Ann. Henri Poincaré 3 (2002) 451 – 482
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1424-0637/02/030451-32 $ 1.50+0.20/0
Annales Henri Poincaré

Multi-Dimensional Semi-Dispersing Billiards: Singularities and the Fundamental Theorem

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Abstract. The fundamental theorem (also called the local ergodic theorem) was introduced by Sinai and Chernov in 1987, see [S-Ch(1987)] and an improved version in [K-S-Sz(1990)]. It provides sufficient conditions on a phase point under which some neighborhood of that point belongs to one ergodic component. This theorem has been instrumental in many studies of ergodic properties of hyperbolic dynamical systems with singularities, both in 2-D and in higher dimensions. The existing proofs of this theorem implicitly use the assumption on the boundedness of the curvature of singularity manifolds. However, we found recently ([B-Ch-Sz-T(2000)]) that, in general, this assumption fails in multidimensional billiards. Here the fundamental theorem is established under a weaker assumption on singularities, which we call Lipschitz decomposability. Then we show that whenever the scatterers of the billiard are defined by algebraic equations, the singularities are Lipschitz decomposable. Therefore, the fundamental theorem still applies to physically important models – among others to hard ball systems, Lorentz gases with spherical scatterers, and Bunimovich-Reháček stadia.

1 Introduction

In contrast to smooth dynamical systems, billiards have singularities which make the application of the classical methods substantially more difficult. One reason is that in the neighbourhood of orbits tangent to the obstacles (the so-called tangent singularities) the derivative of the Poincaré section map diverges. Nevertheless, Sinai's celebrated 1970 result demonstrated that, at least for $d = 2$, the hyperbolicity caused by the strictly convex scatterers overcomes the harmful effect of singularities. In fact, he showed that $2D$ dispersing billiards, i. e. those with strictly convex obstacles, are ergodic and even K-mixing [S(1970)].

Multidimensional geometry is, however, essentially richer so it is not surprising that it had taken 17 years until Chernov and Sinai [S-Ch(1987)] could extend Sinai's original result to multidimensional dispersing billiards. This remarkable achievement was a corollary of their *local ergodicity theorem*, often called the *fundamental theorem*, formulated for semi-dispersing billiards, i. e. those with convex scatterers. Their theorem got slightly generalized with the clarification of some technical details and conditions by Krámli, Simányi and Szász [K-S-Sz(1990)] in 1990.

The considerations in the proof of the local ergodicity theorem are local. As a matter of fact, by assuming the boundedness (from above) of the curvature of all images of the tangencies, which is a straightforward fact for $d = 2$, it became possible to assume that they are linear objects, at least locally. However, in the recent paper of the present authors, [B-Ch-Sz-T(2000)] it has been discovered that for $d \geq 3$, in the neighbourhoods of tangent orbits the images of tangencies (and of other smooth one-codimensional submanifolds of the phase space) develop a pathological behaviour contradicting the boundedness of the curvatures. Therefore for its own interest but also for its various important consequences it became an absolute necessity to correct the original arguments and this is the sole aim of this work. Indeed, instead of the boundedness property of the aforementioned curvatures we formulate a new condition, the so called *Lipschitz-decomposability condition.* Roughly speaking it requires that the singularities can be decomposed into a finite number of graphs of locally Lipschitz functions with the boundaries of these graphs being not too wild. This assumption, together with the other requirements of the local ergodicity theorem, is already sufficient to save the old proof. The next question is, of course, when this new condition holds. Fortunately, we can verify it under one additional requirement: we assume that the scatterer boundaries are *algebraic.* Luckily enough, the main examples of multidimensional semi-dispersing billiards are all algebraic. Just think – first of all – of hard ball systems [SSz(1999)], [Sim(2001)], of the Lorentz process with spherical scatterers ([H(1974)], [Sz(2000)]), of general algebraic cylindrical billiards [Sz(1994)], [Sim(2002)], and of the multidimensional stadia designed by Bunimovich and Rehacek [B-R(1998)].

For keeping our exposition possibly short, we rely heavily on that of [K-S-Sz(1990)]. In section 2, we summarize the necessary notations and prerequisites from the aforementioned work. Section 3 is devoted to the study of singularities. In particular, in subsection 3.1 we briefly recall the pathological behaviour described in [B-Ch-Sz-T(2000)]. Then, in subsection 3.2 we present the aforementioned Lipschitz decomposability property of the singularities. Based upon this assumption, in section 4 we reformulate the local ergodicity theorem and discuss in detail where and how the classical proof of [S-Ch(1987)] and [K-S-Sz(1990)] should be modified. Finally, in section 5 it is shown that the Lipschitz decomposability property holds for algebraic billiards. Though here we use some simple ideas from algebraic geometry and from geometric measure theory, the arguments are still elementary.

2 Prerequisites

The methods in this paper, though quite elementary, come from different branches of mathematics. Throughout the arguments we try to keep the exposition self-contained. More details on the basic notions from algebra or geometric measure theory can be found in the books [B-C-R(1987)], [Sh(1974)], [St(1973)] and [F(1969)], [Fa(1985)], [Fa(1990)]; respectively.

We would also like to fix one notation: for any subset in a Riemannian manifold $H \subset \mathcal{M}$, $H^{[\delta]}$ shall denote its δ-neighborhood:

$$H^{[\delta]} := \{x \in \mathcal{M} \mid \rho(x, H) \leq \delta\}. \tag{2.1}$$

2.1 Multi-Dimensional Semi-Dispersing Billiards

In this subsection we summarize some basic properties of semi-dispersing billiards. Our aim is to introduce the most important concepts and fix the notation in order to keep the exposition of the paper self-contained. For a more detailed description see the literature, especially [K-S-Sz(1990)].

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$. In general, the boundary of the domain in assumed to be piecewise C^3-smooth, however, later on we impose the further restriction of algebraicity on the billiard (cf. section 5). Inside Q the motion is uniform while the reflection at the boundary ∂Q is elastic. As the absolute value of the velocity is a first integral of motion, the phase space of the billiard flow is fixed as $M = Q \times S^{d-1}$ – in other words, every phase point x is of the form $x = (q, v)$ with $q \in Q$ and $v \in \mathbb{R}^d$, $|v| = 1$. The Liouville probability measure μ on M is essentially the product of the Lebesgue measures, i.e. $d\mu = \text{const.}\, dqdv$. The resulting dynamical system $(M, \{S^t, t \in \mathbb{R}\}, \mu)$ is the billiard flow.

Let $n(q)$ denote the unit normal vector of a smooth component of the boundary ∂Q at the point q, directed inwards Q. Throughout the paper we restrict our attention on *semi-dispersing billiards*: we require that for every $q \in \partial Q$ the second fundamental form $K(q)$ of the boundary component be non-negative.

The boundary ∂Q defines a natural cross-section for the billiard flow. Consider namely

$$\partial M = \{(q, v) \mid q \in \partial Q,\ \langle v, n(q)\rangle \geq 0\}.$$

This set actually has a natural bundle structure (cf. [B-Ch-Sz-T(2000)]). In this paper we use the arising Riemannian metric ρ on ∂M. The *billiard map* is defined as the first return map on ∂M. The invariant measure for the map is denoted by μ_1, and we have $d\mu_1 = \text{const.}\, |\langle v, n(q)\rangle|\, dqdv$. Throughout the paper (except for subsection 5.1) we work with this discrete time dynamical system. Its ergodicity implies that of the flow (see [K-S-Sz(1990)]).

Singularities. Consider the set of tangential reflections, i.e.

$$\mathcal{R} := \{(q, v) \in \partial M \mid \langle v, n(q)\rangle = 0\}.$$

It is easy to see that the map T is not continuous at the set $T^{-1}\mathcal{R}$. As a consequence, the singularity set for a higher iterate T^n is

$$\mathcal{R}^{(n)} = \cup_{i=1}^{n} \mathcal{R}^{-i},$$

where in general $\mathcal{R}^k = T^k\mathcal{R}$. Generally it was assumed in the literature that the set $\mathcal{R}^{(n)}$ is a finite collection of smooth and compact submanifolds of the Poincaré phase space ∂M. However, for multi-dimensional semi-dispersing billiards these manifolds can be treated as submanifolds of ∂M only in a topological sense (see section 3).

Remark 2.1 *Above the (tangential) singularities have been introduced for the Poincaré section map T. But in a part of the proof following remark 5.3 they will also be needed for the flow. In fact, the aim of the aforementioned remark is just to hint how this extension of the singularities is understood.*

For completeness we mention that in case the boundary ∂Q is only piecewise smooth, further singularities – the multiple collisions – arise. At such points neither $n(q)$ and, as a consequence, nor the flow dynamics is uniquely defined, thus we can speak about several "branches" of a trajectory. The singularity set must also be treated with a little more care. For this reason, in all cases we will denote by $\mathcal{R}^+$ the set of all singular phase points, which can be points of $\mathcal{R}$ or multiple collision points supplied with the possible *outgoing* velocities. (See [Sim(2001)] and its references for details). In the present paper we consider only tangential singularities. Multiple collisions can be treated in an analogous way, although the main difficulty – the blow-up of the derivative of the dynamics – does not appear here.

We introduce some more notation. For any $n \in \mathbb{N}$, Δ_n stands for the set of doubly singular phase points up to order n, i.e. $x \in \partial M$ belongs to Δ_n whenever there are indices $k_1 \neq k_2$, $|k_i| \leq n$ such that both $T^{k_1}x$ and $T^{k_2}x$ are elements of $\mathcal{R}$. We are mainly interested in phase points with regular or with at most once singular trajectories, thus we consider the following sets:

$$\begin{aligned} \partial M^0 &:= \quad \partial M \setminus \bigcup_{n\in\mathbb{Z}} \mathcal{R}^n \\ \partial M^* &:= \quad \partial M \setminus \bigcup_{n=1}^{\infty} \Delta_n \\ \partial M^1 &:= \quad \partial M^* \setminus \partial M^0. \end{aligned} \tag{2.2}$$

As to regular and at most once singular phase points of the flow, the sets M^0, M^* and M^1 refer to flow-images of ∂M^0, ∂M^* and ∂M^1, respectively.

Different notions of norms and metrics. In billiard theory several notions of metrics and distances are used. Let us assume that two phase points $x = (q, v)$ and $x' = (q', v')$ and a vector in the tangent plane at x, $w = (\delta q, \delta v)$ are fixed. In all calculations presented in the paper we use the *Euclidean norm* $\|w\| = \sqrt{|\delta q|^2 + |\delta v|^2}$ and the generated Euclidean distance $\rho(x, x')$. The measure on ∂M corresponding to this Riemannian metric (generated by the volume form) is simply the Lebesgue measure const. $dq\, dv$. However, in several other statements referred (see e.g. [K-S-Sz(1990)], especially the Erratum) two other metrics come about. For their definition we fix the notation for two $d-1$ dimensional linear subspaces in $\mathbb{R}^d$: $\mathcal{T}$, the one orthogonal to $n(q)$ and $\mathcal{J}$, the one orthogonal to v. Furthermore we introduce the linear operator $V : \mathcal{J} \to \mathcal{T}$ which is simply the projection parallel to v. (On details see [B-Ch-Sz-T(2000)].)

This way we may define the *invariant norm* of a vector: $\|w\|_i = \sqrt{|V^{-1}\delta q|^2 + |\delta v|^2}$ and the generated invariant distance $\rho_i(x, x')$. The name 'invariant' comes from the fact that the measure corresponding to this Riemannian metric (via the volume form) is the invariant measure $d\mu_1 = \text{const.} |\langle v, n(q)\rangle| \, dq \, dv$. Note that $\|w\||\langle v, n(q)\rangle| \leq \|w\|_i \leq \|w\|$, thus the two distances are equivalent if we can ensure $|\langle v, n(q)\rangle| \geq c$ for some constant c. This happens throughout the proof of the fundamental theorem (cf. section 4) where we work in a neighborhood of an interior point $x \in \partial M$ and thus the two metrics are (locally) equivalent.

The third metric-type quantity is the so-called *p-metric* $\|w\|_p = |V^{-1}\delta q|$. Even though this is a degenerate metric in general (that is the reason for the name 'p' – pseudo), it is non-degenerate when restricted to vectors w corresponding to convex fronts (cf. [K-S-Sz(1990)], [B-Ch-Sz-T(2000)]). Its importance is related to the fact that the most convenient way of handling hyperbolicity issues is in terms of the p-metric (see e.g. Lemma 2.2).

Related to the above mentioned metrics there are two ways of measuring distance of a phase point $x = (q, v)$ from the set of tangential reflections $\mathcal{R}$. $z(x) = \rho_i(x, \mathcal{R})$ is simply the distance in terms of ρ_i. Alternatively we may consider tubular neighborhoods U_r (of radii r) of the flow trajectory starting out of x in the configuration space Q. Then define $z_{tub}(x)$ as the supremum of radii r for which the tube does not intersect the set of singular reflections (see [S-Ch(1987)] and [K-S-Sz(1990)], especially the Erratum). It is not difficult to see that $z(x) \leq z_{tub}(x)$.

Hyperbolicity. Besides the presence of singularities the most important feature of semi-dispersing billiard dynamics is that it is – at least locally and non-uniformly – hyperbolic. A highly important consequence of this fact is the abundance of local invariant manifolds. The notion of a local invariant manifold will be used in the traditional sense, i.e. a C^1-smooth, connected submanifolds $\gamma_s \subset \partial M$ is a *local stable manifold* at $x \in \partial M$ iff

$$\begin{aligned} &(i) && x \in \gamma_s \\ &(ii) && \exists K(\gamma_s), C(\gamma_s) > 0 \text{ such that for any } y_1, y_2 \in \gamma_s \\ & && \rho(T^n y_1, T^n y_2) \leq K \exp(-Cn) \rho(y_1, y_2). \end{aligned} \tag{2.3}$$

Local stable manifolds for the inverse dynamics T^{-1} will be referred to as local unstable manifolds.

The treatment of hyperbolicity is traditionally related to *local orthogonal manifolds* (or *fronts*) and *sufficient phase points*. These objects are defined in the flow phase space the following way.

Let $x = (q, v) \in M \setminus \partial M$ and consider a C^2-smooth codimension 1 sub manifold $\Sigma' \subset Q \setminus \partial Q$ such that $q \in \Sigma'$ and $v = v(q)$ is the normal vector to Σ' at q. Denote by Σ the normal section of the unit tangent bundle on Q restricted to Σ'. Σ is called a local orthogonal manifold or simply a front. A front is said to be (strictly) convex whenever its second fundamental form $B_\Sigma(y) \geq 0$ ($B_\Sigma(y) > 0$) for every $y \in \Sigma$.

Let us consider a nonsingular finite trajectory segment for the flow: $S^{[a,b]}x$, where $a < 0 < b$ and a, b, 0 are not moments of collision.

$\mathcal{N}_0(S^{[a,b]}x)$, the *neutral subspace* at time 0 for the segment $S^{[a,b]}x$ is defined as follows:

$$\begin{aligned}\mathcal{N}_0(S^{[a,b]}x) := \quad \{ \quad & w \in \mathbb{R}^d : \exists(\delta > 0) s.t. \forall \alpha \in (-\delta, \delta) \\ & v(S^a(q(x) + \alpha w, v(x))) = v(S^a x) \& \\ & v(S^b(q(x) + \alpha w, v(x))) = v(S^b x)\}.\end{aligned}$$

Observe that $v(x) \in \mathcal{N}_0(S^{[a,b]}x)$ is always true, the neutral subspace is at least 1 dimensional. Neutral subspaces at time moments different from 0 are defined by $\mathcal{N}_t(S^{[a,b]}x) := \mathcal{N}_0(S^{[a-t,b-t]}(S^t x))$, thus they are naturally isomorphic to the one at 0.

The non-singular trajectory segment $S^{[a,b]}x$ is *sufficient* if for some (and in that case for any) $t \in [a,b]$: $\quad dim(\mathcal{N}_t(S^{[a,b]}x)) = 1$. A point $x \in M^0$ is said to be sufficient if its entire trajectory $S^{(-\infty,\infty)}x$ contains a finite sufficient segment. Singular points are treated by the help of trajectory branches (see [K-S-Sz(1990)]): a point $x \in M^1$ (this precisely means that the entire trajectory contains one singular reflection) is sufficient if both of its trajectory branches are sufficient.

All these concepts have their natural counterparts for the billiard map phase space ∂M. For example, a smooth piece $\Sigma \in \partial M$ of the image of a local orthogonal manifold in M is referred to as a front as well.

Hyperbolicity is related to the following simple phenomena. Near sufficient phase points hyperplanes in Q orthogonal to the flow evolve into strictly convex fronts. Convex fronts remain convex under time evolution. The importance of this is shown by the Lemma below. Before formulating it we introduce one more notation: $D^n_{y,\Sigma}$ is the derivative of the (nth power of the) dynamics T^n restricted to the front Σ.

Lemma 2.2 (Equivalent of Lemma 2.13 from [K-S-Sz(1990)].) *For every $x \in \partial M^0$ for which the trajectory is sufficient there exists a neighborhood $U(x)$ and a constant $0 < \lambda(x) < 1$ such that*

- *through almost every point $y \in U(x)$ there do pass uniformly transversal local stable and unstable manifolds $\gamma^s(y)$ and $\gamma^u(y)$ of dimension $d-1$;*
- *for any $y \in U(x)$ and any convex front $\Sigma^{\pm}$ passing through $\pm y$:*

$$\|(D^{\tau}_{\pm y,\Sigma^{\pm}})^{-1}\|_p < \lambda(x), \tag{2.4}$$

where $\tau \in \mathbb{Z}^+$ is the first return time to $U(x)$.

More details about local hyperbolicity and semi-dispersing billiards in general can be found in [K-S-Sz(1990)].

3 Singularities

In several papers that appeared, singularities were assumed – either explicitly or implicitly – to consist of smooth 1-codim submanifolds of the phase space. Often, even a uniform bound on the curvature was assumed, independent of the order of the singularity. This is true for $2D$ billiards. However, it is not true in higher dimensions. In this section we present a counter-example in a 3-dim dispersing billiard. Already the curvature of $\mathcal{R}^{-2}$ has no upper bound, i.e. the curvature blows up near a point where the singularity manifold is not even differentiable. After this example we propose another property which, in most applications, can replace the bounded curvature assumption. We conjecture that this property: *the Lipschitz decomposability of singularities* holds for multi-dimensional semi-dispersing billiards.

3.1 Counter-example for bounded curvature

In this section we recall our example from [B-Ch-Sz-T(2000)] showing that even in a $3D$ dispersing billiard, already the two-step singularities have no bounded curvature. The proof given in [B-Ch-Sz-T(2000)] was rather implicit. We started with the indirect assumption that the curvature was bounded, and found that

Claim. The two-step singularity intersects the one-step singularity tangentially at every point of their intersection, except for a one-codimensional degeneracy, where the intersection is not tangential.

This claim obviously contradicts the bounded curvature assumption.

We do not repeat here the calculations of [B-Ch-Sz-T(2000)], but rather we present the concrete situation where this pathological behaviour appears.

Since this example deals with a very explicitly given billiard configuration, we will not use the complicated notations of the other sections: we will denote $\mathcal{R}^{-k}$ simply with S_k ($k \geq 0$).

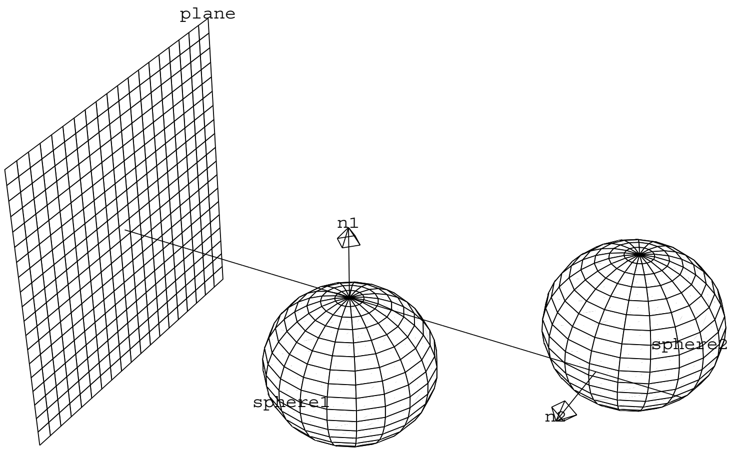

Figure 1: The studied billiard configuration

Consider the situation demonstrated on Figure 1. To present the example as transparent as possible the first scatterer, the surface where the trajectories start out is a plane – thus it is not strictly convex. Nevertheless this modification has no significance.

We are in 3 dimensions, so take a standard 3D Cartesian coordinate system. Let the zeroth 'scatterer' be the $\{z = 0\}$ plane. Let the first scatterer be the sphere with center $O_1 = (0, -1, 1)$ and radius $R = 1$. Let the second scatterer be the sphere with center $O_2 = (1, 0, 2)$ and radius $R = 1$. We look at the component of the phase space corresponding to the zeroth scatterer, near the phase point $(x_0 = 0, y_0 = 0, v_{x0} = 0, v_{y0} = 0)$. Of course, $v_{z0} = 1$, and the trajectory is the z axis. The counterexample mentioned in the Claim is the intersection of S_1, the inverse image of the first scatterer, and S_2, the second inverse image of the second scatterer, both considered on the zeroth scatterer in (the neighbourhood of) the origin. We are mainly interested in the singularity manifolds close to a doubly tangent orbit.

The calculations of [B-Ch-Sz-T(2000)] show that at the origin S_1 and S_2 can not be tangent. This is essentially the consequence of the circumstance that in the two points of tangencies (with the first and second spheres) the two normals of incidence are perpendicular to each other. In all other situations S_1 and S_2 are tangent! Consequently at the origin S_2 is not even differentiable.

Next we recall a much useful paradigm which is a well known object of algebraic singularity theory: the *Whitney umbrella*. It not only illustrates better the pathological situation in three dimensions (rather than our counterexample in dimension 4) but also suggests to find the way out: to substitute the condition on the boundedness of curvatures with the Lipschitz decomposability property.

The Whitney-umbrella. Consider the one-codimensional set in $\mathbb{R}^3$ defined by the polynomial equation:

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 z = y^2\},$$

the Whitney-umbrella. 'One half' of this set (its intersection with the quadrants $\{xy \leq 0\}$) is shown on Figure 2. For simplicity we use the notations: W_2 for this 'half-umbrella' and W_1 for the $\{z = 0\}$ plane. Clearly

- W_2 terminates on W_1 (in the points of the x-axis), thus $W_1 \cap W_2 = \partial W_2$.
- at every point of the x-axis where $x \neq 0$ the intersection of W_2 and W_1 is tangential.
- W_2 has smooth manifold structure in its interior; nevertheless, near the origin its curvature is unbounded as the normal vector changes rapidly (actually, the unit normal vector does not even have a well-defined limit at the origin).

By these properties the geometry of singularities in the counterexample is analogous to Figure 2.[1] W_1 corresponds to S_1, W_2 corresponds to S_2 while the

[1] To be precise, the situation on Figure 2. has one dimension less – in contrast to W_2 the singularities are 3-dimensional manifolds – but this has little significance to the analogy.

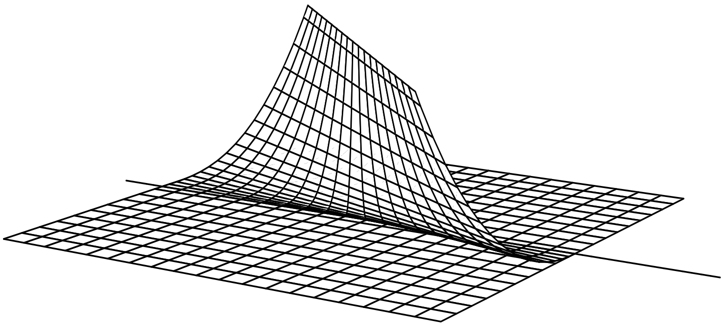

Figure 2: The Whitney Umbrella

origin corresponds to the set of those doubly tangential reflections where the two radii are orthogonal (this set is one-codimensional in $S_1 \cap S_2$).

Lipschitz decomposability of the Whitney-umbrella. This analogy also shows that bounded curvature is not needed for the neighbourhood of a manifold to be small. Indeed, the 'half-umbrella' W_2 can be cut further into two pieces (namely, its intersections with the quadrants $\{x \geq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$), each of which is the graph of a Lipschitz function, when viewed from the appropriate direction. Indeed, easy calculations show that if we choose the direction $(1, 1, 1)$ or $(-1, -1, 1)$ to be 'vertical' (respectively), these 'quarter-umbrellas' become graphs of Lipschitz functions with Lipschitz constant $\sqrt{2}$. So the whole Whitney-umbrella consists of four such graphs plus a one-dimensional tail. This tail (the negative z axis) has no analogue in the singularities of billiards. It's only there because the umbrella was defined in an algebraic way. However, it will not spoil our measure-theoretic estimates because it has one dimension less than the rest of the set.

Generalization I. First let us consider the first-step singularity S_1. By the notations of the previous counterexample (on details see [B-Ch-Sz-T (2000)]) we may characterize the points (x, y, v_x, v_y) belonging to S_1 easily. These are precisely those for which $d(x, y, v_x, v_y) = 1$, where $d(., ., ., .)$ is the distance of the point $O_1 = (0, -1, 1)$ from the line that passes through the point $(x, y, 0)$ and has direction specified by the velocity components v_x, v_y. As d is a smooth function of its variables there is no curvature blow-up for S_1 – and, for first-step singularities in general. Thus S_2 is a *pre-image of a smooth one-codimensional compact submanifolds*, however, *the map* under which the pre-image is taken *has unbounded derivatives and is highly an-isotropic*. Curvature blow-up occurs only at those points of S_2 (near its intersection with S_1) where the map behaves irregularly.

In correspondence with the above observation we conjecture that curvature blow-up is not a peculiar feature of S_2, it is present in the pre-images of one-codimensional smooth submanifolds in general. Consider for example two-step *secondary* singularities Γ_2 – those phase points for which at the second iterate instead of tangentiality the collision term ($\langle n, v\rangle$) is a given constant (see [B-Ch-Sz-T(2000)] for more detail). In the specific example of subsection 3.1 such secondary singular trajectories are precisely those that touch tangentially a sphere of radius R' ($R' < 1$) at the second iterate. It is clear that the geometry of Γ_2 is completely analogous to S_2.

Generalization II. Our calculations in [B-Ch-Sz-T(2000)] do not use any specialty of the explicitly given billiard configuration. Doubly tangential reflections for which the normal vectors of the scatterers at the consecutive collisions are orthogonal can be found in *any multi-dimensional semi-dispersing billiard*. Near such trajectories a similar calculation can be performed.

Generalization III. All in all, the discovered pathology is general. In addition, *the higher step singularities* S_k; ($k \geq 3$) may show even wilder behaviour near their intersections. Nevertheless, we strongly conjecture that a nice geometric characterization – suggested by the analogy with the Whitney-umbrella in the case of S_2 – can be performed.

We have mentioned these generalization to present the reader the picture of singularities we have in mind. Nevertheless, for our further discussion we do not need to verify any of these calculations or generalizations since they are completely independent.

3.2 Lipschitz property of singularities

When treating ergodic or stochastic properties of singular systems, we need to understand the properties of singularities in order to know that their neighbourhood is of small measure. By assuming that the singularities are smooth, e. g. they have bounded curvature, in local considerations one can treat them as planes, by choosing an appropriately small scale. This, of course, implies that the intersection of a (smooth) singularity component and a sphere of radius r has a surface-volume of order r^{m-1} where $m = 2d - 2$ is the dimension of the phase space. Similarly, the δ-neighbourhood of such a singularity-piece has measure of order $r^{m-1}\delta$. These properties have been used in several papers without being checked. We now know that the curvature is in general not bounded, so a more careful investigation is essential.

To ensure that the regularity properties mentioned hold, we (approximately) propose to assume that the singularities have components which are graphs of Lipschitz functions – instead of assuming they have smooth components.

Definition 3.1 *A subset H of $\mathbb{R}^m$ will be called a Lipschitz graph, if we can choose a Cartesian coordinate system so that H becomes the graph of a Lipschitz function: $H = \{(x, f(x)) \mid x \in D\}$ with some (measurable) $D \subset \mathbb{R}^{m-1}$ and $f : \mathbb{R}^{m-1} \to \mathbb{R}$ Lipschitz-continuous.*

Being a Lipschitz graph ensures that H is rectifiable, and that for its surface-volume one has $\mu(H) \leq C\mu(D)$ where the constant C depends only on the Lipschitz constant of f. The main property of Lipschitz graphs is shown by the following very basic

Lemma 3.2 *Let $D \subset \mathbb{R}^{m-1}$ arbitrary, $f : \mathbb{R}^{m-1} \to \mathbb{R}$ Lipschitz-continuous with Lipschitz-constant L. Let $H = \{(x, f(x)) \mid x \in D\} \subset \mathbb{R}^m$. Denote by $\mathcal{L}^m$ the Lebesgue-measure in $\mathbb{R}^m$, and by $\mathcal{L}^{m-1}$ the Lebesgue-measure in $\mathbb{R}^{m-1}$. Denote by $H^{[\delta]}$ the δ-neighbourhood (in $\mathbb{R}^m$) of H. Then*

$$\mathcal{L}^m(H^{[\delta]}) \leq 2\delta\sqrt{L^2+1}\mathcal{L}^{m-1}(\bar{D}) + o(\delta) \tag{3.1}$$

Proof. Just notice that

$$H^{[\delta]} \subset \{(x, y) \mid x \in D^{[\delta]}, |y - f(x)| \leq \delta\sqrt{L^2+1}\},$$

where $D^{[\delta]}$ is the δ-neighbourhood of D in $\mathbb{R}^{m-1}$. This implies

$$\mathcal{L}^m(H^{[\delta]}) \leq 2\delta\sqrt{L^2+1}\mathcal{L}^{m-1}(D^{[\delta]})$$

which gives the lemma, since $\mathcal{L}^{m-1}(D^{[\delta]}) \to \mathcal{L}^{m-1}(\bar{D})$ as $\delta \to 0$. □

To precisely formulate the property that we propose instead of smoothness of the singularities, we need the following two definitions:

Definition 3.3 *(cf. [F(1969)]) A function $f : D \subset \mathbb{R}^{m-1} \to \mathbb{R}$ will be called locally Lipschitz (with Lipschitz constant L), if for any $x \in D$ there exists a neighbourhood $U \subset \mathbb{R}^{m-1}$ of x such that the restricted function $f\Big|_{D\cap U}$ is Lipschitz (with Lipschitz constant L).*

In all our applications D will be open. Notice that in this case, f typically cannot even be extended to $\bar{D}$ in a continuous way.

Definition 3.4 *$H \subset \mathbb{R}^m$ will be called a (one-codimensional) open locally Lipschitz graph (with Lipschitz constant L), if we can choose an appropriate Cartesian coordinate system so that H becomes the graph of a locally Lipschitz function:*

$$H = \{(x, f(x) \mid x \in D\}$$

with some $D \subset \mathbb{R}^{m-1}$ and $f : D \to \mathbb{R}$ locally Lipschitz (with constant L).

We will be mainly interested in the case when the domain D is an open set in $\mathbb{R}^{m-1}$, then

- *H will be called an open locally Lipschitz graph (even though it is not an open set in $\mathbb{R}^m$),*
- *and we will denote by ∂H the boundary of H as of a surface: $\partial H = \bar{H} \setminus H$.*

Now we are able to define the regularity property that should replace the smoothness of singularities. This property, called 'Lipschitz decomposability' will be defined for subsets of $\mathbb{R}^m$ here. For Lipschitz decomposability of subsets of a Riemannian manifold, see Remark 3.6.

Definition 3.5 *Consider $H \subset \mathbb{R}^m$, and $L \in \mathbb{R}$. H will be called 'Lipschitz decomposable' (one-codimensional) subset with constant L if it can be decomposed into a finite number of open locally Lipschitz graphs and a small remainder set in the following way: There exist H^* and $H_1, \ldots, H_K$ such that:*

- $H \subset \bigcup_{i=1}^{K} \bar{H}_i \bigcup H^*$,
- $H_i \cap H_j = \emptyset$ *for any* $i \neq j$,
- *every H_i is a one-codimensional open locally Lipschitz graph (with constant L),*
- $\mathcal{L}^m \left(\left((\bigcup_{i=1}^{K} \partial H_i) \cup H^* \right)^{[\delta]} \right) = o(\delta)$.

The set H^* is included in the decomposition for technical reasons: we want to allow for sets H having parts of strictly higher codimension. This occurs generically if H is an algebraic subvariety of $\mathbb{R}^n$ – cf. subsection 3.1 on the one dimensional tail of the Whitney-umbrella and section 5. Nevertheless we would like to note that such higher codimensional parts are not present in the singularities of semi-dispersing billiards.

Remark 3.6 Lipschitz decomposability in Riemannian manifolds. *Throughout the paper – and in particular in conjecture 3.7 below – subsets of a compact Riemannian manifold $\mathcal{M}$ are considered. For $H \subset \mathcal{M}$ Lipschitz decomposition is understood in terms of coordinate charts.*

To be more precise, let us fix some convention related to the atlas $\{U_t, \psi_t\}_{t=1}^{T}$ for $\mathcal{M}$ first. It is important that $\mathcal{M}$ is compact thus we may consider a finite atlas. We say that the atlas is bi-Lipschitz *if all charts $\psi_t : U_t \to \mathbb{R}^m$ are bi-Lipschitz maps, i.e. both ψ_t and $(\psi_t)^{-1}$ are Lipschitz with some constant $K > 1$. All atlases considered are assumed to be bi-Lipschitz with a fixed constant. This ensures that Euclidean distance on $\mathbb{R}^m$ is comparable to Riemannian metric on the manifold, and thus our metric estimates indeed apply in the arguments of section 4. Note that bi-Lipschitzness – with Lipschitz constant arbitrarily close to one – can always be obtained by choosing the coordinate patches sufficiently small.*

As to the problem of Lipschitz decomposition, we will say that $H \subset \mathcal{M}$ is Lipschitz decomposable whenever a finite bi-Lipschitz atlas can be chosen, such *that for all charts $\psi_t(H \cap U_t)$ is Lipschitz decomposable as a subset of $\mathbb{R}^m$, in the sense of Definition 3.5.*[2]

The precise property that we expect the singularities of semi-dispersing billiards to have is formulated in the form of a conjecture:

Conjecture 3.7 *For any semi-dispersing billiard with a finite horizon there exists an $L \in \mathbb{R}$ such that for any integer N the set $\bigcup_{|n|\le N} \mathcal{R}^n$ of singularities of order up to N is 'Lipschitz decomposable' with constant L.*

It is worth noting that by introducing "transparent walls" (cf. [S-Ch(1987)]) any semi-dispersing billiard can be reduced to one with a finite horizon.

The statement of this conjecture will appear word by word among the conditions of the modified version of the fundamental theorem for semi-dispersing billiards stated in section 4.1. The conjecture will be proven for the utmost important special case of semi-dispersing billiards with algebraic scatterers in section 5.

To help the reader understand why this 'Lipschitz decomposability' property is defined exactly as it is, we present two more lemmas in this section. These are the lemmas through which the decomposability of singularities will be used.

Lemma 3.8 *Let $H \in \mathbb{R}^m$ be a one-codimensional locally Lipschitz graph with $H = \{(x, f(x)) \mid x \in D\}$, $D \subset \mathbb{R}^{m-1}$ open, $f : D \to \mathbb{R}$ locally Lipschitz with constant L. Assume furthermore that $\mathcal{L}^m\left((\partial H)^{[\delta]}\right) = o(\delta)$.*
Let $D' \subset D$ arbitrary, $H' = \{(x, f(x)) \mid x \in D'\}$. Then

$$\mathcal{L}^m\left(H'^{[\delta]}\right) \le 2\delta\sqrt{L^2+1}\,\mathcal{L}^{m-1}\left(\bar{D}'\right) + o(\delta).$$

Proof. Let $x_0 \in D$, $X_0 = (x_0, f(x_0)) \in H$. If $\mathrm{dist}(x_0, \partial D) > \delta$ then

$$B_\delta(X_0) \subset \left\{(x, y) \mid x \in D,\ \mathrm{dist}(x, x_0) \le \delta,\ |y - f(x)| \le \delta\sqrt{L^2+1}\right\}\ .$$

On the other hand, if $d = \mathrm{dist}(x_0, \partial D) \le \delta$ then there exists an $x_1 \in \partial D$ with $\mathrm{dist}(x_0, x_1) = d$.

With this x_1, for every $0 \le t < 1$ $x_t := tx_1 + (1-t)x_0 \in D$, otherwise $\mathrm{dist}(x_0, \partial D) < d$ would hold. The function $g : [0, 1) \to \mathbb{R}$, $g(t) := f(x_t)$ is Lipschitz with constant dL, so $g(1) := \lim_{t\nearrow 1} g(t)$ exists and $|g(1) - g(0)| \le dL$.

Obviously $X_1 := (x_1, g(1)) \in \partial H$ and $\mathrm{dist}(X_0, X_1) \le d\sqrt{L^2+1}$. That is, $B_\delta(X_0) \subset B_{(\sqrt{L^2+1}+1)\delta}(X_1)$. Putting everything together, we have

$$(H')^{[\delta]} \subset \left\{(x, y) \mid x \in (D')^{[\delta]} \cap D, |y - f(x)| \le \delta\sqrt{L^2+1}\right\} \cup (\partial H)^{[(\sqrt{L^2+1}+1)\delta]}. \tag{3.2}$$

[2] The delicate question how sensitive this notion of Lipschitz-decomposition is to the choice of the atlas needs further investigation.

This implies

$$\mathcal{L}^m\left((H')^{[\delta]}\right) \leq 2\delta\sqrt{L^2+1}\mathcal{L}^{m-1}\left((D')^{[\delta]}\right) + \mathcal{L}^m\left((\partial H)^{[(\sqrt{L^2+1}+1)\delta]}\right).$$

This gives the statement of the lemma since $\mathcal{L}^{m-1}\left((D')^{[\delta]}\right) = \mathcal{L}^{m-1}\left(\bar{D}'\right) + o(1)$ and the second term is $o(\delta)$ because of our assumption. □

In the next lemma, π will denote the projection of $\mathbb{R}^m$ to $\mathbb{R}^{m-1}$ parallel to the last axis: $\pi((x,y)) := x$ when $x \in \mathbb{R}^{m-1}$ and $y \in \mathbb{R}$.

Lemma 3.9 *Let $H \subset \mathbb{R}^m$ be a one-codimensional locally Lipschitz graph with $H = \{(x, f(x)) \mid x \in D\}$, $D \subset \mathbb{R}^{m-1}$ open, $f : D \to \mathbb{R}$ locally Lipschitz with constant L. Let $\delta > 0$ and $G \subset \mathbb{R}^m$ be such that* $\operatorname{dist}(G, \partial H) > (\sqrt{L^2+1}+1)\delta$. *Then*

$$\mathcal{L}^m\left(H^{[\delta]} \cap G\right) \leq 2\delta\sqrt{L^2+1}\mathcal{L}^{m-1}\left(\pi(G)\right).$$

Proof. Let $H' = H$. (3.2) holds just like in the previous lemma. Since $\operatorname{dist}(G, \partial H) > (\sqrt{L^2+1}+1)\delta$, this means that

$$H^{[\delta]} \cap G \subset \left\{(x,y) \mid x \in D \cap \pi(G), |y - f(x)| \leq \delta\sqrt{L^2+1}\right\},$$

which gives the statement of the lemma. □

4 The Fundamental Theorem

This section is devoted to the fundamental theorem – or local ergodicity theorem – for semi-dispersing billiards. The two-dimensional case had been settled in Sinai's celebrated work, [S(1970)]. Seventeen years had elapsed until the multidimensional generalization given by Chernov and Sinai appeared, [S-Ch(1987)]. It offered a quite involved, but in essence very transparent formulation of the theorem and a delicate proof. A self-contained exposition of the original ideas with detailed conditions and arguments were provided in [K-S-Sz(1990)] where a slightly more general, the so called "transversal" version of the fundamental theorem was announced — mainly with its application to three-billiards in mind. Several other papers have appeared in the 90s with nice expositions of the theorem, even for classes of dynamical systems more general than the original semi-dispersing billiard setting (eg. Hamiltonian systems with singularities in [L-W(1995)]). However, all of these papers assumed that for all powers of the dynamics the singularity set is a finite collection of one-codimensional smooth and compact submanifolds. Since, as our counterexample shows, this is not the case, it became utmost necessary to replace this assumption.

Throughout the section our main reference is [K-S-Sz(1990)]. Actually, our aim is to demonstrate that it is possible to modify the proof presented there to the case when the singularity sets are not smooth but just finitely Lipschitz-decomposable. After formulating the conditions and the statement of the theorem, we give a sketch of the proof (that goes along the lines of [K-S-Sz(1990)]) and work out those parts in more detail, where the original argument is to be modified. Our notations introduced mainly in section 2.1 coincide in almost all cases with those of [K-S-Sz(1990)] (we have just altered the original conventions at some places for the sake of simplicity).

One more remark: following [K-S-Sz(1990)] the formulation of the theorem presented here (the 'transversal' fundamental theorem) is slightly more general than the one usually referred to in the literature.

4.1 Formulation of the theorem

Before its formulation its is important to point out the **conditions** under which the modified proof of the theorem works. We use the notations introduced in section 2.1.

Condition 4.1 (Chernov-Sinai Ansatz, Condition 3.1 from [K-S-Sz(1990)]). *For $\nu_{\mathcal{R}^+}$-almost every point $x \in \mathcal{R}^+$ we have $x \in \partial M^*$ and, moreover, the positive semi-trajectory of the point x is sufficient.*

What follows below is our new condition – Lipschitz decomposability – on singularities. In the original proof smoothness was assumed, even though it was only formulated as a condition for the set of double singularities – see Condition 3.3 from [K-S-Sz(1990)]).

Condition 4.2 *There exists an $L \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ the singularity set $\bigcup_{|n| \le N} \mathcal{R}^n$ is 'Lipschitz decomposable' with constant L (cf. Conjecture 3.7).*

Some remarks.

- For the set of singular reflections itself the original property remains true: $\mathcal{R}$ is a finite collection of smooth compact manifolds of codimension 1.
- Condition 4.2 can only be satisfied by semi-dispersing billiards with a finite horizon. However, the infinite horizon case can easily be reduced to the finite horizon case (cf. [S-Ch(1987)], [K-S-Sz(1990)].
- As to the original exposition, one more condition was assumed on the geometry of the scatterers (the regularity of the set of degenerate tangencies - Condition 3.2 in [K-S-Sz(1990)]). However, the role of this condition was to guarantee that points belonging to two different smooth components of the singularity set belong to finitely many codimension 2 submanifolds. Now instead of smooth components we have locally Lipschitz graphs and it is enough to require that the δ-neighbourhoods of their boundaries have a volume of $o(\delta)$, which is a little less than being two-codimensional (cf. Definition 3.5).

To formulate the fundamental theorem, we introduce the notion of regular coverings. Note that $m = 2d - 2$ is the dimension of the (Poincaré) phase space ∂M. The next definition will not be absolutely precise, for we omit some technical details for the sake of easier understanding. For a precise formulation please see Definition 3.4 in [K-S-Sz(1990)]).

Definition 4.3 *Let us assume that for a point $x \in \partial M^*$ and its neighborhood $U(x)$ a smooth foliation $U(x) = \cup_{\alpha \in B^{d-1}} \Gamma_\alpha$ is given. The foliae Γ_α are $d-1$-dimensional manifolds uniformly transversal to all possible local stable manifolds (B^{d-1} is the standard $d-1$-dimensional open ball).*

The parameterized family of finite coverings

$$\mathcal{G}^\delta = \{G_i^\delta \mid i = 1, ..., I(\delta)\ \} \qquad 0 < \delta < \delta_0$$

is a family of regular coverings iff:

1. *each G_i^δ is an open parallelepiped of dimension $2d - 2$;*
2. *the $d-1$-dimensional faces of G_i^δ are all cubes with edge-length δ, moreover, they may belong to two different categories: the s-faces are 'parallel with leaves of the stable foliation' while the Γ-faces are 'parallel' with the leaves of the foliation Γ;*[3]
3. *For any point, the maximal number of parallelepipeds covering it is 2^{2d-2};*
4. *if $G_i^\delta \cap G_j^\delta \neq \emptyset$, then*
$$\mu_1(G_i^\delta \cap G_j^\delta) \geq c_1 \delta^{2d-2}$$
with c_1 independent of δ.

Some further convention: Given any G_i^δ its *s-jacket*, $\partial^s(G_i^\delta)$ is the union of those $(2d-3)$-dimensional faces of G_i^δ which contain at least one s-face of it. The Γ-jacket, $\partial^\Gamma(G_i^\delta)$ is defined similarly. Clearly, $\partial(G_i^\delta) = \partial^s(G_i^\delta) \cup \partial^\Gamma(G_i^\delta)$. We say that a stable manifold $\gamma^s(y)$ *intersects G_i^δ correctly* if:

$$\partial(G_i^\delta \cap \gamma^s(y)) \subset \partial^\Gamma(G_i^\delta).$$

Theorem 4.4 (The Fundamental Theorem) *We assume that:*

- *conditions 4.1 and 4.2 are satisfied;*
- *a sufficient phase point $x \in \partial M^*$ in the interior of the phase space is given;*
- *a smooth transversal foliation Γ in a neighborhood U_0 of x is fixed;*
- *a constant $0 < \epsilon_1 < 1$ is chosen.*

[3] More precisely if we consider the center of each parallelepiped $w_i^\delta \in G_i^\delta$, the s- and Γ- faces are parallel with the tangent planes $\mathcal{T}_{w_i^\delta}\gamma^s(w_i^\delta)$ and $\mathcal{T}_{w_i^\delta}\Gamma(w_i^\delta)$, respectively.

Then there is a sufficiently small neighborhood $U_{\epsilon_1}(x)$ such that for any $U(x) \subset U_{\epsilon_1}(x)$ and for any family of regular coverings, the covering $\mathcal{G}^\delta$ can be divided into two disjoint subsets, $\mathcal{G}_g^\delta$ and $\mathcal{G}_b^\delta$ (called 'good' and 'bad'), in such a way that:
(I) For any $G_i^\delta \in \mathcal{G}_g^\delta$ and any s-face E^s of it, the set:

$$\{y \in G_i^\delta \mid \rho(y, E^s) < \epsilon_1 \delta \text{ and } \gamma^s(y) \text{ intersects correctly}\}$$

has positive relative μ_1-measure in G_i^δ.
(II)

$$\mu_1 \left(\bigcup_{G_i^\delta \in \mathcal{G}_b^\delta} G_i^\delta \right) = o(\delta).$$

Remark:
– With suitable modifications of the proof the theorem applies to all sufficient points $x \in \partial M^*$ (see [K-S-Sz(1990)]), however, for simplicity here we restrict ourselves to regular phase points.

4.2 Proof of the Fundamental Theorem

Here we would like to give a sketch of the proof following [K-S-Sz(1990)]. For brevity we do not repeat the whole argument. Our aim is to emphasize the main ideas on the one hand and point out those parts where the original proof is to be modified on the other hand. Several arguments apply word by word, as to these, we do not give an exposition, just refer to the original paper. Those steps that need non-trivial modification are emphasized and worked out in detail.

Throughout the section we think of the sufficient point $x \in \partial M^0$ and its neighborhood U as being fixed. y usually denotes some point in U. Furthermore, a sufficiently small δ is kept fixed - thus we work with one particular covering $\mathcal{G}^\delta$. Of course, for every $G_i^\delta \in \mathcal{G}^\delta$ we have $\operatorname{diam}(G_i^\delta) \leq m\delta$ where $m = 2d - 2$ is the dimension of the phase space. As a preparation for the main argument we state two important Lemmas:

Lemma 4.5 *In correspondence with condition 4.2 let us denote the Lipschitz components of $\cup_{|n| \leq N} \mathcal{R}^n$ with $\mathcal{R}_i$ $(i = 1, ..., K)$, rest with $\mathcal{R}^*$, and the Lipschitz-constant with L. Consider the set*

$$\Delta_{\delta,N} := \{x \mid \exists i, j \leq K, \ , i \neq j, \ \rho(x, \mathcal{R}_i) \leq \delta, \ \rho(x, \mathcal{R}_j) \leq \delta\} \cup \{x \mid \rho(x, \mathcal{R}^*) \leq \delta\}.$$

For all N:

$$\mu_1(\Delta_{\delta,N}) = o(\delta).$$

This Lemma plays essentially the role of Lemma 4.6 from [K-S-Sz(1990)]. However, the proof of it is the first point where the original proof of the fundamental theorem had to be modified.

Proof. Fix an index i and find a coordinate system so that

$$\mathcal{R}_i = \{(x, f_i(x))|\ x \in D_i\}.$$

where $D_i \subset \mathbb{R}^{m-1}$. π_i shall denote the usual projection onto $\mathbb{R}^{m-1}$: $\pi_i((x,y)) := x$, when $x \in \mathbb{R}^{m-1}$ and $y \in \mathbb{R}$. Obviously $\partial D_i = \pi_i(\partial \mathcal{R}_i)$, so $\mathcal{L}^m\left((\partial \mathcal{R}_i)^{[\delta]}\right) \geq 2\delta \mathcal{L}^{m-1}(\partial D_i)$. So the condition $\mathcal{L}^m\left((\partial \mathcal{R}_i)^{[\delta]}\right) = o(\delta)$ implies $\mathcal{L}^{m-1}(\partial D_i) = 0$.

As a consequence for any $\eta > 0$ it is possible to find $\eta' > 0$ such that the (closure of the) open η'-neighbourhood of ∂D_i inside D_i has $\mathcal{L}^{m-1}$-measure smaller than η. Let us denote this open neighborhood by D^i_η and furthermore

$$\Delta^i_\eta = \left\{(x, f_i(x)) \mid x \in D^i_\eta\right\}.$$

Now consider the parts of the singularity far away from the borders of the singularities. For different i-s the sets $\mathcal{R}_i \setminus \Delta^i_\eta$ $(i = 1, ..., K)$ are pairwise disjoint compact sets (as they are continuous images of compact sets). Consequently, for δ small enough the sets $(\mathcal{R}_i \setminus \Delta^i_\eta)^{[\delta]}$ are pairwise disjoint as well. Now for the set mentioned in the Lemma, we can write:

$$\Delta_{\delta,N} \subset \Delta_\eta^{[\delta]} \cup (\mathcal{R}^*)^{[\delta]}$$

where $\Delta_\eta = \bigcup_{i=1}^{K} \Delta^i_\eta$.

Now apply Lemma 3.8 to get

$$\mathcal{L}^m((\Delta^i_\eta)^{[\delta]}) \leq 2\sqrt{L^2+1}\delta\eta + o(\delta)$$

This means that

$$\mathcal{L}^m(\Delta_\eta^{[\delta]}) \leq 2K\sqrt{L^2+1}\delta\eta + o(\delta)$$

stand for every η, meaning that $\mathcal{L}^m(\Delta_\eta^{[\delta]}) = o(\delta)$. Together with $\mathcal{L}^m((\mathcal{R}^*)^{[\delta]}) = o(\delta)$ this gives the statement of the lemma. $\square$

Before formulating the other key Lemma we would like to note that – for Lipschitz-continuous functions are differentiable almost everywhere – in almost every point of a singularity component (i.e. in one particular open locally Lipschitz graph $\hat{\mathcal{R}}$) it makes sense to talk about their (one-codimensional) tangent planes $\mathcal{T}_{\hat{y}}\hat{\mathcal{R}}$. Knowing the behaviour of the tangent plane wherever it exists allows us to think about the "direction" of the whole $\hat{\mathcal{R}}$.

Lemma 4.6 *Given any $x \in \partial M^0$ and any $\epsilon > 0$ there is a neighborhood $U(x) \subset \partial M$ of x such that for every γ^s_1, γ^s_2 and any $(2d-3)$-dimensional Lipschitz component $\hat{\mathcal{R}}$ of some $\mathcal{R}^n$ $(n > 0)$ intersecting $U(x)$ with points y_1, y_2 and $\hat{y}$, lying on the three manifolds, respectively, so that $\mathcal{T}_{\hat{y}}\hat{\mathcal{R}}$ exists:*

$$\begin{aligned} \sphericalangle(\mathcal{T}_{y_1}\gamma^s_1, \mathcal{T}_{y_2}\gamma^s_2) &< \epsilon, \\ \sphericalangle(\mathcal{T}_{y_1}\gamma^s_1, \mathcal{T}_{\hat{y}}\hat{\mathcal{R}}) &< \epsilon. \end{aligned} \tag{4.1}$$

This Lemma is on the parallelization effect and it is exactly the same as Lemma 4.9 in [K-S-Sz(1990)]– the original argument applies. Nevertheless it might be useful to point out what the second inequality in (4.1) means: there is a $(d-1)$-dimensional subspace of the tangent space at almost any point of the $(2d-3)$-dimensional manifold $\hat{\mathcal{R}}$ very close to the stable subspace. Note, however, that $\hat{\mathcal{R}}$ may behave extremely widely – i.e. in a non-smooth manner – in the remaining $(d-2)$ dimensions (in case $d \geq 3$).

Before the proof we should introduce some more notation. The following two quantities measure the hyperbolicity near the point $y \in \partial M^0$. Let

$$\kappa_{n,0}(y) = \inf_{\Sigma} \|(D^n_{-T^n y,\Sigma})^{-1}\|_p^{-1},$$

where the inf is taken over all convex local orthogonal manifolds passing through $-T^n y$. Furthermore denote

$$\kappa_{n,\delta}(y) = \inf_{\Sigma} \inf_{w\in\Sigma} \|(D^n_{w,\Sigma})^{-1}\|_p^{-1}.$$

Here the infimum is taken for the set of convex fronts Σ passing through $-T^n y$ such that (i) T^n is continuous on Σ and (ii) $T^n\Sigma \subset B_\delta(-y)$.

Remark 4.7 (cf. Lemma 5.3 in [K-S-Sz(1990)]and Lemma 2.2 in the present paper) *It is not difficult to see that $\kappa_{n,\delta}(y)$ is an increasing function of n. Furthermore, for sufficient points y clearly:*

$$\lim_{n\to\infty} \kappa_{n,0}(y) = \infty.$$

(Here we do not state in general that $\kappa_{n,0}$ grows exponentially, linear growth – which is obvious for sufficient points y – is enough.)

The following subsets of the neighborhood $U \ni x$ depend on the constant δ.

$$\begin{aligned} U^g &:= \{y \in U \mid \forall n \in \mathbb{Z}_+,\ z_{tub}(T^n y) \geq (\kappa_{n,c_3\delta}(y))^{-1} c_3\delta\}; \\ U^b &:= U \setminus U^g; \\ U^b_n &:= \{y \in U \mid z_{tub}(T^n y) < (\kappa_{n,c_3\delta}(y))^{-1} c_3\delta\} \end{aligned} \tag{4.2}$$

Remark 4.8 *Note that for the points $y \in U^g$ the stable manifold extends to the boundary of $B_{c_3\delta}(y)$, the ball of radius $c_3\delta$ around y (cf. Lemma 5.4 from [K-S-Sz(1990)]). The constant c_3 will be chosen in an appropriate way to guarantee that for any $y \in U^g \cap G_i^\delta$ the stable leaf $\gamma^s(y)$ intersects G_i^δ correctly unless it intersects $\partial^s G_i^\delta$.*

Furthermore, we introduce the class of permitted functions.

Definition 4.9 *A function $F : \mathbb{R}_+ \to \mathbb{Z}_+$ defined in a neighborhood of the origin is permitted whenever $F(\delta) \nearrow \infty$ as $\delta \searrow 0$. For a fixed permitted function $F(\delta)$ we define $U^b_\omega = \cup_{n>F(\delta)} U^b_n$.*

Most of the statements to come hold for any permitted function $F(\delta)$. At one point of the argument we shall fix one particular $F(\delta)$.

Lemma 4.10 (**Tail bound**; Lemma 6.1 from [K-S-Sz(1990)]). *For any permitted function:*

$$\mu_1(U^b_\omega) = o(\delta).$$

The measure estimates in the proof of the Tail Bound are related to $\mathcal{R}$, the set of singular reflections. As already mentioned, this set (in contrast to the higher iterates $\mathcal{R}^n$) is a finite collection of smooth and compact 1-codimensional submanifolds of the phase space. Consequently, there is no need for Lipschitz decomposition here, thus we do not include the proof. Essentially, the original argument from [K-S-Sz(1990)]applies, nevertheless, at the definition of the small set of non-sufficient points a little more care is needed. We would also like to emphasize that the proof of the Tail Bound is the point where the Chernov-Sinai Ansatz (Condition 4.1) is exploited. On more details see [K-S-Sz(1990)].

Remark 4.11 *In what follows we will work with distances defined by the Euclidean metric ρ. However, as the interior point x in ∂M is fixed and its neighborhood $U(x)$ is fixed we have $|\langle v, n(q)\rangle| \geq c$ for some positive constant $c = c(x)$ in this neighborhood. Thus the two distances ρ and ρ_i are equivalent (cf. section 2).*

Now we can start proving the fundamental theorem by telling explicitly how the collection of parallelepipeds $\mathcal{G}^\delta$ is divided into a good and a bad part. We say $G^\delta_i \in \mathcal{G}^\delta_b$ iff

(A) either

- it intersects more than one Lipschitz component of the singularities of $T^{F(\delta)}$,
- or it intersects only one component $\hat{\mathcal{R}}$, but $\rho(G^\delta_i, \partial\hat{\mathcal{R}}) \leq \delta$,
- or it intersects the remaining small set $\mathcal{R}^*$.

(B) or it is not of type (A), but it has an s-face E^s such that

$$\mu_1(G^\delta_i \cap (E^s)^{[\epsilon_1\delta]} \cap U_{ic}) \leq \frac{\epsilon_3}{4}\mu_1(G^\delta_i), \tag{4.3}$$

where ϵ_3 is a positive constant to be defined later and U_{ic} is the set of points in G^δ_i with correctly intersecting local stable manifolds.

Now we choose one particular permitted function $F(\delta)$: by virtue of Lemma 4.5 there definitely exists a permitted function such that:

$$\mu_1(\Delta_{(m+1)\delta,F(\delta)}) = o(\delta). \tag{4.4}$$

As a consequence the overall measure of bad parallelepipeds of type (A) is $o(\delta)$ (such parallelepipeds lie inside the set $\Delta_{(m+1)\delta,F(\delta)}$).

It is time to tell about the choice of our small constants ϵ_i as well. In the formulation of the Fundamental Theorem one particular constant ϵ_1 is given. We shall choose three further constants in the following order: $\epsilon_1 \to \epsilon_3 \to \epsilon_4 \to \epsilon_2$. It is utmost important that all of these choices are independent of δ. (they are chosen in the arguments below, ϵ_3 in 1., ϵ_4 in 2. and ϵ_2 in 3.). After all these choices are made we fix the neighborhood $U_{\epsilon_1}(x)$ (see the formulation of the Fundamental Theorem) in such a way that for all Lipschitz components $\hat{\mathcal{R}}$ of some $\mathcal{R}^n$ $(n > 0)$ that intersect $U_{\epsilon_1}(x)$:

$$\begin{aligned} \sphericalangle(\gamma_1^s, \gamma_2^s) &< \epsilon_2, \\ \sphericalangle(\gamma_1^s, \hat{\mathcal{R}}) &< \epsilon_2. \end{aligned} \tag{4.5}$$

Such a choice is clearly possible by virtue of Lemma 4.6. Here the second inequality is understood at every point of $\hat{\mathcal{R}}$ where it makes sense, that is, where $\hat{\mathcal{R}}$ is differentiable.

One more remark: having fixed the neighborhood $U_{\epsilon_1}(x)$ and the foliation Γ uniformly transversal to the stable foliation, it is possible to uniformly compare two different measures for each product-type set inside $U_{\epsilon_1}(x)$. More precisely there is a constant $c_4 > 0$ such that given any product-type set, the ratio of its μ_1-measure and its measure that arises as a product of measures in the $s-$ and $\Gamma-$directions lies between c_4^{-1} and c_4.

From now on G_i^δ will always denote a bad parallelepiped of type (B). The proof of the Fundamental Theorem follows from the small arguments to come.

1. Let us first give an estimate from below on the measure of $G_i^\delta \cap (E^s)^{[\epsilon_1\delta]}$ where E^s is an s-face for the bad parallelepiped G_i^δ. By the above remark on product measures:

$$\mu_1(G_i^\delta \cap (E^s)^{[\epsilon_1\delta]}) \geq c_4^{-1}(\epsilon_1\delta)^{d-1}\delta^{d-1} \geq c_6\epsilon_1^{d-1}\mu_1(G_i^\delta) \geq \epsilon_3\mu_1(G_i^\delta), \tag{4.6}$$

   in case $\epsilon_3(\epsilon_1)$ is chosen sufficiently small.

2. For estimates from above we fix the constant $\epsilon_4 = \epsilon_4(\epsilon_3)$ sufficiently small. The measure of points near the s-jacket (which consists of $2(d-1)$ faces of dimension $2d-3$), is:

$$\mu_1(G_i^\delta \cap (\partial^s G_i^\delta)^{[\epsilon_4\delta]}) \leq 2(d-1)c_4\epsilon_4\delta\delta^{2d-3} \leq \frac{\epsilon_3}{4}\mu_1(G_i^\delta). \tag{4.7}$$

   We need one more estimate of similar type. *This is the second point where the original proof has to be modified, and the smoothness/Lipschitzness of singularity components is used.* Recall that for a bad parallelepiped of type (B) there is at most one Lipschitz component $\hat{\mathcal{R}}$ of the singularity set for $T^{F(\delta)}$ intersecting it. We are interested in estimating the measure of the $\epsilon_4\delta$-neighborhood of this Lipschitz graph inside the parallelepiped. If $\epsilon_4 <$

$\frac{1}{\sqrt{L^2+1}+1}$ (L is the Lipschitz-constant) then, by the construction of type (A) parallelepipeds, Lemma 3.9 can be applied, and gives

$$\mu_1(G_i^\delta \cap (\hat{\mathcal{R}})^{[\epsilon_4\delta]}) \le c_5\epsilon_4\delta\delta^{2d-3} \le \frac{\epsilon_3}{4}\mu_1(G_i^\delta). \tag{4.8}$$

whenever again $\epsilon_4(\epsilon_3)$ is small enough.

3. Now we choose $\epsilon_2(\epsilon_4)$ small enough, so that by (4.5) stable manifolds and singularity components are 'almost parallel'. Namely, the smallness of ϵ_2 should guarantee that for any $y \in G_i^\delta$ for which $\gamma^s(y)$ does not intersect correctly we have:

$$y \in (G_i^\delta \cap (\hat{\mathcal{R}})^{[\epsilon_4\delta]}) \cup (G_i^\delta \cap (\partial^s G_i^\delta)^{[\epsilon_4\delta]}) \cup U_\omega^b. \tag{4.9}$$

To see that, given a suitable choice of ϵ_2, the above formula is valid we make two remarks.

- First we note that for stable manifolds and singularity components 'not to approach each other too quickly', being 'almost parallel' is enough at almost every point of $\hat{\mathcal{R}}$.
- Recalling the definitions from (4.2) and the various notions of distances from section 2 what we see immediately is that the inclusion of (4.9) is valid with writing $G_i^\delta \cap (\cup_{n\le F(\delta)} U_n^b)$ instead of $G_i^\delta \cap \hat{\mathcal{R}}^{[\epsilon_4\delta]}$ for the first set. Nevertheless, with a suitable choice of ϵ_2 we certainly have $G_i^\delta \cap (\hat{\mathcal{R}}^{[\epsilon_4\delta]}) \subset G_i^\delta \cap (\cup_{n\le F(\delta)} U_n^b)$ as (i) $z_{tub}(x) \ge z(x)$ and (ii) the Euclidean distance ρ and the distance ρ_i (in terms of which $z(x)$ is defined) are equivalent, see Remark 4.11.

We only need some minor considerations to complete the proof. Observe first that for good parallelepipeds the statement (I) evidently holds. As for (II) we have already shown that bad parallelepipeds of type (A) are of measure $o(\delta)$ (recall (4.4)), we shall show the same for those of type (B) as well. Indeed, let us consider a G_i^δ with an s-face E^s for which (4.3) holds. By the arguments 1.-3. above:

$$\mu_1(G_i^\delta \cap U_\omega^b) \ge \mu_1(G_i^\delta \cap (E^s)^{[\epsilon_1\delta]} \cap U_\omega^b) \ge \frac{\epsilon_3}{4}\mu_1(G_i^\delta).$$

Now recall that in a regular covering there are at most 2^{2d-2} parallelepipeds with a non-empty common intersection. Thus:

$$2^{2d-2}\mu_1(U_\omega^b) \ge \sum{}' \mu_1(G_i^\delta \cap U_\omega^b) \ge \frac{\epsilon_3}{4} \sum{}' \mu_1(G_i^\delta),$$

where $\sum'$ denotes the sum over bad parallelepipeds of type (B). By the Tail Bound (Lemma 4.10) we have $\sum' \mu_1(G_i^\delta) = o(\delta)$ thus the proof of Theorem 4.4 is complete.

5 The case of algebraic scatterers

The main aim of this section is to show that the singularity submanifolds of algebraic semi-dispersing billiards satisfy the Lipschitz-decomposability property formulated in Conjecture 3.7. Fortunately, the most important examples of semi-dispersing billiards are algebraic as it has been noted in the introduction. Consequently, the algebraicity condition does not essentially restrict the applicability of the fundamental theorem.

For definiteness we will say that the zero-set of a system of polynomial equations is an *algebraic variety* (we will use these notions over the real ground field). Any (measurable) subset of a k-dimensional algebraic variety will be denoted as a k-dimensional SSAV (for 'subset of an algebraic variety'). As for the dimension of an algebraic variety, see [Sh(1974)]. We also use the following definition.

Definition 5.1 *A semi-dispersing billiard is algebraic if it has finitely many scatterers and the boundary of each of these scatterers is a finite union of one-codimensional SSAV-s (as subsets of $\mathbb{T}^d \subset \mathbb{R}^d$).*

Remark 5.2 *Assume, in general, that we are given a Riemannian manifold $\mathcal{M} = \mathcal{M}_m$ and a subset $A \subset \mathcal{M}$. We say that A is a k-dimensional weakly algebraic subset of $\mathcal{M}$ if it is possible to find an appropriate atlas $\{U_t, \psi_t\}_{t=1}^T$ on $\mathcal{M}$ such that, for every t, $\psi_t(U_t \cap A) \quad (\subset \mathbb{R}^m)$ is a k-dimensional SSAV in $\mathbb{R}^m$. Bi-Lipschitzness of the atlas $\{U_t, \psi_t\}_{t=1}^T$ can always be assumed (cf. Remark 3.6)*

Note that being 'weakly algebraic' is really a weak notion due to the high degree of freedom in the choice of the atlas. For example, every smooth curve is 1-dim. weakly algebraic.

What follows below in three subsections is a proof of Lipschitz decomposability for the singularities $\mathcal{R}^{-n}$ in an algebraic billiard. In subsection 5.1 it is shown that singularities are algebraic as subsets of $\mathbb{R}^{2d}$. This implies that $\mathcal{R}^{-n} \subset \partial M$ is algebraic in the sense of Remark 5.2 as well.[4] The proof is completed in subsections 5.2 and 5.3 where a Lipschitz decomposition is constructed for any (one-codimensional) SSAV of $\mathbb{R}^m$.

5.1 The algebraicity of $\mathcal{R}^{-n}$

Our approach generalizes that of section 3 of [S-Sz(1999)]. Since there a detailed exposition was given, here we are satisfied by referring to the main steps of the complexification of the dynamics. Still we completely explain those parts where our arguments are different.

In a nutshell the picture is the following. In [S-Sz(1999)]

- the authors were only considering quadratic boundaries since hard ball systems are quadratic billiards;

[4] In a small neighbourhood of $y \in \partial M$ identify the tangent plane $\mathcal{T}_y\partial M$ with $\mathbb{R}^m$ and restrict the orthogonal projection $\Pi : \mathbb{R}^{2d} \to \mathcal{T}_y\partial M$ onto ∂M to obtain coordinate charts.

- and for the quadratic case they elaborated a most detailed algebraic analysis of the situation.

Here we do not need such a delicate picture. But on the other hand, we are treating the general algebraic case. The chain of field extensions of [S-Sz(1999)] relied upon the explicit solvability of the arising quadratic equations and applied the related elimination of the square roots. In the general case we rather apply the norm used in Galois theory.

We first fix some notation – slightly different from the usual conventions – at this point. According to the definition above, $\partial Q = \cup_{j=1}^{J} \partial Q_j$, where both the components ∂Q_j and their boundaries are all appropriate dimensional SSAV-s (the decomposition is finer than the one into connected components in $\mathbb{R}^d$). In other words, for each ∂Q_j there is a (non-zero) irreducible polynomial $B_j(q)$ such that

$$\partial Q_j \subset \{q \in \mathbb{R}^d \mid B_j(q) = 0\}.$$

Note that symbolic collision sequences (5.1) are defined in terms of these algebraic boundary components as well.

From this point on it will be suitable to consider orbit segments $S^{[0,T]}x_0, T > 0$ of the billiard flow with T sufficiently large. In fact, it will be useful to also drop the condition $||v|| = 1$. Consequently, the dimension of our phase space will be $2d$ (first the phase space will be $\mathbb{T}^d \times \mathbb{R}^d$ and later just $\mathbb{R}^{2d}$).

The symbolic collision sequence of $S^{[0,T]}x_0$ will be denoted by

$$\sigma = \Sigma(S^{[0,T]}x_0) = (\sigma_1, \sigma_2, \dots, \sigma_n) \qquad (n \geq 0) \tag{5.1}$$

Remark 5.3 *By definition, (q_0, v_0) corresponds to the initial, generally noncollision phase point x_0 of the flow. Furthermore $T^k x_0 = x_k = (q_k, v_k) \in \partial Q_{\sigma_k}$ for every $1 \leq k \leq n$ (we note that for a phase point $x_0 \notin \partial M$ of the flow $Tx_0 \in \partial M$ coincides by definition with the first point where the positive semi-orbit of x_0 reaches the boundary ∂M; in [K-S-Sz(1990)] this map was denoted by T^+). By a slight abuse of notation we will keep denoting by $\mathcal{R}^{-n}$ (introduced in subsection 2.1) the nth inverse image of $\mathcal{R}$ in a $2d$-dimensional neighbourhood of x_0 .*

Having fixed σ, we first explore the algebraic relationship between the consecutive x_ks. For being able to carry out arithmetic operations on our data, we lift the genuine orbit segment to the covering Euclidean space of the torus. This can be done by a straightforward generalization of the trivial Proposition 3.1 of [S-Sz(1999)].

Proposition 5.4 *Let $S^{[0,T]}x_0$ be an orbit segment of the discretized dynamics. Assume that a certain pre-image (Euclidean lifting) $\tilde{q}_0 \in \mathbb{R}^d$ of the position $q_0 \in \mathbb{T}^d$ is given. Then there is a uniquely defined Euclidean lifting $\{\tilde{q}_i \in \mathbb{R}^d | 0 \leq k \leq n\}$ of the given orbit segment which, when considered in continuous time, is a time-continuous extension of the original lifting $\tilde{q}_0$. Moreover, for every collision σ_k*

there exists a uniquely defined integer vector $a_k \in \mathbb{Z}^d$ *— named* the adjustment vector *of* σ_k *— such that*

$$B_{\sigma_k}(\tilde{q}_k - a_k) = 0 \qquad (1 \le k \le n).$$

The orbit segment $\tilde{\omega} = \{\tilde{q}_k | 0 \le k \le n\}$ *is called the* lifted orbit segment *with the system of adjustment vectors* $\mathcal{A} = (a_0, \dots, a_n) \in \mathbb{Z}^{(n+1)d}$.

In the sequel, $<,>$ denotes Euclidean inner product of d-dimensional real vectors. Our next proposition is also a straightforward extension of Proposition 3.3 of [S-Sz(1999)].

Proposition 5.5 *Between the kinetic data corresponding to* σ_{k-1} *and* σ_k *one has the following algebraic relations:*

— the linear collision equation

$$v_k = v_{k-1} - 2 < v_{k-1}, n_k > n_k \qquad (1 \le k \le n) \tag{5.2}$$

where n_k *is the outer unit normal vector of the scatterer* Q_{σ_k} *at the point of impact;*

— the linear free flight equation

$$\tilde{q}_k = \tilde{q}_{k-1} + \tau_k v_{k-1} \qquad (1 \le k \le n) \tag{5.3}$$

— where the time slot $\tau_k = t_k - t_{k-1}$ $(t_0 = 0)$ *in (5.3) is determined by the polynomial equation*

$$B_{\sigma_k}(\tilde{q}_{k-1} + \tau_k v_{k-1} - a_k) = 0. \tag{5.4}$$

Next we turn to the complexification of the billiard ball map T. Given the pair $(\Sigma, \mathcal{A}) = (\sigma_1, \sigma_2, \dots, \sigma_n; a_0, a_1, \dots, a_n)$, the equations (5.2), (5.3), (5.4) make it possible to algebraically characterize the kinetic data $(\tilde{q}_k, v_k)$ by using the preceding data $(\tilde{q}_{k-1}, v_{k-1})$. Since — at the moment — we are dealing with genuine, real orbit segments, in this situation the equations have at least one positive, real root τ_k; in case of several such roots its selection is unique by the geometry of the problem. Our further arguments, however, also use the algebraic closedness of the arising fields and therefore we complexify the dynamics. From this point on, our approach, though related but nevertheless will already be different from that of [S-Sz(1999)].

Definition 5.6 *For* $n = 0$ *the field* $\mathbb{K}_0 = \mathbb{K}(\emptyset; \emptyset)$ *is the transcendental extension* $\mathbb{C}(\mathcal{B})$ *of the coefficient field* $\mathbb{C}$ *by the algebraically independent formal variables*

$$\mathcal{B} = \{(\tilde{q}_0)_j, (v_0)_j | 1 \le j \le d\}$$

Suppose now that the commutative field $\mathbb{K}_{n-1} = \mathbb{K}(\Sigma'; \mathcal{A}')$ *has already been defined, where* $\Sigma' = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}); \mathcal{A}' = (a_0, a_1, \dots, a_{n-1})$. *Consider now the polynomial equation*

$$b_l \tau^l + b_{l-1} \tau^{l-1} + \dots + b_0 = 0 \tag{5.5}$$

arising from (5.4) with k=n. It defines a new field element τ_n to be adjoined to the field $\mathbb{K}_{n-1}$ (of course, $b_0, \ldots, b_l \in \mathbb{K}_{n-1}$). At this point, however, we should be a bit cautious. If the equation 5.5 is irreducible, then all its roots are algebraically equivalent, and τ_n can denote any of them. If (5.4) is reducible, then we should select a particular irreducible factor of its. Indeed, since we are only interested in the images of $\mathcal{R}$, at each step we choose such an irreducible factor of (5.5) which, when its root τ_n gets evaluated for real values of x_0, gives us a real root of (5.5) which is actually the real root specified after (5.4). This irreducible factor defines the extension $\mathbb{K}_n = \mathbb{K}_{n-1}(\tau_n)$.

In such a way we are given a chain of extensions $\mathbb{K}_0, \mathbb{K}_1, \ldots, \mathbb{K}_n$ where for every $k = 1, \ldots, n$ the relation $\mathbb{K}_k = \mathbb{K}_{k-1}(\tau_k)$ holds. By our construction and by the theorem on the prime element of algebra, $\mathbb{K}_n$ can also be expressed as $\mathbb{K}_0(\tilde{\tau}_n)$ for some $\tilde{\tau}_n \in \mathbb{K}_n$ with minimal polynomial $m(\alpha)$ over $\mathbb{K}_0$.

By applying the previous construction we are going to look for an algebraic characterization of $\mathcal{R}^{-n}$.

For every $x_0 \in M_{\Sigma,\mathcal{A}} = \{x \in M \mid \Sigma(S^{[0,T]}x) = \Sigma,\ \mathcal{A}(S^{[0,T]}x) = \mathcal{A}\}$ one has $\tilde{q}_n \in \mathbb{K}_n$. $\tilde{q}_n$ can formally be understood as a function $\tilde{q}_n(x_0, \tau_1, \ldots, \tau_n)$ or (by the theorem on the prime element) simply as a function $\tilde{q}_n(x_0, \tilde{\tau}_n)$ with values in $\mathbb{K}_n$. We will be considering this function exactly in $M_{\Sigma,\mathcal{A}}$, that is, where Σ and $\mathcal{A}$ are constants. Consider Q_{σ_n} at the point $T^n x_0$. At this point the submanifolds $B_{\sigma_n}(\tilde{q}_n - a_n) = 0$ has a normal vector $\vec{n}$ which can be expressed by the partial derivatives of B_{σ_n} at $\tilde{q}_n - a_n$. The condition $\tilde{x}_n \in \mathcal{R}$ just says that $< \vec{n}, v_n >= 0$. Here both $\vec{n}$ and v_n are elements of $\mathbb{K}_n$, i. e. formal functions of x_0 and $\tilde{\tau}_n$. Consequently $< \vec{n}, v_n >= \Phi(\tilde{\tau}_n)$ where Φ is a polynomial whose coefficients are rational functions over $\mathbb{K}_0$. Take now the (Galois-) norm (cf. [St(1973)]) of this element i. e.

$$\|\Phi\| = \Pi\Phi(\tilde{\tau}_n^i)$$

where the product is taken for all roots $\tilde{\tau}_n^i$ of the irreducible polynomial m. This norm does not vanish since it is the product of non-zero elements in the normal hull of $\mathbb{K}_n$. Moreover, it is a symmetric polynomial of the elements $\tilde{\tau}_n^i$. As such it can be expressed as a polynomial of the elementary symmetric polynomials of the variables $\tilde{\tau}_n^i : 1 \leq i \leq l$. These elementary symmetric polynomials can, however, be easily expressed by the coefficients of m which are elements of $\mathbb{K}_0$. As a consequence, we obtain a non-zero element of $\mathbb{K}_0$. The construction just described generalizes the elimination of the square roots method applied in [S-Sz(1999)]. By our construction it remains also true that this polynomial has real coefficients for our real, dynamical orbit. All in all for every fixed Σ and $\mathcal{A}$ the resulting piece of $\mathcal{R}^{-n}$ is an algebraic submanifold. From the finiteness of the horizon it is clear that in the case of our real dynamics only a finite number of Σ and $\mathcal{A}$ provide a non-empty piece of $\mathcal{R}^{-n}$.

In this way we have established

Theorem 5.7 *$\mathcal{R}^{-n}$ is a finite union of one-codimensional SSAV-s in $\mathbb{R}^{2d}$.*

5.2 Dimension and Measure of Algebraic Varieties

The motivation for this section is that we need to estimate the Lebesgue-measure (denoted here by $\mathcal{L}^m$) of the δ-neighbourhood of an algebraic variety . Actually we only need that $\mathcal{L}^m(H^{[\delta]}) = o(\delta)$ if H is (at least) two-codimensional, but our results will be more general than that.

As we will see, this problem is closely related to the box dimension and the so-called Minkowski-content of H (on box dimension, Minkowski-content and their relation to Hausdorff dimension and measure, see section 3.1 in [Fa(1990)]). To start, let us recall some notions and basic facts related to box dimension.

Definition 5.8 *Let H be a bounded subset of $\mathbb{R}^m$, $0 \le d \in \mathbb{R}$. Then the quantities*

$$\overline{\mathcal{M}}^d(H) := \limsup_{\delta \to 0} \frac{\mathcal{L}^m(H^{[\delta]})}{\delta^{m-d}},$$

$$\underline{\mathcal{M}}^d(H) := \liminf_{\delta \to 0} \frac{\mathcal{L}^m(H^{[\delta]})}{\delta^{m-d}}$$

are called the upper and lower d-dimensional Minkowski-content of H.

Definition 5.9 *Let H be again a bounded subset of $\mathbb{R}^m$, $\epsilon > 0$. The set $I \subset H$ is called an ε-net in H if $H \subset I^{[\varepsilon]}$.*

We will always be interested in finite ε-nets I, and we will never use that $I \subset H$.

Some simple facts:

(a) $\dim_H H \le \underline{\dim}_B H \le \overline{\dim}_B H$

(b) $\mathcal{H}^d(H) \le \underline{\mathcal{M}}^d(H) \le \overline{\mathcal{M}}^d(H)$

(c) If $\overline{\mathcal{M}}^d(H) < \infty$, then $\overline{\dim}_B H \le d$

(d) If $\dim_H H < d$ then $\mathcal{H}^d(H) = 0$

(e) If $\overline{\mathcal{M}}^d(H) < \infty$ then $\mathcal{L}^m(H^{[\delta]}) = O(\delta^{m-d})$.

(f) If I is an ε-net (finite) in $H \in \mathbb{R}^m$ then $\mathcal{L}^m(H^{[\varepsilon]}) \le (2\varepsilon)^m |I|$, where $|I|$ is the cardinality of I.

Now we turn to the investigation of algebraic varieties. Our proposition will be an easy corollary of the following lemma. For $0 \le d \in \mathbb{R}$ we will denote the d-dimensional Hausdorff-measure by $\mathcal{H}^d$.

Lemma 5.10 *Let $H = \hat{H} \cap [0,1]^m$, where $\hat{H}$ is an algebraic variety. Let k be the maximum of the degrees of the polynomials defining $\hat{H}$. Let $\varepsilon > 0$, $0 \le d \in \mathbb{Z}$. Let*

$c > 1$ arbitrary. We claim that if $\mathcal{H}^{d+1}(H) = 0$ then, if ε is small enough, there exists a $(d \cdot \varepsilon)$-net I in H with

$$|I| \le N_{m,d,k,\varepsilon} := \sum_{i=0}^{d} c^i \left(\frac{m!}{(m-i)!} \right)^{3/2} k^{m-i} \frac{1}{\varepsilon^i}.$$

Proof. The proof goes by induction on d, and the induction is based on the following

Fact. For every $x \in [0,1]$ let $H_x = H \cap (\{x\} \times [0,1]^{m-1})$. Then $\mathcal{H}^{d+1}(H) = 0$ implies that for Lebesgue almost every $x \in [0,1]$, $\mathcal{H}^d(H_x) = 0$. This is an easy consequence of Theorem 5.8 in [Fa(1985)].

The same is true for subsets of H arising by fixing another (than the first) coordinate: for every $1 \le l \le m$ if $P_x^l := [0,1]^{l-1} \times \{x\} \times [0,1]^{m-l}$ and $H_x^l := P_x^l \cap H$ then we have $\mathcal{H}^d(H_x^l) = 0$ for $\mathcal{L}^1$-a.e. x. We will take advantage of this by choosing ε' arbitrary (later on we will fix $\varepsilon' = \frac{\varepsilon}{\sqrt{m}}$) and fixing $K \le \frac{c}{\varepsilon'}$ points: $0 = x_{l,1} < \ldots < x_{l,K} = 1$, such that $x_{l,j+1} - x_{l,j} \le \varepsilon'$ and $\mathcal{H}^d(H^l_{x_{l,j}}) = 0$ for every j. The $m \cdot K$ hyperplanes $P^l_{x_{l,j}} :\ l = 1, \ldots, m,\ j = 1, \ldots, K$ cut H into blocks of diameter $\le \varepsilon' \sqrt{m}$.

Notice that if H has a point A in any of these blocks, then either it also has one (B) on the surface of the block, so that $\mathrm{dist}(A, B) \le \varepsilon' \sqrt{m}$, or the entire component of H containing A is inside the block.

1.) We start the induction with $d = 0$. The previous construction gives (for any ε') $\mathcal{H}^0(H^l_{x_{l,j}}) = 0$, that is, $H^l_{x_{l,j}} = \emptyset$ for every l, j. That is, the components are points, and we can certainly find the $0 \cdot \varepsilon = 0$-net $I = H$ with $|I| \le k^m = N_{m,0,k,\varepsilon}$, an upper bound for the number of components coming from Bezout's theorem.

2.) Suppose we have the statement for some $d - 1 \ge 0$.

3.) We prove for d. That is, $H \subset [0,1]^m$, $\mathcal{H}^{d+1}(H) = 0$. Apply the previous construction with $\varepsilon' = \frac{\varepsilon}{\sqrt{m}}$. The set $H^l_{x_{l,j}}$ is now an algebraic variety in $[0,1]^{m-1}$, the polynomials defining it can be derived from those defining H by fixing a variable. So the degrees can not grow. We can use the inductive assumption for the $mK \le mc\frac{\sqrt{m}}{\varepsilon}$ sets $H^l_{x_{l,j}}$ with $m \to m-1$ and the same k. Thus taking a $(d-1)\varepsilon$-net on every $H^l_{x_{l,j}}$ according to the inductive assumption, and choosing a point from every component that happens to be entirely inside a block, we get a $d \cdot \varepsilon$-net I in H with $|I| \le mc\frac{\sqrt{m}}{\varepsilon} N_{m-1,d-1,k,\varepsilon} + k^m = N_{m,d,k,\varepsilon}$ □

This lemma leads to the following

Proposition 5.11 *If $H = \hat{H} \cap [0,1]^m$ where $\hat{H} \subset \mathbb{R}^m$ is an algebraic variety, and k is the maximum of the degrees of the polynomials defining $\hat{H}$, then $s := \dim_H(H) = \dim_B(H) \in \mathbb{Z}$ and*

$$0 < \mathcal{H}^s(H) \le \underline{\mathcal{M}}^s(H) \le \overline{\mathcal{M}}^s(H) \le 2^m s^s \left(\frac{m!}{(m-s)!} \right)^{3/2} k^{m-s} < \infty. \tag{5.6}$$

Proof. By (b) we have $\mathcal{H}^s(H) \le \underline{\mathcal{M}}^s(H) \le \overline{\mathcal{M}}^s(H)$. On the other hand, if we choose $d \in \mathbb{Z}$ in such a way that $\mathcal{H}^{d+1}(H) = 0$ then by the Lemma for any $c > 1$:

$$\overline{\mathcal{M}}^d(H) \overset{\text{def}}{=} \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^m(H^{[d\varepsilon]})}{(d\varepsilon)^{m-d}} \overset{(f)}{\le} \limsup_{\varepsilon\to 0} \frac{(2d\varepsilon)^m N_{m,d,k,\varepsilon}}{(d\varepsilon)^{m-d}} \overset{\text{Lemma}}{=} \tag{5.7}$$

$$= \limsup_{\varepsilon\to 0} 2^m d^d c^d \sum_{i=0}^{d} \left(\frac{m!}{(m-i)!}\right)^{3/2} k^{m-i}\varepsilon^{d-i} =$$

$$= c^d 2^m d^d \left(\frac{m!}{(m-d)!}\right)^{3/2} k^{m-d} < \infty.$$

By (c) and (d) this implies $\overline{\dim}_B(H) \le d$ if $s < d+1$ (or even if $\mathcal{H}^{d+1}(H) = 0$). This contradicts (a) unless $\overline{\dim}_B(H) = s \in \mathbb{Z}$ (or even if $\mathcal{H}^s(H) = 0$). Now with $d = s$ (5.7) implies the right end of (5.6). □

Corollary 5.12 *If H is a bounded subset of an (at least) two (algebraic) codimensional algebraic variety in $\mathbb{R}^m$, then $\mathcal{L}^m(H^{[\delta]}) = o(\delta)$.*

Proof. Knowing from [F(1969)] that the algebraic and Hausdorff dimensions coincide, the proposition actually gives $\overline{\mathcal{M}}^{m-2}(H) < \infty$ which means (by (e)) that $\mathcal{L}^m(H^{[\delta]}) = O(\delta^2)$. □

5.3 Lipschitz decomposability of algebraic varieties

In this subsection our aim is to establish the fact that one-codimensional SSAV-s possess the finite Lipschitz decomposability property (in the sense of Definition 3.5). Having already shown the algebraic nature of $\mathcal{R}_{-n}$, this way we find that algebraic billiards satisfy Conjecture 3.7. The main result of the subsection is:

Theorem 5.13 *Any one-codimensional algebraic variety H is Lipschitz decomposable (in the sense of Definition 3.5) with any constant $L > 0$.*

In the following, π shall denote the standard projection of $\mathbb{R}^m$ to $\mathbb{R}^{m-1}$. That is, $\pi(x, y) = x$ for any $x \in \mathbb{R}^{m-1}$, $y \in \mathbb{R}$.

Proof. We construct the decomposition of H. Fix an arbitrary $L > 0$. Let I(H) denote the ring of polynomials vanishing on H. Let H^* be the set of points in H where the gradient of every polynomial in $I(H)$ vanishes. We know from [B-C-R(1987)] that this set at least two (algebraic) codimensional, so Corollary 5.12 ensures that H^* is good (for the purpose of Definition 3.5). For the points $x \in H\setminus H^*$, there is at least one $P \in I(H)$ for which $\mathrm{grad}P(x) \ne 0$ and the gradients of all polynomials in $I(H)$ are parallel to $\mathrm{grad}P(x)$. In the following we will assume $H = \{x|P(x) = 0\}$ for one such P, only for the sake of more transparent notation.

Fix a finite collection of unit vectors $v_1, \dots, v_N$ in $\mathbb{R}^m$, such that for any nonzero vector $v \in \mathbb{R}^d$, there is a v_i for which $tan(\sphericalangle(v, v_i)) < L' < L$. We shall identify those components of H that are Lipschitz graphs as viewed from the direction v_i. We will omit the index i. The construction clearly depends on the vector $v = v_i$. Having fixed v it is possible to choose an orthogonal coordinate system in $\mathbb{R}^m$ such that the mth base vector points in the direction v. For $\arctan(L') < \phi < \arctan(L)$ and $h = cos(\phi)$, consider the following subset of the algebraic variety:

$$\begin{aligned} H^{<\phi} &= \{x \in H \mid \sphericalangle(\mathrm{grad}P(x), v) < \phi\} = \\ &= \left\{x \in H \mid (\frac{\partial}{\partial x_m}P(x))^2 > h^2(\mathrm{grad}P(x))^2\right\}. \end{aligned} \tag{5.8}$$

Note that $H^{<\phi} \cap H^* = \emptyset$, because the inequality in the definition of $H^{<\phi}$ is strict. We claim that for almost every possible ϕ, $\partial H^{<\phi}$ is two-codimensional. Indeed,

$$\partial H^{<\phi} \subset H^{=\phi} := \left\{x \in H \mid (\frac{\partial}{\partial x_m}P(x))^2 = h^2(\mathrm{grad}P(x))^2\right\}.$$

The intersection of $H^{=\phi}$-s corresponding to different ϕ-s is H^*, which is two-codimensional, so its one-codimensional Hausdorff-measure is zero. However, Proposition 5.11 says that the union of all $H^{=\phi}$-s (which is part of H) has a finite one-codimensional Hausdorff-measure. So apart from a countable number of ϕ-s, the one-codimensional Hausdorff-measure of $H^{=\phi}$ is zero. Since $H^{=\phi}$ is algebraic, Proposition 5.11 tells us that almost every $H^{=\phi}$ is two-codimensional.

We fix $H' = H^{<\phi}$ with one such $\arctan(L') < \phi < \arctan(L)$.

We will cut H' into locally Lipschitz graphs. Let $k : \mathbb{R}^{m-1} \to \mathbb{N}$ be the multiplicity of $\pi(H')$. Clearly for every $x \in \pi(H')$ the restriction of P to $\pi^{-1}(x)$ is nonzero, so k is bounded by the degree of P, and the Implicit Function Theorem implies that it is lower semicontinuous. So, the set $D_1 \subset \mathbb{R}^{m-1}$ where k is maximal, is open. Here we can define the finitely many functions $f_{1,1}, \dots, f_{1,k_{max}} : D_1 \to \mathbb{R}$ taking the least, second least, ..., greatest element of $\pi^{-1}(x)$ for some $x \in D_1$. the Implicit Function Theorem implies that these functions are locally Lipschitz with constant L and that their graphs are disjoint.

Now we claim that the boundary of these graphs is two-codimensional. Indeed, $H^{=\phi}$ is two-codimensional and algebraic, so $\pi(H^{=\phi})$ is also part of a one-codimensional algebraic variety in $\mathbb{R}^{m-1}$. The pre-image (by π) of this variety is one-codimensional in $\mathbb{R}^m$, and the boundary of our graphs is on the intersection of this pre-image with H. This intersection is transversal (ensuring two codimensions) at points of $H' \setminus H^*$, and the rest of the boundary is in H^*.

Now erase the closure of these graphs from H'. So the argument can be repeated with k_{max} already at least one less. The procedure ends in finitely many steps, and so finitely many open locally Lipschitz graphs are constructed. Their closures cover H' by construction, and their boundary is two-codimensional.

We carry out this construction for every v_i, and get a covering of the entire $H \setminus H^*$ by finitely many locally Lipschitz graphs. To get the sets H_1,...,H_N in

Definition 3.5 we only need to make these graphs disjoint by subtracting the closure of one from the other. □

Acknowledgments

The authors express their sincere gratitude to Nándor Simányi and András Szűcs for illuminating discussions. Special thanks are due to Károly Böröczky, Lajos Rónyai and Endre Szabó for supplying some of the ideas that were eventually built into the proofs of the paper. We are also greatly indebted to Nándor Simányi for his careful reading of the manuscript and for his most useful remarks. N. Chernov was partially supported by NSF grant DMS-9732728. The financial support of the Hungarian National Foundation for Scientific Research (OTKA), grants T26176 and T32022; and of the Research Group Stochastic of the Hungarian Academy of Sciences, affiliated to the Technical University of Budapest is also acknowledged.

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Communicated by Eduard Zehnder
submitted 07/06/01, accepted 04/02/02