Introduction

In the late nineteenth century, Henri Poincaré created a new branch of mathematics by publishing his famous memoir (Poincaré [1]) on the qualitative theory of ordinary differential equations. Since then, differential topology, one of the principal modern developments of the differential calculus, has provided the proper setting for this theory. The subject has a strong appeal, for it is one of the main areas of cross-fertilization between pure mathematics and the applied sciences. Ordinary differential equations crop up in many different scientific contexts, and the qualitative theory often gives a major insight into the physical realities of the situation. In the opposite direction, substantial portions of many branches of pure mathematics can be traced back, directly or indirectly, to this source.

Suppose that we are studying a process that evolves with time, and that we wish to model it mathematically. The possible states of the system in which the process is taking place may often be represented by points of a differentiable manifold, which is known as the *state space* of the model. For example, if the system is a single particle constrained to move in a straight line, then we may take Euclidean space $\mathbf{R}^2$ as the state space. The point $(x, y) \in \mathbf{R}^2$ represents the state of the particle situated x units along the straight line from a given point in a given direction moving with a speed of y units in that direction. The state space of a model may be finite dimensional, as in the above case, or it may be infinite dimensional. For example, in fluid dynamics we have the velocity of the fluid at infinitely many different points to take into account and so the state space is infinite dimensional. It may happen that all past and future states of the system during the process are completely determined by its state at any one particular instant. In this case we say that the process is *deterministic*. The processes modelled in classical Newtonian mechanics are deterministic; those modelled in quantum mechanics are not.

In the deterministic context, it is often the case that the processes that can take place in the system are all governed by a smooth vector field on the state space. In classical mechanics, for example, the vector field involved is just another way of describing the *equations of motion* that govern all possible motions of the system. We can be more explicit as to what we mean by a vector field governing a process. As the process develops with time, the point representing the state of the system moves along a curve in the state space. The velocity of this moving point at any position x on the curve is a tangent vector to the state space based at x. The process is *governed* by the vector field if this tangent vector is the value of the vector field at x, for all x on the curve.

In the *qualitative* (or *geometric*) *theory*, we study smooth vector fields on differentiable manifolds, focusing our attention on the collection of parametrized curves on the manifold that have the tangency property described above. Our hope is that any outstanding geometrical feature of the curve system will correspond to a significant physical phenomenon when the vector field is part of a good mathematical model for a physical situation. This seems reasonable enough, and it is borne out in practice. We complete this motivational introduction by examining some familiar examples in elementary mechanics from this viewpoint. The remainder of the book is more concerned with the mathematical theory of the subject than with its modelling applications.

I. THE SIMPLE PENDULUM

Consider a particle P of mass m units fixed to one end of a rod of length l units and of negligible mass, the other end Q of the rod being fixed. The rod is free to rotate about Q without friction or air resistance in a given vertical plane through Q. The problem is to study the motion of P under gravity. The mechanical system that we have described is known as the *simple pendulum* and is already a mathematical idealization of a real life pendulum. For simplicity we may as well take $m = l = 1$, since we can always modify our units to produce this end. The first stage of our modelling procedure is completed by the assumption that gravity exerts a constant force on P of g units/sec^2 vertically downwards.

We now wish to find a state space for the simple pendulum. This is usually done by regarding the rotation of PQ about Q as being positive in one direction and negative in the other, and measuring

(i) the angular displacement θ radians of PQ from the downwards vertical through Q, and

(ii) the angular velocity ω radians/sec of PQ (see Figure 0.1).

We can then take $\mathbf{R}^2$ as the state space, with coordinates (θ, ω).

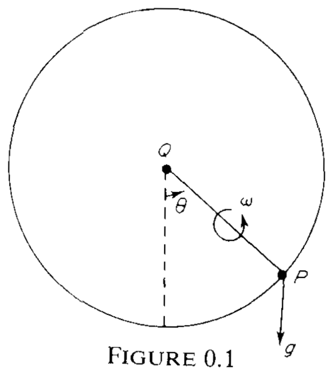

FIGURE 0.1

The equation of motion for the pendulum is

(0.2) $$\theta'' = -g \sin \theta,$$

where $\theta'' = d^2\theta/dt^2$. Using the definition of ω, we can replace this by the pair of first order equations

(0.3) $$\begin{aligned} \theta' &= \omega, \\ \omega' &= -g \sin \theta. \end{aligned}$$

A solution of (0.3) is a curve (called an *integral curve*) in the (θ, ω) plane parametrized by t. If the parametrized coordinates of the curve are $(\theta(t), \omega(t))$ then the tangent vector to the curve at time t is $(\omega(t), -g \sin \theta(t))$, based at the point $(\theta(t), \omega(t))$. We get various integral curves corresponding to various initial values of θ and ω at time $t = 0$, and these curves form the so-called *phase portrait* of the model. It can be shown that the phase portrait looks like Figure 0.4. One can easily distinguish five

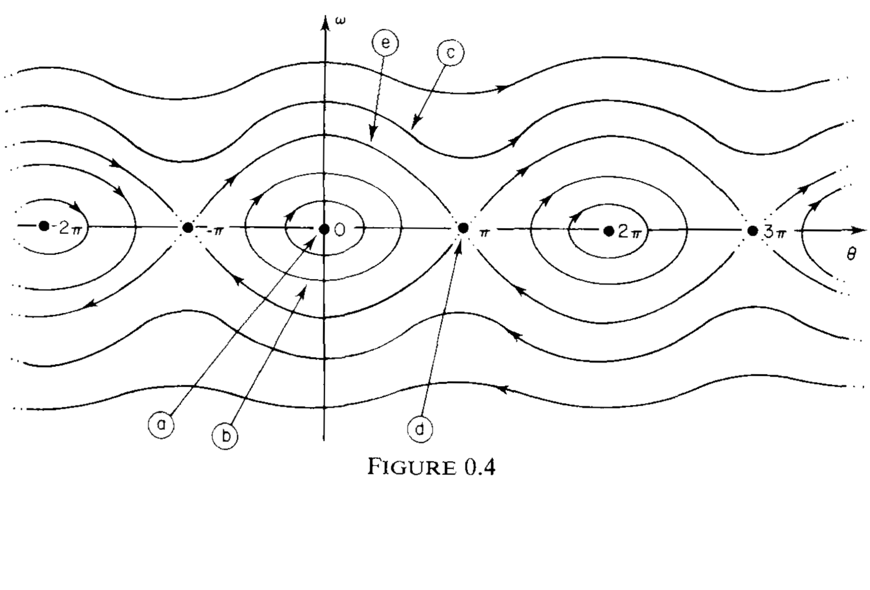

FIGURE 0.4

types of integral curves by their dissimilar appearances. They can be interpreted as follows:

(a) the pendulum hangs vertically downwards and is permanently at rest,
(b) the pendulum swings between two positions of instantaneous rest equally inclined to the vertical,
(c) the pendulum continually rotates in the same direction and is never at rest,
(d) the pendulum stands vertically upwards and is permanently at rest,
(e) the limiting case between (b) and (c), when the pendulum takes an infinitely long time to swing from one upright position to another.

The phase portrait in Figure 0.4 has certain unsatisfactory features. Firstly, the pendulum has only two equilibrium positions, one *stable* (downwards) and one *unstable* (upwards). However, to each of these there correspond infinitely many point curves in the phase portrait. Secondly, solutions of type (c) are periodic motions of the pendulum but appear as nonperiodic curves in the phase portrait. The fact of the matter is that unless we have some very compelling reason to do otherwise we ought to regard $\theta = \theta_0$ and $\theta = \theta_0 + 2\pi$ as giving the same position of the pendulum, since there is no way of instantaneously distinguishing between them. That is to say, the *configuration space*, which is the differentiable manifold representing the spatial positions of the elements of the mechanical system, is really a circle rather than a straight line. To obtain a state space that faithfully describes the system, we replace the first factor $\mathbf{R}$ of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ by the circle S^1, which is the real numbers reduced modulo 2π. Keeping θ and ω as our parameters, we obtain the phase portrait on the cylinder $S^1 \times \mathbf{R}$ shown in Figure 0.5.

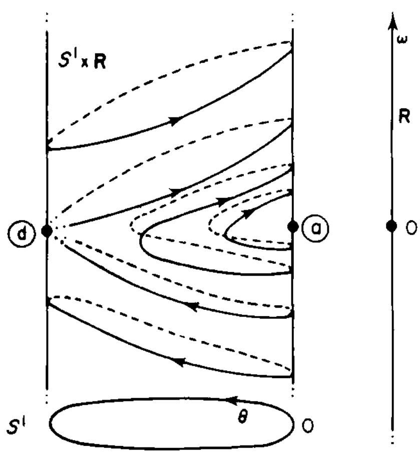

FIGURE 0.5

Consider now the *kinetic energy* T and the *potential energy* V of the pendulum, given by $T(\theta, \omega) = \frac{1}{2}\omega^2$ and $V(\theta, \omega) = g(1 - \cos\theta)$. Writing $E = T + V$ for the *total energy* of the pendulum, we find that equations (0.3) imply that $E' = 0$. That is to say E is constant on any integral curve. In view of this fact, the mechanical system is said to be *conservative* or *Hamiltonian*. In fact, in this example, the phase portrait is most easily constructed by determining the level curves (contours) of E. A pleasant way of picturing the role of E (due to E. C. Zeeman) is to represent the state space cylinder $S^1 \times \mathbf{R}$ as a bent tube in Euclidean 3-space and to interpret E as height. This is illustrated in Figure 0.6. The two arms of the tube contain solutions

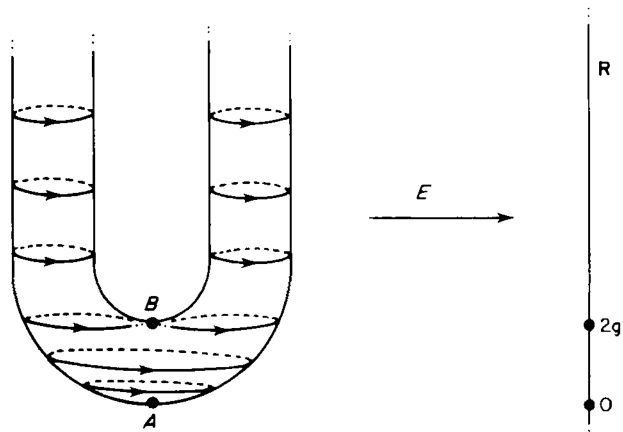

FIGURE 0.6

corresponding to rotations of the pendulum in opposite directions with the same energy E, with $E > 2g$, the potential energy of the unstable equilibrium.

The stability properties of individual solutions are apparent from the above picture. In particular, any integral curve through a point that is close to the stable equilibrium position A remains close to A at all times. On the other hand, there are points arbitrarily close to the unstable equilibrium position B such that integral curves through them depart from a given small neighbourhood of B. Note that the energy function E attains its absolute minimum at A and is stationary at B. In fact it has a saddle point at B.

II. A DISSIPATIVE SYSTEM

The conservation of the energy E in the above example was due to the absence of air resistance and of friction at the pivot Q. We now take these forces into account, assuming for simplicity that they are directly proportional to the angular velocity. Thus we replace equation (0.2) by

(0.7) $$\theta'' = -g \sin \theta - a\theta'$$

for some positive constant a, and (0.3) becomes

(0.8) $$\begin{aligned} \theta' &= \omega, \\ \omega' &= -g \sin \theta - a\omega. \end{aligned}$$

We now find that $E' = -a\omega^2$ is negative whenever $\omega \neq 0$. Thus the energy is dissipated along any integral curve, and the system is therefore said to be *dissipative*. If, as before, we represent E as a height function, the inequality $E' < 0$ implies that the integral curves cross the (horizontal) contours of E downwards, as shown in Figure 0.9.

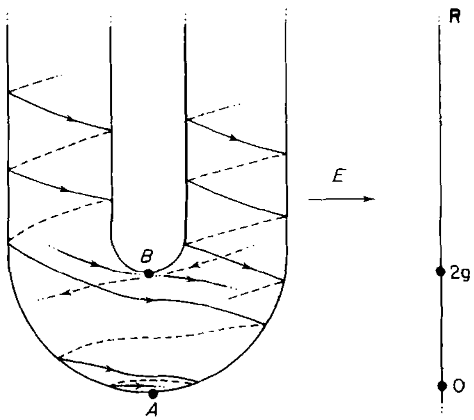

FIGURE 0.9

The reader may care to sketch dissipative versions of Figures 0.4 and 0.5. Notice that the stable equilibrium is now *asymptotically stable*, in that nearby solutions tend towards A as time goes by. We still have the unstable equilibrium B and four strange solutions that either tend towards or away from B. In practice we would not expect to be able to realize any of these solutions, since we could not hope to satisfy the precise initial conditions needed, rather than some nearby ones which do not have the required effect. (One can, in fact, sometimes stand a pendulum on its end, but our model is a poor one in this respect, since it does not take "limiting friction" into account.)

A comparison of the systems of equations (0.3) and (0.8) gives some hint of what is involved in the important notion of *structural stability*. Roughly speaking, a system is structurally stable if the phase portrait remains qualitatively the same when the system is modified by any sufficiently small perturbation of the right-hand sides. By *qualitatively* (or *topologically*) *the same*, we mean that some homeomorphism of the state space maps integral curves of the one onto integral curves of the other. The existence of systems (0.8) shows that the system (0.3) is not structural stable, since the constant a can be as small as we like. To distinguish between the systems (0.3) and (0.8), we observe that most solutions of the former are periodic, whereas the only periodic solutions of the latter are the equilibria. (Obviously this last properly holds in general for any dissipative system, since E decreases along integral curves.) The systems (0.8) are themselves structurally stable, but we do not attempt to prove this fact.

III. THE SPHERICAL PENDULUM

In the case of the simple pendulum, it is desirable, but not essential, to use a state space other than Euclidean space. With more complicated mechanical systems, the need for non-Euclidean state spaces is more urgent; it is often impossible to study them globally using only Euclidean state spaces. We need other spaces on which systems of differential equations can be globally defined, and this is one reason for studying the theory of differentiable manifolds.

Consider, for example, the *spherical pendulum*, which we get from the simple pendulum by removing the restriction that PQ moves in a given plane through Q. Thus P is constrained to lie on a sphere of radius 1 which we may as well take to be the unit sphere $S^2 = \{(x, y, z)\colon x^2 + y^2 + z^2 = 1\}$ in Euclidean 3-space. We use Euler angles θ and ϕ to parametrize S^2, as in Figure 0.10.

The motion of P is then governed by the second order equations

(0.11)
$$\theta'' = \sin\theta\cos\theta(\phi')^2 + g\sin\theta,$$
$$\phi'' = -2(\cot\theta)\theta'\phi',$$

which we replace by the equivalent system of four first order equations

(0.12)
$$\theta' = \lambda,$$
$$\phi' = \mu,$$
$$\lambda' = \mu^2\sin\theta\cos\theta + g\sin\theta,$$
$$\mu' = -2\lambda\mu\cot\theta.$$

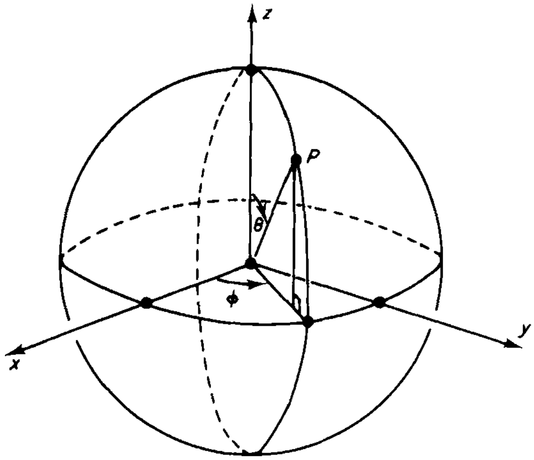

FIGURE 0.10

However, since the parametrization of S^2 by θ and ϕ is not even locally one-to-one at the two poles $(0, 0, \pm 1)$, it is wrong to expect that the four numbers $(\theta, \phi, \lambda, \mu)$ can be used without restriction to parametrize the state space of the system as $\mathbf{R}^4$. They *can* be employed with restrictions (for example $0 < \theta < \pi$, $0 < \phi < 2\pi$) but they do not then give the whole state space. In fact, the state of the system is determined by the position of P on the sphere, together with its velocity, which is specified by a point in the 2-dimensional plane tangent to S^2 at P. The state space is not homeomorphic to $\mathbf{R}^4$, nor even to the product $S^2 \times \mathbf{R}^2$ of the sphere with a plane, but is the tangent bundle TS^2 of S^2. This is the set of all planes tangent to S^2 and it is an example of a non-trivial vector bundle. *Locally*, TS^2 is topologically indistinguishable from $\mathbf{R}^4$, and we can use the four variables θ, ϕ, λ and μ as local coordinates in TS^2, provided that (θ, ϕ) does not represent the north or south pole of S^2.

The system is conservative, so again $E' = 0$ along integral curves, where the energy E is now a real function on TS^2 which, in terms of the above local coordinates, has the form

$$E(\theta, \phi, \lambda, \mu) = \tfrac{1}{2}(\lambda^2 + \mu^2 \sin^2 \theta) + g(1 + \cos \theta).$$

Thus every solution is contained in a contour of E. The contour $E = 0$ is again a single point at which E attains its absolute minimum, corresponding to the pendulum hanging vertically downwards in a position of stable equilibrium. The contour $E = 2g$ again contains the other equilibrium point, where the pendulum stands vertically upright in unstable equilibrium. At this point E is stationary but not minimal. The reader who is acquainted with

Morse theory (see Hirsch [1] and Milnor [3]) will know that for $0<c<2g$ the contour $E^{-1}(c)$ is homeomorphic to S^3, the unit sphere in $\mathbf{R}^4$. In any case, it is not hard to see this by visualizing how the contour is situated in TS^2. For $c>2g$, $E^{-1}(c)$ intersects each tangent plane to S^2 in a circle, and thus can be deformed to the unit circle bundle in TS^2. This can be identified with the topological group $SO(3)$ of orthogonal 3×3 matrices, for (the position vector of) a point of S^2 and a unit tangent vector at this point determine a right-handed orthonormal basis of $\mathbf{R}^3$. Moreover, rather less obviously (see, for example, Proposition 7.12.7 of Husemoller [1]), $SO(3)$ is homeomorphic to real projective space $\mathbf{RP}^3$.

The spherical pendulum is, as a mechanical system, symmetrical about the vertical axis l through the point of suspension Q. By this we mean that any possible motion of the pendulum gives another possible motion if we rotate the whole motion about l through some angle k, and that, similarly, we get another possible motion if we reflect it in any plane containing l. This symmetry shows itself in the equations (0.12), for they are unaltered if we replace ϕ by $\phi+k$ or if we replace ϕ and μ by $-\phi$ and $-\mu$. We say that the orthogonal group $O(2)$ acts on the system as a group of symmetries about the axis l. Symmetry of this sort is quite common in mechanical systems, and it can reveal important features of the phase portrait. In this case, for any c with $0<c<2g$, the 3-sphere $E^{-1}(c)$ is partitioned into a family of tori, together with two exceptional circles. The picture that we have in mind is Figure 0.13 rotated about the vertical straight line m. This decomposes $\mathbf{R}^3$ into a family of tori, together with a circle (through p and q) and the line m. Compactifying with a "point at ∞" (see the appendix to Chapter 2) turns $\mathbf{R}^3$ into a topological 3-sphere and the line m into another (topological) circle.

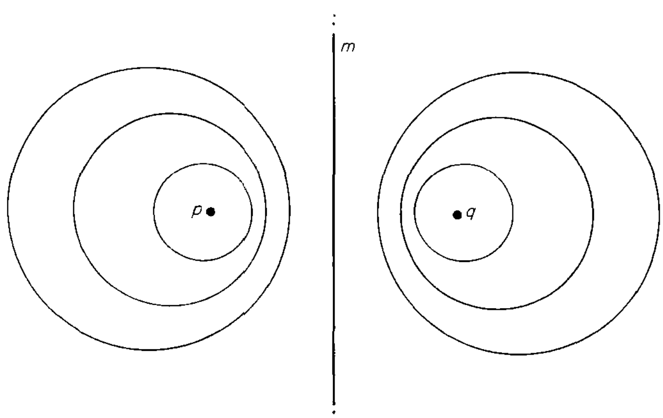

FIGURE 0.13

The submanifolds of this partition are each generated by a single integral curve under the action of $SO(2)$ (i.e. rotations only, not reflections). They are the intersections of $E^{-1}(c)$ with the contours of the angular momentum function on TS^2. The two exceptional circles correspond to the pendulum bob P revolving in a horizontal circle in the two possible directions. Halfway between them comes a torus corresponding to simple pendulum motions in the various planes through l. It is a useful exercise to investigate the similar decomposition of $E^{-1}(c)$ for $c > 2g$, and to see what happens at the critical case $c = 2g$.

IV. VECTOR FIELDS AND DYNAMICAL SYSTEMS

In all the above examples, the dynamical state of the system is represented by a point of the state space, which is the tangent bundle ($S^1 \times \mathbf{R}$ or TS^2) of the configuration space (S^1 or S^2). The equation of motion yields a vector field on the state space. Its integral curves give the possible motions of the mechanical system.

A useful way of visualizing a vector field v on an arbitrary manifold X is to imagine a (compressible) fluid flowing on X. We suppose that the velocity of the fluid at each point x of X is independent of time and equal to the value $v(x)$ of the vector field. In this case the integral curves of v are precisely the paths followed by particles of fluid. Now let $\phi(t, x)$ be the point of X reached at time t by a particle of fluid that leaves the point x at time 0. We can make some rather obvious comments. Firstly $\phi(0, x)$ is always x. Secondly, since velocity is independent of time, $\phi(s, y)$ is the point reached at time $s + t$ by a particle starting at the point y at time t. If we put $y = \phi(t, x)$, so the particle started from the point x at time 0, we deduce that $\phi(s, \phi(t, x)) = \phi(s + t, x)$. Finally we would expect the smoothness of ϕ, regarded as a function of t and x, to be influenced by the smoothness of v.

The map ϕ may not be defined on the whole of the space $\mathbf{R} \times X$, because particles may very well flow off X in a finite time. However, if ϕ is a well defined smooth map from $\mathbf{R} \times X$ to X with the above properties, we call it, in line with the above analogy, a smooth *flow* on X; otherwise we call it a smooth *partial flow* on X. It is said to be the *integral flow* of v or the *dynamical system* given by v.

Smooth vector fields and smooth flows on differentiable manifolds are the main objects of study in this book. If $\phi: R \times X \to X$ is a smooth flow on X, then, for any $t \in \mathbf{R}$, we may define a map $\phi^t: X \to X$ by $\phi^t(x) = \phi(t, x)$, and this is clearly a diffeomorphism, with inverse ϕ^{-t}. If we put $f = \phi^a$ for some $a \in \mathbf{R}$, we have, by induction, that $\phi(na, x) = f^n(x)$ for all integers n. Thus, if

a is small and non-zero, we often get a good picture of the properties of ϕ by studying the iterates f^n of the single map f (just as real events can be described reasonably well by the successive stills of a motion picture). The theory of *discrete dynamical systems* or *discrete flows*, as the study of iterates of a single homeomorphism is called, resembles the theory of flows in many parts, and is sometimes rather easier. We carry the two theories side by side throughout the book, and use the term *dynamical systems* to cover both theories.