Decay of correlations and dispersing billiards

N. Chernov
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294, USA
E-mail: chernov@vorteb.math.uab.edu

April 2, 1998

Abstract

Dispersing billiards (or Sinai billiards) are classical models of dynamical systems that exhibit strong chaotic behavior but are highly nonlinear and contain singularities. It was a long standing conjecture that, due to singularities, the rate of the decay of correlations in dispersing billiards (or the rate of mixing, or the speed of relaxation to equilibrium) is subexponential, i.e. slower than that in Anosov and Axiom A systems. Recently, L.-S. Young disproved this conjecture – she established an exponential decay of correlations for a periodic Lorentz gas with finite horizon. We prove the same result for all the major classes of planar dispersing billiards, including Lorentz gases without horizon and tables with corner points. We also design and prove a general theorem on the exponential decay of correlations for smooth hyperbolic systems with singularities, which is particularly convenient for physical models like billiards.

Keywords: Decay of correlations, Sinai-Ruelle-Bowen measures, dispersing billiards.

1 Introduction

Strong statistical properties – exponential decay of correlations (EDC) and central limit theorem (CLT) – for smooth uniformly hyperbolic dynamical systems, namely Anosov and Axiom A diffeomorphisms, have been proven by Ya. Sinai, D. Ruelle and R. Bowen in the seventies [17, 14, 1]. For piecewise smooth or nonuniformly hyperbolic systems, however, statistical properties are often weaker – the correlations decay slowly or the central limit theorem fails, and, in any case, these properties are very hard to prove.

We concentrate here on systems with uniform hyperbolicity, i.e. such that one step expansion and contraction factors are bounded away from unity, but we do not require smoothness everywhere, i.e. allow singularities. Well studied and physically important systems of this kind are dispersing billiards, in particular the periodic Lorentz gas. L. Bunimovich and Sinai [2] obtained a CLT and a subexponential (stretched exponential) bound on correlations for a planar periodic Lorentz gas with finite horizon in 1981. These results were improved and extended to other planar dispersing billiards in 1991, see [4], and to a multidimensional periodic Lorentz gas with finite horizon in 1994, see [6]. There was much of discussion in physical and mathematical communities in the eighties about the actual rate of the decay of correlations in dispersing billiards, whether it is truly exponential or slower. Numerical experiments produced inconclusive or contradictory estimates, see a resent discussion and further references in [9].

In early nineties, the 'exponential' point of view got the upper hand. In 1992, the EDC was established for piecewise linear hyperbolic 2-D toral automorphisms by the author [5]. In 1994, Liverani [11] established the EDC for 2-D piecewise smooth area-preserving uniformly hyperbolic systems. The singularities in the above papers consisted of a finite number of smooth curves on which the dynamics was discontinuous but had one-sided derivatives (in particular, the derivatives were uniformly bounded). Those classes did not cover billiards, where derivatives are always unbounded. Still the above results showed in principle that singularities did not necessarily slow down the decay of correlations.

A breakthrough occurred in 1996 when Young [18] proved the EDC for quite generic hyperbolic systems with their Sinai-Ruelle-Bowen (SRB) measures[1] under one assumption, the so called exponential tail bound on return times to a selected product-structure hyperbolic set. She verified that assumption for two classes of systems. One was that of 2-D piecewise smooth uniformly hyperbolic maps, and thereby Young extended Liverani's result [11] to non-area-preserving systems. The author recently further extended this result by Young to multidimensional maps [7]. The second class in Young's paper [18] consisted of a planar periodic Lorentz gas with finite horizon. Thus, Young established, for the first time ever, an exponential bound on correlations for a billiard model.

Our goal is to extend Young's result to other classes of dispersing billiards. Before we do that, we prove the EDC for uniformly hyperbolic systems with quite general singularities. Our setup allows countably many singularity manifolds and unbounded derivatives. We find sufficient conditions under which correlations for SRB measures decay exponentially fast. Since we actually prove, under our conditions, Young's exponential tail bound mentioned above, our conditions are more restrictive than Young's. On the other hand, our conditions are a little easier to check than Young's: in particular, they involve only one iteration of the map rather than all its positive iterations. After that, we show that our conditions are mild enough to hold for the major classes of planar dispersing billiards, including Lorentz gases without horizon and tables with corner points. Thus, we establish the EDC for all the main types of planar dispersing billiards. This completely settles the controversy over the decay of correlations in billiards that has attracted so much attention in the physical and mathematical communities. The verification of our

[1] For smooth maps that do not preserve absolutely continuous measures, SRB measures are the most physically relevant invariant measures, see discussions in [18, 7].

conditions for billiards is not so hard a job, by the way, compared to the sophisticated analysis of billiard dynamics done in early papers [4, 6]. In a separate paper, we plan to cover perturbed dispersing billiards (subject to small external fields) and Lorentz gases in any dimensions.

The paper is organized as follows. In Section 2 we state our assumptions and the main theorem. In Sections 3–5 we prove the main theorem. In Sections 6–9 we apply the main theorem to the planar dispersing billiards. In Section 10 we make some remarks and hints on the verification of the assumptions of our main theorem.

2 Statement of the main result

Let M be an open subset in a d-dimensional C^∞ Riemannian manifold, such that $\bar{M}$ is compact (the sets M and $\bar{M}$ are not necessarily connected), and let $\Gamma \subset \bar{M}$ be a closed subset. We consider a map $T : M \setminus \Gamma \to M$, which is a C^2 diffeomorphism of $M \setminus \Gamma$ onto its image.

The set Γ will be referred to as the singularity set for T. For $n \geq 1$ denote by

$$\Gamma^{(n)} = \Gamma \cup T^{-1}\Gamma \cup \cdots \cup T^{-n+1}\Gamma \tag{2.1}$$

the singularity set for T^n. Define

$$M^+ = \{x \in M : T^n x \notin \Gamma,\, n \geq 0\}, \qquad M^- = \cap_{n>0} T^n(M \setminus \Gamma^{(n)})$$

and

$$M^0 = \cap_{n \geq 0} T^n(M^+) = M^+ \cap M^-$$

The sets M^+ and M^- consist, respectively, of points were all the future and past iterations of T are defined, and M^0 is the set of points where all the iterations of T are defined. For any $\delta > 0$ denote by $\mathcal{U}_\delta$ the δ-neighborhood of the closed set $\Gamma \cup \partial M$.

Notation. We denote by ρ the Riemannian metric in M and by m the Lebesgue measure (volume) in M. For any submanifold $W \subset M$ we denote by ρ_W the metric on W induced by the Riemannian metric in M, by m_W the Lebesgue measure on W generated by ρ_W, and by diam$W$ the diameter of W in the ρ_W metric.

Hyperbolicity. We assume that T is fully and uniformly hyperbolic, i.e. there exist two families of cones C^u_x and C^s_x in the tangent spaces $\mathcal{T}_x M$, $x \in \bar{M}$, such that $DT(C^u_x) \subset C^u_{Tx}$ and $DT(C^s_x) \supset C^s_{Tx}$ whenever DT exists, and

$$|DT(v)| \geq \Lambda |v| \qquad \forall v \in C^u_x$$

$$|DT^{-1}(v)| \geq \Lambda |v| \qquad \forall v \in C^s_x$$

with some constant $\Lambda > 1$. These families of cones are continuous on $\bar{M}$, their axes have the same dimensions across the entire $\bar{M}$, and the angles between C^u_x and C^s_x are bound away from zero. Denote by d_u and d_s the dimensions of the axes of C^u_x and C^s_x, respectively. The full hyperbolicity here means that $d_u + d_s = \dim M$.

For any $x \in M^+$ and $y \in M^-$ we set

$$E^s_x = \cap_{n\geq 0} DT^{-n}(C^s_{T^n x}), \qquad E^u_y = \cap_{n\geq 0} DT^{n}(C^u_{T^{-n} y})$$

respectively. It is standard, see, e.g., [13], that the subspaces E^s_x, E^u_x are DT-invariant, depend on x continuously, $\dim E^{u,s}_x = d_{u,s}$, and $E^s_x \oplus E^u_x = \mathcal{T}_x M$ for $x \in M^0$.

As a consequence, there can be no zero Lyapunov exponents on M^0. The space E^u_x is spanned by all vectors with positive Lyapunov exponents, and E^s_x by those with negative Lyapunov exponents.

We call a submanifold $W^u \subset M$ a local unstable manifold (LUM), if T^{-n} is defined and smooth on W^u for all $n \geq 0$, and $\forall x, y \in W^u$ we have $\rho(T^{-n}x, T^{-n}y) \to 0$ as $n \to \infty$ exponentially fast. Similarly, local stable manifolds (LSM), W^s, are defined. Obviously, $\dim W^{u,s} = d_{u,s}$. We denote by $W^u(x)$, $W^s(x)$ local unstable and stable manifolds containing x, respectively.

We primarily work with LUM's, and for brevity we will denote them by just W, suppressing the superscript u. Denote by $J^u(x) = |\det(DT|E^u_x)|$ the jacobian of the map T restricted to $W(x)$ at x, i.e. the factor of the volume expansion on the LUM $W(x)$ at the point x.

We assume the following standard properties of unstable manifolds:

Bounded curvature. The sectional curvature of any LUM W is uniformly bounded by a constant $B \geq 0$.

Distorsion bounds. Let x, y be in one connected component of $W \setminus \Gamma^{(n-1)}$, denote it by V. Then

$$\log \prod_{i=0}^{n-1} \frac{J^u(T^i x)}{J^u(T^i y)} \leq \varphi\left(\rho_{T^n V}(T^n x, T^n y)\right) \tag{2.2}$$

where $\varphi(\cdot)$ is some function, independent of W, such that $\varphi(s) \to 0$ as $s \to 0$.

Absolute continuity. Let W_1, W_2 be two sufficiently small LUM's, such that any LSM W^s intersects each of W_1 and W_2 in at most one point. Let $W_1' = \{x \in W_1 : W^s(x) \cap W_2 \neq \emptyset\}$. Then we define a map $h : W_1' \to W_2$ by sliding along stable manifolds. This map is often called a holonomy map. We assume that it is absolutely continuous with respect to the Lebesgue measures m_{W_1} and m_{W_2}, and its jacobian (at any density point of W_1') is bounded, i.e.

$$1/C' \leq \frac{m_{W_2}(h(W_1'))}{m_{W_1}(W_1')} \leq C' \tag{2.3}$$

with some $C' = C'(T) > 0$.

Non-branching of unstable manifolds. LUM's are locally unique, i.e. for any two LUM's $W^1(x), W^2(x)$ we have $W^1(x) \cap B_\varepsilon(x) = W^2(x) \cap B_\varepsilon(x)$ for some $\varepsilon > 0$. Here $B_\varepsilon(x)$ is the ε-ball centered at x. Furthermore, let $\{W^1_n\}$ and $\{W^2_n\}$ be two sequences of

LUM's that have a common limit point $x \in \bar{M}$, i.e. $\rho(x, W_n^i) \to 0$ as $n \to \infty$ for $i = 1, 2$. Assume also that $\exists \varepsilon > 0$ such that $\rho(x, \partial W_n^i) > \varepsilon$ for all $n \geq 1$ and $i = 1, 2$. Then $\rho_H(W_n^1 \cap B_\varepsilon(x), W_n^2 \cap B_\varepsilon(x)) \to 0$ as $n \to \infty$, where

$$\rho_H(A, B) = \max\{\sup_{x \in A} \rho(x, B), \sup_{y \in B} \rho(y, A)\}$$

is the Hausdorff distance between sets.

u-SRB measures. A unique probability measure ν_W, absolutely continuous with respect to the Lebesgue measure m_W, is defined on any LUM W by the following equation:

$$\frac{\rho_W(x)}{\rho_W(y)} = \lim_{n \to \infty} \prod_{i=1}^{n} \frac{J^u(T^{-i}y)}{J^u(T^{-i}x)} \qquad \forall x, y \in W \tag{2.4}$$

where $\rho_W(x) = d\nu_W/dm_W(x)$ is the density of ν_W with respect to m_W. The existence of the limit in (2.4) is guaranteed by (2.2). We call ν_W the u-SRB measure on W. Observe that u-SRB measures are conditionally invariant under T, i.e. for any submanifold $W_1 \subset TW$, the measure $T_*\nu_W|W_1$ (the image of ν_W under T conditioned on W_1) coincides with ν_{W_1}.

SRB measure. We assume that the map T preserves an ergodic Sinai-Bowen-Ruelle (SRB) measure μ. That is, there is an ergodic probability measure μ on M such that for μ-a.e. $x \in M$ a LUM $W(x)$ exists, and the conditional measure on $W(x)$ induced by μ is absolutely continuous with respect to $m_{W(x)}$. In fact, that conditional measure coincides with the u-SRB measure $\nu_{W(x)}$.

δ_0-LUM's. Let $\delta_0 > 0$. We call W a δ_0-LUM if it is a LUM and $\operatorname{diam} W \leq \delta_0$. For an open subset $V \subset W$ and $x \in V$ denote by $V(x)$ the connected component of V containing the point x. Let $n \geq 0$. We call an open subset $V \subset W$ a (δ_0, n)-subset if $V \cap \Gamma^{(n)} = \emptyset$ (i.e., the map T^n is defined on V) and $\operatorname{diam} T^n V(x) \leq \delta_0$ for every $x \in V$. Note that $T^n V$ is then a union of δ_0-LUM's. Define a function $r_{V,n}$ on V by

$$r_{V,n}(x) = \rho_{T^n V(x)}(T^n x, \partial T^n V(x)) \tag{2.5}$$

Note that $r_{V,n}(x)$ is the radius of the largest open ball in $T^n V(x)$ centered at $T^n x$. In particular, $r_{W,0}(x) = \rho_W(x, \partial W)$.

Flatness and uniformity of LUM's. We will only work with δ_0-LUM's for sufficiently small values of δ_0. For any such δ_0-LUM W the tangent spaces $\mathcal{T}_x W$ are almost parallel at all points $x \in W$. If $n \geq 1$ and $V \subset T^n W$ is another δ_0-LUM, then $T_*^n m_W|V$ (the nth iterate of m_W conditioned on V) has an almost constant density with respect to m_V, due to (2.2). The u-SBR measure ν_W is almost uniform with respect to the Lebesgue measure m_W. The smaller δ_0, the more accurate these approximations are, uniformly in δ_0-LUM's W.

We now turn to the key assumptions on the growth of unstable manifolds that will ensure a fast decay of correlations.

Growth of unstable manifolds. We assume that there are constants $\alpha_0 \in (0,1)$ and $\beta_0, D_0, \kappa, \sigma, \zeta > 0$ with the following property. For any sufficiently small $\delta_0, \delta > 0$ and any δ_0-LUM W there is an open $(\delta_0, 0)$-subset $V_\delta^0 \subset W \cap \mathcal{U}_\delta$ and an open $(\delta_0, 1)$-subset $V_\delta^1 \subset W \setminus \mathcal{U}_\delta$ (one of these may be empty) such that $m_W(W \setminus (V_\delta^0 \cup V_\delta^1)) = 0$ and $\forall \varepsilon > 0$

$$m_W(r_{V_\delta^1,1} < \varepsilon) \leq \alpha_0 \Lambda \cdot m_W(r_{W,0} < \varepsilon/\Lambda) + \varepsilon \beta_0 \delta_0^{-1} m_W(W) \tag{2.6}$$

$$m_W(r_{V_\delta^0,0} < \varepsilon) \leq D_0 \delta^{-\kappa}\, m_W(r_{W,0} < \varepsilon) \tag{2.7}$$

and

$$m_W(V_\delta^0) \leq D_0\, m_W(r_{W,0} < \zeta \delta^\sigma) \tag{2.8}$$

We now state our main result, followed by the necessary definitions.

Theorem 2.1 *Let T satisfy the above assumptions. If the system (T^n, μ) is ergodic for all $n \geq 1$, then the map T has exponential decay of correlations (EDC) and satisfies the central limit theorem (CLT) for Hölder continuous functions on M.*

Let $\mathcal{H}_\eta$ be the class of Hölder continuous functions on M with exponent $\eta > 0$:

$$\mathcal{H}_\eta = \{f : M \to \mathbb{R} \,|\, \exists C > 0 : |f(x) - f(y)| \leq C\rho(x,y)^\eta, \ \forall x, y \in M\}$$

Exponential decay of correlations. We say that (T, μ) has exponential decay of correlations for Hölder continuous functions if $\forall \eta > 0\ \exists \gamma = \gamma(\eta) \in (0,1)$ such that $\forall f, g \in \mathcal{H}_\eta\ \exists C = C(f,g) > 0$ such that

$$\left| \int_M (f \circ T^n) g \, d\mu - \int_M f \, d\mu \int_M g \, d\mu \right| \leq C \gamma^{|n|} \qquad \forall n \in \mathbb{Z}$$

Central limit theorem. We say that (T, μ) satisfies central limit theorem (CLT) for Hölder continuous functions if $\forall \eta > 0, f \in \mathcal{H}_\eta$, with $\int f \, d\mu = 0$, $\exists \sigma_f \geq 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \stackrel{\text{distr}}{\longrightarrow} \mathcal{N}(0, \sigma_f^2)$$

Furthermore, $\sigma_f = 0$ iff $f = g \circ T - g$ for some $g \in L^2(\mu)$

3 Filtrations of unstable manifolds

Existence of LUM's and LSM's. For any $\varepsilon > 0$, let

$$M_{\Lambda,\varepsilon}^{\pm} = \{x \in M^{\pm} : \rho(T^{\pm n}x, \Gamma \cup \partial M) > \varepsilon \Lambda^{-n} \quad \forall n \geq 0\}$$

and

$$M_\Lambda^{\pm} = \cup_{\varepsilon > 0} M_{\Lambda,\varepsilon}^{\pm} \qquad M_\Lambda^0 = M_\Lambda^+ \cap M_\Lambda^-$$

The following fact is standard [13, 18]: $\forall x \in M^-_{\Lambda,\varepsilon}$ there is a LUM $W^u(x)$ such that $\rho(x, \partial W^u(x)) \geq \varepsilon$. Similarly, $\forall x \in M^+_{\Lambda,\varepsilon}$ there is an LSM $W^s(x)$ such that $\rho(x, \partial W^s(x)) \geq \varepsilon$. For $x \in M^-_{\Lambda,\varepsilon}$ we denote by $W^u_\varepsilon(x)$ the LUM which is a ε-ball centered at x in the $\rho_{W^u_\varepsilon(x)}$ metric, i.e. $\rho_{W^u_\varepsilon(x)}(x, y) = \varepsilon$, $\forall y \in \partial W^u_\varepsilon(x)$. Similarly, $W^s_\varepsilon(x)$ is defined $\forall x \in M^+_\Lambda$. We will call $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ stable and unstable *disks* of radius ε through x, respectively.

Z-function. Let W be a δ_0-LUM, $n \geq 0$, and $V \subset W$ an open (δ_0, n)-subset of W. We define the Z-function introduced in [7] by

$$Z[W, V, n] = \sup_{\varepsilon>0} \frac{m_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(W)} \tag{3.1}$$

The supremum here is not necessarily finite. It will be finite if the boundary $\partial T^n V$ is 'regular enough'. In particular, if $\partial T^n V$ is piecewise smooth (i.e., consists of a finite number of smooth compact submanifolds of dimension $\leq d_u - 1$), then $Z[W, V, n] < \infty$, see e.g. [8]. In the case $m_W(W \setminus V) = 0$, the value of $Z[W, V, n]$ characterizes, in a certain way, the 'average size' of the components of $T^n V$ – the larger they are the smaller $Z[W, V, n]$. In particular, the value $Z[W, W, 0]$ characterizes the size of W in the following way.

Examples. Let W be a ball of radius r, then $Z[W, W, 0] \sim r^{-1}$. Let W be a cylinder whose base is a ball of radius r and height $h \gg r$, then again $Z[W, W, 0] \sim r^{-1}$. Let W be a rectangular box with dimensions $l_1 \times l_2 \times \cdots \times l_{d_u}$, then $Z[W, W, 0] \sim 1/\min\{l_1, \ldots, l_{d_u}\}$.

Notation. Let $\delta_{\max} > 0$ be so small that $\alpha := \alpha_0 e^{6\varphi(\delta_{\max})} < 1$. Denote also $\beta := \beta_0 e^{6\varphi(\delta_{\max})}$ and $D := D_0 e^{6\varphi(\delta_{\max})}$. We will always assume that $\delta_0 < \delta_{\max}$, so that $\alpha < 1$. Next, put

$$\bar{\beta} = 2\beta/(1 - \alpha)$$

and

$$a = -(\ln \alpha)^{-1} \quad \text{and} \quad b = \max\{0, -\ln(\delta_0(1 - \alpha)/\beta)/\ln \alpha\}$$

We also put

$$\delta_1 = \delta_0/(2\bar{\beta}) \tag{3.2}$$

Convention of δ's. We will define some small parameters δ_i, $i \geq 1$, so that each δ_i will be a certain function of δ_{i-1}. Still, we can vary all of them together preserving the specified relations between them, like the above (3.2).

δ-Filtration. Let $\delta_0, \delta > 0$ and W be a δ_0-LUM. Two sequences of open subsets $W = W^1_0 \supset W^1_1 \supset W^1_2 \supset \cdots$ and $W^0_n \subset W^1_n \setminus W^1_{n+1}$, $n \geq 0$, are said to make a *δ-filtration* of W, denoted by $\{W^1_n, W^0_n\}$ if[2] $\forall n \geq 0$
(a) the sets W^1_n and W^0_n are (δ_0, n)-subsets of W;
(b) $m_W(W^1_n \setminus (W^1_{n+1} \cup W^0_n)) = 0$.
(c) $T^n W^1_{n+1} \cap \mathcal{U}_{\delta\Lambda^{-n}} = \emptyset$ and $T^n W^0_n \subset \mathcal{U}_{\delta\Lambda^{-n}}$.

[2] In [7], it was called a refined u-filtration.

We put $W^1_\infty = \cap_{n\ge 0} W^1_n$. Observe that $W^1_\infty \subset M^+_{\Lambda,\delta}$, and so a stable disk $W^s_\delta(x)$ of radius δ exists at every point $x \in W^1_\infty$.

Put also $w^1_n = m_W(W^1_n)/m_W(W)$ and $w^0_n = m_W(W^0_n)/m_W(W)$. Observe that $w^1_n = 1 - w^0_0 - \cdots - w^0_{n-1}$ and $w^1_n \searrow w^1_\infty := m_W(W^1_\infty)/m_W(W)$ as $n \to \infty$.

Theorem 3.1 *Let W be a δ_0-LUM and $\delta > 0$. Then there is a δ-filtration $(\{W^1_n\}, \{W^0_n\})$ of W such that*
(i) *$\forall n \ge 1$ and $\forall \varepsilon > 0$ we have*

$$m_W(r_{W^1_n,n} < \varepsilon) \le (\alpha\Lambda)^n \cdot m_W(r_{W,0} < \varepsilon/\Lambda^n) + \varepsilon\beta\delta_0^{-1}(1 + \alpha + \cdots + \alpha^{n-1})\, m_W(W) \quad (3.3)$$

Furthermore, $\forall n \ge 0$ and $\forall \varepsilon > 0$

$$m_W(r_{W^0_n,n} < \varepsilon) \le D\delta^{-\kappa}\Lambda^{\kappa n}\, m_W(r_{W^1_n,n} < \varepsilon) \quad (3.4)$$

and

$$m_W(W^0_n) \le D\, m_W(r_{W^1_n,n} < \zeta\delta^\sigma\Lambda^{-\sigma n}) \quad (3.5)$$

(ii) *we have $\forall n \ge 1$*

$$Z[W, W^1_n, n] \le \alpha^n Z[W, W, 0] + \beta\delta_0^{-1}(1 + \alpha + \cdots + \alpha^{n-1}) \quad (3.6)$$

(iii) *for any $n \ge 0$ we have $Z[W, W^0_n, n] \le D\delta^{-\kappa}\Lambda^{\kappa n} \cdot Z[W, W^1_n, n]$;*
(iv) *for any $n \ge 0$ we have $w^0_n \le D\zeta\delta^\sigma\Lambda^{-\sigma n} \cdot Z[W, W^1_n, n]$.*

Proof of (3.3)-(3.5) goes by induction on n. The bound (3.3) for $n = 1$ and (3.4)-(3.5) for $n = 0$ follow from our assumptions (2.6)-(2.8), respectively, after we set $W^1_1 := V^1_\delta$ and $W^0_0 := V^0_\delta$, since $\alpha_0 < \alpha$, $\beta_0 < \beta$, and $D_0 < D$. Assume now (3.3) for some $n \ge 1$. Denote by $W_{n,j}$, $j \ge 1$, all the connected components of W^1_n. Each $W'_{n,j} := T^n W_{n,j}$ is a δ_0-LUM. So, there are two disjoint open $(\delta_0, 1)$-subsets $V^0_{n,j} \subset W'_{n,j} \cap \mathcal{U}_{\delta\Lambda^{-n}}$ and $V^1_{n,j} \subset W'_{n,j} \setminus \mathcal{U}_{\delta\Lambda^{-n}}$ such that $m_{W'_{n,j}}(W'_{n,j} \setminus (V^0_{n,j} \cup V^1_{n,j})) = 0$ and $\forall \varepsilon > 0$

$$m_{W'_{n,j}}(r_{V^1_{n,j},1} < \varepsilon) \le \alpha_0\Lambda \cdot m_{W'_{n,j}}(r_{W'_{n,j},0} < \varepsilon/\Lambda) + \varepsilon\beta_0\delta_0^{-1} m_{W'_{n,j}}(W'_{n,j})$$

$$m_{W'_{n,j}}(r_{V^0_{n,j},0} < \varepsilon) \le D_0\delta^{-\kappa}\Lambda^{\kappa n}\, m_{W'_{n,j}}(r_{W'_{n,j},0} < \varepsilon)$$

and

$$m_{W'_{n,j}}(V^0_{n,j}) \le D_0\, m_{W'_{n,j}}(r_{W'_{n,j},0} < \zeta\delta^\sigma\Lambda^{-\sigma n})$$

according to (2.6)-(2.8). Using the distorsion bound (2.2) and our definition of α, β, D yields

$$m_{W_{n,j}}(r_{U^1_{n,j},n+1} < \varepsilon) \le \alpha\Lambda \cdot m_{W_{n,j}}(r_{W_{n,j},n} < \varepsilon/\Lambda) + \varepsilon\beta\delta_0^{-1} m_{W_{n,j}}(W_{n,j})$$

$$m_{W_{n,j}}(r_{U^0_{n,j},n} < \varepsilon) \le D\delta^{-\kappa}\Lambda^{\kappa n}\, m_{W_{n,j}}(r_{W_{n,j},n} < \varepsilon)$$

and

$$m_{W_{n,j}}(U^0_{n,j}) \le D\, m_{W_{n,j}}(r_{W_{n,j},n} < \zeta\delta^\sigma\Lambda^{-\sigma n})$$

where $U^1_{n,j} := T^{-n}V^1_{n,j}$ and $U^0_{n,j} := T^{-n}V^0_{n,j}$. Summing up over j gives

$$m_W(r_{W^1_{n+1},n+1} < \varepsilon) \le \alpha\Lambda \cdot m_W(r_{W^1_n,n} < \varepsilon/\Lambda) + \varepsilon\beta\delta_0^{-1} m_W(W^1_n)$$

$$m_W(r_{W^0_n,n} < \varepsilon) \le D\delta^{-\kappa}\Lambda^{\kappa n}\, m_W(r_{W^1_n,n} < \varepsilon)$$

and

$$m_W(W^0_n) \le D\, m_W(r_{W^1_n,n} < \zeta\delta^\sigma\Lambda^{-\sigma n})$$

where $W^1_{n+1} := \cup_j U^1_{n,j}$ and $W^0_n := \cup_j U^0_{n,j}$. The bounds (3.4) and (3.5) for the current value of n are proved. A direct use of (3.3) with ε replaced by ε/Λ, along with the obvious bound $m_W(W^1_n) \le m_W(W)$, gives (3.3) with n replaced by $n+1$. This completes the inductive proof of (3.3). Next, the parts (ii)-(iv) follow directly from (3.3)-(3.5), respectively, upon dividing by $m_W(W)$ and using (2.5). $\square$.

Remark. The proofs of the above theorem would go through even for slightly smaller values of α, β, D: $\alpha = \alpha_0 e^{2\varphi(\delta_0)}$, $\beta = \beta_0 e^{2\varphi(\delta_0)}$, and $D = D_0 e^{2\varphi(\delta_0)}$. Our choice of α, β, D allows us to extend the above theorem to absolutely continuous measures on W whose density with respect to the Lebesgue measure m_W is close enough to a constant. Precisely, if $\tilde{m}_W$ is a measure on W with density $\tilde{\rho}(x) = d\tilde{m}_W/dm_W(x)$, then we can replace m_W with $\tilde{m}_W$ in (3.1) and in the above theorem provided $\tilde{\rho}(x)/\tilde{\rho}(y) \le e^{2\varphi(\delta_0)}$, $\forall x, y \in W$. In particular, this holds for the u-SRB measure $\tilde{m}_W = \nu_W$.

Corollary 3.2 *Let* $\bar{Z}_W = \max\{Z[W,W,0], \bar{\beta}/\delta_0\}$. *Then*
(i) $Z[W,W^1_n,n] \le \bar{Z}_W$ *and* $Z[W,W^0_n,n] \le D\delta^{-\kappa}\Lambda^{\kappa n}\,\bar{Z}_W$ *for all* $n \ge 0$*;*
(ii) $Z[W,W^1_n,n] \le \bar{\beta}/\delta_0 = (2\delta_1)^{-1}$ *for all* $n \ge a\ln Z[W,W,0] + b$*;*
(iii) $w^0_n \le D\zeta\delta^\sigma\Lambda^{-\sigma n}\bar{Z}_W$ *for all* $n \ge 0$*;*
(iv) $w^1_n \ge 1 - D\zeta\delta^\sigma\bar{Z}_W/(1-\Lambda^{-\sigma})$ *for all* $n \ge 1$*;*
(v) $m_W(W^1_\infty) \ge m_W(W)\cdot\left[1 - D\zeta\delta^\sigma\bar{Z}_W/(1-\Lambda^{-\sigma})\right]$

Modified Z-function. The values $Z[W,W^1_n,n]$ and $Z[W,W^0_n,n]$ do not characterize the average size of the components of $T^nW^1_n$ or $T^nW^0_n$, respectively, since W^1_n and W^0_n are not subsets of full measure in W. To characterize the average sizes of the components of any (δ_0,n)-subset $V \subset W$ we will also use the quantity

$$Z[V,n] := \sup_{\varepsilon>0} \frac{m_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(V)} = Z[W,V,n] \times \frac{m_W(W)}{m_W(V)} \tag{3.7}$$

This value depends on V and n, but not on W. It characterizes the average size of the components of T^nV. Accordingly, the values of

$$Z[W^1_n,n] = Z[W,W^1_n,n]/w^1_n \qquad \text{and} \qquad Z[W^0_n,n] = Z[W,W^0_n,n]/w^0_n$$

characterize the average size of the components of $T^n W_n^1$ or $T^n W_n^0$, respectively.

Special case. In our further arguments, the set W_∞^1 will be often very dense in W, so that $w_\infty^1 > 0.9$. We call this a special case, and corollary 3.2 then implies that for all $n \geq a \ln Z[W, W, 0] + b$ we have $Z[W_n^1, n] \leq 0.6/\delta_1$. We will say then that the components of $T^n W_n^1$ are large enough, on the average.

Remark. The values of $Z[W, W_n, n]$, $Z[W, W_n^1, n]$, $Z[W, W_n^0, n]$, w_n^1, and w_n^0 above will certainly not change if we replace the Lebesgue measure m_W by any measure proportional to it. It is also straightforward that all the above results extend to countable disjoint unions of δ_0-LUM's with finite measures that are linear combinations of the Lebesgue measures on individual components. Precisely, let $W = \cup_k W^{(k)}$ be a countable union of pairwise disjoint δ_0-LUM's and let $\hat{m}_W = \sum_k u_k m_{W^{(k)}}$, with some $u_k > 0$, be a finite measure on W. Then $Z[W, V, n]$ is still defined by (3.1), with m_W replaced by $\hat{m}_W$, for any set $V = \cup_k V^{(k)}$, where $V^{(k)}$ are some open (δ_0, n)-subsets of $U^{(k)}$. The definition of δ-filtration and the proof of Theorem 3.1 go through with only minor obvious changes.

Final Remark. Let W' be a δ_0-LUM, $k \geq 1$, and $V' \subset W'$ an open (δ_0, k)-subset. Then $W = T^k V'$ is a finite or countable union of δ_0-LUM's. The measure $\tilde{m}_W := T_*^k m_{W'} | W$ on W is almost uniform (proportional to the Lebesgue measure m_W) on each component of W. Actually, its density differs from a constant by less than $e^{2\varphi(\delta_0)}$, according to (2.2). Due to the remark after Theorem 3.1, all the above results will then apply to $(W, \tilde{m}_W)$, instead of (W, m_W).

The following proposition generalizes the above special case. Its proof goes like the proof of Proposition 4.4 in [7], with obvious modifications.

Proposition 3.3 *Let $(\{W_n^1\}, \{W_n^0\})$ be a δ-filtration of a δ_0-LUM W satisfying Theorem 3.1, such that $w_\infty^1 = p > 0$. Then for all $n \geq a_1(\ln Z[W, W, 0] - \ln p) + b_1$ we have $m_W(W_\infty^1)/m_W(W_n^1) \geq 0.9$ and $Z[W_n^1, n] \leq 0.6/\delta_1$, i.e. the components of $T^n W_n^1$ will be large enough, on the average. Here $a_1 = a + (\sigma \ln \Lambda)^{-1}$ and b_1 is another constant determined by $\alpha, \beta, \delta_0, \Lambda, \zeta, D$.*

Final Remark (Part 2). The above proposition also applies to any pair $(W, \tilde{m}_W)$ described in Final Remark before the proposition. Likewise, some further results stated and proved for δ_0-LUM's W with Lebesgue measures m_W, will also apply to measures $\tilde{m}_W = T_*^k m_{T^{-k}W}$ on W for any $k \geq 1$.

4 Rectangles

Here we mostly repeat, in a brief manner, the constructions of [7], Section 5.

Rectangles and subrectangles. A subset $R \subset M^0$ is called a *rectangle* if $\exists \varepsilon > 0$ such that for any $x, y \in R$ there is an LSM $W^s(x)$ and an LUM $W^u(y)$, both of diameter

$\leq \varepsilon$, that meet in exactly one point, which also belongs in R. We denote that point by $[x, y] = W^s(x) \cap W^u(y)$.

A subrectangle $R' \subset R$ is called a u-subrectangle if $W^u(x) \cap R = W^u(x) \cap R'$ for all $x \in R'$. Similarly, s-subrectangles are defined. We say that a rectangle R' u-crosses another rectangle R if $R' \cap R$ is a u-subrectangle in R and an s-subrectangle in R'.

s-Shadowing and s-distance. Let $x \in M$ and $r \in (0, \delta_0)$. We denote by $S_r(x)$ any s-manifold that is a ball of radius r centered at x in its own metric, $\rho_{S_r(x)}$. By that we mean $\rho_{S_r(x)}(x, y) = r$, $\forall y \in \partial S_r(x)$. We call such $S_r(x)$ an *s-disk*. In order to define s-disks also around points close to ∂M we extend the cone families C^u and C^s continuously beyond the boundaries of M into the δ_0-neighborhood of M. Then s-disks $S_r(x)$ exist $\forall x \in M, \forall r \in (0, \delta_0)$.

Let W be a δ_0-LUM, and $x \in M$. Clearly, any s-disk $S_{\delta_0}(x)$ can meet W in at most one point. We call

$$H_x(W) = \{y \in W : \, y = S_{\delta_0}(x) \cap W \;\text{ for some }\; S_{\delta_0}(x)\}$$

the *s-shadow* of x on W.

We say that a point $x \in M$ is overshadowed by a LUM W if $\forall S_{\delta_0}(x)$ we have $S_{\delta_0}(x) \cap W \neq \emptyset$. We call

$$\rho^s(x, W) = \sup_{S_{\delta_0}(x)} \rho_{S_{\delta_0}(x)}(x, S_{\delta_0}(x) \cap W)$$

the *s-distance* from x to W.

Let W, W' be two δ_0-LUM's. We call

$$H_W(W') = \cup_{x \in W} H_x(W')$$

the s-shadow of W on W'. We say that W' overshadows W if it overshadows every point $x \in W$. In this case we define

$$\rho^s(W, W') = \sup_{x \in W} \rho^s(x, W')$$

the s-distance from W to W'.

We assume that δ_0, and hence $\delta_1 = \delta_0/(2\bar{\beta})$, are small enough, so that

$$A_{\delta_1} \stackrel{\text{def}}{=} \{x \in M : \text{the unstable disk } \; W^u_{\delta_1}(x) \;\text{ exists}\} \neq \emptyset$$

Let $z \in A_{\delta_1}$. Consider $W(z) := W^u_{\delta_1/3}(z)$, the 'central part' of the existing unstable disk $W^u_{\delta_1}(z)$. It is a δ_0-LUM, and a perfect ball in its own metric. It is easy to compute that for a perfect ball W of radius δ in $\mathbb{R}^{d_u}$ one has $Z[W, W, 0] = d_u/\delta$. Since the manifolds $W(z)$, $z \in A_{\delta_1}$, actually have some (bounded) sectional curvature, $Z[W(z), W(z), 0]$ might be larger than $3d_u/\delta_1$, but if δ_1 is small enough, we will have [7]

$$Z[W(z), W(z), 0] \leq 4d_u/\delta_1 \tag{4.1}$$

for all $z \in A_{\delta_1}$.

Now let δ_2 be defined by

$$\frac{\delta_2^\sigma}{\delta_1} = \frac{1-\Lambda^{-\sigma}}{40\, D\zeta d_u} \tag{4.2}$$

For any $z \in A_{\delta_1}$ fix one δ_2-filtration $(\{W_n^1(z)\}, \{W_n^0(z)\})$ of $W(z)$ satisfying Theorem 3.1. Recall that $\forall x \in W_\infty^1(z)$ a stable disk $W_{\delta_2}^s(x)$ exists, cf. Sect. 3. The following lemmas are consequences of (4.1), (4.2) and the parts (ii), (v) of Corollary 3.2, see proofs in [7].

Lemma 4.1 $m_{W(z)}(W_\infty^1(z)) \geq 0.9 \cdot m_{W(z)}(W(z))$.

Lemma 4.2 $\forall n \geq n_0' := a\ln(16d_u) + \max\{1, a\ln[\beta\delta_0^{-1}/(1-\alpha)]\}$ *we have*
(i) $Z[W(z), W_n^1(z), n] < (2\delta_1)^{-1}$ *and* $Z[W_n^1(z), n] < 0.6/\delta_1$*;*
(ii) $m_{W(z)}(x \in W_n^1(z) : r_{W_n^1(z),n}(x) > \delta_1) > 0.4 \cdot m_{W(z)}(W_n^1(z)) > 0.4 \cdot m_{W(z)}(W_\infty^1(z))$.
In other words, (ii) means that at least 40% of the points in $T^nW_n^1(z)$ *(with respect to the measure induced by* $m_{W(z)}$*) lie a distance* $\geq \delta_1$ *away from the boundaries of* $T^nW_n^1(z)$.

Remark. Let $z \in A_{\delta_1}$. For a moment, let $W(z) = W_\varepsilon^u(z)$ be the stable disk of any radius $\varepsilon \in (\delta_1/3, \delta_1)$. That disk $W(z)$ is larger than $W_{\delta_1/3}^u(z)$, and so (4.1) still holds. Therefore, the statements (i) and (ii) of the above lemma hold as well. Furthermore, if, again for a moment, we decrease δ_2 thus making the ratio δ_2^σ/δ_1 even smaller than the one specified by (4.2), then Lemma 4.1 will still hold, and then so will (i) and (ii) of Lemma 4.2.

Let $\delta_3 \ll \delta_2$, to be specified later. The following proposition is proved in [7], Proposition 5.3.

Proposition 4.3 *Let* W *be a* δ_0*-LUM, and* W' *another* δ_0*-LUM that overshadows* W *and* $\rho^s(W, W') \leq \delta_3$*. Let* $(\{W_n^1\}, \{W_n^0\})$ *be a* δ_2*-filtration of* W*. Then* $\forall n \geq 1$ *and any connected component* V *of* W_n^1 *there is a connected domain* $V' \subset W' \setminus \Gamma^{(n)}$ *such that the* δ_0*-LUM* T^nV' *overshadows the* δ_0*-LUM* T^nV*, and* $\rho^s(T^nV, T^nV') \leq \delta_3\Lambda^{-n}$*.*

Canonical rectangles. For any $z \in A_{\delta_1}$ we define a 'canonical' rectangle $R(z)$ as follows: $y \in R(z)$ iff $y = W_{\delta_2}^s(x) \cap W$ for some $x \in W_\infty^1(z)$ and for some LUM W that overshadows $W(z) = W_{\delta_1/3}^u(z)$, and such that $\rho^s(W(z), W) \leq \delta_3$. Observe that if $\delta_3/\delta_2 < c'$, where $c' > 0$ is determined by the minimum angle between the stable and unstable cone families, then every W that overshadows $W(z)$ and is δ_3-close to it in the above sense will meet all stable disks $W_{\delta_2}^s(x)$, $x \in W_\infty^1(z)$. In that case $R(z)$ will be a rectangle, indeed. We fix δ_3/δ_2 now as follows:

$$\delta_3/\delta_2 = \min\{c', 1-\Lambda^{-1}, 1/3\} \tag{4.3}$$

For any connected subdomain $V \subset W(z)$ the set $R_V(z) := \{y \in R(z) : W^s(y) \cap V \neq \emptyset\}$ is an s-subrectangle in $R(z)$ "based on V". For $n \geq 1$, the partition of $W_n^1(z)$ into connected components, $\{V\}$, induces a partition of $R(z)$ into s-subrectangles $\{R_V(z)\}$ that are based on those components. If $R_V(z)$ is one of those s-subrectangles, then Proposition 4.3 implies that $T^nR_V(z)$ is a rectangle.

Lemma 4.4 *For any $\delta_3 > 0$ there is a $\delta_4 > 0$ such that $\forall z, z' \in A_{\delta_1}$ such that $\rho(z, z') < \delta_4$, the LUM $W^u_{\delta_1/2}(z')$ overshadows the LUM $W(z) = W^u_{\delta_1/3}(z)$, and $\rho^s(W(z), W^u_{\delta_1/2}(z')) \leq \delta_3/2$. Likewise, the LUM $W^u_{\delta_1}(z)$ overshadows the LUM $W^u_{\delta_1/2}(z')$, and $\rho^s(W^u_{\delta_1/2}(z'), W^u_{\delta_1}(z)) \leq \delta_3/2$.*

Proof. It is enough to prove the first statement, the second one is completely similar. We actually need to prove that $\forall x \in W(z)$ we have $\rho^s(x, W^u_{\delta_1/2}(z')) \leq \delta_3/2$. Assume that this is not the case, i.e. $\forall \delta_4 > 0\ \exists z, z' \in A_{\delta_1}$ such that $\rho(z, z') < \delta_4$ and $\exists x \in W(z)$ such that $\rho^s(x, W^u_{\delta_1/2}(z')) > \delta_3/2$. We take a sequence $\delta_4 = 1/n$, $n \geq 1$, and the corresponding points z_n, z'_n. Due to the compactness of $\bar{M}$, there is a subsequence n_k such that $\exists z_\infty := \lim_k z_{n_k} = \lim_k z'_{n_k} \in \bar{M}$. This clearly contradicts our assumption on non-branching of unstable manifolds. $\square$

Let $n''_0 = \min\{n \geq 1 : \Lambda^n > 2\}$. The following proposition is proved in [7], Proposition 5.3.

Proposition 4.5 *Let $z \in A_{\delta_1}$ and $n \geq n''_0$. Let V be a connected component of $W^1_n(z)$ and $x \in V$ such that $r_{V,n}(x) > \delta_1$ and $\rho(T^n x, z') < \delta_4$ for some $z' \in A_{\delta_1}$. Then the rectangle $T^n R_V(z)$ u-crosses the rectangle $R(z')$, i.e. $T^n R_V(z) \cap R(z')$ is* (i) *a u-subrectangle in $R(z')$ and* (ii) *an s-subrectangle in $T^n R_V(z)$.*

5 Rectangular structure, return times, and tail bound

The constructions in this section mostly repeat those in [7], Sections 6,7.

Consider the SRB measure μ. Clearly, if δ_0 is small enough, then $\exists z_1 \in A_{\delta_1}$ such that $\mu(R(z_1)) > 0$. We fix such a δ_0 and one such $z_1 \in A_{\delta_1}$. We then denote, for brevity, $R = R(z_1)$, $W = W(z_1)$, $W^1_\infty = W^1_\infty(z_1)$, etc.

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_p\}$ be a finite δ_4-dense subset of A_{δ_1} containing the above point z_1. We call $\mathcal{R} = \cup_i R(z_i)$ the rectangular structure. It is a finite union of rectangles that are likely to overlap and may not cover M or even the support of μ.

We will partition the set W^1_∞ into a countable collection of subsets $W^1_{\infty,k}$, $k \geq 0$, such that for every $k \geq 1$ there is an integer $r_k \geq 1$ such that for the s-subrectangle $R_k \subset R$ based on[3] $W^1_{\infty,k}$ the set $T^{r_k}(R_k)$ will be a u-subrectangle in some $R(z_i)$, $z_i \in \mathcal{Z}$. This fact is considered as a *proper return* (of R_k into $\mathcal{R}$, under r_k iterations of T). We define the return time function $r(x)$ on W^1_∞ by $r(x) = r_k$ for $x \in W^1_{\infty,k}$, $k \geq 1$, and $r(x) = \infty$ for $x \in W^1_{\infty,0}$. We call the sets $W^1_{\infty,k}$ for $k \geq 1$ *gaskets*, cf. [7], and $W^1_{\infty,0}$ the *leftover set.*

The following theorem immediately follows from Young [18], see also [7], Section 6.

Theorem 5.1 *Assume that (T^n, μ) is ergodic for all $n \geq 1$, and $\mu(R) > 0$. If*

$$m_W\{r(x) > n\} \leq C\theta^n \qquad \forall n \geq 1 \tag{5.1}$$

for some $C > 0$, $\theta \in (0, 1)$, then the system (T, μ) satisfies EDC and CLT.

[3] By the s-subrectangle $R_k \subset R$ based on $W^1_{\infty,k}$ we mean the set $R_k = \{x \in R : W^s(x) \cap W^1_\infty \in W^1_{\infty,k}\}$.

The construction of the partition $W^1_\infty = \cup_k W^1_{\infty,k}$ consists of several steps.

Initial growth. First, we take $n_1 = \max\{n_0', n_0''\}$. According to Lemma 4.2, we have
(a) $Z[W, W^1_{n_1}, n_1] < (2\delta_1)^{-1}$ and $Z[W^1_{n_1}, n_1] < 0.6/\delta_1$, i.e. the components of $T^{n_1}W^1_{n_1}$ are large enough, on the average, and
(b) $m_W\{x \in W^1_{n_1} : r_{W^1_n,n}(x) \geq \delta_1\} \geq 0.4\, m_W(W^1_{n_1})$, i.e. at least 40% of the points in $T^{n_1}W^1_{n_1}$ (with respect to the measure induced by m_W) lie a distance $\geq \delta_1$ away from $\partial T^{n_1}W^1_{n_1}$.
(Recall that (b) actually follows from (a).) Let $W^g := T^{n_1}W^1_{n_1}$, and $\tilde{m}_{W^g} = T_*^{n_1} m_W | W^g$ the induced measure on W^g. For every connected component $V \subset W^g$ such that $\exists x_V \in V : \rho_V(x_V, \partial V) \geq \delta_1$ we arbitrarily fix one such point x_V. Then $x_V \in A_{\delta_1}$, and $\exists z_V \in \mathcal{Z}$ such that $\rho(x_V, z_V) < \delta_4$. We fix one such z_V, too. Then we label the set $T^{-n_1}(V \cap R(z_V))$ as one of our gaskets $W^1_{\infty,k}$, and we define $r_k = n_1$ on it. According to Proposition 4.5, $T^{r_k}(R_k)$ is a u-subrectangle in $R(z_V)$. As in [7], we will call the set $V \cap R(z_V)$ a gasket, too.

The next lemma follows from Lemmas 4.1 and 4.2, along with the absolute continuity (2.3), see [7].

Lemma 5.2 *There is a $q = q(T) > 0$ such that, independently of the choice of the points x_V and z_V in the components $V \subset W^g$, the just defined gaskets $W^1_{\infty,k}$ satisfy*

$$m_W\left(\cup W^1_{\infty,k}\right) \geq q\, m_W(W^1_{n_1})$$

In other words, a certain fraction ($\geq q$) of W^g returns at the n_1-th iteration. This is the earliest return. The definition of further returns requires more careful considerations to avoid possible overlaps of gaskets, as it is explained in [7].

Capture. Every connected component V of W^g where a point x_V is picked is now subdivided into two connected sets: $V^c := W^u_{\delta_1/2}(x_V)$ and $V^f := V \setminus V^c$. The gasket $V \cap R(z_V)$ defined above lies wholly in V^c, see [7]. We say that V^c is 'captured' at the n_1-th iteration, and the set V^f, is 'free to move'. Let $W^f = \cup_{V \subset W^g} V^f$. The set W^f contains no points of the previously defined gaskets. For $n \geq 0$, let $W^{f,1}_n := W^f \cap T^{n_1}W^1_{n_1+n}$ and $W^{f,0}_n :=\mathrm{int}\,(W^{f,1}_n \setminus W^{f,1}_{n+1})$ for $n \geq 0$. It is easy to see that the sets $\{W^{f,1}_n, W^{f,0}_n\}$ make a $(\delta_2\Lambda^{-n_1})$-filtration of the manifold W^f and satisfies Theorem 3.1.

It was shown in [7], Sect. 6, that $Z[W^{f,1}_{n_2}, n_2] < 0.6/\delta_1$ for $n_2 := [-\ln 9.6/\ln\alpha] + 1$. In other words, it takes a fixed number of iterations, n_2, to recover the lost average size of the freely moving manifold, $T^n W^{f,1}_n$, $n \geq 0$, after the removal of the captured parts from W^g. As soon as this is done, i.e. at the iteration $n = n_2$, at least 40% of the image $T^n W^{f,1}_n$, will lie a distance $\geq \delta_1$ from its boundary, just as in the claim (b) above.

Next, we inductively repeat the above procedure of picking points x_V, z_V in the large components V of the freely moving manifold, defining new gaskets $V \cap R(z_V)$, capturing disks containing the newly defined gaskets, etc., see [7]. According to Lemma 5.2, the points of the freely moving manifold are being captured at an exponential rate: at least a fraction $q > 0$ of them is captured every n_2 iterations of T. Let $t_0(x)$, $x \in W^1_\infty$, be the number of iterations it takes to capture the image of the point x. Lemma 5.2 implies that

$$m_W(t_0(x) > n)/m_W(W^1_\infty) \le C_0\theta_0^n \tag{5.2}$$

with $\theta_0 = q^{1/n_2} < 1$ and some $C_0 > 0$. In particular, $t_0(x) < \infty$ for a.e. $x \in W^1_\infty$.

Release. Next, we take care of the captured parts of the manifolds $T^n W^1_n$, $n \ge 1$. Let $B^c \subset T^{n_c} W^1_{n_c}$ be a connected part captured at the n_c-th iteration of T, $n_c \ge n_1$. Then B^c is a perfect ball of radius $\delta_1/2$ in some connected component of $T^{n_c} W^1_{n_c}$. It carries the measure $\tilde m_{B^c} = T^{n_c}_* m_W|B^c$. The center x_c of the disk B^c belongs in A_{δ_1}, and there is a point $z_c \in \mathcal{Z}$ such that $\rho(x_c, z_c) < \delta_4$ and such that the set $B^c_R := B^c \cap R(z_c)$ makes a new gasket at the moment of capture. Let $B^c_\infty := B^c \cap T^{n_c} W^1_\infty$.

Denote $B^c_n = B^c \cap T^{n_c} W^1_{n_c+n}$ for $n \ge 0$. Observe that

$$Z[B^c, B^c, 0] \le 4d_u/\delta_1 \qquad \text{and} \qquad Z[B^c_n, n] < 0.6/\delta_1 \ \ \forall n \ge n_0' \tag{5.3}$$

according to the remark after Lemma 4.2. In other words, it takes n_0' iterations of T to make the components of $T^n B^c_n$ large enough, on the average.

In order to define a new gasket in any large component V of $T^n B^c_n$ and avoid possible overlaps with the image $T^n B^c_R$ of the old gasket B^c_R, we will make sure that V contains no points of $T^n B^c_R$. We define a 'point release time', $f(x)$, for points $x \in B^c_\infty \setminus B^c_R$. A point x will be released if $T^{f(x)}(x)$ is sufficiently far from $T^{f(x)} B^c_R$.

The definition of the release time is different for points of different type:

Type I points are such that there is an LSM $W^s(x)$ meeting the manifold $W^u_{\delta_1}(z_c)$ in one point, call it $h(x)$. Then $h(x) \notin W^1_\infty(z_c)$, otherwise x would have belonged in B^c_R. Hence, either $h(x) \in W^u_{\delta_1}(z_c) \setminus W^u_{\delta_1/3}(z_c)$ or $h(x) \in W^0_m(z_c)$ for some $m = m(x) \ge 0$. In the former case, we set $m(x) = 0$ and $\varepsilon(x) = \rho(h(x), W^u_{\delta_1/3}(z_c))$. In the latter case we set $\varepsilon(x) = \rho(T^m h(x), \partial T^m W^0_m(z_c))$. We now define the release time to be $f(x) = m(x) + \log_\Lambda(\delta_0/\varepsilon(x))$, one formula for both cases.

Type II points have no local stable manifolds that extend to $W^u_{\delta_1}(z_c)$. Let $x \in B^c_\infty$ be such a point. According to the second statement in Lemma 4.4, $\rho^s(x, W^u_{\delta_1}(z_c)) \le \delta_3/2$. Hence, no local stable manifold $W^s(x)$ contains a stable disk of radius $\delta_3/2$ around x. Therefore, $x \notin M^+_{\Lambda,\delta_3/2}$, see Section 3. Let then $m = m(x) = \min\{m' > 0 : \rho(T^{m'}x, \Gamma \cup \partial M) \le \delta_3 \Lambda^{-m'}/2\}$. We claim that, on the component of $T^m B^c_m$ containing $T^m x$, there are no points of $T^m B^c_R$ in the $(\delta_2\Lambda^{-m}/2)$-neighborhood of $T^m x$. Indeed, if some point $y \in T^m B^c_R$ were there, its LSM $W^s(y)$ would contain a point $y' \in T^m W^1_\infty(z_c)$, which is at distance $\le \delta_3\Lambda^{-m}$ from y. Then $\rho(y', \Gamma \cup \partial M) \le \delta_2\Lambda^{-m}$, since $\delta_3/\delta_2 \le 1/3$. This, however, contradicts the definition of $W^1_\infty(z_c)$, cf. Section 3. We now define the release time to be $f(x) = 2m(x) + \log_\Lambda(2\delta_0/\delta_2)$.

For any point $x \in B^c_\infty \setminus B^c_R$ of either type and any $n \ge f(x)$ the point $T^n x$ should be at least the distance δ_0 from $T^n B^c_R$ (measured along $T^n B^c_n$), so that the component of $T^n B^c_n$ containing $T^n x$ does not intersect $T^n B^c_R$ at all.

Therefore, we are free to define new gaskets and capture new disks on any component $V \subset T^n B_n^c$ that contains at least one released point, i.e. such that $\exists x \in T^{-n}V:\ f(x) \le n$. We can only define a gasket, however, if $\exists x \in V:\ \rho_V(x, \partial V) \ge \delta_1$, i.e. if V is large enough. Hence the next step.

Growth. To control the size of the components of $T^n B_n^c$, we collect, for every $n \ge 0$, the components $V \subset T^n B_n^c$ released at the n-th iterations. We say that V is released at the n-th iteration if at least one point of V is released at this iteration, and none of the points of the component of $T^i B_i^c$ that contains $T^{-(n-i)}V$ is released at the i-th iteration for any $i = 0, \dots, n-1$. In that case we define another function, the 'component release time', $s(x) = n$, on $B_\infty^c \cap T^{-n}V$. Observe that $s(x)$ is defined for each $x \in B_\infty^c \setminus B_R^c$ and $s(x) \le f(x)$.

For any $s \ge 0$ let

$$\tilde{W} = \tilde{W}(s) = \cup\{V \subset T^s B_s^c:\ s(x) = s \ \ \forall x \in B_\infty^c \cap T^{-s}V\} \tag{5.4}$$

be the union of the components of $T^s B_s^c$ released exactly at the s-th iteration. The manifold $\tilde{W}$ carries the measure $\tilde{m}_{\tilde{W}} = T_*^s \tilde{m}_{B^c} | \tilde{W}$. Consider open sets $\tilde{W}_n^1 := \tilde{W} \cap T^s B_{s+n}^c$ and $\tilde{W}_n^0 := \mathrm{int}\,(\tilde{W}_n^1 \setminus \tilde{W}_{n+1}^1)$, $n \ge 0$. It is easy to see that they make a refined $(\delta_2 \Lambda^{-n_c - s})$-filtration of $\tilde{W}$ satisfying Theorem 3.1. Denote then

$$p(s) = \tilde{m}_{\tilde{W}}(\tilde{W}_\infty^1)/\tilde{m}_{\tilde{W}}(\tilde{W}) = \tilde{m}_{\tilde{W}}(\tilde{W} \cap T^s B_\infty^c)/\tilde{m}_{\tilde{W}}(\tilde{W}) \tag{5.5}$$

If $p(s) = 0$, we can simply disregard such a $\tilde{W} = \tilde{W}(s)$. If $p(s) > 0$, then Proposition 3.3 applies to $(\tilde{W}, \tilde{m}_{\tilde{W}})$, according to Final Remark (Part 2). Hence, $\exists n \ge 1$ such that $Z[\tilde{W}_n^1, n] \le 0.6/\delta_1$, i.e. the components of $T^n \tilde{W}_n^1$ are large enough, on the average. Let g be the minimum of such n's. We call g the 'growth time' and define another function, $g(x) = g$ on $B_\infty^c \cap T^{-s}\tilde{W}$ (note that $g(x)$ is a constant function on $B_\infty^c \cap T^{-s}\tilde{W}$, and it only depends on s, so we will also write it as $g(s)$).

Consider now the manifold $\hat{W} = T^g \tilde{W}_g^1$ and the measure $\tilde{m}_{\hat{W}} = T_*^g \tilde{m}_{\tilde{W}} | \hat{W}$ on it. Denote $\hat{W}_\infty^1 = T^g(\tilde{W}_\infty^1) = T^g(\tilde{W} \cap T^s B_\infty^c)$. According to Proposition 3.3,
(c) $\tilde{m}_{\hat{W}}(\hat{W}_\infty^1) > 0.9\, \tilde{m}_{\hat{W}}(\hat{W})$, and
(d) $Z[\hat{W}, \hat{W}, 0] \le 0.6/\delta_1$, so that at least 40% of the points in $\hat{W}$ (with respect to the measure $\tilde{m}_{\hat{W}}$) lie a distance $\ge \delta_1$ away from $\partial \hat{W}$.
Next, we define new gaskets and capture new disks on the large components of $\hat{W}$, as we did to W^g early in this section, and repeat the procedure 'initial growth' applying it to $\hat{W}$. Let $t(x)$ be the 'capture time' for $x \in \hat{W}_\infty^1$, i.e. the minimum of $t \ge 0$ such that $T^t x$ belongs in a captured disk. The next lemma follows from the properties (c) and (d) of the manifold $\hat{W}$ just like Lemma 5.2 and (5.2) followed from the similar properties of the manifold W^g:

Lemma 5.3 *We have $\tilde{m}_{\hat{W}}(t(x) > n)/\tilde{m}_{\hat{W}}(\hat{W}_\infty^1) \le C_0 \theta_0^n$ with the same constants as in (5.2).*

We emphasize that our construction of gaskets is inductive. For a.e. point $x \in W^1_\infty$, the cycle 'growth→capture→release→growth…' repeats until the point returns to $\mathcal{R}$ at some moment of capture. If it never returns, we put it into the leftover set $W^1_{\infty,0}$ and define $r(x) = \infty$. This concludes our definition of the partition $W^1_\infty = \cup_k W^1_{\infty,k}$ and the return time $r(x)$.

Exponential tail bound. We now turn to the proof of the exponential tail bound (5.1). First, we show that the points of any captured disk B^c are released at an exponential rate.

Lemma 5.4 *There are $C_1 > 0$ and $\theta_1 \in (0,1)$ such that for every captured disk B^c we have $\tilde{m}_{B^c}(f(x) > n)/\tilde{m}_{B^c}(B^c) < C_1\theta_1^n$, $\forall n \geq 0$.*

Proof. We have defined the release time $f(x)$ separately for the captured points of types I and II. First, we take care of points of type I. Recall that for any point x of type I we defined a point $h(x) \in W^u_{\delta_1}(z_c)$ and two numbers, $m(x) \geq 0$ and $\varepsilon(x) > 0$. In view of the absolute continuity (2.3), it is enough to estimate the measure $m_{W^u_{\delta_1}(z_c)}\{h(x) : f(x) > n\}$. Fix an $r \in (0, (1+\kappa)^{-1})$. The measure of the set $\{h(x) : m(x) > rn\}$ is exponentially small in n due to the part (iii) of Corollary 3.2 and (4.1). Next, for every $0 \leq m \leq rn$, we have

$$m_{W^u_{\delta_1}(z_c)}\{h(x) : m(x) = m \ \&\ \varepsilon(x) < \delta_0\Lambda^{-(1-r)n}\} \leq 4d_u D\delta_0\delta_1^{-1}\delta_2^{-\kappa}\Lambda^{-(1-r-r\kappa)n}$$

based on the definition of $Z[W, W^0_m, m]$, the part (i) of Corollary 3.2 and (4.1). Due to our choice of r, the right hand side is exponentially small in n, uniformly in m. Thus, the points of type I obey our claim.

For any point x of type II with $m(x) = m$, observe that $T^m x \in \mathcal{U}_{\delta_3\Lambda^{-m}/2}$ and $T^k x \notin \mathcal{U}_{\delta_3\Lambda^{-k}/2}$ for all $k = 0, \ldots, m-1$. Denote $U = B^c$ and consider a $(\delta_3/2)$-filtration $\{U^1_n, U^0_n\}$ of U satisfying Theorem 3.1. Then $x \in U^0_m$, and $\tilde{m}_{B^c}(U^0_m)$ is exponentially small in m by the part (iii) of Corollary 3.2 and (5.3). $\square$.

The next lemma is proven in [7], Lemma 7.2. It shows that the released components in the images of any captured disk B^c grow at an exponential rate:

Lemma 5.5 *There are $C_2 > 0$ and $\theta_2 \in (0,1)$ such that for every captured disk B^c we have $\tilde{m}_{B^c}(s(x) + g(x) > n) < C_2\theta_2^n$, $\forall n \geq 0$.*

This rest of the proof of the exponential tail bound (5.1) goes by the standard argument developed in [5] (pp. 129–130) and used in [18] (Sublemma 6 in Section 7), as it is explained in [7], Section 7, see also an alternative probabilistic argument there.

Theorem 2.1 is proved.

6 Dispersing billiards: background

In the rest of the paper, we apply our main theorem 2.1 to dispersing billiards. This section contains basic properties of dispersing billiards.

Billiard table. Let Q be the closure of a bounded connected domain in $\mathbb{R}^2$ or on a 2-D torus T^2 with a piecewise C^3 smooth boundary ∂Q. If the boundary ∂Q is strictly concave inward at every point of smoothness, then the billiard system in Q is said to be dispersing, or a Sinai billiard, see detailed studies in [16, 10, 4].

A particularly important class of dispersing billiards is that with entirely smooth boundary (no corner points), i.e. such that ∂Q is a finite disjoint union of C^3 smooth simple closed curves. Such tables only exist on the 2-torus T^2. We restrict ourselves to such tables in Sections 6-8 to avoid technical complications. Billiard tables with corner points are discussed separately in Sect. 9.

Let $M' = \partial Q \times [-\pi/2, \pi/2]$ be the standard cross-section of the billiard flow, $T : M' \to M'$ the first return map (billiard ball map), and $\tau(x) > 0$ the return time (the length of the free path till the next collision), see details in [4]. The coordinates on M' are denoted by (r, φ), where $r \in \partial Q$ is the arc length parameter and $\varphi \in [-\pi/2, \pi/2]$ is the angle of reflection. The map T preserves the smooth measure $d\mu = c_\mu \cos\varphi \, dr \, d\varphi$, where c_μ is the normalizing constant. It is known that μ is an SRB measure, the system (T, μ) is ergodic, mixing, K-mixing and Bernoulli [16, 10].

Theorem 6.1 *The billiard ball map (T, μ) enjoys exponential decay of correlations.*

Discontinuity curves. Let $S_0 = \partial Q \times \{\varphi = \pm\pi/2\}$ be the natural boundary of M'. Put $S_n = T^n S_0$ for all $n \in \mathbb{Z}$, and $S_{m,n} = \cup_{i=m}^{n} S_i$ for $-\infty \le m \le n \le \infty$. Each S_n is a finite union of C^2-curves whose slope, in the r, φ coordinates, is positive for $n \ge 1$ and negative for $n \le -1$. The sets $S_{-n,0}$ and $S_{0,n}$ consist of discontinuity curves for T^n and T^{-n}, respectively. The following *continuation property* is important (see also Sect. 10): each endpoint, x_0, of every smooth curve $\gamma \subset S_{-m,0}$, $m \ge 1$, lies either on $\partial M'$ or on another smooth curve $\gamma' \subset S_{-m,0}$ that itself does not terminate at x_0. Hence, each curve $\gamma \in S_{-m,0}$ can be continued up to $\partial M'$ by other curves in $S_{-m,0}$.

Invariant cones and Alignment. Identifying the tangent space at each point with the (r, φ)-plane, the derivative DT maps the cone $\{r\varphi \ge 0\}$ strictly into itself. We call the DT-image of $\{r\varphi \ge 0\}$ the unstable cone C^u. Similarly, DT^{-1} maps $\{r\varphi \le 0\}$ strictly into itself, and C^s is defined accordingly. These two families of cones are DT-invariant in the sense of Sect. 2. The tangent vectors to the curves in S_m belong in unstable cones for $m \ge 1$ and in stable cones for $m \le -1$, this property is often referred to as Alignment.

Transversality. The angles between stable and unstable cones are bounded away from zero. This follows from the fact that the edges of the cones C^u and C^s are uniformly bounded away from the r- and φ-axes. In other words, for any tangent vector $v = (dr, d\varphi)$ in either stable or unstable cone we have

$$0 < B_1^{-1} \le |d\varphi/dr| \le B_1 < \infty \tag{6.1}$$

Here and further on $B_i = B_i(T) > 0$ mean some positive constants.

More precisely, let $x = (r, \varphi) \in M'$ and $v = (dr, d\varphi)$ be a tangent vector at x. We put

$$\mathcal{B}(x, v) = \frac{1}{\cos\varphi}\left(\frac{d\varphi}{dr} + \mathcal{K}(r)\right) \tag{6.2}$$

where $\mathcal{K}(r) > 0$ is the curvature of the boundary ∂Q at the point of reflection, r. Denote by $\mathcal{K}_{\min} > 0$ and $\mathcal{K}_{\max} > 0$ the minimum and maximum of the curvature of ∂Q. Also, for the class of billiards discussed here, $\tau(x) \geq \tau_{\min} > 0$. The value of $\mathcal{B}(x, v)$ represents the curvature of the orthogonal cross-section of the bundle of the outgoing velocity vectors specified by the points $(r + \varepsilon\, dr, \varphi + \varepsilon\, d\varphi)$, $\varepsilon \approx 0$, see [3, 4].

Put $x_1 = (r_1, \varphi_1) = Tx$ and $v_1 = (dr_1, d\varphi_1) = DT(v)$. It follows from the mirror equation of geometric optics, see [3, 4], that

$$\mathcal{B}(x_1, v_1) = \frac{2\mathcal{K}(r_1)}{\cos\varphi_1} + \frac{1}{\tau(x) + \mathcal{B}^{-1}(x, v)} \tag{6.3}$$

Hence,

$$\frac{d\varphi_1}{dr_1} = \mathcal{K}(r_1) + \cos\varphi_1\left(\tau(x) + \cos\varphi\left(\frac{d\varphi}{dr} + \mathcal{K}(r)\right)^{-1}\right)^{-1} \tag{6.4}$$

This proves (6.1) with $B_1 = \max\{\mathcal{K}_{\min}^{-1}, \mathcal{K}_{\max} + \tau_{\min}^{-1}\}$. Note that for billiard tables with corner points $\tau_{\min} = 0$, and so the upper bound in (6.1) fails, see also Sect. 9.

Hyperbolicity. The expansion and contraction of tangent vectors can be conveniently described in a pseudometric that is loosely called p-metric [4, 18]. If $v = (dr, d\varphi)$ is a vector in either stable or unstable cone, then its p-norm is defined by

$$|v|_p = \cos\varphi\, |dr| \tag{6.5}$$

In this norm, DT is uniformly hyperbolic:

$$|DT(v)|_p \geq \Lambda |v|_p \quad \forall v \in C^u, \qquad \text{and} \qquad |DT^{-1}(v)|_p \geq \Lambda |v|_p \quad \forall v \in C^s \tag{6.6}$$

with some constant $\Lambda > 1$. More precisely, the expansion factor of unstable vectors $v = (dr, d\varphi) \in C_x^u$ is given by [4]

$$\frac{|DT(v)|_p}{|v|_p} = 1 + \tau(x)\mathcal{B}(x, v) = 1 + \frac{\tau(x)}{\cos\varphi}\left(\frac{d\varphi}{dr} + \mathcal{K}(r)\right) \tag{6.7}$$

so we can set $\Lambda = 1 + \tau_{\min}(B_1^{-1} + \mathcal{K}_{\min})$. Clearly, the expansion factor is mainly determined by $\cos\varphi$:

$$\frac{B_2^{-1}\tau_{\min}}{\cos\varphi} \leq \frac{|DT(v)|_p}{|v|_p} \leq \frac{B_2\tau(x)}{\cos\varphi} \qquad \forall v \in C^u \tag{6.8}$$

In particular, the derivative DT in the p-metric is unbounded near S_0, where $\cos\varphi \approx 0$.

"Homegeneity strips" and the definition of M. The unboundedness of DT near S_0 makes the distorsion control in the sense of (2.2) particularly difficult for billiards. One has to partition the neighborhood of S_0 into countably many narrow strips parallel to S_0 in each of which the control is possible. We fix a large $k_0 \geq 1$ and for each $k \geq k_0$ define "homogeneity strips"

$$I_k = \{(r, \varphi) : \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\}$$

and

$$I_{-k} = \{(r, \varphi) : -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\}$$

We put

$$I_0 = \{(r, \varphi) : -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}$$

The exact choice of k_0 will be made later.

Now we define an open subset $M \subset M'$, on which T will satisfy all our assumptions. We put $M = \cup I_k$. Moreover, it is convenient to consider I_k as regions in the (r, φ) plane with disjoint closures (as if we cut M' along the boundaries of I_k and moved the strips I_k apart from each other). The map T restricted on M has the singularity set $\Gamma := S_{-1} \cup T^{-1}(\cup_k \partial I_k)$. Since the boundaries of I_k are parallel to S_0, their preimages under T have tangent vectors in stable cones, so that the above Alignment holds for the curves in Γ, just as for S_{-1}. It also holds for all the curves in $\Gamma^{(n)}$, $n \geq 1$, which are defined by (2.1). We will denote also by T the restriction of the original billiard map T on M.

Stable and unstable fibers. It is known that stable and unstable manifolds, or fibers, for the map T on M (in the sense of Section 2) exist at a.e. point $x \in M$. In [4], they were called 'homogeneous fibers'. The boundedness of the curvature of both stable and unstable fibers, as well as the absolute continuity (2.3) are standard facts, see [4, 18].

The distorsion bound (2.2) requires some extra work. In [4], Proposition A1.1(d), it was established that the left hand side of (2.2) is uniformly bounded above. This is a little less than we now require in (2.2). However, a careful analysis of the argument in [4] reveals that in fact more was proved there:

$$\log \prod_{i=0}^{n-1} \frac{J^u(T^i x)}{J^u(T^i y)} \leq \text{const} \cdot [\text{dist}(T^n x, T^n y)]^a \tag{6.9}$$

for some $a > 0$, in the notation of (2.2). Since the argument in [4] is lengthy, we will not repeat it here, besides all the necessary details are there, in the proof of Proposition A1.1(d) in [4].

Next we prove the non-branching of unstable fibers. Let a sequence of LUM's $\{W_n\}$ have a limit point x, and $\rho(x, \partial W_n) > \varepsilon$ for some $\varepsilon > 0$. Then the curves $W_n \cap B_\varepsilon(x)$ converge in the Hausdorff metric to a LUM of length 2ε through x, see [4]. The uniqueness of the LUM $W(x)$ through x implies the non-branching of unstable fibers.

It is also standard that u-SRB measures on unstable fibers for dispersing billiards exist and the invariant measure μ is SRB measure. It then remains to verify our main assumption, on the growth of unstable fibers. This requires some extra work and switching to a higher iterate of T.

7 Smooth billiards with finite horizon

Here we make an additional assumption on Q, that it has "finite horizon", i.e. the free path between successive collisions is uniformly bounded: $\tau(x) \leq \tau_{\max} < \infty$. For this subclass of dispersing billiards Theorem 6.1 was actually proved by Young [18]. We will prove it here by the techniques developed in the previous sections.

Expansion rates in Euclidean metric. Despite the convenience of the p-metric (6.5), we will work in the Euclidean metric $|v| = (dr^2 + d\varphi^2)^{1/2}$ for the reasons explained below.

First, due to (6.1) for any stable or unstable vector v

$$1 \leq \frac{|v| \cos\varphi}{|v|_p} \leq B_3 < \infty \tag{7.1}$$

with $B_3 = (1 + B_1^2)^{1/2}$. Hence, (6.8) implies

$$\frac{B_4^{-1}}{\cos\varphi_1} \leq \frac{|DT(v)|}{|v|} \leq \frac{B_4}{\cos\varphi_1} \qquad \forall v \in C^u \tag{7.2}$$

with $B_4 = B_2 B_3 \max\{\tau_{\max}, \tau_{\min}^{-1}\}$, here again $(r_1, \varphi_1) = Tx$.

The reason why we prefer the Euclidean metric $|\cdot|$ to the p-metric is related to the specific mechanism of growth of unstable fibers under T in the presence of countably many singularity lines in Γ. Let an unstable fibers W be cut into very many, in the worst case countable many, pieces by the set Γ. Then small pieces of $W \setminus \Gamma$ are mapped into the vicinity of S_0, where $\cos\varphi$ is small. In the p-metric, these pieces will experience strong growth at the next iteration of T, due to (6.8). This will allow them to recover in size, as we will show later. Note, however, a time delay: the recovery occurs one iteration *after* (!) the cutting. For this technical reason, the map T on M in the p-metric has no chance to satisfy the assumption (2.6). In the Euclidean metric, the growth occurs *simultaneously* (!) with cutting, as it follows from (7.2). This makes the verification of (2.6) possible.

The expansion factors for unstable vectors under T in the Euclidean metric are not bounded from 1, however. This is one reason why we need to consider a higher power of T:

Lemma 7.1 *There is an $m_1 \geq 1$ such that for any $m > m_1$ and any point $x \in M$*

$$|DT^m(v)| \geq \Lambda^{m-m_1}|v| \quad \forall v \in C^u, \qquad \text{and} \qquad |DT^{-m}(v)| \geq \Lambda^{m-m_1}|v| \quad \forall v \in C^s$$

where these derivatives exist.

Proof. Combining (7.1), (6.8) and (6.6) yields

$$|DT^m(v)| \geq |DT^m(v)|_p \geq \Lambda^{m-1}|DT(v)|_p \geq \frac{\Lambda^{m-1}\tau_{\min}|v|_p}{B_2 \cos\varphi} \geq \frac{\Lambda^{m-1}\tau_{\min}|v|}{B_2 B_3}$$

Hence it is enough to take any m_1 such that $\Lambda^{m_1-1} > B_2B_3/\tau_{\min}$. The stable vectors are treated similarly. □

Denote by $|W|_{\max}$ the maximal length of LUM's in M.

Accumulation of singularity lines. There are two sources of accumulation of the components of the set Γ that can cut LUM's into arbitrary many pieces.

First, the set $\cup T^{-1}(\cup_k \partial I_k)$ consists of countably many curves stretching approximately parallel to some curves in S_{-1} and approaching them. So, each set $T^{-1}I_k$, $k \neq 0$, is a narrow strip with curvilinear boundaries. The expansion of unstable fibers in these strips can be estimated by (7.2). More precisely, let $W \subset T^{-1}I_k$ be a LUM, for some $k \neq 0$. Then the expansion factor, $J^u(x)$, on W satisfies

$$0 < B_5^{-1} \leq k^{-2}|J^u(x)| \leq B_5 < \infty \quad \forall x \in W \tag{7.3}$$

Second, there might be multiple intersections of the curves in S_{-1}. Denote by K_m the multiplicity of $S_{-m,0}$, i.e. maximal number of smooth curves in $S_{-m,0}$ that intersect or terminate at any one point $x \in M'$. It is known [3, 18] that $K_m \leq Am + B$ for some constants $A, B > 0$. We fix an m_2 such that $Am + B + 1 < \Lambda^{m-m_1}$ for any $m \geq m_2$.

A higher iteration of the map T. It is enough to establish exponential decay of correlations for the system (T^m, μ) with any particular $m \geq 1$, see Proposition 10.1 in Section 10. We now fix $m := \min\{m_1, m_2\} + 1$ and let $T_1 := T^m$. Note that $S_{-m,0}$ is the set of singularity curves for the map T_1 on M'. The map T_1 restricted on M has singularity set $\Gamma_1 := \Gamma^{(m)}$, where $\Gamma^{(m)}$ is defined in terms of Γ by (2.1).

The map T_1 has the same stable and unstable cones and the same LUM's and LSM's as does T. Thus, the Alignment and Transversality hold for T_1 as well. Lemma 7.1 implies that

$$|DT_1(v)| \geq \Lambda_1|v| \ \ \forall v \in C^u, \qquad \text{and} \qquad |DT_1^{-1}(v)| \geq \Lambda_1|v| \ \ \forall v \in C^s$$

with $\Lambda_1 := \Lambda^{m-m_1} > 1$, so the map T_1 is uniformly hyperbolic in the Euclidean metric. Our choice of m also ensures that

$$\Lambda_1 > K_m + 1 \tag{7.4}$$

It remains to verify our main assumption, the one on the growth of unstable fibers, but before we introduce a handy indexing system.

Indexing system. Let $\delta_0 > 0$ and W be a δ_0-LUM. If δ_0 is small enough, then W crosses at most K_m curves of the set S_{-m}, so the set $W \setminus S_{-m}$ consists of at most $K_m + 1$ connected curves, call them $W_1, \ldots, W_p$ with $p \leq K_m + 1$. On each of W_j the map T_1

(as a map on M') is smooth, but any W_j's may be cut into arbitrary many or countably many pieces by other curves in Γ_1, which are the preimages of the boundaries of I_k. Let $\Delta \subset W$ be a connected component of the set $W \setminus \Gamma_1$. It can be uniquely identified with the $(m+1)$-tuple $(k_1, \ldots, k_m; j)$ such that $\Delta \subset W_j$ and $T^i \Delta \subset I_{k_i}$ for $1 \le i \le m$. We will then write $\Delta = \Delta(k_1, \ldots, k_m; j)$. Of course, some strings $(k_1, \ldots, k_m; j)$ may not correspond to any piece of W, for such strings $\Delta(k_1, \ldots, k_m; j) = \emptyset$.

Denote by $J_1^u(x) = J^u(x) \cdots J^u(T^{m-1}x)$ the expansion factor of the unstable subspace E_x^u under DT_1. Let $|\Delta| = m_\Delta(\Delta)$ be the Euclidean length of a LUM Δ. We record two important facts:

(a) For every point $x \in \Delta(k_1, \ldots, k_m; j)$ we have

$$J_1^u(x) \ge L_{k_1,\ldots,k_m} := \max\left\{\Lambda_1,\ B_6 \prod_{k_i \ne 0} k_i^2\right\}$$

where $B_6 = (\max\{B_4, B_5\})^{-m}$. This follows from (7.2) and (7.3).

(b) For each $\Delta(k_1, \ldots, k_m; j)$ we have

$$|\Delta(k_1, \ldots, k_m; j)| \le M_{k_1,\ldots,k_m} := \min\left\{|W|,\ B_7 \prod_{k_i \ne 0} k_i^{-2}\right\}$$

where $B_7 = B_6^{-1}|W|_{\max}$. This follows from the previous fact.

Next, put

$$\theta_0 := 2 \sum_{k=k_0}^{\infty} k^{-2} \le 4/k_0$$

Growth of unstable fibers. Let W be a δ_0-LUM and $\delta > 0$ be small. Due to the Transversality, the angles between W and the curves of Γ_1 that cross W are uniformly bounded away from zero. For each connected component $\Delta \subset W \setminus \Gamma_1$ put $\Delta^0 = \Delta \cap \mathcal{U}_\delta$ and $\Delta^1 = \mathrm{int}(\Delta \setminus \mathcal{U}_\delta)$, where $\mathcal{U}_\delta$ is the δ-neighborhood of $\Gamma_1 \cup \partial M$. Due to the Transversality and Continuation properties, the set Δ^0 consists of two subintervals adjacent to the endpoints of Δ (they may overlap and cover Δ, of course). The set Δ^1 is either empty or a subinterval of Δ. We put $W^1 = \cup_{\Delta \subset W \setminus \Gamma_1} \Delta^1$.

For each Δ^1 the set $T_1(\Delta_1 \cap \{r_{W^1,1} < \varepsilon\})$ is the union of two subintervals of $T_1\Delta^1$ of length ε adjacent to the endpoint of $T_1\Delta^1$. Using the above indexing system gives

$$\begin{aligned} m_W(r_{W^1,1} < \varepsilon) &\le \sum_{k_1,\ldots,k_m,j} 2\varepsilon L_{k_1,\ldots,k_m}^{-1} \\ &\le 2\varepsilon p \left[\Lambda_1^{-1} + B_6(\theta_0 + \theta_0^2 + \cdots + \theta_0^m)\right] \\ &\le 2\varepsilon(K_m + 1)\left(\Lambda_1^{-1} + B_6 m\theta_0\right) \end{aligned} \tag{7.5}$$

We now assume that k_0 is large enough so that

$$\alpha_0 := (K_m + 1)(\Lambda_1^{-1} + B_6 m\theta_0) < 1$$

and thus get

$$m_W(r_{W^1,1} < \varepsilon) \le \min\{|W|, 2\alpha_0\varepsilon\}$$

The first term on the right hand side of (2.6) is equal to

$$\alpha_0\Lambda_1 \min\{|W|, 2\varepsilon/\Lambda_1\} = \min\{\alpha_0\Lambda_1|W|, 2\alpha_0\varepsilon\}$$

Since $\alpha_0\Lambda_1 > 1$, we get

$$m_W(r_{W^1,1} < \varepsilon) \le \alpha_0\Lambda_1 \cdot m_W\left(r_{W,0} < \varepsilon/\Lambda_1\right) \tag{7.6}$$

Next, to obtain an open $(\delta_0, 1)$-subset V_δ^1 of W^1, one needs to further subdivide the intervals $\Delta^1 \subset W$ such that $|T_1\Delta^1| > \delta_0$. Each such LUM $T_1\Delta^1$ we divide into s_Δ equal subintervals of length $\le \delta_0$, with $s_\Delta \le |T_1\Delta^1|/\delta_0$. If $|T_1\Delta^1| < \delta_0$, then we set $s_\Delta = 0$ and leave Δ^1 unchanged. Then union of the preimages under T_1 of the above intervals will make V_δ^1. Now we must estimate the measure of the ε-neighborhood of the additional endpoints of the subintervals of $T_1\Delta^1$. This gives

$$\begin{aligned} m_W(r_{V_\delta^1,1} < \varepsilon) - m_W(r_{W^1,1} < \varepsilon) &\le \sum_{\Delta\subset W\setminus\Gamma_1} 2s_\Delta\varepsilon|B_9\Delta^1|/|T_1\Delta^1| \\ &\le \sum_{\Delta\subset W\setminus\Gamma_1} 2B_9\varepsilon|\Delta^1|/\delta_0 \\ &\le 2B_9\varepsilon\delta_0^{-1}|W| \end{aligned}$$

Here $B_9 = \exp(\text{const} \cdot |W|^a_{\max})$ is an upper bound on distorsions on LUM's, see (6.9). Combining the above bound with (7.6) completes the proof of (2.6) with $\beta_0 = 2B_9$.

We now prove (2.7). It is enough to consider $\varepsilon < |W|/2$, so that the right hand side of (2.7) equals $2D_0\delta^{-\kappa}\varepsilon$. We can put $V_\delta^0 = W \setminus \overline{V_\delta^1}$. Then the left hand side of (2.7) does not exceed $2J_\delta\varepsilon$, where J_δ is the number of nonempty connected components of the set $\overline{V_\delta^0}$, which is at most the number of connected components of $W \setminus \Gamma_1$ of length $> 2\delta$. Hence, clearly $J_\delta \le |W|/\delta \le \delta_0/\delta$. This proves (2.7) with $\kappa = 1$.

Lastly, we prove the inequality (2.8). Again, let Δ be a connected component of $W \setminus \Gamma_1$ and Δ^0, Δ^1 be defined as above, with the set Δ^0 consisting of two subintervals adjacent to the endpoints of Δ. Since the angles between W and curves in $\Gamma_1 \cup \partial M$ are bounded away from zero, each of these subintervals has length between δ and $B_8\delta$, where B_8 depends on the minimum angle between LUM's and curves in $\Gamma_1 \cup \partial M$.

Now, the right hand side of (2.8) equals $D_0 \min\{|W|, 2\zeta\delta^\sigma\}$. So, it is enough to show that $m_W(V_\delta^0) \le B\delta^\sigma$ for some $B, \sigma > 0$. We have

$$\begin{aligned} m_W(V_\delta^0) &\le \sum_{\Delta\subset W\setminus\Gamma_1} \min\{2B_8\delta, |\Delta|\} \\ &\le \sum_{k_1,\dots,k_m,j} \min\{2B_8\delta, M_{k_1,\dots,k_m}\} \\ &\le \text{const}\cdot\delta + \text{const}\cdot \sum_{k_1,\dots,k_m}^{*} \min\left\{\delta, \prod_{k_i\neq 0} k_i^{-2}\right\} \end{aligned}$$

where $\sum^*$ is taken over m-tuples that contain at least one nonzero index $k_i \neq 0$. The following lemma, which is proved in Appendix, completes the proof of (2.8) with $\sigma = (2m)^{-1}$.

Lemma 7.2 *Let $\delta > 0$ and $m \geq 1$. Then*

$$\sum_{k_1,\dots,k_m \geq 2} \min\left\{\delta, (k_1 \cdots k_m)^{-2}\right\} \leq B(m) \cdot \delta^{1/2m}$$

8 Smooth billiards without horizon

Here we relax the assumption on 'finite horizon', i.e. allow arbitrarily long free runs between consecutive collisions. In particular, this is always the case when ∂Q is just one closed curve on T^2.

In this case, the singularity set $S_m \subset M'$ for each $m \neq 0$ is a countable (not finite!) union of smooth curves. These curves accumulate in the vicinities of a finite number of points $\omega_1, \dots, \omega_s \in S_0$, whose trajectories only contain grazing (tangent) reflections at the boundary ∂Q, so that their velocity vectors never change. The finite set $\Omega = \{\omega_1, \dots, \omega_s\}$ is T-invariant. Moreover, for any open set $U \supset \Omega$ there is another open set $V \supset \Omega$ such that $TV \subset U$.

Cell structure of M'. The structure of the singularity curves S_{-1} near the points $\omega_1, \dots, \omega_s$ is described in [3, 4] in great detail. We will need the following facts here:
(a) The curves S_{-1} partition M' into a countable number of connected regions, which we call *cells*. The neighborhood of each point $\omega_j \in \Omega$ contains infinitely many small cells whose sizes decrease as they approach ω_j. Small cells near each ω_j can be naturally labelled $\mathcal{C}_{j,t}$ with $t = 1, 2, \dots$, see [3, 4]. (b) For each cell $\mathcal{C}_{j,t}$ and every $x \in \mathcal{C}_{j,t}$ we have $\mathrm{const}_1 \cdot t \leq \tau(x) \leq \mathrm{const}_2 \cdot t$. Hence, the expansion factor at x satisfies

$$\frac{B_{10}^{-1} t}{\cos\varphi_1} \leq \frac{|DT(v)|}{|v|} \leq \frac{B_{10} t}{\cos\varphi_1} \qquad \forall v \in C^u \tag{8.1}$$

where $(r_1, \varphi_1) = Tx$, as in (7.2). This follows from (6.8) and (7.1).
(c) For each small cell $\mathcal{C}_{j,t}$ and every $x \in \mathcal{C}_{j,t}$ we have $\cos\varphi_1 \leq B_{11} t^{-1/2}$, where again $(r_1, \varphi_1) = Tx$. (A similar bound holds for $\cos\varphi$, but we will not need it.)

Convention. In all that follows, we only consider sufficiently small cells, with numbers $t \geq t_0$, where t_0 is large and will be fixed later. Put $\mathcal{C} = \cup_{j=1}^{s} \cup_{t \geq t_0} \mathcal{C}_{j,t}$. This set is small, and its complement $M' \setminus \mathcal{C}$ makes 'most of ' M'. We need not label any cells in $M' \setminus \mathcal{C}$.

Now, we will repeat the arguments of Section 7, working out the necessary modifications. The bound (7.1) still holds. The bound (7.2) holds for all $x \notin \mathcal{C}$, i.e. in the 'main part' of M', where $\tau(x)$ is bounded. For $x \in \mathcal{C}$ we have the bound (8.1). Lemma 7.1 still holds, because its proof only uses the lower bound on $\tau(x)$.

The analysis of the accumulation of singularity lines has to be supplemented now, since the curves of S_{-1} additionally accumulate near each point $\omega_j \in \Omega$. The bound (7.3) holds for all $x \in (M' \setminus \mathcal{C}) \cap T^{-1}I_k$, $k \neq 0$, whereas for each $x \in \mathcal{C}_{j,t} \cap T^{-1}I_k$, $k \neq 0$, we have

$$0 < B_{12}^{-1} \leq t^{-1}k^{-2}|J^u(x)| \leq B_{12} < \infty \quad \forall x \in W \tag{8.2}$$

and for each $x \in \mathcal{C}_{j,t} \setminus \cup_{k\neq 0} T^{-1}I_k$ we have

$$0 < B_{12}^{-1} \leq t^{-1}|J^u(x)| \leq B_{12} < \infty \quad \forall x \in W \tag{8.3}$$

These follow from (8.1).

The bound $K_m \leq Am + B$ was proved in [3] without assuming finite horizon, so it holds now. The choice of the iteration T^m does not change, so we again arrive at (7.4).

Indexing system. This needs to be more elaborate. Let U be a small neighborhood of Ω, and V another neighborhood of Ω such that $T^iV \subset U$ for all $0 \leq i \leq m$. The set $S_{-m} \setminus V$ consists of a finite number of smooth curves. Hence, $\exists \delta_0 > 0$ such that any δ_0-LUM $W \subset M' \setminus V$ crosses at most K_m curves of the set S_{-m}. In this case we call $W_1, \ldots, W_p$, $p \leq K_m + 1$, the connected components of $W \setminus S_{-m}$. If $W \subset V$, then $T^iW \subset U$ for all $0 \leq i \leq m$, so T^iW can only cross the boundaries of some small cells. In this case we put $W_1 = W$. In either case, each connected component Δ' of the set $W \setminus S_{-m}$ can be uniquely identified with the $(m+1)$-tuple $(l_1, \ldots, l_m; j)$ such that $\Delta' \subset W_j$ and l_i, $1 \leq i \leq m$, is defined by $T^{i-1}\Delta' \subset \mathcal{C}_{j_i,l_i}$ with some $1 \leq j_i \leq s$ if $T^{i-1}\Delta' \subset \mathcal{C}$, and $l_i = 2$ otherwise. Now, each connected component Δ of the set $W \setminus \Gamma_1$, where $\Gamma_1 = \Gamma^{(m)}$ is defined as in sect. 7, can be uniquely identified with the $(2m+1)$-tuple $(l_1, k_1, \ldots, l_m, k_m; j)$ where $(k_1, \ldots, k_m)$ are defined as in Sect. 7. We will then write $\Delta = \Delta(l_1, k_1, \ldots, l_m, k_m; j)$.

We record three important facts:

(d) For every point $x \in \Delta(l_1, k_1, \ldots, l_m, k_m; j)$ we have

$$J_1^u(x) \geq L_{l_1,k_1,\ldots,l_m,k_m} := \max\left\{\Lambda_1,\ B_{13} \prod_{l_i,k_i\neq 0} k_i^2 l_i\right\}$$

This follows from (8.2) and (8.3).

(e) For each $\Delta(l_1, k_1, \ldots, l_m, k_m; j)$ we have

$$|\Delta(k_1, \ldots, k_m; j)| \leq M_{l_1,k_1,\ldots,l_m,k_m} := \min\left\{|W|,\ B_{14} \prod_{l_i,k_i\neq 0} k_i^{-2} l_i^{-1}\right\}$$

(f) For each $k_i \neq 0$ we have $l_i \leq \chi k_i^4$ and for each $k_i = 0$ we have $l_i \leq \chi k_0^4$, with $\chi = 2B_{11}^2$. This follows from the fact (c) above.

We now assume that $t_0 > \chi k_0^4$. Then, in our indexing system, for each $k_i = 0$ we have $l_i = 2$. Next, put

$$\theta_1 : \;=\; 2 \sum_{k=k_0}^{\infty} \sum_{l=[\chi k_0^4]}^{[\chi k^4]} k^{-2} l^{-1}$$

$$
\begin{aligned}
&\leq \text{const} \cdot \sum_{k=k_0}^{\infty} k^{-2} \ln k \\
&\leq \text{const} \cdot k_0^{-1} \ln k_0
\end{aligned}
$$

Growth of unstable fibers. We now proceed with the proofs of (2.6)-(2.8) using the same notation as in Section 7. As in (7.5), we have

$$
\begin{aligned}
m_W(r_{W^1,1} < \varepsilon) &\leq \sum_{l_1,k_1,\dots,l_m,k_m,j} 2\varepsilon L^{-1}_{l_1,k_1,\dots,l_1,k_m} \\
&\leq 2\varepsilon p \left[\Lambda_1^{-1} + B_{13}(\theta_1 + \theta_1^2 + \cdots + \theta_1^m)\right]
\end{aligned}
$$

We now assume that k_0 is large enough so that

$$\alpha_0 := (K_m + 1)(\Lambda_1^{-1} + B_{13} m \theta_1) < 1$$

This fixes our choice of k_0, and hence t_0. After that we complete the proof of (2.6) as in Sect. 7, word by word.

The proof of (2.7) does not change.

To prove (2.8) as in Sect. 7, we note that

$$
\begin{aligned}
m_W(V_\delta^0) &\leq \sum_{l_1,k_1,\dots,l_m,k_m,j} \min\{2B_8\delta, M_{l_1,k_1,\dots,l_m,k_m}\} \\
&\leq \text{const} \cdot \delta + \text{const} \cdot \sum^{*}_{l_1,k_1,\dots,l_m,k_m} \min\left\{\delta, \prod_{l_i,k_i \neq 0} l_i^{-1} k_i^{-2}\right\}
\end{aligned}
$$

where $\sum^*$ is taken over $2m$-tuples that contain at least one nonzero index $k_i \neq 0$. The following lemma, which is proved in Appendix, completes the proof of (2.8) with $\sigma = (6m+1)^{-1}$.

Lemma 8.1 *Let $\delta > 0$ and $m \geq 1$. Then*

$$\sum_{k_1,\dots,k_m \geq 2} \sum_{l_1=2}^{[\chi k_1^4]} \cdots \sum_{l_m=2}^{[\chi k_m^4]} \min\left\{\delta, (l_1 \cdots l_m)^{-1}(k_1 \cdots k_m)^{-2}\right\} \leq B(m) \cdot \delta^{1/(6m+1)}$$

9 Dispersing billiard tables with corner points

In this section we consider dispersing billiard tables $Q \subset \mathbb{R}^2$. They necessarily have corner points, i.e. intersections of smooth curves of ∂Q. We assume, as usual [3, 4], that all such intersections are transversal, i.e. the angle made by the sides of Q at each corner point is positive. By the way, this is widely believed to be a necessary assumption for exponential decay of correlation, because otherwise the decay seems to be polynomial [12].

New singularity lines. Let $\hat{r}_1, \dots, \hat{r}_t$ be the r-cooridanes of the corner points of ∂Q. Put $V_0 = \{(r, \varphi) \in M' : r = \hat{r}_1, \dots, \hat{r}_t\}$. It is convenient to cut M' along the segments $\{r = \hat{r}_i\}$, $1 \le i \le t$, that make V_0 and then think of M' as a union of disjoint rectangles (each bounded by two S_0 segments and two V_0 segments) and cylinders (each bounded by two S_0 closed curves), see [3, 4]. Then $S_0 \cup V_0$ will be the natural boundary of M'. We use the notations $V_m = T^{-m}V_0$ and $V_{m,n}$ in the same way as S_m and $S_{m,n}$, Sect. 6. Then the singularity set for T^m, $m \ge 1$, is $S_{-m,0} \cup V_{-m,0}$. This set has the continuation property, see Sect. 6. The Alignment holds as well, i.e. all the tangent vectors to V_m are in unstable cones for $m > 0$ and in the stable cones for $m < 0$.

Denote by K_m the multiplicity of $S_{-m,0} \cup V_{-m,0}$, i.e. the maximal number of smooth curves of this set that intersect or terminate at any one point of M'. Unlike the previous sections, it is not known for the present class of billiards how fast K_m grows with m. We have to assume that it does not grow too fast. Specifically, there is a large enough m such that

$$K_m < \Lambda_0^{m-m_3} - 1 \tag{9.1}$$

where the constants $\Lambda_0 > 1$ and m_3 are defined below. Similar bounds are commonly assumed in the literature [3, 4, 11, 18]. The bound (9.1) is widely believed to hold for generic billiard tables [3], even though this is not known. There will be no more assumptions on the region Q in this section.

A detailed study of billiard tables with corner points was done in [3, 4]. We will recall the necessary facts.

Corner series. The new phenomenon here is the existence of series of two or more consecutive reflections near a corner point. During those series, the free paths are short, i.e. $\tau(x) \approx 0$, and so the expansion of unstable vectors, even in the p-metric, is weak, due to (6.7). Let us fix a sufficiently small $\varepsilon > 0$, and call a series of consecutive reflections a *corner series* if they all occur in the ε-neighborhood of one corner point. Three facts make the analysis easier:

(a) The number of reflections in any corner series is uniformly bounded above (by some $m_0 \ge 1$). So, there is a constant $\tau'_{\min} > 0$ such that for each $x \in M$ there is an $i \in \{0, \dots, m_0 - 1\}$ such that $\tau(T^i x) \ge \tau'_{\min}$.

(b) Each corner series contains at most one grazing reflection, and that reflection is necessarily the first or the last one in the series. So, there is a constant $c_0 > 0$ such that in each corner series $T^i x = (r_i, \varphi_i)$, $0 \le i \le g$, we have $\cos\varphi_i > c_0$ for all i's, except possibly one, and that exceptional one is either $i = 0$ or $i = g$.

(c) The curvature of LUM's, LSM's and all smooth curves in $S_m \cup V_m$, $m \in \mathbb{Z}$, is uniformly bounded above.

We call a corner series $T^i x$, $0 \le i \le g$, with no grazing reflections (i.e., such that $\cos\varphi_i > c_0$ for all $0 \le i \le g$) a *regular* one. Corner series with the first grazing reflection ($\cos\varphi_0 < c_0$) are said to be *left-singular* and those with the last grazing reflection ($\cos\varphi_g < c_0$) *right-singular*.

"Weaker" tranversality. As in previous sections, the angles between stable and unstable cones are uniformly bounded away from zero, see [3, 4] and below. The lower bound in (6.1) still holds. But the upper bound in (6.1) sometimes fails, and we now describe precisely where and how this happens. Let $T^i x = (r_i, \varphi_i)$, $0 \le i \le g$, be a left-singular corner series. Consider unstable vectors $(dr_i, d\varphi_i) \in C^u_{x_i}$. Only the vector $(dr_0, d\varphi_0)$ satisfies the upper bound in (6.1). For $1 \le i \le g$, we have another bound

$$\frac{B_{15}^{-1}}{t_i + \cos\varphi_0} \le \frac{d\varphi_i}{dr_i} \le \frac{B_{15}}{t_i + \cos\varphi_0} \tag{9.2}$$

where $t_i = \tau(x_0) + \cdots + \tau(x_{i-1})$. This bound easily follows from the detailed estimates in [4], Appendix 1 (A1.3). (Alternatively, (9.2) can be derived from (6.4) by induction on i.) Stable vectors at points $T^i x$, $0 \le i \le g$, satisfy the upper bound in (6.1). Similar facts hold for right-singular series, but now only stable vectors fail to satisfy the upper bound in (6.1). For regular corner series and for reflections away from corner points, the bound (6.1) is never broken. Observe that at no point $x \in M'$ can stable and unstable vectors simultaneously violate the upper bound of (6.1). This is exactly the reason why the angles between stable and unstable cones are bounded away from zero.

Hyperbolicity. The expansion and contraction in the p-metric described by (6.7) is no longer uniform, we just have

$$|DT(v)|_p \ge |v|_p \;\; \forall v \in C^u, \quad \text{and} \quad |DT^{-1}(v)|_p \ge |v|_p \;\; \forall v \in C^s \tag{9.3}$$

However, we still have the uniform hyperbolicity in the sense of (6.6) whenever $\tau(x) > \tau'_{\min} > 0$, with

$$\Lambda := 1 + \tau'_{\min}(B_1 + \mathcal{K}_{\min}) > 1$$

In particular, the expansion and contraction is uniform for the map T^{m_0}:

$$|DT^{m_0}(v)|_p \ge \Lambda_0^{m_0} |v|_p \;\; \forall v \in C^u, \quad \text{and} \quad |DT^{-m_0}(v)|_p \ge \Lambda_0^{m_0} |v|_p \;\; \forall v \in C^s \tag{9.4}$$

with

$$\Lambda_0 := \Lambda^{1/m_0} > 1$$

Next, the homogeneity strips I_k and the region M are defines exactly as in Sect. 6. The properties of stable and unstable fibers and SRB measure described in the end of Sect. 6 are valid without change.

Expansion rates in Euclidean metric. Despite certain deterioration of hyperbolicity in terms of the p-metric, it does not get any worse in terms of the Euclidean metric, as the following lemma shows, cf. (7.2).

Lemma 9.1 *Let $x = (r, \varphi) \in M$ and $Tx = (r_1, \varphi_1) \in M$. Then*

$$\frac{B_{16}^{-1}}{\cos\varphi_1} \le \frac{|DT(v)|}{|v|} \le \frac{B_{16}}{\cos\varphi_1} \qquad \forall v \in C^u \tag{9.5}$$

Proof. We will prove the lower bound, the proof of the upper bound is completely similar. (We will only need the lower bound, anyway.) Denote $v = (dr, d\varphi)$ and $DT(v) = (dr_1, d\varphi_1)$. First of all,

$$\frac{|DT(v)|}{|v|} = \frac{(dr_1^2 + d\varphi_1^2)^{1/2}}{(dr^2 + d\varphi^2)^{1/2}} = \frac{(1 + (d\varphi_1/dr_1)^2)^{1/2}}{(1 + (d\varphi/dr)^2)^{1/2}} \cdot \frac{\cos\varphi}{\cos\varphi_1} \cdot \frac{|DT(v)|_p}{|v|_p}$$

Next, we use the lower bound in (6.1), which always holds, recall that $B_3 = (1 + B_1^2)^{1/2}$, and then substitute (6.7):

$$\begin{aligned}
\frac{|DT(v)|}{|v|} &\geq \frac{d\varphi_1/dr_1}{B_3 \cdot d\varphi/dr} \cdot \frac{\cos\varphi}{\cos\varphi_1} \cdot \frac{|DT(v)|_p}{|v|_p} \\
&\geq \frac{d\varphi_1/dr_1}{B_3 \cdot d\varphi/dr} \cdot \frac{\cos\varphi}{\cos\varphi_1} \cdot \left(1 + \frac{\tau(x)}{\cos\varphi}\frac{d\varphi}{dr}\right) \\
&= \frac{d\varphi_1/dr_1}{B_3 \, \cos\varphi_1} \cdot \left(\left(\frac{d\varphi}{dr}\right)^{-1} \cdot \cos\varphi + \tau(x)\right)
\end{aligned}$$

Now, consider three cases:
Case 1: $\tau(x) > \tau'_{\min}$. Clearly, (9.5) holds with $B_{16} = B_1 B_3/\tau'_{\min}$.
Observe that if $\tau(x) < \tau'_{\min}$, then the points x and Tx belong in one corner series.
Case 2: the corner series containing x and Tx is regular or right-singular. Then $d\varphi/dr \leq B_1$ and $\cos\varphi \geq c_0$, so (9.5) holds with $B_{16} = B_1^2 B_3/c_0$.
Case 3: the points x and Tx belong in a left-singular corner series. Then we have two subcases:
(3a) x is the first point in that corner series. Then $d\varphi/dr \leq B_1$ and $d\varphi_1/dr_1 \geq B_{15}^{-1}(\tau(x) + \cos\varphi)^{-1}$ by (9.2). Hence, (9.5) holds with $B_{16} = B_1 B_3 B_{15}$.
(3b) x is not the first point of the corner series, which then starts at some other point, call it $T^{-j}x = (\tilde{r}, \tilde{\varphi})$, $1 \leq j \leq m_0$. Denote $t = \sum_{i=1}^{j} \tau(T^{-i}x)$. Now,

$$d\varphi_1/dr_1 \geq B_{15}^{-1}(t + \tau(x) + \cos\tilde{\varphi})^{-1} \quad \text{ and } \quad d\varphi/dr \leq B_{15}(t + \cos\tilde{\varphi})^{-1}$$

due to (9.2), and also $\cos\varphi \geq c_0$. Now (9.5) follows with $B_{16} = B_3 B_{15}^2/c_0$.

We now set $B_{16} = \max\{B_1 B_3/\tau'_{\min}, B_1^2 B_3/c_0, B_1 B_3 B_{15}, B_3 B_{15}^2/c_0\}$. The lemma is proved. $\square$

Next, we prove an analogue of Lemma 7.1.

Lemma 9.2 *There is an $m_3 \geq 1$ such that for any $m > m_3$ and any point $x \in M \setminus S_{-m,0} \cup V_{-m,0}$*

$$|DT^m(v)| \geq \Lambda_0^{m-m_3}|v| \qquad \forall v \in C^u$$

A similar bound holds for stable vectors.

Proof. Let $j_1 = 1 + \min\{i \geq 0 : \tau(T^i x) \geq \tau'_{\min}\}$. and $j_2 = 1 + \min\{i \geq j_1 : \tau(T^i x) \geq \tau'_{\min}\}$. Note that $j_1 \leq m_0 + 1$ and $j_2 \leq 2m_0 + 2$. Note also that the points $T^{j_1} x$ and $T^{j_1 - 1} x$ cannot belong in one corner series, so the vector $DT^{j_1} v$ satisfies the upper bounds in (6.1) and (7.1). Due to (9.3) and (9.4), we have

$$|DT^m(v)| \geq |DT^m(v)|_p \geq \Lambda_0^{m - j_2 - m_0} |DT^{j_2}(v)|_p \tag{9.6}$$

Next, we need the following standard estimate for dispersing billiards:

Sublemma 9.3 *Let $x = (r, \varphi) \in M$ and $v = (dr, d\varphi) \in C^u_x$. Then for any $n \geq 1$*

$$\frac{|DT^n(v)|_p}{|v|_p} \geq 1 + \frac{\tau(x) + \cdots + \tau(T^{n-1} x)}{\cos \varphi} \left(\frac{d\varphi}{dr} + \mathcal{K}(r) \right)$$

Proof. For all $0 \leq i \leq n$, denote $T^i x = x_i = (r_i, \varphi_i)$, $\tau_i = \tau(x_i)$ and $\mathcal{B}_i = \mathcal{B}(x_i, DT^i(v))$, cf. (6.2). It follows from (6.3) that $\mathcal{B}_i^{-1} \leq \tau_{i-1} + \mathcal{B}_{i-1}^{-1}$, and so

$$\mathcal{B}_i^{-1} \leq \tau_0 + \cdots + \tau_{i-1} + \mathcal{B}_0^{-1}$$

Now, due to (6.7),

$$\frac{|DT^{i+1}(v)|_p}{|DT^i(v)|_p} = 1 + \tau_i \mathcal{B}_i \geq \frac{\tau_0 + \cdots + \tau_i + \mathcal{B}_0^{-1}}{\tau_0 + \cdots + \tau_{i-1} + \mathcal{B}_0^{-1}}$$

Multiplying this estimate for all $i = 0, \ldots, n-1$ gives

$$\frac{|DT^n(v)|_p}{|v|_p} \geq \frac{\tau_0 + \cdots + \tau_{n-1} + \mathcal{B}_0^{-1}}{\mathcal{B}_0^{-1}}$$

which proves the sublemma. $\square$

We now complete the proof of Lemma 9.2. Let $T^{j_1} x = (r_{j_1}, \varphi_{j_1})$. We subsequently use the sublemma, the lower bound in (6.1), the upper bound in (7.1) for the vector $DT^{j_1}(v)$, and the lower bound in (9.5):

$$\begin{aligned} |DT^{j_2}(v)|_p &\geq \frac{\tau(T^{j_1} x) + \cdots + \tau(T^{j_2 - 1} x)}{B_1 \cos \varphi_{j_1}} |DT^{j_1}(v)|_p \\ &\geq \frac{\tau'_{\min}}{B_1 B_3} |DT^{j_1}(v)| \\ &\geq \frac{\tau'_{\min}}{B_1 B_3 B_{16}^{j_1}} |v| \end{aligned}$$

Recall that $j_1, j_2 \leq m_0 + 1$. So, it is enough to take any m_3 such that $\Lambda_0^{m_3 - 2m_0 - 1} > B_1 B_3 B_{16}^{m_0 + 1} / \tau'_{\min}$. Lemma 9.2 is proved. $\square$

Accumulation of singularity lines and the map T_1. As in Section 7, the boundaries of the regions $T^{-1} I_k$, $k \geq 0$, accumulate near some curves of S_{-1}. For each LUM $W \subset$

$T^{-1}I_k$, $k \geq 0$, we again have the estimate (7.3) (with a different value of B_5, though) since it follows from (9.5).

We now fix a sufficiently large $m > m_3$ for which (9.1) holds. Let $T_1 := T^m$. Then Lemma 9.2 implies that

$$|DT_1(v)| \geq \Lambda_1 |v| \quad \forall v \in C^u \quad \text{and} \quad |DT_1^{-1}(v)| \geq \Lambda_1 |v| \quad \forall v \in C^s$$

with $\Lambda_1 := \Lambda_0^{m-m_3} > 1$. also, (9.1) implies

$$\Lambda_1 > K_m + 1$$

Note also that the set $S_{-m,0} \cup V_{-m,0}$ is a finite union of smooth compact curves.

We are now in exactly the same position as in Section 7. So, the indexing system used in that section and the proofs of (2.6)-(2.7) go through without change.

The proof of (2.8) requires a correction, though, because now some curves in ∂M (specifically, the segments of V_0) are not uniformly transversal to unstable fibers. As a result, for some LUM's W the set $\mathcal{U}_\delta$ may cover on W an interval longer than const$\cdot\delta$, unlike what we had in Sect. 7.

To overcome this problem, we invoke a useful estimate, proved in [3], Lemma 2.7: for any LUM W and any point $x = (r, \varphi) \in W$ the tangent vector $(dr, d\varphi)$ to W satisfies

$$\frac{d\varphi}{dr} \leq \frac{B_{17}}{|r - r_0|^{1/2}}$$

where (r_0, φ_0) is the endpoint of W closets to x. Hence, $|\varphi - \varphi_0| \leq 2B_{17}|r - r_0|^{1/2}$, so that the δ-neighborhood of V_0 can only cover an interval on W of length $\leq B_{18}\delta^{1/2}$. We now finish the proof of (2.8) in a manner similar to that of Sect. 7:

$$\begin{aligned} m_W(V_\delta^0) &\leq \sum_{\Delta \subset W \setminus \Gamma_1} \min\{2B_{18}\delta^{1/2}, |\Delta|\} \\ &\leq \sum_{k_1,\dots,k_m,j} \min\{2B_{18}\delta^{1/2}, M_{k_1,\dots,k_m}\} \\ &\leq \text{const} \cdot \delta^{1/2} + \text{const} \cdot \sum_{k_1,\dots,k_m}^{*} \min\left\{\delta^{1/2}, \prod_{k_i \neq 0} k_i^{-2}\right\} \end{aligned}$$

where $\sum^*$ is taken over m-tuples that contain at least one nonzero index $k_i \neq 0$. The bound (2.8) now follows from Lemma 7.2, but with $\sigma = (4m)^{-1}$ rather than $\sigma = (2m)^{-1}$.

10 Final remarks and discussion

Theorem 2.1 obviously holds for functions that are only Hölder continuous on the connected components of the set $M \setminus \Gamma^{(m)}$ for some $m \geq 1$. Moreover, it can be naturally extended to a wider class of the so called piecewise Hölder continuous functions, as defined in [4, 6].

In applications, it is often enough to prove Theorem 2.1 for any power, T^m, of the map T:

Proposition 10.1 *Let $m \geq 2$. Assume that the map T^l is Hölder continuous (with some exponent $\eta_l > 0$) on every connected component of $M \setminus \Gamma^{(l)}$ for each $l = 1, \ldots, m$. If T^m enjoys exponential decay of correlations, then so does T.*

Proof. Let $n \geq 1$, and $n = km + l$ with some $0 \leq l \leq m-1$. Let $f, g \in \mathcal{H}_\eta$. Then

$$\begin{aligned} \int_M (f \circ T^n) g \, d\mu &= \int_M \left(f \circ T^n - f \circ T^{km} \right) g \, d\mu + \int_M (f \circ T^{km}) g \, d\mu \\ &= \int_M (h_l \circ T^{km}) g \, d\mu + \int_M (f \circ T^{km}) g \, d\mu \end{aligned} \tag{10.1}$$

where $h_l = f \circ T^l - f$. The function h_l is Hölder continuous (with exponent $\eta_l \eta > 0$) on each connected component of $M \setminus \Gamma^{(l)}$. Since l takes a finite number of values, both integrals in (10.1) are exponentially small in k. □

Lastly, we discuss the assumptions of our main theorem 2.1.

First, the assumption on the existence of an ergodic SRB measure μ does not seem to be necessary. Indeed, it can be often proved under various general assumptions similar to ours, see [13, 15, 18], and the proof is normally easier than that of statistical properties of μ. We intentionally left out this problem in the paper, in order to focus on the EDC and CLT. Note, however, that the other assumptions in Section 2 do not logically imply the existence of SRB measures, as the following example shows.

Example. Let $R = \{(x, y) : 0 < x < 1, \, y > 1\}$ be an open strip in $\mathbb{R}^2$, and let $M' = \{(s, t) : 0 \leq s \leq 1, \, 0 \leq t \leq 1\}$ with the identification of $s = 0$ and $s = 1$ be a closed cylinder. Let $T_1 : R \to R$ be given by $(x, y) \to (x/3 + 1/3, 2y - 1)$ and $T_2 : R \to M'$ be defined by $s = y \pmod 1$ and $t = e^{-y} + x(e^{-y-1} - e^{-y})$. Then $M = T_2(R)$ is an open subset of M', and the map $T = T_2 \circ T_1 \circ T_2^{-1}$ takes M to M. It satisfies all the assumptions of Section 2 (other than the existence of an SRB measure), with $\Gamma = \emptyset$, but has no SRB measure.

We now comment on our main assumptions (2.6)–(2.8). They are proved in [7] in the case where $\Gamma \cup \partial M$ was a finite union of smooth compact hypersurfaces, and T had one-sided derivatives on $\Gamma \cup \partial M$.

Assume now that $\Gamma \cup \partial M$ consists of a *countable* number of smooth compact hypersurfaces. Three additional assumptions, all valid for billiard systems, may significantly simplify the proof of (2.6)–(2.8):

Bounded curvature. If the sectional curvature of the smooth components of $\Gamma \cup \partial M$ is uniformly bounded, then one can approximate them by hyperplanes (since they are almost flat on the small scale of our δ_0-LUM's).

Continuation. Assume that each boundary point, x_0, of every smooth component $\gamma \subset \Gamma$ lies either on ∂M or on another smooth component $\gamma' \subset \Gamma$ that itself does not terminate at x_0. Also, assume that for each point $x \in M$ there is a neighborhood $V(x)$ that intersect only a finite number of smooth components of Γ (i.e., infinitely many components of Γ can only accumulate near ∂M).

Transversality. The tangent planes to $\Gamma \cup \partial M$ and unstable cones are uniformly transversal, i.e. the angles between them (properly defined in [7]) are bounded away from zero.

The above properties imply the following. Let W be a LUM, and $x \in W \cap \mathcal{U}_\delta$. Then x lies in a $(B\delta)$-neighborhood of the set $(W \cap \Gamma) \cup \partial W$. Here the constant $B > 0$ is determined by the minimum angle between the tangent planes to $\Gamma \cup \partial M$ and unstable cones. This property allows to work with the $(B\delta)$-neighborhood of the intersection $W \cap \Gamma$ when proving (2.6)–(2.8). This is exactly what we did in Sections 7–8, as well as in [7].

Lastly, in the case $\dim E^u = d_u = 1$, the assumption (2.7) always holds, and our proof in Section 7 applies.

Appendix

Here we provide the proofs of the technical estimates in Lemmas 7.2 and 8.1. We denote by Vol_m the m-dimensional volume in $\mathbb{R}^m$.

Sublemma A.1 *Let $A > 1$ and $m = 1, 2, \ldots$. Consider the region $R_m(A) \subset \mathbb{R}^m$ defined by*

$$R_m(A) = \{x_1, \ldots, x_m \geq 1,\ x_1 \cdots x_m < A\}$$

Then $\mathrm{Vol}_m R_m(A) \leq A(\ln A)^m$.

Proof goes by induction on m. The case $m = 1$ is trivial. For $m \geq 1$, we have

$$\begin{aligned} \mathrm{Vol}_m R_m(A) &= \int_1^A dx_m\, \mathrm{Vol}_{m-1} R_{m-1}(A/x_m) \\ &\leq \int_1^A dx_m\, A x_m^{-1} (\ln A)^{m-1} = A(\ln A)^m \end{aligned}$$

Sublemma A.2 *Let $A, B > 1$ and $m \geq 1$, $k \geq 2$. Consider the region $R_{2m}(A, B, k) \subset \mathbb{R}^{2m}$ defined by*

$$R_{2m}(A, B, k) = \{x_1, y_1, \ldots, x_m, y_m \geq 1,\ x_1 \cdots x_m < A,\ (x_1 \cdots x_m) \cdot (y_1 \cdots y_m)^k < B\}$$

Then $\mathrm{Vol}_{2m} R_{2m}(A, B, k) \leq 4A^{1-1/k} B^{1/k} (\ln A)^m (\ln B)^m$.

Proof. We have

$$\begin{aligned} \mathrm{Vol}_{2m} R_{2m}(A, B, k) &= \int_{R_m(A)} dx_1 \cdots dx_m\, \mathrm{Vol}_m R_m\left(\left[\frac{B}{x_1 \cdots x_m}\right]^{1/k}\right) \\ &\leq \int_{R_m(A)} dx_1 \cdots dx_m\, \left[\frac{B}{x_1 \cdots x_m}\right]^{1/k} \left(\ln B^{1/k}\right)^m \\ &\leq B^{1/k} (\ln B)^m \int_{R_{m-1}(A)} \frac{dx_1 \cdots dx_{m-1}}{(x_1 \cdots x_{m-1})^{1/k}} \int_1^{A/x_1 \cdots x_{m-1}} \frac{dx_m}{x_m^{1/k}} \\ &\leq 2A^{-1/k} B^{1/k} (\ln B)^m \int_{R_{m-1}(A)} \frac{A\, dx_1 \cdots dx_{m-1}}{x_1 \cdots x_{m-1}} \\ &= 2A^{-1/k} B^{1/k} (\ln B)^m \left[\mathrm{Vol}_m R_m(A) + \mathrm{Vol}_{m-1} R_{m-1}(A)\right] \end{aligned}$$

and then use the previous sublemma. □

Proof of Lemma 7.2. Let $Z_\delta \subset Z \subset \mathbb{Z}^m$ be the subsets defined by $Z = \{k_1, \dots, k_m \geq 2\}$ and

$$Z_\delta = \{k_1, \dots, k_m \geq 2,\ (k_1 \cdots k_m)^2 \leq \delta^{-1}\}$$

We estimate the cardinality $|Z_\delta|$ from above. An m-cube with side 1 centered at any point $Q \in Z_\delta$ lies wholly in the region $R_m(C\delta^{-1/2})$ with, say, $C = 2^m$. Therefore,

$$|Z_\delta| \leq \mathrm{Vol}_m R_m(C\delta^{-1/2}) \leq \mathrm{const} \cdot \delta^{-1/2} \left(\ln \delta^{-1/2}\right)^m$$

Next, for any point $(k_1, \dots, k_m) \in Z \setminus Z_\delta$ there is a k_i such that $k_i \geq \delta^{-1/2m}$. Let $K_\delta = [\delta^{-1/2m}]$. Then

$$\begin{aligned} \sum_{Z \setminus Z_\delta} (k_1 \cdots k_m)^{-2} &\leq \sum_{i=1}^{m} \sum_{Z \cap \{k_i \geq K_\delta\}} (k_1 \cdots k_m)^{-2} \\ &\leq \mathrm{const}(m) \cdot K_\delta^{-1} \leq \mathrm{const}(m) \cdot \delta^{1/2m} \end{aligned}$$

Lemma 7.2 is proved. □

Proof of Lemma 8.1. Let $Z'_\delta \subset Z' \subset \mathbb{Z}^{2m}$ be the subsets defined by

$$Z' = \{l_1, k_1, \dots, l_m, k_m \geq 2\} \cap \{l_1 \leq \chi k_1^4, \dots, l_m \leq \chi k_m^4\}$$

and

$$Z'_\delta = \{(l_1, k_1, \dots, l_m, k_m) \in Z' :\ (l_1 \cdots l_m) \cdot (k_1 \cdots k_m)^2 \leq \delta^{-1}\}$$

We estimate the cardinality $|Z'_\delta|$ from above. Observe first that $l_1 \cdots l_m \leq \chi^{m/3}\delta^{-2/3}$ in Z'_δ. Hence, a cube with side 1 centered at any point $Q \in Z'_\delta$ lies wholly in the region $R_{2m}(C_1\delta^{-2/3}, C_2\delta^{-1}, 2)$ with, say, $C_1 = 2^m\chi^{m/3}$ and $C_2 = 2^{3m}$. Therefore,

$$|Z'_\delta| \leq \mathrm{Vol}_{2m} R_{2m}(C_1\delta^{-2/3}, C_2\delta^{-1}, 2) \leq \mathrm{const} \cdot \delta^{-5/6} \left(\ln \delta^{-1}\right)^{2m}$$

Next, for any point $(l_1, k_1, \dots, l_m, k_m) \in Z' \setminus Z'_\delta$ we have $\chi^m (k_1 \cdots k_m)^6 \geq \delta^{-1}$, so there is a k_i such that $k_i \geq \chi^{-1/6}\delta^{-1/6m}$. Let $K'_\delta = [\chi^{-1/6}\delta^{-1/6m}]$. Then

$$\begin{aligned} \sum_{Z' \setminus Z'_\delta} (l_1 \cdots l_m)^{-1}(k_1 \cdots k_m)^{-2} &\leq \sum_{i=1}^{m} \sum_{Z' \cap \{k_i \geq K'_\delta\}} (l_1 \cdots l_m)^{-1}(k_1 \cdots k_m)^{-2} \\ &\leq \mathrm{const}(m) \cdot K_\delta^{-1} \ln K_\delta \\ &\leq \mathrm{const}(m) \cdot \delta^{1/(6m+1)} \end{aligned}$$

Lemma 8.1 is proved. □

References

[1] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes Math., **470**, Springer-Verlag, Berlin, 1975.

[2] L.A. Bunimovich and Ya.G. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Comm. Math. Phys. **78** (1981), 479–497.

[3] L.A. Bunimovich, Ya.G. Sinai and N.I. Chernov, *Markov partitions for two-dimensional hyperbolic billiards*, Russ. Math. Surv. **45** (1990), 105–152.

[4] L.A. Bunimovich, Ya.G. Sinai and N.I. Chernov, *Statistical properties of two-dimensional hyperbolic billiards*, Russ. Math. Surv. **46** (1991), 47–106.

[5] N.I. Chernov, *Ergodic and statistical properties of piecewise linear hyperbolic automorphisms of the 2-torus*, J. Statist. Phys. **69** (1992), 111–134.

[6] N.I. Chernov, *Statistical properties of the periodic Lorentz gas. Multidimensional case*, J. Statist. Phys. **74** (1994), 11–53.

[7] N.I. Chernov, *Statistical properties of piecewise smooth hyperbolic systems in high dimensions*, manuscript, 1997.

[8] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[9] P. Garrido and G. Gallavotti, *Billiards correlation function*, J. Statist. Phys. **76** (1994), 549–585.

[10] G. Gallavotti and D. Ornstein, *Billiards and Bernoulli schemes*, Comm. Math. Phys. **38** (1974), 83–101.

[11] C. Liverani, *Decay of correlations*, Annals of Math. **142** (1995), 239–301.

[12] J. Machta, *Power law decay of correlations in a billiard problem*, J. Statist. Phys. **32** (1983), 555–564.

[13] Ya. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties* Ergod. Th. Dyn. Sys. **12** (1992), 123–151.

[14] D. Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math. **98** (1976), 619–654.

[15] E.A. Sataev, *Invariant measures for hyperbolic maps with singularities*, Russ. Math. Surv. **47** (1992), 191–251.

[16] Ya.G. Sinai, *Dynamical systems with elastic reflections: Ergodic properties of dispersing billiards*, Russ. Math. Surv. **25** (1970), 137–189.

[17] Ya.G. Sinai, *Gibbs measures in ergodic theory*, Russ. Math. Surv. **27** (1972), 21–69.

[18] L.-S. Young, *Statistical properties of systems with some hyperbolicity including certain billiards*, Annals of Math., to appear.