Dynamical systems, simulation, abstract computation.

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Abstract

We survey an area of recent development, relating dynamics to theoretical computer science. We discuss the theoretical limits of simulation and computation of interesting quantities in dynamical systems. We will focus on central objects of the theory of dynamics, as invariant measures and invariant sets, showing that even if they can be computed with arbitrary precision in many interesting cases, there exists some cases in which they can not. We also explain how it is possible to compute the speed of convergence of ergodic averages (when the system is known exactly) and how this entails the computation of arbitrarily good approximations of points of the space having typical statistical behaviour (a sort of constructive version of the pointwise ergodic theorem).

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1 Introduction

The advent of automated computation led to a series of great successes and achievements in the study of dynamical problems.

The use of computers and simulations allowed to compute and forecast the behavior of many important natural phenomena and, on the other hand, led to the discovery of important general aspects of dynamics.

This motivates the huge work that was made by hundred of scientists to improve "practical" simulation and computation techniques.

It also motivates the study of the theoretical limits of simulation and computation techniques, and the theoretical understanding of related problems.

In this paper we want to focus on some of these aspects related to *rigorous* computation and simulation of (discrete time) dynamical systems.

The simulation and investigation of dynamics started with what we call the "naive" approach, in which the user just implements the dynamics without taking rigorous account of numerical errors and roundings. Then he "looks" to the screen to see what happens.

Of course, the sensitivity to initial conditions, and the typical instability of many interesting systems (to perturbations on the map generating the dynamics) implies that what it is seen on the screen could be completely unrelated to what was meant to be simulated.

In spite of this, the naive approach turns out to work "unreasonably" well in many situations and it is still the most used one in simulations. The theoretical reasons why this method works and its limits, in our opinion are still to be understood (some aspects have been investigated in [36],[41],[28] [7, 8] e.g.).

On the opposite side from the naive approach, there is the "absolutely rigorous" approach, which will be the main theme of this paper: the user looks for an algorithm which can give a description of the object which is meant to be computed, up to any desired precision.

In this point of view, for example the constant e is a computable number because there is an algorithm that is able to produce a rational approximation

of e at any given precision (for example finding the right m and calculating $\sum_1^m \frac{1}{n!}$ such that the error is smaller than requested).

In an "absolutely rigorous" simulation (computation) the initial point or some initial distribution is supposed to be known up to any approximation and the transition map is also supposed to be known up to any accuracy (suitable precise definitions will be given below). This allows the evolution of the system to be simulated with any given accuracy, and the question arise, if interesting and important objects related to the dynamics (invariant sets or invariant measures e.g.) can be computed from the description of the system (or maybe adding some additional information).

In this paper we would like to give a survey of a group of results related to these computational aspects, restating and updating them with some new information. We will see that in many cases the interesting objects can be computed, but there are some subtleties, and cases where the interesting object cannot be computed from the description of the system or cannot be computed at all (again, up to any given precision). In particular, this happen case for the computation of invariant measures and Julia sets.

Hence, these results set theoretical limits to such computations.

Computing invariant measures.

An important fact motivating the study of the statistical properties of dynamical systems is that the pointwise long time prediction of a chaotic system is not possible, whereas, in many cases, the estimation or forecasting of averages and other long time statistical properties is. This often corresponds in mathematical terms to computing invariant measures, or estimating some of their properties, as measures contain information on the statistical behavior of the system (X, T) and on the potential behavior of averages of observables along typical trajectories of the system (see Section 3).

An invariant measure is a Borel probability measure μ on X such that for each measurable set A it holds $\mu(A) = \mu(T^{-1}(A))$. They represent *equilibrium* states, in the sense that probabilities of events do not change in time.

Rigorously, compute an invariant measure means to find an algorithm which is able to output a description of the measure (for example an approximation of the measure made by a combination of delta measures placed on "rational" points) up to any prescribed precision.

We remark that once an interesting invariant measure is computed, it is possible to deduce from it several other important information about the dynamics: Lyapunov exponents, entropy, escape rates, etc... For example, in dimension one, once an ergodic invariant measure μ has been approximated, the Lyapunov exponent λ_μ can be estimated using the formula $\lambda_\mu = \int_0^1 \log_2 T' d\mu$, where T' denotes the derivative of the map generating the dynamics. In higher dimensions, similar techniques can be applied (see e.g. [21] for more examples of derivation of dynamical quantities from the computation of the invariant measure).

Before giving more details about the computation of invariant measures, we remark that, since there are only countably many "algorithms" (computer programs), whatever we mean by "approximating a measure by an algorithm" would

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imply that only countable many measures can be computed whereas, in general, a dynamical system may have uncountably many invariant measures (usually an infinite dimensional set). So, a priori most of them will not be algorithmically describable. This is not a problem because we should put our attention on the most "meaningful" ones. An important part of the theory of dynamical systems is indeed devoted to the understanding of "physically" relevant invariant measures. Informally speaking, these are measures which represent the asymptotic statistical behavior of "many" (positive Lebesgue measure) initial conditions (see Section 3 for more details).

The existence and uniqueness of physical measures is a widely studied problem (see [47]), which has been solved for some important classes of dynamical systems. These measures are some of the good candidates to be computed.

The more or less *rigorous* computation of such measures is the main goal of a part of the literature related to computation and dynamics. A main role here is played by the transfer operator induced by the dynamics. Indeed, the map T defining the dynamics, induces a dynamics L_T on the space of probability measures on X, $L_T : PM(X) \to PM(X)$. L_T is called the transfer operator associated to T (definition and basic results about this are recalled in Section 3). Invariant measures are fixed points of this operator. The main part of the methods which are used to compute invariant measures deals with suitable finite dimensional approximation of this operator. In Section 3 we will review briefly some of these methods, and some references.We then consider the problem from an abstract point of view, and give some general result on rigorous computability of the physical invariant measure. In particular we will see that the transfer operator is computable up to any approximation in a general context (see Thm 15). The invariant measure is computable, provided we are able to give a description of a space of "regular" measures where the physical invariant measure is the only invariant one (see Thm 16 and following corollaries).

We will also show how the description of the space can be obtained from the one of the system in a class of examples (piecewise expanding maps) by using the Lasota-Yorke inequality.

After such general statements one could conjecture that all computable dynamical systems should always have a computable invariant measure. We will see that, perhaps surprisingly, this is not true. Not all systems that can be described explicitly (the dynamics can be computed up to any prescribed approximation) have computable invariant measures (see Section 3.5). The existence of such examples reveals some subtleties in the computation of invariant measures.

To further motivate these results, we finally remark that from a technical point of view, computability of the considered invariant measure is a requirement in several results about relations between computation, probability, randomness and pseudo-randomness (see Section 6 and e.g. [3, 23, 24, 25])

Computability in Complex Dynamics

Polynomial Julia sets have emerged as the most studied examples of fractal sets generated by a dynamical system. One of the reasons for their popularity is the beauty of the computer-generated images of such sets. The algorithms

used to draw these pictures vary; the naïve approach in this case works by iterating the center of a pixel to determine if it lies in the Julia set. There exists also more sophisticated algorithms (using classical complex analysis, see [42]) which work quite well for many examples, but it is well known that in some particular cases computation time will grow very rapidly with increase of the resolution. Moreover, there are examples, even in the family of quadratic polynomials, where no satisfactory pictures of the Julia set exist.

In the rigorous approach, a set is computable if, roughly speaking, its image can be generated by a computer with an arbitrary precision. Under this notion of computability, the question arise if Julia sets are always computable. In a series of papers ([5, 4, 11, 12]) it was shown that even though in many cases (hyperbolic, parabolic) the Julia set is indeed computable, there exists quadratic polynomials which are computable (again, in the sense that all the trajectories can be approximated by an algorithm at any desired accuracy), and yet the Julia set is not.

So we can not simulate the set of limits points on which the chaotic dynamics takes place, but, what about the statistical distribution?. In fact, it was shown by Brolin and Luybich that there exists a unique invariant measure which maximizes entropy, and that this measure is supported on the Julia set. The question of whether this measure can be computed has been recently solved in [6], where it is proved that the Brolin-Lyubich measure is always computable. So that even if we can not visualize the Julia set as a spatial object, we can approximate its distribution at any finite precision.

Computing the speed of convergence and pseudorandom points.
In several questions in ergodic theory The knowledge of the speed of convergence to ergodic behavior is important to deduce other practical consequences. In the computational framework, the question turn out to be the effective estimation of the speed of convergence in the ergodic theorems[1]. From the numerical-practical point of view this has been done in some classes of systems, having a spectral gap for example. In this case a suitable approximation of the transfer operator allows to compute the rate of decay of correlations [22][37] and from this, other rates of convergence can be easily deduced.

Other classes of systems could be treated joining the above spectral approach, with combinatorial constructions (towers, see [38] e.g.), but the general case need a different approach.

In [2] it was shown that much more in general, *if the system can be described effectively, then the rate of convergence in the pointwise ergodic theorem can be effectively estimated*. We give in section 5 a very short proof of a statement of this kind (see Theorem 39) for ergodic systems, and show some consequences. Among these, a *constructive* version of pointwise ergodic theorem. If the system is computable (in some wide sense that will be described) then, it is possible to compute points having typical statistical behavior. Such points could be hence called *pseudorandom* points in the system (see Section 6).

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[1]Find a N such that $\frac{1}{n} \sum f \circ T^n$ differs from $\int f \, d\mu$ less than a given error for each $n \geq N$.

Since the computer can only handle computable initial conditions, any simulation can start only from these points. Pseudorandom initial conditions are hence in principle good points where to start a simulation.

We remark that it is widely believed that naive computer simulations very often produce correct statistical behavior. The evidence is mostly heuristic. Most arguments are based on the various "shadowing" results (see e.g. [33] chapter 18). In this kind of approach (different from ours), it is possible to prove that in a suitable system every pseudo-trajectory, as the ones which are obtained in simulations with some computation error, is close to a real trajectory of the system. However, even if we know that what we see in a simulation is near to some real trajectory, we do not know if this real trajectory is typical in some sense. A limit of this approach is that shadowing results hold only in particular systems, having some strong hyperbolicity, while many physically interesting systems are not like this. In our approach we consider real trajectories instead of "pseudo" ones and we ask if there is some computable point which behaves as a typical point for the dynamics.

2 Computability on metric spaces

To have formal results and precise assumptions on the computability (up to any given error) of continuous objects, we have to introduce some concepts.

We have to introduce some recursive version of open and compact sets, and characterize the functions which are well suited to operate with those sets (computable functions). We explain this theory in this section trying the explanation to be to be as much as possible simple and self contained.

2.1 Computability

The starting point of recursion theory was to give a mathematical definition making precise the intuitive notions of *algorithmic* or *effective procedure* on symbolic objects. Several different formalizations have been independently proposed (by Church, Kleene, Turing, Post, Markov...) in the 30's, and have proved to be equivalent: they compute the same functions from $\mathbb{N}$ to $\mathbb{N}$. This class of functions is now called the class of *recursive functions*. As an algorithm is allowed to run forever on an input, these functions may be *partial*, i.e. not defined everywhere. The *domain* of a recursive function is the set of inputs on which the algorithm eventually halts. A recursive function whose domain is $\mathbb{N}$ is said to be *total*.

We now recall an important concept from recursion theory. A set $E \subseteq \mathbb{N}$ is said to be ***recursively enumerable (r.e.)*** if there is a (partial or total) recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ enumerating E, that is $E = \{\varphi(n) : n \in \mathbb{N}\}$. If

$E \neq \emptyset$, φ can be effectively converted into a total recursive function ψ which enumerates the same set E.

2.2 Algorithms and uniform algorithms

Strictly speaking, recursive functions only work on natural numbers, but this can be extended to the objects (thought as "finite" objects) of any countable set, once a numbering of its elements has been chosen. We will use the word *algorithm* instead of *recursive function* when the inputs or outputs are interpreted as finite objects. The operative power of algorithms on the objects of such a numbered set obviously depends on what can be effectively recovered from their numbers.

More precisely, let X and Y be numbered sets such that the numbering of X is injective (it is then a bijection between $\mathbb{N}$ and X). Then any *recursive function* $\varphi : \mathbb{N} \to \mathbb{N}$ induces an *algorithm* $\mathcal{A} : X \to Y$. The particular case $X = \mathbb{N}$ will be much used.

For instance, the set $\mathbb{Q}$ of rational numbers can be injectively numbered $\mathbb{Q} = \{q_0, q_1, \ldots\}$ in an *effective* way: the number i of a rational a/b can be computed from a and b, and vice versa. We fix such a numbering: from now and beyond the rational number with number i will be denoted by q_i.

Now, let us consider computability notions on the set $\mathbb{R}$ of real numbers, introduced by Turing in [44].

Definition 2 *Let x be a real number. We say that:*

- *x is **lower semi-computable** if the set $\{i \in \mathbb{N} : q_i < x\}$ is r.e.*

- *x is **upper semi-computable** if the set $\{i \in \mathbb{N} : q_i > x\}$ is r.e.*

- *x is **computable** if it is lower and upper semi-computable.*

Equivalently, a real number is computable if and only if there exists an algorithmic enumeration of a sequence of rational numbers converging exponentially fast to x. That is:

Proposition 3 *A real number is **computable** if there is an algorithm $\mathcal{A} : \mathbb{N} \to \mathbb{Q}$ such that $|\mathcal{A}(n) - x| \leq 2^{-n}$ for all n.*

Uniformity. Algorithms can be used to define computability notions on many classes of mathematical objects. The precise definitions will be particular to each class of objects, but they will always follow the following scheme:

An object O is ***computable*** if there is an algorithm

$$\mathcal{A} : X \to Y$$

which computes O in some way.

Each computability notion comes with a uniform version. Let $(O_i)_{i \in \mathbb{N}}$ be a sequence of computable objects:

O_i is computable **_uniformly in i_** if there is an algorithm

$$\mathcal{A} : \mathbb{N} \times X \to Y$$

such that for all i, $\mathcal{A}_i := \mathcal{A}(i, \cdot) : X \to Y$ computes O_i.

For instance, the elements of a sequence of real numbers $(x_i)_{i \in \mathbb{N}}$ are uniformly computable if there is a algorithm $\mathcal{A} : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that $|\mathcal{A}(i, n) - x_i| \leq 2^{-n}$ for all i, n.

In other words a set of objects is computable uniformly with respect to some index if they can be computed with a "general" algorithm *starting* from the value of the index.

In each particular case, the computability notion may take a particular name: computable, recursive, effective, r.e., etc. so the term "computable" used above shall be replaced.

2.3   Computable metric spaces

A computable metric space is a metric space with an additional structure allowing to interpret input and output of algorithms as points of the metric space. This is done in the following way: there is a dense subset (called ideal points) such that each point of the set is identified with a natural number. The choice of this set is compatible with the metric, in the sense that the distance between two such points is computable up to any precision by an algorithm getting the names of the points as input. Using these simple assumptions many constructions on metric spaces can be implemented by algorithms.

Definition 4 *A **computable metric space** (CMS) is a triple $\mathcal{X} = (X, d, S)$, where*

(i) *(X, d) is a separable metric space.*

(ii) *$S = \{s_i\}_{i \in \mathbb{N}}$ is a dense, numbered, subset of X called the set of **ideal points**.*

(iii) *The distances between ideal points $d(s_i, s_j)$ are all computable, uniformly in i, j (there is an algorithm $\mathcal{A} : \mathbb{N}^3 \to \mathbb{Q}$ such that $|\mathcal{A}(i, j, n) - d(s_i, s_j)| < 2^{-n}$).*

S is a numbered set, and the information that can be recovered from the numbers of ideal points is their mutual distances. Without loss of generality, we will suppose the numbering of S to be injective: it can always be made injective in an effective way.

Definition 5 *We say that in a metric space (X, d), a sequence of points $(x_n)_{n \in \mathbb{N}}$ converges* recursively *to a point x if there is an algorithm $D : \mathbb{Q} \to \mathbb{N}$ such that $d(x_n, x) \leq \epsilon$ for all $n \geq D(\epsilon)$.*

Definition 6 *A point $x \in X$ is said to be* **computable** *if there is an algorithm $\mathcal{A} : \mathbb{N} \to S$ such that $(\mathcal{A}(n))_{n \in \mathbb{N}}$ converges recursively to x.*

We define the set of ***ideal balls*** to be $\mathcal{B} := \{B(s_i, q_j) : s_i \in S, 0 < q_j \in \mathbb{Q}\}$ where $B(x, r) = \{y \in X : d(x, y) < r\}$ is an open ball. We fix a numbering $\mathcal{B} = \{B_0, B_1, \ldots\}$ which makes the number of a ball effectively computable from its center and radius and vice versa. $\mathcal{B}$ is a countable basis of the topology.

Definition 7 (Effective open sets) *We say that an open set U is* effective *if there is an algorithm $\mathcal{A} : \mathbb{N} \to \mathcal{B}$ such that $U = \bigcup_n \mathcal{A}(n)$.*

Observe that an algorithm which diverges on each input n enumerates the empty set, which is then an effective open set. Sequences of uniformly effective open sets are naturally defined. Moreover, if $(U_i)_{i \in \mathbb{N}}$ is a sequence of uniformly effective open sets, then $\bigcup_i U_i$ is an effective open set.

Definition 8 (Effective G_δ-set) *An* **effective G_δ-set** *is an intersection of a sequence of uniformly effective open sets.*

Obviously, an uniform intersection of effective G_δ-sets is also an effective G_δ-set.

Let $(X, S_X = \{s_1^X, s_2^X, \ldots\}, d_X)$ and $(Y, S_Y = \{s_1^Y, s_2^Y, \ldots\}, d_Y)$ be computable metric spaces. Let also B_i^X and B_i^Y be enumerations of the ideal balls in X and Y. A computable function $X \to Y$ is a function whose behavior can be computed by an algorithm up to any precision. For this it is sufficient that the pre-image of each ideal ball can be effectively enumerated by an algorithm.

Definition 9 (Computable Functions) *A function $T : X \to Y$ is* **computable** *if $T^{-1}(B_i^Y)$ is an effective open set, uniformly in i. That is, there is an algorithm $\mathcal{A} : \mathbb{N} \times \mathbb{N} \to \mathcal{B}^X$ such that $T^{-1}(B_i^Y) = \bigcup_n \mathcal{A}(i, n)$ for all i.*
A function $T : X \to Y$ is **computable on** *$D \subseteq X$ if there are uniformly effective open sets U_i such that $T^{-1}(B_i^Y) \cap D = U_i \cap D$.*

Remark 10 *Intuitively, a function T is computable (on some domain C) if there is a computer program which computes $T(x)$ (for $x \in C$) in the following sense: on input $\epsilon > 0$, the program, along its run, asks the user for approximations of x, and eventually halts and outputs an ideal point $s \in Y$ satisfying $d(T(x), s) < \epsilon$. This idea can be formalized, using for example the notion of oracle computation. The resulting notion coincides with the one given in the previous definitions.*

Recursive compactness is an assumption which will be needed in the following. Roughly, a compact set is recursively compact if the fact that it is covered by a finite collection of ideal balls can be tested algorithmically (for equivalence with the ϵ-net approach and other properties of recursively compact set see [26]).

Definition 11 *A set $K \subseteq X$ is* **recursively compact** *if it is compact and there is a recursive function $\varphi : \mathbb{N}^* \to \mathbb{N}$ such that $\varphi(i_1, \ldots, i_p)$ halts if and only if $(B_{i_1}, \ldots, B_{i_p})$ is a covering of K.*

3 Computing invariant measures

3.1 Invariant measure and statistical properties

Let X be a metric space, $T : X \mapsto X$ a Borel measurable map and μ a T-invariant Borel probability measure. A set A is called T-invariant if $T^{-1}(A) = A$ (*mod* 0). The system (X, T, μ) is said to be ergodic if each T-invariant set has total or null measure. In such systems the famous Birkhoff ergodic theorem says that time averages computed along μ-typical orbits coincides with space average with respect to μ. More precisely, for any $f \in L^1(X, \mu)$ it holds

$$\lim_{n \to \infty} \frac{S_n^f(x)}{n} = \int f \, \mathrm{d}\mu, \tag{1}$$

for μ almost each x, where $S_n^f = f + f \circ T + \ldots + f \circ T^{n-1}$.

This shows that in an ergodic system, the statistical behavior of observables, under typical realizations of the system is given by the average of the observable made with the invariant measure.

We say that a point x belongs to the basin of an invariant measure μ if (1) holds at x for each bounded continuous f. In case X is a manifold (possibly with boundary), a physical measure is an invariant measure whose basin has positive Lebesgue measure (for more details and a general survey see [47]). Computation of such measures will be the main subject of this section.

3.1.1 The transfer operator

A function T between metric spaces naturally induces a function L_T between probability measure spaces. This function L_T is linear and is called transfer operator (associated to T). Measures which are invariant for T are fixed points of L_T.

Let us consider a computable metric space X and a measurable function $T : X \to X$. Let us also consider the space $PM(X)$ of Borel probability measures on X.

Let us define the linear function $L_T : PM(X) \to PM(X)$ by duality in the following way: if $\mu \in PM(X)$ then $L_T(\mu)$ is such that

$$\int f \, \mathrm{d}L_T(\mu) = \int f \circ T \, \mathrm{d}\mu.$$

The computation of invariant measures (and many other dynamical quantities) very often is done by computing the fixed points (or the spectrum) of this operator in a suitable function space. The most applied and studied strategy is to find a suitable finite dimensional approximation of L_T (restricted to

a suitable function space) so reducing the problem to the computation of the corresponding relevant eigenvectors of a finite matrix.

An example of this is done by discretizing the space X by a partition A_i and replacing the system by a (finite state) Markov Chain with transition probabilities

$$P_{ij} = \frac{m(A_i \cap A_j)}{m(A_i)}$$

where m is the Lebesgue measure on X (see e.g. [20][21][37]), then, taking finer and finer partitions it is possible to obtain in some cases that the finite dimensional model will converge to the real one (and its natural invariant measure to the physical measure of the original system). In some case there is an estimation for this speed of convergence (see eg. [21] for a discussion), but a rigorous bound on the error (and then a real rigorous computation) is known only in a few cases (piecewise expanding or expanding maps, see [37]).

Similar approaches consists in applying a kind of Faedo-Galerkin approximation to the transfer operator by considering a complete Hilbert base of the function space and truncating the operator to the action on the first elements (see [46]).

Another approach is to consider a perturbation of the system by a small noise. The resulting transfer operator has a kernel. This operator then can be approximated by a finite dimensional one, again by Faedo-Galerkin method and relevant eigenvectors are calculated (see e.g. [17][16]) then, if we prove that the physical measure of the original system can be obtained as a limit when the size of the noise tends to zero (this happen on uniformly hyperbolic system for example) we have a method which in principle can rigorously compute this measure.

Variations on the method of partitions are given in [18, 19], while in [40] a different method, fastly converging, based on periodic points is exploited for piecewise analytic Markov maps. Another strategy to face the problem of computation of invariant measures consist in following the way the measure μ can be constructed and check that each step can be realized in an effective way. In some interesting examples we can obtain the physical measure as limit of iterates of the Lebesgue measure $\mu = \lim_{n \to \infty} L_T^n(m)$ (recall that m is the Lebesgue measure). To prove computability of μ the main point is to explicitly estimate the speed of convergence to the limit. This sometimes can be done using the spectral properties of the system ([24]).

Concluding, if the goal is to rigorously compute an invariant measure, most of the results which are in the today literature are partial. Indeed, beside being applied to a quite restricted class of systems, to compute the measure those methods need additional information. For example, the way to compute rigorously the finite dimensional approximation is often not done effectively, or the rate of convergence of the approximation is computed up to some constants depending on the system, which are not estimated.

In the remaining part of the section we present some results, mainly from [26] explaining some result about rigorous computations of invariant measures.

These results have the advantage to give in principle an effective method for the rigorous computation of an invariant measure and give a quite general result, where all the needed assumptions are explicited.

They have the disadvantage to not optimize computation time. So it is not clear if they can be implemented and used in practice.

The rigorous framework into they are proved, however allows to see them as an investigation about the theoretical limits of rigorous computation of invariant measures.

Moreover by this, also negative results can be proved. In particular, we give examples of *computable* systems having no computable invariant measures.

3.2 Computability of measures

In this section we explain precisely what we mean by computing a measure. This means, having an algorithm who is able to approximate the measure by "simple measures" up to any given approximation.

Let us consider the space $PM(X)$ of Borel probability measures over X. Let $C_0(X)$ be the set of real-valued bounded continuous functions on X. We recall the notion of weak convergence of measures:

Definition 12 μ_n *is said to be* **weakly convergent** *to* μ *if* $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ *for each* $f \in C_0(X)$.

Let us introduce the Wasserstein-Kantorovich distance between measures. Let μ_1 and μ_2 be two probability measures on X and consider:

$$W_1(\mu_1, \mu_2) = \sup_{f \in 1\text{-Lip}(X)} \left| \int f \, \mathrm{d}\mu_1 - \int f \, \mathrm{d}\mu_2 \right|$$

where $1\text{-Lip}(X)$ is the space of functions on X having Lipschitz constant less than one. The distance W_1 has the following useful properties

Proposition 13 (see [1] Prop 7.1.5)

1. W_1 *is a distance and if X is bounded, separable and complete, then $PM(X)$ with this distance is a separable and complete metric space.*

2. *If X is bounded, a sequence is convergent for the W_1 metrics if and only if it is convergent for the weak topology.*

3. *If X is compact $PM(X)$ is compact with this metrics.*

Item (1) has an effective version: $PM(X)$ inherits the computable metric structure of X. Indeed, given the set $\S_X$ of ideal points of X we can naturally define a set of ideal points $\S_{PM(X)}$ in $PM(X)$ by considering finite rational convex combinations of the Dirac measures δ_s supported on ideal points $s \in S_X$. This is a dense subset of $PM(X)$. The proof of the following proposition can be found in ([31]).

Proposition 14 *If X bounded then $(PM(X), W_1, \S_{PM(X)})$ is a computable metric space.*

A measure μ is then computable if there is a sequence $\mu_n \in \S_{PM(X)}$ converging recursively fast to μ in the W_1 metric (and hence for the weak convergence).

3.3 Computable invariant "regular" measures

Invariant measures can be found as fixed points of the transfer operator, i.e. as solutions of the equation $W_1(\mu, L_T(\mu)) = 0$. There is an abstract theorem allowing the computation of isolated fixed points (see [26] Corollary 3) of computable maps. To apply it is necessary to consider a space where the transfer transfer operator is computable (this space hopefully will contain the physical invariant measure).

We remark that if T is not continuous then L_T is not necessarily continuous (this can be realized by applying L_T to some delta measure placed near a discontinuity point) hence not computable. Still, we have that L_T is continuous (and its modulus of continuity is computable) at all measures μ which are "far enough" from the discontinuity set D. This is technically expressed by the condition $\mu(D) = 0$.

Proposition 15 *Let X be a computable metric space and $T : X \to X$ be a function which is computable on $X \setminus D$. Then L_T is computable on the set of measures*

$$PM_D(X) := \{\mu \in PM(X) : \mu(D) = 0\}. \tag{2}$$

From a practical point of view this proposition provides sufficient conditions to rigorously approximate the transfer operator by an algorithm.

The above tools allow us to ensure the computability of L_T on a large class of measures. This will enable us to apply a general result on computation of fixed points and obtain

Theorem 16 ([26], Theorem 3.2) *Let X be a computable metric space and T a function which is computable on $X \setminus D$. Suppose there is a recursively compact set of probability measures $V \subset PM(X)$ such that for every $\mu \in V$, $\mu(D) = 0$ holds. Then every invariant measure isolated in V is computable. Moreover the theorem is uniform: there is an algorithm which takes as inputs finite descriptions of T, V and an ideal ball in $M(X)$ which isolates[2] an invariant measure μ, and outputs a finite description of μ.*

A trivial and general consequence of Theorem 16 is the following:

Corollary 17 *If a system as above is uniquely ergodic and its invariant measure μ satisfies $\mu(D) = 0$, then it is a computable measure.*

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[2]If the invariant measure is unique in V the isolating ball is not necessary.

The main problem in the application of Theorem 16 is the requirement that the invariant measure we are trying to compute be isolated in V. In general the space of invariant measures in a given dynamical system could be very large (an infinite dimensional convex subset of $PM(X)$) and physical measures have often some kind of particular regularity. To isolate a particular measure we can restrict and consider a subclass of "regular" measures.

Let us consider the following *seminorm*:

$$\|\mu\|_\alpha = \sup_{x \in X, r > 0} \frac{\mu(B(x,r))}{r^\alpha}.$$

If α and K are computable and X is recursively compact then

$$V_{\alpha,K} = \{\mu \in PM(X) : \|\mu\|_\alpha \leq K\} \tag{3}$$

is recursively compact ([26]). This implies

Proposition 18 *Let X be recursively compact and T be computable on $X \setminus D$, with $dim_H(D) < \infty$. Then any invariant measure isolated in $V_{\alpha,K}$ with $\alpha > dim_H(D)$ is computable. Once again, this is uniform in T, α, K.*

The above general propositions allow us to obtain as a corollary the computability of many absolutely continuous invariant measures. For the sake of simplicity, let us consider maps on the interval.

Proposition 19 *If $X = [0,1]$, T is computable on $X \setminus D$, with $dim_H(D) < 1$ and (X, T) has a unique absolutely continuous invariant measure μ with bounded density, then μ is computable (starting from T and a bound for the L^∞ norm of the invariant density).*

Similar results hold for maps on manifolds (again see [26]).

3.3.1 Computing the measure from a description of the system in the class of piecewise expanding maps

As it is well known, interesting examples of systems having a unique absolutely continuous invariant measure (with bounded density as required) are topologically transitive *piecewise expanding maps* on the interval or *expanding maps* on manifolds.

We show how to find a bound for the invariant density on piecewise expanding maps. By this, the invariant measure can be calculated starting from a description of T.

Definition 20 *A nonsingular function $T : ([0,1], m) \to ([0,1], m)$ is said piecewise expanding if*[3]

---

[3]For the seek of simplicity we will consider the simplest setting in which we can work and give precise estimations. Such class was generalized in several ways, we hence warn the rader that now in the current literature by piecewise expanding maps it it meant something slightly more general than the ones defined here.

1. *There is a finite set of points $d_1 = 0, d_2, ..., d_n = 1$ such that $T|_{(d_i, d_{i+1})}$ is C^2 and can be extended to a C^2 map on $[d_i, d_{i+1}]$.*

2. *$\inf(T') > 1$ on the set where it is defined.*

3. *T is topologically mixing.*

It is now well known that such maps have an unique ergodic invariant measure with bounded variation density.

Such density is also the unique fixed point of the (Perron Frobenius) operator[4] $L : L^1[0, 1] \to L^1[0, 1]$ defined by

$$[Lf](x) = \sum_{y \in T^{-1}x} \frac{f(y)}{T'(y)}.$$

We now show how to find a bound for such density, starting from the description of the system, and then compute the associated invariant measure.

The following proposition was proved in the celebrated paper [35] (Thm. 1 and its proof) and it is now called Lasota-Yorke inequality. We give a precise statement where the constants are explicited.

Proposition 21 *Let T a piecewise hyperbolic map. If f is of bounded variation in $[0, 1]$. Let $d_1, ..., d_n$ be the discontinuity points of T.*
If $\lambda = \inf\limits_{x \in [0,1] - \{d_1, ..., d_n\}} T'(x)$. Then

$$Var(Lf) \leq 2\lambda Var(f) + B||f||_1$$

where

$$B = \frac{\sup\limits_{x \in [0,1] - \{d_1, ..., d_n\}} \left(\left|\frac{T''(x)}{(T'(x))^2}\right|\right)}{\inf\limits_{x \in [0,1] - \{d_1, ..., d_n\}} \left(\frac{1}{T'(x)}\right)} + \frac{2}{\min(d_i - d_{i+1})}.$$

The following is an elementar fact about the behavior of real sequences

Lemma 22 *If a real sequence a_n is such that $a_{n+1} \leq la_n + k$ for some $l < 1, k > 0$, then*

$$\sup(a_n) \leq \max(a_0, \frac{k}{1 - l}).$$

Proposition 23 *If f is the density of the physical measure of a piecewise expanding map T as above and $\lambda > 2$. Then*

$$Var(f) \leq \frac{B}{1 - 2\lambda}$$

where B is defined as above.

---

[4]Note that this operator corresponds to the above cited transfes operator, but it acts on densities instead of measures.

Proof. (sketch) The topological mixing assumption implies that the map has only one invariant physical measures (see [45]). Let us use the above results iterating the constant density corresponding to the Lebesgue measure. Proposition 21 and Lemma 22 give that the variation of the iterates is bounded by $\frac{B}{1-2\lambda}$. Suppose that the limit measure has density f. By compactness of BV in L^1, the above properties give a bound on the variation of f (see [35] proof of Thm. 1). ∎

The following is a trivial consequence of the fact that $||f||_\infty \leq Var(f) + \int f d\mu$.

Corollary 24 *In the above situation* $||f||_\infty \leq \frac{B}{1-2\lambda} + 1$.

The bound on the density of the invariant measure, together with Corollary 19 gives the following *uniform* result on the computation of invariant measures of such maps.

Theorem 25 *Suppose a piecewise expanding map T and also its derivative is computable on $[0, 1] - \{d_1, ..., d_n\}$, and also its extensions to the closed intervals $[d_i, d_{i+1}]$ are computable. Then, the physical measure can be computed starting from a description of the system (the program computing the map and its derivative).*

Proof. (sketch) Since T and T' are computable on each interval $[d_i, d_{i+1}]$ we can compute a number λ such that $1 < \lambda \leq \inf(T')$ (see [26] Proposition 3) If we consider the iterate T^N instead of T the associated invariant density will be the same as the one of T, and if $\lambda^N > 2$, then T^N will satisfy all the assumptions needed on Proposition 23. Then we have a bound on the density and we can apply Corollary 19 to compute the invariant measure. ∎

3.4 Unbounded densities, non uniformly hyperbolic maps

The above results ensure computability of some absolutely continuous invariant measure with bounded density. If we are interested in situations where the density is unbounded, we can consider a new norm, "killing" singularities.

Let us hence consider a computable function $f : X \to \mathbb{R}$ and

$$\|\mu\|_{f,\alpha} = \sup_{x \in X, r > 0} \frac{f(x)\mu(B(x,r))}{r^\alpha}.$$

If α and K are computable and X is recursively compact then

$$V_{\alpha,K} = \{\mu \in PM(X) : \|\mu\|_{f,\alpha} \leq K\} \tag{4}$$

is recursively compact, and 18 also hold for this norm. If f is such that $f(x) = 0$ when $\lim_{r \to 0} \frac{\mu(B(x,r))}{r^\alpha} = \infty$ this can let the norm be finite when the density diverges.

As an example, where this can be applied, let us consider the Manneville Pomeau type maps on the unit interval. These are maps of the type $x \to x + x^z (\mathrm{mod} \ 1)$.

When $1 < z < 2$ the dynamics has a unique absolutely continuous invariant measure μ_z. This measure has a density $e_z(x)$ which diverges at the origin, and $e_z(x) \asymp x^{-z+1}$ and is bounded elsewhere (see [32] Section 10 and [45] Section 3 e.g.). If we consider the norm $\|.\|_{f,1}$ with $f(x) = x^2$ we have that $\|\mu_z\|_{f,1}$ is finite for each such z. By this it follows that for each such z the measure μ_z is computable.

3.5 Computable systems without computable invariant measures

We have seen several techniques to establish the computability of many physical invariant measures. This raises naturally the following question: a computable systems does necessarily have a computable invariant measure? what about ergodic physical measures?

The following is an easy example showing that this is not true in general even in quite simple systems, hence the whole question of computing invariant measures has some subtlety.

Let us consider a system on the unit interval given as follows. Let $\tau \in (0,1)$ be a lower semi-computable real number which is not computable. There is a computable sequence of rational numbers τ_i such that $\sup_i \tau_i = \tau$. For each i consider $T_i(x) = \max(x, \tau_i)$ and

$$T(x) = \sum_{i \geq 1} 2^{-i} T_i.$$

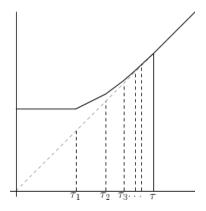The functions T_i are uniformly computable so T is also computable.

Figure 1: The map T.

The system $([0,1], T)$ is hence a computable dynamical system. T is non-decreasing, and $T(x) > x$ if and only if $x < \tau$.

This system has a physical ergodic invariant measure which is δ_τ, the Dirac measure placed on τ. The measure is physical because τ attracts all the interval at its left. Since τ is not computable then δ_τ is not computable. We remark that coherently with the previous theorems δ_τ is not isolated.

It is easy to prove, by a simple dichotomy argument, that a computable function from $[0,1]$ to itself must have a computable fixed point. Hence it is not possible to construct a system over the interval having no computable invariant measure (we always have the δ over the fixed point). With some more work we will see that such an example can be constructed on the circle. Indeed the following can be established

Proposition 26 *There is a computable, continuous map T on the circle having no computable invariant probability measure.*

For the description of the system and for applications to reverse mathematics see [26] (Proposition 12).

4 Computability in Complex Dynamics

Let K be a compact subset of the plane. Informally speaking, in order to draw the set K on a computer screen we need a program which is able to decide, given some precision ϵ, if pixel p has to be colored or not. By representing pixel p by a ball $B(p, \epsilon)$ where $(p, \epsilon) \in \mathbb{Q}^2$, the question one would like to answer is: does $B(p, \varepsilon)$ intersects K ? The following definition captures exactly this idea:

Definition 27 *A compact set $K \subset \mathbb{R}^2$ is said to be **computable** if there is an algorithm $\mathcal{A}$ such that, upon input (p, ϵ):*

- *halts and outputs "yes" if $K \cap B(p, \epsilon) \neq \emptyset$,*

- *halts and outputs "no" if $K \cap \overline{B(p, \epsilon)} = \emptyset$,*

- *run forever otherwise.*

We remark that if a compact set is computable under the definition above, then it is recursively compact in the sense of Definition 11. The converse is however false.

4.1 Julia sets and the Brolin-Lyubich measure

The question of whether Julia sets are computable under this definition has been completely solved in a series of papers by Binder, Braverman and Yampolsky. See [4, 5, 11, 12, 13]. Here we review some of their results. For simplicity, we give the definition of the Julia set only for quadratic polynomials. Consider a quadratic polynomial
$$P_c(z) = z^2 + c : \mathbb{C} \to \mathbb{C}.$$

18

Obviously, there exists a number M such that if $|z| > M$, then the iterates of z under P will uniformly scape to ∞. The **filled Julia set** is then defined by:

$$K_c = \{z \in \mathbb{C} \ \sup_n |P^n(z)| < \infty\}.$$

That is, the set of points whose orbit remains bounded. For the filled Julia set one has the following result:

Theorem 28 ([13]) *The filled Julia set is always computable.*

The **Julia set** can be now defined by:

$$J_c = \partial K_c,$$

where $\partial(A)$ denotes the boundary of A. The Julia set is the *repeller* of the dynamical system generated by P_c. For all but finitely many points, the limit of the n-th preimages $P_c^{-n}(z)$ coincides with the Julia set J_c. The dynamics of P_c on the set J_c is chaotic, again rendering numerical simulation of individual orbits impractical. Yet Julia sets are among the most drawn mathematical objects, and countless programs have been written for visualizing them.

In spite of this, the following result was shown in [11].

Theorem 29 *There exist* computable *quadratic polynomials* $P_c(z) = z^2 + c$ *such that the Julia set J_c is* not computable.

This phenomenon of *non-computability* is rather subtle and rare. For a detailed exposition, the reader is referred to the monograph [12].

Thus, we cannot accurately simulate the set of limit points of the preimages $(P_c)^{-n}(z)$, but what about their statistical distribution? The question makes sense, as for all $z \neq \infty$ and every continuous test function ψ, the averages

$$\frac{1}{2^n} \sum_{w \in (P_c)^{-n}(z)} \psi(w) \xrightarrow[n \to \infty]{} \int \psi d\lambda,$$

where λ is the *Brolin-Lyubich probability measure* [14, 39] supported on the Julia set J_c. We can thus ask whether the the Brolin-Lyubich measure is computable. Even if $J_c = \text{Supp}(\lambda)$ is not a computable set, the answer does not *a priori* have to be negative. In fact, the following result holds:

Theorem 30 ([6]) *The Brolin-Lyubich measure is always computable.*

The proof of the previous result does not involve (perhaps surprisingly) much analytic machinery, but it follows from some general principles in the same spirit of the ideas introduced in Section 3.3.

4.2 The Mandelbrot set

Another important object in complex dynamics is the Mandelbrot set, which is defined as follows:

$$\mathcal{M} := \{c \in \mathbb{C} : 0 \in K_c\}.$$

That is, the set of parameters of quadratic polynomials for which the orbit of 0 remains bounded. As for Julia sets, there exists many computer programs to visualize it, and the question arise whether this set is actually computable in a rigorous sense. This is, up to date, an open question. However, some partial results have been obtained. For instance, in [29] it is proved that the Mandelbrot set is recursively compact in the sense of Definition 11. There exists also analytical questions about the Mandelbrot set that remain unsolved. For instance, it is unknown whether it is *locally connected*. It has been conjectured that this is indeed the case. Computability of the Mandelbrot set is closely related to this conjecture, as it has been observed in [29]:

Theorem 31 ([29]) *If the Mandelbrot set is locally connected, then it is computable.*

It is known that local connectivity implies another well known conjecture, namely the *hyperbolicity conjecture*, which in turn also implies computability. For a detailed exposition in this subject the reader is referred to [15].

5 Computing the speed of ergodic convergence

As recalled before, the Birkhoff ergodic theorem tells that, if the system is ergodic, there is a full measure set of points for which the averages of the values of the observable f along its trajectory (time averages) coincides with the spatial average of the observable f. Similar results can be obtained for the convergence in the L^2 norm, and others. Many, more refined results are linked to the speed of convergence of this limit. And the question naturally arise, if there is a possibility to compute this speed of convergence in a sense similar to Definition 5.

In the paper [2] some abstract results imply that in a computable ergodic dynamical system, the speed of convergence of such averages can be algorithmically estimated. On the other hand it is also shown that there are non ergodic systems where this kind of estimations are not possible. In [27] a very short proof of this result for ergodic systems is shown. We expose the precise result (Theorem 39) and the short proof in this section.

5.0.1 Convergence of random variables

We first precise what is meant by "compute the speed" in some pointwise a.e. convergence.

Definition 32 *A* **random variable** *on* (X, μ) *is a measurable function* $f : X \to \mathbb{R}$.

Definition 33 *Random variables* f_n **effectively converge in probability** *to* f *if for each* $\epsilon > 0$, $\mu\{x : |f_n(x) - f(x)| < \epsilon\}$ *converges effectively to* 1, *uniformly in* ϵ. *That is, there is a computable function* $n(\epsilon, \delta)$ *such that for all* $n \geq n(\epsilon, \delta)$, $\mu\{|f_n - f| \geq \epsilon\} < \delta$.

Definition 34 *Random variables* f_n **effectively converge almost surely** *to* f *if* $f'_n = \sup_{k \geq n} |f_k - f|$ *effectively converge in probability to* 0.

Definition 35 *A* **computable probability space** *is a pair* (X, μ) *where* X *is a computable metric space and* μ *a computable Borel probability measure on* X.

Let Y be a computable metric space. A function $(X, \mu) \to Y$ is ***almost everywhere computable*** (a.e. computable for short) if it is computable on an effective G_δ-set of measure one, denoted by $\mathrm{dom} f$ and called the *domain of computability of* f.

A ***morphism*** of computable probability spaces $f : (X, \mu) \to (Y, \nu)$ is a morphism of probability spaces which is a.e. computable.

Remark 36 *A sequence of functions* f_n *is uniformly a.e. computable if the functions are uniformly computable on their respective domains, which are uniformly effective* G_δ-*sets. Observe that in this case, intersecting all the domains provides an effective* G_δ-*set on which all* f_n *are computable. In the following we will apply this principle to the iterates* $f_n = T^n$ *of an a.e. computable function* $T : X \to X$, *which are uniformly a.e. computable.*

Remark 37 *The space* $L^1(X, \mu)$ *(resp.* $L^2(X, \mu)$*) can be made a computable metric space, choosing some dense set of bounded computable functions as ideal elements. We say that an integrable function* $f : X \to \overline{\mathbb{R}}$ *is* $L^1(X, \mu)$-*computable if its equivalence class is a computable element of the computable metric space* $L^1(X, \mu)$. *Of course, if* $f = g$ μ-*a.e., then* f *is* $L^1(X, \mu)$-*computable if and only if* g *is. Basic operations on* $L^1(X, \mu)$, *such as addition, multiplication by a scalar, min, max etc. are computable. Moreover, if* $T : X \to X$ *preserves* μ *and* T *is a.e. computable, then* $f \to f \circ T$ *(from* L^1 *to* L^1*) is computable (see [30]).*

Let us call (X, μ, T) a ***computable ergodic system*** if (X, μ) is a computable probability space where T is a measure preserving morphism and (X, μ, T) is ergodic. Let $||f||$ denote the L^1 norm or the L^2 norm.

Proposition 38 *Let* (X, μ, T) *be a computable ergodic system. Let* f *be a computable element of* $L^1(X, \mu)$ *(resp.* $L^2(X, \mu)$*).*

The L^1 *convergence (resp.* L^2 *convergence) of the Birkhoff averages* $A_n = (f + f \circ T + \ldots + f \circ T^{n-1})/n$ *is effective. That is: there is an algorithm* $n : \mathbb{Q} \to \mathbb{N}$ *such that for each* $m \geq n(\epsilon)$ $||A_m - \int f d\mu|| \leq \epsilon$. *Moreover the algorithm depends effectively on* T, μ, f.

Proof. Replacing f with $f - \int f d\mu$, we can assume that $\int f d\mu = 0$. The sequence $||A_n||$ is computable (see Remark 37) and converges to 0 by the ergodic theorems.

Given $p \in N$, we write $m \in N$ as $m = np + k$ with $0 \leq k < p$. Then

$$A_{np+k} = \frac{1}{np+k} \left(\sum_{i=0}^{n-1} p A_p \circ T^{pi} + k A_k \circ T^{pn} \right)$$

$$||A_{np+k}|| \leq \frac{1}{np+k} (np||A_p|| + k||A_k||)$$

$$\leq ||A_p|| + \frac{||A_k||}{n}$$

$$\leq ||A_p|| + \frac{||f||}{n}.$$

Let $\epsilon > 0$. We can compute some $p = p(\epsilon)$ such that $||A_p|| < \epsilon/2$. Then we can compute some $n(\epsilon) \geq \frac{2}{\epsilon}||f||$. The function $m(\epsilon) := n(\epsilon)p(\epsilon)$ is computable and for all $m \geq m(\epsilon)$, $||A_m|| \leq \epsilon$. $\square$

5.1 Effective almost sure convergence

Now we use the above result to find a computable estimation for the a.s. speed of convergence.

Theorem 39 *Let (X, μ, T) be a computable ergodic system. If f is $L^1(X, \mu)$-computable, then the a.s. convergence is effective. Moreover, the rate of convergence can be computed as above starting from T, μ, f.*

This will be proved by the following

Proposition 40 *If f is $L^1(X, \mu)$-computable as above, and $||f||_\infty$ is bounded, then the almost-sure convergence is effective (uniformly in f and a bound on $||f||_\infty$ and on T, μ).*

To prove this we will use the Maximal ergodic theorem:

Lemma 41 (Maximal ergodic theorem) *For $f \in L^1(X, \mu)$ and $\delta > 0$,*

$$\mu(\{\sup_n |A_n^f| > \delta\}) \leq \frac{1}{\delta}||f||_1.$$

The idea is simple: compute some p such that $||A_p^f||_1$ is small, apply the maximal ergodic theorem to $g := A_p^f$, and then there is n_0, that can be computed, such that A_n^f is close to A_n^g for $n \geq n_0$.

Proof (of Proposition 40) Let $\epsilon, \delta > 0$. Compute p such that $||A_p^f|| \leq \delta\epsilon/2$. Applying the maximal ergodic theorem to $g := A_p^f$ gives:

$$\mu(\{\sup_n |A_n^g| > \delta/2\}) \leq \epsilon. \tag{5}$$

Now, A_n^g is not far from A_n^f: expanding A_n^g, one can check that

$$A_n^g = A_n^f + \frac{u \circ T^n - u}{np},$$

where $u = (p-1)f + (p-2)f \circ T + \ldots + f \circ T^{p-2}$. $||u||_\infty \leq \frac{p(p-1)}{2}||f||_\infty$ so if $n \geq n_0 \geq 4(p-1)||f||_\infty/\delta$, then $||A_n^g - A_n^f||_\infty \leq \delta/2$. As a result, if $|A_n^f(x)| > \delta$ for some $n \geq n_0$, then $|A_n^g(x)| > \delta/2$. From (5), we then derive

$$\mu(\{ \sup_{n \geq n_0} |A_n^f| > \delta\}) \leq \epsilon.$$

As n_0 can be computed from δ and ϵ, we get the result. $\square$

Remark 42 *This result applies uniformly to a uniform sequence of computable $L^\infty(X,\mu)$ observables f_n.*

We now extend this to $L^1(X,\mu)$-computable functions, using the density of $L^\infty(X,\mu)$ in $L^1(X,\mu)$.

Proof. (of Theorem 39) Let $\epsilon, \delta > 0$. For $M \in \mathbb{N}$, let us consider $f'_M \in L^\infty(X,\mu)$ defined as

$$f'_M(x) = \left\{ \begin{array}{ll} \min(f, M) & if \ f(x) \geq 0 \\ \max(f, -M) & if \ f(x) \leq 0. \end{array} \right.$$

Compute M such that $||f - f'_M||_1 \leq \delta\epsilon$. Applying Proposition 40 to f'_M gives some n_0 such that $\mu(\{\sup_{n \geq n_0} |A_n^{f'_M}| > \delta\}) < \epsilon$. Applying Lemma 41 to $f''_M = f - f'_M$ gives $\mu(\{\sup_n |A_n^{f''_M}| > \delta\}) < \epsilon$. As a result, $\mu(\{\sup_{n \geq n_0} |A_n^f| > 2\delta\}) < 2\epsilon$. $\square$

Remark 43 *We remark that a bounded a.e. computable function, as defined in Definition 35 is a computable element of $L^1(X,\mu)$ (see [30]). Conversely, if f is a computable element of $L^1(X,\mu)$ then there is a sequence of uniformly computable functions f_n that effectively converge μ-a.e. to f.*

6 Computing pseudorandom points, constructive ergodic theorem

Let X again be a computable metric space and μ a computable probability measure on it. Suppose X is complete.

Points satisfying the above recalled pointwise ergodic theorem for an observable f, will be called *typical for f* .

Points which are typical for each f which is continuous with compact support are called *typical for the measure μ* (and for the dynamics).

The set of computable points in X (see Definition 6) being countable, is a very small (invariant) set, compared to the whole space. For this reason, a

computable point could be rarely be expected to be typical for the dynamics, as defined before. More precisely, the Birkhoff ergodic theorem and other theorems which hold for a full measure set, cannot help to decide if there exist a computable point which is typical for the dynamics. Nevertheless computable points are the only points we can use when we perform a simulation or some explicit computation on a computer.

A number of theoretical questions arise naturally from all these facts. Due to the importance of forecasting and simulation of a dynamical system's behavior, these questions also have some practical motivation.

Problem 44 *Since simulations can only start with computable initial conditions, given some typical statistical behavior of a dynamical system, is there some computable initial condition realizing this behavior? how to choose such points?*

Such points could be called *pseudorandom* points, and a result showing its existence and their computability from the description of the system could be seen as a *constructive* version of the pointwise ergodic theorem.

Meaningful simulations, showing typical behaviors of the dynamics can be performed if computable, pseudorandom initial conditions exist (and can be computed from the description of the system). Thanks to a kind of effective Borel-Cantelli lemma, in [24] the above problem was solved affirmatively for a class of systems which satisfies certain technical assumptions which includes systems whose decay of correlation is faster that $C \log^2(time)$. After the results on the estimation of the rate of convergence given in the previous section we can remove the technical assumption on the speed of decay of correlations. We will moreover show how the result is uniform also in T and μ (the pseudorandom points will be calculated from a description of the system).

The following result ([24], Theorem 2 or [27]) shows that given a sequence f_n which converges effectively a.s. to f and given its speed of convergence then there it is possible to compute points x_i for which $f_n(x_i) \to f(x_i)$.

Theorem 45 *Let X be a complete metric space. Let f_n, f be uniformly a.e. computable random variables. If f_n effectively converges almost surely to f then the set $\{x : f_n(x) \to f(x)\}$ contains a sequence of uniformly computable points which is dense in $\mathrm{Supp}(\mu)$. Moreover, the effective sequence found above depends algorithmically on f_n and on the function $n(\delta, \epsilon)$ giving the rate of convergence*

Since by the results of the previous section, $n(\delta, \epsilon)$ can be calculated starting from T, μ and f. This result hence can be directly used to find typical points for the dynamics. Indeed the following holds (see [27] for the details)

Theorem 46 *If (X, μ, T) is a computable ergodic system, f is $L^1(X, \mu)$ and a.e. computable then there is a uniform sequence x_i of computable points which is dense on the support of μ such that for each i*

$$\lim_{n \to \infty} \frac{1}{n} \sum f(T^n(x_i)) = \int f \, \mathrm{d}\mu.$$

Moreover this sequence can be computed starting from a description of T, μ and f.

Since the above result is uniform in f and it is possible to construct a r.e. set of computable observables which is dense in the space of compactly supported continuous functions we can also obtain the following (see again [27] for the details)

Theorem 47 *If (X, μ, T) is a computable ergodic system then there is a uniform sequence x_n of computable points which is dense on the support of μ such that for each n, x_n is $\mu-$typical. Moreover this sequence can be computed starting from a description of T and μ*

Now, as an application we get together the uniform results about computation of invariant measures for the class of piecewise expanding functions shown in Subsection 3.3.1 to show that in that class the pseudorandom points can be calculated starting from a description of the map. Indeed since in that class the physical invariant measure μ can be calculated starting from T (see Theorem 25). We hence obtain

Corollary 48 *Each piecevise expanding map, with computable derivative, as in the assumptions of Theorem 25 has a sequence of pseudorandom points which is dense in the support of the measure. Moreover this sequence can be computed starting from the description of the system.*

Hence establishing some kind of constructive versions of the ergodic theorem.

7 Conclusions and directions

In this article we have reviewed some results about the rigorous computation of invariant measures, invariant sets and typical points. Here, the sentence *rigorous computation* means "computable by a Turing Machine". Thus, this can be seen as a theoretical study of which infinite objects in dynamics can be arbitrarily well approximated by a modern computer (in an absolute sense), and which cannot. In this line, we presented some general principles and techniques that allow the computation of the relevant objects in several different situations. On the other hand, we also presented some examples in which the computation of the relevant objects is not possible at all, stating some theoretical limits to the abilities of computers when used to simulate dynamical systems.

The examples of the second kind, however, seem to be rather rare. An important question is therefore whether this phenomenon of non-computability is robust or prevalent in any sense, or if it is rather exceptional. For example, one could ask whether the non-computability is destroyed by small changes in the dynamics or whether the non-computability occurs with small or null probability.

Besides, in this article we have not considered the efficiency (and therefore the feasibility) of any of the algorithms we have developed. An important (and

more difficult) remaining task is therefore the development of a *resource bounded* version of the study presented in this paper. In the case of Julia sets, for instance, it has been shown in [9, 43, 10] that hyperbolic Julia sets, as well as some Julia sets with parabolics, are computable in polynomial time. On the other hand, in [5] it was shown that there exists computable Siegel quadratic Julia sets whose time complexity is arbitrarily high.

For the purpose to compute the invariant measure form the description of the system, in section 3.3.1 we had to give explicit estimations on the constants in the Lasota Yorke inequality. This step is important also when techiques different from ours are used (see [37] e.g.). Similar estimations could be done in other classes of systems, following the way the Lasota Yorke inequality is proved in each class (although, sometimes this is not a completely trivial task and requires the "effectivization" of some step in the proof). A more general method to have an estimation for the constants or other ways to get information on the regularity of the invariant measure would be useful.

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