CHAPTER 3

Integration of Vector Fields

At the end of the introduction we said a few words about the motion of a liquid whose velocity at any point of space is independent of time. This example was used to motivate the axioms for a flow. Conversely, given a flow ϕ on a space X, we can recover a notion of *velocity at a point* provided that X has the structure of a differentiable manifold and ϕ has a certain degree of smoothness. The velocity of ϕ is a *vector field* on X (a fact that is often obscured in Euclidean space, which is commonly identified with its own tangent bundle). In this chapter we discuss existence and uniqueness of a flow whose velocity is a prescribed vector field. The local problem is, using a chart, equivalent to the problem of existence and uniqueness of solutions (integral curves) of a system of ordinary differential equations.

In terms of the moving liquid model, an integral curve is the path of a given particle of fluid. Picard's theorem deals with the local existence of such integral curves. The idea behind its proof is as follows. One chooses at random a curve γ_0 (a continuous map from time to position in space). Taking the velocity of the liquid at $\gamma_0(t)$ for each t gives a vector valued function v of time. Now consider a journey with the same starting point as γ_0 but with velocity $v(t)$ at time t. Let γ_1 be the path of this journey (see Figure 3.1). Applying this process again with γ_0 replaced by γ_1 we obtain a new curve γ_2, and so on. The curves form an infinite sequence (γ_i).

At each stage we modify the previous curve to fit in with the velocity of the fluid along it, so it is not surprising that, for large i, γ_i approaches an integral curve. What we have described is rather like holding onto one end of a light thread while the thread gradually drifts into streamline position, but it is a discrete rather than a continuous process.

Convergence of such a sequence of paths may be thought of in terms of convergence of the sequence $(\gamma_i(t))$ of positions, for all times t. A more fruitful approach, however, is to consider convergence in an infinite

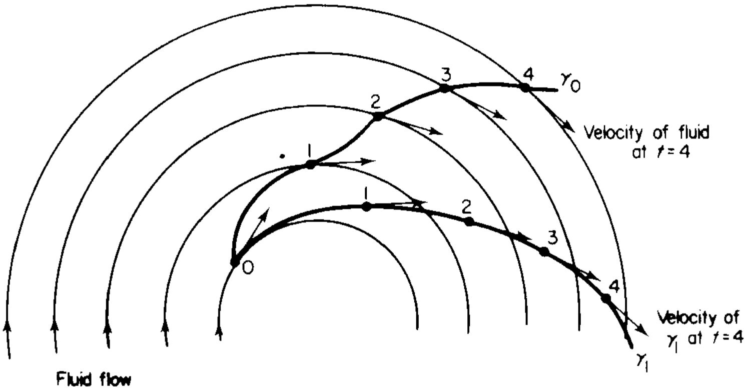

FIGURE 3.1

dimensional Banach space, each of whose points is a continuous map from time to position. We shall make use of map spaces at many crucial stages in this and in later chapters. This is not sophistication for its own sake; it leads to considerable simplifications of proofs. For example, most of the complications in the classical proof that the integral of a smooth vector field is a smooth function of initial position vanish once we have at our disposal a few elementary results on map spaces. We describe the requisite theory in Appendices B and C. The latter contains a version of Banach's contraction mapping theorem. This extremely useful tool is applied to avoid long-drawn-out successive approximation arguments, for example in Picard's theorem and in other parts of the integration theory below.

I. VECTOR FIELDS

Let X be a differentiable manifold, and let I be a real interval. A *vector field* on X is a map $v : X \to TX$ associating with each point x of X a vector $v(x)$ in the tangent space T_xX to X at x. We think of T_xX as the space of all possible velocities at x of a particle moving along paths on the manifold X. Thus we have in mind a picture like Figure 3.2, where X is the submanifold S^2 of $\mathbf{R}^3$ and T_xX is embedded in $\mathbf{R}^3$ as a plane geometrically tangent to S^2 at x. However we really need an intrinsic definition of T_xX, independent of any particular embedding of the manifold X in Euclidean space. This is given in Appendix A, together with further notes on vector fields. Notice that a particle moving along I can have any real velocity at $t \in I$, and so T_tI is a copy of the real line. In fact one may identify T_tI with $\{t\} \times \mathbf{R}$ in $I \times \mathbf{R}$, and hence

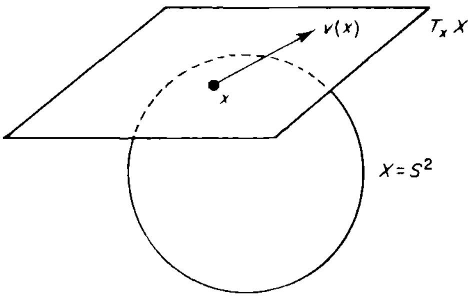

FIGURE 3.2

identify TI with $I \times \mathbf{R}$. Similarly we show in Example A.31 that if U is an open subset of Euclidean space $\mathbf{R}^n$ then TU may be identified with $U \times \mathbf{R}^n$. In this case, if v is a C^r vector field on U then the first component of $v(x)$ with respect to the product is always x, signifying that $v(x)$ is in the tangent space at x. Thus $v(x) = (x, f(x))$ for some C^r map $f: U \to \mathbf{R}^n$. Conversely, any C^r map $f: U \to \mathbf{R}^n$ corresponds to a C^r vector field (id, f) on U. The map f that determines and is determined by v in this way is called the *principal part* of v. It is common, and usually perfectly harmless, to blur the distinction between vector fields and principal parts. We shall resist the temptation to do so, however, at least for the rest of this chapter. Note that all the above remarks hold equally well when $\mathbf{R}^n$ is replaced by any Banach space $\mathbf{E}$.

(3.3) Example. (*Gradient vector fields*). Let $f: X \to \mathbf{R}$ be a C^{r+1} function $(r > 0)$, and suppose that X has a Riemannian structure (see Appendix A). Then we may associate with the linear tangent map $Tf: TX \to T\mathbf{R}$ a C^r vector field ∇f (or grad f) on X, called the *gradient vector field* of f. Example A.57 of Appendix A describes in detail the construction of ∇f. If $X = \mathbf{R}^n$, $f(x)$ has principal part $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. Thus, for example, if $f(x_1, x_2) = x_1^2 x_2$, then $\nabla f(x_1, x_2)$ has principal part $(2x_1x_2, x_1^2)$. If $X = S^2$, embedded as

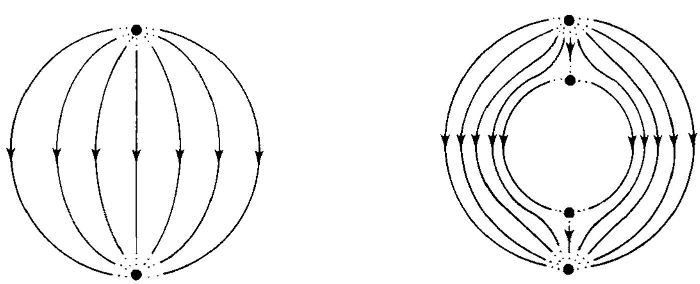

FIGURE 3.3

the unit sphere in $\mathbf{R}^3$, and f is the *negative height function* $f(x_1, x_2, x_3) = -x_3$, then $\nabla f(x)$ is tangent to the path taken by a raindrop on S^2, that is to say the *line of steepest descent* through x, which is a meridian of longitude. Figure 3.3 illustrates this, and also ∇f for f the negative height function on the embedding of T^2 as a tyre inner tube.

(3.4) Example. Two rather trivial vector fields needed later are the *zero vector field* on X, $x \mapsto 0_x$, where 0_x is the zero vector in T_xX (0_x is identified with the point x itself in Figure 3.2), and the *unit vector field* on I, $t \mapsto 1_t$, where 1_t is the vector $(t, 1)$ in $T_tI = \{t\} \times \mathbf{R}$. Now consider the product manifold $I \times X$. At any point (t, x), the velocity of a moving particle has components in the I and X directions, so we may identify $T(I \times X)$ with $TI \times TX$ (see Exercise A.33). We denote by u the *unit vector field* on $I \times X$ in the positive t direction. That is to say, $u(t, x) = (1_t, 0_x)$ (see Figure 3.4).

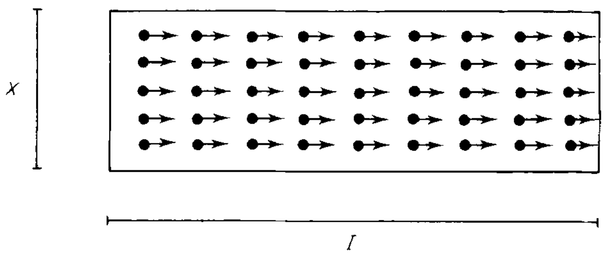

FIGURE 3.4

II. VELOCITY VECTOR FIELDS AND INTEGRAL FLOWS

Let $\phi: I \times X \to X$ be any map such that $\phi^t: X \to X$ is a homeomorphism for all $t \in I$. Given $x \in X$ and $t \in I$, let $y = (\phi^t)^{-1}(x)$. Then $\phi_y(t) = \phi(t, y) = x$. If we regard $\phi_y: I \to X$ as determining the position of a particle moving on X, then the particle reaches the point $x \in X$ at time t. Now suppose that the function ϕ_y is differentiable at t. Then the velocity of the particle when it reaches x is the vector $\phi_y'(t)$ of T_xX (see Example A.37). We call this vector the *velocity of ϕ at the point x at time t.* Thus, for example, if ϕ is defined for $t > 0$ and $x \in \mathbf{R}$ by $\phi(t, x) = \sinh(tx)$ then the above y is $(1/t)\sinh^{-1} x$, and differentiating ϕ with respect to t at (t, y) gives the velocity of ϕ at x at time t as the vector $(1/t)\sinh^{-1} x \cosh(\sinh^{-1} x)$ in the tangent space to $\mathbf{R}$ at x. Note that if ϕ is C^1 then its velocity at x at time t is $(T\phi)u(t, y)$, where u is as above.

We shall show below (Theorem 3.9) that of ϕ is a flow on X then its velocity at x is the same at all times t. Thus, in this case, ϕ gives rise to a vector field v, called the *velocity vector field* of f, where $v(x)$ is the velocity of ϕ at x at any time t. Since $\phi_x(0) = x$, the simplest formula for v is $v(x) = \phi_x'(0)$. For example, if $X = \mathbf{R}$ and ϕ is the flow $t \,.\, x = (x^{1/3} + t)^3$, we differentiate with respect to t at $t = 0$, and obtain $v(x) = (x, 3x^{2/3})$. If ϕ is C^1 then $v(x) = (T\phi)u(0, x)$, so the diagram

$$\begin{tikzcd} T(\mathbf{R}\times X) \arrow[r, "T\phi"] & TX \\ \mathbf{R}\times X \arrow[u, "u"] \arrow[r, "\phi"'] & X \arrow[u, "v"'] \end{tikzcd}$$

commutes. If ϕ is $C^r (r > 1)$, then v is C^{r-1}.

(3.5) Exercise. Find the velocity vector fields of the flows in Examples 1.14–1.17 and Exercise 1.18.

Conversely, if v is a given vector field on X, we call any flow ϕ on X an *integral flow* of v if v is the velocity vector field of ϕ, and say that v is *integrable* if such a flow exists.

(3.6) Example. Let $X = \mathbf{R}$ and let $v: X \to TX = \mathbf{R}^2$ be the vector field $v(x) = (x, -x)$. Then $\phi: \mathbf{R} \times X \to X$ given by $\phi(t, x) = x\, e^{-t}$ is an integral flow of ϕ.

(3.7) Exercise. Show that the zero vector field on any manifold X has the trivial flow as its unique integral flow.

(3.8) Exercise. Find an integral flow of the unit vector field u on $\mathbf{R} \times X$ defined above.

(3.9) Theorem. *Let ϕ be a flow on X such that, for all $x \in X$, the map $\phi_x : \mathbf{R} \to X$ is differentiable. Then the velocity of ϕ at any point is independent of time. Thus ϕ has a well defined velocity vector field.*

Proof. We prove that the velocity at x at time t equals the velocity at x at time 0. Let $y = \phi^{-t}(x)$. Then, by the basic property of flows $\phi_y(u) = \phi_x(-t + u)$ for all $u \in \mathbf{R}$. Differentiating with respect to u at $u = t$, using the chain rule for the right-hand side, we obtain $\phi_y'(t) = \phi_x'(0)$, as required. □

We wish to find conditions on a vector field v on X that ensure its integrability. Our main global theorem (Theorem 3.43) states that if X is compact and v is $C^r (r \geq 1)$ then v has a unique integral flow ϕ, which is itself

C^r. The technique used in the proof of this result can be visualized as gluing ϕ together from fragments constructed locally at each point (t, x) of $\mathbf{R} \times X$. These fragments of map coincide on the intersections of their domains of definition by a uniqueness theorem (Theorem 3.34). We need some local definitions to get started on this programme.

An *integral curve* of v is a C^1 map $\gamma : I \to X$, where I is any real interval, satisfying $\gamma'(t) = v\gamma(t)$ for all $t \in I$. If $0 \in \operatorname{int} I$, we call γ an integral curve *at* $\gamma(0)$. Thus if ϕ is a flow on X and v is the velocity vector field of ϕ, then the orbit of ϕ through $x \in X$ is the image of an integral curve of v at x. A *local integral* of v is a map $\phi : I \times U \to X$, where I is an interval neighbourhood of 0 and U is a non-empty open subset of X, such that, for all $x \in U$, $\phi_x : I \to U$ is an integral curve of v at x (see Figure 3.10). For all $x \in U$, we call ϕ a local integral *at* x, and say that v is *integrable at* x if such a local integral exists. If it does so, and is C^1, then the diagram

$$\begin{array}{ccc} T(I \times U) & \xrightarrow{T\phi} & TX \\ \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle v} \\ I \times U & \xrightarrow[\phi]{} & X \end{array}$$

commutes. If further $I = \mathbf{R}$ and $U = X$ then the local integral is, as we shall show later (Corollary 3.38) a flow on X, and is hence an integral flow of v. The term *local integral* should not be confused with the term *local first integral*. See the appendix to this chapter for a discussion of *first integrals*.

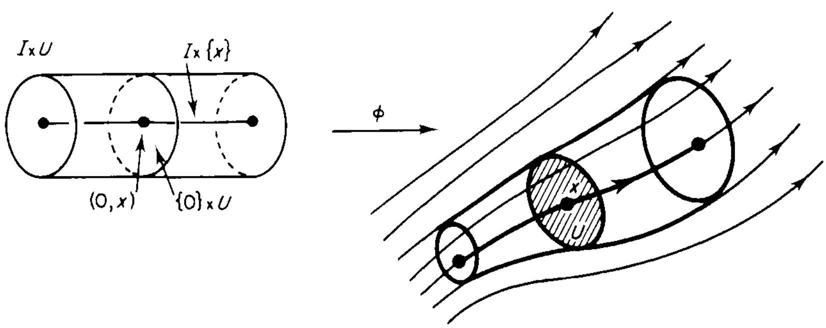

FIGURE 3.10

In investigating the existence of local integrals, we may restrict our attention to vector fields on (open subsets of) Banach spaces. For suppose that we wish to show that a vector field v on a manifold X is integrable at a point x. We take an admissible chart $\xi: V \to V' \subset \mathbf{E}$ at x, where X is modelled on $\mathbf{E}$. The C^∞ diffeomorphism ξ induces, from $v|V$, a vector field $\xi_*(v)$ on V', which is integrable at $\xi(x)$ if and only if v is integrable at x. This is a consequence of the following result:

(3.11) Theorem. *Let v be a vector field on X, and let $h:X \to Y$ be a C^1 diffeomorphism. If γ is an integral curve of v then $h\gamma$ is an integral curve of the vector field $h_*(v) = (Th)vh^{-1}$ induced on Y from v by h. If $\phi: I \times U \to X$ is a local integral of v at x, then $h\phi(id \times (h|U)^{-1})$ is a local integral of $h_*(v)$ at $h(x)$.*

Proof. The diagram

$$\begin{tikzcd} TI \arrow[r, "T\gamma"] & TX \arrow[r, "Th"] & TY \\ I \arrow[u, "\iota"] \arrow[r, "\gamma"'] & X \arrow[u, "v"] \arrow[r, "h"'] & Y \arrow[u, "h_*(v)"'] \end{tikzcd}$$

commutes, where ι is the positive unit vector field on I, and $T(h\gamma) = ThT\gamma$. Thus $h\gamma$ is an integral curve of $h_*(v)$. The proof of the second statement is similar. □

III. ORDINARY DIFFERENTIAL EQUATIONS

Let V be an open subset of a Banach space $\mathbf{E}$, and let v be a C^r vector field on $V (r \geq 0)$. Suppose that $\phi: I \times U \to V$ is a local integral of v. Let us make explicit the relation so established between ϕ and the principal part f of v. According to the definition of local integral, for all $(t, x) \in I \times U$,

(3.12) $$\gamma'(t) = v\phi(t, x) = (\phi(t, x), f\phi(t, x)).$$

Hence $D_1\phi(t, x)(1) = f\phi(t, x)$. We make the usual abuse of notation that identifies a linear map from $\mathbf{R}$ to $\mathbf{E}$ with its value at 1, and write

(3.13) $$D_1\phi = f\phi.$$

Conversely, any map $\phi: I \times U \to V$ satisfying (3.13) also satisfies (3.12). We are led to make definitions of local integrals and integral curves of any map $f: V \to \mathbf{E}$ in the obvious manner. That is to say, $\phi: I \times U \to V$ is a *local integral* of f at $x \in U$ if ϕ^0 is the inclusion and $D_1\phi = f\phi$ on $I \times U$. An *integral curve* of f is a C^1 map $\gamma: I \to V$ such that $\gamma' = f\gamma$ on I. We say that γ is an integral curve *at* x if $0 \in \operatorname{int} I$ and $\gamma(0) = x$. Trivially:

(3.14) Proposition. *A map is a local integral (resp. integral curve) of a vector field v on V if and only if it is a local integral (resp. integral curve) of the principal part of v.* □

We observe that the relation $\gamma' = f\gamma$ implies that any integral curve of a C^0 vector field is C^1. More generally, we obtain, by induction, the following result:

(3.15) Proposition. *Any integral curve of a C^r vector field is C^{r+1}.* □

In classical notation, with $\mathbf{E} = \mathbf{R}^n$, the relations $D_1\phi = f\phi$ and $\gamma' = f\gamma$ are both written in the form $dx/dt = f(x)$, which is called an *autonomous system of first order ordinary differential equations.* Here the term "autonomous" refers to the fact that the variable t is not explicitly present on the right-hand side. It is a "system of equations" since there are n scalar equations of the form $dx_i/dt = f_i(x_1, \ldots, x_n)$. The function γ is called a solution of the equations, and ϕ is described as a *solution regarded as a function of initial conditions,* or *general solution.*

(3.16) Example. Let v be the vector field on $\mathbf{R}^2$ with principal part $f: \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x, y) = (-y, x)$. The corresponding pair of scalar differential equations is

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x.$$

The solution $x = \cos t$, $y = \sin t$ of these equations is, in our parlance, the integral curve $\gamma: \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) = (\cos t, \sin t)$. The general solution $x = A \cos t - B \sin t$, $y = A \sin t + B \cos t$, where A and B are "arbitrary constants", is in our parlance, the local integral (and, indeed, integral flow) $\phi: \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ given by

$$\phi(t, (x, y)) = (x \cos t - y \sin t, x \sin t + y \cos t).$$

(3.17) *Note. Non-autonomous systems.* In the present context, we lose no generality by restricting our attention to autonomous differential equations. Any non-autonomous system $dx/dt = g(t, x)$, where g maps an open subset V of $\mathbf{R} \times \mathbf{E}$ to $\mathbf{E}$, gives rise to an autonomous system $dy/dt = f(y)$, where $f: V \to \mathbf{R} \times \mathbf{E}$ is defined by $f(u, x) = (1, g(u, x))$. In other words, we are replacing the given differential equation by $du/dt = 1$, $dx/dt = g(u, x)$. Any solution of $dy/dt = f(y)$ yields a solution of $dx/dt = g(t, x)$. Specifically, suppose that we want a solution δ of $dx/dt = g(t, x)$ satisfying the boundary condition $\delta(t_0) = p$. Suppose that we have found an integral curve $\gamma: I \to V$ of f at (t_0, p). Then we assert that $\delta(t) = \gamma_2(t - t_0)$ defines a suitable solution δ, where the index denotes the component in the product $\mathbf{R} \times \mathbf{E}$. For observe that $\delta(t_0) = \gamma_2(0) = p$ and

$$\delta'(t) = \gamma_2'(t - t_0) = f_2\gamma(t - t_0) = g(\gamma_1(t - t_0), \delta(t)).$$

But, since $\gamma_1'(t) = f_1\gamma(t) = 1$ and $\gamma_1(0) = t_0$, $\gamma_1(t) = t + t_0$, and so $\delta'(t) = g(t, \delta(t))$.

We should also point out that an mth order ordinary differential equation in "normal form"

$$\frac{d^m y}{dt^m} = h\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{m-1}y}{dt^{m-1}}\right)$$

may be reduced by the substitutions

$$x_1 = y, x_2 = \frac{dy}{dt}, \ldots, x_m = \frac{d^{m-1}y}{dt^{m-1}}$$

to the form $dx/dt = g(t, x)$, where $x = (x_1, \ldots, x_m) \in \mathbf{E}^m$, and

$$g(t, x) = (x_2, x_3, \ldots, x_m, h(t, x_1, \ldots, x_m)).$$

Moreover, the implicit mapping theorem (Exercise C.14) can often be used locally to reduce more general mth order equations to the above normal form.

(3.18) Example. The simple pendulum equation $\theta'' = -g \sin\theta$ of the introduction was reduced to $\theta' = \omega$, $\omega' = -g \sin\theta$ by the substitution $\omega = \theta'$.

(3.19) Example. In conservative mechanics, *Lagrange's equations* of motion are usually written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

where q_i $(i = 1, 2, \ldots, n)$ are "generalized coordinates", the *Lagrangian L* is $T - V$, T is the *kinetic energy* and V is the *potential energy*. If we choose q_i such that the positive definite quadratic form

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i (q_i')^2,$$

where the m_i are "generalized masses", then the equations of motion take the form

$$m_i q_i'' = -\frac{\partial V}{\partial q_i}.$$

The substitution $m_i q_i' = p_i$, the "generalized momentum", converts the equations of motion to $p_i' = -\partial V/\partial q_i$, and the substitution and equations of motion may be written together as *Hamilton's equations*

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i},$$

where the *Hamiltonian* $H = T + V$ is the total energy as a function of p_i and q_i.

(3.20) Example. *Van der Pol's equation.* This is, amongst other things, an important mathematical model for electronic oscillators. It has the form

$$x'' - \alpha x'(1 - x^2) + x = 0$$

where α is a positive constant. This is equivalent to the pair of first order equations

$$x' = y, \qquad y' = -x + \alpha y(1 - x^2)$$

where α is a positive constant. The phase portrait of this vector field on $\mathbf{R}^2$ is topologically equivalent to the reverse flow of the flow in Example 1.27. It has a unique closed orbit, which is the ω-set of all orbits except the fixed point at the origin (see Figure 3.20; a proof of this assertion may be found,

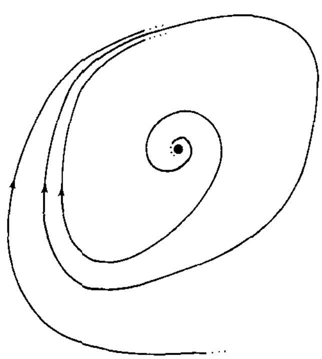

FIGURE 3.20

for example, in Hirsch and Smale [1] and in Simmons [1]). The system is said to be *auto-oscillatory*, since all solutions (except one) tend to become periodic as time goes by.

IV. LOCAL INTEGRALS

Let p be a point of a Banach space $\mathbf{E}$. Suppose that we have a vector field v with principal part f defined on some neighbourhood B' of p. We wish to show that for some neighbourhood B of p there are unique integral curves at each point of B. We must first make some comments. To stand any chance of proving uniqueness of an integral curve at the point x, we must fix the interval on which the curve is to be defined (since restricting to a smaller interval gives, strictly speaking, a different integral curve). If we wish to fit the integral curves together to form a local integral ϕ at p, we may as well define them all on the same interval I. We cannot have I arbitrarily large, since the orbit of p may not be wholly contained in B'. Similarly, however small I, we usually need B strictly inside B', since points x near the frontier of B' will leave B' at some time in I.

Actually, we adopt a slightly different approach, for ease of presentation. We start with given sets B' and B, balls with centre p in $\mathbf{E}$. We insist that the radius d of B is strictly less than the radius d' of B'. Note, however, that the argument works equally well with $B = B' = \mathbf{E}$, so we include this as an exceptional possibility $d = d' = \infty$. We prove that there is some interval I on which the above situation holds (see Figure 3.21). For this we need to make

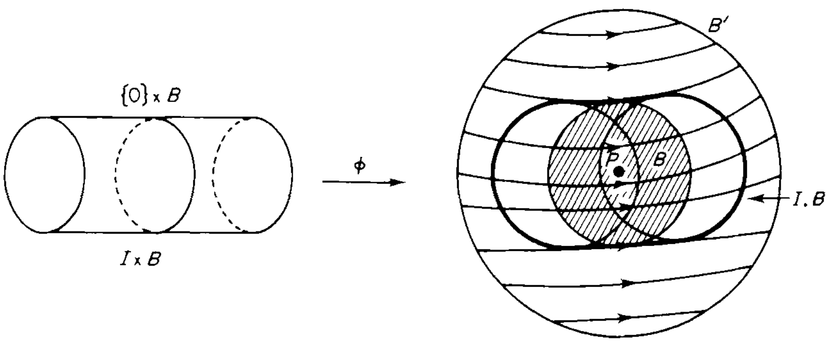

FIGURE 3.21

some assumption about the smoothness of f. Continuity of f ensures existence of integral curves, but not their uniqueness. For uniqueness we need something stronger, and the relevant property is what is known as a *Lipschitz condition* (see Appendix C). This is about half way between continuity and differentiability; it implies continuity and it is certainly satisfied by any C^1 map on B'.

Any Lipschitz map is bounded on a ball of finite radius. From now on we shall be making full use of the map space notation and theory of Appendix B and the contraction mapping theorem of Appendix C. In particular, $|f|_0$ denotes $\sup\{|f(x)|: x \in B'\}$. Our version of Picard's theorem is:

(3.22) Theorem. *Let $f: B' \to \mathbf{E}$ be Lipschitz with constant κ, and let I be the interval $[-a, a]$, where $a < (d'-d)/|f|_0$ if $d' < \infty$. Then for each $x \in B$, there exists a unique integral curve $\phi_x: I \to B'$ of f at x. Let $\phi: I \times B \to B'$ send (t, x) to $\phi_x(t)$. Then for all $t \in I$, $\phi^t: B \to B'$ is uniformly (in t) Lipschitz, and is C^r iff f is C^r.*

Proof. We give a proof under the additional restriction $a < 1/\kappa$, which may be removed by an easy technical modification (see Exercise 3.24 below). We define a map $\chi: B \times C^0(I, B') \to C^0(I, B')$ by, for all $t \in I$,

$$\chi(x, \gamma)(t) = x + \int_0^t f\gamma(u)\, du, \tag{3.23}$$

where $C^0(I, B')$ is as defined in Appendix B. We must check, of course, that $\chi(x, \gamma)$ is indeed in $C^0(I, B')$. It is continuous by elementary analysis, and, by the inequalities

$$\left|x + \int_0^t f\gamma(u)\, du - p\right| \leq |x-p| + |t|\,|f|_0 \leq d + a|f|_0 \leq d',$$

its image is in B'.

Now, for all $x \in B$ and $\gamma, \tilde{\gamma} \in C^0(I, B')$,

$$\begin{aligned}|\chi^x(\gamma) - \chi^x(\tilde{\gamma})|_0 &= \sup\left\{\left|\int_0^t (f\gamma(u) - f\tilde{\gamma}(u))\, du\right| : t \in I\right\} \\ &\leq \sup\left\{\left|\int_0^t |f\gamma(u) - f\tilde{\gamma}(u)|\, du\right| : t \in I\right\} \\ &\leq \kappa a |\gamma - \tilde{\gamma}|_0.\end{aligned}$$

By the contraction mapping theorem (Theorem C.5) there exists a unique map $g: B \to C^0(I, B')$ such that, writing $g(x) = \phi_x$ for all $x \in B$,

$$\phi_x(t) = x + \int_0^t f\phi_x(u)\, du$$

for all $t \in I$, or (equivalently) such that ϕ_x is an integral curve of f at x. The uniqueness of g gives the uniqueness of the integral curve at x, since another integral curve at x would give a possible alternative for the value $g(x)$.

Since χ is uniformly Lipschitz on the first factor, with constant 1, it follows by Theorem C.7 that g is Lipschitz. Thus ϕ^t, which is g composed with the continuous linear evaluation map $ev^t: C^0(I, B') \to B'$, is Lipschitz, where

$ev^t(\gamma) = \gamma(t)$. Moreover, since $|ev^t| = 1$, the Lipschitz constant of g is also a Lipschitz constant for each map ϕ^t.

Suppose, finally, that f is C^r. We observe that χ may be expressed as the sum of two maps. The first, taking (x, γ) to the constant map with value x, is continuous linear (and hence C^∞). The second is the composite of projection onto the second factor, the map

$$\gamma \mapsto f\gamma : C^0(I, B') \to C^0(I, \mathbf{E})$$

and the map

$$\mu \mapsto \left(t \mapsto \int_0^t \mu(u)\, du\right) : C^0(I, \mathbf{E}) \to C^0(I, \mathbf{E}).$$

The outside maps of the composite are continuous linear, while the middle one is C^r, by Theorem B.10. Thus, by Theorem C.7, g, and hence ϕ^t, is C^r. □

(3.24) Exercise. The assumption $a < 1/\kappa$ in the proof of Theorem 3.22 was made to ensure that the maps χ^x were contractions. They lose this property for larger a, but regain it if they are conjugated with a suitable automorphism of the space $C^0(I, \mathbf{E})$. In fact, the map $\gamma \mapsto (t \mapsto \operatorname{sech} \alpha t \,.\, \gamma(t))$ has the desired effect for $\alpha > \kappa$. The exercise, then, is to eliminate the assumption $a < 1/\kappa$, by considering instead of χ, the map $\psi : B \times C \to C$ defined, for all $t \in I$, by

$$\psi(x, \delta)(t) = \operatorname{sech} \alpha t \left(x + \int_0^t f(\cosh \alpha u \,.\, \delta(u))\, du\right),$$

where C is the closed subset $\{\delta \in C^0(I, \mathbf{E}) : \delta(t) \cosh \alpha t \in B' \text{ for all } t \in I\}$, the image of $C^0(I, B')$ under the above mentioned automorphism.

(3.25) Exercise. Derive a local integrability theorem for non-autonomous ordinary differential equations (see Note 3.17).

(3.26) Exercise. Show that the map $f: \mathbf{R} \to \mathbf{R}$ sending x to $x^{2/3}$ is not Lipschitz on any neighbourhood of 0, but that nevertheless f has a continuous local integral at 0.

The way in which the local integral ϕ depends separately on time t and initial position x emerges very clearly and easily from Proposition 3.15 and Theorem 3.22. One is naturally led to ask how ϕ depends on t and x together. Here the answers are just as simple. As we shall now prove, the map ϕ is locally Lipschitz, and it is as smooth as the function f.

(3.27) Theorem. *If $d' < \infty$ then the map ϕ of Theorem 3.22 is Lipschitz. In any case, it is locally Lipschitz.*

Proof. For all (t, x) and $(t', x') \in I \times B$,

$$|\phi(t', x') - \phi(t, x)| \leq \lambda |x' - x| + |\phi_x(t') - \phi_x(t)|, \tag{3.28}$$

where λ is, for all $t \in I$, a Lipschitz constant for ϕ^t (which is uniformly Lipschitz by Theorem 3.22). When d' is finite, f is bounded (being Lipschitz) and thus so is $D\phi_x (= f\phi_x)$. Hence ϕ_x is uniformly (in x) Lipschitz, and we deduce from (3.28) that ϕ is Lipschitz. If d' is not finite, then (3.28) nevertheless gives that ϕ is continuous. Thus $f\phi$ is locally bounded, ϕ_x is locally uniformly Lipschitz and hence ϕ is locally Lipschitz. □

(3.29) Exercise. Find an example of a Lipschitz map $f: \mathbf{E} \to \mathbf{E}$ for which there is no corresponding Lipschitz $\phi: I \times \mathbf{E} \to \mathbf{E}$, for any I.

(3.30) Exercise. Suppose that $\gamma: [0, \tau] \to \mathbf{R}$ is continuous, and that, for all $t \in [0, \tau]$,

$$\gamma(t) - \kappa \int_0^t \gamma(u)\, du \leq A + Bt + C e^{\alpha t},$$

where $\kappa > 0$, $\alpha \neq \kappa$ and where A, B and C are all constants. Prove that

$$\gamma(t) \leq A\, e^{\kappa t} + B\kappa^{-1}(e^{\kappa t} - 1) + C(\alpha - \kappa)^{-1}(\alpha\, e^{\alpha t} - \kappa\, e^{\kappa t}).$$

(This deduction is called "integrating the inequality".) Show that if $\phi: I \times U \to V$ is a local integral of a map $f: V \to \mathbf{E}$ with Lipschitz constant κ, then, for all $t \in I$, ϕ^t satisfies the inequality

$$|\phi^t(x) - \phi^t(x')| - \kappa \left| \int_0^t |\phi^u(x) - \phi^u(x')| du \right| \leq |x - x'|.$$

Deduce that ϕ^t is Lipschitz with constant $e^{\kappa|t|}$.

We now turn to the situation when f is smooth. The only substantial point is to show that the partial derivatives of the local integral ϕ with respect to x are continuous in t and x together. We deal with this first.

(3.31) Lemma. *Let f be C^r and let ϕ be the map of Theorem 3.22. Then the partial derivatives $(D_2)^s\phi: I \times B \to L_s(\mathbf{E}, \mathbf{E})$ exist and are continuous for $1 \leq s \leq r$.*

Proof. Since we may "slow down" f by multiplying it by a small positive constant α, we may assume that the Lipschitz constant κ of f is less than 1. The local integral for f is recovered by "speeding up" the local integral of αf. More precisely, we multiply the time parameter t by $1/\alpha$ and then apply the local integral of αf. This procedure does not affect the smoothness class of the integral. We leave the details as an exercise. We also continue to assume that $a < 1/\kappa$, having already indicated in Exercise 3.24 how to deal with this point.

We observe that the relation (3.23) defines a map

$$\chi : B \times C^1(I, B') \to C^1(I, B')$$

which is still a uniform contraction on the second factor. The extra calculation needed is

$$|D(\chi^x(\gamma) - \chi^x(\tilde{\gamma}))|_0 = |f\gamma - f\tilde{\gamma}|_0 \leq \kappa|\gamma - \tilde{\gamma}|_0,$$

and, by assumption, $\kappa < 1$. Moreover, χ_γ is still uniformly Lipschitz, and χ is itself C^r. To prove this latter assertion, consider the inclusion ι of $C^1(I, B')$ in $C^0(I, \mathbf{E})$, followed by $f_* : \operatorname{im} \iota \to C^0(I, \mathbf{E})$. Since f_* is composition with f, $f_*\iota$ is C^r, by Theorem B.10. Also integrating back into $C^1(I, \mathbf{E})$ is a continuous linear operation, so $\chi : B \times C^1(I, B') \to C^1(I, B')$ is C^r as claimed. Thus the fixed point map $g : B \to C^1(I, B')$ corresponding to our new χ is C^r. As in Theorem 3.22, we define ϕ by $\phi(t, x) = g(x)(t)$. By the uniqueness of integral curves in Theorem 3.22, the present map ϕ is the map ϕ of Theorem 3.22.

Now for all $t \in I$, ϕ^t is the composite of $g : B \to C^1(I, B')$ and the continuous linear map $ev^t : C^1(I, B') \to B'$. Thus, for $1 \leq s \leq r$, $(D_2)^s\phi$ exists, and is given by

$$(D_2)^s\phi(t, x) = ev^t \circ D^s g(x)$$

(recall that $D^s g(x)$ is an s-linear map from $\mathbf{E}$ to $C^1(I, \mathbf{E})$ so we may talk of its composite with $ev^t : C^1(I, \mathbf{E}) \to \mathbf{E}$). Thus $(D_2)^s\phi = \operatorname{comp}(ev^{\cdot} \times D^s g)$, where $ev^{\cdot} : I \to L(C^1(I, \mathbf{E}), \mathbf{E})$ is the continuous map of Corollary B.17 and comp is the continuous bilinear composition map from $L(C^1(I, \mathbf{E}), \mathbf{E}) \times L_s(\mathbf{E}, C^1(I, \mathbf{E}))$ to $L_s(\mathbf{E}, \mathbf{E})$. Thus $(D_2)^s\phi$ is continuous. □

Our main theorem on smoothness of local integrals now follows easily.

(3.32) Theorem. *If the map f of Theorem 3.22 is $C^r (r \geq 1)$, then the local integral ϕ is C^r. Moreover, $D_1(D_2)^r\phi$ exists and equals*

$$(D_2)^r(f\phi) : I \times \mathbf{E} \to L_s(\mathbf{E}, \mathbf{E}).$$

Proof. Let f be C^r. We prove by induction on s the statement that ϕ is C^s, for $0 \leq s \leq r$. By Theorem 3.27, ϕ is locally Lipschitz, and hence C^0. Now assume that ϕ is C^j, for some j with $0 \leq j < s$. We prove that ϕ is C^{j+1}, by showing that its jth partial derivatives are C^1. There are two cases.

(i) With the exception of $(D_2)^j\phi$, all jth partial derivatives can be expressed, by the symmetry of $D^j\phi$, as $(j-1)$st partial derivatives of $D_1\phi$. Since $D_1\phi = f\phi$, which is C^j, these partial derivatives are C^1.

(ii) By Lemma 3.31 we know that $(D_2)^j\phi$ is continuous, and also that $(D_2)^{j+1}\phi$ exists and is continuous. Since $(D_2)^j D_1\phi = (D_2)^j(f\phi)$ is continuous, $D_1(D_2)^j\phi$ exists and equals $(D_2)^j D_1\phi$. (To see this, observe that

$$(D_2)^j D_1\phi(t,x) = D_1 \int_0^t (D_2)^j D_1\phi(u,x)\,du$$

$$= D_1(D_2)^j \int_0^t D_1\phi(u,x)\,du = D_1(D_2)^j\phi(t,x)\,.)$$

Hence, by induction, ϕ is C^s, and hence C^r. Finally, since $(D_2)^r D_1\phi = (D_2)^r(f\phi)$ is continuous, $D_1(D_2)^r\phi$ exists and equals $(D_2)^r(f\phi)$. □

Since any C^1 vector field is locally Lipschitz (Proposition C.2), we obtain the following local result for manifolds:

(3.33) Corollary. *Any C^r vector field $(r \geq 1)$ on a manifold has a C^r local integral at each point of the manifold.* □

A very natural question arises at this stage. How does the local integral ϕ depend on the vector field v? If we make small perturbations of f and its derivatives, are the corresponding alterations in ϕ and its derivatives small? We deal with this problem in the appendix to this chapter.

V. GLOBAL INTEGRALS

Our main tool for extending local integrability results to global ones is the uniqueness of integral curves proved in Theorem 3.22. We first extend this into a global uniqueness theorem.

(3.34) Theorem. *Let v be a locally Lipschitz vector field on a manifold X, and let $\alpha: J \to X$ and $\beta: J \to X$ be integral curves of v. If, for some $t_0 \in J$, $\alpha(t_0) = \beta(t_0)$, then $\alpha = \beta$.*

Proof. Let $\alpha(t_0) = \beta(t_0)$, where $t_0 \in J$. We prove that $\alpha(t) = \beta(t)$ for all $t \in J$ with $t \geq t_0$; the proof for $t \leq t_0$ follows by reversing v (that is, replacing v by $-v$). Let $\tau = \sup\{t \in J : \alpha = \beta \text{ on } [t_0, t]\}$, and suppose that $\tau \in \operatorname{int} J$. By the continuity of α and β, $\alpha(\tau) = \beta(\tau) = p$, say. Now identify a neighbourhood U of p in X with an open subset U' of the model space $\mathbf{E}$ of X, by a chart $\xi: U \to U'$. Choose balls B and B' in U with centre p and obtain a corresponding interval $I = [-a, a]$ as in Theorem 3.22. We may take a small enough for $\tau + a$ to be in J. If $\tilde{\alpha}: I \to X$ and $\tilde{\beta}: I \to X$ take t to $\alpha(t+\tau)$ and $\beta(t+\tau)$ respectively then $\tilde{\alpha} = \tilde{\beta}$ by Theorem 3.22, since both are integral curves at p with domain I. Thus $\alpha = \beta$ on $[t_0, \tau + a]$, contrary to the definition of τ. Thus τ is the right end-point of J. If $\tau \in J$ then, as above, $\alpha(\tau) = \beta(\tau)$. □

(3.35) Exercise. Let $f: \mathbf{R} \to \mathbf{R}$ take x to $x^{2/3}$. Find two different integral curves of f at 0 with the same domain.

(3.36) Exercise. Prove that all integral curves of a locally Lipschitz vector field v at a point p are constant functions if and only if p is a singular point of v (p is a *singular point*, or *zero*, of v if $v(p) = 0_p$).

Earlier in the chapter we commented that the velocity of a flow is independent of time, and so may be regarded as a vector field. Conversely, a local integral of a vector field is locally a flow in the following sense:

(3.37) Theorem. *Let $\phi: I \times U \to X$ be a local integral of a locally Lipschitz vector field on a manifold X. Then for all $x \in U$ and all $s, t \in I$ such that $s + t \in I$ and $\phi(t, x) \in U$, $\phi(s, \phi(t, x)) = \phi(s + t, x)$.*

Proof. Apply Theorem 3.34 to integral curves $\alpha: [0, s] \to X$ and $\beta: [0, s] \to X$ defined by $\alpha(u) = \phi(u, \phi(t, x))$ and $\beta(u) = \phi(u + t, x)$. Since α and β agree at 0, they agree at s. □

(3.38) Corollary. *If $\phi: \mathbf{R} \times X \to X$ is a local integral of a locally Lipschitz vector field on a manifold X, then it is a flow on X.* □

(3.39) Theorem. *Any Lipschitz vector field v on a Banach space* **E** *has a unique integral flow ϕ. If v is C^r then ϕ is C^r.*

Proof. For all $a > 0$, v has a unique local integral $\phi: [-a, a] \times \mathbf{E} \to \mathbf{E}$, by Theorem 3.22. Any two such agree on the intersection of their domains. Hence the collection of all such local integrals defines a local integral $\phi: \mathbf{R} \times \mathbf{E} \to \mathbf{E}$, which is a flow by Corollary 3.38. If v is C^r then ϕ is C^r by Theorem 3.32 and uniqueness. □

(3.40) Exercise. (i) Prove that the vector field on $\mathbf{R}$ with principal part $f(x) = x^2$ does not have an integral flow. (*Hint*: The vector field is so large that the orbits escape to infinity in finite time.)

(ii) Prove that any C^1 vector field v on a manifold X gives rise to a partial flow $\phi: D \to X$ on X, such that, for all $x \in X$, $\phi_x: D_x \to X$ is the maximal integral curve of v at x.

We have been at some pains to prove that the local integrals that we have obtained in our existence theorems are as smooth as the vector fields we integrate. This behaviour is completely general, as we now show.

(3.41) Theorem. *Any local integral of a C^r vector field ($r \geq 1$) on a manifold X is C^r. Similarly any integral of a locally Lipschitz vector field is locally Lipschitz.*

Proof. Let v be a C^r vector field on X, and let $\phi: I \times U \to X$ be a local integral of v. Let $a > 0$ be the right end-point of I (possibly $a = \infty$). Fix $x_0 \in U$, and let $T = \{t > 0: \phi$ is C^r at (u, x_0) for all $u \in [0, t]\}$. By Corollary 3.33, T is non-empty. Let $\tau = \sup T$. It is enough to show that $\tau = a$, and that $a \in T$ if $a \in I$, for then a similar argument for negative t completes a proof that ϕ is C^r at all points of $I \times \{x_0\}$.

Suppose that $\tau \in \operatorname{int} I$. By Corollary 3.33 there exists a C^r local integral $\psi:]-b, b[\times V \to X$ of v at $\phi(\tau, x_0)$, for some $b > 0$. By continuity of ϕ_{x_0} at τ, there is some $t_0 \in T$ with $t_0 > \tau - b$ and $\phi(t_0, x_0) \in V$. By continuity of ϕ^{t_0} at x_0 there is some neighbourhood W of x_0 in U with $\phi^{t_0}(W) \subset V$. Let $c = \min\{b, a - t_0\}$, so that $t_0 + c > \tau$. Then, for all $x \in W$ and all u with $0 \leqslant u < c$, $\phi(t_0 + u, x) = \psi(u, \phi(t_0, x))$. To check this, we have only to fix x and apply Theorem 3.34 to integral curves α and β whose values at $u \in [0, c[$ are respectively $\phi(t_0 + u, x)$ and $\psi(u, \phi(t_0, x))$. Since ψ is C^r and ϕ^{t_0} is C^r at x_0, it follows that ϕ is C^r at $(t_0 + u, x_0)$ for all $u \in]0, c[$, which contradicts the definition of τ. Thus $\tau = a$.

If $a \in I$, we obtain ψ, t_0 and W as above, and observe that, for all $x \in W$ and all u with $0 \leqslant u \leqslant a - t_0$, $\phi(t_0 + u, x) = \psi(u, \phi(t_0, x))$. This shows that ϕ is C^r at (a, x_0), as required.

An identical approach works for the locally Lipschitz case. □

To complete our integration programme, we prove that every smooth vector field on a compact manifold has an integral flow. We make use of a lemma which is of some interest in its own right.

(3.42) Lemma. *Let v be a locally Lipschitz vector field on a manifold X. Suppose, that for some $a > 0$ and for all $x \in X$, v has an integral curve at x with domain $[-a, a]$. Then v has a unique integral flow ϕ, which is locally Lipschitz. Furthermore, if v is $C^r (r \geqslant 1)$ then ϕ is C^r.*

Proof. We have to show that for all $x \in X$ there is an integral curve $\phi_x: \mathbf{R} \to X$ of v at x. Suppose that for some x this is not the case. Reversing v if necessary, we have that

$$\tau = \sup\{t \in \mathbf{R}: v \text{ has an integral curve at } x \text{ with domain } [0, t]\}$$

is finite. By hypothesis $\tau > 0$. Choose t_0 with $\tau - a < t_0 < \tau$. Then there is an integral curve $\gamma: [0, t_0] \to X$ at x, and, by hypothesis, an integral curve $\delta: [-a, a] \to X$ at $\gamma(t_0)$. Now $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) = \delta(t - t_0)$ is an integral curve on $[t_0 - a, t_0 + a]$. Using Theorem 3.34, we may extend the domain of γ to $[0, t_0 + a]$ by defining $\gamma = \tilde{\gamma}$ on $[t_0 - a, t_0 + a]$. This contradicts the definition of τ. We have, then, a local integral $\phi: \mathbf{R} \times X \to X$, and its various properties come from Theorem 3.34, Corollary 3.38 and Theorem 3.41. □

(3.43) Theorem. *Any locally Lipschitz vector field v on a compact manifold*

X has a unique integral flow ϕ, which is locally Lipschitz. If v is C^r $(r \geq 1)$ then ϕ is C^r.

Proof. For all $x \in X$, there is, by Theorem 3.22, a local integral at x. Let this be $\phi_x : I_x \times U_x \to X$, where $I_x = [-a_x, a_x]$, with $a_x > 0$, and U_x is an open neighbourhood of x in X. Choose a finite subcovering $U_{x_1}, \ldots, U_{x_n}$ of the open covering $\{U_x : x \in X\}$ of X, and let $a = \min\{a_{x_1}, \ldots, a_{x_n}\}$. Then any point x of X is in some U_{x_i}, and the corresponding ϕ_{x_i} provides an integral curve at x with domain $[-a_{x_i}, a_{x_i}]$, and hence, by restriction, one with domain $[-a, a]$. The result now follows from Lemma 3.42. □

(3.44) Exercise. One does not normally expect a vector field on a non-compact manifold to have an integral flow, since integral curves may not be defined on the whole of $\mathbf{R}$ (see Exercise 3.40). However, we can remedy this situation without affecting the orbit structure as follows. Let X be a (paracompact) manifold admitting C^r partitions of unity $(r \geq 1)$. Let v be a C^r vector field on X. Prove that for some positive C^r function $\alpha : X \to \mathbf{R}$, the vector field $x \to \alpha(x)v(x)$ has an integral flow $\phi : \mathbf{R} \times X \to X$ with the same orbits as v. (Here the *orbit* of v *through* x is defined to be the image of the integral curve of v at x with maximal domain.) A solution to this problem may be found in Renz [1].

By virtue of the correspondence between velocity vector fields and integral flows, the qualitative theories of smooth vector fields and smooth flows are one and the same subject. For example we define two smooth vector fields to be *flow equivalent* if their integral flows are flow equivalent and *topologically equivalent* if, after the modification in Exercise 3.44 if necessary, they have topologically equivalent integral flows. It is sometimes easier to describe the theory in terms of vector fields, sometimes in terms of flows. We are free to use whichever terminology we find the more convenient in any given situation.

Appendix 3

I. INTEGRALS OF PERTURBED VECTOR FIELDS

We have already found spaces of maps to be a great technical convenience, and we now put them to a new use. We discuss the map that takes vector fields, regarded as points in a map space, to their local integrals, similarly interpreted. The properties of this map are clearly of considerable interest. For example, its continuity at a point (= vector field) corresponds to small perturbations in the vector field producing small perturbations in the integral. In this context, the precise nature of a "small perturbation" depends, of course, on the topology that we put on the map spaces. We do not attempt a comprehensive treatment of the subject, but give instead a couple of typical theorems that can be proved along these lines.

Suppose that notations are as in Theorem 3.22. For the purposes of the following theorem $N^r(r \geq 0)$ denotes the subset of $C^r(B', \mathbf{E})$ consisting of Lipschitz maps h with $|h|_0 < ((d'-d)/a) - |f|_0$ when $d' < \infty$. Thus if f is C^r and $h \in N^r$, $f+h$ has, by Theorem 3.32 a C^r integral $\psi: I \times B \to B'$, say.

(3.45) Theorem. *Suppose that f is $C^r(r \geq 0)$, with Df C^{r-1}-bounded if $r \geq 1$. Then $g^h - g^0$ is C^r-bounded, where $g^h : B \to C^0(I, B')$ denotes the map taking x to ψ_x. If, further, f is uniformly C^r then the map $\alpha : N^r \to C^r(B, C^0(I, \mathbf{E}))$ taking h to $g^h - g^0$ is continuous at 0. In this case, for all $t \in I$, the map $\beta : N^r \to C^r(B, \mathbf{E})$ taking h to $\psi^t - \phi^t$ is continuous at 0.*

Proof. We may assume that the Lipschitz constant of $f+h$ is strictly less than $1/a$ (otherwise we apply the technique of Exercise 3.24). We define a map

$$\chi: N^r \times B \times C^0(I, B') \to C^0(I, \mathbf{E})$$

by

$$\chi(h, x, \gamma)(t) = x + \int_0^t (f+h)\gamma(u)\, du,$$

for $t \in I$. For each $h \in N^r$, we have, as in the proof of Theorem 3.22, a fixed point map $g^h : B \to C^0(I, B')$. Now $\chi^h - \chi^0$ is bounded by $a|h|_0$, and $D\chi^h$ is C^{r-1}-bounded, by Corollary B.11. Thus, by Theorem C.10, $g^h - g^0$ is C^r-bounded. We may express $\chi^h - \chi^0$ as a composite

$$B \times C^0(I, B') \longrightarrow C^0(I, B') \xrightarrow{h_*} C^0(I, \mathbf{E}) \longrightarrow C^0(I, \mathbf{E}),$$

where the first map is the product projection, and the third is the continuous linear integration map sending γ to $(t \mapsto \int_0^t \gamma(u)\, du)$. By Corollary B.12, the map for N^r to $C^r(C^0(I, B'), C^0(I, \mathbf{E}))$ taking h to h_* is continuous linear. Hence by Lemmas B.4 and B.13, so is the map from N^r to

$$C^r(B \times C^0(I, B'), C^0(I, \mathbf{E}))$$

taking h to $\chi^h - \chi^0$. Suppose now that f is uniformly C^r. Then χ^0 is uniformly C^r, by Theorem B.10. Thus we may apply Theorem C.10 to deduce that α is continuous at 0. So also is $\beta = (ev^t)_* \alpha$, where

$$ev^t : C^0(I, B') \to B'$$

is the continuous linear evaluation map.

The situation with purely Lipschitz perturbations is not so clear. However, we shall only make use of the following results, which gives continuity at f of the operation of taking local integrals in the case when f is linear.

(3.46) Theorem. *Let f be a continuous linear endomorphism of $\mathbf{E}$ and let $h: V \to \mathbf{E}$ be Lipschitz, where V is open in $\mathbf{E}$. Suppose that $\phi: I \times U \to V$ and $\psi: I \times U \to V$ are local integrals of f and $f + h$ respectively. Then, for all $t \in I$, $\psi^t - \phi^t$ is Lipschitz, with a constant that tends to zero with the Lipschitz constant of h.*

Proof. We may suppose that $t > 0$ (otherwise reverse the vector fields). Let θ^t denote $\psi^t - \phi^t$. For all $x, x' \in U$,

$$\begin{aligned}|\theta^t(x) - \theta^t(x')| &= \left| \int_0^t ((f+h)\psi^u(x) - f\phi^u(x) - (f+h)\psi^u(x') + f\phi^u(x'))\, du \right| \\ &\leq \kappa \int_0^t |\theta^u(x) - \theta^u(x')|\, du + \lambda \int_0^t |\psi^u(x) - \psi^u(x')|\, du,\end{aligned}$$

where κ and λ are positive Lipschitz constants for f and h respectively. Let $\mu = \kappa + \lambda$. Then μ is a Lipschitz constant for $f + h$, and thus, by Exercise 3.30, ψ^u is Lipschitz with constant $e^{\mu u}$. Thus

$$|\theta^t(x) - \theta^t(x')| - \kappa \int_0^t |\theta^u(x) - \theta^u(x')|\, du \leq \mu^{-1}(e^{\mu t} - 1)\lambda\, |x - x'|.$$

Integrating this inequality (see Exercise 3.30 again), we obtain

$$|\theta^t(x)-\theta^t(x')|\leq \mu^{-1}\left(\frac{\mu e^{\mu t}-\kappa e^{\kappa t}}{\mu-\kappa}-e^{\kappa t}\right)\lambda|x-x'|=(e^{\mu t}-e^{\kappa t})|x-x'|,$$

and so θ^t is Lipschitz with constant $e^{\mu t}-e^{\kappa t}$. The proof is now complete, since $\mu\to\kappa$ as $\lambda\to 0$. □

II. FIRST INTEGRALS

Let v be a vector field on a differentiable manifold X and let $g: X\to R$ be a C^r function ($r\geq 1$). The *derivative of* g in *the direction of* v is the C^{r-1} function $L_v g$ defined by

$$Tg(v(x))=(g(x), L_v g(x)).$$

If γ is an integral curve of v at x, then $L_v g(x)$ is, by the chain rule, the derivative with respect to t at $t=0$ of $f\gamma(t)$. In fact, it is the rate of change of f at x at any time t as we move along an integral curve through x with the prescribed velocity v. If X has a given Riemannian structure $\langle\ ,\ \rangle$ then the gradient vector field ∇g of g is defined as in Example 3.3, and in terms of this $L_v g(x)=\langle v(x), \nabla g(x)\rangle$. Thus, if X is Euclidean space $\mathbf{R}^n$,

$$L_v g(x)=v_1(x)\frac{\partial g}{\partial x_1}+\cdots+v_n(x)\frac{\partial g}{\partial x_n}.$$

This explains the notation

$$v_1(x)\frac{\partial}{\partial x_1}+\cdots+v_n(x)\frac{\partial g}{\partial x_n}$$

sometimes used for a vector field on $\mathbf{R}^n$. The notation is justifiable since we can recover v from L_v by applying L_v to the coordinate functions $x\mapsto x_i$. In fact, the notion of a vector field is quite commonly defined in terms of the directional derivative property (see, for example, Warner [1]).

A *first integral* of v is a C^1 function $g: X\to\mathbf{R}$ such that $L_v g=0$, the zero function. Thus a first integral is constant along any integral curve of v. Of course all constant functions on X are trivially first integrals of any vector field X. For many vector fields these are the only first integrals. When a non-constant first integral g does exist, it is of considerable interest. The reason for this is that, by Sard's theorem the level sets of g are usually submanifolds of X of codimension 1 (see Theorem 7.3 and Example A.7, below, and Hirsch [1]). Since the integral curve at any point lies in the level set through that point, $v(x)$ is tangential to the submanifold at every point x. Thus for each non-critical value of g, we obtain a vector field on a manifold of dimension 1 less than X (assuming X to be finite dimensional). Since this ought to be easier to integrate than the original vector field v on X, we have made a reasonable first attempt at integrating v, hence the name *first integral* given to g.

(3.47) Exercise. Prove that the vector field on $\mathbf{R}$ with principal part $f(x) = x$ has no non-constant first integral.

(3.48) Exercise. Find a non-constant first integral for the vector field on $\mathbf{R}^2$ with principal part (i) $f(x, y) = (x, -y)$, (ii) $f(x, y) = (-y, x)$.

(3.49) Exercise. Prove that $g: \mathbf{R}^2 \to \mathbf{R}$ defined by $g(x, y) = x^2 + y^2$ is a first integral for the vector fields on $\mathbf{R}^2$ with principal part

(i) $f_1(x, y) = (-y(x^2 + y^2), x(x^2 + y^2))$,
(ii) $f_2(x, y) = (-xy^2, x^2y)$,
(iii) $f_3(x, y) = (-x^2y^3, x^3y^2)$,
(iv) $f_4(x, y) = (x^3y^4, -x^4y^3)$.

Sketch the phase portraits of the vector fields. Are any of the vector fields topologically equivalent?

(3.50) Exercise. Let $U = \{(x, y) \in \mathbf{R}^2 : x \neq 0\}$. Prove that the function $f: U \to \mathbf{R}$ defined by $f(x, y) = (x^2 + y^2)/x$ is a first integral for the vector field on $\mathbf{R}^2$ with principal part $v(x, y) = (2xy, y^2 - x^2)$. Sketch the phase portrait of the vector field.

Prove that the function $g = \mathbf{R}^3 \to \mathbf{R}$ defined by $g(x, y, z) = x^2 + y^2 + z^2$ is a first integral of the vector field on $\mathbf{R}^3$ with principal part

$$w(x, y, z) = 2xyz, (y^2 - x^2)z, -(x^2 + y^2)y).$$

By comparing v and w, write down another non-constant first integral $h: U \times \mathbf{R} \to \mathbf{R}$ of w. Sketch the phase portrait of w on the hemisphere $x^2 + y^2 + z^2 = 1$, $x \geq 0$, as viewed from a point $(a, 0, 0)$ for large positive a.

(3.51) Example. The Hamiltonian $H = T + V$ is a first integral of the Hamiltonian vector field of conservative mechanics (see Example 3.19). The *principle of conservation of energy* is the observation that H is constant along integral curves of the vector field.

Although a vector field may have no non-constant global first integrals, it always has non-constant *local first integrals* in the neighbourhood of any non-singular point. This is an easy consequence of the rectification theorem (Theorem 5.8 below).