PERIODIC ORBITS FOR HYPERBOLIC FLOWS.

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We shall study the periodic orbits of a class of flows defined by S. Smale [23]. Let $\psi_t\colon M \to M$ be an Axiom A flow (see Section 1), $X \subset M$ a basic set of ψ_t, and $\phi_t = \psi_t \mid X$. Let $\nu_\epsilon(t)$ be the number of closed orbits with a period $\tau \leqq t$ and $\tau \in [t - \epsilon, t + \epsilon]$ respectively. Our main results on the flow $\Phi = \{\phi_t\}$ are:

A. Either X is a point, (ϕ_t, X) is the suspension of a homeomorphism, or (ϕ_t, X) is C-dense (i.e. every (strong) unstable set is dense in X).

B. The topological entropy $h(\Phi) = h(\phi_1)$ satisfies

$$h(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \nu(t).$$

If X is neither a point nor a single closed orbit, then $h(\Phi) > 0$. If X is C-dense, then, for each $\epsilon > 0$,

$$h(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \nu_\epsilon(t).$$

C. The periodic orbits of Φ are equidistributed (as the period approaches ∞) with respect to a Φ-invariant Borel measure μ_Φ.

D. (Φ, μ_Φ) is ergodic. The measure theoretic entropy $h_{\mu_\Phi}(\phi_t) = h(\phi_t)$. If X is C-dense, then (Φ, μ_Φ) is weak mixing.

The main fact behind these results is the specification theorem 3.8; it produces for us a periodic orbit when we specify in advance where it should approximately be at certain times. All the results proved here are analogous of ones known for Axiom A diffeomorphisms [4]. In Section 6 we shall discuss some background material and further problems. This paper benefited from talks with P. Walters and C. Bones.

1. Definitions. Let $\Psi = \{\psi_t\colon M \to M\}_{t \in R}$ be a differentiable flow on a compact Riemannian manifold. The *nonwandering* set is

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$$\Omega = \{x \in M\colon \text{for every neighborhood } U \text{ of } x \text{ and every } T > 0,\ U \cap \bigcup_{t \geqq T} \psi_t(U) \neq \phi\}.$$

One notices that the closed orbits of Ψ are in Ω, Ω is closed, and $\psi_t(\Omega) = \Omega$. A ψ-invariant subet Λ is *hyperbolic* for Ψ if the tangent bundle restricted to Λ, $T_\Lambda(M)$, can be written as the Whitney sum of three $D\psi_t$-invariant subbundles $T_\Lambda(M) = E + E^s + E^u$ where E is the 1-dimensional bundle tangent to the flow and there are constants c, $\lambda > 0$ so that

(a) $\| D\psi_t(v) \| \leqq ce^{-\lambda t} \| v \|$ for $v \in S^s$, $t \geqq 0$

and

(b) $\| D\psi_{-t}(u) \| \leqq ce^{-\lambda t} \| u \|$ for $u \in E^u$, $t \geqq 0$.

(1.1) *Definition* [23, p. 803]. Ψ is an *Axiom A flow* if $\Omega = F \cup \Lambda$ where

(a) F is the set of fixed points of Ψ and is finite; each fixed point is hyperbolic.

(b) Λ is the closure of the set of closed orbits and is hyperbolic for Ψ.

and

(c) $F \cap \Lambda = \phi$.

We recall the

(1.2) *Spectral Decomposition Theorem* [23, p. 803], [16]. If ψ_t satisfies Axiom A, then Ω can be written uniquely as a disjoint union $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ where each Ω_i is closed, invariant and each $\psi_t\colon \Omega_i \to \Omega_i$ is topologically transitive.

Standing Hypothesis. For the whole of this paper X shall denote one of the basic sets Ω_i which is larger than a single point (e.e. $X \subset \Lambda$); $\phi_t = \psi_t \mid X$.

$\phi_t\colon X \to X$ is studied through various stable and unstable sets: for $x \in X$ and $\epsilon > 0$ let

$$\begin{aligned} W^s(x) &= \{y \in X\colon d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to +\infty\} \\ W^s_\epsilon(x) &= \{y \in W^s(x) : d(\phi_t(x), \phi_t(y)) \leqq \epsilon \text{ for all } t \geqq 0\} \\ W^u(x) &= \{y \in X\colon d(\phi_{-t}(x), \phi_{-t}(y)) \to 0 \text{ as } t \to +\infty\} \\ W^u_\epsilon(x) &= \{y \in W^u(x) : d(\phi_{-t}(x), \phi_{-t}(y)) \leqq \epsilon \text{ for all } t \geqq 0\}. \end{aligned}$$

Our notation differs somewhat from that of other papers. First these sets are all subsets of X; in other places one looks at all $y \in M$ rather than just

$y \in X$. Second, our $W^s(x)$ goes with the strong table set $W^{ss}(x)$ of [10] (there $W^s(x)$ refers to a larger set).

(1.3) PROPOSITION [10]. *There are constants* c, $\lambda > 0$ *so that, for small* ϵ,

(a) $d(\phi_t(x), \phi_t(y)) \leqq ce^{-\lambda t} d(x,y)$ *when* $y \in W^s_\epsilon(x)$ *and* $t \geqq 0$

and

(b) $d(\phi_{-t}(x), \phi_{-t}(y)) \leqq ce^{-\lambda t} d(x,y)$ *when* $y \in W^u_\epsilon(x)$ *and* $t \geqq 0$.

(1.4) *Canonical Coordinates* [16], [23]. There are δ, $\gamma > 0$ for which the following is true: If $x, y \in X$ and $d(x,y) \leqq \delta$, then there is a unique $v = v(x,y)$ with $|v| \leqq \gamma$ so that

$$W^s_\gamma(\phi_v(x)) \cap W^u_\gamma(y) \neq \phi.$$

This set consists of a single point, which we denote $\langle x, y \rangle$. The maps v and $\langle\ ,\ \rangle$ are continuous on the set $\{(x,y) : d(x,y) \leqq \delta\} \subset X \times X$.

Proof [16]. Notice that we are always inside X, even though the proof of this statement is not.

(1.5) LEMMA. *For any* $\eta > 0$ *there is an* $\alpha > 0$ *so that* $|v(x,y)| \leqq \eta$ *and*

$$\langle x, y \rangle \in W^s_\eta(\phi_v(x)) \cap W^u_\eta(y)$$

whenever $d(x,y) \leqq \alpha$.

Proof. Now $\langle x, x \rangle = x$ and $v(x,x) = 0$. By uniform continuity, given any $\beta > 0$, for $d(x,y)$ small enough

$$d(\langle x, y \rangle, x), \quad d(\langle x, y \rangle, y) \leqq \beta$$

and $v = v(x,y)$ is so small that

$$d(\phi_v(x), x) \leqq \beta.$$

Then $d(\langle x, y \rangle, \phi_v(x)) \leqq 2\beta$. Since $\langle x, y \rangle \in W^s_\gamma(\phi_v(x))$, for $t \geqq 0$:

$$d(\phi_t \langle x, y \rangle, \phi_t(y)) \leqq ce^{-\lambda t}(2\beta).$$

Take $\beta < \eta/2c$; then $\langle x, y \rangle \in W^s_\eta(\phi_v(x))$. As $\langle x, y \rangle \in W^u_\gamma(y)$, for $t \geqq 0$

$$d(\phi_{-t}\langle x, y \rangle, \phi_{-t}(y)) \leqq ce^{-\lambda t}\beta < \eta$$

and so $\langle x, y \rangle \in W^u_\eta(y)$.

We prove an "expansive-type" statement.

(1.6) PROPOSITION. *For each $\eta > 0$ there is an $\alpha > 0$ with the following property. If, $x, y \in X$ and $s\colon R \to R$ continuous with $s(0) = 0$ satisfy*

$$d(\phi_{t+s(t)}(y), \phi_t(x)) \leqq \alpha \text{ for } |t| \leqq L,$$

then $|s(t)| \leqq 3\eta$ for all $|t| \leqq L$, $|v(x,y)| \leqq \eta$ and

$$d(\phi_s(y), \phi_s\phi_v(x)) \leqq c\gamma e^{-\lambda(L-|s|)}$$

for any $|s| \leqq L$. In particular

$$d(y, \phi_v(x)) \leqq c\gamma e^{-\lambda L}.$$

Proof. We may assume η is so small that $\eta < \gamma/8$ and

$$\sup\{d(\phi_u(x), x) : x \in X, |u| \leqq 4\eta\} \leqq \gamma/8.$$

Let $\alpha > 0$ be as in Lemma 1.5. Consider x, y as above. Since $s(0) = 0$, $d(x,y) \leqq \alpha$. Define $w = \langle x, y\rangle = W^s{}_\eta(\phi_v(x)) \cap W^u{}_\eta(y)$ and the sets

$$A = \{t \in [0,L] : |s(t)| \geqq 3\eta \text{ or } d(\phi_t(y), \phi_t(w)) \geqq \tfrac{1}{2}\gamma\}$$
$$B = \{t \in [0,L] : |s(-t)| \geqq 3\eta \text{ or } d(\phi_{-t+v}(x), \phi_{-t}(w)) \geqq \tfrac{1}{2}\gamma\}.$$

Suppose $A \neq \phi$. Let $t_1 = \inf A \in A$. Then $d(\phi_{t_1-u}(y), \phi_{t_1-u}(w)) \leqq \frac{1}{2}\gamma$ for all $u \geqq 0$. Since $|s(t_1)| \leqq 3\eta$ (remember $s(0) = 0$ and s is continuous), using the triangule inequality we get

$$d(\phi_{t_1+s(t_1)-u}(y), \phi_{t_1+s(t_1)-u}(w)) < \frac{3\gamma}{4} \text{ for } u \geqq 0;$$

hence $\phi_{t_1+s(t_1)}(w) \in W^u{}_\gamma(\phi_{t_1+s(t_1)}(y)$. Now $d(\phi_u(w), \phi_{v+u})x)) \leqq \eta < \frac{\gamma}{8}$ for $u \geqq 0$. Since $|s(t_1)| \leqq 3\eta$, the triangle inequality gives us (let $u = t_1 + p$)

$$d(\phi_{t_1+s(t_1)+p}(w), \phi_{t_1+s(t_1)+v+p}(x)) \leqq \frac{3\gamma}{8} \text{ for } p \geqq 0;$$

hence $\phi_{t_1+s(t_1)}(w) \in W^s{}_\gamma(\phi_{s(t_1)+v}(\phi_{t_1}(x)))$. We have shown

$$\phi_{t_1+s(t_1)}(w) \in W^s{}_\gamma(\phi_{s(t_1)+v}(\phi_{t_1}(x))) \cap W^u(\phi_{t_1+s(t_1)}(y)).$$

Since $|s(t_1) + v| \leqq |s(t_1)| + |v| \leqq 4\eta < \gamma$ and $d(\phi_{t_1+s(t_1)}(y), \phi_{t_1}(x)) \leqq \alpha$, we have

$$v(\phi_{t_1}(x), \phi_{t_1+s(t_1)}(y)) = s(t_1) + v(x,y)$$

and

$$\phi_{t_1+s(t_1)}(w) = \langle \phi_{t_1}(x), \phi_{t_1+s(t_1)}(y)\rangle.$$

By Lemma 1.5, $|s(t_1)+v| \leqq \eta$ and $d(\phi_{t_1+s(t_1)}(w), \phi_{t_1+s(t_1)}(y)) \leqq \eta$. Because $|s(t_1)| \leqq 2\eta$ we get $d(\phi_{t_1}(w), \phi_{t_1}(y)) \leqq \eta + 2(\frac{\gamma}{8}) \leqq \frac{3\gamma}{8}$. Also

$$|s(t_1)| \leqq |s(t_1)+v| + |v| \leqq 2\eta.$$

These statements contradict $t_1 \in A$. Hence $A = \phi$.

One similarly shows $B = \phi$. Now $A = \phi$ implies $\phi_L(w) \in W^u{}_{\frac{1}{2}\gamma}(\phi_L(y))$. For $|s| \leqq L$ 1.3 gives us

$$d(\phi_s(w), \phi_s(y)) \leqq \tfrac{1}{2}\gamma c e^{-\lambda(L-|s|)}$$

$B = \phi$ implies $\phi_{-L}(w) \in W^s{}_{\frac{1}{2}\gamma}(\phi_{-L+v}(x))$ and

$$d(\phi_s(w), \phi_{s+v}(x)) \leqq \tfrac{1}{2}\gamma c e^{-\lambda(L-|s|)}$$

These two inequalities combine:

$$d(\phi_s(y), \phi_{s+v}(x)) \leqq \gamma c e^{-\lambda(L-|s|)}.$$

$A \cup B = \phi$ also gives $|s(t)| \leqq 3\eta$ for $t \in [-L, L]$.

The above proposition implies that Φ is *flow-expansive* [3]: for every $\eta > 0$ there is an $\alpha > 0$ so that $y = \phi_v(x)$ for some $|v| \leqq \eta$ whenever $s: R \to R$ is continuous with $s(0) = 0$ and $d(\phi_{t+s(t)}(y), \phi_t(x)) \leqq \alpha$ for all $t \in R$. This property in turn implies ϕ is *entropy-expansive* [8].

Statements similar to 1.6 or flow-expansiveness have occurred in [1], [10], [17], and [26]. This section was helped by conversations with M. Hirsch, C. Pugh, and M. Shub and P. Walters.

We fix some notation: CO is the set of all periodic orbits of Φ, $CO(t)$ those with t a period and $CO_\epsilon(t)$ those with some period in the interval $[t-\epsilon, t+\epsilon]$. $CO^*(t)$ and $CO^*{}_\epsilon(t)$ stands for the points lying on orbits in these sets. For reference we state some easy facts:

(1.7) LEMMA. (a) *If* $x \in W^u{}_\epsilon(z)$ *then* $\phi_{-t}(x) \in W^u{}_{c\epsilon e^{-\lambda t}}(\phi_{-t}(z))$ *for* $t \geqq 0$.

(b) $\phi_t W^u(p) = W^u(\phi_t(p))$

(c) *For any* $Y \subset V$ *let* $W^u{}_\epsilon(Y) = \bigcup_{x \in Y} W^u{}_\delta(x)$. *Then*

$$W^u{}_\delta W^u{}_\epsilon(x) \subset W^u{}_{\delta+\epsilon}(x).$$

2. An approximation theorem. In this section we look for points whose orbits are given approximately in advance.

(2.1) *Definition.* (T, Γ) is an *L-specification if*

(a) $\Gamma = \{x_i\}^{+\infty}_{i=-\infty}$ where $x_i \in X$

and

(b) $T = \{t_i\}^{\infty}_{i=-\infty}$ where $t_i \in R$ satisfy $t_i - t_{i-1} \geqq L$ for every $i \in Z$.

Furthermore, (T, Γ) is called δ-*possible* if

$$d(\phi_{t_i}(x_i), \phi_{t_i}(x_{i-1})) \leqq \delta$$

for all i.

Notice that, if $x_i = x$ for all $i \in Z$, then (T, Γ) is δ-possible for any $\delta > 0$. If $s: R \to R$ we denote

$$U_\epsilon(s, T, \Gamma) = \{y \in X : d(\phi_{t+s(t)}(y), \phi_t(x_i)) \leqq \epsilon \text{ for } t \in (t_i, t_{i+1}), i \in Z\}$$

$$STEP_\epsilon(T) = \{s : s \text{ is constant on } (t_i, t_{i+1}), s(t_i) = s(t_i + 0) \text{ or } s(t_i - 0), |s(t_0)| \leqq \epsilon \text{ and } |s(t_i + 0) - s(t_i - 0)| \leqq \epsilon\}$$

$$U^*_\epsilon(T, \Gamma) = \cup \{U_\epsilon(s, T, \Gamma) : s \in STEP_\epsilon(T)\}.$$

(2.2) APPROXIMATION THEOREM. *For $\epsilon > 0$ there are $L, \delta > 0$ so that $U^*_\epsilon(T, \Gamma) \neq \phi$ whenever (T, Γ) is a δ-possible L-specification.*

Proof. Let $\delta_1 > 0$ be a number to be determintd later; choose $\delta > 0$ small enough so that $\delta < \delta_1$ and

$$W^u_{\delta_1}(\phi_t(x)) \cap W^s_{\delta_1}(y) \neq \phi$$

for some $|t| \leqq \delta_1$ whenever $d(x, y) \leqq 2\delta$ (see 1.5). Pick L so that $L^* = L - \delta_1$ satisfies (c and λ as in 1.3)

$$ce^{-\lambda L^*}\delta_1 < \delta$$

and

$$\sum_{k=1}^{\infty} ce^{-\lambda L^* k} = \frac{ce^{-\lambda L^*}}{1 - e^{-\lambda L^*}} < 1.$$

Suppose that (T, Γ) is a δ-possible L-specification. We may assume the indexing is such that $0 \in [t_0, t_1)$. Let $z_0 = x_0$ and define z_n, $n > 0$, recursively as follows. Having z_n with

$$d(\phi_{t_{n+1}-t_n}(z_n), \phi_{t_{n+1}}(x_{n+1})) \leqq 2\delta,$$

choose

$$z_{n+1} \in W^u_{\delta_1}(\phi_{t_{n+1}-t_n+\epsilon_{n+1}}(z_n)) \cap W^s_{\delta_1}(\phi_{t_{n+1}}(x_{n+1}))$$

with $|\epsilon_{n+1}| < \delta_1$. Then, by 1.3,

$$d(\phi_{t_{n+2}-t_{n+1}}(z_{n+1})), \phi_{t_{n+2}}(x_{n+1})) \leqq \delta_1 c e^{-\lambda(t_{n+2}-t_{n+1})} \leqq \delta_1 c e^{-\lambda L} < \delta.$$

Since (T, Γ) is δ-possible,

$$d(\phi_{t_{n+2}}(x_{n+1}), \phi_{t_{n+2}}(x_{n+2})) \leqq \delta$$

and so

$$d(\phi_{t_{n+2}-t_{n+1}}(z_{n+1}), \phi_{t_{n+2}}(x_{n+2})) \leqq 2\delta.$$

Hence we can proceed to z_{n+2}, and so forth.

Let $r_{n+1} = t_{n+1} - t_n + \epsilon_{n+1} \geqq L - \delta_1 \geqq L^*$. Then, by 1.7(a),

$$\phi_{-r_{n+1}}(z_{n+1}) \in W^u{}_{\delta_1 c e^{-\lambda r_{n+1}}}(z_n).$$

Using 1.7(c) as well,

$$\begin{aligned} \phi_{-r_{n+1}-r_n}(z_{n+1}) &\in W^u{}_{\delta_1 c e-\lambda(r_{n+1}+r_{n+1}+r_n)}(\phi_{-r_n}(z_n)) \\ &\subset W^u{}_{\delta_1 c e-\lambda(r_{n+1}+r_n)} W^u{}_{\delta_1 c e-\lambda r_n}(z_{n-1}) \\ &\subset W^u{}_{\delta_1 c(e-\lambda(r_{n+1}+r_n)+e-\lambda r_n)}(z_{n-1}) \end{aligned}$$

Inductively we get, for $0 \leqq j \leqq n$,

$$\begin{aligned} u_{n,j} &= \phi_{-(r_n+r_{n-1}+\cdots+r_{j+1})}(z_n) \\ &\in W^u{}_{\delta_1 c(e-\lambda(r_n+\cdots+r_j+)+\cdots+e-\lambda r_{j+1})}(z_j) \\ &\subset W^u{}_{\delta_1 c \sum_{k=1}^{\infty} e^{-\lambda L^* k}}(z_j) \subset W^u{}_{\delta_1}(z_j). \end{aligned}$$

LEMMA. *For fixed j, $v_j = \lim_{n\to\infty} u_{n,j}$ exists. Furthermore, $v_j \in W^u{}_{\delta_1}(z_j)$ and $v_{j+1} = \phi_{r_{j+1}}(v_j)$.*

Proof. For $n \geqq j + k$ we have $u_{n,j+k} \in W^u{}_{\delta_1}(z_{j+k})$ and

$$d(u_{n,j}, u_{j+k,j}) = d(\phi_{-r_{j+k}\cdots-r_{j+1}}(u_{n,j+k}), \phi_{-r_{j+k}-\cdots-r_{j+1}}(z_{j+k}) \leqq c\delta_1 e^{-\lambda L^* k}.$$

By the triangle inequality, for m, $n \geqq j + k$,

$$d(u_{n,j}, u_{m,j}) \leqq 2c\delta_1 e^{-\lambda L^* k}.$$

Letting $k \to \infty$, we see the sequence $\{u_{n,j}\}^\infty{}_{n=j}$ is Cauchy. As it is contained in the compact set $W^u{}_{\delta_1}(z_j)$, it has a limit v_j in that set. Since

$$u_{n,j} = \phi_{-r_{j+1}}(u_{n,j+1}),$$

by continuity we have $v_j = \phi_{-r_{j+1}}(v_{j+1})$, Define the function s on $[t_0, +\infty)$ by $s([t_0, t_1)) = 0$ and

$$s([t_i, t_{i+1})) = \epsilon_1 + \cdots + \epsilon_i.$$

Let $y_1 = \phi_{-t_0}(v_0)$. For $t \in (t_i, t_{i+1})$,

$$\begin{aligned}\phi_{t+s(t)}(y_1) &= \phi_{(t-t_i)+(t_i-t_{i-1}+\epsilon_i)+\cdots+(t_1-t_0+\epsilon_1)+t_0}(y_1)\\ &= \phi_{(t-t_i)+r_i+\cdots+r_1}(v_0) = \phi_{t-t_i}(v_i)\\ &= \phi_{t-t_i-r_{i+1}}(v_{i+1}).\end{aligned}$$

Hence

$$\begin{aligned}d(\phi_{t+s(t)}(y_1), \phi_t(x_i)) &\leqq d(\phi_{t-t_i-r_{i+1}}(v_{i+1}), \phi_{t-t_i-r_{i+1}}(z_{i+1}))\\ &+ d(\phi_{t-t_i-r_{i+1}}(z_{i+1}), \phi_{t-t_i}(z_i)) + d(\phi_{t-t_i}(z_i), \phi_t(x_i)).\end{aligned}$$

Now $t - t_i - r_{i+1} = t - t_{i+1} - \epsilon_{i+1} \leqq \delta_1$ since $t \leqq t_{i+1}$ and $|\epsilon_{i+1}| < \delta_1$. Let $\delta_2 > 0$ be a number yet to be determined. Make $\delta_1 < \frac{\delta_2}{3}$ small enough to that

$$x, y \in X, d(x, y) \leqq \delta_1, |u| \leqq \delta_1 \Rightarrow d(\phi_u(x), \phi_u(y)) \leqq \frac{\delta_2}{3}.$$

If $t - t_1 - r_{i+1} \leqq 0$, then $v_{i+1} \in W^u{}_{\delta_1}(z_{i+1})$, implies

$$d(\phi_{t-t_i-r_{i+1}}(v_{i+1}), \phi_{t-t_i-r_{i+1}}(z_{i+1})) \leqq \delta_1 \leqq \frac{\delta_2}{3}.$$

If $0 \leqq t - t_i - r_{i+1} \leqq \delta_1$, then again this term is $\leqq \frac{\delta_2}{3}$. As $z_i \in W^s{}_{\delta_1}(\phi_{t_i}(x_i))$, the last term is $\leqq \delta_1$. Finally, consider the second term above. Recall that $z_{i+1} \in W^u{}_{\delta_1}(\phi_{r_{i+1}}(z_i))$. As $t - t_i - r_{i+1} \leqq \delta_1$, we get

$$d(\phi_{t-t_i-r_{i+1}}(z_{i+1}), \phi_{t-t_i-r_{i+1}+r_{i+1}}(z_i)) \leqq \frac{\delta_2}{3}.$$

Thus $d(\phi_{t+s(t)}(y_1), \phi_t)) \leqq \delta_2$ for $t \in (t_i, t_{i+1})$, $i \geqq 0$

Repeating the construction above on the flow $\beta_t = \phi_{-t}$ we can find y_2 and extend s to all of R, $s \in STEP_{\delta_1}(T)$ so that $d(\phi_{t+s(t)}(y_2), \phi_t(x_i)) \leqq \delta_2$ for all $t \in (t_i, t_{i+1}), i \leqq 0$. Now

$$d(\phi_{t_0}(y_1). \phi_{t_0}(y_2)) = \lim_{t \to t^+_0} d(\phi_t(y_1), \phi_t(y_2)) \leqq 2\delta_2.$$

Choose $\eta > 0$ so small that $\eta < \frac{\epsilon}{8}$ and

$$\sup\{d(x, \phi_t(x)) : x \in X, |t| \leqq \eta\} < \frac{\epsilon}{4}.$$

Make sure $\delta_2 < \frac{\epsilon}{4}$ is so small that

$$W^u{}_\eta(\phi_u(x)) \cap W^s{}_\eta(y) \neq \phi$$

for some $|u| \leqq \eta$ whenever $d(x,y) \leqq 2\delta$ (see 1.5). Let

$$y' \in W^u{}_\eta(\phi_u(\phi_{t_0}(y_2))) \cap W^s{}_\eta(\phi_{t_0}(y_1)).$$

Set $y = \phi_{-t_0}(y')$ and define $s' \in STEP_{\delta_1+\eta}(T) \subset STEP_\epsilon(T)$ by

$$s'(t) = \begin{cases} s(t) & \text{for } t \geqq t_0 \\ s(t) - u & \text{for } t < t_0. \end{cases}$$

We claim $y \in U_\epsilon(s', T, \Gamma)$. For $t \in (t_i, t_{i+1})$, with $i \geqq 0$, we have $t + s(t) \geqq t_0$ and

$$\begin{aligned} d(\phi_{t+s'(t)}(y), \phi_t(x_i)) &\leqq d(\phi_{t+s(t)}(y), \phi_{t+s(t)}(y_1)) \\ &\quad + d(\phi_{t+s(t)}(y_1), \phi_t(x_i)) \\ &\leqq \eta + \delta_2 < \epsilon. \end{aligned}$$

For $t \in (t_i, t_{i+1})$ with $i < 0$,

$$\begin{aligned} d(\phi_{t+s'(t)}(y), \phi_t(x_i)) &\leqq d(\phi_{t+s'(t)}(y), \phi_{t+s(t)}(y)) \\ &\quad + d(\phi_{t+s(t)}(y), \phi_{t+s(t)}(y_2)) + d(\phi_{t+s(t)}(y_2), \phi_t(x_i)) \\ &\leqq \eta + (\eta + \tfrac{\epsilon}{2}) + \delta_2 < \epsilon. \end{aligned}$$

(2.3) LEMMA. *For any $\beta > 0$ one can find $\epsilon > 0$ so that the following is true: if $y_1, y_2 \in U^*{}_\epsilon(T, \Gamma)$ where (T, Γ) is an L-specification, and $\frac{\epsilon}{L} < 1$, then $y_1 = \phi_\alpha(y_2)$ for some $|\alpha| \leqq \beta$.*

Proof. Let $y_k \in U_\epsilon(s_k, T, \Gamma)$ with $s_k \in STEP_\epsilon(T)$, $k = 1, 2$. Define $s^*{}_k\left(\frac{t_i + t_{i+1}}{2}\right) = s_k\left(\frac{t_i + t_{i+1}}{2}\right)$ and extend $s^*{}_k$ linearly between these points. One sees that $s^*{}_k$ has Lipschitz constant at most $\frac{\epsilon}{L}$ and that $|s_k(t) - s^*{}_k(t)| \leqq \epsilon$. Since $\frac{\epsilon}{L} < 1$, $g_k(t) = t + s^*{}_k(t)$ is a homsomorphism of R onto itself

Let $\epsilon' = \sup\{d(\phi_u(x), x) : x \in X, |u| \leqq \epsilon\}$. Then, for $t \in (t_i, t_{i+1})$

$$\begin{aligned} d(\phi_{t+s^*_1(t)}(y_1), \phi_{t+s^*_2(t)}(y_2)) &\leqq d(\phi_{t+s^*_1(t)}(y_1), \phi_{t+s_1(t)}(y_1)) \\ &\quad + d(\phi_{t+s_1(t)}(y_1), \phi_t(x_i)) + d(\phi_t(x_i), \phi_{t+s_2(t)}(y_2)) \\ &\quad + d(\phi_{t+s_2(t)}(y_2), \phi_{t+s^*_2(t)}(y_2)) \\ &\leqq 2\epsilon + 2\epsilon'. \end{aligned}$$

For $w \in R$, define $s'(w)$ by

$$w + s'(w) = g^{-1}{}_2(w) + s^*{}_1(g^{-1}(w)).$$

Then

$$d(\phi_{w+s'(w)}(y_1), \phi_w(y_2)) \leqq 2\epsilon + 2\epsilon'.$$

Consider $w_0 = g_2(t_0) = t_0 + s^*{}_2(t_0)$. Then $w_0 + s'(w_0) = t_0 + s^*{}_1(t_0)$ and

$$\begin{aligned} |s'(w_0)| &\leqq |t_0 - w_0| + |s^*{}_1(t_0)| \\ &\leqq |s^*{}_2(t_0.| + |s^*{}_1(t_0)| \leqq 4\epsilon. \end{aligned}$$

Set $y'_1 = \phi_{w_0+s'(w_0)}(y_1), y'_2 = \phi_{w_0}(y_2)$ and $s''(u) = s'(w_0 + u) - s'(w_0)$. Then

$$d(\phi_{u+s''(u)}(y'_1), \phi_u(y'_2)) = d(\phi_{w_0+u+s'(w_0+u)}(y_1), \phi_{w_0+u}(y_2)) \leqq 2\epsilon + 2\epsilon'.$$

Since $\epsilon' \to 0$ as $\epsilon \to 0$ for small ϵ, flow expansiveness gives us

$$y'_1 = \phi_{\alpha'}(y'_2) \text{ with } |\alpha'| \leqq \tfrac{1}{2}\beta.$$

Then $y_1 = \phi_{-s'(w^0)+\alpha'}(y_2)$. As $|s'(w_0)| \leqq 4\epsilon$ we get our result for small enough ϵ.

(2.4) Closed Orbit Theorem. *For any $\beta > 0$ there are δ, $L > 0$ for which the following is true: if $d(\phi_r(x), x) \leqq \delta$ and $r \geqq L$, then there are $y \in X$ and r' so that $\phi_{r'}(y) = y$, $|r' - r| \leqq \beta$ and*

$$d(\phi_t(y), \phi_t(x)) \leqq \beta \text{ for } 0 \leqq t \leqq r.$$

Proof. Let $t_i = ri$ and $x_i = \phi_{-t_i}(x)$. Then $d(\phi_{t_{i+1}}(x_i), \phi_{t_{i+1}}(x_{i+1})) = d(\phi_r(x), x) \leqq \delta$; (T, Γ) is a δ-possible L-specification. Let $\epsilon < \beta$ be as in 2.3. Pick δ, L as in 2.2 with L big enough so that $\dfrac{\epsilon}{L} < 1$.

Let $y \in U_\epsilon(s, T, \Gamma)$ by 2.2; the proof there shows we may take $s(0) = 0$. For $t \in (t_i, t_{i+1})$

$$\epsilon \geqq d(\phi_{t+r+s(t+r)}(y), \phi_{t+r}(x_{i+1})) = d(\phi_{t+s(t+r)}\phi_r(y), \phi_t(x_i)).$$

Now $s_1(t) = s(t+r)$ defines $s_1 \in STEP_\epsilon(T)$ and

$$\phi_r(\mathrm{y}) \in U_\epsilon(s_1, T, \Gamma) \subset U^*{}_\epsilon(T, \Gamma).$$

By 2.3 $y = \phi_{r+\alpha}(y)$ for some $|\alpha| \leqq \beta$. The last statement follows from $y \in U_\epsilon(s, T, \Gamma)$, $s(0) = 0$.

3. *C*-density. Suppose $f\colon Y \to Y$ is a homeomorphism of a compact space and $\tau > 0$. Let $\mathrm{Sus}_\tau(f)$ be the space obtained from $Y \times [0,\tau]$ by identifying (x,τ) with $(f(x),0)$ for each $x \in Y$. There is a natural flow $S_t(f)$ induced on $\mathrm{Sus}_\tau(f)$ by projecting the partial flow

$$\alpha_t(x,s) = (x, t+s) \text{ for } s, t \geqq 0,\ t+s \leqq \tau$$

onto $\mathrm{Sus}_\tau(f)$ (see [23, p. 797]). The flow $(S_t(f), \mathrm{Sus}_\tau(f))$ is called the *time τ suspension* of f.

(3.1) *Definition.* We say that X is *C-dense* if $X \subset \Lambda$ (i.e. X is not a single fixed point) and $X = \overline{W^u(p)}$ for each $p \in CO^*$.

(3.2) THEOREM. *Exactly one of the following is true:*

(a) *X is a fixed point.*

(b) *For some τ, Φ is the time τ suspension of a homoemorphism satisfying* Axiom A* (see [4]).

(c) *X is C-dense.*

Remark. For Anosov flows with invariant Lebesgue measure this was first proved in [1]. In this case $f\colon Y \to Y$ is an Anosov diffeomorphism. J. Plante [15] has proved this without assuming the measure.

Proof. The three conditions are disjoint. Notice that, if $\pi\colon Y \times [0,\tau] \to \mathrm{Sus}_\tau(f)$ is the projection, then $W^u(\pi(x,s)) \subset \pi(X \times s)$ is not dense. We assume $Y = \overline{W^u(p)} \neq X$ with $\phi_T(p) = p$ and we shall find a $\tau > 0$ and $f\colon Y \to Y$ which work. This will be done by a permutation of Smale's proof of Special Decomposition Theorem [23, p. 782].

(3.3) LEMMA. *For any $\eta > 0$, $D_\eta = \bigcup_{|t| \leqq \eta} \phi_t(Y)$ is a compact neighborhood of Y. Furthermore, $X = \bigcup_{0 \leqq t \leqq T} \phi_t(Y)$.*

Proof. Choose $\alpha > 0$ as in 1.15. We prove that

$$D_\eta \supset B_\alpha(Y) = B_\alpha(\overline{W^u(p)}) = B_\alpha(W^u(p))$$

(remember that everything is inside X). As CO^* is dense in X, it is enough to show $D_\eta \supset B_\alpha(W^u(p)) \cap CO^*$.

Suppose $\phi_r(y) = y$, $x \in W^u(p)$ and $d(y,x) < \alpha$. Consider

$$z = \langle x, y \rangle \in W^u{}_\eta(\phi_t(x)) \cap W^s{}_\eta(y)$$

where $|t| < \eta$. Then $z \in W^u(\phi_t(x)) \subset \phi_t W^u(x) \subset \phi_t(Y)$. For integral k,

$$\begin{aligned}\phi_{t+kT}(Y) &= \phi_{t+kT}(\overline{W^u(p)}) = \overline{\phi_{t+kT}(W^u(p)} \\ &= \overline{W^u(\phi_{t+kT}(p))} = \overline{W^u(\phi_t(p))} = \phi_t(Y).\end{aligned}$$

Hence $\phi_{kT}(z) \in \phi_t(Y)$. Let k_n be an increasing sequence of integers so that $k_nT \to 0 \pmod r$. Then

$$d(y, \phi_{k_nT}(y)) = d(y, \phi_{k_nT-m_nr}(y)) \to 0$$

where m_n is an integer so that $0 \leqq kT - m_nr < r$. Since $z \in W^s(y)$ we get

$$d(y, \phi_t(Y)) \leqq d(y, \phi_{k_nT}(z)) \leqq d(y, \phi_{k_nT}(y)) + d(\phi_{k_nT}(y), \phi_{k_nT}(z)) \to 0.$$

Thus $y \in \phi_t(Y)$.

D_η is compact since is the image of $[-\eta, \eta] \times Y$ under $(t, x) \to \phi_t(x)$. Finally, set $A = \bigcup_{0 \leqq t \leqq T} \phi_t(Y)$. Since $\phi_{t+kT}(Y) = \phi_t(Y)$, we see that $A = \phi_s(A) = \bigcup_{t \in R} \phi_t(Y)$ for $s \in R$. $A = D_T$ is closed and a neighborhood of Y; hence $A = \phi_s(A)$ is a neighborhood of $\phi_s(Y)$ for all s and it follows that A is open. As A is a nonempty, open, closed Φ-invariant set and Φ is transitive, we must have $A = X$.

(3.4) LEMMA. *If $\phi_t(Y) \cap Y \neq \phi$, then $\phi_t(Y) = Y$.*

Proof. Notice first that $\bigcap_{\eta > 0} D_\eta = Y$ follows from the continuity of Φ.

Suppose there is a $v \in \phi_t(Y)$ with $v \notin Y$. As Y is closed and $\phi_t(Y) = \overline{W^u(\phi_t(p))}$ we may assume $v \in W^u(\phi_t(p))$. Since $v \notin Y$, $v \notin D_\eta$ for some $\eta > 0$. As $D_{\eta/2}$ is a neighborhood of Y and

$$\overline{W^u \mathrm{A} \phi_t(p))} \cap Y = \phi_t(Y) \cap Y \neq \phi,$$

pick $w \in W^u(\phi_t(p)) \cap D_{\eta/2}$. Since $v, w \in W^u(\phi_t(p))$, $d(\phi_{-kT}(v), \phi_{-kT}(w)) \to 0$ as $k \to \infty$. Now $\phi_{-kT}(w) \in {}_{-kT}(D_{\eta/2}) = D_{\eta/2}$, so

$$d(\phi_{-kT}(v), D_{\eta/2}) \to 0.$$

Now D_η is a neighborhood of $D_{\eta/2}$ because $D_\eta \supset \phi_s D_{\eta/2}$ is a neighborhood of each $\phi_s(Y)$ with $|s| \leqq \eta/2$ by 3.3. Hence $\phi_{-kT}(v) \in D_\eta$ for large k. This contradicts $v \notin D_\eta$ and $\phi_{kT} D_\eta = D_\eta$, proving $\phi_t(Y) \subset Y$. Proving $\phi_t(Y) \supset Y$ is similar.

(3.5) LEMMA. *There is a smallest $t > 0$ with $\phi_t(Y) = Y$; call it τ. The t with $\phi_t(Y) = Y$ are the multiples of τ; $X = \bigcup_{0 \leqq t < \tau} \phi_t(Y)$.*

Proof. Suppose $\phi_{t_n}(Y) = Y$ with $t_n \to 0$. Then $\phi_{kt_n}(Y) = Y$ for $k \in Z$. Since $t_n \to 0$, $\phi_r(Y) = Y$ for a dense set of $r \in R$. As $\phi_r(p) \in Y$ is a closed condition, $\phi_r(p) \in Y$ for all r. Then $\phi_r(p) \in Y \cap \phi_r(Y)$ and so $\phi_r(Y) = Y$. Hence $Y = \bigcup_{r \in R} \phi_r(Y) = X$, contradicting the original assumption that $Y \neq X$. This contradiction means there must be such a $\tau > 0$. The rest of the lemma is easy.

Proof of 3.2 (continued). Consider the surjective continuous map $H: Y \times [0, \tau] \to X$ given by $H(x,t) = \phi_t(x)$. It is injective on $Y \times [0, t)$ because $Y \cap \phi_t(Y) = \phi$ for $t \in (0, \tau)$. Also $H(x, \tau) = H(y, 0)$ iff $\phi_\tau(x) = y$. It follows that (Φ, X) is isomorphic to $(S_t(f), \mathrm{Sus}_\tau(f))$ where $f = \phi_\tau \mid Y$. It is straightforward to check that the properties needed for f to be Axiom A* follow from things we know about Φ (see [4]). For instance

$$W^s{}_f(x) = \{y \in Y:\ d(\phi_{n\tau}(x), \phi_{n\tau}(y)) \to 0 \text{ as } n \to \infty\}$$
$$= W^s{}_\Phi(x) \cap Y = W^s{}_\Phi(x).$$

(3.6) LEMMA. *Suppose X is C-dense and $\delta > 0$. There is a T so that $B_\delta(\phi_t W^u{}_\delta(x)) = X$ whenever $t \geqq T$ and $x \in X$.*

Proof. $B_\delta(Z) = \{y: d(y, Z) < \delta\}$. We omit the proof of this lemma; it is similar to Lemma 2.3 of [4].

(3.7) PROPOSITION. *Suppose X is C-dense and $\epsilon > 0$. There is an N such that, for any N-sepecification (T, Γ), one can find $y \in X$ and $s \in STEP_\epsilon(T)$ so that*

$$d(\phi_{t+s(t)}(y), \phi_t(x_i)) \leqq \epsilon$$

for $t \in [t_i, t_{i+1} - N]$, $i \in Z$.

Proof. Let δ and L be as in the approximation theorem 2.2, but for $\frac{1}{2}\epsilon$ instead of ϵ. Make sure $\delta \leqq \epsilon$. Let $N \geqq L$ be the T of Lemma 3.6, for $\frac{1}{2}\delta$ instead of δ. Pick $y_i \in \phi_N W^u{}_{\frac{1}{2}\delta}(\phi_{t_{i+1}-N}(x_i)) \cap B_{\frac{1}{2}\delta}(\phi_{t_{i+1}}(x_{i+1}))$. Define $\Gamma' = \{x'_i\}$ by $x'_i = \phi_{-t_{i+1}}(y_i)$. Then $d(\phi_t(x_i), \phi_t(x'_i)) \leqq \frac{1}{2}\delta$ for $t \in [t_i, t_{i+1} - N]$. As

$$d(\phi_{t_{i+1}}(x'_i) . \phi_{t_{i+1}}(x'_{i+1})) \leqq d(\phi_{t_{i+1}}(x'_i), \phi_{t_{i+1}}(x_{i+1}))$$
$$+ d(\phi_{t_{i+1}}(x_{i+1}), \phi_{t_{i+1}}(x'_{i+1})) \leqq \delta,$$

(T, Γ') is δ-possible. By 2.2 there is a $y \in X$ and $s \in STEP_{\frac{1}{2}\epsilon}(T)$ so that

$$d(\phi_{t+s(t)}(y), \phi_t(x'_i)) \leqq \tfrac{1}{2}\epsilon \text{ for } t \in [t_i, t_{i+1}].$$

Our result follows by applying the triangle inequality.

(3.8) SPECIFICATION THEOREM. *Let X be C-dense. For any $\alpha > 0$ and $n \geqq 1$ there is an $N = N_{\alpha,n}$ for which the following is true: if $z_0, \cdots, z_n \in X$ and $t_0, \cdots, t_{n+1} \in R$ with $t_{k+1} \geqq t_k + N$, then there is an $x \in CO^*_\alpha(t_{n+1} - t_0)$ with*

$$d(\phi_{t_k+u}(x), \phi_u(z_k)) < \alpha \text{ for } 0 \leqq u \leqq t_{k+1} - t_k - N,\ 0 \leqq k \leqq n.$$

Proof. Let L, δ be as in 2.4 for $\beta = \frac{\alpha}{2}$. Make sure $\delta < \alpha$. Choose $\epsilon < \frac{\delta}{4}$ so small that

$$\sup\{d(\phi_u(x), x) : x \in X, |u| \leqq (n+2)\epsilon\} < \frac{\delta}{4}.$$

Let $N \geqq L$ be the N of 3.7 for this ϵ, $z_{n+1} = z_0$, and $x_i = \phi_{-t_i}(z_i)$.

On can define x_i and t_i for $i \notin [0, n+1]$ in various ways so that $T = \{t_i\}$ and $\Gamma = \{x_i\}$ give us an L-specification (T, Γ). By 3.7 there is a $y \in X$ and $s \in STEP_\epsilon(T)$ so that

$$d(\phi_{t+s(t)}(y), \phi_t(x_k)) \leqq \epsilon < \frac{\delta}{4}$$

for $t \in [t_k, t_{k+1} - N]$. For such a t and $k \in [0, n+1]$, we have $|s(t)| \leqq (n+z)\epsilon$ and so

$$d(\phi_{t+s(t)}(y), \phi_t(y)) \leqq \frac{\delta}{2}$$

Hence $d(\phi_t(y), \phi_t(x_k)) < \frac{\delta}{2}$ for $t \in [t_k, t_{k+1} - L]$, $0 \leqq k \leqq n+1$.

Now $$d(\phi_{t_{n+1}}(y), \phi_{t_0}(y)) \leqq d(\phi_{t_{n+1}}(y), \phi_{t_{n+1}}(x_{n+1})) + d(\phi_{t_0}(x_0), \phi_{t_0}(y)) \leqq \frac{\delta}{2} + \frac{\delta}{2} \leqq \delta.$$

As $t_{n+1} - t_0 \geqq nN \geqq L$, 2.4 gives us a point $x \in CO^*_{\alpha/2}(t_{n+1} - t_0)$ so that

$$d(\phi_t(x), \phi_t(y)) \leqq \frac{\alpha}{2} \text{ for } t_0 \leqq t \leqq t_{n+1}.$$

Thus we get

$$d(\phi_t(x), \phi_t(x_k)) \leqq \frac{\alpha}{2} + \frac{\delta}{2} < \alpha$$

for $t \in [t_k, t_{k+1} - N]$, $0 \leqq k \leqq n$. Note that $\phi_t(x_k) = \phi_{t-t_k}(z_k)$.

4. Topological entropy and counting orbits. We shall now investigate the growth of $\nu(t)$, $\nu_\epsilon(t)$ and certain related numbers. For the suspension case this is the same as studying the periodic points of f. This has been done extensively, often in terms of the zeta function [4], [9], [12], [23], [25]. For this reason we only need to look at the C-dense case really.

A subset $E \subset X$ is *(t,ϵ)-separated* for Φ if, for $x \neq y$ in E, $d(\phi_s(x), \phi_s(y)) > \epsilon$ for some $s \in [0,t]$. Let $M_\epsilon(t)$ be the maximum cardinality for any (t,ϵ)-separated set for Φ. Because Φ is flow expansive, the topological entropy is given by [8, Theorem 2.4]

$$h(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log M_\epsilon(t)$$

for small $\epsilon > 0$ ([8] uses "spanning" sets instead of "separated" sets, but Lemma 1 of [7] shows this makes no difference).

(4.1) LEMMA. *Suppose E to be (t,ϵ)-separated set with $M_\epsilon(t)$ elements. Then, for any $x \in X$, there is a $y \in E$ so that*

$$d(\phi_s(x), \phi_s(y)) \leqq \epsilon \text{ for all } 0 \leqq s \leqq t.$$

Proof. If $x \in E$, take $y = x$. If $x \notin E$, then $E \cup \{x\}$ is not (t,ϵ)-separated as it has too many elements. This means there is such a y.

(4.2) LEMMA. *For small α, $\epsilon > 0$ there are constants $C_1 = C_1(\alpha,\epsilon)$ and $D = D(\alpha,\epsilon) > 0$ so that*

$$M_\alpha(t + C_1) \geqq DM_\epsilon(t) \text{ for all } t \geqq 0.$$

Proof. Let E be a set so that $\phi_{-\frac{1}{2}t}(E)$ is (ϵ,t)-separated and F have $M_\alpha(t+C_1)$ elements with $\phi_{-\frac{1}{2}(t+C_1)}(F)$ being $(\alpha, t+C_1)$-separated. For $x \in E$ choose $g(x) \in F$ (by 4.1) so that

$$d(\phi_u(x), \phi_u(g(x))) \leqq \alpha$$

for all $|u| \leqq \frac{1}{2}(t+C_1)$.

Suppose $g(x) = g(y)$. Then $d(\phi_u(x), \phi_u(y)) \leqq 2\alpha$ for $|u| \leqq \frac{1}{2}(t+C_1)$. By 1.5, for small α there is an $v < \frac{1}{3}$ so that

$$d(\phi_p(y), \phi_p\phi_v(x)) \leqq c\gamma e^{-\lambda(C_1-2)/2}$$

for $|p| \leqq \frac{1}{2}t + 1$ and some $|v| \leqq \eta$. Choose C_1 so large that $c\gamma e^{-\lambda(C_1-2)/2} < \alpha/3$.

Let $\{x_0 = y, x_1, \cdots, x_m\} = g^{-1}(g(y))$. Set $v_k = v(x_k, y)$. Then

$$d(\phi_p\phi_{v_k}(x_k), \phi_p\phi_{v_j}(x_j)) < \tfrac{2}{3}\alpha$$

for $|p| \leqq \frac{1}{2}t + 1$. As $|v_k| \leqq \eta < 1$ we get

$$d(\phi_p(x_k), \phi_{p+v_j-v_k}(x_j)) < \tfrac{2}{3}\alpha$$

for $|p| = t$. Choose $\beta > 0$ so small that

$$\sup\{d(\phi_s(x), x) : x \in X, |s| \leqq \beta\} < \tfrac{1}{3}\alpha.$$

If $|v_j - v_k| \leqq \beta$, then $d(\phi_p(x_k), \phi_p(x_j)) < \alpha$ for all $|p| \leqq \frac{1}{2}t$; $k = j$ since $\phi_{-\frac{1}{2}t}(E)$ is (t, α)-separated. Hence $v_0, \cdots, v_m$ are numbers in $[-1, 1]$, any two of which differ by at least β. This shows $m \leqq 2/\beta$; so

$$\operatorname{card} E \geqq \left(\frac{\beta}{2} + 1\right) \operatorname{card} F.$$

(4.3) LEMMA. $M_\epsilon(t_1 + \cdots + t_n) \leqq M_{\frac{1}{2}\epsilon}(t_1) M_{\frac{1}{2}\epsilon}(t_2) \cdots M_{\frac{1}{2}\epsilon}(t_n)$.

Proof. Let E be $(t_1 + \cdots + t_n, \epsilon)$-separated, E_k maximal $(t_k, \frac{1}{2}\epsilon)$-separated. By 4.1 map $g : E \to \prod_k E_k$ by $g_k(x)$ satisfying

$$d(\phi_{u+t_1+\cdots+t_{k-1}}(x), \phi_u g_k(x)) \leqq \tfrac{1}{2}\epsilon$$

for $0 \leqq u \leqq t_k$. g is injective.

(4.4) LEMMA. *For small $\epsilon > 0$ and any L there is a $C_2 = C_2(\epsilon, L)$ so that $M_\epsilon(t + L) \leqq C_2 M_\epsilon(t)$ for all $t \geqq 0$.*

Proof. It is enough to find C_2 working for large t. By Lemma 4.2 there are C_1 and $D > 0$ so that $M_\epsilon(t + C_1) \geqq D M_{\frac{1}{2}\epsilon}(t)$. By 4.3

$$\begin{aligned} M_\epsilon(t + L) &\leqq M_{\frac{1}{2}\epsilon}(t - C_1) M_{\frac{1}{2}\epsilon}(C_1 + L) \\ &\leqq \frac{1}{D} M_\epsilon(t) M_{\frac{1}{2}\epsilon}(C_1 + L). \end{aligned}$$

(4.5) LEMMA. *For α, ϵ small there is a $C_3 = C_3(\alpha, \epsilon)$ so that $C_3 M_\alpha(t) \geqq M_\epsilon(t)$ for all $t \geqq 0$.*

Proof. With $C_1 = C_1(\alpha, \epsilon)$ as in 4.2:

$$M_\alpha(t + C_1) \geqq D M_\epsilon(t).$$

By 4.4 there is a $C_2 = C_2(\epsilon, C_1)$ with $M_\alpha(t + C_1) \leqq C_2 M_\alpha(t)$.

(4.6) LEMMA. *For small ϵ there is a $C_4 = C_4(\epsilon) > 0$ so that $M_\epsilon(t + s) \leqq C_4 M_\epsilon(t) M_\epsilon(s)$ for all $s, t \geqq 0$.*

Proof. $M_\epsilon(t + s) \leqq M_{\frac{1}{2}\epsilon}(t) M_{\frac{1}{2}\epsilon}(s)$. Also $M_{\frac{1}{2}\epsilon}(s) \leqq C_3 M_\epsilon(s)$ and $M_{\frac{1}{2}\epsilon}(t) \leqq C_3 M_\epsilon(t)$ where $C_3 = C_3(\epsilon, \frac{1}{2}\epsilon)$.

(4.7) LEMMA. *Suppose X is C-dense. For α small and $n \geqq 1$ there is a $C_5 = C_5(\alpha, n) > 0$ so that*

$$M_\alpha(s_0 + \cdots + s_n) \geqq C_5 M_\alpha(s_0) M_\alpha(s_1) \cdots M_\alpha(s_n)$$

when the s_i are sufficiently large.

Proof. Let $N = N_{\alpha,n}$ be as in the specification Theorem 3.8. Assume $s_i \geqq N$. Let E_i be $(s_i - N, 3\alpha)$-separated with $M_{3\alpha}(s_i - N)$ members. Let $t_0 = 0$ and $t_k = s_0 + \cdots + s_{k-1}$ for $1 \leqq k \leqq n+1$. For

$$(z_0, \cdots, z_n) \in E_0 \times \cdots \times E_n$$

the specification theorem gives us a point $x(z_0, \cdots, z_n)$ so that

$$d(\phi_{t_k+u}(x(z_0, \cdots, z_n)), \phi_u(z_k)) < \alpha$$

for $0 \leqq u \leqq s_k - N$. Using the triangle inequality we see that

$$\{x(z_0, \cdots, z_n)\}_{z_0, \cdots, z_n}$$

is $(s_0 + \cdots + s_n, \alpha)$-separated. Hence

$$M_\alpha(s_0 + \cdots + s_n) \geqq M_{3\alpha}(s_0 - N) \cdots M_{3\alpha}(s_n - N).$$

Setting $C_3 = C_3(3\alpha, \alpha)$ and $C_2 = C_2(\alpha, N)$,

$$M_{3\alpha}(s_k - N) \geqq C^{-1}{}_3 M_\alpha(s_k - N) \geqq C^{-1}{}_3 C^{-1}{}_2 M_\alpha(s_k).$$

Take the product

(4.8) *Definition.* For $\gamma \in CO$ let $\tau(\gamma)$ be it minimaum. Set

$$N_\epsilon(t) = \sum_{\gamma \in CO_\epsilon(t)} \tau(\gamma).$$

(4.9) Lemma. *Suppose X is not a single point. There is a constant $C_6 > 0$ so that*

$$C_6 \nu_\epsilon(t) \leqq N_\epsilon(t) \leqq (t + \epsilon) \nu_\epsilon(t).$$

Proof. Let C_6 be the smallest period of any closed orbit; $C_6 > 0$ since X has no fixed points. For any $\gamma \in CO_\epsilon(t)$

$$C_6 \leqq \tau(\gamma) \leqq t + \epsilon.$$

(4.10) Lemma. *Suppose X is C-dense. For small $\alpha, \epsilon > 0$ there are $C_7 = C_7(\alpha, \epsilon) > 0$ and $C_8 = C_8(\alpha, \epsilon) > 0$ so that*

$$C_7 M_\alpha(t) \geqq N_\epsilon(t) \geqq C_8 M_\alpha(t)$$

for all large t.

Proof. We find C_8 first. Let $\delta = \min\{\epsilon, \frac{1}{3}\alpha\}$. Let $N = N_{\delta,1}$ as in 3.8. Let E be a $(t - N, \alpha)$-separated set. By 3.8, for $z \in E$ find $x(z) \in CO^*{}_\epsilon(t)$ so that

$$d(\phi_s(z), \phi_s(x(z))) < \tfrac{1}{3}\alpha \text{ for } 0 \leqq s \leqq t - N.$$

By the triangle inequality, if $z \neq z'$ then for some s

$$d(\phi_s x(z), \phi_s x(z')) > \tfrac{1}{3}\alpha.$$

Choose $\beta > 0$ so that

$$\sup\{d(x, \phi_u(x)) : x \in X, |u| \leqq 3\beta\} < \tfrac{1}{3}\alpha.$$

For $y \in \phi_{[-3\beta,3\beta]}(x(z))$ we have

$$d(\phi_s(y), \phi_s(x(z))) < \tfrac{1}{3}\alpha \text{ for all } s.$$

Hence $x(z') \notin \phi_{[-3\beta,3\beta]}x(z)$ for $z \neq z'$; hence $\phi_{[-\beta,\beta]}x(z) \cap \phi_{[-\beta,\beta]}x(z') = \phi$. From this we get $N_\epsilon(t) \geqq 2\beta \operatorname{card} E = 2\beta M_\alpha(t - N)$. Lemma 4.4 finishes it.

Suppose that $x, y \in CO^*_\epsilon(t)$ satisfy $d(\phi_s(x), \phi_s(y)) \leqq \alpha$ for $0 \leqq s \leqq t$ and that ϵ and α are small. Define a specification (T, Γ) by $t_i = it$ and $x_i = \phi_{-ti}(x)$. Let $\phi_\tau(x) = x$ where $\tau \in [t - \epsilon, t + \epsilon]$. Define $s_1 \in STEP_\epsilon(T)$ by $s_1[t_i, t_{i+1}) = i(\tau - t)$. Then $x \in U_\epsilon(s_1, T, \Gamma) \subset U^*_\epsilon(T, \Gamma)$. Let $\phi_{\tau'}(y) = y$ where $\tau' \in [t - \epsilon, t + \epsilon]$. Define $s_2 \in STEP_\epsilon(T)$ by $s_2[t_i, t_{i+1}) = i(\tau' - t)$. For $u + t_i \in (t_i, t_{i+1})$, $0 < u < t$, we have

$$\phi_{t_i + u + s(t_i + u)}(y) = \phi_{it + u + i(\tau' - t)}(y) = \phi_u(y)$$

and $d(\phi_u(y), \phi_{t_i+u}(x_i)) = d(\phi_u(x)) \leqq \alpha$. To find a C_7, Lemma 4.5 shows that we may use any α; in particular we may assume $\alpha \leqq \epsilon$. Then $y \in U^*_\epsilon(T, \Gamma)$. Now Lemma 2.3 implies, since ϵ is small, that $x = \phi_u(y)$ for some $|u| \leqq \beta$ where β does not depend on x, y or t (so long as $t > \epsilon$).

Now we can find a set $E \subset CO^*_\epsilon(t)$ with $\operatorname{card} E \geqq N_\epsilon(t)/2\beta$ so that $x \notin \phi_{[-\beta,\beta]}(y)$ whenever x, y are distinct points in E. By the preceding paragraph, E is (t, α)-separated. Thus $M_\alpha(t) \geqq N_\epsilon(t)/2\beta$.

(4.11) THEOREM. $h(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \nu(t)$. *If X is C-dense and $\epsilon > 0$ small, then* $h(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \nu_\epsilon(t)$.

Proof. Suppose X is C-dense. Using 4.10 and 4.9

$$\begin{aligned} h(\Phi) &= \lim_{t\to\infty} \frac{1}{t} \log M_\alpha(t) = \lim_{t\to\infty} \frac{1}{t} \log N_\epsilon(t) \\ &= \lim_{t\to\infty} \frac{1}{t} \log \nu_\epsilon(t). \end{aligned}$$

Because

$$\nu(t) \geqq \nu_\epsilon(t - \epsilon)$$

and

$$\nu(t) \leqq \nu_\epsilon(t) + \nu_\epsilon(t - 2\epsilon) + \cdots + \nu_\epsilon(0)$$

(there are at most $t/2\epsilon$ terms here), we also get the other formula.

For the suspension case the formula follows from the analogous one for Axiom A* homeomorphisms [4, Theorem 4.5]; see statements 18 and 21 of [7].

(4.12) THEOREM. *$h(\Phi) > 0$ unless X is a point or a single closed orbit.*

Proof. For the suspension case this follows from the corresponding fact about Axiom A* homeomorphisms [8a, Theorem 4.6]. Assume X is C-dense.

Choose $p \neq q$ and $\epsilon < d(p,q)/6$. Consider $(a_0, a_1, \cdots, a_n)$ with each a_i equal to p or q. Imitating the proof of 2.2, we can find points $z(a_0, \cdots, a_n)$ and small numbers $\epsilon(a_0, \cdots, a_n)$ so that (L is a big number)

$$z(a_0, \cdots, a_n) \in W^s{}_\epsilon(a_n)$$

and

$$z_k(a_0, \cdots, a_n) = \phi_{-(kL + \sum_{j=n-k+1}^{n} \epsilon(a_0, \cdots, a_j))} z(a_0, \cdots, a_n) \in W^u{}_\epsilon(z(a_0, \cdots, a_{n-k})).$$

For $(a_0, \cdots, a_n) \neq (a'_0, \cdots, a'_n)$ let i be the smallest integer with $a_i \neq a'_i$. Set $t = iL + \sum_{j=1}^{i} \epsilon(a_0, \cdots, a_j)$ and $t' = iL + \sum_{j=1}^{i} \epsilon(a'_0, \cdots, a'_j)$. Then

$$\phi_t z_n(a_0, \cdots, a_n) = z_i(a_0, \cdots, a_n) \in B_\epsilon(z(a_0, \cdots, a_i)) \subset B_{2\epsilon}(a_i)$$

and $\phi_{t'} z(a'_0, \cdots, a'_n) \in B_{2\epsilon}(a'_i)$. Now make sure that L is so big that we can insure $|\epsilon(a_0, \cdots, a_n)| \leqq \delta$ where δ is small enough that

$$\sup\{d(x, \phi_s(x)) : x \in X, |s| \leqq 2\delta\} < \epsilon.$$

As $(a_0, \cdots, a_{i-1}) = (a'_0, \cdots, a'_{i-1})$,

$$|t - t'| = |\epsilon(a_0, \cdots, a_i) - \epsilon(a'_0, \cdots, a'_i)| \leqq 2\delta$$

and so $d(\phi_t z_n(a'_0, \cdots, a'_n), \phi_{t'} z_n(a'_0, \cdots, a'_n)) < \epsilon$. Since $d(a_i, a'_i) > 6\epsilon$, the triangle inequality gives $d(\phi_t z_n(a_0, \cdots, a_n), \phi_t z_n(a'_0, \cdots, a'_n)) > \epsilon$

We have shown that $\{z_n(a_0, \cdots, a_n)\}$ is an $(n(L+\delta), \epsilon)$-separated set.

$$M_\epsilon(n(L+\delta)) \geqq 2^{n+1}$$

$$h(\Phi) \geqq \lim_{n\to\infty} \frac{1}{n(L+\delta)} \log 2^{n+1} = \frac{1}{L+\delta} \log 2 > 0.$$

5. Invariant measures and the equidistribution of closed orbits. Let $\mathfrak{M}$ be the space of Φ-invariant normalized Borel measures on X; $\mathfrak{M}$ is a compact metrizable space under the weak topology [28]. In this space μ_n converges to μ if $\mu(E) = \lim \mu_n(E)$ for every Borel set E with $\mu(\partial E) = 0$ (iff $\liminf \mu_n(G) \geqq \mu(G)$ for every open set $G \subset X$ iff $\limsup \mu_n(F) \leqq \mu(F)$ for every closed set F). From a closed orbit $\gamma \in CO$ one can get an element of $\mathcal{M}$ as follows: let ω_γ be the measure induced on γ from Lebesgue measure on $[0, \tau(\gamma))$ by the map $t \to \phi_t(x)$ where $x \in \gamma$ and $\tau(\gamma)$ is the minimum period of γ. Then $\frac{1}{\tau(\gamma)} \omega_\gamma \in \mathcal{M}$.

Provided $CO_\epsilon(t) \neq \phi$, one can consider the measure

$$\omega_{\epsilon,t} = \frac{1}{N_\epsilon(t)} \sum_{\gamma \in CO_\epsilon(t)} \omega_\gamma.$$

Fixing $\epsilon > 0$, if $\{t_i\}^\infty{}_{i=1}$ is an increasing sequence of real numbers with $t_i \to \infty$ so that the measures ω_{ϵ,t_i} are defined and ω_{ϵ,t_i} converges in $\mathcal{M}$, we shall denote the limit by

$$\omega_{\epsilon,\{t_i\}} = \lim_{i\to\infty} \omega_{\epsilon,t_i}.$$

Notice that 4.11 and 4.12 together imply that $\omega_{\epsilon,t}$ is defined in the C-dense case whenever t is large enough (depending on ϵ). In the suspension case we need that t be approximately a multiple of the time of suspension τ.

We shall show that all the limits $\omega_{\epsilon,\{t_i\}}$ are in fact identical; we shall denote this measure by μ_Φ. The closed orbits of Φ are equidistributed according to this measure; we also show μ_Φ is orgodic.

K. Sigmund [18], [19] studied the generic properties of elements of $\mathcal{M}$ for the Axiom A diffeomorphism case. The specification theorem 3.8 should allow much of his work to be carried over to the flow case.

(5.1) LEMMA. *For small θ, ϵ, $\eta > 0$ there is a $P = P_{\theta,\epsilon,\eta} > 0$ for which the following is true. For $V \subset X$ one can find a finite set $A \subset V \cap CO^*(t)$ and $R_x \subset [-\eta, \eta]$ for each $x \in A$ so that*

(1) $\phi_{R_x}(x) \subset V$.

(2) $\phi_s(A)$ *is (t,θ)-separated for any s,*

and

(3) $\omega_{\epsilon,t}(\bigcup_x \phi_{R_x}(x)) = \dfrac{1}{N_\epsilon(t)} \sum_x m(R_x) \geqq P\omega_{\epsilon,t}(V)$.

Proof. Choose q so that $x, y \in CO^*{}_\epsilon(t)$ and $x \notin \phi_{[-q,q]}(y)$ implies x and y are (t,θ)-separated (proof of 4.10). Let $\eta' = \min\{\eta, q\}$. Divide $\gamma \in CO_\epsilon(t)$

into consecutive closed segments $I_1, \cdots, I_m$ of equal (time)length l with $\frac{1}{2}\eta' < l < \eta'$. Let S be an integer greater than $2q/\eta'$. Suppose $|i-j| > S$ (mod m); then

$$I_j \cap \phi_{[-q,q]}(I_i) \subset I \cap \bigcup_{|k-i| \leqq S} I_k = \phi$$

We can divide the segments into $2(S+1)$ groups $E_1, \cdots, E_{2(S+1)}$ so that $|i-j| > S$ for $I_i, I_j \in E_k$ with $i \neq j$. For some k,

$$\omega_{\epsilon,t}(V \cap \gamma \cap \bigcup_{I_i \in E_k} I_i) \geqq \frac{1}{2(S+1)} \omega_{\epsilon,t}(V \cap \gamma).$$

Form A_γ by picking one point from each $V \cap I_i$ with $I_i \in E_k$. For $x \in A_\gamma \cap I_i$ set

$$R_x = \{t : \phi_t(x) \in I_i\} \subset [-\eta, \eta]$$

Let $A = \cup \{A_\gamma : \gamma \in CO_\epsilon(t)\}$.

For $R = (r_1, \cdots, r_n)$ an increasing sequence of positive integers let $I(R) = \min_{0 \leqq k < n} (r_{k+1} - r_k)$ (here $r_0 = 0$; $I(R) = \infty$ when $n = 0$). For $E_0, \cdots, E$ subsets of X and $\beta > 0$ let

$$V_{R,\beta}(E_0, \cdots, E_n) = \{x \in E_0 : \phi_{[r_i - \beta, r_i + \beta]}(x) \cap E_i \neq \phi \text{ for } 0 < i \leqq n\}.$$

(5.2) PROPOSITION. *Assume X is C-dense. For small α, β, $\epsilon > 0$ and $n \geqq 0$ there is a $Q = Q_{\alpha,\beta,\epsilon,n} > 0$ so that the following holds. If $V_0, \cdots, V_n$ are closed and $U_i \supset V_i$ open, we can find t^* and N^* so that*

$$\omega_{\alpha,t'} V_{R,\beta}(U_0, \cdots, U_n) \geqq Q \prod_{i=0} \omega_{\epsilon,t}(V_i)$$

whenever $t \geqq t^*$, $I(R) \geqq t + N^*$ and $t' \geqq r_n + t + N^*$.

Proof. Choose $\eta > 0$ so small that $4\eta < \beta$ and

$$\sup\{d(x, \phi_v(x)) : |v| \leqq 4\eta\} < \alpha.$$

Choose $\alpha^* > 0$ with $\alpha^* \leqq \alpha$ small enough so that for every $\psi > 0$ there is a t^* for which $d(\phi_s(x), \phi_s(y)) \leqq \alpha^*$ for all $|s| \leqq t^*$ implies $d(x, \phi_u(y)) \leqq \psi$ for some $|u| \leqq \eta$; see 1.6. Let $N = N_{\alpha^*,n}$ as in the Specification Theorem 3.8.

Now suppose $U_i \supset V_i$ as above. As the V_i are compact, $U_i \supset B_\rho(V_i)$ for some $\rho > 0$. Choose $\psi > 0$ so that

$$\sup\{d(\phi_s(x), \phi_s(y)) : d(x, y) \leqq \psi, |s| \leqq \eta\} < \rho.$$

Let t^* be as above for this ψ.

Consider $t \geqq 2t^*$. Let A_i be as in Lemma 5.1 for $V = V_i$ with $\theta = 3\alpha, \epsilon, \eta$. For $x \in A_0$ let R_x be as in 5.1 as well. Consider R with $I(R) \geqq t + 2N$ and $t' \geqq r_n + t + 2N$. Set $t_{2i} = r_i - t^*$ and $t_{2i+1} = r_i - t^* + t + N$ for $0 \leqq t \leqq n$. Also, set $t_{2n+2} = t' - r_0 + t^*$. For each $0 \leqq i \leqq n$, let B_i be

$$(t_{2i+2} - t_{2i+1} - N, 3\alpha)\text{-separated}$$

with maximum cardinality. By the specification theorem we can map $g\colon E = A_0 \times B_0 . \times A_1 \times \cdots \times A_n \times B_n \to CO^*{}_\alpha(t')$ so that

$$d(\phi_{t_{2i}+u} g(x_0, y_0, \cdots, x_n, y_n), \phi_u \phi_{-t^*}(x_i)) < \alpha^* \quad \text{for } 0 \leqq u \leqq t_{2i+1} - t_{2i} - N = t,$$

and

$$d(\phi_{t_{2i+1}+u} g(x_0, y_0, \cdots, x_n, y_n), \phi_u(y_i)) < \alpha^* \quad \text{for } 0 \leqq u \leqq t_{2i+2} - t_{2i+1} - N.$$

If $(x_0, y_0, \cdots, x_n, y_n) \neq (x , y_0, \cdots, x_n, y_n)$, then $x_i \neq x'_i$ or $y_i \neq y'_i$ for some i. If $x_i \neq x'_i$, because $\phi_{-t^*}(A_i)$ is $(t, 3\alpha)$ separated, for some $0 \leqq u \leqq t$

$$\begin{aligned} d(\phi_{t_{2i}+u} g(x_0, y_0, \cdots, x_n, y_n), \phi_{t_{2i}+u} g(x'_0, y'_0, \cdots, x'_n, y'_n)) \\ \geqq d(\phi_{t_{2i}+u} g(x_i), \phi_{t_{2i}+u}(x'_i)) \\ - d(\phi_{t_{2i}+u} g(x_0, \cdots, y_n), \phi_{t_{2i}+u}(x_i)) \\ - d(\phi_{t_{2i}+u} g(x'_0, \cdots, y'_n), \phi_{t_{2i}+u}(x'_i)) \\ \geqq 3\alpha - \alpha^* - \alpha^* \geqq \alpha. \end{aligned}$$

A similar thing happens if $y_i \neq y'_i$ as B_i is $(t_{2i+2} - t_{2i+1} - N, 3\alpha)$-separated. We have shown that $\phi_{-t^*} g(E)$ is (t', α) separated.

Now for any $\omega = (x_0, y_0, \cdots, y_n) \in E$ we have

$$d(\phi_s(x_0), \phi_s g(\omega)) < \alpha^* \text{ for all } |s| \leqq t^*.$$

By the choice of t^* this implies $d(x, \phi_{u(\omega)} g(\omega)) \leqq \psi$ for some $|u(\omega)| \leqq \eta$. By the choice of η

$$\phi_{u(\omega)+R_{x0}} g(\omega) \subset B_\rho \phi_{R_{x_0}}(x_0) \subset B_\rho(V_0) \subset U_0.$$

Also, for $0 < i \leqq n$ we have

$$d(\phi_s(x_i), \phi_{r_i+s} g(\omega)) < \alpha^* \text{ for } |s| \leqq t^*$$

Hence $\phi_{r_i} g(\omega) \in \phi_{[-\eta,\eta]} B_\psi(V_i) \subset \phi_{[-\eta,\eta]} U_i$. As $|u(\omega)| \leqq \eta$ and $R_{x_0} \subset [-\eta, \eta]$

$$\phi_{r_i} \phi_{u(\omega)+R_{x0}} g(\omega) \subset \phi_{[-3\eta,3\eta]} U_i$$

As $\beta \geqq 3\eta$ we get that

$$V_{R,\beta}(U_0, \cdots, U_n) \supset \bigcup_\omega \phi_{u(\omega)+R_{x0(\omega)}} g(\omega).$$

Suppose $\phi_{u(\omega)+R_{x_0}}g(\omega) \cap \phi_{u(\omega')+R_{x'_0}}g(\omega') \neq \phi$. Then $g(\omega) = \phi_v(\omega')$ for some $|v| \leqq 4\eta$. By the definition of η, for any s,

$$d(\phi_s g(\omega), \phi_s g(\omega')) = d(\phi_v \phi_s g(\omega'), \phi_s g(\omega')) < \alpha$$

If $\omega \neq \omega'$, this would contradict the fact that $\phi_{-t^*}g(E)$ is (t', α)-separated. Thus $\bigcup_{\omega} \phi_{u(\omega)+R_{x_0}(\omega)}g(\omega)$ is a disjoint union and

$$\begin{aligned}
\omega_{\alpha,t'}V_{R,\beta}(U_0,\cdots,U_n) &\geqq \frac{1}{N_\alpha(t')}\sum_{\omega\in E} m(R_{x_0(\omega)}) \\
&\geqq \frac{\operatorname{card}(B_0\times A_1\times\cdots\times B_{n-1}\times A_n)}{N_\alpha(t')}\sum_x m(R_x) \\
&\geqq \frac{\prod_{i=0}^n \operatorname{card}(B_i)\prod_{i=1}^n \operatorname{card}(A_i)}{N_\alpha(t')} N_\epsilon(t) P_{3\alpha,\epsilon,\eta}\omega_{\epsilon,t}(V_0).
\end{aligned}$$

Now $2\eta \operatorname{card}(A_i) \geqq N_\epsilon(t) P_{3\alpha,\epsilon,\eta}\omega_{\epsilon,t}(V_i)$ by (3) of 5.1. Also (Lemma 4.10) $N_\epsilon(t) \geqq C_7 M_\epsilon(t)$ for large enough t (make sure t^* wasn't too small). Thus

$$\operatorname{card} A_i \geqq W M_\epsilon(t)\omega_{\epsilon,t}(V_i)$$

where W depends only on α, ϵ and β but not the U_i or V_i (remember η depends on α, ϵ and β). By lemmas 4.3 and 4.5 there is a $C = C_{\epsilon,n} > 0$ with

$$M_{3\epsilon}(t)^n \prod_{i=1}^n \operatorname{card} B_i \geqq C M_{3\epsilon}(nt + \sum_{i=1}^n (t_{2i+2} - t_{2i+1} - N)).$$

But $t_{2i+1} - t_{2i} - N = t$, so

$$\begin{aligned}
nt + \sum_{i=1}^n (t_{2i+2} - t_{2i+1} - N) &= \sum_{k=0}^{2n+1} (t_{k+1} - t_k - N) \\
&= (t_{2n+2} - t_0) - (2n+1)N = t' - (2n+1)N.
\end{aligned}$$

By lemmas 4.10 and 4.6 we get

$$\frac{M_{3\epsilon}(t' - (2n+1)N)}{N_\alpha(t')} \geqq C^* > 0$$

for some C^* and big t'. We finally get

$$\begin{aligned}
\omega_{\alpha,t'}V_{R,\beta}(U_0,\cdots,U_n) &\geqq \frac{\prod_{i=0}^n \operatorname{card}(B_i)\prod_{i=0}^n W M_\epsilon(t)\omega_{\epsilon,t}(V_i)}{N_\alpha(t')} \\
&\geqq \frac{W^n M_{3\epsilon}(t' - (2n+1)N)}{N_\alpha(t')}\prod_{i=0}^n \omega_{\epsilon,t}(V_i).
\end{aligned}$$

We prove a weak type of "mixing" statement next.

(5.3) PROPOSITION. *Assume X is C-dense. For any $\omega_{\alpha,\{t'_i\}}$, $\omega_{\epsilon,\{t_i\}}$ and Borel sets $E_0,\cdots,E_n\subset X$*

$$\liminf_{I(R)\to\infty}\omega_{\alpha,\{t'_k\}}V_{R,\beta}(E_0,\cdots,E_n)\geqq Q\prod_{i=0}^{n}\omega_{\epsilon,\{t_j\}}(E_j)$$

where Q is as in 5.2.

Proof. Suppose $Y_i\supset Z_i$ closed and Y_i open. Find open sets U_i and W_i so that

$$Y_i\supset\bar U_i\supset U_i\supset\bar W_i\supset W_i\supset Z_i.$$

We apply 5.2 to the U_i and $V_i=\bar W_i$. For $t_j\geqq t^*$, R fixed with $I(R)\geqq t_j+N^*$ we have

$$\omega'_{\alpha,t_k}V_{R,\beta}(\bar U_0,\cdots,\bar U_n)\geqq\omega'_{\alpha,t_k}V_{R,\beta}(U_0,\cdots,U_n)\geqq Q\prod_{i=0}^{n}\omega_{\epsilon,t_j}(\bar W_i)$$

for k large. As $V_{R,\beta}(\bar U_0,\cdots,\bar U_n)$ is closed, letting $k\to\infty$ we get

$$\begin{aligned}\omega_{\alpha,\{t'_k\}}V_{R,\beta}(Y_0,\cdots,Y_n)&\geqq\omega_{\alpha,\{t'_k\}}V_{R,\beta}(\bar U_0,\cdots\ \bar U_n)\\&\geqq\limsup_{k\to\infty}\omega_{\alpha,t'_k}V_{R,\beta}(\bar U_0,\cdots,\bar U_n)\\&\geqq Q\prod_{i=0}^{n}\omega_{\epsilon,t_j}(\bar W_i).\end{aligned}$$

Letting $I(R)\to\infty$,

$$\liminf_{I(R)\to\infty}\omega_{\alpha,\{t'_k\}}(V_{R,\beta}(Y_0,\cdots,Y_n))\geqq Q\prod_{i=0}^{n}\omega_{\epsilon,t_j}(\bar W_i).$$

Letting $j\to\infty$ we have

$$\begin{aligned}\liminf_{j\to\infty}\omega_{\epsilon,t_j}(\bar W_i)&\geqq\liminf_{j\to\infty}\omega_{\epsilon,t_j}(W_i)\\&\geqq\omega_{\epsilon,\{t_j\}}(W_i)\geqq\omega_{\epsilon,\{t_j\}}(Z_i).\end{aligned}$$

Thus

$$\liminf_{I(R)\to\infty}\omega_{\alpha,\{t'_k\}}V_{R,\beta}(Y_0,\cdots,Y_n)\geqq Q\prod_{i=0}^{n}\omega_{\epsilon,\{t_j\}})Z_i)$$

Now keep the Z_i fixed and for each i take a sequence $Y^1{}_i\supset Y^2{}_i\supset\cdots$ of open sets with $\bigcap_{m>0}Y^m{}_i=Z_i$. Then $\bigcap_{m>0}\phi_{[-\beta,\beta]}Y^m{}_i=\phi_{[-\beta,\beta]}Z_i$. For any $a>0$ we can find an m with $\omega_{\alpha\{t'_k\}}(\phi_{[-\beta,\beta]}Y^m{}_i\backslash\phi_{[-\beta,\beta]}Z_i)<a$ for all $0\leqq i\leqq n$. For any R

$$V_{R,\beta}(Y^m{}_0,\cdots,Y^m{}_n)\backslash V_{R,\beta}(Z_0,\cdots,Z_m)\subset\bigcup_{i=0}^{n}\phi_{-r_i}(\phi_{[-\beta,\beta]}Y^m{}_i\backslash\phi_{[-\beta,\beta]}Z_i)$$

and so, as $\omega_{\alpha,\{t'_k\}}$ is Φ-invariant,

$$\omega_{\alpha,\{t'_k\}}V_{R,\beta}(Z_0,\cdots,Z_m) \geqq \omega_{\alpha,\{t'_k\}}V_{R,\beta}(Y^m{}_0,\cdots,Y^m{}_n) - (n+1)a$$

So we get

$$\liminf_{I(R)\to\infty} \omega_{\alpha,\{t'_k\}}V_{R,\beta}(Z_0,\cdots,Z_m) \geqq Q\prod_{i=0}^{n}\omega_{\epsilon,\{t_j\}}(Z_i) - (n+1)a.$$

Let $a\to 0$:

$$\liminf_{I(R)\to\infty} \omega_{\alpha,\{t'_k\}}V_{R,\beta}(Z_0,\cdots,Z_n) \geqq Q\prod_{i=0}^{n}\omega_{\epsilon,\{t_j\}}(Z_i).$$

Now let $E_0,\cdots,E_n$ be sets measurable with respect to both $\omega_{\alpha,\{t'_k\}}$ and $\omega_{\epsilon,\{t_j\}}$. Let $Z^m{}_0, \cdot\cdot, Z^m{}_n$ be closed with $Z^m{}_i \subset E_i$ and

$$\omega_{\epsilon,\{t_i\}}(E_i) - \omega_{\epsilon,\{t_j\}}(Z^m{}_i) \to 0$$

as $m\to\infty$. Since $V_{R,\beta}(E_0,\cdots,E_n) \supset V_{R,\beta}(Z^m{}_0,\cdots,Z^m{}_n)$, applying the above inequality and letting $m\to\infty$ we get our result.

(5.4) THEOREM. *Assume X is C-dense. There is a $\mu_\Phi \in \mathfrak{M}_\Phi$ so that $\mu_\Phi = \lim_{t\to\infty}\omega_{\epsilon,t}$ in $\mathfrak{M}_\Phi$ for any small ϵ. μ_Φ is ergodic for Φ.*

Proof. Consider any two measures $\omega_{\alpha,\{t'_k\}}$ and $\omega_{\epsilon,\{t_j\}}$. Proposition 5.3 applied to $n=0$ gives us a $Q_{\alpha,\epsilon,n} > 0$ (note that β does not enter in for $n=0$) with

$$\omega_{\alpha,\{t'_k\}}(E) \geqq Q\omega_{\epsilon,\{t_j\}}(E)$$

for all Borel sets E. This implies that the two msasures are equivalent, i. e. they have the same measurable sets and the same sets of measure 0 (see [4, 5.3] for instance).

Also, any $\nu = \omega_{\epsilon\{,t_j\}}$ is ergodic. Otherwise there is a ν-measurable set E so that $0 < \nu(E) < 1$ and $\phi_t(E) = E$ for all t. Apply 5.3 to $E_0 = E$ and $E_1 = X\backslash E$ (we showed there we didn't actually need E Borel):

$$\liminf_{I(R)\to\infty} \nu(V_{R,\beta}(E, X\backslash E)) \geqq Q\nu(E)\nu(X\backslash E) > 0.$$

But $V_{R,\beta}(E,X\backslash E) = \phi$ because $\phi_t(E) = E$ for all t.

Any two $\omega_{\alpha,\{t'_k\}}$ and $\omega_{\epsilon,\{t_j\}}$ are equal, for any two ergodic equivalent measures are equal. Let μ_Φ be their common value. Suppose $\mu_\Phi \neq \lim_{t\to\infty}\omega_{\epsilon,t}$. Then for some sequence $t_j\to\infty$, the sequence ω_{ϵ,t_j} would have a limit in $\mathfrak{M}_\Phi$ other than μ_Φ. But its limit would be $\omega_{\epsilon,\{t_j\}} = \mu_\Phi$.

(5.5) *Equidistribution of Closed Orbits.* For X C-dense, the closed orbits of Φ are equidistributed with respect to μ_Φ as the period tends to $+\infty$. More precisely, for any small $\epsilon > 0$ and any Borel set E with $\mu_\Phi(\partial E) = 0$,

$$\mu_\Phi(E) = \lim_{t\to+\infty} \omega_{\epsilon,t}(E).$$

Proof. $\omega_{\epsilon,t}(E)$ says what proportion of the closed orbits with period $\tau \in [t-\epsilon, t+\epsilon]$ lie in E. The precise statement is the condition for $\mu_\Phi = \lim \omega_{\epsilon,t'}$.

(5.6) PROPOSITION. *For X C-dense, (Φ, μ_Φ) is weak mixing.*

Proof. Otherwise there is an $f \in L^2(\mu_\Phi)$ not equivalent to a constant and a $\theta \neq 0$ so that, for each t,

$$f(\phi_t(x)) = e^{i\theta t} f(x) \text{ a.e.}$$

By [11, p. 27] we may find f so that actually

$$f(\phi_t(x)) = e^{i\theta t} f(x) \text{ for all } x \text{ and } t.$$

As f is not constant, we can find a closed disk B and a t_0 so that $0 < \mu(f^{-1}(B)) < 1$ and $B \cap e^{i\theta t_0} B = \phi$. For some small $\beta > 0$

$$B \cap e^{i\theta[t_0-\beta, t_0+\beta]} B = \phi.$$

Then

$$f^{-1}(B) \cap \phi_{2\pi m/\theta + t_0} \phi_{[-\beta,\beta]} f^{-1}(B) = \phi.$$

But this set is $V_{R_m,\beta}(f^{-1}(B), f^{-1}(B))$ where $R_m = (0, \frac{2\pi m}{\theta} + t_0)$. Proposition 5.3 gives us a contradiction (remember that it actually applies to μ_Φ-measurable sets).

(5.7) THEOREM. *Suppose X is a time τ suspension of a C-dense Axiom A* homeomorphism. Then $\mu_\Phi = \lim_{k\to\infty} \omega_{k\tau} \in \mathfrak{M}$ exists. (Φ, μ_Φ) is ergodic (but not weak mixing).*

Proof. That the homeomorphism be C-dense is equivalent to picking the minimum $\tau > 0$. The theorem follows from corresponding facts about Axiom A* homeomorphism [4, 6.6].

(5.8) PROPOSITION. *$\mu_\Phi(W) > 0$ for $W \neq \phi$ open.*

Proof. For X a suspension it follows from the corresponding fact for homeomorphism [4, 5.4]. Suppose X is C-dense. Choose open U, $V \neq \phi$ so that $W \supset \bar{U} \supset U \supset \bar{V}$. As $\overline{CO^*} = X$, $\omega_{\epsilon,t}(V) > 0$ for arbitrarily large t's. Applying 5.2 to $U \supset \bar{V}$:

$$\liminf_{t'\to+\infty} \omega_{\alpha,t'}(U) \geqq Q\omega_{\epsilon,t}(\bar{V})$$

where $t \geqq t^*$ so that $\omega_{\epsilon,t}(\bar{V}) > 0$. As $\omega_{\alpha,t'} \to \mu_\Phi$

$$\mu_\Phi(W) \geqq \mu_\Phi(\bar{U}) \geqq \liminf \omega_{\alpha,t'}(\bar{U}) > 0.$$

(5.9) PROPOSITION. *For $\epsilon > 0$ small there is a $K > 0$ so that: if E is measurable and* $\operatorname{diam} \phi_s(E) \leqq \epsilon$ *for all* $s \in [0, L]$, *then* $\mu_\Phi(E) \leqq K/M_\epsilon(L)$.

Proof. Assume X is C-dense. By enlarging E, we may assume E is open and $\operatorname{diam} \phi_s(E) \leqq 2\epsilon$. Apply 5.1 to $\theta = 2\epsilon$, ϵ and $\eta = \epsilon$. For each t there is a set $A_t \subset E$ so that

(a) A_t is $(t, 2\epsilon)$-separated

and

(b) $P\omega_{\epsilon,t}(E) \leqq \dfrac{2\eta}{N_\epsilon(t)} \operatorname{card}(A_t)$.

Since $\operatorname{diam} \phi_s(A_t) \leqq 2\epsilon$ for $s \in [0, L]$, $\phi_L(A_t)$ is $(t - L, 2\epsilon)$-separated and hence $\operatorname{card} A_t \leqq M_{2\epsilon}(t - L)$. Thus, using 4.10 and 4.6,

$$\omega_{\epsilon,t}(E) \leqq \frac{2\eta M_\epsilon(t-L)}{PN_\epsilon(t)} \leqq \frac{K}{M_\epsilon(L)}$$

for some constant $K > 0$. Since E is open and $\omega_{\epsilon,t} \to \mu_\Phi$

$$\mu_\Phi(E) \leqq \liminf \omega_{\epsilon,t}(E) \leqq \frac{K}{M_\epsilon(L)}.$$

For X a suspension the statement follows from an analogous one for Axiom A homeomorphisms [4, 3.9(v)].

(5.10) COROLLARY. *Unless X is a single point or a single closed orbit,* $\mu_\Phi(\phi_R(x)) = 0$ *for each* $x \in X$.

Proof. As μ_Φ is σ-additive, it is enough to prove $\mu_\Phi(\phi_{[0,a]}(x)) = 0$ for small $a > 0$. But

$$\operatorname{diam} \phi_s(\phi_{[0,a]}(x)) < \epsilon$$

for all s, provided a is small. Therefore $\mu_\Phi(\phi_{[0,a]}(x)) \leqq K/M_\epsilon(L)$ for all L. For X larger than a point or a single closed orbit, $h(\Phi) > 0$ by 4.12, hence $M_\epsilon(L) \to \infty$ as $L \to \infty$ by 4.11 and $\mu_\Phi(\phi_{[0,a]}(x)) = 0$.

We now calculate the measure theoretic entropy.

(5.11) THEOREM. *For any t,*

$$h_{\mu_\Phi}(\phi_t) = h(\phi_t) = th(\Phi).$$

Proof. We refer the reader to [30] for a definition of h_μ. $h(\phi_t) \geqq h_{\mu_\Phi}(\phi_t)$ is a case of Goodwyn's theorem [29]. Let $\epsilon > 0$ be small and choose $\beta > 0$ so that $d(x,y) \leqq \beta$ implies $d(\phi_s(x), \phi_s(y)) \leqq \epsilon$ for all $s \in [0,t]$. Let $\boldsymbol{B} = \{B_1, \cdots, B_m\}$ be a measurable partition of X with $\operatorname{diam} B_i \leqq \beta$. Finally, let C_n denote the collection of nonempty sets of the form

$$V = \bigcap_{k=0}^{n-1} \phi_{-kt}(B_{i_k}).$$

From the definition of β we see

$$\operatorname{diam} \phi_u(V) \leqq \epsilon \text{ for } u \in [0, nt].$$

By 5.9

$$\mu_\Phi(V) \leqq K/M_\epsilon(nt).$$

Now $h_{\mu_\Phi}(\phi_t) \geqq h_{\mu_\Phi}(\phi_t, \boldsymbol{B}) = \lim_{n\to\infty} \frac{1}{n} \int h_n d\mu_\Phi$ where

$$h_n = \sum_{V \in C_n} (-\log \mu_\Phi(V)) \chi_V$$

and χ_V is the characteristic function. As $h_n \geqq \log M_\epsilon(nt) - \log K$,

$$h_{\mu_\Phi}(\phi_t) \geqq \lim \left(\frac{1}{n} \log M_\epsilon(nt)\right) - \frac{1}{n} \log K) \geqq th(\Phi)$$

by 4.11.

6. Conclusion. An important example of an Axiom A flow is the geodesic flow on a compact manifold of negative curvature. For this case Sinai [20] studied the asymptotic growth of $\nu(t)$ and obtained

$$(n-1)K_2 \leqq \liminf_{t\to\infty} \frac{1}{t} \log \nu(t) \leqq \limsup_{t\to\infty} \frac{1}{t} \log \nu(t) \leqq (n-1)K_1$$

where n is the dimension of the manifold and $-K^2{}_1$ and $-K^2{}_2$ are the lower and upper limits of the two-dimensional curvature. This is a case of Theorem 4.11 as one can show directly that here $(n-1)K_2 \leqq h(\Phi) \leqq (n-1)K_1$. Margulis [13] states a much stronger results for this geodesic flow case:

$$\nu(t) \sim \frac{ce^{dt}}{t}$$

where c and d are constants. By 4.11 we have $d = h(\Phi)$.

Problem 1. For general Axiom A ϕ_t is there *a* c so that

$$\nu(t) \sim \frac{ce^{h(\Phi)t}}{t}?$$

For an Anosov flow with invariant Lebesgue measure, ergodicity questions have bben studied by Anosov and Sinai [1], [2], [21]. These works suggest the following.

Problem 2. For X C-dense, is (Φ, μ_Φ) a K-flow?
Recent work by Ornstein [14] and Katznelson [30] suggest in fact:

Problem 3. For X C-dense, is (ϕ_t, μ_Φ) Bernoulli for (almost) all t?

In the Anosov case one cannot generally expect μ_Φ to be a smooth measure. One would hope at least for a positive answer to the following.

Problem 4. If Φ is an Anosov flow induced from a 1-parameter subgroup of a Lie group [24], is μ_Φ the measure induced by Haar measure?

This would imply in particular that the closed geodesics on a surface of constant negative curvature are equidistributed with respect to the usual measure. Problem 4 might best be answered by proving a stronger statement:

Problem 5. Is μ_Φ the only Φ-invariant measure ρ with $h_\rho(\Phi) = h(\Phi)$?

A number of these problems may benefit from a study of Markov partitions and the associated symbolic dynamics, as the analogous problems did for Axiom A diffeomorphims [22], [5], [6].

REFERENCES.

[1] D. Anosov, "Geodesic flows on closed Riemannian manifolds of negative curvature," *Proc. Steklov Inst.*, vol. 90 (Amer. Math. Soc. translation).
[2] ——— and Ya. G. Sinai, "Some smooth ergodic systems," *Russian Mathematical Surveys*, vol. 22 (1967), No. 5, pp. 103-167.
[3] R. Bowen and P. Walters, "Expansive one-parameter flows," *Jour. Diff. Eqns.*
[4] R. Bowen, "Periodic points and measures for Axiom A diffeomorphisms," *Transactions, American Mathematical Society*, vol. 154 (1971), pp. 377-397.
[5] ———, "Markov partitions for Axiom A diffeomorphisms," *American Journal of Mathematics*, vol. 92 (1970), pp. 725-747.
[6] ———, "Markov partitions and minimal sets for Axiom A diffeomorphisms," *American Journal of Mathematics*, vol. 92 (1970), pp. 907-918.
[7] ———, "Entropy for group endomorphisms and homogeneous spaces," *Transactions, American Mathematical Society*, vol. 153 (1971), pp. 401-414.
[8] ———, "Entropy-expansive maps," *Trans. Amer. Math. Soc.*, vol. 164 (1972).
[8a] ———, "Topological entropy and Axiom A," *Proc. Symp. Pure Math.*, vol. 14 (1970), pp. 23-41.

[9] J. Guckenheimer, "Axiom A and no cycles, implies $\zeta_f(t)$ rational," *Bulletin of the American Mathematical Society*, vol. 76 (1970), pp. 592-594.

[10] M. Hirsch, C. Pugh and M. Shub, "Invariant manifolds," *Bull. Amer. Math. Soc.*, vol. 76 (1970), pp. 1015-1019.

[11] E. Hopf, *Ergodentheorie*, Springer, 1937.

[12] A Manning, "Axiom A diffeomorphisms have rational zeta functions," *Bull. London Math. Soc.*, vol. 3 (1971), pp. 215-220.

[13] G. Margulis, "Applications of ergodic theory to the investigation of manifolds of negative curvature," *Func. Anal. and its Appl.*, vol. 4 (1970), no. 1, pp. 55-67.

[14] D. Ornstein, "Imbedding Bernoulli shifts in flows," to appear.

[15] J. Plante, "Anosov flows," to appear.

[16] C. Pugh and M. Shub, "The Ω-stability theorem for flows," *Inventiones Math.*, vol. 11 (1970), pp. 150-158.

[17] M. Shub, "Periodic orbits of hyperbolic diffeomorphism and flows," *Bulletin of the American Mathematical Society*, vol. 75 (1969), pp. 57-58.

[18] K. Sigmund, "Generic properties of unvariant measures for Axiom A diffeomorphisms," *Inventiones Math.*, vol. 11 (1970), pp. 99-109.

[19] ———, "Ergodic averages for Axiom A diffeomorphisms," *Z. Wahrscheinlichkeitstheorie verw. Geb.*, vol. 20 (1971), pp. 319-324.

[20] Ya. G. Sinai, "The asymptotic behavior of the number of closed orbits on a compact manifold of negative curvature," *American Mathematical Society Translations* (2), vol. 73, pp. 227-250.

[21] ———, "Classical dynamical systems with countably Lebesgue spectrum II," *American Mathematical Society Translations* (2), vol. 68, pp. 34-88.

[22] ———, "Markov partitions and C-diffeomorphisms," *Func. Anal. and its Appl.*, vol. 2 (1968), no. 1, pp. 64-89.

[23] S. Smale, "Differentable dynamical systems," *Bulletin of the American Mathematical Society*, vol. 73 (1967), pp. 747-817.

[24] P. Tomter, "Anosov flows on infrahomogeneous spaces," *Proc. Symp. Pure Math.*, vol. 14 (1970), pp. 229-328.

[25] R. Williams, "The zeta function in global analysis," *Proc. Symp. Pure Math.*, vol. 14 (1970), pp. 335-339.

[26] M. Morse, "Symbolic Dynamics," lecture notes, Institute for Advanced Study, Princeton.

[27] G. Margulis, "Certain measures associated with U-flows on compact manifolds," *Func. Analy, and its Appl.*, vol. 4 (1970), no. 1, pp. 55-67.

[28] K. Parathasarathy, *Probability measures on metric spaces*, Academic Press, 1967.

[29] L. Goodwyn, "Topological entropy bounds measure theoretic entropy," *Proceedings of the American Mathematical Society*, vol. 23 (1969), pp. 679-691.

[30] P. Billingsley, *Ergodic Theory and Information*, John Wiley and Sons, N. Y., 1965.

[31] Y. Katznelson, "Ergodic automorphisms of T^n are Bernoulli shifts," to appear.