APPENDIX B

Map Spaces

In proving theorems about dynamical systems in the text of the book we have often applied the contraction mapping theorem to map spaces. Typically, we have had a collection of contractions indexed by some parameter. We have deduced smoothness results for the original dynamical systems by showing that the fixed point of the contraction depends smoothly on the parameter. This technique requires familiarity with the differential calculus in map spaces, and we outline some of the theory below. The proofs are rather straightforward and uninteresting, the main problem being how to organize the results in the least indigestible way. We give one approach; the reader may prefer a more economical treatment that appears in Franks [1].

The basic theorem is to the effect that if $g: Y \to Z$ is a C^{r+s} map then the map $g_*: C^r(X, Y) \to C^r(X, Z)$ defined by $g_*(f) = gf$ is C^s, where $C^r(X, Y)$ is the space of C^r maps from X to Y. For more intricate applications we need results of the form that the map $\text{comp}: C^r(X, Y) \times C^{r+s}(Y, Z) \to C^r(X, Z)$ sending (f, g) to gf is C^s. Unfortunately, it does not seem to be practicable to develop the theory in a sufficiently general context to cover all applications simultaneously. One needs, in the proofs, a certain element of uniformity in some of the maps concerned, and this can be introduced in various ways. For example, one can make certain spaces compact, or others finite dimensional, or one can restrict one's attention to spaces of uniform maps. Trying to cope with all these tactics simultaneously would further complicate what is already a not particularly appealing piece of theory. We prefer to stick to one approach in our exposition and to indicate possible modifications, some of which are actually applied in the text.

We first deal with maps between Banach spaces, and later come on to the more general theory of sections of vector bundles. In the final section we define topologies for spaces of dynamical systems on a compact manifold. As we have previously commented, these spaces can actually be given Banach manifold structures, but we do not do so here.

Throughout $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, etc. are real Banach spaces, with norm written $| \ |$.

I. SPACES OF SMOOTH MAPS

Let X be any set. The set $C_b(X, \mathbf{F})$ of all bounded maps from X to $\mathbf{F}$ inherits a vector space structure from $\mathbf{F}$. That is to say, we define $f+g$ and αf by

$$(f+g)(x)=f(x)+g(x), \qquad (\alpha f)(x)=\alpha f(x)$$

where $f, g \in C_b(X, \mathbf{F})$, $x \in X$ and $\alpha \in \mathbf{R}$. We define a norm $|\ |_0$ on $C_b(X, F)$ by

$$|f|_0=\sup\{|f(x)|: x \in X\},$$

and this makes $C_b(X, \mathbf{F})$ into a Banach space. If X is a topological space, the subspace $C^0(X, \mathbf{F})$ of all bounded continuous maps is closed in $C_b(X, \mathbf{F})$ and hence is itself a Banach space.

The vector space $L(\mathbf{E}, \mathbf{F})$ of all continuous linear maps from $\mathbf{E}$ to $\mathbf{F}$ has a Banach space structure with norm $|\ |$ defined by

$$|T|=\sup\{|T(x)|: |x| \leqslant 1\}=\sup\{|T(x)|/|x|: x \neq 0\}.$$

If X is an open subset of $\mathbf{E}$, we say that $f: X \to \mathbf{F}$ is *(Frechet) differentiable* at $x \in X$ if, for some map $T \in L(\mathbf{E}, \mathbf{F})$

$$|f(x+h)-f(x)-T(h)|=\mathrm{o}(|h|)$$

as $h \to 0$. If T exists it is unique, and we call it $Df(x)$, the *differential* of f at x. If $Df(x)$ exists for all $x \in X$, we say that f is *differentiable*. The map $Df: X \to L(\mathbf{E}, \mathbf{F})$ is called the *derivative* of f. Higher derivatives are defined inductively by $D^r f=D(D^{r-1} f)$. We say that f is C^0 if f is continuous, C^r if $D^r f$ is continuous and C^∞ if f is C^r for all $r \geqslant 0$. Strictly speaking $D^r f$ is a map from X to $L(\mathbf{E}, L(\mathbf{E}, \ldots L(\mathbf{E}, \mathbf{F}) \ldots))$, but, as usual, we identify the latter space with the space $L_r(\mathbf{E}, \mathbf{F})$ of all r-linear maps from $\mathbf{E}^r$ to $\mathbf{F}$, putting $S=T$ when $S(x_1)(x_2) \ldots (x_r)=T(x_1, x_2, \ldots, x_r)$ for all $(x_1, x_2, \ldots, x_r) \in \mathbf{E}^r$. Thus $L_r(\mathbf{E}, \mathbf{F})$ has norm $|\ |$ given by

$$|T|=\sup\{|T(x_1, \ldots, x_r)|: |x_i| \leqslant 1, 1 \leqslant i \leqslant r\}.$$

If f is C^r, $D^r f(x)$ is a symmetric r-linear map. Note that continuous multilinear maps are themselves C^∞.

If X is any subset of $\mathbf{E}$ with $X \subset \overline{\operatorname{int} X}$, we shall say that $f: X \to \mathbf{F}$ is C^r $(0 \leqslant r \leqslant \infty)$ if it has a C^r extension $\bar{f}$ to some open neighbourhood of X in $\mathbf{E}$. In this case the differentials $D^i f(x)$, $1 \leqslant i \leqslant r$ at points x of the frontier ∂X of X are independent of the choice of the extension $\bar{f}$, and we define $D^i f(x)$ to be $D^i \bar{f}(x)$ for such x. *We make the standing assumption that whenever a map $f: X \to Y$ is said or implied to be differentiable, its domain X always satisfies $X \subset \overline{\operatorname{int} X}$.*

If X is a subset of $\mathbf{R}$, one commonly writes $Df(x)$ or $f'(x)$ instead of $Df(x)(1)$. The context decides whether $Df(x)$ is an element of $\mathbf{F}$ or $L(\mathbf{R}, \mathbf{F})$.

This remark also applies to partial derivatives $D_i f(x)(=\partial f/\partial x_i)$ when $\mathbf{E}$ is a product of Banach spaces $\mathbf{E}_1 \times \cdots \times \mathbf{E}_n$ and the ith factor $\mathbf{E}_i$ is $\mathbf{R}$. Recall that the jth *partial derivative* of f at $a=(a_1,\ldots,a_n)\in X$ is defined to be the differential of the partial map

$$x_j \mapsto f(a_1,\ldots,a_{j-1},x_j,a_{j+1},\ldots,a_n)$$

at a_j. It is an element of $L(\mathbf{E}_j,\mathbf{F})$.

The C^r map $f: X\to \mathbf{F}$ is *C^r-bounded* if the number

$$|f|_r = \sup\{|D^i f(x)|: x\in X, 0\leq i\leq r\}$$

exists and is finite.

(B.1) Exercise. Prove that if $f: X\to\mathbf{F}$ and $g:\mathbf{F}\to\mathbf{G}$ are maps with Df and Dg C^r-bounded then $D(gf)$ is C^r-bounded. Prove that if, in addition g is C^0-bounded then gf is C^{r+1}-bounded.

The set $C^r(X,\mathbf{F})$ of all C^r-bounded maps from X to $\mathbf{F}$ has the vector space structure of $C^0(X,\mathbf{F})$, and we give it the norm $|\ |_r$. In some cases, $C^r(X,\mathbf{F})$ inherits completeness from $\mathbf{F}$; we need to know this in the following cases:

(B.2) Exercise. Prove that if X is open in E then $C^r(X,\mathbf{F})$ is a Banach space. Prove that if I is a compact real interval then $C^1(I,\mathbf{F})$ is a Banach space. (This is basically a well known theorem on uniform convergence; see, for example (8.6.4) of Dieudonne [1]).

When Y is a subset of $\mathbf{F}$, $C^r(X, Y)$ is (identified with) the subset of $C^r(X,\mathbf{F})$ consisting of maps taking values in Y. If Y is closed in $\mathbf{F}$ then $C^r(X, Y)$ is closed in $C^r(X,\mathbf{F})$. If X is compact and Y is open in F, then $C^r(X, Y)$ is open in $C^r(X,\mathbf{F})$. If $X=Y$, we abbreviate $C^r(X,X)$ to $C^r(X)$.

If Z is a subset of X, we say that $f: X\to\mathbf{F}$ is *uniformly C^r at Z* ($r\geq 0$) if given $\varepsilon>0$ there exists $\delta>0$ such that, for all $x\in X$ and for all $z\in Z$ with $|x-z|<\delta$, $\sup\{|D^i f(x)-D^i f(z)|: 0\leq i\leq r\}<\varepsilon$. We say that f is uniformly C^r if it is uniformly C^r at X. We denote by $UC^r(X,\mathbf{F})$ the closed subspace of $C^r(X,\mathbf{F})$ consisting of all uniformly C^r maps.

(B.3) Exercise. Let $f: X\to Y$ and $g: Y\to\mathbf{G}$ be uniformly C^r maps. Prove that if Df and Dg are C^{r-1}-bounded then the composite gf is uniformly C^r.

II. COMPOSITION THEOREMS

Most of the following theorems are proved by induction. When the integer inducted upon is the degree of smoothness r of the map space $C^r(\ ,\ )$ concerned, the inductive step always depends upon the trivial relation

$|f|_{k+1} = \max\{|f|_k, |Df|_k\}$ for $0 \leq k < r$. Since these proofs run to a pattern, we tend to cut down on the detail. Throughout Y is a subspace of $\mathbf{F}$, and X is a topological space if $r = 0$ and a subset of $\mathbf{E}$ if $r > 0$. The first two lemmas are special cases of a general result about left composition with a continuous multilinear map.

(B.4) Lemma. *If $T \in L(\mathbf{F}, \mathbf{G})$ then, for all $f \in C^r(X, \mathbf{F})$, $|Tf|_r \leq |T|\,|f|_r$. Thus $T_*(f) = Tf$ defines a continuous linear map $T_*: C^r(X, \mathbf{F}) \to C^r(X, \mathbf{G})$.*

Proof. By induction on r.

$r = 0$. $|Tf|_0 \leq |T|\,|f|_0$, trivially.

Inductive step. Assume the inequality holds for $r = k$. Let $\tau: L(\mathbf{E}, \mathbf{F}) \to L(\mathbf{E}, \mathbf{G})$ be the continuous linear map $S \mapsto TS$. Then $D(Tf) = \tau Df$, and so, by hypothesis, $|D(Tf)|_k \leq |\tau|\,|Df|_k = |T|\,|Df|_k$. Combined with $|Tf|_k \leq |T|\,|f|_k$, this gives $|Tf|_{k+1} \leq |T|\,|f|_{k+1}$. □

(B.5) Lemma. *If $B: \mathbf{F} \times \mathbf{G} \mapsto \mathbf{H}$ is continuous bilinear, then for all $f \in C^r(X, \mathbf{F})$ and $g \in C^r(X, \mathbf{G})$ the map $B_*(f, g) \in C^r(X, \mathbf{H})$ defined by $B_*(f, g)(x) = B(f(x), g(x))$ satisfies*

$$|B_*(f, g)|_r \leq 2^r |B|\,|f|_r |g|_r.$$

Thus $B_*: C^r(X, \mathbf{F}) \times C^r(X, \mathbf{G}) \to C^r(X, \mathbf{H})$ is continuous bilinear.

Proof. By induction on r.

Inductive step. This uses $|D(B_*(f, g)|_k = |\beta_*(f, Dg) + \gamma_*(Df, g)|_k$ where $\beta: \mathbf{F} \times L(\mathbf{E}, \mathbf{G}) \to L(\mathbf{E}, \mathbf{H})$ and $\gamma: L(\mathbf{E}, \mathbf{F}) \times \mathbf{G} \to L(\mathbf{E}, \mathbf{H})$ are the continuous bilinear maps $\beta(y, T) = (x \mapsto B(y, T(x))$ and $\gamma(S, z) = (x \mapsto B(S(x), z))$. Note that $|\beta| = |\gamma| = |B|$. □

We are particularly concerned with the case when B is the composition map $B: L(\mathbf{F}, \mathbf{G}) \times L(\mathbf{G}, \mathbf{H}) \to L(\mathbf{F}, \mathbf{H})$ taking (S, T) to TS. Note that $|B| = 1$. In this case we simplify notation by writing $g \cdot f$ for the so-called *compositional product* $B_*(f, g)$ of $f \in C^r(X, L(\mathbf{F}, \mathbf{G}))$ and $g \in C^r(X, L(\mathbf{G}, \mathbf{H}))$. Thus Lemma B.5 becomes

(B.6) $$|g \cdot f|_r \leq 2^r |f|_r |g|_r.$$

The space $L(\mathbf{R}, \mathbf{G})$ is commonly identified with $\mathbf{G}$, equating maps with their value at 1, and in this case B becomes the *evaluation map* $(x, g) \mapsto g(x)$.

(B.7) Lemma. *For all $r \geq 0$, there is a constant A (independent of X, Y, $\mathbf{G}$) such that, for all $f: X \to Y$ with Df C^{r-1}-bounded and for all $g \in C^r(Y, \mathbf{G})$,*

$$|gf|_r \leq A|g|_r M_r(f),$$

where $M_r f = \max\{1, (|Df|_{r-1})^r\}$.

Proof. By induction on r.

$r=0$. $|gf|_0 \leq |g|_0$.

Inductive step. $|D(gf)|_k = |(Dg)f \cdot Df|_k \leq 2^k |(Dg)f|_k |Df|_k$ by (B.6). □

If X is compact and $g: Y \to \mathbf{G}$ is C^r then g induces the *left composition map* $g_*: C^r(X, Y) \to C^r(X, G)$ defined by $g_*(f) = gf$. This map may also exist in other circumstances, for example if g is C^r-bounded (by Lemma B.7).

(B.8) Lemma. *Let X be compact and $g: Y \to \mathbf{G}$ be C^r. Then $g_*: C^r(X, Y) \to C^r(X, \mathbf{G})$ is continuous. If g is uniformly continuous then $g_*: C^0(X, Y) \to C^0(X, \mathbf{G})$ is uniformly continuous.*

Proof. By induction on r.

$r=0$. Let $f_0 \in C^0(X, Y)$. Then g is uniformly continuous at the compact subset $f_0(X)$. Thus $|g_*(f) - g_*(f_0)|_0$ is small for f C^0-near f_0. Moreover if g is uniformly continuous, $|g_*(f) - g_*(f_0)|_0$ is uniformly small (as f_0 varies).

Inductive step. For all $f, f_0 \in C^{k+1}(X, Y)$, where $0 \leq k \leq r$,

$$\begin{aligned} |D(g_*(f) - g_*(f_0))|_k &= |(Dg)_*(f) \cdot Df - (Dg)_*(f_0) \cdot Df_0|_k \\ &\leq 2^k(|(Dg)_*(f) - (Dg)_*(f_0)|_k |Df|_k \\ &\quad + |(Dg)_*(f_0)|_k |Df - Df_0|_k). \end{aligned}$$

□

(B.9) Exercise. Prove that if g is uniformly C^r and Dg is C^{r-1}-bounded then $g_*: C^r(X, Y) \to C^r(X, \mathbf{G})$ exists and is uniformly continuous at subsets $\mathscr{F}$ of $C^r(X, Y)$ such that $\sup\{|Df|_{r-1} : f \in \mathscr{F}\} < \infty$.

We now come to a result dealing with smoothness of the map g_*. In the applications the domain $C^r(X, Y)$ of g_* is usually open (e.g. if $Y = \mathbf{F}$, or if X is compact and Y open). However, occasionally it is not. In order to have $C^r(X, Y) \subset \overline{\operatorname{int} C^r(X, Y)}$ (to fit in with our standing assumption) we need some other set of conditions on X and Y. The only one that need concern us here is X compact and Y a closed ball in $\mathbf{F}$.

(B.10) Theorem. *Let X be compact, Y be open or a closed ball, and $g: Y \to \mathbf{G}$ be C^{r+s}. Then $g_*: C^r(X, Y) \to C^r(X, \mathbf{G})$ is C^s, with Dg_* given, for all $f \in C^r(X, Y)$ and $\eta \in C^r(X, \mathbf{F})$ by*

$$Dg_*(f)(\eta) = (Dg)f \cdot \eta.$$

If g is uniformly C^s, and $r = 0$, then g_* is uniformly C^s.

Proof. We first prove that if $s \geq 1$ g_* is differentiable, by induction on r. We may assume that Y is open (otherwise g has a C^{r+1} extension to an open neighbourhood Y' of Y, inducing an extension of g_* to $C^r(X, Y')$).

$r=0$. Let $f_0 \in C^0(X, Y)$. For all sufficiently C^0-small $\eta \in C^0(X, \mathbf{F})$ (i.e. such that, for all $x \in X$, the line segment $L_x = [f_0(x), f_0(x)+\eta(x)]$ is in Y),

$$\begin{aligned}
&|g_*(f_0+\eta)-g_*(f_0)-(Dg)f_0 \cdot \eta|_0 \\
&\quad = \sup\{|g(f_0(x)+\eta(x))-gf_0(x)-D_g(f_0(x))(\eta(x))| : x \in X\} \\
&\quad \leqslant |\eta|_0 \sup\{|Dg(y)-Dg(f_0(x))| : y \in L_x, x \in X\}
\end{aligned}$$

by the mean value theorem (Corollary 2 in § 4 of Chapter 5 of Lang [2]). Since Dg is uniformly continuous at $f_0(X)$, the right-hand side is $0(|\eta|_0)$ as $|\eta|_0 \to 0$.

Inductive step. Let $f_0 \in C^{k+1}(X, Y)$, for $0 \leqslant k < r$. For all C^{k+1}-small $\eta \in C^{k+1}(X, \mathbf{F})$,

$$\begin{aligned}
&|D(g_*(f_0+\eta)-g_*(f_0)-(Dg)f_0 \cdot \eta)|_k \\
&\quad = |Dg(f_0+\eta) \cdot (Df_0+D\eta)-(Dg)f_0 \cdot Df_0 \\
&\qquad -(D^2g)f_0 \cdot \eta \cdot Df_0-(Dg)f_0 \cdot D\eta|_k \\
&\quad \leqslant 2^k(|(Dg)_*(f_0+\eta)-(Dg)_*(f_0)-(D^2g)f_0 \cdot \eta|_k|Df_0+D\eta|_k \\
&\qquad +2^{2k}|D^2g)f_0|_k|\eta|_k|D\eta|_k).
\end{aligned}$$

The right-hand side is $0(|\eta|_{k+1})$ as $|\eta|_{k+1} \to 0$.

The remaining results follow by induction on s. The case $s=0$ is Lemma B.8. If the theorem holds for $s=k$ and g is C^{r+k+1} then Dg_* is clearly C^k since it is $(Dg)_*$ followed by the continuous linear map λ from $C^r(X, L(\mathbf{F}, \mathbf{G}))$ to $L(C^r(X, \mathbf{F}), C^r(X, \mathbf{G}))$ that takes ζ to $(\eta \mapsto \zeta \cdot \eta)$. Similarly for the uniformity result. □

(B.11) Corollary. *Given $r \geqslant 0$ and $s \geqslant 1$, there exists a constant A (independent of X, Y and $\mathbf{G}$) such that if X and Y are as above, $f \in C^r(X, Y)$ and g is c^{r+s} with Dg C^{r+s-1}-bounded, then*

$$|D^s g_*(f)| \leqslant A|Dg|_{r+s-1}M_r(f),$$

where, again, $M_r(f) = \max\{1, (|Df|_{r-1})^r\}$.

Proof. By induction on s.

$$s=1. \quad |Dg_*(f)| = \sup\{|(Dg)f \cdot \eta|_r/\eta|_r : \eta \neq 0 \in C^r(X, \mathbf{F})\}$$
$$\leqslant 2^r B|Dg|_r M_r(f) \qquad \text{by Lemma B.7.}$$

Inductive step.
$$\begin{aligned}
|D^{k+1}g_*(f)| &= |D^k(\lambda(Dg)_*)(f)| \qquad (\lambda \text{ as in Theorem B.10}) \\
&\leqslant |\lambda||D^k(Dg)_*(f)| \\
&\leqslant 2^k|D^k(Dg)_*(f)|.
\end{aligned}$$
□

(B.12) Corollary. *Let X and Y be as above. For all C^s maps g with Dg C^{s-1}-bound the map $g_*: C^0(X, Y) \to C^0(X, \mathbf{G})$ is C^s-bounded. Moreover the map from $C^s(Y, \mathbf{G})$ to $C^s(C^0(X, Y), C^0(X, \mathbf{G}))$ taking g to g_* is continuous linear.* □

The above theory is sufficient for most applications. However, we now move on to smoothness of the map $\text{comp}: C^r(X, Y) \times C^{r+s}(Y, \mathbf{G}) \to C^r(X, \mathbf{G})$ defined by $\text{comp}(f, g) = gf$, and we approach this via its partial derivatives. We have already dealt with differentiability of the partial map g_*. We also have to consider *right composition* maps of the form $f^*: C^{r+s}(Y, \mathbf{G}) \to C^r(X, \mathbf{G})$, where $f \in C^r(X, Y)$ and $f^*(g) = gf$. In fact, such maps are defined for all C^r maps f such that Df is C^{r-1}-bounded, by Lemma B.7. Since they are trivially linear, Lemma B.7 also gives:

(B.13) Lemma. *The maps f^* are continuous linear.* □

However, this remark is not sufficient for our purposes. We now know that $Df^*: C^{r+s}(Y, \mathbf{G}) \to L(C^{r+s}(Y, \mathbf{G}), C^r(X, \mathbf{G}))$ is the constant map with value $(\zeta \mapsto \zeta f)$. We need to know how smoothly this value depends on f. The necessary result is:

(B.14) Lemma. *Let X be compact and Y be open or a closed ball. Then for $s \geqslant 1$, the map $\theta: C^r(X, Y) \to L(C^{r+s}(Y, \mathbf{G}), C^r(X, \mathbf{G}))$ defined by $\theta(f) = (\zeta \mapsto \zeta f)$ is C^{s-1}.*

Proof. By induction on s. We may assume that Y is open in $\mathbf{F}$, and so $C^r(X, Y)$ is open.

$s = 1$. Let $f_0 \in C^r(X, Y)$. For all sufficiently C^r-small $\eta \in C^r(X, \mathbf{F})$,

$$\begin{aligned}|\theta(f_0 + \eta) - \theta(f_0)| &= |\zeta \mapsto (\zeta(f_0 + \eta) - \zeta f_0)| \\ &= \sup\{|\zeta_*(f_0 + \eta) - \zeta_*(f_0)|_r / |\zeta|_{r+1} : \zeta \neq 0 \in C^{r+1}(Y, \mathbf{G})\} \\ &\leqslant \sup\{|D\zeta_*(f_0 + t\eta)||\eta|_r / |\zeta|_{r+1} : 0 \leqslant t \leqslant 1, \zeta \neq 0\}\end{aligned}$$

we have by the mean value theorem applied to ζ_*. By Corollary B.11, $|D\zeta_*(f_0 + t\eta)| \leqslant A|\zeta|_{r+1} M_r(f_0 + t\eta)$, and since $M_r(f_0 + t\eta)$ is bounded for C^r-small η, $\theta(f_0 + \eta) \to \theta(f_0)$ as $\eta \to 0$.

Inductive step. We assume that the theorem holds for $s = k \geqslant 1$ and prove it for $s = k + 1$. Let $f_0 \in C^r(X, Y)$ and $\eta \in C^r(X, \mathbf{F})$. We assert that θ is differentiable with derivative given by $D\theta(f_0)(\eta) = (\zeta \mapsto (D\zeta) f_0 \cdot \eta)$. This is because, for sufficiently C^r-small η,

$$\begin{aligned}&|\theta(f_0 + \eta) - \theta(f_0) - (\zeta \mapsto (D\zeta) f_0 \cdot \eta)| \\ &\quad = |\zeta \mapsto (\zeta_*(f_0 + \eta) - \zeta_*(f_0) - D\zeta_*(f_0)(\eta))|\end{aligned}$$

$$= \sup\{|\zeta_*(f_0+\eta)-\zeta_*(f_0)-D\zeta_*(f_0)(\eta)|_r/|\zeta|_{r+k+1} : \zeta \neq 0\}$$
$$\leqslant \sup\{|D\zeta_*(f_0+t\eta)-D\zeta_*(f_0)||\eta|_r/|\zeta|_{r+k+1} : 0\leqslant t\leqslant 1, \zeta\neq 0\}$$
$$\leqslant \sup\{|D^2\zeta_*(f_0+ut\eta)|t(|\eta|_r)^2/|\zeta|_{r+k+1} : 0\leqslant t\leqslant 1, 0\leqslant u\leqslant 1, \zeta\neq 0\}$$

by two applications of the mean value theorem, and this is $o(|\eta|_r)$ as $|\eta|_r\to 0$, using Corollary B.11 again. Thus $D\theta$ is the composite of $\theta: C^r(X, Y)\to L((C^{r+k}(Y, L(\mathbf{F}, \mathbf{G})), C^r(X, L(\mathbf{F}, \mathbf{G}))$, which is C^{k-1} by hypothesis, and the continuous linear map ϕ from the target of θ to

$$L(C^r(X, \mathbf{F}), L(C^{r+k+1}(Y, \mathbf{G}), C^r(X, \mathbf{G})))$$

defined by

$$\phi(T) = (\eta \mapsto (\zeta \mapsto T(D\zeta)\,.\,\eta)). \qquad \square$$

We can now prove:

(B.15) Theorem. *Let X be compact and Y be open or a closed ball. Then comp: $C^r(X, Y)\times C^{r+s}(Y, \mathbf{G})\to C^r(X, \mathbf{G})$ is C^s.*

Proof. By induction on s.

$s=0$. This uses the inequality

$$|\mathrm{comp}\,(f, g)-\mathrm{comp}\,(f_0, g_0)|_r \leqslant |(g-g_0)f|_r+|g_0f-g_0f_0|_r$$
$$\leqslant A|g-g_0|_r M_r(f)+|g_{0*}(f)-g_{0*}(f_0)|$$

and Lemma B.8.

Induction step. We know that for $s=k+1\geqslant 1$ comp has partial derivatives given by

$$D_1\,\mathrm{comp}\,(f, g) = (\eta \mapsto (Dg)f\,.\,\eta) \in L(C^r(X, \mathbf{F}), C^r(X, \mathbf{G}))$$
$$D_2\,\mathrm{comp}\,(f, g) = (\zeta \mapsto \zeta f) \in L(C^{r+s}(Y, \mathbf{G}), C^r(X, \mathbf{G})).$$

Thus D_2 Comp is C^k by Lemma B.14. We break down D_1 Comp as a composite of three maps

$$(f, g) \overset{1}{\mapsto} (f, Dg) \overset{2}{\mapsto} (Dg)f \overset{3}{\mapsto} (\eta \mapsto (Dg)f\,.\,\eta).$$

The first and third are continuous linear, and the second is comp: $C^r(X, Y)\times C^{r+k}(Y, L(\mathbf{F}, \mathbf{G})\to C^r(X, L(\mathbf{F}, \mathbf{G}))$, which is C^k by inductive hypothesis. $\square$

By putting $X=$ a singleton $\{x\}$ and identifying $C^0(X, Y)$ with Y(f with $f(x)$) we have:

(B.16) Corollary. *If Y is open or a closed ball, then the evaluation map* ev: $Y\times C^s(Y, \mathbf{G})\to \mathbf{G}$ *defined by $ev(y, g)=g(y)$ is C^2.*

We also need the following result, obtained similarly from Lemma B.14.

(B.17) Corollary. *If Y is open or a closed ball, the map* ev˙ *from Y to $L(C^s(Y, \mathbf{G})), G)$ defined by* $\mathrm{ev}^{\cdot}(y) = ev^y = (g \mapsto g(y))$ *is C^{s-1}.* □

This is as far as we wish to take the theory with X compact. Let us just recall how that hypothesis was used. Firstly, we needed it in the case $s \geqslant 1$, together with Y open or a closed ball, in order to ensure $C^r(X, Y) \subset \overline{\text{int } C^r(X, Y)}$. Secondly, we used it in Lemma B.8 and Theorem B.10 to infer that the continuous maps g and Dg were uniformly continuous at $f_0(X)$. How can we dispense with compactness of X? If $s \geqslant 1$, we can put $Y = \mathbf{F}$ so that $C^r(X, Y)$ becomes the whole space $C^r(X, \mathbf{F})$. If $\mathbf{F}$ is finite dimensional, we can enclose $f_0(X)$ in a compact set K (since f_0 is C^0-bounded) and the proof proceeds as before. Actually, if $r > 0$, we also need $\mathbf{E}$ finite dimensional, so that $L(\mathbf{E}, \mathbf{F})$ is finite dimensional, or the induction will not work. We leave it to the reader to reformulate the main theorems in this finite dimensional case.

Another possible course of action is to build in the uniformity by restricting attention to maps g that, together with their derivatives, are uniformly continuous. We have used theorems of this sort in the text, so we state them below, leaving proofs to the reader. We are able to consider a slightly more general situation than before. Let $f_0 : X \to \mathbf{F}$ be a fixed C^r map with C^{r-1}-bounded derivative Df_0.

(B.18) Theorem. *Let $g \in UC^{r+s}(\mathbf{F}, \mathbf{G})$. The map from $C^r(X, \mathbf{F})$ to $C^r(X, \mathbf{G})$ taking f to $g(f + f_0)$ is C^s (and uniformly continuous if $s = 0$). For $s > 0$ its differential at f is $(\eta \mapsto Dg(f + f_0) \,.\, \eta)$.* □

(B.19) Theorem. *The map from $C^r(X, \mathbf{F}) \times UC^{r+s}(\mathbf{F}, \mathbf{G})$ to $C^r(X, \mathbf{G})$ taking (f, g) to $g(f + f_0)$ is C^s.* □

In the text, in the course of proving the stable manifold theorem, we made use of spaces of sequences. These may be interpreted as map spaces, and so the above theory may be invoked. Perhaps we should be more explicit. Let $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ and let $Y^{\mathbf{N}}$ be the set of all sequences in Y. As before Y is a subset of the Banach space $\mathbf{F}$, so $Y^{\mathbf{N}}$ is a subset of the vector space $\mathbf{F}^{\mathbf{N}}$. We denote by $\mathscr{B}(\mathbf{F})$ the Banach space $C^0(\mathbf{N}, \mathbf{F})$ of bounded sequences in $\mathbf{F}$, and by $\mathscr{S}(\mathbf{F})$ the closed subspace consisting of convergent sequences. Notice that $\mathscr{S}(\mathbf{F})$ may be identified with $C^0(\bar{\mathbf{N}}, \mathbf{F})$, where $\bar{\mathbf{N}}$ is the one point compactification of $\mathbf{N}$. Let $\mathscr{B}(Y) = C^0(\mathbf{N}, Y)$ and $\mathscr{S}(Y) = C^0(\bar{\mathbf{N}}, Y)$.

Any bounded map (or any Lipschitz map) $g: Y \to \mathbf{G}$ induces a map $g_*: \mathscr{B}(Y) \to \mathscr{B}(\mathbf{G})$ taking γ to $g\gamma$. Similarly, any continuous map $g: Y \to \mathbf{G}$ induces $g_*: \mathscr{S}(Y) \to \mathscr{S}(\mathbf{G})$. We obtain results about these spaces by putting $r = 0, X = \bar{\mathbf{N}}$ or, when possible, $\mathbf{N}$ in the above composition theorems. In the text, it was convenient to use $\mathscr{S}_0(G)$, which is the closed subspace of $\mathscr{S}(\mathbf{G})$

consisting of series converging to 0 in $\mathbf{G}$. The relevant theorems are:

(B.20) Theorem. *If Y is open or a closed ball and if $g: Y \to \mathbf{G}$ is C^s, with $g(0) = 0$, then $g_*: \mathscr{S}_0(Y) \to \mathscr{S}_0(\mathbf{G})$ is C^s. If, further, g is uniformly C^s then so is g_*.* □

(B.21) Theorem. *If Y is open or a closed ball then the map* comp *from $\mathscr{S}_0(Y) \times C_0^s(Y, \mathbf{G})$ to $\mathscr{S}_0(\mathbf{G})$ is C^s, where* comp $(f, g) = gf$ *and $C_0^s(Y, \mathbf{G})$ is the subspace of $C^s(Y, \mathbf{G})$* consisting of maps g with $g(0) = 0$. □

Finally in Exercise 6.11 of the text, and its applications to hyperbolic closed orbits, we need the following generalization of the previous pair of theorems:

(B.22) Exercise. Let α be a positive real number, and let $\tilde{\alpha} : \mathbf{F}^{\mathbf{N}} \to \mathbf{F}^{\mathbf{N}}$ be the isomorphism taking γ to $(n \mapsto \alpha^n \gamma(n))$. Let ${}_{\alpha}|\ |$ denote the norm induced by $\tilde{\alpha}$ on $\tilde{\alpha}(\mathscr{B}(\mathbf{F}))$ from that of $\mathscr{B}(\mathbf{F})$ (so ${}_{\alpha}|\delta| = \sup\{\alpha^{-n}|\delta(n)|: n \in \mathbf{N}\}$). Suppose that Y is a ball with centre 0 in $\mathbf{F}$ and that $g: Y \to \mathbf{G}$ is a Lipschitz map with $g(0) = 0$. Prove that g induces a Lipschitz map $g_* : \tilde{\alpha}(\mathscr{S}_0(Y)) \to \tilde{\alpha}(\mathscr{S}_0(\mathbf{G}))$ (defined by $g_*(\gamma) = g\gamma$).

Now suppose that $\alpha \leq 1$. Show that g_* is C^s when g is C^s ($s \geq 1$) and uniformly C^s when g is uniformly C^s. Prove also that the map comp $: \tilde{\alpha}(\mathscr{S}_0(Y)) \times C^s(Y, \mathbf{G}) \to \tilde{\alpha}(\mathscr{S}_0(\mathbf{G}))$ is C^s. Investigate the situation when α is greater than 1.

III. SPACES OF SECTIONS

Let $\pi: B \to X$ be a C^r vector bundle with fibre $\mathbf{F}$. *We suppose throughout for simplicity that X is compact.* Of course for $r > 0$ X is a C^r manifold. Recall (Appendix A) that $C^r(\pi)$ (or, sometimes, $C^r(B)$) denotes the vector space of all C^r sections of π. We wish to give $C^r(\pi)$ a norm. We say that an admissible chart $\tilde{\xi} : \tilde{U} \to U' \times \mathbf{F}$ on π is a *norm chart* if it is the restriction of an admissible chart $\tilde{\eta}: \tilde{V} \to V' \times \mathbf{F}$ where $U' \subset K \subset V'$ for some compact K. A *norm atlas* is a finite atlas of norm charts. Let $\mathscr{A} = \{\tilde{\xi}_i: 1 \leq i \leq n\}$ be such an atlas for π, and let σ be a C^r section of π. Then, for each i, we have a local representative $(id, \sigma_i): U_i' \to U_i' \times \mathbf{F}$ for σ. The definition of norm chart ensures that $\sigma_i : U_i' \to \mathbf{F}$ is C^r-bounded. That is to say, $|\sigma_i|_r$ exists, as defined in the last section. We define a norm $|\ |_{\mathscr{A},r}$ on $C^r(\pi)$ by

$$|\sigma|_{\mathscr{A},r} = \max\{|\sigma_i|_r: 1 \leq i \leq n\}.$$

It is not hard to prove that, with this norm, $C^r(\pi)$ is a Banach space.

Moreover if $\mathscr{B}$ is another norm atlas for π then $|\ |_{\mathscr{A},r}$ and $|\ |_{\mathscr{B},r}$ are equivalent norms. We suppose from now on that some atlas $\mathscr{A}$ has been chosen, and we abbreviate $|\ |_{\mathscr{A},r}$ to $|\ |_r$.

(B.23) Exercise. Show that norm atlases exist, and fill in the details of the previous paragraph.

Notice that there is a canonical norm-preserving isomorphism between $C^r(U_i', \mathbf{F})$ and $C^r(\pi_i)$, where $\pi_i: U_i' \times \mathbf{F} \to U_i'$ is the trivial bundle. This points to the way in which spaces of sections generalize the map spaces of the preceding sections. The generalization is a substantial one for, whereas above we composed elements of the map space with C^{r+s} maps of $\mathbf{F}$, we now compose with maps of the total space of the vector bundle satisfying a weaker differentiability condition. Clearly we have to introduce some new condition, because it does not usually make sense to talk of a C^{r+s} map of the total space of a C^r vector bundle. The condition that we introduce occurs naturally because of the fundamental distinction between base and fibre coordinates in vector bundles.

Let $\pi: B \to X$ and $\rho: C \to X$ be C^r vector bundles, and let $g: B \to C$ be a fibre preserving map (over the identity). Thus g maps each fibre B_x to C_x, not necessarily linearly. Recall (Example A.27 of Appendix A) that the linear map bundle $L(\pi, \rho)$ is a vector bundle with base space X and fibre $L(B_x, C_x)$ over x. Its total space is denoted by $L(B, C)$. We define inductively $L^n(\pi, \rho) = L(\pi, (L^{n-1}(\pi, \rho))$. Suppose that, for all $x \in X$, the restriction $g_x: B_x \to C_x$ of g is n times differentiable. We say that g is *n times fibre differentiable* and define its nth *fibre derivative* $F^n g: B \to L^n(B, C)$ by

$$F^n g(v) = D^n g_x(v)$$

where $v \in B_x$. We say that g is of class ${}^rF^s$ if $F^n g$ exists and is C^r for $0 \leq n \leq s$. We write F^s for ${}^0F^s$. Thus ${}^rF^s$ is a stronger condition than C^r but weaker than C^{r+s}. In terms of local coordinates, ${}^rF^s$ means that all partial derivatives up to order s in the fibre direction exist and are C^r (as functions of all coordinates together, both fibre and base). We may also use this local criterion to define what we mean by a map $g: B \to Y$ being ${}^rF^s$, where B is the total space of a vector bundle and Y is any smooth manifold.

In the text we used the following results, which may be given proofs closely resembling those of analogous theorems in the last section (see Eliasson [1] and Foster [1] for details).

(B.24) Theorem. *Let $\pi: B \to X$ and $\rho: C \to X$ be C^r vector bundles and let $g: B \to C$ be ${}^rF^s$. Then $g_*: C^r(\pi) \to C^r(\rho)$ defined by $g_*(\sigma) = g\sigma$ is C^s. For*

$s \geq 1$ *its derivative* Dg_* *is defined by*

(B.25) $$Dg_*(\sigma)(\tau) = (Fg)\sigma.\tau$$

for all σ *and* τ *in* $C^r(\pi)$, *where, as in the last section,* . *denotes compositional product.* □

(B.26) Remarks. The above theorem may be modified in various ways. For example, the domain of g may be an open neighbourhood N of the image of some given section of π, rather than the whole of B. Of course g_* is then only defined on sections taking values in N. Again, we may (and in fact, in the proof of the generalized stable manifold theorem, do) replace $C^0(\pi)$ by $C_b(\pi)$, the space of bounded but not necessarily continuous sections. We may in this case prove that $g\,F^s$ implies $g_*\,C^s$, with Dg_* as in (B.25), provided that the fibres of π and σ are finite dimensional (this condition being required for uniformity arguments in the proof). Finally, if one gives the space of ${}^rF^s$ maps from B to C a natural Banach space structure, one may prove an analogue of Theorem B.15 above (see Theorem 15 of Foster [1]).

Theorem B.24 has a partial converse. This is very useful when, as in the proof of the generalized stable manifold theorem, one attempts to establish results about a subset of a manifold by applying differential calculus to the space of sections of the tangent bundle of the manifold over the subset. In this situation, one needs a tool for transferring results from the space of sections back down to the manifold again, and the following theorem (due to Foster [2]) serves this purpose:

(B.27) Theorem. *Let* $\pi: B \to X$ *and* $\rho: C \to X$ *be* C^0 *vector bundles and let* $g: B \to C$ *be a fibre preserving map. If* $g_*: C^0(\pi) \to C^0(\rho)$ *is* C^s *then* g *is* C^s.

Before proving this theorem, we establish two preliminary results. The first tells us that small open sets in the total space of a vector bundle may be embedded nicely in the space of sections.

(B.28) Lemma. *Let* $\pi: B \to X$ *be a* C^0 *vector bundle, and let* $x_0 \in X$. *There exists an open neighbourhood* U *of* x_0 *in* X *and a continuous map* $\sigma.\ \tilde{U} \to C^0(\pi)$ (*where* $\tilde{U} = \pi^{-1}(U)$) *with value at* p *denoted* σ_p *such that*

(i) *for all* $x \in U$, $\sigma.|B_x$ *is continuous linear,*

(ii) *for all* $p \in \tilde{U}$, σ_p *takes the value* p *at* $\pi(p)$,

(iii) *for all* $p \in \tilde{U}$, $|\sigma_p|_0 = |p|$ *where* $|\ \ |$ *is the norm induced on the fibres of* $\tilde{U}$ by some chart of $\mathscr{A}$.

Proof. Let $\tilde{\xi}: \tilde{W} \to W' \times \mathbf{F}$ be a norm chart in $\mathscr{A}$ with $x_0 \in W$. Choose open neighbourhoods U and V of x_0 in X with $\bar{U} \subset V$ and $\bar{V} \subset W$. Let $\lambda: W \to [0, 1]$ be a continuous function which takes the value 1 on U and 0 on $W \backslash V$

(λ exists by Urysohn's theorem). For all $p \in \tilde{U}$, define $\sigma_p \in C^0(\pi)$ by

$$\sigma_p(x) = \begin{cases} 0_x & \text{if } x \in X \backslash V \\ \tilde{\xi}^{-1}(\xi(x), \lambda(x)\tilde{\xi}(p)_2) & \text{if } x \in V. \end{cases}$$

Then $\sigma. = (p \mapsto \sigma_p)$ has the required properties. □

(B.29) Lemma. *The formula $\psi(\gamma)(\sigma) = \gamma \cdot \sigma$ defines a continuous linear map $\psi: C^0(L(\pi, \rho)) \to L(C^0(\pi), C^0(\rho))$ which is injective and has a closed image.*

Proof. We take local representatives with respect to norm charts for π and ρ and the associated chart for $L(\pi, \rho)$ (see Example A.27). In terms of these charts, ψ sends (id, δ) to $((id, \tau) \to (id, \delta \cdot \tau))$, where $\delta \in C^0(U', L(\mathbf{E}, \mathbf{F}))$, $\tau \in C^0(U', \mathbf{E})$ and $\mathbf{E}$, $\mathbf{F}$ are the fibres of π, ρ. It is now clear that ψ is well defined and continuous linear. For injectivity we need to show that if $\gamma \cdot \sigma = 0 \in C^0(\rho)$ for all $\sigma \in C^0(\pi)$ then $\gamma = 0 \in C^0(L(\pi, \rho))$. This follows providing that for any $p \in B$ there is a C^0 section $\sigma \in C^0(\pi)$ taking the value p at $\pi(p)$, and this is so by Lemma B.28. Similarly, closure of Im ψ amounts to showing that if $\sup\{|\gamma_m \cdot \sigma - \gamma_n \cdot \sigma|_0 : |\sigma|_0 \leq 1\}$ is arbitrarily small for sufficiently large m, n then (γ_n) is a Cauchy sequence, and this again uses Lemma B.28. □

Proof of Theorem B.27. We prove the theorem by induction on s. Let $\sigma. : \tilde{U} \to C^0(\pi)$ be as in Lemma B.28. We write g on $\tilde{U}$ as the composite

$$\tilde{U} \xrightarrow{(\pi, \sigma.)} U \times C^0(\pi) \xrightarrow{id \times g_*} U \times C^0(\rho) \xrightarrow{\text{ev}} C.$$

Thus continuity of g_* implies continuity of g (the evaluation map ev is continuous, essentially by Corollary B.16). If we restrict g to a single fibre B_x of U we have the composite $\text{ev}_x\, g_*\, \sigma.$, so, since ev_x and $\sigma.$ are continuous linear, differentiability of g_* implies the existence of Fg. From the definition of derivatives, we deduce that Fg satisfies the relation $Dg_* = \psi(Fg)_*$, where ψ is the map of Lemma B.29. Now assume that the theorem holds for $s = t \geq 0$ and that g_* is C^{t+1}. Then Dg_* is C^t, and hence, by the above relation and Lemma B.29, $(Fg)_*$ is C^t. Thus by the inductive hypothesis Fg is F^t. We deduce that g is F^{t+1}. □

IV. SPACES OF DYNAMICAL SYSTEMS

Let X be a compact differentiable manifold. We wish to describe a topology for the set of all C^r dynamical systems on X which takes into account all derivatives of the system up to order $r (r \geq 1)$. For vector fields on

X we already have such a topology. A C^r vector field on X is just a C^r section of the tangent bundle π_X of X, and we have seen in the last section how to give the space $C^r(\pi_X)$ a C^r norm $|\ |_r$. As we commented above, the norms corresponding to any two norm atlases are equivalent, so we may sensibly talk of the C^r *topology* on the space. We usually write $\Gamma^r(X)$ for $C^r(\pi_X)$.

The situation for the set $\text{Diff}^r(X)$ of all C^r diffeomorphisms of X is rather more complicated. The easiest way to link it with what we have done so far is to note that if X has a Riemannian metric then right composition with the exponential map $\exp: TX \to X$ takes $0 \in \Gamma^r(X)$ to the identity map $id_X \in \text{Diff}^r(X)$, and induces a bijection from a small enough neighbourhood U of 0 in $\Gamma^r(X)$ to a subset V of $\text{Diff}^r(X)$ containing id_X. We define a basic system of neighbourhoods of id_X in $\text{Diff}^r(X)$ to be the subset V as U ranges over all discs in $\Gamma^r(X)$ with centre 0 and with radius (small) $\varepsilon > 0$. We may now obtain a basic system of neighbourhoods at any $f \in \text{Diff}^r(X)$ by composition (either right or left) with f.

An equivalent and more straightforward way to define a basic system of neighbourhoods at f is as follows: Take any pair of finite atlases $\mathcal{A} = \{\xi_i : U_i \to U_i'\}$ and $\mathcal{B} = \{\eta_j : V_j \to V_j'\}$ with the property that, for all i, $f(U_i) \subset V_{j(i)}$ for some $j(i)$, and define $W_\varepsilon(f)$ to be the set of all $g \in \text{Diff}^r(X)$ such that, for all i,

$$\overline{g(U_i)} \subset V_{j(i)}$$

and

$$\sup_i |\eta_{j(i)} g \xi_i^{-1} - \eta_{j(i)} f \xi_i^{-1}|_r < \varepsilon.$$

Then $\{W_\varepsilon(f): \varepsilon > 0\}$ is a basic system of neighbourhoods of f in the C^r-topology. Yet another description of the C^r topology is in terms of jet bundles (see, for example, § 2.1 of Hirsch [1]).

We may easily define topologies for $\Gamma^\infty(X)$ and $\text{Diff}^\infty(X)$ by taking as a basis the C^r topologies for all finite r. For example U is open in $\text{Diff}^\infty(X)$ if and only if for all $f \in U$ there is an open subset V of $\text{Diff}^r(X)$, for some r, with $f \in V \subset U$.

One may now consider the case when X is finite dimensional but non-compact. There is more than one candidate for the C^r topology. See § 2.1 of Hirsch [1] for a good discussion of the *weak* and *strong* (*Whitney*) topologies. One may also extend these ideas to X with boundary, and even to infinite dimensional X.