4 it follows that $\pi \mid R_j \leqslant \xi_j$. Hence, by Proposition 4.3, on $R = R_1 \cap R_2$

$$\pi \mid R \leqslant \xi_1 \wedge \xi_2 = \nu \mid R.$$

Thus, we have proved that R is contained mod 0 in some element of π. Since π is completely invariant and f preserves ν, from the ergodicity it follows that all the elements of π have the same positive measure, and that f permutes these elements cyclically. Let $N = N(\Lambda_i)$ be the number of elements of π. It is evident that $f^N \mid C_\pi$ is a K-automorphism.

COROLLARY 7.2. *If a diffeomorphism f on Λ_i, $i > 0$, has a continuous spectrum* (see [25], §2), *then $f \mid \Lambda_i$ is a K-automorphism.*

§8. The Bernoulli property

THEOREM 8.1. $F = f^{n_i}|\Lambda_i^1$ $(i = 1, 2, \ldots)$ (see Theorem 7.9) *is a Bernoulli automorphism.*

Our proof is a generalization of the proof of a similar assertion in [41] for U-automorphisms of the two-dimensional torus. In this subsection we use the definitions and notation of §2.5. We also assume that on Λ_i^1 the measure is normalized. Let x be a density point of Λ_i^1. We denote by α' any finite measurable partition of M every element of which has a piecewise smooth boundary. We put $\alpha = \alpha'|\Lambda_i^1$. On the basis of Theorem 2.1 it is sufficient to prove that α is a VWB partition. (It is not difficult to construct an increasing sequence of partitions α_n' of M, $\alpha_n' \to \varepsilon_0$, such that the elements of α_k' have a piecewise smooth boundary.) We fix an $\varepsilon > 0$ and select an integer $l > 0$ such that

$$\nu(\Lambda_i^1 \cap \tilde{\Lambda}_{k,s}^l)(\nu(\Lambda_i^1))^{-1} \geqslant 1 - \varepsilon. \tag{8.1}$$

In Ornstein's proof of the fact that a U-automorphism of the two-dimensional torus is Bernoullian an essential role is played by the concept of a parallelogram with sides on the fibres of contracting and expanding foliations. The construction of such a parallelogram in this case is not difficult. We start by introducing the concept of a δ-parallelipiped on $\tilde{\Lambda}_{k,s}^l$, which is a natural generalization to the n-dimensional case of the concept of a parallelogram. Here we have to overcome the difficulty caused by the fact that $\tilde{\Lambda}_{k,s}^l$ is, so to speak, "perforated". We define a δ-parallelipiped axiomatically by a number of conditions. Then we show how to construct such a parallelipiped (see Lemma 8.1) and also how to partition $\tilde{\Lambda}_{k,s}^l$ into parallelipipeds (see Lemmas 8.2 and 8.3).

We put $l_1 = \psi(l)$ (the function ψ: $\mathbf{Z}^+ \to \mathbf{Z}^+$ was constructed in Theorem 6.4).

A measurable set π is said to be a δ-parallelipiped at the point $w \in \bar{\tilde{\Lambda}}_{k,s}^l$ if it satisfies the following conditions:

$$w \in \Pi \subset \Lambda_{k,s}^{l_1} \cap B(w, \delta); \tag{8.2}$$

$$V^-(y) \cap V^+(z) \in \Pi \quad \text{for any } y, z \in \Pi. \tag{8.3}$$

LEMMA 8.1. *For every δ, $0 < \delta \leqslant \frac{1}{8}\delta_s^{l_1}$ and any $w \in \bar{\tilde{\Lambda}}_{k,s}^l$ there is an $r > 0$ independent of w, and a δ-parallelipiped Π at w such that*

$$\bar{\tilde{\Lambda}}_{k,s}^l \cap B(w, r) \subset \Pi. \tag{8.4}$$

PROOF. We fix a δ, $0 < \delta \leqslant \frac{1}{8}\delta_s^l$. There is an $r > 0$ such that for any $y_1, y_2 \in \tilde{\Pi} = B(w, r) \cap \tilde{\Lambda}_{k,s}^l$

$$V^-(y_1) \cap V^+(y_2) \in B(w, \delta)$$

(see Theorem 4.2.3 and Proposition 4.7). We set $\Pi = \{y \in M$: there are $y_1, y_2 \in \widetilde{\Pi}$ such that $y = V^-(y_1) \cap V^+(y_2)\}$.

Now (8.2) follows from Theorem 7.4 and the definition of $\widetilde{\Pi}$, and (8.4) is obvious. To prove (8.3) we choose points $y_1, y_2, z_1, z_2 \in \widetilde{\Pi}$ for which

$$y = V^-(y_1) \cap V^+(y_2), \quad z = V^-(z_1) \cap V^+(z_2).$$

Hence it follows from Theorem 4.2 that

$$V^-(y) \cap V^+(z) = V^-(y_1) \cap V^+(z_2) \in \Pi.$$

LEMMA 8.2. *For any two parallelipipeds* Π *and* Π' *we can find sets* $\Pi_1, \ldots, \Pi_5$ *such that*

1) $\Pi_i \cap \Pi_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{5} \Pi_i = \Pi \cup \Pi'$;

2) *if* $\Pi_i \cap \widetilde{\overline{\Lambda}}{}^l_{k,s} \neq \emptyset$, *then* Π_i *is a parallelipiped.*

PROOF. We may assume that $\Pi \cap \Pi' \neq \emptyset$. We put

$$\widetilde{\Pi} = \bigcup_{y \in \Pi \cap \Pi'} V^+(y)$$

and write

$$\Pi_1 = \Pi \cap \Pi', \quad \Pi_2 = \Pi \setminus \widetilde{\Pi}, \quad \Pi_3 = \Pi' \setminus \widetilde{\Pi},$$

$$\Pi_4 = (\widetilde{\Pi} \cap \Pi) \setminus (\Pi' \cap \Pi), \quad \Pi_5 = (\widetilde{\Pi} \cap \Pi') \setminus (\Pi' \cap \Pi).$$

Now 1) is obvious. Let $x \in \Pi_i \cap \widetilde{\overline{\Lambda}}{}^l_{k,s}$. We claim that Π_i is a parallelipiped. Let $y, z \in \Pi_i$ and $w \in V^-(y) \cap V^+(z)$. Since $\Pi_i \subset \Pi$ (or $\Pi_i \subset \Pi_i'$), we see that $w \in \Pi$ (or $w \in \Pi'$). We restrict our attention to the cases $i = 1, 2, 4$. The remaining ones are analyzed similarly.

1) $i = 1$; since $y, z \in \Pi'$, we have $w \in \Pi'$, consequently, $w \in \Pi \cap \Pi' = \Pi_1$;

2) $i = 2$; since $z \in \widetilde{\Pi}$, we have $V^+(z) \not\subset \widetilde{\Pi}$, consequently, $w \in \Pi \setminus \widetilde{\Pi} = \Pi_2$;

3) $i = 4$; since $z \in \widetilde{\Pi}$, we have $w \in V^+(z) \subseteq \widetilde{\Pi}$, consequently, either $w \in \Pi_4$ or $w \in \Pi_1$. Let $w \in \Pi_1$. Since $y \in \widetilde{\Pi}$, we can find a $y_1 \in \Pi \cap \Pi' = \Pi_1$ such that $y \in V^+(y_1)$. Therefore, $y = V^-(w) \cap V^+(y_1) \in \Pi_1$ (since $y_1, w \in \Pi_1$), which is impossible because $y \notin \Pi_1$. This proves the lemma.

LEMMA 8.3. *For every* δ, $0 < \delta \leqslant \frac{1}{8}\delta_s^l$, *there are sets* Π_i $(i = 1, \ldots, m)$ *such that*

1) *if* $\widetilde{\Pi}_i = \Pi_i \cap \widetilde{\overline{\Lambda}}{}^l_{k,s} \neq \emptyset$, *then* Π_i *is a* δ*-parallelipiped*;

2) $\Pi_i \cap \Pi_j = \emptyset$, $i \neq j$, *and* $\bigcup_{i=1}^{m} \widetilde{\Pi}_i = \widetilde{\overline{\Lambda}}{}^l_{k,s}$.

PROOF. We choose a δ with $0 \leqslant \delta \leqslant \frac{1}{8}\delta_s^l$ and $r > 0$ in accordance with Lemma 8.1. Since $\widetilde{\overline{\Lambda}}{}^l_{k,s}$ is closed, there is a finite covering of this set by balls $B(w_i, r)$, $w_i \in \bar{\tilde{\Lambda}}^l_{k,s}$ $(i = 1, \ldots, m)$. We consider the δ-parallelipipeds at w_i constructed in Lemma 8.1. By (8.4), their union contains $\tilde{\Lambda}^l_{k,s}$. Subdividing consecutively every pair of intersecting parallelipipeds in accordance with Lemma 8.2, we construct sets Π_i satisfying 1) and 2).

Let $A \subset M$ be a measurable set and Π a parallelipiped. We say that A intersects Π layerwise if

$$V^+(w) \cap \Pi \subset A \cap \Pi \quad \text{for every} \quad w \in A \cap \Pi. \tag{8.5}$$

LEMMA 8.4. *Let Π be a δ-parallelipiped and $\beta > 0$. There is a $N_1 > 0$ such that for any $N' \geqslant N \geqslant N_1$ and β-almost every element $A \in \bigvee_N^{N'} F^k\alpha$ we can find a set $E \subset A$, intersecting Π layerwise, for which*

$$\nu(E)(\nu(A))^{-1} \geqslant 1 - \beta. \tag{8.6}$$

PROOF. Let D be an element of α. We denote by B^k_D the subset of the element $F^k(D)$ of F^k_α that intersects Π layerwise. We consider an element D' of α' for which $D' | \Lambda^l_i = D$. Since $\Pi \subset \Lambda^l_{k,s}$, by the choice of α and by (4.41), (4.16), and (8.5), the distance of any point of $D \cap F^{-k}(\Pi) \backslash F^{-k}(B^k_D)$ from the boundary of D' is

$$d_k \leqslant C_1(\varkappa_s)^k, \tag{8.7}$$

where $C_1 = C_1(l, s)$ and $\varkappa_s$ are constants.

We set $G_k = \bigcup_{D \in \alpha} (F^k(D) \backslash B^k_D)$. From (8.7) it follows that $\nu(G_k) \leqslant C_2(\varkappa_s)^k$ where $C_2 = C_2(l, s)$ is a constant. Let $G = \bigcup_{k=N_1}^{\infty} G_k$. We choose an integer N_1 large enough so that $\nu(G) \leqslant \sum_{k=N_1}^{\infty} \nu(G_k) \leqslant \beta^2$. It is easy to see that $\nu(A \cap G) \leqslant \beta\nu(A)$ for any $N' \geqslant N \geqslant N_1$ and β-almost every element $A \in \bigvee_N^{N'} F^k\alpha$, and we can put $E = A \backslash A \cap G$.

To prove that α is a VWB-partition we use Theorem 2.2. First we construct the map θ of an arbitrary set E that intersects Π layerwise, onto the parallelipiped. Then, using the preceding lemma, the K-property, and the partition into parallelipipeds constructed above, we construct θ in Theorem 2.2.

LEMMA 8.5. *For every $\delta > 0$ there is a δ_1, $0 < \delta_1 \leqslant \delta$, such that for any δ_1-parallelipiped Π and any set $E \subset \Pi$, $\nu(E) > 0$, that intersects Π layerwise, we can find a bijective map θ: $E \overset{\text{onto}}{\to} \Pi$* (see Remark 2.1) *for which*

1) *the Jacobian $J(\theta)(y)$ of θ at $y \in E$ satisfies the condition*

(8.8) $$| J(\theta)(y) - 1 | \leqslant \delta;$$

2) $$\rho(F^k(y),\ F^k(\theta(y))) \leqslant \delta,\ k \in \mathbf{Z}^+,\ y \in E.$$

PROOF. Let Π be a δ-parallelipiped at w and $w_1 \in \Pi$. We consider the succession map p_{w,w_1} of a measurable subset in $V^-(w)$ onto a subset of $V^-(w_1)$, realized by means of the local stable manifolds $V^+(y)$, $y \in \Pi$ (see §§4.5, 4.7). If δ_1 is sufficiently small, then by (4.28) and Theorem 4.1.4 the Jacobian $J(p_{w,w_1})$ satisfies the condition

(8.9) $$| J(p_{w,w_1}) - 1 | \leqslant \frac{1}{3}\delta.$$

Decreasing δ_1, if necessary, we may assume on the basis of (4.14) that for any y_1, $y_2 \in V^-(w_1)$, $k \in \mathbf{Z}^+$

(8.10) $$\rho(F^k(y_1),\ F^k(y_2)) \leqslant \delta.$$

Let E be a measurable set of positive measure that intersects Π layerwise. We choose any bijective map preserving ν_w^-

$$\theta_0\colon E \cap V^-(w) \overset{\text{onto}}{\longrightarrow} V^-(w) \cap \Pi.$$

It is easy to see that such a map exists. Let $y \in E$. Then

$$z = V^+(y) \cap V^-(w) \in \Pi \cap V^-(w).$$

Furthermore, $z \in E$, consequently, $z \in E \cap V^-(w)$. For $y \in E$ we put

$$\theta(y) = V^+(\theta_0(z)) \cap V^-(y) = p_{w,y} \circ \theta_0 \circ p_{w,y}^{-1}(y).$$

The first assertion follows from (8.9), and the second from (8.10) and the condition $\theta(y) \in V^-(y)$ for $y \in E$.

LEMMA 8.6. *For any* ε', $0 < \varepsilon' < \varepsilon$, *there is an integer* $N_2 > 0$ *such that for any* $N' > N \geqslant N_2$ *and* ε*-almost every element* $A \in \bigvee_N^{N'} F^k\alpha$ *we can find a set* $E \subset A$ *and a bijective map* $\theta\colon E \to \Lambda_l^1$ (see Remark 2.1) *for which*

1) $\nu(E)(\nu(A))^{-1} > 1 - 2\varepsilon$;

2) θ *is* 13ε*-measure preserving*;

3) $\rho(F^k(y),\ F^k(\theta(y))) \leqslant \varepsilon'$, $k \in \mathbf{Z}^+$, $y \in E$.

PROOF. For a given $\varepsilon' > 0$ we choose a δ_1 in accordance with Lemma 8.5 and let $\eta = \{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ be the partition of Λ_l^1 formed from the sets Π_j ($j = 1, \ldots, m$), which were constructed in Lemma 8.3 with reference to δ_1 and $\Pi_0 = \Lambda_l^1 \setminus \bigcup_{j=1}^{m} \Pi_j$. By (8.1)

(8.11) $$\nu(\Pi_0) < \varepsilon.$$

Replacing δ in (2.1) by $\varepsilon \times \min_{1 \leqslant j \leqslant m} \nu(\Pi_j)$, we choose an N_0 in accordance with Theorem 2.3. Applying Lemma 8.4 consecutively to every Π_j ($j = 1, \ldots, m$) and taking $\delta = \delta_1$, $\beta = \frac{\varepsilon}{m} \min_{1 \leqslant j \leqslant m} \nu(\Pi_j)$, we choose the numbers N_{1j} ($j = 1, \ldots, m$). We put $N_2 = \max_{1 \leqslant j \leqslant m} \{N, N_{1j}\}$. From Lemma 8.4 and (8.6), (8.11) it follows that for ε-almost every element $A \in \bigvee_N^{N'} F^k\alpha$ there is a set $E \subset A$ that satisfies the first assertion of the lemma and intersects Π_j, $1 \leqslant j \leqslant m$, layerwise, and such that $E \cap \Pi_0 = \varnothing$. Therefore, by Lemma 8.4,

$$\nu((A \setminus E) \cap \Pi_j) \leqslant \beta\nu(A) \leqslant \varepsilon\nu(A)\nu(\Pi_j) \leqslant \varepsilon\nu(E)\nu(\Pi_j)(1 + 3\varepsilon).$$

From this and (2.1) it follows that

$$\left| \frac{\nu(E \cap \Pi_j)}{\nu(E)} - \nu(\Pi_j) \right| \leqslant 6\varepsilon\nu(\Pi_j). \tag{8.12}$$

Let $\theta_j \colon E \cap \Pi_j \xrightarrow{\text{onto}} \Pi_j$ be the map constructed in Lemma 8.5, $1 \leqslant j \leqslant m$. We define the map $\theta \colon E \to \Lambda_l^1$ by putting $\theta(y) = \theta_j(y)$ for $y \in E \cap \Pi_j$. Let $B \subset E$ be an arbitrary measurable set. We put $B_j = B \cap \Pi_j$, $E_j = E \cap \Pi_j$. From (8.8) it follows that

$$\left| \frac{\nu(\theta_j(B_j))\,\nu(E_j)}{\nu(\Pi_j)\,\nu(B_j)} - 1 \right| \leqslant \varepsilon'.$$

Therefore, by (8.12),

$$\left| \frac{\nu(\theta_j(B_j))\,\nu(E)}{\nu(B_j)} - 1 \right| \leqslant \varepsilon' + (1 + \varepsilon')\,6\varepsilon \leqslant 13\varepsilon.$$

From this and (8.11) it follows that θ is 13ε-measure preserving, and the second assertion is proved. The third follows directly from the Lemma 8.5.2.

We now complete the proof of the theorem. Let $N' > N \geqslant N_2$ and suppose that $A \in \bigvee_N^{N'} F^k\alpha$ does not belong to some exceptional set of measure less than ε. (This set and N_2 are chosen in accordance with Lemma 8.6.) We consider the set E and the map θ constructed in Lemma 8.6. Since ε' can be chosen arbitrarily small, from Lemma 8.6.3 and the assumption that the elements of α' have a piecewise smooth boundary we obtain the conditions of Theorem 2.2. From this theorem it follows that

$$\bar{d}(\{F^{-i}\alpha\}_1^n,\ \{F^{-i}\alpha \mid A\}_1^n) \leqslant 300\varepsilon.$$

Since ε is arbitrary, this means that α is a VWB-partition.

§9. Flows

9.1. Let f^t be a ν-measure preserving flow on a manifold M, given by a vector field X of class C^r, $r \geqslant 1$. We consider the set

$$(9.1)\quad \Lambda = \{x \in M\colon \chi^+(x, v) \neq 0 \text{ for every } v \in T_xM \setminus \{\alpha X(x)\},\ \alpha \in \mathbf{R}\},$$

where χ^+ is the characteristic Lyapunov exponent of the dynamical system f^t (see §3.2). On the basis of what was said in §3, Λ is measurable and f^t-invariant. We assume that $\nu(\Lambda) > 0$. It is easy to see that $\chi^+(x, v) = 0$ for any $x \in M$, $v = \alpha X(x)$. We consider the set of distinct values of χ^+ at $x \in M$, arranged in increasing order,

$$\chi_1(x) < \ldots < \chi_{s(x)}(x), \quad 1 \leqslant s(x) \leqslant \dim M,$$

and also the set $\widetilde{\Lambda}$ of regular points in Λ. Let $k(x)$ be the number of distinct negative values of χ^+ at x. It is obvious that $\chi^+(x, v) < 0$ for any $v \in L_{k(x)}(x)$ (see (3.3)). We put ($x \in \widetilde{\Lambda}$)

$$E_{1x} = \bigoplus_{j=1}^{k(x)} E_j(x), \quad E_{2x} = \bigoplus_{j=k(x)+2}^{s(x)} E_j(x), \quad E_{0x} = E_{k(x)+1}(x),$$

$$\lambda(x) = e^{\chi_{k(x)}(x)}, \quad \mu(x) = e^{\chi_{k(x)+2}(x)}.$$

Here $E_j(x)$ ($j = 1, \ldots, s(x)$) are the subspaces constructed in Theorem 3.4. E_{0x} is generated by the vector $X(x)$. It is easy to see that the measurable functions $\lambda(x)$ and $\mu(x)$ satisfy (4.24) and (4.2) and that for any $x \in M$

$$T_xM = E_{1x} \oplus E_{0x} \oplus E_{2x}, \quad df^t E_{ix} = E_{if^t(x)}, \quad t \in \mathbf{R} \qquad (i = 0, 1, 2).$$

Further, $\chi_{k(x)+1}(x) = 0$, $\chi^+(x, v) > 0$ for any $v \in E_{2x}$. Applying Theorem 4.1 to every diffeomorphism f^t (t fixed) we construct a family of measurable functions $C_t(x, \varepsilon)$ and $K_t(x, \varepsilon)$. Since for some $a > 0$ and $b > 0$

$$a \leqslant \max_{1/2 \leqslant t \leqslant 1} \| df^t \| \leqslant b,$$

according to [22] (see §2) the following functions exist:

$$(9.2)\quad C(x, \varepsilon) = \max_{1/2 \leqslant t \leqslant 1} C_t(x, \varepsilon) < \infty, \quad K(x, \varepsilon) = \min_{1/2 \leqslant t \leqslant 1} K_t(x, \varepsilon) > 0.$$

It is easy to see that $C(x, \varepsilon)$ and $K(x, \varepsilon)$ satisfy (4.4). Moreover, for any $t \in \mathbf{R}$ the inequalities (4.5) hold with n replaced by t, and the angle $\gamma_{i,j}(x)$ between E_{ix} and E_{jx} ($i, j = 0, 1, 2$, $i \neq j$) has the lower bound $\gamma_{i,j}(x) \geqslant K(x, \varepsilon)$.

9.2. The definitions of the sets $\widetilde{\Lambda}_s$, $\widetilde{\Lambda}_s^l$, $\widetilde{\Lambda}_{k,s}^l$ etc., and also of the measurable functions $\varepsilon(x)$ and $\varkappa(x)$ (see §§4.2, 4.3, and 4.4) carry over to flows verbatim. We emphasize that these functions do not depend on t, because they are defined in terms of the functions $\lambda(x)$ and $\mu(x)$, which do not depend on t.

9.3. We now construct local stable and unstable manifolds for flows. We fix $t \in \mathbf{R}$ and by applying Theorem 4.1 to f^t we construct a measurable function $\delta_t(x)$ and a family, depending measurably on x of maps $\varphi_t(x)$: $B^1(\delta_t(x)) \to E_{0x} \oplus E_{2x}$ of class C^{r-1}, given by the family of submanifolds $V_t(x) = \{\exp_x(v, \varphi_t(x)v)$: $v \in B^1(\delta_t(x))\}$ of class C^{r-1}. From Theorem 4.1. 4) and 5) it follows that for any t_1 and t_2 with $1/2 \leqslant t_1 \leqslant t_2 \leqslant 1$

$$V_{t_1}(x) \cap U(x, \widetilde{\delta}(x)) = V_{t_2}(x) \cap U(x, \widetilde{\delta}(x)), \tag{9.3}$$

where $\widetilde{\delta}(x) = \min\{\delta_{t_1}(x), \delta_{t_2}(x)\}$. According to §9.2 and Theorem 4.1

$$\inf_{1/2 \leqslant t \leqslant 1} \delta_t(x) = \delta(x) > 0, \quad x \in \widetilde{\Lambda}. \tag{9.4}$$

We set

$$V^-(x) = V_t(x) \cap U(x, \delta(x)), \quad 1/2 \leqslant t \leqslant 1. \tag{9.5}$$

By (9.3) and (9.4) this is well-defined. It is easy to see that $\delta(x)$ and $V^-(x)$ satisfy Theorem 4.1. 1)–8) (with n replaced by t). The local unstable manifolds $V^+(x)$ are similarly defined.

9.4. We fix $\tau > 0$ and write for $x \in \widetilde{\Lambda}$

$$V^{-0}(x) = \bigcup_{|t|<\tau} V^-(f^t(x)), \quad V^{+0}(x) = \bigcup_{|t|<\tau} V^+(f^t(x)).$$

From what we have said above and Theorems 4.1 and 4.2 we derive the next result.

THEOREM 9.1. *There is a measurable function $\delta_t(x)$, $x \in \Lambda$, such that* $0 < \delta_\tau(x) \leqslant \delta(x)$,

$$\delta_\tau(f^t(x)) \geqslant e^{-15\varepsilon(x)t}\delta_\tau(x), \quad \inf_{x \in \widetilde{\Lambda}_s^l} \delta_\tau(x) = \delta_{\tau,s}^l > 0,$$

and for any $x \in \widetilde{\Lambda}$

$$\bigcup_{y \in V^-(x) \cap U(x,\, \delta_\tau(x))} \Big(\bigcup_{|t|<\tau} f^t(y)\Big) = V_\tau^{-0}(x) \subset V^{-0}(x),$$

$$\bigcup_{y \in V^+(x) \cap U(x,\, \delta_\tau(x))} \Big(\bigcup_{|t|<\tau} f^t(y)\Big) = V_\tau^{+0}(x) \subset V^{+0}(x).$$

Let ν_x^-, ν_x^+, ν_x^{-0}, ν_x^{+0}, respectively, be the measures induced on $V^-(x)$, $V^+(x)$, $V^{-0}(x)$, $V^{+0}(x)$ by the Riemannian metric. It is not difficult to see that the following assertion holds.

PROPOSITION 9.1. *Let $N \subset v^-(x)$ (or $N \subset V^+(x)$) and let* $\nu_x^-(N) = 0$ $(\nu_x^+(N) = 0)$. *Also, let* $N_1 = \bigcup_{y \in N,\, |t|<\tau} f^t(y)$. *Then* $\nu_x^{-0}(N_1) = 0$ $(\nu_x^{+0}(N_1) = 0)$.

9.5. Let x be a density point of $\Lambda_{k,s}^l$ and $r \leqslant \frac{1}{8}\min(\delta_{\tau,s}^l, \tau)$. The set

$$P_{k,s}^l(x, r) = \bigcup_{y \in \Lambda_{k,s}^l \cap U(x,r)} (V^+(y) \cup V_\tau^{-0}(y))$$

is called the lattice of local stable manifolds at x. We note that if r is sufficiently small, then for any points $w_1, w_2 \in \Lambda_{k,s}^l \cap U(x, r)$ the submanifolds $V^+(w_1)$ and $V_\tau^{-0}(w_2)$ intersect transversally.

We put $Q(x) = \bigcup_{-\infty<t<\infty} f^t(P^l_{k,s}(x, r))$.

THEOREM 9.2. *The flow $f^t | Q(x)$ is ergodic.*

The proof of this theorem makes use of Proposition 9.1 and proceeds like that of Theorem 7.1.

This theorem permits us to describe the partition into ergodic components for the flow $f^t | \Lambda$.

THEOREM 9.3. *There are measurable sets $\Lambda_n \subset \Lambda$ ($n = 0, 1, 2, \ldots$) such that*

1) $\bigcup_{n\geqslant 0} \Lambda_n = \Lambda$, $\Lambda_n \cap \Lambda_m = \emptyset$ *if* $n \neq m$;

2) $\nu(\Lambda_0) = 0$, $\nu(\Lambda_n) > 0$ *if* $n > 0$;

3) $f^t(\Lambda_n) = \Lambda_n$;

4) *the flow $f^t | \Lambda_n$ is ergodic for $n > 0$.*

9.6. The results for diffeomorphisms obtained in §§7.2 and 7.3 carry over to flows. We restrict ourselves only to stating these results, since the changes that have to be made in the proofs are obvious.

THEOREM 9.4. $\nu_x^-(V^-(x)\setminus\Lambda_{k,s}) = 0$ *for almost all points* $x \in \Lambda_{k,s}$. *A similar statement holds for* $V^+(x)$, $V_\tau^{-0}(x)$, *and* $V_\tau^{+0}(x)$.

We put (see (6.1))

$$W^-(x) = \bigcup_{-\infty<t<\infty} f^{-t}(V^-(f^t(x))), \quad W^+(x) = \bigcup_{-\infty<t<\infty} f^{-t}(V^+(f^t(x))). \tag{9.6}$$

The sets $W^-(x)$ and $W^+(x)$ have the properties stated in Theorems 6.1 and 6.2 (when considering the sets $W^+(x)$ we have to reverse the direction of time).

We consider $\Lambda_{k,s}$ and denote by W^- and W^+ the partitions of this set into the sets $W^-(x)$ and $W^+(x)$. The "measurable foliation" W^- is "integrable" with respect to the foliation Z formed by the trajectories of the flow (see [1]). This means that the sets

$$W^{-0}(x) = \bigcup_{z\in W^-(x)} \bigcup_{-\infty<t<\infty} f^t(z)$$

form a partition of $\Lambda_{k,s}$, denoted by W^{-0}. We remark that by Theorem 6.1.4 (its analogue for flows)

$$W^{-0}(x) = \bigcup_{-\infty<t<\infty} W^-(f^t(x)).$$

The partition W^{+0} is constructed similarly.

THEOREM 9.5. *The assertions in Theorems* 7.5, 7.6, 7.7, *and* 7.8 *hold for the flow f^t on $\Lambda_{k,s}$.*

9.7. In this subsection we state the results that establish the K-property and the Bernoullian property for $f^t | \Lambda$.

THEOREM 9.6 (see [24], Theorem 2.1). *Suppose that $f^t | \Lambda_n$, $n > 0$, has a continuous spectrum* (see [25], §2; the set Λ_n is constructed in

Theorem 9.3). *Then it is a K-flow.*

Here is a sketch of a proof. First, by combining the method of proof of Theorem 8.1 and those of Anosov (see [1], Lemmas 21.1 and 21.3) we show that the partition $\xi = W^-|\Lambda_n$ is metrically transitive (that is, $\nu(\xi) = \nu$). From this and Theorem 6.3 we deduce that the diffeomorphism $f^1|\Lambda_n$ has the K-property. Using Rudolph's result [42] we conclude that so has $f^t|\Lambda_n$.

THEOREM 9.7 (see [24], Theorem 3.1). *Under the conditions of Theorem 9.6 $f^t|\Lambda_n$ is isomorphic to a Bernoulli flow.*

The proof of this theorem is a simple modification of that of Theorem 8.1.

THEOREM 9.8 (see [24], Theorem 9.6). *Suppose that*

1. $\Lambda_{k,s} = M(\text{mod } 0)$ *for some k and s.*
2. *there are continuous foliations $\widetilde{W}^-$ and $\widetilde{W}^+$ of M such that $\widetilde{W}^-(x) = W^-(x)$ and $\widetilde{W}^+(x) = W(x)$ for almost all $x \in \Lambda_{k,s}$.*

Then any measurable eigenfunction of f^t is mod 0 *continuous.*

The next theorem, which follows from Theorems 9.6 and 9.8, is an analogue to the theorem on the alternative for U-flows (see [1], Theorem 14).

THEOREM 9.9 (see [24], Theorem 9.7). *Under the conditions of Theorem 9.8 f^t is either isomorphic to a Bernoulli flow or can be represented as a suspension over a diffeomorphism of a compact manifold having almost everywhere non-zero characteristic Lyapunov exponents.*

§10. Geodesic flows on closed Riemannian manifolds without focal points

10.1. For the convenience of the presentation we introduce in this subsection some concepts and results concerning Riemannian manifolds without conjugate and without focal points. More details about them can be found, for example, in [11] or [29].

We consider an n-dimensional manifold equipped with a Riemannian metric of class C^3. A Jacobi field is a vector field Y along a geodesic γ satisfying Jacobi's equation

$$Y'' + R_{XY}X = 0, \tag{10.1}$$

where the dashes denote covariant differentiation along the geodesic, R is the curvature tensor, and $X = \dot{\gamma}(t)$ is the unit tangent vector field along γ. We denote by $J(\gamma)$ the $2n$-dimensional space of Jacobi fields along γ.

Let $\{e_i(t)\}$ $(i = 1, \dots, n)$ be a system of vector fields obtained by parallel displacement of an orthonormal system at $\gamma(0)$, where $e_n(t) = \dot{\gamma}(t)$. Then (10.1) can be rewritten in the matrix form

$$\frac{d^2}{dt^2} Y(t) + R(t) Y(t) = 0, \tag{10.2}$$

where $R(t) = (R_{ij}(t))$, $R_{ij}(t) = \langle R_{e_n(t)e_i(t)}e_n(t), e_j(t)\rangle$ $(i, j = 1, \ldots, n)$. We denote by $\pi: TM \to M$ the natural projection and by $K: T(TM) \to TM$ the map of Riemannian connectivity. For every $v \in TM$

$$T_vTM = \operatorname{Ker} d\pi \oplus \operatorname{Ker} K.$$

In T_vTM we introduce a scalar product by setting

$$\langle\xi, \eta\rangle = \langle d\pi\xi, d\pi\eta\rangle_{\pi(v)} + \langle K\xi, K\eta\rangle_{\pi(v)}.$$

In this metric $\operatorname{Ker} d\pi$ and $\operatorname{Ker} K$ are orthogonal.

Let $v \in TM$, $\xi \in T_vTM$, and γ_v the geodesic with the initial vector v. We define the Jacobi field Y_ξ by the initial conditions

$$Y_\xi(0) = d\pi\xi, \quad Y'_\xi(0) = K\xi. \tag{10.3}$$

The map $\xi \to Y_\xi$ is a linear isomorphism of T_vTM onto $J(\gamma_v)$ (see [35], §1).

Two points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are said to be conjugate if there is a Jacobi field $Y \not\equiv 0$ along γ such that $Y(t_1) = Y(t_2) = 0$.

Two points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are said to be focal if there is a Jacobi field Y along γ such that

$$Y(t_1) = 0, \; Y'(t_1) \neq 0, \quad \frac{d}{dt}(\|Y(t)\|^2)\big|_{t=t_2} = 0.$$

We say that a Riemannian manifold does not have conjugate (or focal) points if no two points on any geodesic are conjugate (or focal).

If a Riemannian manifold does not have focal points, then it does not have conjugate points. If it has non-positive curvature, then it has no focal points. The universal Riemannian covering H of a manifold without conjugate points is diffeomorphic to $\mathbf{R}^n$. Any two geodesics on H intersect at most in one point (see [11], [29], [36]).

10.2. In this subsection we state Eberlein's results, which describe the structure of the equation in variations for a geodesic flow on a Riemannian manifold without conjugate points. Proofs of these results are in [35].

A geodesic flow acts on TM according to the formula $f^t(v) = \dot\gamma_v(t)$ and is given by a vector field V of class C^2 on TM. The submanifold $SM \subset TM$ of unit linear elements of dimension $2n-1$ is invariant under f^t, and the vector field $V\,|\,SM$ defines a flow on SM. We consider the Jacobi matrix equation corresponding to (10.2)

$$\frac{d^2}{dt^2}D(t) + R(t)\,D(t) = 0. \tag{10.4}$$

PROPOSITION 10.1 (see [35], §2). *Let $D_s(t)$, $s \in \mathbf{R}^+$, be the solution of* (10.4) *with boundary conditions $D_s(0) = I$, $D_s(s) = 0$. Then this equation has a solution $D^-(t)$ for which* ,

$$D^-(0) = I, \quad D^-(t) = \lim_{s\to+\infty} D_s(t), \quad (D^-(0))' = \lim_{s\to+\infty}(D_s(0))',$$

$\det(D^-(t)) \neq 0$ *for every* $t \in \mathbf{R}$.

$D^-(t)$ is called the negative limit solution of (10.4). Similarly we construct the positive limit solution $D^+(t)$.

For every $v \in SM$ we set

$$X^-(v) = \{\xi \in T_vSM\colon \langle \xi, V(v) \rangle = 0, \ Y_\xi(t) = D^-(t)d\pi\xi\},$$
$$X^+(v) = \{\xi \in T_vSM\colon \langle \xi, V(v) \rangle = 0, \ Y_\xi(t) = D^+(t)d\pi\xi\}.$$

$X^-(v)$ and $X^+(v)$ are called the stable and unstable subspaces of T_vSM, respectively.

PROPOSITION 10.2 (see [35], Propositions 2.4, 2.6, 2.11).

1. $X^-(v)$ *and* $X^+(v)$ *for any* $v \in SM$ *are vector subspaces of* T_vSM *of dimension* $n-1$.
2. $d\pi(X^-(v)) = d\pi(X^+(v)) = \{w \in T_{\pi(v)}M\colon w$ *is orthogonal to* $v\}$.
3. *If* $\tau\colon SM \to SM$ *is an involution,* $\tau(v) = -v$, *then*

$$X^+(-v) = d\tau X^-(v) \ u \ X^-(-v) = d\tau X^+(v).$$

4. *Suppose that the curvature of* M *in any two-dimensional direction is greater than or equal to* $-a^2$, $a > 0$. *Then for any* $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$

$$\| K\xi \| \leqslant a \, \| d\pi\xi \|. \tag{10.5}$$

PROPOSITION 10.3 (see [35], §3). *Suppose that a Riemannian manifold* M *has no focal points. Then for any Jacobi field* Y_ξ, $\xi \in X^-(v)$ ($\xi \in X^+(v)$) *the function* $\| Y_\xi(t)\|$ *is not increasing (decreasing).*

PROPOSITION 10.4 (see [35], Proposition 1.7). *Let* $v \in SM$, $\xi \in T_vSM$.

1. $Y_\xi(t) = d\pi \circ df^t\xi$, $Y_\xi'(t) = K \circ df^t\xi$.
2. $\| df^t\xi \|^2 = \| Y_\xi(t) \|^2 + \| Y_\xi'(t) \|^2$.
3. *If* $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$, *then* $Y_\xi(t) \neq 0$.

PROPOSITION 10:5 (see [35], Propositions 2.4, 2.12). *Let* $v \in SM$.

1. $df^tX^-(v) = X^-(f^t(v))$, $df^tX^+(v) = X^+(f^t(v))$.
2. $\xi \in X^-(v)$ ($\xi \in X^+(v)$) *if and only if* $\langle \xi, V(v) \rangle = 0$ *and* $\| d\pi \circ df^t v \| \leqslant \text{const.}$ *for* $t > 0$ ($t < 0$).

10.3. From here on we assume that M is a compact manifold without conjugate points and that the parameter on a geodesic is the arc length. Two geodesics γ_1 and γ_2 on the universal Riemannian covering H are said to be asymptotic for $t > 0$ if there is a constant $C > 0$ such that $\rho(\gamma_1(t), \gamma_2(t)) \leqslant C$ for all $t > 0$.

Similarly we can define asymptotic geodesics for $t < 0$. Being asymptotic for $t > 0$ ($t < 0$) is an equivalence relation. A class of equivalent elements is called a point at infinity, and the set of equivalence classes is called the absolute and is denoted by $H(\infty)$. The class of geodesics asymptotic to $\gamma(t)$ for $t > 0$ ($t < 0$) is denoted by $\gamma(+\infty)(\gamma(-\infty))$.

We say that M satisfies the axiom of uniform visibility (see [36]) if for any $\varepsilon > 0$ there is an $R = R(\varepsilon)$ such that from every point $x \in H$ any geodesic segment γ for which $\rho(x, \gamma) \geqslant R$ is visible under an angle less than ε.

PROPOSITION 10.6 (see [36], Theorems 4.2, 5.1). *If* $\dim M = 2$ *and the genus of* M *is at least* 2, *then* M *satisfies the axiom of uniform visibility.*

PROPOSITION 10.7 (see [36], Proposition 1.13). *If a Riemannian manifold* M *satisfies the axiom of uniform visibility, then for any two geodesics* $\gamma_1(t)$ *and* $\gamma_2(t)$ *there is a geodesic* $\gamma(t)$ *such that* $\gamma(+\infty) = \gamma_1(+\infty)$, $\gamma(-\infty) = \gamma_2(-\infty)$.

On the absolute we can introduce a topology and construct a homeomorphic map of the closed unit ball in $\mathbf{R}^n$ onto the set $H \cup H(\infty)$ that associates H with the interior of the ball and $H(\infty)$ with S^{n-1}.

10.4. In [45] Eberlein has constructed limit spheres (horospheres) for a geodesic flow on a Riemannian manifold with non-positive curvature. His results can be generalized to manifolds that satisfy a certain very weak condition (the so-called "axiom of being asymptotic"; see [24], §12), in particular, on manifolds without focal points (see [24]).

THEOREM 10.1 (see [24], §6, 7). *If a compact manifold* M *does not have conjugate points and satisfies the axiom of uniform visibility, or does not have focal points, then the distributions* X^- *and* X^+ *are integrable and their integral manifolds form continuous* f^t*-invariant foliations* $\mathfrak{S}^-$ *and* $\mathfrak{S}^+$ *of* SM (see [1]).

The distributions X^- and X^+ and the foliations $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of SM can be "lifted" to SH. The resulting distributions and foliations in SH are denoted by $\mathring{X}^-$, $\mathring{X}^+$, $\mathring{\mathfrak{S}}^-$, and $\mathring{\mathfrak{S}}^+$, respectively. A fibre $\mathfrak{S}^-(v)$ ($\mathfrak{S}^+(v)$) of $\mathfrak{S}^-$ ($\mathfrak{S}^+$) is called the stable (unstable) horosphere passing through the linear element $v \in SM$. The set $L(x, p) = \pi(\mathring{\mathfrak{S}}^-(v))$ is called the limit sphere with centre at $p = \gamma_v(+\infty) \in H(\infty)$ passing through $x = \pi(v)$.

The fundamental group $\pi_1(M)$ of M acts by isometries on H. This action can be extended to the absolute $H(\infty)$. Let $p = \gamma_v(+\infty) \in H(\infty)$ and $\varphi \in \pi_1(M)$. Then $\varphi(p)$ is the class of geodesics asymptotic to $\varphi(\gamma_v(t))$.

THEOREM 10.2 (see [24], §7). *Suppose that a Riemannian manifold* M *does not have conjugate points and satisfies the axiom of uniform visibility, or that it does not have focal points.*

1. *For any* $x \in H$, $p \in H(\infty)$, *there is a unique limit sphere* $L(x, p)$ *with centre at* p *passing through* x.

2. *The fibre* $\mathring{\mathfrak{S}}^-(v)$ *is the equipment of the limit sphere* $L(x, p)$ ($x = \pi(v)$, $p = \gamma_v(+\infty)$) *with orthogonal unit vectors having the same direction as* v.

The fibre $\mathring{\mathfrak{S}}^+(v)$ *is the equipment of the limit sphere* $L(x, q)$ ($q = \gamma_v(-\infty) = \gamma_{-v}(+\infty)$) *with orthogonal unit vectors having the same direction as* v.

3. *If* $\varphi \in \pi_1(M)$ *then* $\varphi(L(x, p)) = L(\varphi(x), \varphi(p))$,

$$d\varphi\mathring{\mathfrak{S}}^-(v) = \mathring{\mathfrak{S}}^-(d\varphi v), \qquad d\varphi\mathring{\mathfrak{S}}^+(v) = \mathring{\mathfrak{S}}^+(d\varphi v).$$

4. *For any v, $w \in SH$ such that $\gamma_v(+\infty) = \gamma_w(+\infty) = p$, the geodesic $\gamma_w(t)$ intersects the limit sphere $L(\pi(v), p)$ at some point.*

For $v \in SM$ we denote by $Z(v)$ the one-dimensional subspace of T_vSM generated by the vector $V(v)$. Also, let $\mathfrak{S}^0$ denote the smooth foliation of SM (see [1]) formed by the trajectories of the flow.

THEOREM 10.3 (see [24], §7). *The pair of foliations $\mathfrak{S}^-$ and $\mathfrak{S}^0$ is integrable in the sense of* [1], *and the fibres of the corresponding foliation (denoted by $\mathfrak{S}^{-0}$ are integral manifolds of the distribution $X^- \oplus Z$. The foliation $\mathfrak{S}^{-0}$ is invariant under f^t. Moreover, $w \in \mathfrak{S}^{-0}$ if and only if the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ are asymptotic. The pair of foliations $\mathfrak{S}^+$ and $\mathfrak{S}^0$ has similar properties (we denote the corresponding foliation by $\mathfrak{S}^{+0}$).*

The foliations $\mathfrak{S}^{-0}$ and $\mathfrak{S}^{+0}$ can be "lifted" to foliations on SH, which we denote by $\mathring{\mathfrak{S}}^{-0}$ and $\mathring{\mathfrak{S}}^{+0}$, respectively.

Let $v \in SM$. We consider an orthonormal system of parallel vector fields along $\gamma_v(t)$ and a vector w orthogonal to v. We put

$$\text{(10.6)} \qquad \begin{cases} w(t) = \dfrac{D^-(t)(w)}{\|D^-(t)(w)\|}, \\ K_{v,w}(t) = \langle R_{\dot\gamma_v(t)w(t)}\dot\gamma_v(t),\ w(t)\rangle, \end{cases}$$

where $D^-(t)$ is the solution of (10.4) constructed in Proposition 10.1.

A linear element $v \in SH$ is called an element of non-uniqueness if there is a vector $w \in SH$ such that $\gamma_v(+\infty) = \gamma_w(+\infty)$, $\gamma_v(-\infty) = \gamma_w(-\infty)$. The other vectors $v \in SH$ are called elements of uniqueness.

We denote by $\rho_{\mathfrak{S}^-(v)}$ $(\rho_{\mathring{\mathfrak{S}}^-(v)})$ the distance induced on $\mathfrak{S}^-(v)$ $(\mathring{\mathfrak{S}}^-(v))$ by the Riemannian metric.

THEOREM 10.4. *Suppose that a Riemannian manifold M does not have focal points. Then the following assertions hold.*

1. *If v is an element of uniqueness, then for every $w \in \mathring{\mathfrak{S}}^-(v)$ the function $\rho_{\mathring{\mathfrak{S}}^-(f^{-t}(v))}(f^{-t}(v), f^{-t}(w))$ is monotone increasing and tends to $+\infty$ as $t \to \infty$.*

2. *If v is an element of non-uniqueness, then there is a vector $w \in SH$ orthogonal to v such that $K_{v,w}(t) \equiv 0$ for all $t \in R$, and the field $D^-(t)w$ is obtained by parallel displacement of w along the geodesic $\gamma_v(t)$.*

10.5. As is well known (see [3]), a geodesic flow f^t acting on SM has a smooth invariant measure, which we denote by μ. We consider the set

(10.7) $\Lambda_0 = \Big\{ v \in SM$: for every $w \in SM$ orthogonal to v,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t K_{v,\,w}(s)\,ds < 0 \Big\}.$$

The set Λ is measurable and invariant under f^t.

Let χ^+ be a characteristic exponent of the dynamical system f^t (see §3).

THEOREM 10.5. *Suppose that a Riemannian manifold M does not have focal points. Then*

1. $\chi^+(v, \xi) < 0$ *for any* $v \in \Lambda_0$, $\xi \in X^-(v)$;
2. $\chi^+(v, \xi) > 0$ *for any* $v \in \Lambda_0$, $\xi \in X^+(v)$.

PROOF. For any continuous function ψ: $\mathbf{R}^+ \to \mathbf{R}$ we define

$$\overline{\psi} = \overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \psi(s)\,ds, \quad \underline{\psi} = \underline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \psi(s)\,ds, \quad \widetilde{\psi} = \underline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \psi^2(s)\,ds.$$

LEMMA 10.1. *Let* ψ: $\mathbf{R}^+ \to \mathbf{R}$ *be a continuous function and let* $C = \sup |\psi(t)| < \infty$.

1) *If* $\psi(t) \leqslant 0$ *and* $\widetilde{\psi} > 0$, *then* $\overline{\psi} < 0$;

2) *if* $\psi(t) \geqslant 0$ *and* $\widetilde{\psi} > 0$, *then* $\underline{\psi} > 0$.

PROOF. It is obvious that $\overline{\psi} \leqslant 0$. On the other hand,

$$\frac{1}{C}(\underline{|\psi|}) = \left|\underline{\left(\frac{1}{C}\psi\right)}\right| \geqslant \widetilde{\left(\frac{1}{C}\psi\right)} = \frac{1}{C^2}\widetilde{\psi} > 0.$$

Therefore, $\overline{\psi} < 0$. Part 2) is proved similarly.

Let us now prove the theorem. Let $v \in \Lambda$, $\xi \in X^-(v)$. We consider the function $\varphi(t) = \|Y_\xi(t)\|^2$, which satisfies the second order differential equation

$$\frac{d^2}{dt^2}\varphi(t) = -K(t)\varphi(t) + \|Y'_\xi(t)\|^2,$$

where $K(t) = K_{Y_\xi(t)\dot\gamma_{d\pi\xi(t)}(t)}(t)$. By Propositions 10.3 and 10.2, $\varphi(t) \neq 0$ and $\frac{d}{dt}\varphi(t) < 0$. We set $z(t) = \left(\frac{d}{dt}\varphi(t)\right)\varphi^{-1}(t)$. The function $z(t)$ satisfies the differential equation

$$\frac{d}{dt}z(t) + z^2(t) - \frac{\|Y'_\xi(t)\|^2}{\varphi(t)} = 0. \tag{10.8}$$

We estimate $|z(t)|$. By Proposition 10.4 and (10.5)

$$\left|\frac{d}{dt}\varphi(t)\right| = \left|\frac{d}{dt}\langle Y_\xi(t),\ Y_\xi(t)\rangle\right| = |2\langle Y_\xi(t),\ Y'_\xi(t)\rangle| =$$
$$= |\langle d\pi \circ df^t\xi,\ K \circ df^t\xi\rangle| \leqslant 2a\,\|d\pi \circ df^t\xi\|^2 = 2a\varphi(t),$$

hence $\sup\limits_{t\geqslant 0} |z(t)| \leqslant 2a$. We integrate (10.8) over $[0, t]$. From what has been said above and the definition of Λ_0 it follows that

$$\varliminf_{t\to\infty}\frac{1}{t}\int_0^t z^2(s)\,ds>0.$$

Hence, by Lemma 10.1,

$$\varlimsup_{t\to\infty}\frac{1}{t}\int_0^t z(s)\,ds<0.$$

Using 1) and 3) in Proposition 10.4 and (10.5) we find that

$$\chi^+(v,\xi)=\varlimsup_{t\to\infty}\frac{1}{t}\log\|df^t\xi\|=\varlimsup_{t\to\infty}\frac{1}{t}\ln\|d\pi\circ df^t\xi\|=$$
$$=\varlimsup_{t\to\infty}\frac{1}{t}\log\|Y_\xi(t)\|=\frac{1}{2}\varlimsup_{t\to\infty}\frac{1}{t}\int_0^t z(s)\,ds<0.$$

This proves 1), and 2) is proved similarly.

10.6. In this subsection we consider two-dimensional manifolds without focal points. For $x\in M$ we denote by $K(x)$ the curvature of M at x. The following assertion was proved by Kramli (see [12]).

THEOREM 10.6. *Suppose that*

$$\int_M K(x)\,d\nu<0. \tag{10.9}$$

Then $\mu(\Lambda_0)>0$.

PROOF. Let $v,w\in SM$ be two orthogonal vectors. It is easy to see that

$$\varlimsup_{t\to\infty}\frac{1}{t}\int_0^t K_{v,w}(s)\,ds=\varlimsup_{t\to\infty}\frac{1}{t}\int_0^t K(\pi(f^s(v)))\,ds.$$

From Birkhoff's theorem it follows that $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t K(\pi(f^s(v)))ds=\Phi(v)$ exists for almost every $v\in SM$; moreover, $\int_{SM}\Phi(v)d\mu(v)=\int_M K(\pi(v))d\nu(\pi(v))$. Hence, by the conditions of the theorem, the set of those vectors v for which $\Phi(v)<0$ has positive measure. This proves the theorem.

By the Gauss–Bonnet formula (see [19]), $\frac{1}{2\pi}\int_M K(x)d\nu(x)$ is equal to the Euler characteristic of M. Hence, (10.9) is equivalent to the fact that the genus of M is greater than 1.

THEOREM 10.7. *A geodesic flow on a two-dimensional compact manifold of genus greater than* 1 *and without focal points is isomorphic to a*

Bernoulli flow.

PROOF. We consider the set Λ_0 defined by (10.7). Since the genus of M is greater than 1, by Theorem 10.6 $\mu(\Lambda_0) > 0$, and by Theorem 10.5 $\mu(\Lambda) > 0$ (see (9.1)). We consider the local stable and unstable manifolds $V^-(v)$ and $V^+(v)$, $v \in \widetilde{\Lambda}$, constructed in §9. We denote by $\varkappa\colon H \to M$ the covering map. In what follows, the sign "$\circ$" over symbols of vectors, sets etc. means that they are taken on the universal Riemannian covering H or in SH. Thus, $\varkappa(\overset{\circ}{v}) = v$, $\varkappa(\overset{\circ}{V}{}^-(\overset{\circ}{v})) = V^-(v)$, $\varkappa(\overset{\circ}{\Lambda}) = \Lambda$.

LEMMA 10.2. $\overset{\circ}{V}{}^-(\overset{\circ}{v}) \subset \overset{\circ}{\mathfrak{S}}{}^-(\overset{\circ}{v})$, $\overset{\circ}{V}{}^-(\overset{\circ}{v}) \subset \overset{\circ}{\mathfrak{S}}{}^+(\overset{\circ}{v})$ *for any* $\overset{\circ}{v} \in \overset{\circ}{\widetilde{\Lambda}}$.

PROOF. Let $\overset{\circ}{w} \in \overset{\circ}{V}{}^-(\overset{\circ}{v})$. By (4.14),

$$\rho(\pi(f^t(\overset{\circ}{v})), \pi(f^t(\overset{\circ}{w}))) \to 0 \quad \text{as} \quad t \to \infty. \tag{10.10}$$

From this it follows, in particular, that $\gamma_{\overset{\circ}{v}}(+\infty) = \gamma_{\overset{\circ}{w}}(+\infty)$. Suppose that $\overset{\circ}{w} \in \overset{\circ}{\mathfrak{S}}{}^-(\overset{\circ}{v})$. We consider the limit sphere $L(\pi(\overset{\circ}{v}), \gamma_{\overset{\circ}{v}}(+\infty))$. Let $\overset{\circ}{z}$ be the point of intersection of $\gamma_{\overset{\circ}{w}}(t)$ with $L(\pi(\overset{\circ}{v}), \gamma_{\overset{\circ}{v}}(+\infty))$. Then by Theorems 10.2 and 10.3,

$$\rho(\pi(f^t(\overset{\circ}{v})), \pi(f^t(\overset{\circ}{w}))) \geqslant \rho(\pi(\overset{\circ}{w}), \overset{\circ}{z}) > 0,$$

which contradicts (10.10). The lemma is now proved.

Since $\dim M = 2$ and the genus of M is greater than 1, by Proposition 10.6 M satisfies the axiom of uniform visibility. As Eberlein has proved (see [36], Theorem 3.7), in this case the geodesic flow f^t is topologically transitive in SM. Therefore, from Lemma 10.2 and Theorem 9.5 it follows that $f^t | \Lambda$ is ergodic. Since M is not homeomorphic to the two-dimensional torus (because its genus is greater than 1), by Arnol'd's theorem (see [1], §23) f^t has no continuous eigenfunctions. From Theorem 9.5 (see also Theorem 7.7) it follows that $\mathfrak{S}^-(v) = W^-(v)$, $\mathfrak{S}^+(v) = W^+(v)$ for almost every $v \in \widetilde{\Lambda}$ (the sets $W^-(v)$ and $W^+(v)$ are defined by (9.6)). Hence, from Theorem 9.8 and what was said above it follows that $f^t | \Lambda$ has a continuous spectrum, consequently, by Theorems 9.6 and 9.7, that it is isomorphic to a Bernoulli flow. It remains to show that $\Lambda = SM (\mathrm{mod}\ 0)$.

We consider a linear element $\overset{\circ}{v} \in \overset{\circ}{\widetilde{\Lambda}}{}^l_{1,s}$ and a segment Δ on the limit sphere $L(\overset{\circ}{x}, \overset{\circ}{p})$, $\overset{\circ}{x} = \pi(\overset{\circ}{v})$, $\overset{\circ}{p} = \gamma_{\overset{\circ}{v}}(+\infty)$. We denote the end-points of this segment by $\pi(\overset{\circ}{w}_1)$ and $\pi(\overset{\circ}{w}_2)$, where $\overset{\circ}{w}_1, \overset{\circ}{w}_2 \in \overset{\circ}{\mathfrak{S}}{}^+(-\overset{\circ}{v})$. We identify the set $H \cup H(\infty)$ with the closed unit circle in $T_{\overset{\circ}{x}}H$ (see Proposition 10.7) and consider the set $\overset{\circ}{A}_t \subset H$, $t > 0$, bounded by Δ, segments of the geodesics $\gamma_{\overset{\circ}{w}_1}(s)$ and $\gamma_{\overset{\circ}{w}_2}(s)$, $0 \leqslant s \leqslant t$, and the segment $\Delta_t \subset L(\gamma_{\overset{\circ}{v}}(t), \overset{\circ}{p})$ with the end-points $\gamma_{\overset{\circ}{w}_1}(t)$ and $\gamma_{\overset{\circ}{w}_2}(t)$. If we choose Δ sufficiently small so that $\Delta \subset \pi(\overset{\circ}{V}{}^-(\overset{\circ}{v}))$, then, as follows from Theorems 10.4 and 4.3, any linear element $\overset{\circ}{w} \in \overset{\circ}{\mathfrak{S}}{}^+(-\overset{\circ}{v})$ for which $\pi(\overset{\circ}{w}) \in \Delta$ is an element of uniqueness. Therefore, the geodesics $\gamma_{\mathring{w}_1}(t)$ and $\gamma_{\mathring{w}_2}(t)$ diverge. Hence, for sufficiently large t (depending on the length of Δ) the set $\mathring{A}_t$ contains a fundamental domain in H. Let $\Delta_\infty \subset H(\infty)$ be the segment with the end-points $\gamma_{\mathring{w}_1}(+\infty)$ and $\gamma_{\mathring{w}_2}(+\infty)$. We put $\mathring{B}_t(\mathring{v}) = \{ \mathring{w} \in SH\colon \pi(\mathring{w}) \in \mathring{A}_t,\ \gamma_{\mathring{w}}(+\infty) \in \Delta_\infty\}$. We claim that for any $t \in [0, \infty)$

$$\mathring{B}_t(\mathring{v}) \subset \mathring{\Lambda}\,(\mathrm{mod}\ 0). \tag{10.11}$$

We fix a $t > 0$ and set

$$\mathring{C}_t(\mathring{v}) = \bigcup_{-t\leq s\leq t}\ \bigcup_{\mathring{w}\in \mathring{V}^+(-\mathring{v})} f^s(\mathring{w}),\quad \mathring{D}_t(\mathring{v}) = \bigcup_{\mathring{w}\in\mathring{C}_t(\mathring{v})} \mathring{\mathfrak{S}}^-(\mathring{w}).$$

It is easy to see that $\mathring{B}_t(\mathring{v}) \subset \mathring{B}_\infty(\mathring{v}) \subset D_\infty(v)$.

LEMMA 10.3. $\mathring{D}_\infty(\mathring{v}) \subset \Lambda\,(\mathrm{mod}\ 0)$ *for almost all* $\mathring{v} \in \mathring{\Lambda}^l_{1,s}$.

PROOF. From Theorem 9.5 (see also Theorem 7.7) it follows that $\varkappa(\mathring{D}_t(\mathring{v})) = \bigcup_{w\in\varkappa(\mathring{C}_t(\mathring{v}))} W^-(v) \subset \Lambda$ mod 0. Hence the required assertion follows.

We choose an open ball $U \subset H$ and an open segment $\Delta \subset H(\infty)$. We consider the open set $\mathring{R} = \{\mathring{w} \in SH\colon \pi(\mathring{w}) \in U,\ \gamma_{\mathring{w}}(+\infty) \in \Delta_\infty\}$. From results of [36] it follows that if t is sufficiently large, then for any $\mathring{v} \in SH$ there is an isometry φ of H such that $\varphi(\mathring{R}) \subset B(\mathring{v})$. Hence, on the basis of Lemma 10.3 and (10.11), $\mathring{R} \subset \mathring{\Lambda}\,(\mathrm{mod}\ 0)$, and the theorem is proved.

Let

$A = \{v \in SM\colon$ no two points on the geodesic $\gamma_v(t)$ are focal$\}$.

THEOREM 10.8. *Suppose that* $\dim M = 2$, *that the genus of* M *is greater than* 1, *and that the Riemannian metric on* M *does not have conjugate points. If* $\nu(A \cap \Lambda_0) > 0$, *then a geodesic flow is isomorphic to a Bernoulli flow. Moreover,* $\Lambda_0 = SM\,(\mathrm{mod}\ 0)$ *and* $A = SM\,(\mathrm{mod}\ 0)$.

PROOF. Reasoning as in the proof of Theorem 10.7, it is easy to show that $f^t|A \cap \Lambda$ is ergodic, so that $A = \Lambda\,(\mathrm{mod}\ 0)$, and Λ is open mod 0. Next we remark that for almost every $v \in \Lambda$ almost every point $w \in V^-(v)$ lies in A. This permits us to repeat the final part of the proof of Theorem 10.7 verbatim.

REMARK 10.1. Theorem 10.7 can be generalized to the n-dimensional case if we assume, in addition, that $\nu(\Lambda_0) > 0$ and that M satisfies the axiom of uniform visibility (see [24]).

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