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Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties

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Abstract. We introduce a class of dynamical systems on a Riemannian manifold with singularities having attractors with strong hyperbolic behavior of trajectories. This class includes a number of famous examples such as the Lorenz type attractor, the Lozi attractor and some others which have been of great interest in recent years. We prove the existence of a special invariant measure which is an analog of the Bowen–Ruelle–Sinai measure for classical hyperbolic attractors and study the ergodic properties of the system with respect to this measure. We also describe some topological properties of the system on the attractor. Our results can be considered a dissipative version of the theory of systems with singularities preserving the smooth measure.

0. *Introduction*

At present there is a rather widespread opinion that instability is one of the main reasons for stochasticity in completely deterministic dynamical systems. This opinion is based on rigorous results in the study of the stochasticity of hyperbolic and some quasihyperbolic attractors (such as the Lorenz attractor, the Lozi attractor, etc). It is also based on a numerical investigation of some physical origins where the assurance both in stochasticity of the limit set and in instability of the trajectories in its neighborhood takes place. Moreover, the linear approximation exhibits in general, a rather strong instability. Usually the models of such a type are described by systems of ordinary differential equations and it is convenient for the study to pass from the phase flow to the first-return time map (the Poincare map) of a certain cross-section surface. This map is, as a rule, discontinuous which creates additional complications for the investigations.

In this paper we will introduce and study a new class of maps having 'generalized hyperbolic attractors'. They are rather strongly unstable. In the linear approximation their instability is as strong as it is in classical hyperbolic attractors. However, the maps considered here are discontinuous on some closed subset (which is usually

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the union of a finite number of submanifolds). There are trajectories which very often approach 'anomalously near' the discontinuity set. Although the set of such trajectories is 'small enough' their existence implies weakening of hyperbolicity. In fact the hyperbolicity of our maps is as weak as one encounters in systems with nonzero Lyapunov exponents. Thus the systems with generalized hyperbolic attractors describe a rather widespread way for the appearance of stochasticity.

Our class of maps is described by a collection of axioms. In particular, the hyperbolicity conditions are given by means of an invariant system of cones (i.e. by requirements on the differentials of maps). Such an approach is due to Alekseev [**1**], Anosov [**2**] and Sinai [**21**] and was developed for the attractors by Afraimovich *et al.* [**4**]. There are also some conditions for estimating the rate of growth of the differential of the system in a neighborhood of the discontinuity set. They have the same meaning as the analogous requirements in the definition of general systems with singularities (cf [**10**] and also [**8**]).

The aim of our work is to describe the ergodic properties of the dynamical systems having the generalized hyperbolic attractors. In particular, we will prove the existence of Gibbs u-measures (cf Theorem 1). They are analogous to Bowen-Ruelle-Sinai measures for classical hyperbolic attractors (cf [**16**]). Our approach for the construction of such measures is a generalization of the method given in [**19**]. We will give the description of ergodic properties of the systems with respect to Gibbs u-measure (cf Theorems 2-4, 6, 7).

We will also study some topological properties of maps on generalized hyperbolic attractors. In particular, we will prove (under certain additional conditions) an analog of the theorem of spectrum decomposition on basic sets for axiom A diffeomorphisms (cf § 9). It is worthwhile to notice two circumstances. First, in our case a number of components of topological transitivity is in general countable. Secondly, for a typical point (with respect to the Riemannian volume) in a basin of a generalized hyperbolic attractor (whose Riemannian volume is positive) its trajectory can be considered to be 'quite stochastic' (cf Theorem 3). However, this basin is not in general a neighborhood of the attractor. One can happen that there exists a subset of a positive Riemannian volume in a small neighborhood of the attractor consisting of points whose trajectories go to the attractor but are not 'stochastic'.

We will consider some examples of generalized hyperbolic attractors. Among them there are Lorenz type attractors (described in [**6**]), generalized Lozi attractors (introduced in [**22**]), Belykh attractors (cf [**7**]). The metric and topological properties of Lorenz type attractors have been rather well established (cf [**4-6**, **9**, **23**]). In the case of generalized Lozi attractors only the existence of Bowen-Ruelle-Sinai measures has been proved in [**22**]. As for the Belykh attractor its topological structure and ergodic properties of the map acting on it have been almost unknown until now, in spite of the very simple form of this map.

1. *Definition of generalized hyperbolic attractors: local properties*

1.1. Let M be a smooth p-dimension Riemannian manifold, $K \subset M$ an open bounded connected subset with compact closure, $N \subset K$ a closed subset. Let also $f: K\setminus N \to K$ be a map, satisfying the following hypotheses:

(H1) f is a C^2-diffeomorphism from the open set $K\setminus N$ onto its image $f(K\setminus N)$;
(H2) there exist $C_i > 0$, $\alpha_i \geq 0$, $i = 1, 2$ such that

$$\|d^2 f_x\| \leq C_1 \rho(x, N^+)^{-\alpha_1} \quad \text{for any } x \in K\setminus N,$$
$$\|d^2 f_x^{-1}\| \leq C_2 \rho(x, N^-)^{-\alpha_2} \quad \text{for any } x \in f(K\setminus N)$$

where ρ is the Riemannian distance in M, $N^+ = N \cup \partial K$ is the 'discontinuity set' for f and

$$N^- = \{y \in K: \text{there exist } z \in N^+ \text{ and } z_n \in K\setminus N^+ \text{ such that } z_n \to z, f(z_n) \to y\}$$

is the 'discontinuity set' for f^{-1} (the image of the discontinuity set for f).

Define

$$K^+ = \{x \in K: f^n(x) \notin N^+, n = 0, 1, 2, \ldots\}$$

and set

$$D = \bigcap_{n \geq 0} f^n(K^+), \Lambda = \bar{D}.$$

We call Λ the attractor for f. It has the following properties which follow easily from the definitions.

PROPOSITION 1.
(1) $D = \Lambda \setminus (\bigcup_{n \in \mathbb{Z}} f^n(N^+))$;
(2) *The maps* f, f^{-1} *are defined on* D *and* $f(D) = D$, $f^{-1}(D) = D$.

1.2. We set for arbitrary $\varepsilon > 0$ and $l = 1, 2, \ldots$

$$\hat{D}^+_{\varepsilon,l} = \{z \in K^+: \rho(f^n(z), N^+) \geq l^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \ldots\},$$
$$D^-_{\varepsilon,l} = \{z \in \Lambda: \rho(f^{-n}(z), N^-) \geq l^{-1} e^{-\varepsilon n}, n = 0, 1, 2, \ldots\},$$
$$D^+_{\varepsilon,l} = \hat{D}^+_{\varepsilon,l} \cap \Lambda, D^0_{\varepsilon,l} = D^-_{\varepsilon,l} \cap D^+_{\varepsilon,l},$$
$$D^\pm_\varepsilon = \bigcup_{l \geq 1} D^\pm_{\varepsilon,l}, D^0_\varepsilon = \bigcup_{l \geq 1} D^0_{\varepsilon,l}.$$

It is easy to see that the sets $\hat{D}^+_{\varepsilon,l}$, $D^\pm_{\varepsilon,l}$, $D^0_{\varepsilon,l}$ are closed; $D^0_\varepsilon = D^+_\varepsilon \cap D^-_\varepsilon$; D^+_ε is f-invariant, D^-_ε is f^{-1}-invariant; D^0_ε is both f and f^{-1}-invariant. Besides, $D^0_\varepsilon \subset D$ for any ε.

We say that Λ is *regular* if

(H3) $D^0_\varepsilon \neq \varnothing$ for all small enough $\varepsilon > 0$.

We will give an additional condition on Λ to be regular. Consider the function $\varphi(z) = \rho(z, N^+)$ and define for $z \in D$

$$\chi^\pm_\varphi(z) = \overline{\lim_{n \to \pm\infty}} \ln \varphi(f^n(z)).$$

It is obvious that functions $\chi^\pm_\varphi(z)$ are both f- and f^{-1} invariant on D and $\chi^\pm_\varphi(z) \leq 0$ for $z \in D$. It follows directly from the definition of $\chi^\pm_\varphi(z)$ that for any $\varepsilon > 0$ and $z \in D$ there exists $K(\varepsilon, z) > 0$ such that for any $n \in \mathbb{Z}$

$$\rho(f^n(z), N) = \varphi(f^n(z)) \leq K(\varepsilon, z) \exp((\chi^\pm_\varphi(z) + \varepsilon)|n|). \tag{1}$$

Define $\tilde{D}^{\pm}=\{z\in D$: there exists the limit

$$\chi_{\varphi}^{\pm}(z)=\lim_{n\to\pm\infty}\frac{1}{|n|}\ln\varphi(f^n(z))=0\},$$

$\tilde{D}^0=\tilde{D}^+\cap\tilde{D}^-$. It is easy to see that these sets are both f and f^{-1}-invariant. Moreover for any $\varepsilon>0$ and $z\in\tilde{D}^+$ (respectively $z\in\tilde{D}^-$ or $\tilde{D}^0$) there exists $C(\varepsilon, z)$ such that for any $n>0$ (respectively $n<0$ or $n\in\mathbb{Z}$)

$$\rho(f^n(z), N)=\varphi(f^n(z))\geq C(\varepsilon, z)\exp(-\varepsilon|n|). \tag{2}$$

It follows from here that $\tilde{D}^{\pm}\subset D_{\varepsilon}^{\pm}$, $\tilde{D}^0\subset D_{\varepsilon}^0$ for any $\varepsilon>0$.

Remark. Let $z\in\tilde{D}^0$ and $\tilde{C}(\varepsilon, z)=\sup\{C(\varepsilon, z)$ for which (2) holds for all $n\in\mathbb{Z}\}$. One can show (cf [14]) that for any $m\in\mathbb{Z}$

$$\tilde{C}(\varepsilon, f^m(z))\geq\tilde{C}(\varepsilon, z)\exp(-\varepsilon|m|). \tag{3}$$

For $A\subset\Lambda$ denote by $f^{-1}(A)=\{z\in\Lambda\backslash N^+: f(z)\in A\}$. A measure μ on Λ is called *f-invariant* if $\mu(A)=\mu(f^{-1}(A))$ for any $A\subset\Lambda$. Let M_f be a collection of normalized Borel f-invariant measures on Λ.

PROPOSITION 2. *Assume that there exists $\mu\in M_f$ such that*

$$(1)\quad \mu(D)>0; \qquad (2)\quad \left|\int_{\Lambda}\ln\varphi(z)\,d\mu(z)\right|<\infty.$$

Then $\mu(\tilde{D}^0)>0$ and, in particular, Λ is regular.

Proof. One can assume that μ is ergodic (otherwise one should consider its arbitrary ergodic component). It follows from the conditions (1) and (2) in Proposition 2, that for μ-almost every $z\in D$ there exist the limits

$$\chi_{\varphi}^{\pm}(z)=\lim_{n\to\pm\infty}\frac{1}{|n|}\ln\varphi(f^n(z))$$

and $\chi_{\varphi}^{+}(z)=\chi_{\varphi}^{-}(z)$. Hence $\tilde{D}^+=\tilde{D}^-=\tilde{D}^0$ (μ-mod 0). We will show that $\mu(\tilde{D}^+)>0$. Assuming the contrary we have by virtue of (1) that for μ-almost every $z\in D$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k(z))=0.$$

Therefore $\int_D\varphi\,d\mu=0$. But it is impossible because $\mu(D)>0$. □

We will give another condition for the regularity of Λ. Denote by $U(\varepsilon, N^+)$ the ε-neighborhood (in K) of N^+ and by ν the Riemannian volume in K.

PROPOSITION 3. *Assume that there exist $C>0$, $q>0$ such that for any $\varepsilon>0$ and $n>0$*

$$\nu(f^{-n}(U(\varepsilon, N^+)\cap f^n(K^+)))\leq C\varepsilon^q. \tag{4}$$

Then Λ is regular.

Proof. Set

$$\nu_n=f_*^n\nu, \qquad \mu_n=\frac{1}{n}\sum_{k=0}^{n-1}\nu_k.$$

It is easy to see that the sequence of measures μ_n is compact in a weak topology in $\overline{K}$ and let μ be a limit measure. It is obvious that μ is concentrated on Λ and is invariant under f. Take arbitrary $\gamma > 0$, $\varepsilon > 0$. We have from the definition of $\hat{D}^+_{\varepsilon,l}$ that for $l > \gamma^{-1}$

$$K \backslash \hat{D}^+_{\varepsilon,l} \subset \{x \in K: \text{there exists } m \in \mathbb{Z}^+ \text{ such that } f^m(x) \in U(\gamma e^{-\varepsilon m}, N^+)\}$$

Therefore the condition (2) implies that

$$\begin{aligned} \nu_m(K \backslash \hat{D}^+_{\varepsilon,l}) &\le \sum_{m=0}^{\infty} \nu(f^{-m}(U(\gamma e^{-\varepsilon m}, N^+) \cap K_m)) \\ &\le C\gamma^q \sum_{m=0}^{\infty} e^{-q\varepsilon m} \le C_1 \gamma^q, \end{aligned}$$

where $C_1 > 0$ is a constant independent of n. This means that

$$\sqrt{\mu_m}(\hat{D}^+_{\varepsilon,l}) \ge 1 - C_1 \gamma^q$$

for all large enough l. Taking into consideration that the sets $\hat{D}^+_{\varepsilon,l}$ are closed we obtain from here that

$$\mu(\hat{D}^+_{\varepsilon,l}) \ge 1 - C_1 \gamma^q.$$

As $\mu(\Lambda) = 1$ we have $\mu(D^+_{\varepsilon,l}) \ge 1 - C_1 \gamma^q$. This implies that $\mu(D^+) = 1$ and consequently $\mu(N^+) = 0$. Since μ is f-invariant this means by virtue of Proposition 1 that $\mu(D) = 1$. It follows from (1) that given $\varepsilon > 0$ small enough and any $n > 0$

$$\begin{aligned} \left| \int_K \ln \varphi(z)\, d\mu_n(z) \right| &< \left| \int_{U(\varepsilon, N^+)} \ln \varphi(z)\, d\mu_n(z) \right| + \left| \int_{K \backslash U(\varepsilon, N^+)} \ln \varphi(z)\, d\mu_n(z) \right| \\ &\le C + \ln(1/\varepsilon) < \infty. \end{aligned}$$

We have from here

$$\left| \int_\Lambda \ln(z)\, d\mu(z) \right| < \infty.$$

Now the desired result follows from Proposition 2. □

Remark. Condition (4) formulated for the case of dissipative systems is analogous to the corresponding condition for conservative systems with singularities (cf [**10**]).

1.3. Denote by $C(z, \alpha, P)$ a cone at point $z \in K$ ($\alpha > 0$ is a number, P is a subspace in T_zM) consisting of all $v \in T_zM$ such that

$$\measuredangle(v, P) \overset{\text{def}}{=} \min_{w \in P} \measuredangle(v, w) \le \alpha.$$

We say that Λ is a *generalized hyperbolic attractor* if there exist $C > 0$, $0 < \lambda < 1$, a function $\alpha(z)$ and two fields of subspaces $P^{(s)}(z)$, $P^{(u)}(z) \subset T_zM$, $\dim P^{(s)}(z) = q$, $\dim P^{(u)}(z) = p - q$ ($p = \dim M$) $z \in K \backslash N^+$ such that the cones $C^{(s)}(z) = C^{(s)}(z, \alpha(z), P^{(s)}(z))$ and $C^{(u)}(z) = C(z, \alpha(z), P^{(u)}(z))$ satisfy the following conditions:

(1) the angle between $C^{(s)}(z)$ and $C^{(u)}(z)$ is greater than *const* > 0 (uniformly over $z \in K \backslash N^+$); in particular, $C^{(s)}(z) \cap C^{(u)}(z) = 0$;

(2) $df(C^{(u)}(z)) \subset C^{(u)}(f(z))$ for any $z \in K \setminus N^+$; $df^{-1}(C^{(s)}(z)) \subset C^{(s)}(f^{-1}(z))$ for any $z \in f(K \setminus N^+)$;
(3) for any $n > 0$
  (a) let $z \in K^+$, $v \in C^{(u)}(z)$ then $\|df^n v\| \geq C\lambda^{-n}\|v\|$;
  (b) let $z \in f^n(K^+)$, $v \in C^{(s)}(z)$ then $\|df^{-n}v\| \geq C\lambda^{-n}\|v\|$.

For $z \in D$ we set

$$E^{(s)}(z) = \bigcap_{n \geq 0} df^{-n}C^{(s)}(f^n(z)), \qquad F^{(u)}(z) = \bigcap_{n \geq 0} df^n C^{(u)}(f^{-n}(z)).$$

Repeating the arguments given in **[15]** one can show that for any $z \in D$
(1) $E^{(s)}(z)$, $E^{(u)}(z)$ are subspaces in T_zM; $\dim E^{(s)}(z) = q$, $\dim E^{(u)}(z) = p - q$;
(2) $T_zM = E^{(s)}(z) \oplus E^{(u)}(z)$, $E^{(s)}(z) \cap E^{(u)}(z) = 0$;
(3) the angle between $E^{(s)}(z)$ and $E^{(u)}(z)$ is greater than *const* > 0 (uniformly over z);
(4) for any $n \geq 0$

$$\|df^n v\| \leq C\lambda^n \|v\|, \qquad v \in E^{(s)}(z),$$
$$\|df^{-n}v\| \geq C^{-1}\lambda^{-n}\|v\|, \qquad v \in E^{(u)}(z).$$

1.4. It follows from the above assertions that subspaces $E^{(s)}(z)$ and $E^{(u)}(z)$ define the uniform hyperbolic structure on D. It means that $f|\Lambda$ is the smooth map of a hyperbolic type with singularities. Our aim is to construct an invariant measure for this map with 'good' ergodic properties. The first step is to define *local stable manifolds* $V^{(s)}(z)$ at every point $z \in D^+$ and *local unstable manifolds* $V^{(u)}(z)$ at every point $z \in D^-$. The discontinuous character of f implies that the 'size' of the local manifolds is only a measurable (but not continuous) function on D despite the fact that the hyperbolic structure on D is uniform. Therefore the situation is similar to one arising in the systems with non-zero Lyapunov exponents. We can study it using the methods presented in **[10, 11, 15, 18, 20]**. It is worthwhile to emphasize the following important circumstance. In the general theory of maps with non-zero Lyapunov exponents (both smooth and smooth with singularities) and preserving a measure equivalent to the Riemannian volume the set of regular points in the sense of Lyapunov plays the crucial role. It is the set where one can construct local stable and unstable manifolds whose size can decrease along trajectories but at most with a small exponential rate (in comparison with the rates of contraction and expansion). In our case the same role belongs to the set D^0_ε (especially in view of (3), cf also statement (5) in Proposition 4).

1.5. We give the exact formulations.

PROPOSITION 4. (cf **[11, 18, 20]**.) *There exists $\varepsilon > 0$ such that the following statements hold*:
(1) *there exist $\delta_l > 0$ and maps*

$$\varphi^{(s)}(z): B^{(s)}(\delta_l) \to E^{(u)}(z),\ z \in D^+_{\varepsilon,l},$$
$$\varphi^{(u)}(z): B^{(u)}(\delta_l) \to E^{(s)}(z),\ z \in D^-_{\varepsilon,l}$$

($B^{(s)}(\delta_l)$, $B^{(u)}(\delta_l)$ are balls in $E^{(s)}(z)$ and $E^{(u)}(z)$ respectively, of radius δ_l centered at zero) such that the sets

$$V^{(s)}(z)=\{\exp_z(u,\varphi^{(s)}(z)(u)) \colon u\in B^{(s)}(\delta_l)\},\ z\in D^-_{\varepsilon,l},$$
$$V^{(u)}(z)=\{\exp_z(u,\varphi^{(u)}(z)(u)) \colon u\in B^{(u)}(\delta_l)\},\ z\in D^+_{\varepsilon,l},$$

are C^1-submanifolds in K and

(2)
$$T_zV^{(s)}(z)=E^{(s)}(z),\qquad T_zV^{(u)}(z)=E^{(u)}(z);$$
$$f(V^{(s)}(z))\subset V^{(s)}(f(z)),\ z\in D^+_{\varepsilon,l},$$
$$f^{-1}(V^{(u)}(z))\subset V^{(u)}(f^{-1}(z)),\ z\in D^-_{\varepsilon,l};$$

(3) *there exist $\mu\in(\lambda,1)$, $C^{(1)}_l>0$ such that for any $n\geq 0$:*

$$\rho(f^n(y),f^n(z))\leq C^{(1)}_l\mu^n\rho(y,z)\quad \text{for } z\in D^+_{\varepsilon,l},\ y\in V^{(s)}(z), \tag{5}$$
$$\rho(f^{-n}(y),f^{-n}(z))\leq C^{(1)}_l\mu^n\rho(y,z)\quad \text{for } z\in D^-_{\varepsilon,l},\ y\in V^{(u)}(z); \tag{5'}$$

(4) *if $z\in D^+_{\varepsilon,l}$, $y\in D^+_{\varepsilon,l}\cap\exp_z(B^{(s)}(\delta_l)\times B^{(u)}(\delta_l))$ and $\rho(f^n(y),f^n(z))\to 0$ when $n\to\infty$ then $y\in V^{(s)}(z)$;*
if $z\in D^-_{\varepsilon,l}$, $y\in D^-_{\varepsilon,l}\cap\exp_z(B^{(s)}(\delta_l)\times B^{(u)}(\delta_l))$ and $\rho(f^{-n}(y),f^{-n}(z))\to 0$ when $n\to\infty$ then $y\in V^{(u)}(z)$;

(5) *there exists $C^{(2)}_l>0$ such that*

$$d(T_{y_1}V^{(s)}(z),\ T_{y_2}V^{(s)}(z))\leq C^{(2)}_l\rho(y_1,y_2) \tag{6}$$

for any $z\in D^+_{\varepsilon,l}$, $y_1,y_2\in V^{(s)}(z)$ and

$$d(T_{y_1}V^{(u)}(z),\ T_{y_2}V^{(u)}(z))\leq C^{(2)}_l\rho(y_1,y_2) \tag{6'}$$

for any $z\in D^-_{\varepsilon,l}$, $y_1,y_2\in V^{(u)}(z)$ (d is a distance between subspaces in TM);

(6) *there exists $\gamma>0$ such that for any $m\in\mathbb{Z}$*

$$\delta_{l+m}\geq\delta_l\exp(-\gamma|m|).$$

(7) *$V^{(s)}(z)$ (respectively $V^{(u)}(z)$) depends continuously on $z\in D^+_{\varepsilon,l}$ (respectively $z\in D^-_{\varepsilon,l}$).*

Remark. If $f\in C^{r+\alpha}$, $r\geq 1$ then $\varphi^{(s)}(z)$, $\varphi^{(u)}(z)\in C^r$ (for a fixed z); if $f\in C^{1+\alpha}$ then instead of (6), (6′) we have

$$d(T_{y_1}V^{(s)}(z),\ T_{y_2}V^{(s)}(z))\leq C^{(2)}_l\rho(y_1,y_2)^\alpha,$$
$$d(T_{y_1}V^{(u)}(z),\ T_{y_2}V^{(u)}(z))\leq C^{(2)}_l\rho(y_1,y_2)^\alpha.$$

Everywhere in the following we will write $D^\pm$ and $D^\pm_l$ instead of $D^\pm_\varepsilon$ and $D^\pm_{\varepsilon,l}$.

PROPOSITION 5. *$V^{(u)}(z)\subset D^-$ for any $z\in D^-$.*

Proof. Let $z\in D^-_l$ for some $l\geq 1$. Consider a ball B_n centered at $f^{-n}(z)$ of radius $z_n=rC^{(1)}_l\exp(-\varepsilon|n|)$ (ε is taken according to Proposition 4). We have that $B_n\cap N=\phi$ for small enough $\varepsilon>0$ and any $n\geq 0$. Define the sets $G_{\kappa,n}$, $\kappa=0,1,\dots,n$ by induction setting

$$G_{1,n}=B_n,\quad G_{\kappa,n}=B_{n-\kappa}\cap f(G_{\kappa-1,n}),\quad \kappa=1,2,\dots,n.$$

We have $G_{n-1,n}\subset f^n(K^+_n)$. It follows from [18] that

$$\bigcup_{n\leq 0}G_{n-1,n}\supset V^{(u)}(z).$$

Hence, $V^{(u)}(z) \subset D$. On the other hand Proposition 4 implies the existence of $m = m(l) > 0$ such that if $y \in V^{(u)}(z)$, then $y \in D_m^-$. The desired result follows from here. □

Let $A \subset \Lambda$. Define

$$\hat{f}(A) = f(A \backslash N^+), \qquad \hat{f}^{-1}(A) = \hat{f}^{-1}(A \backslash N^-).$$

The sets $\hat{f}^n(A)$ and $\hat{f}^{-n}(A)$ for $n > 1$ are defined in the same way. We set for $z \in D^+$

$$W^{(s)}(z) = \bigcup_{n \geq 0} \hat{f}^{-n}(V^{(s)}(f^n(z))),$$

and for $z \in D^-$

$$W^{(u)}(z) = \bigcup_{n \geq 0} \hat{f}^n(V^{(u)}(f^{-n}(z))).$$

The set $W^{(s)}(z)$ is a smooth imbedded, but possibly non-connected, submanifold in K of the dimension q. It is called the *global stable manifold* at z. If $y \in W^{(s)}(z)$ then all images $f^n(y)$ for $n \geq 0$ are correctly defined. For $y \in W^{(s)}(z)$ we denote by $B^{(s)}(y, r)$ a ball in $W^{(s)}(z)$ of radius r centered at y (with respect to the natural distance in $W^{(s)}(z)$; we restrict ourself to a connected component in $W^{(s)}(z)$). The assertions are true for $W^{(u)}(z)$. It is called the *global unstable manifold* at z. We will use the same notation.

Fix $r > 0$ and take $y \in W^{(s)}(z)$, $w \in B^{(s)}(y, r)$, $n \geq 0$ (respectively $y \in W^{(u)}(z)$, $w \in B^{(u)}(y, r)$, $n \leq 0$). We have by virtue of Proposition 4 (cf (5), (5')) that

$$\rho^{(s)}(f^n(y), f^n(w)) \leq C\mu^n \rho^{(s)}(y, w) \tag{7}$$

(respectively

$$\rho^{(u)}(f^{-n}(y), f^{-n}(w)) \leq C\mu^n \rho^{(u)}(y, w)) \tag{7'}$$

where $C = C(r) > 0$ is a constant.

2. *Gibbs u-measures: existence and ergodic properties*

2.1. We give the definition of Gibbs u-measures. Denote by $J^{(u)}(z)$ the Jacobian of the map $df \mid E^{(u)}(z)$ at a point $z \in D$. Fix $l > 0$, points $z \in D_l^-$, $y \in W^{(u)}(z)$, $n > 0$ and set

$$\kappa_n(z, y) = \prod_{j=0}^{n-1} [J^{(u)}(f^{-j}(z))][J^{(u)}(f^{-j}(y))]^{-1}.$$

One can derive the following assertion from Proposition 4 (cf (5'), (6')) and conditions (H1) and (H2).

PROPOSITION 6.

(1) *For any $l \geq 1$, $z \in D_l^-$, $y \in W^{(u)}(z)$ there exists a limit*

$$\kappa(z, y) = \lim_{n \to \infty} \kappa_n(z, y) > 0.$$

Moreover there is $r_l^1 > 0$ such that for any $\varepsilon > 0$, $r \in (0, r_l^1)$ one can find $N = N(\varepsilon, r)$ such that

$$\max_{z \in D_l^-} \max_{y \in \bar{B}^{(u)}(z,r)} |\kappa_n(z, y) - \kappa(z, y)| \leq \varepsilon$$

for any $n \geq N$.

(2) *The function $\kappa(z, y)$ is continuous on D_l^-; more precisely, for any $z \in D_l^-$, $y \in B^{(u)}(z, r)$ and any two sequences of points $\{z_n\}$, $\{y_n\}$ such that $z_n \to z$, $y_n \to y$, $z_n \in D_l^-$, $y_n \in B^{(u)}(z_n, r)$ we have*

$$\lim_{n\to\infty} \kappa(z_n, y_n) = \kappa(z, y).$$

(3) *For any $z \in D_l^-$, $y_1, y_2 \in W^{(u)}(z)$*

$$\kappa(z, y_1)\kappa(y_1, y_2) = \kappa(z, y_2).$$

Fix $l \geq 1$, a point $z \in D_l^-$ and let $B(z, r)$ be a ball in K centered at z of radius r.

PROPOSITION 7. *There exist $r_l^2 > r_l^3 > r_l^4 > 0$ such that the following is true. Let*

$$W(z) = \exp_z \{u \in E^{(s)}(z) \colon \|u\| \leq r_l^2\}. \tag{8}$$

Then for any $y \in B(z, r_l^4) \cap D_l^-$ the intersection $V^{u}(y) \cap W(z)$ is not empty and consists of a single point (which we denote by $[y, z]$); besides

$$V^{(u)}(y) \supset B^{(u)}([y, z], r_l^{(3)}). \tag{9}$$

Proof. This follows from Propositions 4 and 6 (cf also [**15**]). □

Take $r \leq r_l^4$ and set

$$\Pi = \Pi(z, r) = \bigcup_{y \in B(z,r) \cap D_l^-} B^{(u)}([y, z], r_l^{(3)}).$$

It is called the rectangle at z. Denote by $\nu^{(u)}$ the Riemannian volume on $W^{(u)}(z)$, $z \in D^-$ induced by the Riemannian metric in M; by $\xi = \xi(\Pi)$ the partition of $\Pi(z, r)$ by the sets $C_\xi(y) = B^{(u)}([y, z], r_l^{(3)})$, $y \in B(z, r) \cap D_l^-$. This partition is continuous and measurable with respect to any Borel measure on Λ.

Let μ be a Borel probability measure on Λ. Consider $z \in D_l^-$ and a rectangle $\Pi = \Pi(z, r)$ at z. Assume that $\mu(\Pi) > 0$ and denote by $\mu_\xi(y)$, $y \in B(z, r) \cap D_l^-$ the conditional measures. We say that μ is *the Gibbs u-measure* (or, simply, *u-measure*) if for any $l \geq 0$, $z \in D_l^-$ and $\Pi = \Pi(z, r)$, $(r \leq r_l^4)$ with $\mu(\Pi) > 0$

$$d\mu_\xi(y') = r(y)\kappa([z, y], y')\, d\nu^{(u)}(y').$$

Here $y \in B(z, r) \cap D_l^-$, $y' \in B^{(u)}([z, y], r_l^{(3)})$ and $r(y)$ is the 'normalizing factor' given by the formula

$$r(y) = \left(\int_{B^{(u)}([y,z], r_l^{(3)})} \kappa([z, y], y')\, d\nu^{(u)}(y') \right)^{-1}.$$

Denote by $\tilde{M}_f$ the class of measures $\mu \in M_f$ for which $\mu(D^0) = 1$ and by $M_f^{(u)}$ the class of Gibbs u-measures $\mu \in \tilde{M}_f$.

Any $\mu \in \tilde{M}_f$ is the measure with non-zero Lypunov exponents $\chi^{(1)}(x), \dots, \chi^{(p)}(x)$ at every $x \in D^0$. In addition, if μ is ergodic the functions $\chi^{(i)}(x)$, $i = 1, \dots, p$ are constant μ-almost everywhere. We denote the corresponding values by $\chi_\mu^{(i)}$ and assume that

$$\chi_\mu^{(1)} \geq \cdots \geq \chi_\mu^{(q)} > 0 > \chi_\mu^{(q+1)} \geq \cdots \geq \chi_\mu^{(p)}.$$

2.2. Fix $z \in D_l^-$, $r \leq r_l^{(3)}$ (cf (9)) and set

$$U_0 = B^{(u)}(z, r), \quad \tilde{U}_0 = U_0, \quad \tilde{U}_n = f(U_{n-1}), \quad U_n = \tilde{U}_n \setminus N^+ \tag{10}$$

and further

$$\bar{\bar{C}}_0=1, \qquad \bar{\bar{C}}_n=\left(\prod_{k=0}^{n-1} J^{(\mathrm{u})}(f^k(z))\right)^{-1}$$

Define the measures $\tilde{\nu}_n$ on U_n by the formula

$$d\tilde{\nu}_n(y)=\bar{\bar{C}}_n\kappa(f^n(z),y)d\nu^{(\mathrm{u})}(y), \quad n\geq 0,$$

and the measures ν_n on Λ putting for any Borel set $A\subset\Lambda$

$$\nu_n(A)=\tilde{\nu}_n(A\cap U_n), \quad n\geq 0. \tag{11}$$

PROPOSITION 8. *Assume that there exists a point $z\in D^-$ such that for any $k>0$*

$$\nu^{(\mathrm{u})}(W^{(\mathrm{u})}(f^k(z))\cap N^+)=0.$$

Then for any $n>0$ and any Borel set $A\subset\Lambda$

$$\nu_n(A)=\nu_0(f^{-n}(A))$$

(here measures ν_n are constructed with respect to z).

Proof. (Compare [19], Proposition 3.) For any $n\geq 0$ and any Borel set $A\subset\Lambda$ we have $A\cap\tilde{U}_n=f^n(f^{-n}(A)\cap U_0)$. It follows from here and the condition of Proposition 8 that

$$\begin{aligned}\nu_n(A)&=\tilde{\nu}_n(A\cap U_n)=\tilde{\nu}_n(A\cap\tilde{U}_n)\\&=\tilde{\nu}_n(f^n(f^{-n}(A)\cap U_0))\\&=\int_{f^{-n}(A)\cap U_0}\bar{\bar{C}}_n\kappa(f^n(z),f^n(y))\prod_{k=0}^{n-1}J^{(\mathrm{u})}(f^k(y))\,d\nu^{(\mathrm{u})}(y)\\&=\int_{f^{-n}(A)\cap U_0}\bar{\bar{C}}_n\kappa(f^n(z),f^n(y))\prod_{k=0}^{n-1}J^{(\mathrm{u})}(f^k(y))[\kappa(z,y)]^{-1}\,d\tilde{\nu}_0(y)\\&=\int_{f^{-n}(A)\cap U_0}\bar{\bar{C}}_n\prod_{k=0}^{n-1}J^{(\mathrm{u})}(f^k(z))\,d\tilde{\nu}_0(y)=\int_{f^{-n}(A)\cap U_0}d\tilde{\nu}_0(y)\\&=\tilde{\nu}_0(f^{-n}(A)\cap U_0)=\nu_0(f^{-n}(A)).\end{aligned}$$

□

2.3. We say that the attractor Λ has the property (H4) if there exist a point $z\in D^-$ and $C>0$, $t>0$, $\varepsilon_0>0$ such that for any $\varepsilon, 0<\varepsilon\leq\varepsilon_0$ and $n\geq 0$

$$\nu^{(\mathrm{u})}(V^{(\mathrm{u})}(z)\cap f^{-n}(U(\varepsilon,N^+)))\leq C\varepsilon^t,$$

where $U(\varepsilon, N^+)$ is the ε-neighborhood of N^+ in M.

It is easy to see that if Λ has the property (H4) then it satisfies the condition of Proposition 8.

THEOREM 1. *Assume that Λ is a generalized hyperbolic regular attractor having property (H4). Then there exists a measure $\mu\in M_f^{(\mathrm{u})}$ concentrated on D which satisfies conditions (1) and (2) of Proposition 2.*

Proof. Let $z\in D^-$ be the point mentioned in the property (H4) and ν_k be the measures on Λ constructed by (11) with respect to z. Consider the sequence of measures on Λ

$$\mu_n=\frac{1}{n}\sum_{k=0}^{n-1}\nu_k. \tag{12}$$

It is easy to see that it is weakly compact. We will show that there exists a subsequence of measures μ_{n_t} which converges (in the weak topology) to a f-invariant Gibbs u-measure on Λ. The crucial point in this approach is to prove that some limit measure μ for the sequence of measures μ_n is concentrated on D (e.a. $\mu(D)=1$; this is the measure that is the Gibbs u-measure). The proof of this fact is essentially based on the property (H4). Now we give the exact formulations.

LEMMA 1. *For any $\gamma>0$ there exists $l_0=0$ such that for any $n>0$ and $l\geq l_0$*

$$\mu_n(D_l^-)\geq 1-\gamma.$$

Proof. Fix $\gamma>0$, $\alpha>0$, $n>0$ and let U_n be the sets defined by (10). It follows from the definition of D_l^- that for large enough l (depending only on α)

$$U_n\backslash D_l^-\subset\{x\in U_n\text{: there exists } k, 0\leq k\leq n \text{ such that } f^{-k}(x)\in U(\alpha\, e^{-\varepsilon k}, N^-)\}$$

(here ε is taken according to Proposition 6). Therefore

$$\nu_n(U_n\backslash D_l^-)\leq\sum_{k=0}^{n}\nu^{(u)}(V^{(u)}(z)\cap f^{-(n-k)}(U(\alpha e^{-\varepsilon k}, N^-))).$$

By virtue of (H4) we have from here that

$$\nu_n(U_n\backslash D_l^-)\leq C\alpha^t(1-e^{-\varepsilon t})^{-1}.$$

The right-hand side of this inequality will be less than γ if α is small enough (respectively l is large enough). □

It follows directly from Lemma 1 that the sequence of the restrictions of measures μ_n to the closed set D_l^- (for a fixed and large enough l) is weakly compact. Let $\tilde{\mu}^{(l)}$ be a limit measure. It is easy to see that $\tilde{\mu}^{(l)}(D_l^-)>0$.

LEMMA 2. *$\tilde{\mu}^{(l)}$ is a Gibbs u-measure on D_l^-.*

Proof. (cf [17], Theorem 4.) Let $z\in D_l^-$ be a Lebesque point for $\tilde{\mu}^{(l)}$, $\Pi=\Pi(z, r)$ a rectangle at z with $r\leq r_l^{(4)}(y)$, $y\in B(z, r)\cap D_l^-$. We identify $\bar{\Pi}/\xi$ with a closed subset W in $W(z)$ (cf (8)) consisting of points $[z, y]$ where $y\in B(z, r)\cap D_l^-$. Set

$$A_n=\{y\in B(z, r)\cap D_l^-\colon V^{(u)}(y)\cap U_n\neq\varnothing\},$$
$$B_n=\{y\in B(z, r)\cap D_l^-\colon V^{(u)}(y)\cap\partial U_n\neq\varnothing\},$$
$$C_n=A_n\backslash B_n,\ D_n=\bigcup_{y\in B_n}B^{(u)}(y, r).$$

It is easy to see that $B_n\subset A_n$ and sets A_n, B_n, C_n have a finite number of elements. We have from the definitions of measures $\tilde{\nu}_n$ and numbers $\bar{\bar{C}}_n$ and from Propositions 4 and 6 that

$$\begin{aligned}\tilde{\nu}_n(D_n)&\leq\bar{\bar{C}}_n\int_{D_n}\kappa(f^n(z), y)\, d\nu^{(u)}(y)\\ &=\bar{\bar{C}}_n\int_{f^{-n}(D_n)}\kappa(f^n(z), f^n(y'))\prod_{k=0}^{n-1}J^{(u)}(f^k(y'))\, d\nu^{(u)}(y')\\ &=\int_{f^{-n}(D_n)}\kappa(z, y')\, d\nu^{(u)}(y')\leq K_1\nu^{(u)}(f^{-n}(D_n))\leq 2K_2r\mu^{-n},\end{aligned}$$

where $K_1>0$, $K_2>0$ are some constants. Denote by δ_n the measure on C_n such that

$$\delta_n(Z)=N(Z)/N(C_n),$$

where $Z\subset C_n$ and $N(Z)$, $N(C_n)$ are numbers of elements in Z and C_n. Let h be a continuous function on D_l^- with a support in Π, and $d=\max_{y\in D_l^-}|h(y)|$. We have

$$\begin{aligned}\int_{D_l^-} h(y)\,d\nu_n(y)&=\int_{\Pi} h(y)\,d\nu_n(y)=\sum_{y\in A_n}\int_{V^{(u)}(y)\cap U_n} h(w)\,d\tilde{\nu}_n(w)\\&=\sum_{y\in C_n}\int_{V^{(u)}(y)\cap U_n} h(w)\,d\tilde{\nu}_n(w)+\sum_{y\in B_n}\int_{V^{(u)}(y)\cap U_n} h(w)\,d\tilde{\nu}_n(w)\\&=I_1^{(n)}+I_2^{(n)}.\end{aligned}$$

It easily follows from what was said above that

$$I_2^{(n)}\leq K_3\nu^{(u)}(D_n)\leq 2K_3K_2r\mu^{-n}, \tag{13}$$

where $K_3>0$ is a constant. It follows from Proposition 6 and the definition of measures ν_n that

$$\begin{aligned}I_1^{(k)}&=C_n\sum_{y\in C_n}\kappa(f^n(z),y)\int_{V^{(u)}(y)} h(w)\kappa(y,w)\,d\nu^{(u)}(w)\\&=\int_{\bar{W}} C_n\kappa(f^n(z),y)\beta(y)\,d\delta_n(y)\int_{V^{(u)}(y)} h(w)\frac{\kappa(y,w)}{\beta(y)}\,d\nu^{(u)}(w),\end{aligned}$$

where

$$\beta(y)=\int_{V^{(u)}(y)}\kappa(y,w)\,d\nu^{(u)}(w)$$

is a normalizing factor. Now the result we need follows from (12) and Lemma 13 in [**19**]. □

For $l\geq l_0$ consider the sequence of measures on Λ

$$\mu_{n,l}(A)=\mu_n(A\cap D_l^-),\quad A\subset\Lambda.$$

There exists a sequence of numbers n_i such that the subsequence of measures μ_{n_i} weakly converges to a measure μ on Λ and for any fixed $l\geq l_0$ the subsequence of measures $\mu_{n_i,l}$ weakly converges to a measure $\tilde{\mu}^{(l)}$. It is easy to verify that $\tilde{\mu}^{(l)}(A)\leq\mu(A)$ for any $l\geq l_0$ and any Borel set $A\subset\Lambda$. This means that $\tilde{\mu}^{(l)}$ is absolutely continuous with respect to μ and $0\leq d\tilde{\mu}^{(l)}/d\mu\leq 1$.

LEMMA 3. *μ is a Gibbs u-measure on Λ.*

Proof. Fix arbitrary $\beta,\varepsilon\in(0,1)$, β is sufficiently close to 1 and ε is small enough and let $\gamma=\varepsilon(1-\beta)<1$. Take $l\geq l_0$ so large that $\tilde{\mu}^{(l)}(D_l^-)\geq 1-\gamma$ (cf Lemma 1) and denote by $A=D_l^-$,

$$B=\{z\in A: 0\leq d\tilde{\mu}^{(l)}/d\mu(z)\leq\beta\}.$$

Let $\mu(A)=a$, $\mu(B)=b$. We have that

$$a=\mu(A)\geq\tilde{\mu}^{(l)}(A)\geq 1-\gamma.$$

Further

$$\tilde{\mu}^{(l)}(B)\leq\beta\mu(B),\ \tilde{\mu}^{(l)}(A\backslash B)\leq\mu(A\backslash B)=a-b.$$

It follows from here that $\tilde{\mu}^{(l)}(A) \le \tilde{\mu}^l(B) + \tilde{\mu}^l(A \setminus B) \le a - b(1-\beta)$ and therefore $b(1-\beta) \le a - 1 + \gamma \le \gamma$, hence $b \le \varepsilon$. This means that μ coincides up to a factor β to the measure $\tilde{\mu}^{(l)}$ on a set of μ-measure $\gtrsim 1 - \gamma - \varepsilon$. It implies what we need. □

LEMMA 4. *μ is f-invariant.*

Proof. We have that $\mu(D^-) = 1$. Let $A \subset D^-$ be a closed subset. Then $f^{-1}(A) \subset D^-$ and there exist a number α and an α-neighborhood of A in K such that

$$U(A, \alpha) \cap N = \varnothing \quad \text{and} \quad f^{-1}(U(A, \alpha)) \cap N = \varnothing.$$

Let φ be a continuous function being equal to 1 on A and to 0 out of $U(A, \alpha)$. Since μ is the weak limit of μ_{n_i} it follows from the Proposition 8 that

$$\int_{D^-} \varphi(x)\, d\mu(x) = \int_{D^-} \varphi(f^{-1}(x))\, d\mu(x).$$ □

As $\mu(D^-) = 1$ we have that $\mu(N^+) = 0$. Taking into consideration that μ is f-invariant we conclude that $\mu(D) = 1$. Repeating the arguments given in the proof of Proposition 3 one can show that

$$\left| \int_\Lambda \ln \varphi(z)\, d\mu(z) \right| < \infty$$

(recall that $\varphi(z) = \rho(z, N^+)$). This means that μ satisfies conditions (1) and (2) of Proposition 2. □

2.4. In the following we will assume that Λ is a generalized hyperbolic attractor satisfying condition (H4) and $\mu \in M_f^{(\mathrm{u})}$, $\mu(D_0) = 1$. We will describe the ergodic properties of μ. One can derive the next assertion from the definition of Gibbs u-measure and the Fubini theorem.

PROPOSITION 9. *For μ-almost every $z \in D^-$*

$$\nu^{(\mathrm{u})}(D_0 \cap V^{(\mathrm{u})}(z)) = 1. \tag{14}$$

Fix $z \in D^-$ for which (14) takes place and choose l such that

$$\nu^{(\mathrm{u})}(D_l^+ \cap V^{(\mathrm{u})}(z)) > 0.$$

Let W be a smooth submanifold in a small neighborhood of $V^{(\mathrm{u})}(z)$ taking in the form

$$W = \{\exp_z (y, \varphi(y)), y \in I \subset E^{(\mathrm{u})}(z)\},$$

where I is an open subset and $\varphi : I \to E^{(\mathrm{s})}(z)$ is a diffeomorphism. W has the same dimension as $V^{(\mathrm{u})}(z)$ and is transversal to $V^{(\mathrm{s})}(y)$ for all $y \in D_l^+ \cap V^{(\mathrm{u})}(z)$.Consider the map p: $D_l^+ \cap V^{(\mathrm{u})}(z) \to W$ where $p(y)$ is the point of the intersection $V^{(\mathrm{s})}(y)$ and W. Denote by ν_W the measure in W induced by the Riemannian metric in W (considered as a smooth submanifold in M). One can prove the following result using the arguments presented in [**18**] (cf proof of Theorem 4.4).

PROPOSITION 10. *The measure $p_* \nu^{(\mathrm{u})}$ is absolutely continuous with respect to ν_W.*

Fix $z \in D^-$ and for $l > 0$ denote by

$$Q(l, z) = \bigcup_{y \in D_l^+ \cap V^{(u)}(z)} V^{(s)}(y) \cap \Lambda.$$

PROPOSITION 11. *For ν-almost every $z \in \Lambda$ and any large enough $l > 0$ we have* (1) $\mu(Q(l, z)) > 0$; (2) *the set* $Q = \bigcup_{n \in \mathbf{Z}} f^n(Q(l, z))$ *is the ergodic component of a positive measure for the map* $f \mid \Lambda$ *(with respect to* μ*).*

Proof. For any continuous function φ on Λ we set

$$\varphi^+(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^k(z)),$$

$$\varphi^-(z) = \lim_{n \to \infty} \frac{1}{k} \sum_{k=-n}^{0} \varphi(f^k(z)),$$

where $z \in D$. It follows from Birkhoff's ergodic theorem that φ^+ and φ^- are well-defined and $\varphi^+(z) = \varphi^-(z)$ almost everywhere except a set A of measure zero. It follows from the Fubini theorem and the definition of Gibbs u-measure that $\nu^{(u)}(V^{(u)}(z) \cap N) = 0$ for μ-almost every $z \in D_0$. For such points z Proposition 10 implies that $\mu(Q(l, z)) > 0$ if l is arbitrary large enough. Moreover for μ-almost every point $y \in Q(l, z)$ we have $V^{(u)}(z) \cap V^{(s)}(y) = w \notin N$. Because φ is continuous and $\rho(f^n(y), f^n(w)) \to 0$ when $n \to \infty$ it is easy to see that $\varphi^+(y) = \varphi^+(w)$. The same arguments show that $\varphi^-(z) = \varphi^-(w)$. Hence, $\varphi^-(y) = \varphi^+(y) = \varphi^+(w) = \varphi^-(w) = \varphi^-(z) = \varphi^+(z)$. Now one can prove the desired result by repeating arguments presented in [3] (cf proof of Theorem 4.4). □

Remark. One can also prove by a slight modification of the above arguments that any ergodic component of a positive μ-measure can be written in the form

$$Q = \bigcup_{n \in \mathbf{Z}} f^n(R(z)), \tag{15}$$

where z is a typical point (with respect to μ) and

$$R(z) = \bigcup_{y \in D_l^+ \cap B(z, \delta_l)} V^{(u)}(y)$$

for some large enough l (in particular it points out on a symmetry in a structure of an ergodic component with respect to local stable and unstable leaves).

2.5. Now we can present a description of the ergodic properties of the map $f \mid \Lambda$ with respect to a Gibbs u-measure μ. It can be obtained from Proposition 11 by arguments in [**10, 11, 15**].

THEOREM 2. *Let $\mu \in M_f^{(u)}$. Then there exist sets $\Lambda_i \subset \Lambda$, $i = 0, 1, 2, \ldots$, such that*
(1) $\Lambda = \bigcup_{i \geq 0} \Lambda_i$, $\Lambda_i \cap \Lambda_j = \varnothing$ *for* $i \neq j$, $i, j = 0, 1, 2, \ldots$;
(2) $\mu(\Lambda_0) = 0$, $\mu(\Lambda_i) > 0$ *for* $i > 0$;
(3) *for* $i > 0$: $\Lambda_i \subset D$, $f(\Lambda_i) = \Lambda_i$, $f \mid \Lambda_i$ *is ergodic*;
(4) *for* $i > 0$: *there exists a composition*:

$$\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_i^j, \quad n_i \in \mathbb{Z}^+$$

where

(a) $\Lambda_i^{j1} \cap \Lambda_i^{j2} = \varnothing$ *for* $j_1 \neq j_2$;

(b) $f(\Lambda_i^j) = \Lambda_i^{j+1}$ *for* $j = 1, 2, \ldots, n_i - 1$, $f(\Lambda_i^{n_i}) = \Lambda_i^1$;

(c) $f^{n_i} | \Lambda_i^1$ *is isomorphic to a Bernoulli automorphism*;

(5) *for the metric entropy* $h_\mu(f | \Lambda)$ *the following formula takes place*

$$h_\mu(f | \Lambda) = \int_\Lambda \sum_{i=1}^{s(x)} \chi_i(x) \, d\mu(x)$$

where $\{\chi_i(x)\}$, $i = 1, \ldots, s(x)$ *is the collection of all positive values of the Lyapunov exponent at* x;

(6) *there exists a partition* η *of* Λ *with the following properties*:

(a) *for* μ*-almost every* $x \in \Lambda$ *the element* $C_\eta(x)$ *of the partition* η *is an open subset in* $W^{(u)}(x)$;

(b) $f\eta \geq \eta$, $\bigvee_{k \geq 0} f^k \eta = \varepsilon$, $\bigwedge_{k \geq 0} f^k \eta = \nu(W^{(u)})$
(*here* $\nu(W^{(u)})$ *is a measurable hull of the partition of* Λ *consisting of single leaves* $W^{(u)}(x)$ *if* $x \in D^-$ *and single points* $\{x\}$ *if* $x \in \Lambda \backslash D^-$);

(c) $h(f | \Lambda, \eta) = h_\mu(f | \Lambda)$ *where* $h(f | \Lambda, \eta)$ *is the entropy of* $f | \Lambda$ *with respect to* η.

Denote by

$$W^{(s)}(\Lambda) = \bigcup_{z \in D^+} W^{(s)}(z).$$

The next assertion follows directly from Proposition 9 and Theorem 2.

THEOREM 3. *Let* $\mu \in M_f^{(u)}$. *Then for any set* Λ, *constructing in Theorem* 2 *with* $i > 0$ *the following holds*:

(1) mes $(W^{(s)}(\Lambda_i)) > 0$ (*where* mes *denotes the Riemannian volume in* M);

(2) *there exists* $A_i \subset \Lambda$ *such that* $\mu(A_i) = \mu(\Lambda_i)$ *and for any* $z \in W^{(s)}(A_i)$ *and any continuous function* φ *in* M *there exists*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) = \frac{1}{\mu(\Lambda_i)} \int_{\Lambda_i} \varphi \, d\mu.$$

Using the above results we can now describe the whole class of Gibbs u-measures on Λ.

THEOREM 4. *There exist sets* Λ_n, $n = 0, 1, 2, \ldots$ *and measures* $\mu_n \in M_f^{(u)}$, $n = 1, 2, \ldots$ *such that*:

(1) $\Lambda = \bigcup_{n \geq 0} \Lambda_n$, $\Lambda_n \cap \Lambda_m = \varnothing$ *for* $n \neq m$;

(2) mes $(W^{(s)}(\Lambda_n) \cap W^{(s)}(\Lambda_m)) = 0$ *for* $n \neq m$, $n, m > 0$;

(3) *for* $n > 0$: $\Lambda_n \subset D$, $f(\Lambda_n) = \Lambda_n$, $\mu_n(\Lambda_n) = 1$, $f | \Lambda_n$ *is ergodic with respect to* μ_n;

(4) *for* $n > 0$: *there exist* $k_n > 0$ *and subset* $A_n \subset \Lambda_n$ *such that*

(a) *the sets* $A_{n,i} = f^i(A_n)$ *are disjoint for* $i = 1, \ldots, k_n - 1$ *and* $A_{n,k_n} = A_{n,1}$, $\Lambda = \bigcup_{i=1}^{k_n - 1} A_{n,i}$;

(b) $f^{k_n} | A_{n,1}$ *is isomorphic to a Bernoulli automorphism* (*with respect to* μ_n);

(5) *for any* $\mu \in M_f^{(u)}$

$$\mu = \sum_{n > 0} \alpha_n \mu_n, \qquad \alpha_n \geq 0, \ \sum_{n > 0} \alpha_n = 1;$$

(6) *If ν is a measure in K being absolutely continuous with respect to the Riemannian volume and $\nu_n = \nu \mid W^{(s)}(\Lambda_n)$, $n > 0$ then*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} f^i_* \nu_n = \mu_n.$$

Proof. Let $\mu_1, \mu_2 \in M_f^{(u)}$ be two ergodic measures and $A_1, A_2 \subset \Lambda$ be two subsets for which $\mu_1(A_1) = 1$ and $\mu_2(A_2) = 1$. It follows from Theorem 3 that

$$\operatorname{mes}(W^{(s)}(A_1) \cap W^{(s)}(A_2)) = 0.$$

This implies the statements (1)-(5). The last statement follows from assertion (2) of Theorem 3. □

In conclusion we mention a connection between Gibbs u-measures and property (H4). First of all let us notice that any accumulation point of the sequence of measures (12) is a Gibbs u-measure (this follows essentially from Theorem 1). We describe a special property of such measures.

PROPOSITION 12. *If μ is the Gibbs u-measure constructed in Theorem* 1 *then there is $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon = (0, \varepsilon_0]$ and any $n > 0$*

$$\mu(U(\varepsilon, N^+)) \le C\varepsilon^t, \tag{16}$$

where $C > 0$, $t > 0$ are constant independent of ε, n. (The constants C, t, ε_0 can depend on the initial point z using in the construction of μ.)

Proof. Let h be a continuous function $|h(x)| \le 1$ for any $x \in M$ which is equal to 1 on $U(\varepsilon, N^+)$ and is equal to 0 outside $U(2\varepsilon, N^+)$. We have $\int h\, d\mu_{n_k} \to \int h\, d\mu$ for some sub-sequence $n_k \to \infty$. It follows from (H4), and the definition of measures μ_n and Proposition 8 that

$$\left| \int h\, d\mu_{n_k} \right| \le \nu^u(V^{(u)}(z) \cap f^{-n_k}(U(2\varepsilon, N^+))) \le C(2\varepsilon)^t.$$

This implies the desired result. □

We have seen that property (H4) is sufficient to prove the existence of an f-invariant Gibbs u-measure on a generalized hyperbolic attractor. Now we will show that it is 'almost' necessary.

PROPOSITION 13. *Let $\mu \in M_f^{(u)}$ (e.a. μ is the Gibbs u-measure on Λ invariant under f and $\mu(D_0) = 1$) satisfy* (16) *with some constants C, t, ε. Then for μ-almost every point $z \in D_0$ there exists $\varepsilon(z) > 0$ such that condition (H4) is true with respect to z and any $\varepsilon \in (0, \varepsilon(z)]$ (and the same C, t).*

Proof. Since μ is f-invariant we have from (16) for any $l > 1$, any rectangle $\Pi \subset D_l$ of a positive μ-measure, and any $n > 0$ that

$$\mu(f^{-n}(U(\varepsilon, N^+)) \cap \Pi) \le C\varepsilon^t.$$

It follows from the definition of Gibbs u-measures and the Fubini theorem that for any $\delta > 0$ small enough there exist a set Π_δ of measure $\ge 1 - \delta$ such that condition (H4) holds with respect to any $z \in \Pi_\delta$ and any $\varepsilon \in (0, \varepsilon(l, \delta))$ (for some $\varepsilon(l, \delta)$). It implies the desired result. □

3. *Ergodicity and topological properties of the map* $f\,|\,\Lambda$

3.1. Our approach to the study of the ergodicity and topological properties of generalized hyperbolic attractors is based on the above description of the metric properties and some information about a topological structure of the stable lamination $W^{(s)}$. Roughly speaking we require it to be continuous in a weak sense (in general it is only measurable). This requirement holds if $W^{(s)}(x)$ are 'parts' of some continuous foliation and are expanding 'in all directions' under f^{-n} (the last demand can be omitted in the two-dimensional case).

We represent some general definitions closely following the approach presented in [17]. Let μ be a Borel measure in M, $X \subset M$ a Borel subset with $\mu(X) > 0$.

We say that W is a *measurable lamination* on X if W is a partition of X and for μ-almost every $x \in X$ the element of this partition containing x has a form $W(x) \cap X$ where $W(x)$ is an immersed C^1-submanifold in M passing through X. $W(x)$ is called the *global manifold* at x.

Given $r > 0$ the connected part of intersection $W(x) \cap B(x, r)$ containing x is denoted by $V(x, r)$. If $r = r(x)$ is small enough then $V(x) = V(x, r(x))$ is a graph of a C^1-map $\varphi_x : U_x \to M$ where $U_x \subset T_x W(x)$ is an open neighborhood of zero and $\varphi(0) = x$, $d\varphi(0) = T_x V(x)$. $V(x)$ is called the *local manifold* at x (of course, it is not uniquely defined).

We say that W is a *continuous* (*mod* 0) $r(x)$-*lamination* on X if it is a measurable lamination and for μ-almost every $x \in X$ the maps $\varphi(y)$ and $d_u\varphi(y)(u)$ are continuous over $y \in X \cap B(x, r(x))$.

Let W be a continuous (mod 0) $r(x)$-lamination on X. Fix $x \in X$ and assume that Π_1, Π_2 are two submanifolds in $B(x, r(x))$ uniformly transversal to any $V(y)$, $y \in X \cap B(x, r(x))$. Let

$$Z_i = \{z \in \Pi_i : \text{there exists } y \in X \cap B(x, r(x)) \text{ such that } z = Z_i \cap V(y)\},\ i = 1, 2.$$

Consider the projection map $\chi : Z_1 \to Z_2$ where $\chi(z_1) = z_2$ and $z_i = X \cap V(y)$ for some $y \in X \cap B(x, r(x))$.

A continuous (mod 0) $r(x)$-lamination W on X is called *absolutely continuous* if for almost every $x \in X$ and any Π_1, Π_2 the projection map χ is absolutely continuous (e.a. the measure $\chi_*\mu_1$ is absolutely continuous with respect to the measure μ_2 where μ_i is a Riemannian volume on Π_i considered to be a smooth submanifold in M).

A measurable lamination W on X is called *expanding* if there exist $r > 0$ and a measurable function $n(x)$ on X such that for μ-almost every $x \in X$ and any $n \geq n(x)$

$$f^{-n}(V(x)) \supset B_W(f^{-n}(x), r),$$

where $B_W(x, r)$ is a ball on $W(x)$ of radius r centered at x.

3.2. Let Λ be a generalized hyperbolic attractor for map f satisfying (H1)-(H4). It is not difficult to see that the partition of D^+ (respectively D^-) on stable (respectively unstable) leaves is a measurable lamination on Λ with respect to any Borel measure μ concentrated on Λ for which $\mu(D^+) = 1$ (respectively $\mu(D^-) = 1$). We denote them by $W^{(s)}$, $W^{(u)}$. In the general case they are neither continuous nor expanding with respect to f^{-1} or f) and we give the additional conditions for continuity and expansivity.

Let μ be a Gibbs u-measure on Λ. The following results can be proved by the arguments given in [15].

THEOREM 5.
(1) *If $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination then it is absolutely continuous.*
(2) *Let W be a continuous (mod 0) $r(x)$-lamination on Λ and $W^{(s)}(x) \subset W(x)$ for μ-almost every $x \in \Lambda$. Assume that $W^{(s)}$* is expanding (with respect to f^{-1}). *Then $W^{(s)}$ is the continuous (mod 0) $\tilde{r}(x)$-lamination with $\tilde{r}(x) = \min(r(x), r)$.*
(3) *If* $\dim W^{(s)}(x) = 1$ *for μ-almost every $x \in \Lambda$ then $W^{(s)}$ is expanding (with respect to f^{-1}).*

Now we give some additional conditions for the ergodicity of the map $f \mid \Lambda$ with respect to a measure $\mu \in M_f^{(u)}$.

THEOREM 6. *Assume that $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination on Λ (with respect to μ). Then any ergodic component $Q \subset \Lambda$ for f of a positive μ-measure is open (mod 0) (with respect to μ and the induced topology in Λ).*

Proof. By virtue of Proposition 5 for μ-almost every $x \in \Lambda$ we have that $V^{(u)}(x) \subset \Lambda$ and the balls $B_{W^{(s)}}(y, r(y))$ are correctly defined for $\nu^{(u)}$-almost every $y \in V^{(u)}(x)$ (e.a. on a set $A^{(u)}(x) \subset V^{(u)}(x)$). Consider the set

$$Q(x) = \left(\bigcup_{y \in A^{(u)}(x)} B_{W^{(s)}}(y, r(y)) \right) \cap \Lambda \cap B(x, r(x)).$$

It is easy to see that $\mu(Q(x)) > 0$. One can show using the Theorem 5 (see statement (1); compare with Proposition 11) that the set

$$Q = \bigcup_{n \in \mathbf{Z}} f^n(Q(x))$$

is an ergodic component of a positive measure for the map $f \mid \Lambda$. We will show that $Q(x)$ is open (mod 0) in the induced topology in Λ. Assuming, on the contrary, that we can find a set $A \subset \Lambda \cap B(x, r(x))$ of a positive μ-measure such that $A \cap Q(x) = \varnothing$. Then there exists a point $y \in A$ such that

$$\nu^u(A \cap V^{(u)}(y)) > 0.$$

It follows from Theorem 6 that the set $\chi(A \cap V^{(u)}(y)) \subset V^{(u)}(x)$ has a positive $\nu^{(u)}$-measure (recall that χ is the projection map). However, it is impossible, because

$$\chi(A \cap V^{(u)}(y)) \cap A^{(u)}(x) = \varnothing. \qquad \square$$

The next statement is a direct consequence of Theorem 6.

THEOREM 7. *Assume that $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination on Λ with respect to μ and $f \mid \Lambda$ is topologically transitive. Then the map $f \mid \Lambda$ is ergodic.*

3.3. Now we give a description of topological properties of $f \mid \Lambda$. Let us notice that if $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination on Λ with respect to some measure $\mu \in M_f^{(u)}$ then it can happen that it is not the same with respect to some other measure $\nu \in M_f^{(u)}$. This motivates the following definition. We say that $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination with respect to the class of measures $M_f^{(u)}$ if $W^{(s)}$ is the same with respect to every $\mu \in M_f^{(u)}$. For example if W is a continuous C^1-lamination on Λ (for a definition, see [16]) and $W^{(s)}(x) \subset W(x)$ for every $x \in D^+$ then $W^{(s)}$ is continuous (mod 0) $r(x)$-lamination with respect to $M_f^{(u)}$ for some function $r(x)$.

The following result is a direct consequence of Theorem 7 and describes the topological properties of the map $f \mid \Lambda$.

THEOREM 8. *Assume that $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination on Λ with respect to $M_f^{(u)}$. Let Λ_n, $n = 0, 1, 2, \ldots$ and μ_n, $n = 1, 2, \ldots$ be respectively the sets and the measures constructed in Theorem 4. Then for $n > 0$*

(1) *Λ_n is an open subset in Λ (mod 0) (with respect to μ_n and induced topology in Λ);*
(2) *$f \mid \Lambda_n$ is topologically transitive;*
(3) *$f^{k_n} \mid A_{n,1}$ is topologically mixing (the number k_n and the set $A_{n,1} \subset \Lambda_n$ are constructed in Theorem 4).*

The result is similar to the theorem on a spectrum decomposition for axiom A diffeomorphisms with Λ_n considered to be the basic sets (cf [16]). But in our case it is possible that f would have a countable (but not finite) number of components of topological transitivity (cf, example below in § 5). E. A. Sataev gave additional conditions on f to have a finite number of components of topological transitivity (unpublished).

Let us also notice that in Theorem 8 it is described the topological properties of f only on, so to speak, 'essential' part of Λ: a topological behavior of f on the set Λ_0 is unknown (and in general can be arbitrary).

The next statement follows directly from Theorem 8 and is an important addition to Theorem 7.

THEOREM 9. *Assume that $W^{(s)}$ is a continuous (mod 0) $r(x)$-lamination on Λ and $f \mid \Lambda$ is topologically transitive. Then f has only one Gibbs u-measure on Λ.*

It follows from Proposition 10 that mes $(W^{(s)}(\Lambda)) > 0$. Therefore if ν is an arbitrary smooth initial distribution in a small neighborhood U of Λ then one can describe the evolution of the measure $\nu \mid W^{(s)}(\Lambda)$. Namely, if the conditions of Theorem 9 hold the limit distribution is the uniquely defined Gibbs u-measure on Λ. In the general case the limit distribution is described by Theorem 4. However the evolution of ν on the other part of U is unknown. In view of this it is interesting to know whether the set $W^{(s)}(\Lambda)$ is open or not. We give now a sufficient condition for this.

THEOREM 10. *Assume that W is a continuous and absolutely continuous foliation in U and $W^{(s)}(x) \subset W(x)$ for any $x \in D^+$. Then the set $W^{(s)}(\Lambda_n)$ for any $n > 0$ is open (mod 0) (with respect to* mes*); in particular, the set $W^{(s)}(\Lambda)$ is open (mod 0) too.*

Proof. See arguments given in the proof of Theorem 6.

In order to complete a description of the topological properties of the map $f \mid \Lambda$ we consider the problem of the existence of periodic points.

THEOREM 11. *Let Λ be a generalized hyperbolic regular attractor for f satisfying conditions $(H1)$-$(H4)$. Then the periodic hyperbolic points are everywhere dense in Λ.*

Proof. Take an arbitrary measure $\mu \in M_f^{(u)}$. Its Lebesgue points are everywhere dense in Λ and let x be one of them. There exists $l > 0$ such that $x \in D_l^0$ and $\mu(D_l^0) > 0$. Fix $\alpha > 0$ and find $n > 0$ such that $y = f^n(x) \in B(x, \alpha) \cap D_l^0$ (we can chose n arbitrary large). If α is sufficiently small the intersection $V^{(s)}(x) \cap V^{(u)}(y)$ is not empty and consists of the single point $z_1 = [x, y]$. We have that $y_1 = f^n(z_1) \in V^{(s)}(y)$ and

$$\rho(y_1, y) \leq C\lambda^n \rho(x, z_1),$$

where $C > 0$, $0 < \lambda < 1$. First we will show that there is an unstable manifold passing through y_1 and having 'almost the same' size as $V^{(u)}(x)$. It is easy to see that $f^k(x) \in D_{l(k)}^+$ where $l(k)$ is a function such that $l(0) = l$, $l(n) = l$ and (cf statement (6) in Proposition 4)

$$l(k) - t \leq l(k+1) \leq l(k) + t$$

for some $t > 0$. Besides, we have that

$$\rho(f^k(x), f^k(z_1)) \leq C_1 \lambda^n \rho(x, z_1),$$

where $C_1 > 0$ is a constant. It follows from here that

$$\rho(f^k(z_1), N^+) \geq C_2 l^{-1} e^{-\varepsilon k},$$

where $C_2 > 0$ is a constant. This means that $z_1 \in D_{l+m}^+$ for some m and moreover $f^k(z_1) \in D_{l(k)+m}$. If α is small enough it implies that

$$\delta(y_1) \geq C_3 \delta(x)$$

with some constant $C_3 > 0$. It follows from here that the intersection $V^{(s)}(x) \cap V^{(u)}(y_1)$ is not empty and consists of the single point $z_2 = [x, y_1]$. One can show (cf [**15**]) that

$$\rho(z_1, z_2) \leq C_4 \rho(y_1, y),$$

where $C_4 > 0$ is a constant. Besides, $y_2 = f^n(z_2) \in V^{(s)}(y)$ and

$$\rho(y_2, y_1) = C\lambda^n \rho(z_1, z_2).$$

Continuing this procedure we construct a sequence of points $z_m \in V^{(s)}(x)$ which converges (if n is large enough) to a point $z \in V^{(s)}(x)$. This point is in an arbitrary small neighborhood of x (if α is taken small enough) and has the property that the unstable manifold passing through it is fixed under f^n. Now consider the intersection $V^{(u)}(x) \cap V^{(s)}(y)$ which is also not empty and consists of the single point $w = [y, x]$. Repeating the above arguments with respect to this point we construct a sequence of points $w_m \in V^{(u)}(x)$ converging to a point $v \in V^{(u)}(x)$ and having the same properties as z. It is easy to see that the intersection $V^{(s)}(v) \cap V^{(u)}(z)$ is not empty and consists of a single point which is hyperbolic and periodic of the period n. □

4. *Generalized partially hyperbolic attractors*

Let M be a smooth p-dimensional Riemannian manifold, $K \subset M$ an open boundered connected subset with the compact closure, $N \subset K$ a closed subset and $f: K \setminus N \to K$

a map satisfying (H1) and (H2). As in § 1 one can construct the attracting limit set D and the attractor $\Lambda = \bar{D}$ for f. Then one can define the sets $\hat{D}^{+}_{\varepsilon,l}$, $D^{\pm}_{\varepsilon,l}$, $D^{0}_{\varepsilon,l}$, $D^{\pm}_{\varepsilon}$, D^{0}_{ε} for arbitrary $\varepsilon > 0$ small enough and $l = 1, 2, \ldots$. The attractor Λ is called regular if $D^{0}_{\varepsilon} \neq \varnothing$ for any sufficiently small ε. Following [19] we say that Λ is generalized partially hyperbolic if D is a partially hyperbolic invariant set for f. This means that for any $x \in D$ there exist subspaces $E^{(s0)}(x), E^{(u)}(x) \subset T_xM$ such that

(1) $E^{(s0)}(x) \oplus E^{(u)}(x) = T_xM$;

(2) $dfE^{(u)}(x) = E^{(u)}(f(x)), dfE^{(s0)}(x) = E^{(s0)}(f(x))$;

(3) there exist $C > 0$, $0 < \mu < \lambda < \infty$, $\mu < 1$ independent of x such that for any $n > 0$

$$\|df_x^{-n} v\| \geq C^{-1} \lambda^{-n} \|v\|, \quad v \in E^{(u)}(x),$$
$$\|df_x^{n} v\| \leq C \mu^{n} \|v\|, \quad v \in E^{(s0)}(x).$$

(4) $E^{(s0)}(x), E^{(u)}(x)$ are uniformly continuous on D (in particular, the angle $\measuredangle(E^{(s0)}(x), E^{(u)}(x) \geq const > 0)$.

As in § 1 one can construct local unstable manifolds $V^{(u)}(x)$, $x \in D^{-}_{\varepsilon}$ having the properties formulated in Proposition 4. At last, one can define global unstable manifolds $W^{(u)}(x)$, $x \in D^{-}$. The definition of Gibbs u-measures is transformed on this case without any changes. The following theorem is proved as Theorem 1.

THEOREM 12. *If Λ is a generalized partially hyperbolic regular attractor having property* $(H4)$ *then there exists a Gibbs u-measure $\mu \in M_f$ concentrated on D and satisfying conditions* (1) *and* (2) *of Proposition* 2.

The next theorem describes ergodic properties of the map $f \mid \Lambda$ with respect to a Gibbs u-measure and can be proved as Theorem 2.

THEOREM 13. *Let $\mu \in M_f^{(u)}$. Then*: (1) *there exists a partition η of Λ such that*

(a) *for μ-almost every $x \in \Lambda$ the element $C_\eta(x)$ of η containing x is an open subset in $W^{(u)}(x)$;*

(b) $f\eta \geq \eta$, $\bigvee_{k \geq 0} f^k \eta = \varepsilon$, $\bigwedge_{k \geq 0} f^k \eta = \nu(W^{(u)})$;

(c) *the entropy $\eta_\mu(f \mid \Lambda)$ admits the following estimation from below*

$$\eta_\mu(f \mid \Lambda) \geq \int_\Lambda \sum_{i=1}^{s(x)} \chi_i(x)\, d\mu(x)$$

where $\{\chi_i(x)\}$, $i = 1, \ldots, s(x)$ is the collection of values of the Lyapunov exponent for vectors in $E^{(u)}(x)$.

5. *Examples*

5.1. We consider a number of examples of maps with generalized hyperbolic attractors in the two-dimensional case (e.a. M is a two-dimensional manifold). First we formulate some general assumptions which guarantee the validity of hypotheses (H3) and (H4). Let f be a map satisfying condition (H1). Suppose that

(H5) $K = \bigcup_{i=1}^{m} K^{(i)}$, $K^{(i)}$ is closed, int $K^{(i)} \cap$ int $K^{(j)} = \varnothing$

$$\partial K^{(i)} = \left(\bigcup_{j=1}^{r_i} N_{ij} \right) \cup \left(\bigcup_{j=1}^{q_i} M_{ij} \right)$$

where N_{ij}, M_{ij} are smooth curves and

$$N=\bigcup_{i=1}^{m}\bigcup_{j=1}^{r_i} N_{ij}, \qquad \partial K=\bigcup_{i=1}^{m}\bigcup_{j=1}^{q_i} M_{ij};$$

(H6) f is continuous and differentiable map in $K^{(i)}$, $i=1,\dots,m$;
(H7) f possesses two families of stable and unstable cones $C^{(s)}(z)$, $C^{(u)}(z)$, $z\in K\setminus\bigcup_{i=1}^{m}\partial K^{(i)}$, which satisfy conditions (1)-(3) in § 1.3.
(H8) the unstable cone $C^{(u)}(z)$ at z depends continuously on $z\in K^{(i)}$ and there exists $\alpha>0$ such that for any $z\in N_{ij}\setminus\partial N_{ij}$, $v\in C^{(u)}(z)$ and any vector w tangent to N_{ij} we have that $\measuredangle(v,w)\geq\alpha$;
(H9) there exists $\tau>0$ such that $f^k(N)\cap N=\varnothing$, $k=0,\dots,\tau$ and $a^\tau>2$ where

$$a=\inf_{z\in K\setminus N}\ \inf_{v\in C^{(u)}(z)}|dfv|>1.$$

THEOREM 14. *If f satisfies conditions $(H1)$, $(H5)$-$(H9)$ then it satisfies condition $(H4)$ for any $z\in D_0$ with constants C, 1, ε_0 (C does not depend on z, ε_0 may depend on z) and has property* (4); (*in particular, f satisfies condition* $(H3)$).

Proof. First of all it is sufficient to prove this result for the map f^τ instead of f which still satisfies properties (H1), (H5)-(H9) with $\tau=1$. Therefore we can assume, without loss of generality, that $\tau=1$. We say that $\gamma:[0,b]\to M$ is a u-curve if γ is smooth and $\gamma(t)\in C^{(u)}(\gamma(t))$, $t\in[0,b]$ (b is a positive number; we assume that t is the length of the curve). We shall prove condition (H4) for an arbitrary u-curve. By virtue of the continuity of unstable cones it implies property (4) and, hence, condition (H3) (cf Proposition 3). First let us notice that if γ is a u-curve then by virtue of (H5)-(H9) the curve $f(\gamma)$ for any $n>0$ consists of a finite number of u-curves, $f^n(\gamma)=\bigcup_i\gamma_{i,n}$. Moreover, for each i there exist $j=j(i,n)$, $l=l(i,n)$ such that $\gamma_{i,n}=f(\gamma_{j,n-1}\cap K^{(l)})$. There exists a constant $C>0$ such that if γ is a u-curve in one of the sets $K^{(i)}$ with the length $\leq C$ then $f(\gamma)$ can intersect only one component N_{pq}. Let $d=\min\{C,\text{ length }(\gamma)\}$. Fix a number $n>0$ and consider a curve $\gamma_{i,n}$ for some i. We say that it is long if length $(\gamma_{i,n})\geq d$. Otherwise it will be called a short curve. Consider two cases.

(1) $\gamma_{i,n}$ is a long curve. Denote by $\tilde\gamma_{i,n}=\gamma_{i,n}\cap U(N,\varepsilon)$. Let us notice that for any $k=0,\dots,n$ the curve $f^{-k}(\gamma_{i,n})$ lies entirely in one of the sets $K^{(j)}$ for some $j=j(i,n,k)$. This allows us to write that

$$\begin{aligned}\text{length }(\gamma_{i,n})&=\int_{f^{-n}(\gamma_{i,n})}|df^n(f^{-n}(\gamma_{i,n}(t))|\,dt=|df^n(z_{i,n})|\text{ length }(f^{-n}(\gamma_{i,n})),\\ \text{length }(\tilde\gamma_{i,n})&=\int_{f^{-n}(\tilde\gamma_{i,n})}|df^n(f^{-n}(\tilde\gamma_{i,n}(t))|\,dt=|df^n(y_{i,n})|\text{ length }(f^{-n}(\tilde\gamma_{i,n})),\end{aligned}\tag{17}$$

where $z_{i,n}\in f^{-n}(\gamma_{i,n})$, $y_{i,n}\in f^{-n}(\tilde\gamma_{i,n})$ are some points. First we write

$$\frac{|df^n(y_{i,n})|}{|df^n(z_{i,n})|}=\prod_{k=0}^{n=1}\frac{|df(f^k(y_{i,n}))|}{|df(f^k(z_{i,n}))|}.$$

Taking into consideration that $f^{-(n-k)}(\gamma_{i,n})$ is a u-curve we have the estimation from (5′) for the distance between points $f^k(y_{i,n})$, $f^k(z_{i,n})$, $k=0,\dots,n-1$. This implies by virtue of condition (H6) that there is $C_1>0$ independent of n and γ such that

$$C_1^{-1}\leq |df^n(y_{i,n})|/|df^n(z_{i,n})|^{-1}\leq C_1. \tag{18}$$

It follows from (H7) and (H8) that there is $C_2>0$ independent of γ such that

$$\text{length}\,(\tilde{\gamma}_{i,n})\leq 2C_2 m\varepsilon\ \text{length}\,(\gamma_{i,n}), \tag{19}$$

for any ε small enough. Now (17), (18) and (19) imply that

$$\text{length}\,(f^{-n}(\tilde{\gamma}_{i,n}))\leq C_3\varepsilon\ \text{length}\,(f^{-n}(\gamma_{i,n})), \tag{20}$$

where $C_3>0$ is a constant independent of i, n, ε.

(2) $\gamma_{i,n}$ is a short curve. In this case there exists k, $1\leq k\leq n$ such that the following is true. The curve $f^{-1}(\gamma_{i,n})$ can be stuck together with not more than one curve $f^{-1}(\gamma_{j,n})$ where $\gamma_{j,n}$ is a short curve. This means that the curves $f^{-1}(\gamma_{i,n})$ and $f^{-1}(\gamma_{j,n})$ have a common end and two ends of the curve

$$\gamma_{i,j,n}=f^{-1}(\gamma_{i,n})\cup f^{-1}(\gamma_{j,n})$$

are 'free'. We have that

$$\text{length}\,(\gamma_{i,j,n})\leq 2a^{-1}c\leq c.$$

So, it is a short curve. Moreover, the measure of the set

$$\tilde{\gamma}_{i,j,n}=f^{-1}(\gamma_{i,n}\cap U(N,\varepsilon))\cup f^{-1}(\gamma_{j,n}\cap U(N,\varepsilon))$$

does not exceed $2a^{-1}C_4\varepsilon$ where $C_4>0$ is a constant independent of i, j, n, ε. The curve $\gamma_{i,j,n}$ has the same property as the curve $\gamma_{i,n}$ and this process can be carried out $(k-1)$-times. At the end we construct a short curve γ'_k such that the measure of the set

$$\tilde{\gamma}'_k=\{x\in\gamma'_k: f^{k-1}(x)\in U(N,\varepsilon)\}$$

does not exceed $(2a^{-1})^kC_4\varepsilon$. On the kth step of this process the curve $f^{-1}(\gamma'_k)$ will be stuck together with a long curve. Thus for any k, $1\leq k\leq n$ and any long curve $\gamma_{l,n-k}\in f^{n-k}(\gamma)$ there exist not more than two curves γ'_k, γ''_k constructed above which are stuck together with it. According to (17) and (18) the contribution from the corresponding sets $f^{-n+k-1}(\tilde{\gamma}'_k)$, $f^{n+k-1}(\tilde{\gamma}''_k)$ will not exceed $(2a^{-1})^{n-k-1}C_5\varepsilon$ where $C_5>0$ is a constant independent of k, n, ε. It follows from what was said above that the total contribution from all of the curves $\tilde{\gamma}'_k$ constructed above for all $k=1,\ldots,n$ is less than

$$C_5\varepsilon\sum_{k=1}^{n}(2a^{-1})^{n-k-1}\ \text{length}\,(\gamma)\leq C_6\varepsilon\ \text{length}\,(\gamma), \tag{21}$$

where $C_6>0$ is a constant independent of n, ε. Now the result we need follows from (20) and (21). □

Remark. Assume that f satisfies conditions (H1), (H2), (H5), (H6), (H8) and (instead of (H6) and (H9)) the following condition holds: f is differentiable map in $K^{(i)}\backslash\partial K^{(i)}$ and

$$\text{(H6')}\qquad \rho(f^k(N),N)\geq A\exp(-\gamma k),\ k=1,2,\ldots, \tag{22}$$

where $A>0$ is a constant and $\gamma>0$ is small enough (in comparison with λ; in particular, $f^k(N)\cap N=\varnothing$, $k=1,2,\dots$). Then f satisfies condition (H4) for any $z\in D_0$ with some constants C, 1, ε_0 (C does not depend on z, ε_0 may depend on z); in particular, f satisfies condition (H3).

5.2. Now we will represent a number of two-dimensional maps with generalized hyperbolic attractors.

1. Lorenz type attractors. Let $I=(-1,1)$, $K=I\times I$. Let also $-1=a_0<a_1<\cdots<a_q<a_{q+1}=1$. Set

$$P_i=I\times(a_i,a_{i+1}),\ i=0,\dots,q,\ l=I\times\{a_0,a_1,\dots,a_1,a_{q+1}\}.$$

Let $T:K\backslash l\to K$ be an injective map,

$$T(x,y)=(f(x,y),g(x,y)),\ x,y\in I \tag{23}$$

where the functions f and g satisfy the following conditions:

(L1) f, g are continuous in $\bar P_i$ and

$$\lim_{y\uparrow a_i} f(x,y)=f_i^-,\qquad \lim_{y\uparrow a_i} g(x,y)=g_i^-,$$
$$\lim_{y\downarrow a_i} f(x,y)=f_i^+,\qquad \lim_{y\downarrow a_i} g(x,y)=g_i^+,$$

where $f_i^\pm$ and $g_i^\pm$ do not depend on x, $i=1,2,\dots,q$;

(L2) f, g have two continuous derivatives in P_i and for $(x,y)\in P_i$, $i=1,\dots,q$,

$$\left.\begin{aligned} df(x,y)&=B_i^1(y-a_i)^{-\nu_i^1}(1+A_i^1(x,y))\\ dg(x,y)&=C_i^1(y-a_i)^{-\nu_i^2}(1+D_i^1(x,y))\end{aligned}\right\}\ \text{if } y-a_i\le\gamma$$
$$\left.\begin{aligned} df(x,y)&=B_i^2(a_{i+1}-y)^{-\nu_i^3}(1+A_i^2(x,y))\\ dg(x,y)&=C_i^2(a_{i+1}-y)^{-\nu_i^4}(1+D_i^2(x,y))\end{aligned}\right\}\ \text{if } a_{i+1}-y\le\gamma,$$

where $\gamma>0$ is a small enough constant, $B_i^1, B_i^2, C_i^1, C_i^2$ are some positive constants, $0\le\nu_i^1,\nu_i^2,\nu_i^3,\nu_i^4<1$, $A_i^1(x,y)$, $A_i^2(x,y)$, $D_i^1(x,y)$, $D_y^2(x,y)$ are continuous functions, which tend to zero when $y\to a_i$ or $y\to a_{i+1}$ uniformly over x.

Besides, $\|f_{xx}\|$, $\|f_{xy}\|$, $\|g_{xy}\|$, $\|g_{xx}\|\le const$;

(L3) the following inequalities take place

$$\|f_x\|<1,\qquad \|g_y^{-1}\|<1,$$
$$1-\|g_y^{-1}\|\,\|f_x\|>2\sqrt{\|g_y^{-1}\|\,\|g_x\|\,\|g_y^{-1}f_y\|},$$
$$\|g_y^{-1}\|\,\|g_x\|<(1-\|f_x\|)(1-\|g_y^{-1}\|)$$

where $\|\cdot\|=\max_{i=0,\dots,q}\sup_{(x,y)\in P_i}|\cdot|$.

The class of maps satisfying L1–L3 was introduced in [**6**]. It includes the famous geometric model of Lorenz attractor [**4, 5, 23, 24**]. It is described as follows.

THEOREM 15. (cf [4].) *Assume that $l=I\times\{0\}$, $K=I\times I$ and $T:K\backslash l\to K$ is a map of form* (23) *where the functions f, g are given by the equalities*

$$f(x,y)=(-B|y|^{\nu_0}+Bx\operatorname{sgn} y|y|^{\nu}+1)\operatorname{sgn} y$$
$$g(x,y)=((1+A)|y|^{\nu_0}-A)\operatorname{sgn} y.$$

If $0<A<1$, $0<B<\frac12$, $\nu>1$, $1/(1+A)<\nu_0<1$ then T satisfies conditions L1–L3.

Using the results in [5] one can prove that the class of maps introduced above is rather representative.

THEOREM 16. *On an arbitrary smooth compact Riemannian manifold of dimension* ≥ 3 *there exists a vector field* X *having the following property: there is a smooth submanifold* S *such that the first-return time map* T *induced on* S *by the flow given by* X *satisfies conditions* $L1$–$L3$.

We will describe the ergodic and topological properties of maps with Lorenz type attractors.

THEOREM 17.

(1) *A map* T *with properties* $L1$–$L3$ *satisfies conditions* $(H1)$ *and* $(H2)$ *and the attractor* Λ *for* T *is a regular generalized hyperbolic; the stable (unstable) cone at any point* $z \in K$ *is the set of vectors having an angle* $\leq 30°$ *with the horizontal (respectively vertical) line.*

(2) *The stable lamination* $W^{(s)}$ *can be continued up to a continuous* C^1*-foliation in* K.

(3) *If one of the following conditions holds*

   (a) $\nu_i^j = 0$, $i = 1, \ldots, q$, $j = 1, 2, 3, 4$;

   (b) $\rho(T^n(f_i^{\pm}, g_i^{\pm}), l) \geq C_i \exp(-\gamma n)$ *for any* $n \geq 0$, $i = 1, \ldots, q$ $(C_i > 0$ *are constants independent of* n; γ *is small enough*$)$;

   then T *satisfies conditions* $(H5)$–$(H9)$, (*or* $(H5)$, $(H6')$, $(H7)$ *and* $(H8)$). *In particular it satisfies condition* $(H4)$ *for any* $z \in D_0$ *with constants* C, 1, ε_0 (ε_0 *may depend on* z) *and has property* (4).

(4) *Under the conditions* (*a*) *and* (*b*) T *possesses at most a countable number of ergodic Gibbs u-measures; the ergodic properties of* T *with respect to any of them are described by assertions* (4), (5) *and* (6) *of Theorem* 2 *and by Theorems* 3 *and* 4;

(5) *Under the conditions* (*a*) *and* (*b*) T *possesses at most a countable number of components of topological transitivity for which assertions* (1)–(3) *of Theorem* 8 *are true; the periodic hyperbolic points for* T *are everywhere dense in* Λ.

E. A. Sataev has informed me that he could prove statements (3) and (4) without the additional conditions (a) or (b) and also has shown that a number of ergodic Gibbs measures are at most finite.

Proof. Conditions (H1) and (H2) follow directly from L1–L3. It was shown in [5] that Λ has a hyperbolic periodic point $z \notin l$. This means that Λ is regular. Statement (2) is proved in [6]. Statements (3), (4) and (5) follow directly from the remark to Theorem 14 and Theorems 2, 3, 4, 8 and 11. □

Remarks.

(1) The existence of Gibbs u-measures for the classical geometric model of Lorenz attractors (when K is a square, l consists of a single interval) was shown in [9]. Their proof is based on Markov partitions (a construction of Markov partitions in the general case is described in [6]). If the stable foliation $W^{(s)}$ is smooth (it takes place, for example, when g does not depend on x) the existence of a Gibbs u-measure follows from the well-known result in the theory of one-dimensional mappings (one

can show that Λ is isomorphic to the inverse limit of a one-dimensional piece-wise expanding mapping for which (a_i, a_{i+1}), $i=0,\ldots,q$ are intervals of monotonicity; for details and references see [**6**]).

(2) We give an example of the Lorenz type attractor for which the discontinuity set consists of a countable number of intervals. We will see that the corresponding map has a countable number of components of topological transitivity. Consider a one-dimensional map $g(y)$, $y\in[0,1]$ given by formula

$$g(y)=\frac{1}{n+2}+\frac{2}{2n+1}y \quad \text{if}\ \frac{1}{n+1}\le y<\frac{2n+1}{2(n+1)}$$

$$g(y)=\frac{2n+1}{2(n+1)}+\frac{1}{2(n+1)}y \quad \text{if}\ \frac{2n+1}{2(n+1)}\le y<\frac{1}{n}$$

for $n=1,2,3,\ldots$.

One can show that there exists a function $f(x,y)$ such that the map $T(x,y)=(f(x,y), g(y))$ satisfies condition L1-L3. However it is easy to see that the set

$$\Lambda\cap I\times\left[\frac{1}{n+1},\frac{1}{n}\right]$$

is a component of topological transitivity for T.

2. Generalized Lozi attractors. Let $c>0$, $I=(0,c)$, $K=I\times I$ and $0=a_0<a_1<\cdots<a_q<a_{q+1}=c$. Set $l=\{a_0,a_1,\ldots,a_q,a_{q+1}\}\times I$ and let $T:K\to K$ be an injective continuous map

$$T(x,y)=(f(x,y), g(x,y)),\quad x,y\in I$$

satisfying the following conditions:

Loz 1. $T|(K\backslash l)$ is a C^2-diffeomorphism and the second derivatives of the maps T and T^{-1} are bounded from above;

Loz 2. $Jac(T)<1$;

Loz 3. $\inf\{(|\partial f/\partial x|-|\partial f/\partial y|)-(|\partial g/\partial x|+|\partial g/\partial y|\}\ge 0$;

Loz 4. $\inf\{|\partial f/\partial x|-|\partial f/\partial y|\}\overset{\text{def}}{=}u>1$;

Loz 5. $\sup\{(|\partial f|\partial x|+|\partial g|\partial y|)/(|\partial f|\partial x)-|\partial f/\partial y|)^2\}<1$;

Loz 6. there exists $N>0$ such that $T^k(l)\cap l=\varnothing$ for $1\le k\le N$ and $u^N>2$.

This class of maps was introduced in [**22**]. It includes the map

$$T(x,y)=(1+by-a|x|, x) \tag{27}$$

which is obtained from the well-known Lozi map by a change of coordinates (the definition of this map and its properties, see for example, in [**12–14**, **17**, **25**]). It is easy to verify that there exist open intervals of a and b such that (27) takes some square $[0,c]\times[0,c]$ into itself and satisfies Loz 1-Loz 6.

THEOREM 18.

(1) *A map T with properties Loz 1-Loz 6 satisfies conditions $(H1)$, $(H5)$-$(H9)$ and the attractor Λ for T is regular generalized hyperbolic; the stable (unstable) cone at any point $z\in K$ has a vertical (respectively, horizontal) line as the center line.*

This map also satisfies condition $(H4)$ *for any* $z \in D_0$ *with some constants* C, t, ε_0 *independent of* z *and has property* (4).

(2) *The stable lamination* $W^{(s)}$ *can be continued up to a continuous* C^0*-foliation in* K.

(3) T *possesses at most a countable number of ergodic Gibbs u-measures*; *ergodic properties of* T *with respect to any of them are described by assertions* (4), (5) *and* (6) *of Theorem* 2 *and by Theorems* 3 *and* 4.

(4) T *possesses at most a countable number of components of topological transitivity for which assertions* (1)–(3) *of Theorem* 8 *are true*; *the periodic hyperbolic points for* T *are everywhere dense in* Λ.

Proof. It is not difficult to verify that Loz 1–Loz 6 imply the conditions (H1), (H5)–(H9). Hence, statement (1) follows from Theorem 14. Statement (2) is proved in [14]. Statements (3), and (4) follow from Theorems 2, 3, 4, 8, 11 and 14. □

3. Belykh attractor. Let $I=[-1,1]$, $K=I\times I$ and $l=\{(x,y): y=kx\}$. Consider the map T

$$T(x,y)=\begin{cases}(\lambda_1(x-1)+1,\ \lambda_2(y-1)+1) & \text{for } y>kx\\ (\mu_1(x+1)-1,\ \mu_2(y+1)-1) & \text{for } y<kx.\end{cases}$$

In the case $\lambda_1=\mu_1$, $\lambda_2=\mu_2$ this map was introduced in [7] and was used as the simplest model in the so-called phase synchronization theory.

THEOREM 19.

(1) *Assume that*

$$0<\lambda_1<\frac{1}{2},\quad 0<\mu_1<\frac{1}{2},\quad 1<\lambda_2<\frac{2}{1-|k|},\quad 1<\mu_2<\frac{2}{1-|k|},\quad |k|<1.$$

Then T *is a map from* $K\backslash l$ *into* K *satisfying conditions* $(H1)$, $(H5)$–$(H8)$ *and the attractor* Λ *for* T *is generalized hyperbolic* (*the stable and unstable one-dimensional subspaces at any point* $z\in D_0$ *are respectively horizontal and vertical lines*; *the stable and unstable cones at any point* $z\in K$ *are the set of vectors having an angle* $\leq 45°$ *with the horizontal or vertical lines*).

(2) *The stable foliation* $W^{(s)}$ *is continued up to a continuous* $(\delta(z),1)$*-lamination in* K *consisting of intervals* $I(z)\ni z$ *on the corresponding horizontal line* (*passing through* z) *with an endpoint on* ∂K *and* $l(I(z))=\rho(z,l)$.

(3) *If* $\lambda_2>2$, $\mu_2>2$ *then* T *satisfies condition* $(H9)$ *and, hence, condition* $(H4)$ *for any* $z\in D_0$ *with some constants* C, 1, ε_0; T *also has property* (4);

(4) T *possesses at most a countable number of ergodic Gibbs u-measures*; *ergodic properties of* T *with respect to any of them are described by assertions* (4), (5) *and* (6) *of Theorem* 2 *and by Theorems* 3 *and* 4;

(5) T *possesses at most a countable number of components of topological transitivity for which assertions* (1)–(3) *of Theorem* 8 *are true*; *the periodic hyperbolic points for* T *are everywhere dense in* Λ.

Proof. Repeats arguments given in the proof of Theorem 18. □

4. Let M, N be smooth compact Riemannian manifolds of dimension ≥ 3, X, Y, respectively, smooth vector fields on M and N, and φ_X^t, φ_Y^t the flows given by them. Assume that the first-return time maps T_X, T_Y induced by φ^t_X, φ^t_Y on respectively S_X, S_Y have Lorenz type attractors Λ_X, Λ_Y. Denote by $N_X \subset S_X$, $N_Y \subset S_Y$ the discontinuity sets for T_X, T_Y and consider the map

$$T:(S_X \backslash N_X) \times (\Lambda_Y \backslash N_Y) \to S_X \times \Lambda_Y$$

given by the formula

$$T(x, y) = (T_X(x), \varphi_Y^{\tau(x)}(y)),\ x \in S_X \backslash N_X,\ y \in \Lambda_Y \backslash N_Y$$

where $\tau(x)$ is the first-return time for the trajectory $\varphi^t_X(x)$ to S_X.

THEOREM 20. *The map T satisfies $(H1)$ and $(H2)$ and the attractor Λ for T is regular generalized partially hyperbolic with property $(H4)$ for any point $z \in D_0$.*

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I have announced the main results of this paper in [**17**]. In 1987 I prepared the whole manuscript. I translated it into English and added some new results two years later during my two months stay on the Gruppo Nazionale di Fisica Matematica CNR program as visiting professor at the Department of Mathematics, University of Rome 'La Sapienze'. I would like to thank L. Tedeschini Lalli and C. Boldrighini for their kind hospitality in Rome. At last I finished it during my visit in spring 1990 at the Department of Mathematics, University of Chicago. I am very glad to thank R. Zimmer and P. May for their great help and support given me and my family in Chicago.

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