CHAPTER 2

Equivalent Systems

The *classification* of dynamical systems is one of the main goals of the subject. One begins by placing an equivalence relation upon the set of all dynamical systems. This relation should be a natural one, in the sense that it is based on qualitative resemblances of the systems. One then attempts to list the equivalence classes, distinguishing between them by numerical and algebraic *invariants* (quantities that are associated with all systems and that are equal for all systems in the same equivalence class). If one cannot achieve a classification of the set of all dynamical systems, one would at any rate like to classify a large (in some sense) subset of it. A good classification requires a careful choice of basic equivalence relation, with tractable invariants. There are several "obvious" relations to be considered and we discuss these in some detail.

The search for invariants involves a study not only of the structure of individual orbits of a system, but also of the topological relationships between orbits. This results in the formulation of such concepts as *limit set*, *minimal set* and *non-wandering set*.

I. TOPOLOGICAL CONJUGACY

Let $f: X \to X$ and $g: Y \to Y$ be homeomorphisms of topological spaces X and Y. A *topological conjugacy* from f to g is a homeomorphism $h: X \to Y$ such that $hf = gh$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[g]{} & Y \end{array}$$

The homeomorphisms f and g (and the discrete flows that they generate) are said to be *topologically conjugate* if such a homeomorphism h exists. Trivially, topological conjugacy is an equivalence relation.

(2.1) Exercise. Show that a topological conjugacy maps orbits onto orbits, periodic points to periodic points, and preserves periods. Prove that if h is a topological conjugacy from f to g, and if $f^n(x) \to a$ as $n \to \infty$, then $g^n(y) \to b$ as $n \to \infty$, where $y = h(x)$ and $b = h(a)$.

(2.2) Remark. In this book we are mainly concerned with the situation where X and Y are differentiable manifolds and f and g are diffeomorphisms. In this context, it might seem natural to require the map h to be a diffeomorphism, rather than just a homeomorphism. This modification gives the notion of *differentiable conjugacy*. Differentiable conjugacy is a stronger relation than topological conjugacy and, correspondingly, there are, in general, many more equivalence classes with respect to it. This does not inevitably mean that a differentiable classification is harder to make than a topological one (see Exercise 2.3 below). However, we do find that, with differentiable conjugacy, *stable* diffeomorphisms (ones which stay in the same equivalence class when slightly perturbed) are very rare. We also have to class as non-equivalent diffeomorphisms which most people would feel are qualitatively the same (for example, the contractions $x \mapsto \frac{1}{2}x$ and $x \mapsto \frac{1}{3}x$ of the real line $\mathbf{R}$). For these reasons, topological conjugacy continues as the basic equivalence relation when we restrict ourselves to the differentiable category.

(2.3) Exercise. Prove, by differentiation at the origin, that if two linear automorphisms of $\mathbf{R}^n$ are differentiably conjugate then they are similar. (Thus the differentiable classification of $GL(\mathbf{R}^n)$ is the classical theory of (real) Jordan canonical form (see Chapter 4). In contrast, the topological classification is very hard, and not completely solved at the time of writing (see Kuiper and Robbin [1] and Chapter 4).) Prove, more generally, that if f and g are diffeomorphisms of a differentiable manifold X and if a differentiable conjugacy h from f to g takes a fixed point p to q then T_pf is similar to T_qg.

II. HOMEOMORPHISMS OF THE CIRCLE

To get some feeling for the difficulties involved in the classification problem, we discuss briefly the situation when X is the circle S^1. The results here (due to Poincaré and Denjoy) are too complete to be typical of compact manifolds in general. As is usual in the one-dimensional case, the ordering of the real line plays a distinctive role. We sketch the ideas behind the theory, and leave some of the details as Exercise 2.4 below. A more complete account is given in Chapter 1 of Nitecki [1].

Let $f: S^1 \to S^1$ be a homeomorphism. We suppose that f *preserves orientation*, in that, when any point moves clockwise on S^1 then so does its image under f (see Exercise 2.4 and the Appendix to this chapter for more accurate definitions). We focus our attention on some point $x \in S^1$, and measure the average angle $\theta(n, x)$ turned through by this point in n successive applications of the map f. The reader will immediately, and correctly, protest that this average is not well defined, since the individual angles are all ambiguous by integer multiples of 2π. However, we shall see in the exercise that it is possible, by moving from S^1 to its covering space $\mathbf{R}$, to resolve the ambiguities well enough to overcome this objection. We now let n tend to ∞. For any $y \in S^1$, the sequence $\theta(n, x) - \theta(n, y)$ will tend to zero, since otherwise, intuitively, one point will overtake the other under successive applications of f. Similarly, but perhaps rather less clearly, it can be argued that $\theta(n, x)$ converges, since otherwise x will overtake itself. Thus we are led to believe in a *rotation number* $\rho(f)$, defined modulo 1, such that, for all $x \in S^1$, the average angle turned through by x under infinitely many successive applications of the map f is $2\pi\rho(f)$. It is clear that this number is an invariant of orientation preserving topological conjugacy, for, if g and h are orientation preserving homeomorphisms of S^1 such that $g = hfh^{-1}$, then $g^n = hf^nh^{-1}$. For large n, the single maps h and h^{-1} make a negligible contribution to the average rotation, so $\rho(g) = \rho(f)$.

If f has a periodic point x of period s, then $\rho(f)$ is rational, of form r/s. This is immediate, since f^s rotates x through an angle $2\pi r$ for some integer r. Rather less obviously, the converse also holds. Thus, for example, $\rho(f) = 0$ if and only if f has a fixed point. It is clear that the rotation number does not by itself classify homeomorphisms of S^1. For instance, given any closed subset E of S^1, one may construct a homeomorphism f of S^1 whose fixed point set is precisely E. The complement $S^1 \backslash E$ is essentially a countable collection of open real intervals and one merely defines f as an increasing homeomorphism on each such interval. (Moreover one can, if one is careful, construct f to be as smooth as one wishes.) Of course two homeomorphisms with non-homeomorphic fixed point sets are not topologically conjugate (see Exercise 2.1). A rather similar situation holds for $\rho(f)$ any rational number r/s, since then f^s has fixed points.

The case $\rho(f)$ irrational is in complete contrast. Here, provided the maps concerned are C^2, $\rho(f)$ *is* a complete set of invariants of orientation preserving topological conjugacy. That is to say, two C^2 diffeomorphisms f and g with irrational rotation numbers are topologically conjugate if and only if $\rho(f)=\rho(g)$ or $\rho(f)=1-\rho(g)$. If we allow maps which are not C^1 with bounded variation, we introduce new conjugacy classes for each irrational rotation number. These have been classified by Markley [1].

(2.4) Exercise. Let $S^1=\mathbf{R}/\mathbf{Z}$ and let $p:\mathbf{R}\to S^1$ be the quotient map. Let $f:S^1\to S^1$ be a homeomorphism. A *lifting* of f is a continuous map $\tilde{f}:\mathbf{R}\to\mathbf{R}$ such that $p\tilde{f}=fp$. Note that p maps every open interval of length 1 in $\mathbf{R}$ homeomorphically onto the complement of a point in S^1. Using this fact, obtain a lifting of f by first constructing it on $]0, 1[$ and then extending it to $\mathbf{R}$ in the only way possible. Prove that any two liftings of f differ by an integer constant function. We say that f is *orientation preserving* if its liftings are increasing functions, and *orientation reversing* if they are decreasing.

Now let f be orientation preserving, and let $\tilde{f}$ be a lifting of f. Prove that, for all $n>0$, $\tilde{f}^n-id$ is periodic of period 1. Let α_n be the greatest integer $\leqslant\tilde{f}^n(0)$. Prove that, for all $m>0$ and $x\in[0, 1]$, $m\alpha_n\leqslant\tilde{f}^{mn}(x)\leqslant m(\alpha_n+2)$, and hence that

$$\left|\frac{1}{mn}\tilde{f}^{mn}(0)-\frac{1}{n}\tilde{f}^n(0)\right|\leqslant\frac{2}{n}.$$

Deduce that $(1/n)\tilde{f}^n(0)$ is a Cauchy sequence. Prove that, for all $x, y\in\mathbf{R}$, $|\tilde{f}^n(x)-\tilde{f}^n(y)|\leqslant|x-y|+1$. Deduce that, as $n\to\infty$, $(1/n)\tilde{f}^n(x)$ converges to a limit which is independent of x. Since two liftings differ by an integer, the congruence class of this limit modulo 1 depends only on the underlying map f, and is called the *rotation number* $\rho(f)$ of f.

Prove that $\rho(f)$ is an invariant of orientation preserving topological conjugacy, and that if g is topologically conjugate to f by an orientation reversing conjugacy then $\rho(g)=1-\rho(f)$. Prove further that $\rho(f)$ is rational of form r/s if and only if f has a periodic point of period s(for necessity, show that f^s has a fixed point). Finally, suppose that $\rho(f)$ is irrational, and prove that, for all $x\in\mathbf{R}$, the map sending $n\rho(f)+m$ to $\tilde{f}^n(x)+m$ is an increasing map of the subset $\{m+n\rho(f)\colon m, n\in\mathbf{Z}\}$ of $\mathbf{R}$ into $\mathbf{R}$. Deduce that if the orbit of $p(x)$ is dense in S^1 then f is topologically conjugate to the rotation map $p(y)\mapsto p(y+\rho(f))$.

III. FLOW EQUIVALENCE AND TOPOLOGICAL EQUIVALENCE

Let ϕ and ψ be flows on topological spaces X and Y respectively. We say that $h:X\to Y$ is a *flow map from ϕ to ψ* if it is continuous and if there exists an increasing continuous homomorphism $\alpha:\mathbf{R}\to\mathbf{R}$ such that the diagram

$$\begin{tikzcd}
\mathbf{R}\times X \arrow[r, "\phi"] \arrow[d, "\alpha\times h"'] & X \arrow[d, "h"] \\
\mathbf{R}\times Y \arrow[r, "\psi"'] & Y
\end{tikzcd}$$

commutes. Recall that α is just multiplication by a positive constant. If, further, h is a homeomorphism we call the pair (α, h), or the single map h in case $\alpha = id$, a flow equivalence from ϕ to ψ. We say that ϕ is *flow equivalent* to ψ if there exists such a pair (α, h). Thus, in this case, ψ is the flow induced on Y from ϕ by (α, h) (see Example 1.26).

(2.5) Exercise. Let (α, h) be a flow equivalence from a flow ϕ on X to a flow ψ on Y. Show that h is a topological conjugacy from $f = \phi^1$ to $g = \psi^t$, where $t = \alpha(1)$.

Although the notion of flow equivalence seems very natural, it is rather too strong for the qualitative theory of flows. For example, it preserves ratios of periods of closed orbits, and flows may differ in this aspect and yet have a very similar appearance (in fact they may have identical phase portraits!). We now define a weaker equivalence relation which is usually regarded as the basic one in the subject. We say that $h: X \to Y$ is a *topological equivalence* from ϕ to ψ if it is a homeomorphism which maps each orbit of ϕ onto an orbit of ψ, and preserves orientation of orbits. Intuitively this last requirement means that h takes the direction of increasing t on each orbit of ϕ to the direction of increasing t on the corresponding orbit of ψ (see Figure 2.6). More precisely, *h preserves the orientation* of $\mathbf{R}.x$ if there exists an increasing homeomorphism $\alpha: \mathbf{R} \to \mathbf{R}$ such that, for all $t \in \mathbf{R}$, $h\phi(t, x) =$

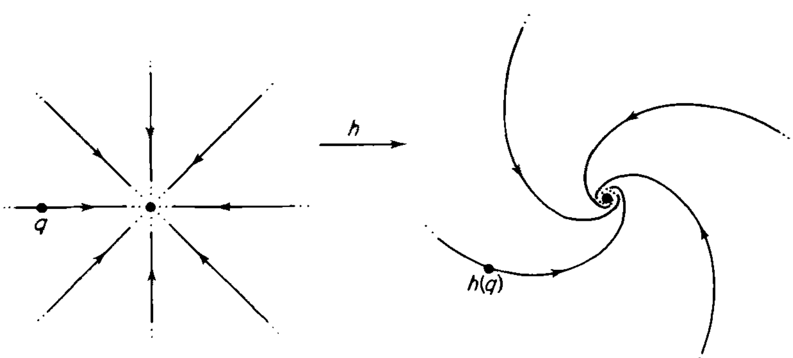

FIGURE 2.6

$\psi(\alpha(t), h(x))$. Similarly, h *reverses the orientation* of $\mathbf{R}.x$ if there exists a decreasing homeomorphism $\alpha : \mathbf{R} \to \mathbf{R}$ satisfying this equation. It is straightforward to check that these properties are independent of the choice of x on the orbit, for if $x' \in \mathbf{R}.x$ then $\phi_x^{-1}\phi_{x'} : \mathbf{R} \to \mathbf{R}$ is a translation.

(2.7) Remarks. (a) Remark 2.2 made the point that it is undesirable to strengthen "topological" to "smooth" in the definition of conjugacy of diffeomorphisms. This applies equally to the definition of equivalence of smooth flows, for the same reasons.

(b) The above definitions of a homeomorphism preserving and reversing orientation of orbits are perfectly satisfactory, but they leave various questions unanswered. For example, can a homeomorphism of one orbit onto another fail to either preserve or reverse the orientation? We deal with this and similar points in the appendix to this chapter.

(c) The reader may have noticed that topological conjugacy is the immediate analogue of flow equivalence and wondered about the discrete analogue of topological equivalence of flows. We say that homeomorphisms f and g of a topological space X are *topologically equivalent* if there is a homeomorphism h of X taking orbits of f onto orbits of g. This relation has not been widely studied. It is different from topological conjugacy. For example, rotations of S^1 through $[\frac{1}{5}]$ and $[\frac{2}{5}]$ are topologically equivalent (with $h = id$), but, as we have seen above, they are not topologically conjugate. However, quite often topological equivalence of f and g is the same as topological conjugacy of f and either g or g^{-1}. This is certainly the case if the complement of Per f in X is path connected and dense in X, a result due to Kupka [2]. For let f and g be related by a homeomorphism h as above. Any two points x and y in $X\backslash\text{Per} f$ may be joined by a continuous curve $\gamma : I \mapsto X\backslash\text{Per} f$, where $I = [0, 1]$. If we denote by $C_n\{t \in I : g^n h\gamma(t) = hf\gamma(t)\}$ for all $n \in \mathbf{Z}$, then it is easy to see that the set C_n are mutually disjoint, closed and have union I. It follows from Lemma 2.49 below that $I = C_n$ for some $n \in \mathbf{Z}$, and hence that $g^n h = hf$ on $X\backslash\text{Per} f$. We deduce that $n = \pm 1$, since otherwise images of orbits of f under h are not whole orbits of g. Finally, by continuity, $g^n h = hf$ on X.

(2.8) Exercise. Prove that flow equivalence and topological equivalence are equivalence relations, and that the former implies the latter. Find a flow on $\mathbf{R}^2$ that is topologically equivalent but not flow equivalent to Example 1.16. Prove that Examples 1.15 and 1.16 are topologically equivalent to their reverse flows. Prove that any topological equivalence induces a homeomorphism of orbit spaces.

(2.9) Example. *Rational and irrational flows on the torus.* Our task here is to compare the suspension and product constructions for rational and irrational flows given in Chapter 1, and to classify the flows obtained. Recall that the suspension of a rotation through $[\theta] \in S^1 = \mathbf{R}/\mathbf{Z}$ is a flow on the space $X = (\mathbf{R} \times S^1)/\sim$, where $\sim$ is defined by $(x, [y]) \sim (x', [y'])$ if and only if $x = x' + m$, $m \in \mathbf{Z}$, and $[y'] = [y + m\theta]$. The flow is defined by $t.[x, [y]] = [t + x, [y]]$, where the outer brackets denote equivalence classes with respect to $\sim$. It is not completely clear at first that X is the torus. We define a map h from X to $T^2 = S^1 \times S^1$ by $h([x, [y]]) = ([x], [y + \theta x])$. It is well defined since, all m and $n \in \mathbf{Z}$,

$$h([x - m, [y + n + m\theta]]) = ([x - m], [y + n + m\theta + \theta(x - m)]) = ([x], [y + \theta x]).$$

It is continuous since the quotient maps concerned are local homeomorphisms and h is induced by them from a linear isomorphism of $\mathbf{R}^2$. Moreover, h is a homeomorphism, since it has a well defined continuous inverse given by $h^{-1}([x], [y]) = [x, [y - \theta x]]$. Finally note that h is a flow equivalence from the suspension flow to the product of a pair of rotation flows with speeds 1 and θ. This is because

$$\begin{aligned} t.h([x, [y]]) &= t.([x], [y + \theta x]) = ([x + t], [y + \theta x + \theta t]) = h([x + t, [y]]) \\ &= h(t.[x, [y]]). \end{aligned}$$

Thus:

(2.10) Theorem. *For any $\theta \in \mathbf{R}$, the suspension of a rotation of S^1 through $[\theta]$ is flow equivalent to the produce of two rotation flows on S^1 with speeds 1 and θ.* □

The maps h and h^{-1} are, in fact, C^ω, so the two constructions are equivalent in the strongest possible sense. For the rest of the section then, we may think in terms of product flows as we proceed towards a classification.

We begin with some trivial remarks. For any pair of flows ϕ and ψ, the product flow $\phi \times \psi$ is flow equivalent to $\psi \times \phi$, by the homeomorphism which interchanges factors. Thus if we consider a product of two rotation flows and if the product is non trivial we may assume that the first factor is non trivial. Moreover, by speeding up or slowing down both factors simultaneously, we may assume that the speeds of the factors are respectively 1 and θ. Note also that by Theorem 2.10 we get a flow equivalent flow if we replace θ by any $\theta' \in [\theta]$.

Before we go any further, we must say a few words about homeomorphisms of T^2, and introduce some notation. Let $\pi : \mathbf{R}^2 \to T^2$ be the quotient map $\pi(x, y) = ([x], [y])$, and let $h : T^2 \to T^2$ be any homeomorphism satisfying $h\pi(0, 0) = \pi(0, 0)$. Then h lifts to a unique continuous map $H : \mathbf{R}^2 \to \mathbf{R}^2$ such that $H(0, 0) = (0, 0)$ and the diagram commutes. This is because π is a *covering* (see Greenberg [1]); we have homeomorphisms of the form $(\pi|V)^{-1}h(\pi|U)$ for any open square U of side 1 in $\mathbf{R}^2$ and for any connected component V of the set $\pi^{-1}(h\pi(U))$, and continuity determines how we

$$\begin{tikzcd} \mathbf{R}^2 \arrow[r, "H"] \arrow[d, "\pi"'] & \mathbf{R}^2 \arrow[d, "\pi"] \\ T^2 \arrow[r, "h"] & T^2 \end{tikzcd}$$

should stick some of these homeomorphisms together to form H. The map H is a homeomorphism, its inverse H^{-1} being the lifting of h^{-1}. It has the property that there exist integers p, q, r and s such that, for all $(x, y) \in \mathbf{R}^2$ and for all $(m, n) \in \mathbf{Z}^2$,

$$H(x+m, y+n) = H(x, y) + (pm + rn, qm + sn).$$

In particular, H maps $\mathbf{Z}^2$ linearly into itself with matrix

$$A = \begin{bmatrix} p & r \\ q & s \end{bmatrix}.$$

In fact, $H|\mathbf{Z}_2$ is just the map h_* (or $H_1(h)$): $H_1(T^2) \to H_1(T^2)$ of homology theory (see, for example, Greenberg [1]). It describes the fact that h wraps the circle $S^1 \times \{0\}$ p times round the first factor of $S^1 \times S^1$ and q times round the second factor, with corresponding numbers r and s for $\{0\} \times S^1$. Since H^{-1} also has an associated integer matrix, B say, and $HH^{-1} = id$, we deduce that B is the inverse of A, and so $A \in GL_2(\mathbf{Z})$. The matrix A defines a linear automorphism, denoted by L_h, of $\mathbf{R}^2$, and this covers a homeomorphism denoted by l_h, of T^2. Notice that if

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in GL_2(\mathbf{Z})$$

then p and q are coprime, because any common factor also divides the determinant of A, and this is ± 1. Conversely if p and q are coprime integers then there exist r and s such that

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in GL_2(\mathbf{Z}).$$

To see this, assume $0 \leq p \leq q$, since

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in GL_2(\mathbf{Z})$$

if

$$\begin{bmatrix} q & s \\ p & r \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -p & -r \\ -q & -s \end{bmatrix} \in GL_2(\mathbf{Z}).$$

We observe that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in GL_2(\mathbf{Z}),$$

and if $0 < p < q$, we may choose for s any integer with $s(q-p) \equiv 1 \bmod q$ and put $r = (1 + ps/q)$.

The situation as regards classification should now be somewhat clearer. The reason that the rational flows corresponding to $\theta = 0$ and $\theta = \frac{2}{3}$ look different is that we are not mentally allowing for homeomorphisms of T^2 that alter the homology of the generators $S^1 \times \{0\}$ and $\{0\} \times S^1$; we are letting our picture of T^2 as a submanifold of $\mathbf{R}^3$ dominate and limit our imagination. For each circle of rational slope q/p in T^2, there is a linear automorphism of $\mathbf{R}^2$ covering a homeomorphism of T^2 that maps the generator $S^1 \times \{0\}$ onto the circle. Thus the homeomorphism equates the $\theta = 0$ case to the $\theta = q/p$ case. On the other hand, since there are only countably many such homeomorphisms, they cannot equate the uncountable infinity of irrational flows. Of course there are uncountably many other homeomorphisms h of T^2 to consider as possible equivalences, but there is a countable restriction placed upon them by the associated maps l_h, and this turns out to be vital. We now fill in the details. We first prove:

(2.11) Theorem. *All rational flows on T^2 are flow equivalent.*

Proof. It is enough to give a flow equivalence from the product of two rotation flows with speeds 1 and θ to the product of two rotation flows with speeds p and q, where p and q are coprime integers with $p \neq 0$. Choose

$$A = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in GL_2(\mathbf{Z}),$$

and let $h: T^2 \to T^2$ be the homeomorphism covered by the linear automorphism of $\mathbf{R}^2$ with matrix A. Then h is a flow equivalence, since

$$\begin{aligned} h(t.([x],[y])) &= h([x+t], y) \\ &= ([px + pt + ry], [qx + qt + sy]) \\ &= t.([px + ry], [qx + sy]) \\ &= t.h([x],[y]). \end{aligned}$$

□

In classifying irrational flows it is, as we have seen above, sufficient to consider products of two rotation flows with speeds 1 and θ.

(2.12) Theorem. *For any irrational numbers α and β, the $\theta = \alpha$ and $\theta = \beta$ cases are topologically equivalent if and only if there exists $A \in GL_2(Z)$ such that $A\begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix}$.*

Proof. Suppose such a matrix A exists. The corresponding linear automorphism of $\mathbf{R}^2$ takes lines of slope α to lines of slope β, and so induces a homeomorphism h of T^2 taking orbits of the $\theta=\alpha$ flow to orbits of the $\theta=\beta$ flow. In fact, h is a flow equivalence, since

$$\begin{aligned} h(t.([x],[y]) &= h([x+t],[y+\alpha t]) \\ &= h([x],[y]) + h([t],[\alpha t]) \\ &= h([x],[y]) + ([t],[\beta t]) \\ &= t.h([x],[y]). \end{aligned}$$

Now suppose, conversely, that $h: T^2 \to T^2$ is a topological equivalence from the $\theta=\alpha$ flow to the $\theta=\beta$ flow. We may suppose that $h([0],[0]) = ([0],[0])$, since any translation of $\mathbf{R}^2$ induces an orbit preserving homeomorphism of T^2. Consider the homeomorphism $f = l_h^{-1}h$ of T^2. Let ϕ be the flow on T^2 induced from the $\theta=\alpha$ flow by f. The orbits of ϕ lift to the family of all lines of slope γ in $\mathbf{R}^2$, where $(1,\gamma) = l_h^{-1}(1,\beta)$. Let S be the circle $\{0\}\times S^1$ in T^2, and let $T = f(S)$. The time 1 map of the $\theta=\alpha$ flow maps S homeomorphically onto itself. Let g denote this homeomorphism of S. Then g is covered by the translation $t\mapsto t+\alpha$ of $\mathbf{R}$, so the rotation number $\rho(g)$ of g is α (see Exercise 2.4). Since f is a flow equivalence from the $\theta=\alpha$ flow to ϕ, $f|S$ conjugates g with the time 1 homeomorphism $\phi^1|T$. Notice that $\phi^1|T$ is the *first return homeomorphism* of T with respect to the flow ϕ; that is to say, for any $x\in T$, $\phi^1(x)$ may be described as $t.x$ for the first $t>0$ for which $t.x\in T$.

Let $F:\mathbf{R}^2\to\mathbf{R}^2$ be the lifting of f such that $F(0,0)=(0,0)$. Then F is a homeomorphism satisfying

$$F(x+m, y+n) = F(x,y) + (m,n)$$

for all $(x,y)\in\mathbf{R}^2$ and all $(m,n)\in\mathbf{Z}^2$. Let $\mathbf{R}_m$ denote the line $\{m\}\times\mathbf{R}$ in $\mathbf{R}^2$. Thus $\pi^{-1}(S) = \bigcup\{\mathbf{R}_m : m\in\mathbf{Z}\}$, and correspondingly $\pi^{-1}(T) = \bigcup\{F(\mathbf{R}_m) : m\in\mathbf{Z}\}$. Each of these continuous arcs $F(\mathbf{R}_m)$ is the translate of $F(\mathbf{R}_0)$ by the vector $(m,0)$. Since F maps lines of slope α onto lines of slope γ, and each line of slope α intersects $\mathbf{R}_0$ in a single point, each line of slope γ intersects $F(\mathbf{R}_0)$ in a single point. Thus there is a continuous map $\lambda:\mathbf{R}\to\mathbf{R}$, which is periodic of period 1, such that $F(\mathbf{R}_0) = \{(\lambda(t), t+\gamma\lambda(t)) : t\in\mathbf{R}\}$. This enables us to define a homeomorphism, k say, from $S^1 = \mathbf{R}/\mathbf{Z}$ to T by $k([t]) = ([\lambda(t)], [t+\gamma\lambda(t)])$. Now the portion of the orbit of ϕ between $([\lambda(t)], [t+\gamma\lambda(t)])$ and the point of its next return to T lifts to the line segment joining $(\lambda(t), t+\gamma\lambda(t))$ in $F(\mathbf{R}_0)$ to $(1+\lambda(t+\gamma), t+\gamma+\lambda(t+\gamma))$ in $F(\mathbf{R}_1)$. This latter point projects to the same point of T as does $(\lambda(t+\gamma), t+\gamma+\lambda(t+\gamma))$, and so k^{-1} conjugates $\phi^1|T$ with the map of S^1 taking $[t]$ to $[t+\gamma]$. Since this has rotation number γ and is conjugate to f, we deduce that $[\gamma]=[\alpha]$. Hence

$$\begin{bmatrix}1\\ \gamma\end{bmatrix}=\begin{bmatrix}1 & 0\\ n & 1\end{bmatrix}\begin{bmatrix}1\\ \alpha\end{bmatrix}$$

for some $n \in \mathbf{Z}$. But $(1, \beta) = l_h(1, \gamma)$, and the matrix of l_h is in $GL_2(\mathbf{Z})$. Therefore

$$\begin{bmatrix}1\\ \beta\end{bmatrix}=A\begin{bmatrix}1\\ \alpha\end{bmatrix}$$

for some $A \in GL_2(\mathbf{Z})$, as required. □

IV. LOCAL EQUIVALENCE

It is often possible to modify the definition of a global property or relation by inserting the word "local" and paraphrasing it as "in some neighbourhood of a given point". Even if the new definition does not immediately make perfect sense, it often points quite clearly towards a sensible concept. In many situations we need to handle topological conjugacy, topological equivalence and flow equivalence in a local form. There is an element of doubt as to whether the adaptation process described above produces the most useful definitions of local equivalence for these relations. For example, it is arguable that, since the theory of discrete dynamical systems is essentially a study of homeomorphisms with respect to their orbit structure, we should in no circumstances call periodic points of different periodicities locally equivalent. Similarly, in the flow case, two points may have neighbourhoods on which the phase portraits are identical and may yet be distinguished by certain types of recurrence involving orbits that leave the said neighbourhoods but return at some later time. In spite of these objections we find the straightforwardly adapted definitions useful, with the reservation that the equivalence relations that they give are very weak for regular points (see the first section of Chapter 5 below; a point is *regular* if it is not a fixed point).

Let U and V be open subsets of topological spaces X and Y respectively, and let $f: X \to X$ and $g: Y \to Y$ be homeomorphisms. We say (by abuse of notation) that $f|U$ is *topologically conjugate* to $g|V$ if there is a homeomorphism $h: U \cup f(U) \to V \cup g(V)$ such that $h(U) = V$ and, for all $x \in U$, $hf(x) = gh(x)$. If $p \in X$ and $q \in Y$, we say that f is *topologically conjugate at p to g at q* if there exist open neighbourhoods U of p and V of q such that $f|U$ is topologically conjugate to $g|V$ by a conjugacy h taking p to q. We also say, by a further abuse, that $f|p$ is *topologically conjugate* to $g|q$, or even that p is *topologically conjugate* to q.

We now turn to flows. Let $\phi : D \to X$ be a continuous map, where X is a topological space and D is a neighbourhood of $\{0\} \times X$ in $\mathbf{R} \times X$. We write $t.x$ for $\phi(t, x)$ and D_x for the set $\{t \in \mathbf{R}: (t, x) \in D\}$. We say that ϕ is a *partial flow* on X if, for all $x \in X$,

(i) D_x is an interval,
(ii) $0.x = x$,
(iii) for all $t \in D_x$ with $s \in D_{t.x}$, $(s+t).x = s.(t.x)$,
(iv) for all $t \in D_x$, $D_{t.x} = \{s - t : s \in D_x\}$.

Thus, as the name suggests, ϕ is a flow that is not defined for all time. Condition (iv) implies that ϕ is maximal; it cannot be extended without offending the preceding conditions. We show this in the following proposition, which gives a few useful properties of partial flows.

(2.13) Proposition. *Let $\phi : D \to X$ be a partial flow on X. Then*

(i) *D is open in $\mathbf{R} \times X$,*
(ii) *if $D_x =]a, b[$ with $b < \infty$, ϕ cannot be extended to a continuous map of $D \cup \{(b, x)\}$ into X,*
(iii) *if x is a fixed point or a periodic point of ϕ then $D_x = \mathbf{R}$.*

Proof. Let $(t, x) \in D$. Then for some neighbourhood U of $t.x$ and for some $\varepsilon > 0$, $]-\varepsilon, \varepsilon[\times U \subset D$, since D is an open neighbourhood of $\{0\} \times X$ in $\mathbf{R} \times X$. By continuity of ϕ, there is some neighbourhood V of x such that $t.y \in U$ for all $y \in V$. Since for all such y, $]-\varepsilon, \varepsilon[\subset D_{t.y}$ it follows by (iv) of the definition that $]t-\varepsilon, t+\varepsilon[\subset D_y$. Thus D contains $]t-\varepsilon, t+\varepsilon[\times V$, and hence D is open in $\mathbf{R} \times X$.

To see (ii), note that if ϕ could be extended by putting $b.x = y$, then $t.x$ would necessarily converge to y as $t \to b-$. But, as in the first part, this would imply that for some $\varepsilon > 0$, $]-\varepsilon, \varepsilon[\subset D_{t.x}$ for t near b, and thus that $]t-\varepsilon, t+\varepsilon[\subset D_x$. This contradicts the definition of b when t is larger than $b - \varepsilon$. In fact, we have shown that $(t.x)$ has no cluster points in X as $t \to b-$.

Part (iii) of the proposition is immediate from (iv) of the definition. □

We may define flow equivalence and topological equivalence for partial flows in the obvious way. Thus if ψ is a partial flow on a topological space Y then a flow equivalence from ϕ to ψ is a pair (α, h) where $h : X \to Y$ is a homeomorphism, $\alpha : \mathbf{R} \to \mathbf{R}$ is multiplication by a positive constant, and $h(t.x) = \alpha(t).h(x)$ for all $(t, x) \in D$. A *topological equivalence* from ϕ to ψ is a homeomorphism $h : X \to Y$ that maps all orbits $D_x.x$ of ϕ onto orbits of ψ and preserves their orientation by increasing t.

(2.14) Proposition. *Flow equivalence and topological equivalence are equivalence relations on the set of all partial flows on topological spaces.*

The only non trivial point in the proof of this proposition is to establish the symmetry of flow equivalence. This requires the following lemma:

(2.15) Lemma. *Let (α, h) be a flow equivalence from ϕ to ψ. Then $\alpha \times h$ maps the domain D of ϕ homeomorphically onto the domain E of ψ.*

Proof. By the definition of flow equivalence $(\alpha \times h)(D) \subset E$, and it suffices to show that $(\alpha \times h)(D) = E$. But (α, h) induces a partial flow $\tilde{\psi}: (\alpha \times h)(D) \to Y$ on Y, and $\tilde{\psi} = \psi$ on $(\alpha \times h)(D)$. Since by Proposition 2.13 (ii) partial flows cannot be extended, the domain of $\tilde{\psi}$ equals the domain of ψ. $\square$

Now let ϕ be any flow (or, indeed, partial flow) on X and let U be an open subset of X. Then, for all $x \in X$, $\phi_x^{-1}(U)$ is open in $\mathbf{R}$ and hence a countable union of disjoint open intervals. Let D_x be the one containing 0. We call $\phi_x(D_x)$ the *orbit component of ϕ through x (in U)* (see Figure 2.16). We

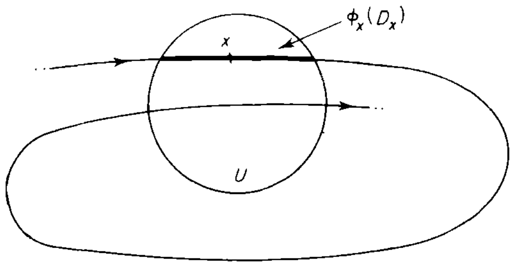

FIGURE 2.16

denote by D the subset $\bigcup_{x \in U} D_x \times \{x\}$ of $\mathbf{R} \times U$ (see Figure 2.17), and define a map $\phi|U: D \to U$ by $(\phi|U)(t, x) = \phi(t, x)$. Of course this is an abuse of notation, since U is not in the domain of ϕ. By the same abuse, we call $\phi|U$ the *restriction of the flow ϕ to the subset U*. It is very easy to check that $\phi|U$ is a partial flow on U. Thus if ψ is a flow on a topological space Y and V is an

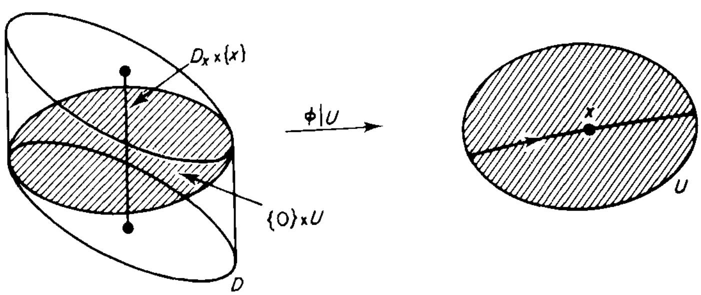

FIGURE 2.17

open subspace of Y, then to say that $\phi|U$ is *flow* or *topologically equivalent to* $\psi|V$ means that they are equivalent as partial flows. If $p \in X$ and $q \in Y$, we say that ϕ is equivalent at p to ψ at q if there exist open neighbourhoods U of p and V of q and an equivalence from $\phi|U$ to $\psi|V$ taking p to q. Once again, we write that *$\phi|p$ is equivalent to $\psi|q$*, or even that *p is equivalent to q*.

(2.18) Proposition. *Flow equivalence and topological equivalence are equivalence relations on $\{(\phi, p) : \phi$ is a flow, $p \in$ the phase space of $\phi\}$. If $\phi|p$ is flow equivalent to $\psi|q$ then $\phi|p$ is topologically equivalent to $\psi|q$.* □

V. LIMIT SETS OF FLOWS

We now begin an investigation of properties of orbits of flows that are preserved under topological equivalence. An obvious example is the topological types of the orbits; for instance if the sets of fixed points of two flows have different cardinalities then we can say immediately that the flows are not topologically equivalent.

Let ϕ and ψ be flows on topological spaces X and Y. Suppose that h is a topological equivalence from ϕ to ψ. Then h maps the closure $\bar{\Gamma}$ in X of each orbit Γ of ϕ onto the closure $\overline{h(\Gamma)}$ of $h(\Gamma)$ in Y. Consequently, h maps the set $\bar{\Gamma}\backslash\Gamma$ onto the set $\overline{h(\Gamma)}\backslash h(\Gamma)$. Compare, for instance, Examples 1.14 and 1.15 which are distinguished topologically by the number of orbits Γ with $\bar{\Gamma}\backslash\Gamma$ empty (see Figure 2.19).

This observation takes no account of the fact that topological equivalence must preserve orientation of orbits. Example 1.14 is not topologically equivalent to its reverse flow, and the difference obviously lies in the way that the orbits begin at the origin in Example 1.14 but end there in

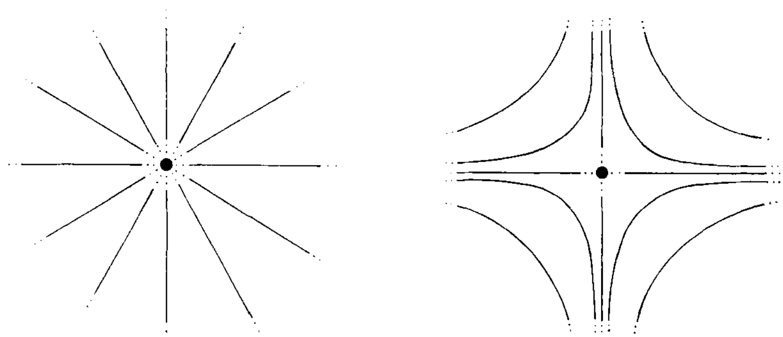

FIGURE 2.19

the reverse flow. In order to handle these differences, we must analyse closures of orbits more carefully, and pick out those parts corresponding to "large positive t" and "large negative t". These *limit sets* will be the main objects of study for the rest of this chapter.

Let I_t denote the closed half-line $[t, \infty[$. The *ω-set* (or *ω-limit set*) $\omega(x)$ of a point $x \in X$ (with respect to the flow ϕ) is defined by

$$\text{(2.20)} \qquad \omega(x) = \bigcap_{t \in \mathbf{R}} \overline{\phi_x(I_t)}.$$

Intuitively, $\omega(x)$ is the subset of X that $t.x$ approaches as $t \to \infty$. For instance, in the saddle point picture of Example 1.15, if x is on either of the two vertical orbits then $t.x$ approaches the origin as $t \to \infty$, and this is reflected in the fact that $\omega(x) = \{0\}$. It follows immediately from the definition of flow that, for all $t \in \mathbf{R}$, $\omega(t.x) = \omega(x)$. Thus we may define the *ω-set* $\omega(\Gamma)$ of any orbit Γ of ϕ by $\omega(\Gamma) = \omega(x)$ for any $x \in \Gamma$. Notice that if Γ is a fixed point or periodic orbit then $\phi_x(I_t) = \Gamma$ for all $x \in \Gamma$ and $t \in R$, and so $\omega(\Gamma) = \Gamma$. Thus $\omega(\Gamma)$ is not necessarily part of $\bar{\Gamma} \backslash \Gamma$.

(2.21) Exercise. Determine the ω-sets for the orbits of the examples of flows in Chapter 1. For instance, show that any orbit of an irrational flow on T^2 has T^2 as ω-set.

Notice that in the definition (2.20) one could equally well have taken the intersection over any subset of $\mathbf{R}$ that is unbounded above, for example $N = \{1, 2, 3, \ldots\}$. Notice also that a point y is in $\omega(x)$ if and only if there is a real sequence (t_n) such that $t_n \to \infty$ and $t_n.x \to Y$ as $n \to \infty$.

Similarly, we define the α-set $\alpha(x)$ of a point $x \in X$ by

$$\alpha(x) = \bigcap_{t \in \mathbf{R}} \overline{\phi_x(J_t)}$$

where $J_t =]-\infty, t]$, and the *α-set* $\alpha(\Gamma)$ of an orbit Γ by $\alpha(\Gamma) = \alpha(x)$ for any $x \in \Gamma$. The α-set is the subset that $t.x$ approaches as $t \to -\infty$. To avoid continual repetition, we confine attention in the results that follow to ω-sets, since the corresponding results for α-sets are always exactly analogous in statement and proof. In fact, we can give a precise formulation for this correspondence in terms of the reverse flow ϕ^- of ϕ described in Example 1.26, as follows:

(2.22) Proposition. *Let Γ by an orbit of ϕ and let $\alpha^-(\Gamma)$ and $\omega^-(\Gamma)$ denote the α-set and ω-set of Γ as an orbit of ϕ^-. Then $\alpha^-(\Gamma) = \omega(\Gamma)$ and $\omega^-(\Gamma) = \alpha(\Gamma)$.*

We now derive a few simple properties of ω-sets (of a flow ϕ on a topological space X unless otherwise stated). First we show that any topological equivalence maps ω-sets onto ω-sets.

(2.23) Theorem. *Let $h : X \to Y$ be a topological equivalence from ϕ to ψ. Then, for each orbit Γ of ϕ, h maps $\omega(\Gamma)$ onto $\omega(h(\Gamma))$, the ω-set of the orbit $h(\Gamma)$ of ψ.*

Proof. Let $\lambda : \mathbf{R} \to \mathbf{R}$ be an increasing homeomorphism such that $h\phi_x = \psi_{h(x)}\lambda$ where $x \in \Gamma$. Then

$$h(\omega(\Gamma)) = h\left(\bigcap_{t\in\mathbf{R}} \overline{\phi_x(I_t)}\right) = \bigcap_{t\in\mathbf{R}} h\overline{\phi_x(I_t)} \qquad \text{(since } h \text{ is injective)}$$

$$= \bigcap_{t\in\mathbf{R}} \overline{h\phi_x(I_t)} \qquad \text{(since } h \text{ is a homeomorphism)}$$

$$= \bigcap_{t\in\mathbf{R}} \overline{\psi_{h(x)}\lambda(I_t)} = \bigcap_{t\in\mathbf{R}} \overline{\psi_{h(x)}(I_{\lambda(t)})} \qquad \text{(since } \lambda \text{ is increasing)}$$

$$= \omega(h(\Gamma)). \qquad \square$$

(2.24) Proposition. *Let Γ be an orbit of ϕ. Then $\omega(\Gamma)$ is a closed subset of X, and $\omega(\Gamma) \subset \bar{\Gamma}$.* $\square$

(2.25) Exercise. Show that if Γ is an orbit of a flow on X, then $\bar{\Gamma} = \Gamma \cup \alpha(\Gamma) \cup \omega(\Gamma)$. Deduce that if $\alpha(\Gamma) = \omega(\Gamma) = \varnothing$, then Γ is a closed subset of X.

(2.26) Proposition. *Let Γ and Δ be orbits of ϕ such that $\Gamma \subset \omega(\Delta)$. Then $\omega(\Gamma) \subset \omega(\Delta)$.*

Proof. By Proposition 2.24, $\omega(\Gamma) \subset \bar{\Gamma}$, and so $\omega(\Gamma) \subset \overline{\omega(\Delta)}$. But $\overline{\omega(\Delta)} = \omega(\Delta)$, by Proposition 2.24. $\square$

Any union of orbits of a dynamical system is called an *invariant set* of the system (a subset is *invariant under a map* if the map takes it onto itself; a union of orbits is invariant under the maps ϕ^t for all $t \in \mathbf{R}$).

(2.27) Proposition. *Any ω-set of ϕ is an invariant set of ϕ.*

Proof. We have to show that, for all $s \in \mathbf{R}$ and for all $p \in \omega(\Gamma)$, $s.p \in \omega(\Gamma)$, or, equivalently, that $\phi^s(\omega(\Gamma)) = \omega(\Gamma)$. Now, for any $x \in \Gamma$,

$$\phi^s(\omega(\Gamma)) = \phi^s\left(\bigcap_{t\in\mathbf{R}} \overline{\phi_x(I_t)}\right) = \bigcap_{t\in\mathbf{R}} \overline{\phi^s\phi_x(I_t)}$$

$$= \bigcap_{t\in\mathbf{R}} \overline{\phi_x(I_{s+t})} = \omega(\Gamma). \qquad \square$$

Thus, for instance, if $\omega(\Gamma)$ is a single point q, then q is a fixed point of ϕ.

As we have seen in the examples of Chapter 1, an orbit may have an empty ω-set. This phenomenon seems to be associated with the orbit "going to infinity" so it is reasonable to suppose that if we introduce some compactness condition we can ensure non-empty ω-sets. What can we say about connectedness of ω-sets? At first sight it seems plausible that $\omega(\Gamma)$ inherits connectedness from Γ. However, with a little imagination one can visualize a flow on $\mathbf{R}^2$ having orbits with non-connected ω-sets. If we wish to ensure connected ω-sets, the simplest answer is, once again, a compactness condition. In considering these, and later questions, it is convenient to quote two purely topological lemmas, whose statements and proofs are given in the appendix to this chapter.

(2.28) Proposition. *Let K be a compact subset of X, such that, for all $n \in \mathbf{N}$, $\overline{\phi_x(I_n)} \cap K \neq \varnothing$. Then $\omega(x) \cap K \neq \varnothing$.*

Proof. For all $n \in \mathbf{N}$, $F_n = \overline{\phi_x(I_n)} \cap K$ is a closed subset of the compact subset K and hence is compact. By Lemma 2.44 of the appendix, $\bigcap F_n$ is non-empty. □

(2.29) Proposition. *Let K be a compact subset of X, such that, for some $r \in N$, $\phi_x(I_r) \subset K$. Then $\omega(x)$ is a non-empty compact subset of K. If in addition X is Hausdorff, then $\omega(x)$ is connected.*

Proof. By definition $\omega(x) \subset \overline{\phi_x(I_r)} \subset K$. Also $\omega(x)$ is closed in X, by Proposition 2.24, hence closed in K and hence compact. By Proposition 2.28, $\omega(x)$ is non-empty. Finally, if X is Hausdorff, the connectedness of $\omega(x)$ follows from Lemma 2.44 of the appendix with $F_n = \overline{\phi_x(I_{n+r})}$. □

(2.30) Corollary. *If X is compact Hausdorff then, for any orbit Γ, $\omega(\Gamma)$ is non-empty, compact and connected.* □

(2.31) Proposition. *Let X be compact. Then for any neighbourhood U of $\omega(x)$, there exists $n \in \mathbf{N}$ such that $\phi_x(I_n) \subset U$.*

Proof. We may suppose that U is open, whence $X \backslash U$ is closed and compact. The result now follows immediately from Proposition 2.28 with $K = X \backslash U$. □

This is one of several results that we shall improve upon in the appendix when we discuss compactification (see Exercise 2.61).

(2.32) ***Note.*** Suppose that U and V are open subsets of X and Y respectively and that $h: U \to V$ is a topological equivalence from $\phi|U$ to $\psi|V$. We should like to know that h takes ω-sets of orbit components of ϕ onto ω-sets of orbit components of ψ. There are various awkward points to contend with. For example, an orbit in U may have part of its ω-set outside U. Again, an orbit wholly in U may be mapped by h to a component of an orbit that leaves V. However, suppose that X and Y are Hausdorff, that K is a compact (and hence closed) subset of U, and let Γ be an orbit of ϕ such that, for some $x \in \Gamma$ and $t \in \mathbf{R}$, $\phi_x(I_t) \subset K$. Thus, by Proposition 2.29, $\omega(\Gamma)$ is a non-empty subset of K. One may prove that if, by abuse of notation, we denote by $h(\Gamma)$ the orbit of ψ containing $h\phi_x(I_t)$, then the orbit component of $h(\Gamma)$ containing $h\phi_x(I_t)$ contains $\psi_{h(x)}(I_{t'})$ for some $t' \in \mathbf{R}$, and $\omega(h(\Gamma)) = h(\omega(\Gamma))$.

We commented earlier that if Γ is a fixed point or periodic orbit then $\omega(\Gamma) = \Gamma$. We now investigate this property more thoroughly.

(2.33) Lemma. *Let X be Hausdorff and let Γ be a compact orbit of ϕ. Then $\omega(\Gamma) = \Gamma$.*

Proof. The set Γ is closed in X. Hence, by Proposition 2.29, $\omega(\Gamma)$ is a non-empty subset of Γ. Since it is an invariant set, it must be the whole of Γ. □

(2.34) Proposition. *Let X be compact and Hausdorff. Then an orbit Γ of ϕ is closed in X if and only if $\omega(\Gamma) = \Gamma$.*

Proof. Necessity is a special case of Lemma 2.33. If $\omega(\Gamma) = \Gamma$ then, by Proposition 2.24, Γ is closed. □

(2.35) Theorem. *Let X be Hausdorff. Then an orbit Γ of ϕ is compact if and only if it is a fixed point or a periodic orbit.*

Proof. Sufficiency is immediate. Suppose, then, that Γ is compact and that Γ is neither a fixed point nor a periodic orbit. Then, for any $x \in \Gamma$, ϕ_x is injective. For each $n \in \mathbf{N}$, $C_n = \phi_x([-n, n])$ is a compact subset of X, and hence closed. Moreover, $\omega(\Gamma) = \Gamma$ by Lemma 2.33, and so, for all $p \in \Gamma$, each neighbourhood of p contains points in $\phi_x(I_{n+1}) \subset \Gamma \backslash C_n$. We may therefore apply Lemma 2.45 of the appendix with $Y = A = \Gamma$, and deduce that Γ is not locally compact, which is a contradiction. □

(2.36) Corollary. *Let X be compact and Hausdorff. Then the following three conditions on an orbit Γ of ϕ are equivalent*:

(i) Γ *is a closed subset of* X,
(ii) Γ *is a fixed point or a periodic orbit,*
(iii) $\omega(\Gamma) = \Gamma$. □

As we commented in Chapter 1, this result partially justifies the use of the term "closed orbit" as a synonym for "periodic orbit". See Exercise 2.63 of the appendix for a generalization of Corollary 2.36 to non-compact X.

(2.37) Exercise. A *minimal* set of a dynamical system is a non-empty closed invariant set that does not contain any closed invariant proper subset. Use Zorn's Lemma (see Lang [2]) to prove that if, for any orbit Γ of ϕ, $\bar{\Gamma}$ is compact then it contains a minimal set. Give an example to show that Γ may have a non-empty ω-set that is not minimal.

VI. LIMIT SETS OF HOMEOMORPHISMS

The theory of α- and ω-limit sets may also be developed in the context of discrete dynamical systems. If f is a homeomorphism of a topological space, then the ω-set $\omega(x)$ of $x \in X$ with respect to f is defined by

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^r(x) : r \geqslant n\}}.$$

The α-set $\alpha(x)$ of x is the ω-set of x with respect to f^{-1}. All results of the previous section have analogues, with the one obvious exception that ω-sets of homeomorphisms need not be (and seldom are) connected.

VII. NON-WANDERING SETS

The fundamental equivalence relations in the theory of dynamical systems are indisputably topological equivalence (for flows) and topological conjugacy (for homeomorphisms). However, when the going gets rough in the classification problem, one tends to cast about for a new (but still natural) equivalence relation with respect to which classification may be easier. We shall describe some attempts in this direction in Chapter 7. One of these, generally accepted to be the most important, is concerned with a certain invariant set known as the *non-wandering set*, and now is a suitable time to explain this concept. The definition is due to David Birkhoff [1] and the logic behind it is as follows. If one compares phase portraits of dynamical systems, for example the two in Figure 2.38, it seems that certain parts are qualitatively more important than others. If one were asked to pick out the significant features of the left hand picture, one would inevitably begin by mentioning the fixed points and closed orbit. Generally speaking, qualitative features in a phase portrait of a dynamical system ϕ can usually be traced

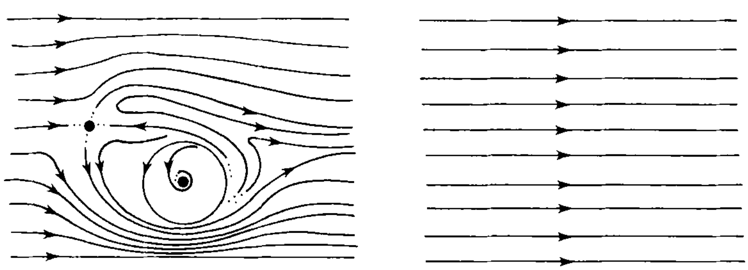

FIGURE 2.38

back to sets of points that exhibit some form of recurrence. The strongest form of recurrence is periodicity, where a point resumes its original position arbitrarily often, but there are weaker forms that are also important. One uses the technical term *recurrent point* for a point that belongs to its own ω-limit set. For example, all points of the torus T^2 are recurrent with respect to an irrational flow (Example 1.25), although none are periodic. By definition, a point is recurrent if and only if, for all neighbourhoods U of x, $t.x \in U$ for arbitrarily large $t \in G$. We define x to be *non-wandering* if, for all neighbourhoods U of x, $(t.U) \cap U$ is non-empty for arbitrarily large $t \in G$. Thus we have a form of recurrence that is weaker than technical recurrence. To see that it is strictly weaker, observe that any point of the non-recurrent orbit Γ in Figure 2.39 is a non-wandering point. Rather more importantly

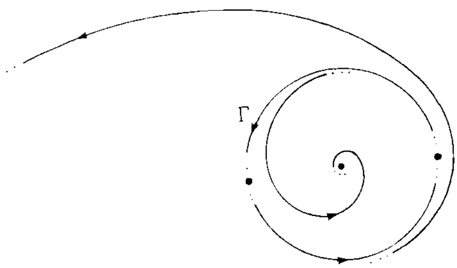

FIGURE 2.39

(because it is in a stabler configuration) consider any point $x \neq 0$ on the stable manifold of the fixed point 0 of the hyperbolic toral automorphism (Example 1.30). It is clearly not recurrent, since $\omega(x) = \{0\}$, but it is non-wandering, since any neighbourhood of it contains periodic points.

The term "non-wandering point" is an unhappy one, since not only may the point wander away from its original position but, as we have seen, it may never come back again. K. Sigmund has put forward the attractive alternative *nostalgic point*, for although the point itself may wander away, its thoughts (represented by U) keep coming back. Non-wandering points are also called Ω-*points*, and the set of all non-wandering points of ϕ, the *non-wandering set* of ϕ, is denoted $\Omega(\phi)$ (or $\Omega(f)$ if ϕ is discrete and $f = \phi^1$).

(2.40) Exercise. Prove that for any homeomorphism f of X, $\Omega(f^{-1}) = \Omega(f)$. Note the distinction between ω-sets and the Ω-set. All limit points are Ω-points, but the converse is false, as the following exercise shows:

(2.41) Exercise. Prove that if $y \in \omega(x)$ for some $x \in X$, then $y \in \Omega(\phi)$. Sketch the phase portrait of a flow on $\mathbf{R}^2$ such that (i) $\Omega(\phi) = \mathbf{R}^2$, but (ii) for some $x \in \mathbf{R}^2$, x is neither an α-limit point nor an ω-limit point of any orbit of ϕ.

The following result sums up some elementary properties of Ω-sets:

(2.42) Theorem. *For any dynamical system ϕ on X, $\Omega(\phi)$ is a closed invariant subset of X, and is non-empty if X is compact. Topological conjugacies and equivalences preserve Ω-sets.*

Proof. The complement of $\Omega(\phi)$ is open in X, for if x has an open neighbourhood U such that $(t\,.\,U)\cap U$ is empty for all sufficiently large t, then so has every point of U. Thus $\Omega(\phi)$ is closed in X. Moreover, for all $s \in G$, x has such a neighbourhood U if and only if $s.x$ has a neighbourhood (namely $s.U$) with a similar property. Thus $X\backslash\Omega(\phi)$, and hence also $\Omega(\phi)$, is an invariant set. If X is compact, then, by Proposition 2.29, any orbit of ϕ has a non-empty ω-set, and by Exercise 2.41, this is part of $\Omega(\phi)$.

Finally, suppose that $h: X \to Y$ is a topological conjugacy or equivalence from ϕ to a dynamical system ψ on Y. Let $p \in \Omega(\phi)$. Let V be a neighbourhood of $q = h(p)$ in Y and let $t_0 \in G$. We have to prove that, for some $t \geqq t_0$, and some $y \in V$, $t.y \in V$. This is trivially true if $V \cap (t_0\,.\,V)$ is non-empty, so suppose that it is empty. Let $h^{-1}(t_0.q) = t_1.p$. By continuity of ϕ, there is a neighbourhood U of p in $h^{-1}(V)$ such that $t_1\,.\,U \subset h^{-1}(t_0\,.\,V)$. Since $p \in \Omega(\phi)$, there exists $x \in U$ and $t_2 \in G$ with $t_2 > t_1$, such that $t_2\,.\,x \in U$. Let $h(t_1.x) = t_0.y$. Then $y \in V$. Also, since h is orientation preserving, $h(t_2.x) = t\,.\,y$ for some $t > t_0$, and, since $t_2\,.\,x \in U$, $t\,.\,y \in V$ (see Figure 2.42). $\square$

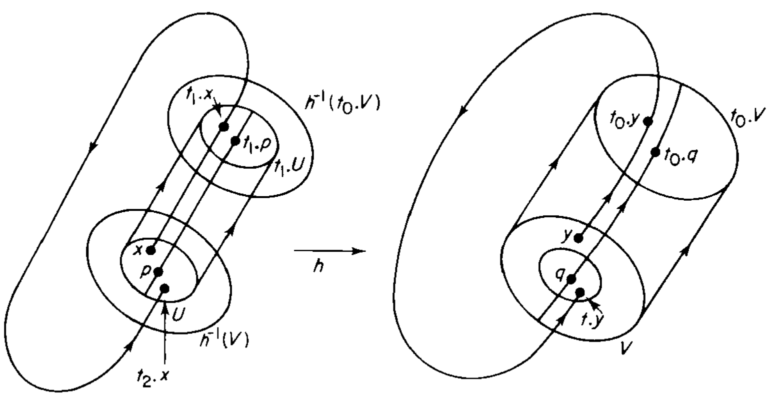

FIGURE 2.42

The new equivalence relations mentioned at the beginning of the section, are called *Ω-equivalence* (for flows) and *Ω-conjugacy* (for homeomorphisms). They are just the old ones, topological equivalence and conjugacy, restricted to Ω-sets. Thus if $\phi|\Omega(\phi)$ denotes the restriction of the flow ϕ to $\Omega(\phi)$, defined by $(\phi|\Omega(\phi))(t, x) = \phi(t, x)$ for all $(t, x) \in \mathbf{R} \times \Omega(\phi)$, then ϕ is

Ω-equivalent to ψ if and only if $\phi \mid \Omega(\phi)$ is topologically equivalent to $\psi \mid \Omega(\psi)$. Similarly homeomorphisms f and g are *Ω-conjugate* if and only if their restrictions $f|\Omega(f)$ and $g|\Omega(g)$ are topologically conjugate. By Theorem 2.42 topological equivalence (resp. conjugacy) is stronger than Ω-equivalence (resp. conjugacy).

(2.43) Exercise. Prove that topological equivalence is strictly stronger than Ω-equivalence. That is to say, give examples of flows that are Ω-equivalent but not topologically equivalent.

Appendix 2

In this appendix we prove the two topological results quoted in the main body of the chapter, we further discuss some points arising from orientation of orbits, and we extend results about limit sets from compact to locally compact spaces by the process of compactification.

I. TWO TOPOLOGICAL LEMMAS

Let X be a topological space. A sequence $(F_n)_{n\in\mathbf{N}}$ of subsets of X is *decreasing* if, for all m, $n \in \mathbf{N}$, $m \geqslant n$ implies $F_m \subset F_n$. For the definition of *increasing*, reverse the inclusion. Thus, in the above context of ω-sets, $(\overline{\phi_x(I_n)})_{n\in\mathbf{N}}$ is a decreasing sequence.

(2.44) Lemma. *Let (F_n) be a decreasing sequence of closed compact, non-empty subsets of X. Then $F = \bigcap_{n\in\mathbf{N}} F_n$ is non-empty. If further, X is Hausdorff and each F_n is connected, then F is connected.*

Proof. Without loss of generality we may assume that $X = F_1$, and hence that X is compact. Suppose that $F = \varnothing$. Then $\{X\backslash F_n\}_{n\in\mathbf{N}}$ is an open cover of X. Since X is compact, this cover has a finite subcover. But $X\backslash F_m \supset X\backslash F_n$ for all $m \geqslant n$. Hence $X = X\backslash F_{n_0}$ for some $n_0 \in N$, and so $F_{n_0} = \varnothing$, which is a contradiction.

Now let X be Hausdorff, and let each F_n be connected. Suppose F is not connected. Then $F = G \cup H$, where G and H are non-empty disjoint sets that are closed in the closed set F, and hence closed in X. Now X, being compact and Hausdorff, is also normal. Thus, there exist disjoint open subsets U and V of X containing G and H respectively. For all $n \in \mathbf{N}$, $F_n \cap U$ and $F_n \cap V$ are non-empty, and hence $F_n \not\subset U \cup V$, because F_n is connected. Let $K_n = F_n \cap (X\backslash(U \cup V))$. Applying the first part to (K_n), we deduce that $K = \bigcap_{n\in\mathbf{N}} K_n$ is non-empty. But $K \subset X\backslash(U \cap V)$ and also $K \subset F \subset U \cap V$. Therefore $K = \varnothing$, which is a contradiction. $\square$

Our second lemma is a version of the Baire category theorem. A topological space is a *Baire space* if the intersection of any countable sequence of dense open subsets of the space is a dense subset. Baire's theorem asserts that locally compact Haudsorff spaces (and complete metric spaces) are Baire spaces. We give a direct proof of an equivalent but slightly more technical statement which is particularly convenient for our applications.

(2.45) Lemma. *Let the topological space Y be the union of a finite or countably infinite sequence $(C_n)_{n>0}$ of closed subsets. Let A be a non-empty subset of Y such that, for all $a \in A$ and for all $n > 0$, $a \in \overline{A \backslash C_n}$. Then Y is not locally compact Hausdorff.*

Proof. Suppose that Y is locally compact Hausdorff, and hence regular. We choose inductively a sequence (a_n) of points of A and a decreasing sequence (F_n) of compact closed neighbourhoods F_n of a_n, such that, for all $n > 0$, $C_n \cap F_n$ is empty. Then, by Lemma 2.44, $\bigcap F_n$ is non-empty, and yet, by construction, it contains no point of $Y = \bigcup C_n$, which is a contradiction.

We may as well start the induction by taking $C_0 = \varnothing$, a_0 = any point of A and F_0 = any compact closed neighbourhood of a_0. Suppose that a_{n-1} and F_{n-1} have been constructed. Then int F_{n-1} contains a point a_n of $A \backslash C_n$. Choose F_n as any closed neighbourhood of a_n in (int $F_{n-1}) \cap (Y \backslash C_n)$. □

II. ORIENTED ORBITS IN HAUSDORFF SPACES

Let ϕ and ψ be flows on topological spaces X and Y respectively, and let $h: X \to Y$ be a homeomorphism mapping orbits of ϕ onto orbits of ψ. Recall that h preserves the orientation of an orbit Γ of ϕ if there is, for some $x \in \Gamma$, an increasing homeomorphism $\alpha: \mathbf{R} \to \mathbf{R}$, such that, for all $t \in \mathbf{R}$, $h(t.x) = \alpha(t).h(x)$. Similarly, h reverses the orientation of Γ if there is a decreasing homeomorphism $\beta: \mathbf{R} \to \mathbf{R}$ such that $h(t.x) = \beta(t).h(x)$. According to these definitions, h may both preserve and reverse the orientation of Γ.

(2.46) Exercise. Prove that h both preserves and reverses the orientation of Γ if and only if Γ is a fixed point.

A more worrying possibility is that h may neither *preserve* nor *reverse* the orientation of Γ.

(2.47) Example. Let ϕ be the flow given by $\phi(t, x) = x + t$ on the space $\mathbf{R}^\flat$ of real numbers with indiscrete topology. Then the topological equivalence $h: \mathbf{R}^\flat \to \mathbf{R}^\flat$ from ϕ itself defined by

$$h(x) = x \quad \text{for} \quad x \neq \pm 1, \qquad h(\pm 1) = \mp 1$$

neither preserves nor reverses the orientation of the unique orbit of ϕ.

The situation revealed by this example is, of course, pathological, and we shall show that the phenomenon cannot occur in Hausdorff spaces. The proof is rather tricky; a much easier result of the same type is:

(2.48) Exercise. Prove that if Γ is a closed orbit of ϕ then h either preserves or reserves the orientation of Γ. (*Hint*: See the first part of Exercise 2.4.)

We begin by proving a slightly off-beat property of real intervals (or, more generally, of connected, locally connected, locally compact, Hausdorff spaces).

(2.49) Lemma. *Let I be a real interval. If I is the union of a sequence (C_n) of two or more disjoint non-empty closed subsets, then the sequence is uncountably infinite.*

Proof. Suppose that $(C_n)_{n>0}$ is a finite or countably infinite sequence with the given property. Let A_n be the frontier of C_n in I. Then A_n is a non-empty (because I is connected) subset of C_n. Let $A = \bigcup A_n$, and let $a \in A_n$ for some n. Then, for all $m \neq n$, $a \in A \backslash A_m$. We prove that $a \in \overline{A \backslash A_n}$. Let V be any neighbourhood of a in I. Then V meets C_n and also C_m for some $m \neq n$. We may assume that V is connected, in which case, since V meets both C_m and its complement in I, V meets the frontier A_m of C_m. Since $A_m \subset A \backslash A_n$, we conclude that $a \in \overline{A \backslash A_n}$. Lemma 2.45 now gives a contradiction. □

(2.50) Theorem. *If X is a Hausdorff space and the homeomorphism $h: X \to Y$ maps an orbit Γ of ϕ onto an orbit of ψ, then h either preserves or reverses the orientation of Γ.*

Proof. The theorem is trivial if Γ is a fixed point, and is just the result of Exercise 2.48 if Γ is a closed orbit. It remains to prove the result when ϕ_x is a continuous injection. In this case, by Theorem 2.35, $\psi_{h(x)}$ is also a continuous injection, and therefore it induces a continuous bijection from $\mathbf{R}$ onto $h(\Gamma)$. Let $\mu: h(\Gamma) \to \mathbf{R}$ be the inverse of this bijection. The map μ is not necessarily continuous; nevertheless, we assert that $\lambda = \mu h \phi_x : \mathbf{R} \to \mathbf{R}$ is a homeomorphism. This latter statement implies, of course, that λ is increasing or that λ is decreasing, and the conclusion of the theorem follows immediately.

To prove our assertion about λ, it is enough to show that λ is continuous at each $t \in \mathbf{R}$. Let J be a compact interval such that $t \in \operatorname{int} J$. Then $h\phi_x(J)$ is a compact subset of the Hausdorff space Y, and is therefore closed in Y. Since $\psi_{h(x)}$ is continuous, $\lambda(J) = K$, say, is closed in $\mathbf{R}$. In fact, we shall prove that K is compact. This implies that $\psi_{h(x)}$ maps K homeomorphically onto $h\phi_x(J)$, hence that λ maps J homeomorphically onto K, and in particular that λ is continuous at t.

Suppose first that k contains an unbounded interval, say $[a, \infty[$. Then, by Proposition 2.29, $\omega(h(\Gamma))$ is non-empty and contained in $h\phi_x(J)$. Hence, using Proposition 2.27, $h\phi_x(J) = h(\Gamma) = h\phi_x(\mathbf{R})$. But $h\phi_x$ is injective and

$J \neq \mathbf{R}$, so we have a contradiction. Similarly, K can contain no unbounded interval of the form $]-\infty, b[$. Thus there exists an increasing sequence $(a_n)_{n\in\mathbf{Z}}$ of points of $\mathbf{R}\backslash K$ such that $a_n \to \infty$ as $n \to \infty$ and $a_n \to -\infty$ as $n \to -\infty$. Thus K is the union of the sequence $(K \cap [a_{n-1}, a_n])_{n\in\mathbf{Z}}$ of disjoint compact sets. These are mapped homeomorphically by λ^{-1} into compact sets whose union is J. It follows from Lemma 2.49 that K is a single compact set contained in $[a_{n-1}, a_n]$ for some n. □

Note that a similar theorem holds when $h: U \to V$ is a homeomorphism between open subsets U of X and V of Y, and both Γ and $h(\Gamma)$ are orbit components.

The difficulty in proving the above theorems stems from the fact that the continuous bijection $\phi_x : \mathbf{R} \to X$ is not necessarily an embedding. To see that it is not, consider any irrational flow on the torus (Example 1.25). One has, in fact, the following simple criterion for ϕ_x to be an embedding.

(2.51) Theorem. *Let X be Hausdorff, and let ϕ have stabilizer $\{0\}$ at x. Then $\phi_x : R \to X$ is an embedding if and only if its image Γ is locally compact.*

Proof. We may as well take $X = \Gamma$. Necessity is trivial. For sufficiency, we have to prove that the inverse of the continuous bijection $\phi_x : \mathbf{R} \to \Gamma$ is continuous. Suppose not. Let F be a closed subset of $\mathbf{R}$ such that $\phi_x(F)$ is not closed. Let p be a point of $\overline{\phi_x(F)}\backslash\phi_x(F)$. For all $n > 0$, ϕ_x maps $F \cap [-n, n]$ homeomorphically onto a closed subset. Hence p is in the closure of either $\phi_x(F \cap [n, \infty[)$ or $\phi_x(F \cap]-\infty, -n])$. We deduce that p is in either $\alpha(\Gamma)$ or $\omega(\Gamma)$, say the latter. Thus $\omega(\Gamma) = \Gamma$. This is impossible, by the proof of Theorem 2.35. □

(2.52) Exercise. Make the suggested generalization of Lemma 2.49.

(2.53) Exercise. Let G be a *second countable* (i.e. having a countable basis for its topology), locally compact topological group acting on a Hausdorff space X. Prove that, for any $x \in X$, the map $G/G_x \to G.x$ taking gG_x to $g.x$ is a homeomorphism if and only if $G.x$ is locally compact. (*Hint*: either generalize the proof of Theorem 2.51 or prove directly that the map is open, using a version of Baire's theorem, as, for example, in § 3 of Chapter 2 of Helgason [1]).

(2.54) Exercise. Let G and H be second countable, locally compact, locally connected topological groups acting *freely* (i.e. with trivial stabilizers) and *transitively* (i.e. with only one orbit) on a Hausdorff space X. Prove that, for all $x \in X$, the map $\psi_x^{-1}\phi_x : G \to H$ is a homeomorphism (where ϕ and ψ are the actions). The ω-set argument in the proof of Theorem 2.50 does not seem to generalize easily, but it can be replaced, as in Irwin [1].

III. COMPACTIFICATION

Let X be a non-compact topological space. There is a standard procedure for associating with X a compact topological space, called the *one-point compactification* X^* of X. Let ∞ denote some point not in X. We define the set X^* to be $X \cup \{\infty\}$. To turn X^* into a topological space, we define a subset U of X^* to be open in X^* if and only if either U is an open subset of X or $X^* \backslash U$ is a closed compact subset of X. It is an easy exercise to verify that X^* is indeed compact, and that it is Hausdorff if and only if X is locally compact and Hausdorff (see Proposition 8.2 of Chapter 3 of Hu [1]).

(2.55) Example. Let $X = \mathbf{R}^n$. Then X^* is homeomorphic to S^n, the unit sphere in $\mathbf{R}^{n+1}$. A particular homeomorphism h may be constructed by mapping $\mathbf{R}^n$, identified with the hyperplane $x_{n+1} = -1$ of $\mathbf{R}^{n+1}$, onto $S^n \backslash \{e_{n+1}\}$ by stereographic projection from e_{n+1}, and ∞ to e_{n+1}, where $e_{n+1} = (0, \ldots, 0, 1)$. See Figure 2.55.

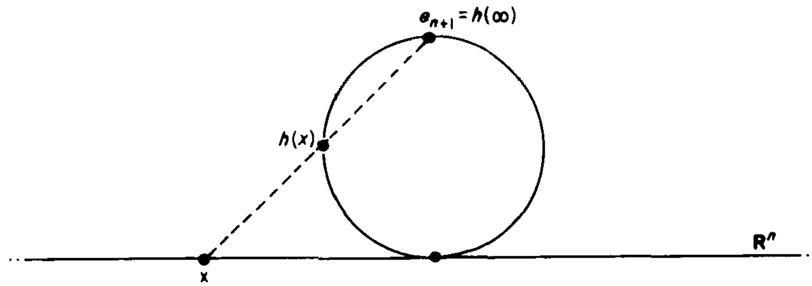

FIGURE 2.55

Now let ϕ be a dynamical system on X. We define the *one-point compactification* of ϕ to be the map $\phi^* : G \times X^* \to X^*$ defined by $\phi^*(g, x) = \phi(g, x)$ if $x \neq \infty$, and $\phi(g, \infty) = \infty$. For instance the phase portraits of ϕ^* for the flows ϕ of Examples 1.8 and 1.9 are illustrated in Figure 2.56 (identifying $\mathbf{R}^*$ with S^1, as above).

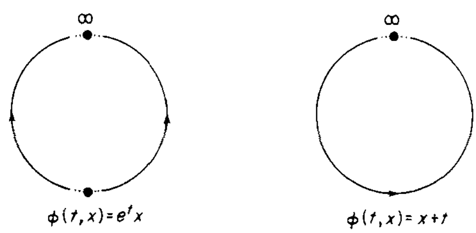

FIGURE 2.56

(2.57) Exercise. Sketch the phase portrait of ϕ^* near ∞ for the flows ϕ of Examples 1.14–1.17.

We now prove that ϕ^* is a dynamical system on X^*. This is trivial if $G = \mathbf{Z}$, so we consider the flow case.

(2.58) Lemma. *Let $\Phi: \mathbf{R} \times X \to \mathbf{R} \times X$ be defined by $\Phi(t, x) = (t, \phi(t, x))$. Then Φ is a homeomorphism.*

Proof. The inverse Φ^{-1} of Φ is given by $\Phi^{-1}(t, y) = (t, \phi(-t, y))$. Moreover if π_1 and π_2 denote projection of $\mathbf{R} \times X$ onto its factors, then $\pi_1\Phi = \pi_1$ and $\pi_2\Phi = \phi$. Since both π_1 and ϕ are continuous, it follows that Φ is continuous. Similarly, Φ^{-1} is continuous. □

(2.59) Theorem. *The one-point compactification ϕ^* of a dynamical system on X is a dynamical system on X^*.*

Proof. It is immediate that ϕ^* satisfies conditions 1.1. We have to prove that ϕ^* is continuous. More precisely, we prove that, for all $(t, x) \in \mathbf{R} \times X^*$ and for any open neighbourhood V of $\phi^*(t, x)$ in X^*, there exists an open neighbourhood U of (t, x) in $\mathbf{R} \times X^*$ such that $\phi^*(U) \subset V$. There are two cases to consider.

(i) Suppose $x \in X$. Then we may assume that V is open in X. Thus $U = (\phi^*)^{-1}(V) = \phi^{-1}(V)$ is open in $\mathbf{R} \times X$ and hence in $\mathbf{R} \times X^*$.

(ii) Suppose that $x = \infty$. The set V is an open neighbourhood of ∞ in X^*, and so $F = X^* \backslash V$ is a compact closed subset of X. Let

$$\begin{aligned} K &= \phi^{-1}(F) \cap ([t-1, t+1] \times X) \\ &= \Phi^{-1}([t-1, t+1] \times F), \end{aligned}$$

where Φ is the homeomorphism of Lemma 2.58. Hence K is a compact closed subset of $\mathbf{R} \times X$, and $\pi_2(K)$ is a compact subset of X. Moreover, since $\pi_1(K)$ is compact, $\pi_2 | \pi_1(K) \times X$ is proper, and so $\pi_2(K)$ is closed. Let $W = X^* \backslash \pi_2(K)$. Then W is an open neighbourhood of ∞ in X^*. Finally, we take $U =]t-1, t+1[\times W$, and check that $\phi^*(U) \subset V$. □

One-point compactifications of spaces and dynamical systems are useful in extending results on compact spaces to locally compact spaces. One applies the known theory to ϕ^* and deduces properties of ϕ. It is immediate that any orbit Γ or ϕ is also an orbit of ϕ^*. It may happen, however, that $\omega^*(\Gamma)$ (the ω-set of Γ regarded as an orbit of ϕ^*) contains the point ∞. For instance, this is always so when $\omega(\Gamma) = \varnothing$.

(2.60) Exercise. Prove that $\omega(\Gamma) \subset \omega^*(\Gamma) \subset \omega(\Gamma) \cup \{\infty\}$. Deduce that if $\omega(\Gamma) = \varnothing$ then $\omega^*(\Gamma) = \{\infty\}$.

(2.61) Exercise. Let X be locally compact Hausdorff and let $\omega(\Gamma)$ be compact and non-empty. Prove that $\omega^*(\Gamma) = \omega(\Gamma)$. Deduce that for any $x \in \Gamma$ and any open neighbourhood U of $\omega(\Gamma)$ there exists $t \in \mathbf{R}$ such that $\phi_x(I_t) \subset U$.

(2.62) Exercise. Prove that if X is locally compact Hausdorff and if $\omega(\Gamma)$ has a compact connected component W, then $\omega(\Gamma) = W$.

(2.63) Exercise. Prove that if X is locally compact Hausdorff then the following four statements are equivalent:

(i) $\omega(\Gamma) = \Gamma$,
(ii) $\omega^*(\Gamma) = \Gamma$,
(iii) Γ is compact,
(iv) Γ is a fixed point or a periodic orbit.