a priori and, consequently, cannot possibly be obtained other than by constructing an initial uniform distribution inside the cell and by shifting it subsequently with the help of dynamics. All these questions require further investigations.

§2. *Exponential instability in dispersing billiards*

One of Krylov's most brilliant achievements is the discovery of exponential instability in dynamical systems with elastic collisions. We shall now explain this property using the simplest example of the so-called dispersing billiards. Let us consider a region Q on a plane the boundary of which is formed by several arcs that are convex inwards. Let us suppose that the curvature of the boundary is a continuous strictly positive function of the point with the unit normal vectors directed inwards Q. Figure 1 shows examples of the regions Q on each arc. We shall consider a

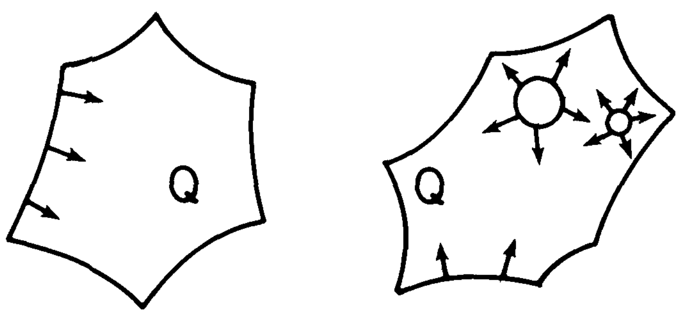

Fig. 1

dynamical system generated by a material point moving inside Q at a unit velocity, with reflections from the boundaries of Q governed by the law: "the angle of arrival is equal to the angle of reflection." The phase space M of the system consists of pairs (q, v) where $q \in Q$ and v is a unit velocity vector that can obviously be characterized by an angle ϕ measured with respect to a certain fixed direction. Let us introduce in M a measure μ for which $d\mu = dq_1\, dq_2\, d\phi$. It is well known that this measure is invariant with respect to the dynamical system under consideration.

Fig. 2

We shall show that the system possesses the property of exponential instability. Let us take a smooth curve $\tilde{l}$ in Q and a continuous set l of vectors normal to $\tilde{l}$. Then l is a smooth curve in M. It is evident that two curves l correspond to every curve $\tilde{l}$ according to the choice of a field of normal vectors. After fixing the curve l we can speak about the curvature of the curve $\tilde{l}$. Therefore it is more correct, in general, to speak about the curvature of the curve l. We shall say that the curve l is convex when its curvature is positive throughout. Figure 2 shows a convex curve.

Let l_0 be a convex curve inside Q and $\kappa(x_0)$ —its curvature at a point $x_0 \epsilon l_0$. We shall take t so small, that no point of the curve l_0 could reach the boundary $\partial\phi$ during the time between 0 and t. If l_t is the curve obtained from l_0 by the time shift t, then the curvature at the point $x_t \epsilon l_t$ obtained from x_0 can be written as $\kappa(x_t) = \kappa(x_0)(1+t\kappa(x_0))^{-1}$ or $\dfrac{1}{\kappa(x_t)} = \dfrac{1}{\kappa(x_0)} + t$. The latter formula becomes particularly clear if we take into account that $1/\kappa$ is the curvature radius, which in the process of motion grows linearly with respect to t. From the equalities given above it follows that if $\kappa(x_0) > 0$ then $\kappa(x_t) > 0$; that is, if l_0 is a convex curve, then l_t is a convex curve, too. Thus in the process of motion inside Q, the curvature decreases but its sign remains unchanged. It is equally easy to see that if ds_0 is an element of the arc l_0 and ds_t is an element of the arc $\tilde{l}_t$ * obtained from ds_0, then $ds_t = ds_0(1+t\kappa(x_0))$ or $ds_t\kappa(x_t) = ds_0\kappa(x_0)$. In other words, in the process of motion inside Q the dimensions of $\tilde{l}_t$ increase linearly with respect to t and the curvature decreases according to the 1/t law.

* $\tilde{l}_t$ is an arc in Q obtained by the natural projection of l_t onto Q.

Let us see now what happens upon reflection from the boundary of Q. We find that when reflected from a boundary, convex inwards, a convex curve remains convex and its curvature acquires an additional positive component. More exactly, let l_0 be a convex curve of sufficiently small size and let t be such that in the time between 0 and t every point of l_0 is reflected from the boundary only once. Then l_t remains a convex curve. In order to obtain its curvature at a point x_t we shall introduce the time $\tau(x_0)$ of the motion of the point x_0 before it is reflected from the boundary. It follows from the above that

$$\lim_{s\to\tau(x_0)-0} \frac{1}{\kappa(x_s)} = \frac{1+\tau\kappa(x_0)}{\kappa(x_0)} \quad \text{or} \quad \lim_{s\to\tau(x_0)-0} \kappa(x_s) = \kappa_- = \kappa(x_0)(1+\tau\kappa(x_0))^{-1},$$

$$\lim_{s\to\tau(x_0)+0} \frac{1}{\kappa(x_s)} = \frac{1-(t-\tau(x_0))\kappa(x_t)}{\kappa(x_t)} \quad \text{or}$$

$$\lim_{s\to\tau(x_0)+0} \kappa(x_s) = \kappa_+ = \kappa(x_t)(1-(t-\tau(x_0))\kappa(x_t))^{-1}.$$

It can be shown, using direct geometrical arguments, that $\kappa_+ = \kappa_- + \dfrac{\kappa(q_\tau)}{\cos\phi(x_\tau)}$, where $\kappa(q_\tau)$ is the curvature of the boundary at the point of reflection and $\phi(x_\tau)$ is the angle between the incident ray and the normal, with $\cos\phi(x_\tau) \geq 0$. It can be seen from this formula that $\dfrac{\kappa(q_\tau)}{\cos\phi(x_\tau)}$ is no less than the minimal curvature of the boundary and it assumes large values when $\phi(x_\tau)$ is small. Thus, reflection from a convex boundary has the effect of making every convex beam approaching the boundary with a decreased curvature become distorted and turn into a beam with a large curvature. From this we can already derive the exponential instability of our system. To do this we shall fix an initial point x_0 and shift it back in time; that is, we shall consider a point $x_{-t} = (q_{-t}, v_{-t})$ for large t's. We shall let out a trajectory beam from q_{-t} in directions close to v_{-t} which will move for a time t. Then we shall obtain a curve l'_t (see

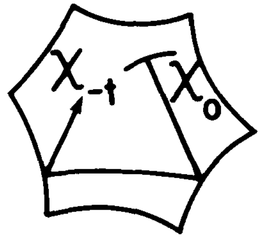

Fig. 3

Figure 3), which, by virtue of the above considerations, will be a convex one. Now we shall take an initial beam $l_t^{(0)}$ so small that all its trajectories will have the same number of reflections from the boundary. If $\Delta\phi_n$ is the angular value of the beam $l_t^{(0)}$, then it follows from the above that

$$s(\tilde{l}'_t) = \int_{l_t^{(0)}} d\phi \cdot \tau_1(y)(1+\tau_2^{(y)}\kappa_1^{(y)})(1+\tau_3(y)\kappa_2(y))\cdots(1+\tau_{n-1}(y)\kappa_{n-1}(y))(1+(t-\tau_n)\kappa_n(y)).$$

Here y is the point of the beam $l_t^{(0)}$ corresponding to the value of the angle ϕ and $\tau_1(y), \tau_2(y), \cdots, \tau_n(y)$ are the time intervals between successive reflections from the boundary; the numbers $\kappa_i(y)$ are related to each other by the recurrence relation $\kappa_{i+1}(y) = \kappa_i(1+\tau_i\kappa_i)^{-i} + \dfrac{\kappa(q^{(i)}}{\cos\phi_i}$; $\kappa(q^{(i)})$ is the curvature of the boundary at the point of i-th reflection and ϕ_i is the angle of arrival for the i-th reflection. For $t \to \infty$ the number of reflections from the boundary increases for a typical point in proportion to t. Therefore the number of factors under the integral symbol is proportional to t and each factor is uniformly greater than 1. If $s(l'_t)$ does not tend to zero with $t \to \infty$, then we find $\Delta\phi_n$ to be exponentially small. In other words, a trajectory beam of size ε coming out of a single point will have a size $\varepsilon\, e^{\text{const}\, t}$ after a time t. Evidently ε and t must be restricted by the condition that the quantity $\varepsilon\, e^{\text{const}\, t}$ must be of the order of 1. The last-mentioned relation serves precisely to express the exponential instability of the system.

Using this example, one can understand the meaning of the Hadamard-Perron theorem formulated above. One can see that for almost every point $x_0(q_0, v_0)$ a complex zigzag curve can be constructed, which passes through q_0, has singularities of a cusp-type and is such that, if one takes the vectors normal to this curve, the trajectories determined by those vectors will, with $t \to \infty$, come exponentially close to the trajectory of the point x_0. It is impossible to write analytic expressions for such curves, but there is a natural procedure leading to their construction. The singularities are due to the trajectories that touch the boundary or pass through the vertices of the boundary. The mixing and other ergodic properties of the system stem from the properties of these curves.

The type of dynamical system described above was first considered in [28]. A simpler version of the proof of the main theorem is given in [8]. Kubo's work [19] also discusses that case and some of its generalizations at great length; Gallavotti in his work [11] gives a detailed proof of the important property of absolute continuity of transversal foliations. We shall also mention the work by Gallavotti and Ornstein [12], which shows that the corresponding dynamical system is of Bernoulli type. An interesting work by L. A. Bunimovitch [9] shows that billiards in a stadium-like region is also exponentially unstable with the result that it is ergodic and mixing. The boundary of the stadium in this case consists of two semicircles and two parallel segments of straight lines. A similar result is false for billiards with boundaries of class C^2 (cf. [20], which under general assumptions, proves the existence of a set of caustics of a positive measure).

§3. *System of hard spheres*

This section describes a new method of analyzing systems of particles with elastic collisions and their generalizations—systems of billiard type. We shall show, using this method, that the system of $r \leq 5$ elastic disks on the torus, the center of gravity of which is at rest, is ergodic. If we make our arguments more exact, it can be shown that the system is mixing and a K-system. A reader who is familiar with the methods of the entropy theory of dynamical systems (in particular, with the works [22], [24], [28]) can do it himself. We shall point out that the new method makes it possible to obtain very quickly the results of the works [8] and [28].

1. *The configuration space and the phase space. Reduction to a billiard-type system.*

Let us consider r identical disks of diameter d on a unit two-dimensional torus, T^2. The configuration space $Q^{(r)}$ of such a system is the region obtained by removing from the $2r$-dimensional torus T^{2r}, the interior of the $\frac{r(r-1)}{2}$ cylinders

$$\tilde{Q}_{ij} = \{q : (q_1^{(i)} - q_1^{(j)})^2 + (q_2^{(i)} - q_2^{(j)})^2 = d^2\}.$$

Here $(q_1^{(i)}, q_2^{(i)})$ are the coordinates of the center of the i-th disk with $q = \{q_p^{(i)}\}$, $i = 1, \cdots, r$; $p = 1, 2$.

Each cylinder $\tilde{Q}_{ij}$ is the direct product of a one-dimensional circle and a $(2r-2)$-dimensional torus. From the geometric point of view, $\tilde{Q}_{ij}$ is a skew product the base of which is a circle and whose leaves are $(2r-2)$-dimensional flat generators of the cylinder. When the torus is mapped onto the Euclidean space these generators are transferred into $(2r-2)$-dimensional planes which, under reverse mapping, are folded into $(2r-2)$-dimensional tori. An essential feature of our problem is the fact that there is a two-dimensional vector subspace of the form $e = \{e_p^{(i)}\}$, $e_1^{(i)} = e_1$, $e_2^{(i)} = e_2$, which is parallel to all those generators. So $Q^{(r)}$ is a $2r$-dimensional manifold with a boundary and the boundary is formed by the cylinders $\tilde{Q}_{ij}$. An individual cylinder $\tilde{Q}_{ij}$ corresponds to the contiguity of the i-th and j-th disks. The intersection of several cylinders corresponds to the simultaneous contiguity of several disks. We shall refer to points belonging to several cylinders $\tilde{Q}_{ij}$ as singular points of the boundary.

The phase space $\mathfrak{M}^{(r)}$ of the system under consideration is the unit tangent bundle over $Q^{(r)}$. A point $x \in \mathfrak{M}^{(r)}$ is a pair $x = (q, v)$, where $q \in Q^{(r)}$ and $v = \{v_p^{(i)}\}$, $i = 1, \cdots, r$; $p = 1, 2$. The vector $v^{(i)} = \{v_1^{(i)}, v_2^{(i)}\}$ is the velocity vector of the i-th disk. The normalization $\sum_{i,p} (v_p^{(i)})^2 = 1$ is obviously not important. Let us consider the free motion of the r disks with elastic collisions. It is easy to see that the elastic collision laws in this case manifest themselves in the fact that the motion of our disk system is expressed in the phase space $\mathfrak{M}^{(r)}$ as a billiard-type motion in the course of which the point moves uniformly and rectilinearly inside $Q^{(r)}$ at a velocity $\vec{v}$ and is reflected from the boundary of $Q^{(r)}$ according to the law: "the angle of arrival is equal to the angle of reflection." In a multidimensional case this means that the tangent component of the velocity is retained and the normal component of the velocity changes its sign. Some uncertainty arises at the points of intersection of several cylinders, but this is of no importance to us: multiple collisions occur on manifolds of smaller dimension and, consequently, have a probability zero. Therefore we can disregard these cases in ergodic theory problems. We shall denote by $\{S^t\}$ the one-parameter group of shifts generated by the billiard system under consideration. Let us introduce in $\mathfrak{M}^{(r)}$ a natural measure μ, $d\mu = \prod_{\substack{1 \le i \le r \\ p=1,2}} dq_p^{(i)} d\omega$, where $d\omega$ is a uniform measure on the $(2r-1)$-dimensional unit sphere of vectors tangent to $Q^{(r)}$ at a point q. It is easy to demonstrate that the group $\{S^t\}$ conserves the measure μ.

The full momentum of the system $I = (I_1, I_2)$, $I_1 = \sum_i v_1^{(i)}$, $I_2 = \sum_i v_2^{(i)}$ is the first integral of the group $\{S^t\}$. The vector $\dot{I}$ is proportional to the velocity vector of the center of gravity of the system. We shall consider an invariant submanifold of the phase space on which $\dot{I} = 0$. We shall also fix the position of the center of gravity of the system. As a result, we shall obtain a $(4r-5)$-dimensional submanifold $\mathfrak{M}_0^{(r)}$ invariant with respect to the group $\{S^t\}$.

2. *Construction of transversal leaves*

An essential peculiarity of the configurational space $Q^{(r)}$ is the fact that the boundary of $Q^{(r)}$ is convex inwards, and the cylinders $\tilde{Q}_{ij}$ are oriented with the help of the external (with respect to them) unit normal vectors. This imparts a dispersing nature to a billiard in $Q^{(r)}$: the trajectories coming out in various directions from any point $q \in Q^{(r)}$ diverge. However, the cylinders $\tilde{Q}_{ij}$ have flat generators and are not, therefore, strictly convex. If they were strictly convex, adjacent trajectories would diverge at an exponential velocity just like the geodesic curves on manifolds of negative curvature (cf. [14]). In our case, however, an exponential divergence on $\mathfrak{M}_0^{(r)}$ will still take place though it will be due to the "nonparallelism" of the cylinders $\tilde{Q}_{ij}$ (we should keep in mind that the coordinates corresponding to the center of gravity are fixed), but this fact requires a much more complex analysis.

The method of studying systems with exponential divergence (in the smooth case: of Anosov's systems) is based on the construction of transversal leaves and on the investigation of their properties. We shall describe the construction of these leaves in our case. The idea is again borrowed from the theory of geodesic flow in spaces of negative curvature and goes back to G. Hedlund [13] and E. Hopf [14].

We shall fix $x^{(0)} = (q^{(0)}, v^{(0)})$ and let $x^{(t)} = (q^{(t)}, v^{(t)}) = S^t x^{(0)}$. We shall let out, from the point $q^{(t)}$, trajectory segments of length t in directions close to $-v^{(t)}$. If the semitrajectory of the point $x^{(0)}$ does not pass through the singular points of the boundary, then for every $t > 0$ a $(2r-1)$-dimensional submanifold $\tilde{\Sigma}^{(t)}$ arises in the neighborhood of the point $q^{(0)}$, which is formed by the ends of the trajectory segments constructed. It is obvious that $q^{(0)} \in \tilde{\Sigma}^{(t)}$ for all $t > 0$. By $\Sigma^{(t)} \subset \mathfrak{M}^{(r)}$ we shall denote the $(2r-1)$-dimensional submanifold made up of unit vectors normal to $\tilde{\Sigma}^{(t)} (q^{(0)}, -v^{(0)}) \in \Sigma^{(t)}, \Sigma_0^{(t)} = \Sigma^{(t)} \cap \mathfrak{M}_0^{(r)}$. It can easily be seen that $\dim \Sigma_0^{(t)} = 2r - 3 \leq 7$.

The submanifolds $\tilde{\Sigma}^{(t)}$, $\Sigma^{(t)}$, $\Sigma_0^{(t)}$ and their behavior for $t \to \infty$ will now be the object of our study. It is convenient, in general, to investigate submanifolds of codimension one with the help of the second fundamental form. Let us recall the corresponding definitions. Let $\tilde{\Sigma}$ be an arbitrary submanifold of codimension one in the Euclidean space R^k and Σ a continuous field of unit vectors normal to $\tilde{\Sigma}$. For any point $q \epsilon \tilde{\Sigma}$ we shall take a vector dq from the plane $\tilde{\tau}_q$ tangent to $\tilde{\Sigma}$ at the point q. Then, if $n(q) \epsilon \Sigma$ is a unit normal vector at the point q, we can write $n(q+dq) = n(q) + \mathcal{B}(q)dq$, where $\mathcal{B}(q)$ is a linear self-adjoint operator acting on the plane $\tilde{\tau}_q$. The quadratic form defined by $\mathcal{B}(q)$ is called the second fundamental form. The submanifold Σ is called convex (strictly convex), if $\mathcal{B}(q) \geq 0$ $(\mathcal{B}(q) > 0)$ for any point $q \epsilon \tilde{\Sigma}$. The submanifold Σ is called concave (strictly concave), if $\mathcal{B}(q) \leq 0$ $(\mathcal{B}(q) < 0)$ for any point $q \epsilon \tilde{\Sigma}$. It is evident that the definitions of convexity and concavity depend on the way in which the unit normal vectors are oriented for the submanifold $\tilde{\Sigma}$. By changing the orientation to the opposite one we cause concave submanifolds to be transformed into convex submanifolds and vice versa.

An important note. Let $Q_{ij} \subset \mathfrak{M}^{(r)}$ be a field of unit vectors normal to $\tilde{Q}_{ij}$, directed inwards $Q^{(r)}$ and hence outwards $\tilde{Q}_{ij}$. Then Q_{ij} is a convex submanifold. The second fundamental form of Q_{ij} at the point $q \epsilon \tilde{Q}_{ij}$, which we shall denote by $K^{(0)}(q)$, is equal to zero on the (2r–2)-dimensional subspace tangent to the flat generator of the cylinder $\tilde{Q}_{ij}$ and is equal to d^{-1} on the one-dimensional orthogonal subspace $\tilde{\mathcal{E}}^{(+)}(q)$. If we take new coordinates in the neighborhood of the point $q \epsilon \tilde{Q}_{ij}$ keep the old values for the coordinates $q_1^{(k)}, q_2^{(k)}$, $k \neq i,j$ and suppose $\tilde{\tilde{q}}_p^{(i,j)} = \dfrac{q_p^{(i)} + q_p^{(j)}}{2}$, $p = 1,2$; $q_p^{(i,j)} = \dfrac{q_p^{(i)} - q_p^{(j)}}{2}$, then the subspace $\tilde{\mathcal{E}}^{(+)}(q)$ in these coordinates will consist of vectors for which $q_1^{(k)} = q_2^{(k)} = 0$, $k \neq i,j$; $\tilde{q}_1^{(i,j)} = \tilde{q}_2^{(i,j)} = 0$. The remaining two coordinates are chosen so that the two-dimensional vector obtained should be proportional to the vector $\tilde{\tilde{q}}^{(i,j)} = \{\tilde{\tilde{q}}_1^{(i,j)}, \tilde{\tilde{q}}_2^{(i,j)}\}$.

Going back to the situation under consideration we shall,for any $t > 0$, take a point $q^{(t)}$, and we shall, for any s, $0 \leq s < t$, let out from $q^{(t)}$ trajectory segments of length $t-s$ in directions close to $-v^{(t)}$. The ends of these segments form a submanifold $\tilde{\Sigma}_s^{(t)}$, which contains the point $q^{(s)}$. We shall choose unit vectors normal to $\tilde{\Sigma}_s^{(t)}$ so that the vector $-v^{(s)}$ should be among them. Then we shall obtain a submanifold $\Sigma_s^{(t)} \subset \mathfrak{M}^{(r)}$. By $\mathcal{B}_s^{(r)}(q^{(s)})$ we shall denote the second fundamental form $\Sigma_s^{(t)}$ at a point $q^{(s)}$.

The above constructions must be made more exact if s is the moment of reflection of the trajectory from the boundary. We shall discuss this case a little later. Now let s_1 and s_2 be such that $s_1 < s_2$ and no reflections from the boundary take place between s_1 and s_2. Then the planes tangent to $\tilde{\Sigma}_{s_1}^{(t)}, \tilde{\Sigma}_{s_2}^{(t)}$ are orthogonal to $v^{(s_1)}, v^{(s_2)}$ respectively and, since these two vectors are parallel, the tangent planes are parallel and they can naturally be identified with each other. Therefore the operators $\mathcal{B}_{s_1}^{(t)}(q^{(s_1)}), \mathcal{B}_{s_2}^{(t)}(q^{(s_2)})$ can also be thought to be acting in the same space.

LEMMA 1. $\mathcal{B}_{s_2}^{(t)}(q^{(s_2)}) = \mathcal{B}_{s_1}^{(t)}(q^{(s_1)})(I + (s_2 - s_1)\mathcal{B}_{s_1}^{(t)}(q^{(s_1)}))^{-1}$,

The proof of this lemma is exceedingly simple and we shall not give it here. It follows from the lemma that if $\Sigma_{s_1}^{(t)}$ is convex (strictly convex), $\Sigma_{s_2}^{(t)}$ is convex (strictly convex), too.

Now we shall consider the case when $s = s_0$ is the moment of reflection of the trajectory from the boundary. It is convenient to assume that we have at that moment two velocity vectors $v_-^{(s_0)}$, $v_+^{(s_0)}$ before and after the reflection, respectively, and $\tilde{\tau}_-^{(s_0)}$, $\tilde{\tau}_+^{(s_0)}$ are the planes orthogonal to $v_-^{(s_0)}, v_+^{(s_0)}$. By $\mathcal{B}_{s_0-}^{(t)}$ ($\mathcal{B}_{s_0+}^{(t)}$) we shall denote the limit of the operator $\mathcal{B}_s^{(t)}$ for $s \uparrow s_0$ ($s \downarrow s_0$). To obtain the relation between $\mathcal{B}_{s_0-}^{(t)}$

and $\mathcal{B}^{(t)}_{s_0+}$ we shall introduce the plane $\tilde{\tau}_0(s_0)$, tangent to the cylinder Q_{ij} at the point $q^{(s_0)}$, and the following operators:

1) the isometric operator U which maps $\tilde{\tau}_-^{(s_0)}$ onto $\tilde{\tau}_+^{(s_0)}$ in a direction parallel to the vector $n(q^{(s_0)})$ normal to the boundary Q_{ij} at point $q^{(s_0)}$;

2) the operator V which maps $\tilde{\tau}_+^{(s_0)}$ onto $\tilde{\tau}_0^{(s_0)}$ in a direction parallel to the vector $v_+^{(s_0)}$ and an adjoint operator V^* which maps $\tilde{\tau}_0^{(s_0)}$ onto $\tilde{\tau}_+^{(s_0)}$ in a direction parallel to the vector $n(q^{(s_0)})$.

LEMMA 2. $\mathcal{B}^{(t)}_{s_0^+} = U^{-1}\mathcal{B}^{(t)}_{s_0^-}U - 2(v_+^{(s_0)}, n(q^{(s_0)}))V^*K^{(0)}_{(q^{(s_0)})}V$.

The proof of this lemma can be arrived at by direct reasoning and we shall not give it here, either.

As $-(v_+^{(s_0)}, n(q^{(s_0)})) > 0$, $K^{(0)} \geq 0$, the second term in the formula is a non-negative operator. Besides, from the properties of $K^{(0)}$ it follows that the operator $V^*K^{(0)}(q^{(s_0)})V$ is equal to zero on a (2r–2)-dimensional subspace and has a single eigenvector with the eigenvalue $|v^{(rel)}|^2/(v_+^{(s_0)}, n(q(s_0)))^2$, where $v^{(rel)}$ is the relative velocity vector of the disks brought into collision. Consequently, the corresponding eigenvalue of the operator $-2(v_+^{(s_0)}, n(q(s_0)))V^*K^{(0)}(q^{(s_0)})V$ is equal to $-\dfrac{2|v^{(rel)}|^2}{v_+^{(s_0)}, n(q^{(s_0)})}$.

From Lemma 2 it follows that if $\mathcal{B}^{(t)}_{s_0-} \geq 0$, then $\mathcal{B}^{(t)}_{s_0+} \geq 0$. We can now derive an important conclusion from Lemmas 1 and 2: $\Sigma_s^{(t)}$ for all $s > 0$ is convex. The foregoing statement can also be made stronger in the following way: let Σ be a field of unit vectors tangent to a submanifold $\tilde{\Sigma}$ which is itself a convex submanifold. Then $S_t\Sigma$ for any $t > 0$ is also a convex submanifold.

Let us return to the point $x^{(0)} = (q^{(0)}, v^{(0)}) \in \mathfrak{M}^{(r)}$. For any $t > 0$ we have a submanifold $\Sigma_0^{(t)} = \Sigma^{(t)}$ passing through $-x^{(0)} = (q^{(0)}, -v^{(0)})$. Leaving aside for the time being the question of the convergence of $\Sigma^{(t)}$ for $t \to \infty$, we shall find the expression for the second fundamental form of the final submanifold. Let $0 < s_1 < s_2 < s_3 < \cdots < s_k < \cdots$ be the moments of the successive reflections of the semitrajectory of the point $x^{(0)}$ from the boundary, $s_k \to \infty$ for $k \to \infty$, and let $\tau_i = s_i - s_{i-1}$, $q_i \in \partial Q^{(r)}$ be the point of the boundary at which the i-th reflection occurs, and let $v_+^{(i)}, v_-^{(i)}$ be the velocities directly before and after the i-th reflection, and let $\cos\phi_i = -(v_+^{(i)}, n(q^{(s_i)})) \geq 0$, U_i be the isometric operator which maps $\tilde{\tau}_-^{(s_i)}$ onto $\tilde{\tau}_+^{(s_i)}$ in a direction parallel to the normal vector $n(q^{(s_i)})$ at the point of the i-th reflection, and let $K_i^{(0)} = K^{(0)}(q^{(s_i)})$, v_i be the operator which maps $\tilde{\tau}_-^{(s_i)}$ onto $\tilde{\tau}^{(s_i)}$ in the direction parallel to the vector $v_-^{(s_i)}$, and let the adjoint operator V_i^* map $\tilde{\tau}_0^{(s_i)}$ onto $\tilde{\tau}_-^{(s_i)}$ in the direction parallel to the vector $n(q^{(s_i)})$. We shall now write formally an infinite continued fraction:

$$(1)\quad \mathfrak{B}(x^{(0)}) = \cfrac{I}{s_1 I + \cfrac{I}{2\cos\phi_1 V_1^* K_1^{(0)} V_1 + U_1^{-1} \cfrac{I}{s_2 I + \cfrac{I}{2\cos\phi_2 V_2^* K_2^{(0)} V_2 + U_2^{-1}\cdots}} U_1}}$$

where I is the identity operator. In principle, such a fraction can be written for any point $x^{(0)}$, the semitrajectory of which does not pass through the singular points of the boundary even if we do not assume that $s_i \to \infty$. But we shall now demonstrate that with $s_i \to \infty$ the continued fraction written above converges.

The proof is similar to that in the case of continued fractions of ordinary numbers. Since all the operators $\tau_i I$, $2\cos\phi_i V_i^* K_i^{(0)} V_i$ are non-negative, the convergents of the continued fraction of even order form a decreasing sequence and the convergents of the continued fraction of odd order form an ascending sequence of non-negative operators. We shall use the notation

$$a_1 = s_1 I,\ a_2 = 2\cos\phi_1 V_1^* K_1^{(0)} V_1,\ a_3 = \tau_2 I,\ a_4 = V_2^* K_2^{(0)} V_2 \quad \text{and so on.}$$

and let

$$A_n = \cfrac{I}{a_1 + \cfrac{I}{a_2 + U_1^{-1} \cfrac{I}{a_3 + \cfrac{I}{a_4 + U_2^{-1} \cfrac{I}{a_s + \cdots} U_2}} U_1}} \qquad \frac{I}{a_n}\,,$$

if n is even and

$$A_n = \cfrac{I}{a_1 + \cdots \cfrac{I}{U^{-1}_{\frac{n-1}{2}}\, a_n^{-1}\, U_{\frac{n-1}{2}}}}\,,$$

if n is odd. For even n's we have

$$A_{n+1} - A_n = \cfrac{I}{a_1 + \cfrac{I}{\cdots \cfrac{I}{a_n + U^{-1}_{n/2}\, a^{-1}_{n+1}\, U_{n/2}}}} \left(\cfrac{I}{a_2 + \cdots \cfrac{}{U^{-1}_{n/2}\, a^{-1}_{n+1}\, U_{n/2}}} - \right.$$

$$\left. - \cfrac{I}{a_2 + \cdots \cfrac{}{a_n^{-1}}} \right) \cdot \cfrac{I}{a_1 + \cfrac{I}{\cdots a_n^{-1}}} =$$

$$= (I + (a_2 + U_1^{-1} \frac{I}{\cdots} U_1)\, a_1)^{-1} U_1^{-1} \left(\frac{1}{a_3 + \cdots} - \frac{1}{a_3 + \cdots} \right) U_1 (I + (a_2 + U_1 \frac{I}{\cdots} U_1)\, a_1)\,.$$

The expression $\left(\frac{1}{a_3 + \cdots} - \frac{1}{a_3 + \cdots} \right)$ has the same structure as the original expression. Therefore, we can apply the same transformation to it, and so on. It should be noted that $\|(I + (a_2 + U_1^{-1} \frac{I}{\cdots} U_1) a_1)^{-1}\| \le 1$ as well as the norms of all the other factors of that form. The operators a_i with odd i's are proportional to the identity operator. Hence

$$\left(a_2 + \left(U_1^{-1} \frac{I}{a_3 + \frac{I}{\cdots}} U_1\right)\right) a_1 \ge$$

$$U_1^{-1} \frac{a_1}{a_3 + \frac{I}{\cdots}} U_1 \ge \frac{a_1}{a_3 + \cdots + a_{n/2}} = \frac{s_1}{s_1 + \tau_2 + \cdots + \tau_n} I = \frac{s_1}{s_{n/2}} I .$$

Analogous estimates are true for the factors that follow. Thus we can write

$$\|A_{n+1} - A_n\|^2 \le \left(1 + \frac{s_1}{\tau_2 + \cdots + \tau_{n/2}}\right)^{-2} \cdot \left(1 + \frac{\tau_2}{\tau_3 + \cdots + \tau_{n/2}}\right)^{2} \cdot \cdots$$

$$\cdot \left(1 + \frac{\tau_{n/2} - 1}{\tau_{n/2-1} + \tau_n}\right)^{-2} = \frac{1}{s_{n/2}^2} .$$

Consequently, if $\sum_n \frac{1}{s_n^2} < \infty$, the continued fraction (1) converges.

Let us analyze in greater detail the condition $\sum \frac{1}{s_n^2} < \infty$. We shall call a point q of the boundary $\partial\tilde{Q}^{(r)}$ open if there exists an open set of unit tangent vectors at the point q directed inwards $Q^{(r)}$.

Assumption A. Every point of the boundary ∂Q is an open point. Assumption A can prove invalid for some exceptional value of d. For example, it can be false for $d = \frac{1}{k}$, where k is an integer. For general values of d it is true. In connection with Assumption A the following proposition has been proved by the present author: there exist such numbers ε_0 and k_0 that for any trajectory that does not pass through the singular points of the boundary in any series of k_0 successive reflections the time interval between at least two successive reflections is no less then ε_0. G. Galperin and L. Wasserstein obtained recently a number of strong results connected with the problem [31].

From the above proposition it follows that if Assumption A is true for any point $x^{(0)}$ the continued fraction $\mathfrak{B}(x^{(0)})$ exists.

It is evident that $\mathfrak{B}(x^{(0)}) \geq 0$. However, there undoubtedly exists a two-dimensional subspace $\mathfrak{S}$ on which $\mathfrak{B}(x^{(0)})$ becomes equal to zero. The existence of this subspace is a consequence of the conservation of the total momentum I. Let the point $x^{(0)}$ be such that the total momentum $I = 0$. Then any vector of the form $e = \{e_1^{(i)} = e_1, e_2^{(i)} = e_2\}$ lies in the plane $\tau^{(0)}$ passing through $q^{(0)}$ orthogonal to $v^{(0)}$. In the process of parallel translation along the trajectory* the vector remains all the time parallel to itself. It is easy to see that $V_i^* K_i^{(0)} V_i e = 0$ for all i's, from which our statement follows directly. In a similar way, analogous two-dimensional subspaces can be constructed for any $x^{(0)} \epsilon \mathfrak{M}^{(r)}$. For our present purposes it is sufficient to confine ourselves to $x^{(0)} \epsilon \mathfrak{M}_0^{(r)} \cap \mathfrak{M}^{(r)}$. The following theorem is true for arbitrary r but its proof is rather complex. Special considerations can be used for $r \leq 5$.

THEOREM 1. *For almost every point* $x^{(0)} \epsilon \mathfrak{M}_0^{(r)}$ *the operator* $\mathfrak{B}(x^{(0)})$ *is strictly positive definite on the orthogonal complement,* $\overline{\mathfrak{S}}(x^{(0)})$, *of* $\mathfrak{S}$.

Let us trace the course of the reasoning we use in the proof. It should be pointed out that $\dim \overline{\mathfrak{S}}(x^{(0)}) = 2r-3 = 1, 3, 5, 7$ for $r = 2, 3, 4, 5$ respectively. Then it is easy to understand that, if for some series of reflections with numbers between i and $i+k$ the operator

$$2\cos\phi_i V_i^* K_i^{(0)} V_i + U_i^{-1} \cfrac{I}{\tau_{i+1} I + \cfrac{I}{2\cos\phi_{i+1} V_{i+1}^* K_{i+1}^{(0)} V_{i+1} + U_{i+1}^{-1} \cfrac{}{\cdots} U_i}} U_i$$

$$\overline{2\cos\phi_{i+k} V_{i+k}^* K_{i+k}^{(0)} V_{i+k}}$$

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*Parallel displacement of the vector is understood here as an ordinary Euclidean parallel displacement during the time between reflections and transformation under the effect of U_i at the point $q^{(s_i)}$ of the reflection.

is positive definite on the subspace $\overline{\mathcal{E}}(x_+^{(s_i)})$, then the operator $\mathcal{B}(x^{(0)})$ is positive definite, too. Therefore it is sufficient to find a series of reflections for which the above should hold.

Let us consider some series of reflections with numbers between i and $i+k$. We shall associate with the $(i+k)$-th reflection a partition of the entire set of disks into N single point subsets. Let us suppose now that a partition a_j of the entire set of disks is associated with the j-th reflection. If the $(j-1)$-th reflection takes place between disks belonging to the same element of the partition a_j, we shall take $a_{j-1} = a_j$. If, on the contrary, the $(j-1)$-th reflection occurs between disks of different elements of the partition a_j, we shall take a_{j-1} equal to a partition which is obtained by uniting two elements of the partition a_j, containing the colliding disks, into one element. We shall call the series of reflections $i, i+1, \cdots, i+k$ sufficient if a_i is a trivial partition ν, where the entire set of disks is the only partition element and i is the largest number with that property. The length of all sufficient series of reflections is no less than $r-1$. Indeed, if $f(r)$ is the smallest length of a sufficient series of reflections of a set of r disks, then $f(r_1+r_2) \geq f(r_1) + f(r_2) + 1$ for any $r_1 \geq 1$, $r_2 \geq 1$. Since $f(1) = 0$, $f(2) = 1$ we find, using induction, that $f(r) \geq r-1$.

Let us consider a sufficient series of reflections $i, i+1, \cdots, i+k$. We shall take a point $q^{(s_{i+k})}$, a small flat area orthogonal to $-v_+^{(s_{i+k})}$ and a field Σ of unit vectors normal to the area and directed in the same manner as $-v_+^{(s_{i+k})}$. Then $S^{-s_{i+k}+s_i}\Sigma$, in accordance with Lemmas 1 and 2, will be a convex submanifold, that is, a convex orientation of some surface in the coordinate space. In the general case the surface will be strictly convex in the $(r-1)$-th direction and automatically flat in the direction of $\mathcal{E}^{(0)}$. Therefore to obtain strict convexity in all directions from $\overline{\mathcal{E}}$ two sufficient series of reflections are necessary in the general case. Indeed, it can be shown that if a point $x^{(0)}$ does not belong to a union of a countable number of submanifolds of smaller dimension, then from the existence of two sufficient groups of reflections it follows that $\mathfrak{B}$ is positive.

For $r = 2$, to obtain positivity, one reflection is sufficient ($\dim \overline{\mathfrak{E}} = 1$); for $r = 3$ it is sufficient to have a series of reflections, the first and the last of which correspond to the collision of one disk, to be more definite, the first one, with one of the two remaining disks, and all the internal reflections correspond to the collisions of the second and the third disks with each other. Similarly, for $r = 4, 5$ it is easy to describe all the reflection series leading to the positiveness of $\mathfrak{B}$.

If we analyzed three-dimensional spheres instead of two-dimensional disks, then $\dim \overline{\mathfrak{E}} = 3r-3$ and it would be necessary to have three sufficient groups of reflections.

From the above discussion it follows that Theorem 1 can be strengthened: if a point $x^{(0)}$ does not belong to a union of a countable number of submanifolds of smaller dimension and if the trajectory of the point is such that the set of disks cannot be divided into two groups in such a way that, from a certain moment of time, the disks of the two different groups no longer interact with each other, then the operator $\mathfrak{B}$ is positive definite on the subspace $\overline{\mathfrak{E}}$.

From Theorem 1 the existence of transversal leaves can be derived by using standard methods. The reader is referred to the recent work by Ja. Pesin [21], where those methods are described in a form sufficient for the applications required. Let us state here the necessary result. Let $\mathfrak{M}_{00}^{(r)}$ be a subset of $\mathfrak{M}_0^{(r)}$ consisting of points $x^{(0)}$ such that the disks cannot be divided into two subsets for which such t_0 can be found that either for all $t > t_0$, or for all $t < t_0$, the disks of the different subsets do not come into collision at all.

THEOREM 2. *For almost every point* $x^{(0)} = (q^{(0)}, v^{(0)}) \in \mathfrak{M}_{00}^{(r)}$ *a concave (convex) submanifold* $\Sigma^{(s)}(x^{(0)})(\Sigma^{(u)}(x^{(0)})), x^{(0)} \in \Sigma^{(s)}(x^{(0)})\, x^{(0)} \in \Sigma^{(u)}(x^{(0)})$ *can be constructed in such a way that*

$$\lim_{t\to\infty}\left[-\frac{\ln \operatorname{diam} S^t(\Sigma^{(s)}(x^{(0)}))}{t}\right] > 0\,, \quad \lim_{t\to\infty}\left[-\frac{\ln \operatorname{diam} S^{-t}(\Sigma^{(u)}(x^{(0)}))}{t}\right] > 0\,.$$

$\mathcal{B}(x^{(0)})\,(\mathcal{B}(q^{(0)}, v^{(0)}))$ *is in this case the second fundamental form of the submanifold* $\Sigma^{(s)}(x^{(0)})\,(\Sigma^{(u)}(x^{(0)}))$.

In other words, $\Sigma^{(s)}(x^{(0)})$ consists of such y, that $\operatorname{dist}(S^t x^{(0)}, S^t y)$ tends to zero exponentially with $t \to \infty$ and $\Sigma^{(u)}(x^{(0)})$ consists of such y that $\operatorname{dist}(S^t x^{(0)}, S^t y)$ tends to zero exponentially with $t \to -\infty$.

In the theory of geodesic flows in spaces of negative curvature the roles of $\Sigma^{(s)}, \Sigma^{(u)}$ are played by horospheres (cf. [13], [14], [3], [4]). The existence of $\Sigma^{(s)}, \Sigma^{(u)}$ in the billiard systems under consideration is, in fact, the best proof of the analogy between these systems and geodesic flow in spaces of negative curvature, pointed to by N. S. Krylov.

The following corollary stems from Theorem 2:

COROLLARY. *The ergodic components of the set* $\mathfrak{M}_{00}^{(r)}$ *have a positive measure and, consequently, there are at most countably many of them. The flow* $\{S^t\}$ *on each ergodic component is a* K-*system and, consequently, a mixing flow.*

This corollary can be derived from Theorem 2 and from the so-called property of absolute continuity of transversal leaves (cf. [4], [21], [24]). Its proof is the same as in [4].

Now we shall use induction with respect to the number r of the disks. For $r = 2$ it is known that the two-disk system is ergodic and mixing (cf. [28]). For $r = 3$ let us consider a dynamical system in which two disks come into collision and the third disk does not interact with the first two and moves in an irrational direction. The system may be thought of as a direct product of two dynamical systems, one of which is mixing and the other, ergodic. Therefore the entire system is ergodic and almost each of its trajectories is dense throughout. In particular, it passes through the interior of any of the cylinders $\tilde{Q}_{ij}$ and, consequently, cannot be a trajectory of a billiard system of three disks. Hence we have that $\mu(\mathfrak{M}_{00}^{(3)}) = 1$. In points 3.3 and 3.4, from the equality $\mu(\mathfrak{M}_{00}^{(3)} = 1$ we shall derive the ergodicity of a billiard system for $r = 3$. As has already been said, the property of mixing is derived from ergodicity with the use of standard considerations. Thus, we have described the first step of the induction. Similar arguments show that from the ergodicity and the mixing property of a system of r' disks follow the ergodicity and the mixing property of a system of r disks for $r' < r$.

3. *The main theorem*

For $r = 2$, $\Sigma^{(s)}(x^{(0)})$ consists of unit normal vectors which form a concave curve (cf. §2). We shall form expressions

$$\Sigma_{gl}^{(s)}(x^{(0)}) = \bigcup_{t \geq 0} S^{-t}(\Sigma^{(s)}(S^t x^{(0)})), \quad \Sigma_{gl}^{(u)}(x^{(0)}) = \bigcup_{t \geq 0} S^t(\Sigma^{(u)}(S^{-t} x^{(0)}))$$

and we shall call $\Sigma_{gl}^{(s)}(x^{(0)})$ and $\Sigma_{gl}^{(u)}(x^{(0)})$ the stable and unstable full leaf of a point $x^{(0)}$ respectively. Due to the discontinuous nature of a billiard system each leaf has singularities at the points of intersection with the trajectories passing through the singular points of the boundary. For $r = 2$, $\Sigma_{gl}^{(s)}$ and $\Sigma_{gl}^{(u)}$ are projected onto the coordinate space as complex zigzag curves with irregular changes of direction (cf. §2 and Figure 4). For these reasons the usual methods of deriving the ergodicity from Theorem 2, which are effective in the case of Anosov system, are inapplicable.

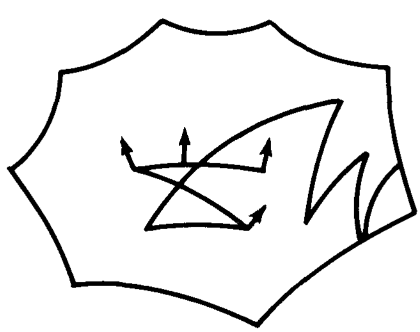

Fig. 4

Now we shall describe a new (as compared with [28], [8], [11], [19]) method of proving the ergodicity of the system $\{S^t\}$. First we shall introduce a number of concepts. An approach is often used in the ergodic theory that makes it possible to pass from a dynamical system with continuous time to a dynamical system with discrete time. Such a transition is particularly easy to achieve in the case of billiard systems. Let us consider a space $\mathfrak{M}_1^{(r)}$ consisting of such $x = (q, v) \in \mathfrak{M}_0^{(r)}$ that q is a non-singular point of the boundary $\partial\tilde{Q}^{(r)}$ and the vector v is directed inwards $Q^{(r)}$, that is, $(v, n(q)) \geq 0$. We shall let a trajectory come out of q in the direction v and proceed till it is reflected for the first time. Let q' be the point at which that reflection occurs and let v' be the velocity after the reflection. Then $(q', v') \in \mathfrak{M}_1^{(r)}$ and we can introduce the transformation $T: (q, v) \to (q', v')$ of the set $\mathfrak{M}_1^{(r)}$. It is easy to demonstrate that T conserves the measure μ for which $d\mu(x) = (n(q), v)\, dq, d\omega$, where dq is a natural measure on the cylinder $\tilde{Q}_{ij}$ containing q. Then the ergodicity of the original billiard system is equivalent to the ergodicity of T.

Let $\Sigma \subset \mathfrak{M}_1^{(r)}$ be an open $(2r-3)$-dimensional submanifold. We shall call it convex (concave) if for every point $x \in \Sigma$ we can find $\tau > 0$ and a convex (concave) open submanifold Σ', $S^{+\tau}x \in \Sigma'$, such that the natural projection of Σ' onto $\mathfrak{M}_1^{(r)}$ with the help of a motion with respect to t in a negative direction will give some neighborhood of x on Σ. For convex and concave submanifolds the second fundamental forms can be introduced as the limits, with $\tau \to 0$, of the second fundamental forms for the submanifolds Σ'. It follows from Lemmas 1 and 2 that any convex (concave) submanifold is transformed under the effect of $T(T^{-1})$ into a convex (concave) submanifold. If for $x \in \mathfrak{M}_1^{(r)}$ the time before the boundary is again reached is $\tau(x)$, then according to Lemmas 1 and 2 we have

$$\mathcal{B}_{T\Sigma}(x') = u^{-1}\mathcal{B}_{\Sigma}(x)(1+\tau(x)\mathcal{B}_{\Sigma}(x))^{-1}u + 2(n(q'),v')V^*K^{(0)}(q')V, \; Tx = (x',q').$$

A similar formula can be written for T^{-1}.

Let $\mathfrak{A} \subset \mathfrak{M}_1^{(r)}$. We shall call $\mathfrak{A}$ a parallelogram if measurable partitions $\xi^{(u)}$ and $\xi^{(s)}$ of the set $\mathfrak{A}$ are defined with convex and concave submanifolds, respectively, being the elements of these partitions and with every two elements $\gamma^{(u)}$ and $\gamma^{(s)}$ of the partitions $\xi^{(u)}$ and $\xi^{(s)}$ intersecting in only one point (an analogous concept is defined for Anosov systems in [5], [25]). Any parallelogram $\mathfrak{A}$ can be written in the form $\mathfrak{A} = \bigcup_{x \epsilon \gamma^{(u)}} \gamma^{(s)}(x) = \bigcup_{x \epsilon \gamma^{(s)}} \gamma^{(u)}(x)$, where $\gamma^{(u)}(\gamma^{(s)})$ is an arbitrary element of the partition $\xi^{(u)}(\xi^{(s)})$ and $\gamma^{(s)}(x)(\gamma^{(u)}(x))$ is an element of the partition $\xi^{(s)}(\xi^{(u)})$ passing through x.

We shall use the term "the boundary of a parallelogram $\mathfrak{A}$" for the set $\partial\mathfrak{A} = \bigcup_{x \epsilon \gamma^{(u)}} \partial\gamma^{(s)}(x) \cup \bigcup_{x \epsilon \gamma^{(s)}} \partial\gamma^{(u)}(x) = \partial\mathfrak{A}^{(s)} \cup \partial\mathfrak{A}^{(u)}$. A parallelogram $\mathfrak{A}_1$ will be termed s-inserted (u-inserted) in a parallelogram $\mathfrak{A}_2$ if $\mathfrak{A}_2 \subset \mathfrak{A}_1$ and $\partial\mathfrak{A}_2^{(s)} \subseteq \partial\mathfrak{A}_1^{(s)}(\partial\mathfrak{A}_2^{(u)} \subseteq \partial\mathfrak{A}_1^{(u)})$.

Let the point $x^{(0)} = (q_0, v_0) \epsilon \mathfrak{M}^{(r)}$ be such that all the $T^n x_0$ are defined (that is, no multiple collisions occur along the trajectory of the point $x^{(0)}$) and $\mathfrak{B}^{(s)}(x^{(0)}) > 0, \mathfrak{B}^{(u)}(x^{(0)}) > 0$. We shall now construct neighborhoods of the point $x^{(0)}$ of a special type. To do this we shall consider $S^\tau x^{(0)} = x^{(\tau)}(q^{(\tau)}, v^{(\tau)})$ and draw an ellipsoid $W^{(u)}$ through $q^{(\tau)}$ tangent at the point $q^{(\tau)}$ to the plane orthogonal to $v^{(\tau)}$ in such a way as to make its main axes coincide with the eigenvalues of the operator $\mathfrak{B}^{(u)}(x^{(\tau)}) = \mathfrak{B}^{(u)}(x^{(0)}) \cdot (I - \tau \mathfrak{B}^{(u)}(x^{(0)}))$ as well as to make the second fundamental form at the point $q^{(\tau)}$ coincide with $\mathfrak{B}^{(u)}(x^{(\tau)})$ (with unit normals oriented in $\tilde{W}^{(u)}$ so that $v^{(\tau)}$ should enter into that framing). We shall denote this framing by $W^{(u)}$ and let $V^{(u)} \subset W^{(u)}$ be a neighborhood of $x^{(0)}$ on that framing. Now we shall in a similar way draw an ellipsoid $\tilde{W}^{(s)}(y)$ through every point $y \epsilon V^{(u)}$ with second fundamental form at a point $q_y(y = (q_y, v_y))$ equal to $\mathfrak{B}^{(s)}(x_0)$, and we shall use the notation $V^{(s)}(y)$ to designate a neighborhood of y on $W^{(s)}(y)$ which is a framing of $W^{(s)}(y)$, containing v_y. We shall assume $U' = \bigcup_{y \epsilon V^{(u)}} V^{(s)}(y)$ and project u' onto $\mathfrak{M}_1^{(r)}$ by means of the shift along the trajectory of the dynamical system. Then we shall obtain a neighborhood U of a point $x^{(0)}$ on $\mathfrak{M}_1^{(r)}$. It is clear that by choosing sufficiently smooth boundaries $V^{(u)}$ and $V^{(s)}$ we shall be able to ensure that U will have a piecewise smooth boundary. We shall call U a rectangular neighborhood of $x^{(0)}$.

After fixing a rectangular neighborhood U, we shall construct regular δ-coverings of U. By this term we shall mean open coverings $\mathfrak{A}^{(\delta)} = \{\mathfrak{A}_1, \cdots \mathfrak{A}_p\}$ possessing the following properties:

1) if $\mu(\mathfrak{A}_i \cap \mathfrak{A}_j) > 0$ then there exists a number const which does not depend on δ and is such that $\mu(\mathfrak{A}_i \cap \mathfrak{A}_j) \geq \text{const}\, \mu(\mathfrak{A}_i)$; $\mu(\mathfrak{A}_i \cap \mathfrak{A}_j) \geq \text{const}\, \mu(\mathfrak{A}_j)$;

2) there exists a number const which does not depend on δ and is such that every point $x \in U$ belongs to no more than const elements of the covering $\mathfrak{A}^{(\delta)}$;

3) every element $\mathfrak{A}_i$ of the covering $\mathfrak{A}^{(\delta)}$ is a parallelogram, the leaves $\gamma^{(s)}$ and $\gamma^{(u)}$ of which are open submanifolds with a piece-wise smooth boundary and the diameter of $\gamma^{(s)}$ and $\gamma^{(u)}$ does not exceed const δ;

4) if $\mathfrak{B}_{\gamma^{(s)}}(x)$, $\mathfrak{B}_{\gamma^{(u)}}(x)$ are the second fundamental forms of the leaves $\gamma^{(s)}_{(x)}$ and $\gamma^{(u)}_{(x)}$, respectively, then $\sup_{x \in U} \|\mathfrak{B}_{\gamma^{(s)}}(x) - \mathfrak{B}^{(s)}(x)\|$, $\sup_{x \in U} \|\mathfrak{B}_{\gamma^{(u)}}(x) - \mathfrak{B}^{(u)}(x)\| \to 0$ with $\delta \to 0$.

Regular δ-coverings exist and it is easy to construct them explicitly.

Now we can formulate the main theorem. Let $x_0 \in \mathfrak{M}_1^{(r)}$ be such that $T^n x_0$, $-\infty < n < \infty$ does not pass through any singular points of the boundary and $\mathfrak{B}^{(s)}(x_0) > 0$, $\mathfrak{B}^{(u)}(x_0) > 0$. For any sufficiently small $\lambda > 0$ we shall use $U_\lambda(x_0)$ to denote a rectangular neighborhood of a point x_0 with a diameter which does not exceed λ, and use $A^{(\delta)} = \{\mathfrak{A}_1, \cdots, \mathfrak{A}_p\}$ to denote a regular δ-covering of $U_\lambda(x_0)$.

THE MAIN THEOREM. *Let $\omega > 0$ be given. Then there exist $\beta = \beta(\omega) > 0$, $\lambda = \lambda(\omega) > 0$, a family of regular δ-coverings, $A^{(\delta)}$, of the neighborhood $U_\lambda(x_0)$, and a function $\phi(\delta)$, $\phi(\delta) \downarrow 0$ as $\delta \downarrow 0$ with the following properties:*

For any $\alpha > 0$ *and all sufficiently small* δ $(\delta < \delta(\alpha,\omega))$, *there are two sequences,* $\{\alpha_m\}$ *and* $\{\beta_m\}$, *and a subset* $W^{(\delta)}$ *of elements of* $A^{(\delta)}$ *such that every* $A_i \notin A^{(\delta)}$ *contains sets* $\tilde{G}^{(u)}_{mi}$, $m = 0, 1, 2, \cdots$, *which are the union of disjoint parallelograms* $\tilde{G}^{(u)}_{mij}$, $j = 1, \cdots, N_{mi}$. *These satisfy*

1) $\mu\left(\bigcup_{A_i \in W^{(\delta)}} A_i\right) \leq \delta\phi(\delta)$

2) *For* $A_i \notin W^{(\delta)}$, *no more than two parallelograms,* $\tilde{G}^{(u)}_{i1}, \tilde{G}^{(u)}_{i2}$, *are* s-*inserted into* A_i *and*

$$\mu(\tilde{G}^{(u)}_{i1} \cup \tilde{G}^{(u)}_{i2}) \geq (1-\omega)\mu(A_i)$$

3) *For every* $A_i \notin W^{(\delta)}$, *and* $\tilde{G}^{(u)}_{mi}$, *the sequence of* $\tilde{G}^{(u)}_{mij}$, $j = 1, \cdots, N_{mi}$ *satisfies*

a1) $N_{0i} = 1$, $\tilde{G}^{(u)}_{0i1} = A_i$;

$N_{1i} = 2$, $\tilde{G}^{(u)}_{1i} = \tilde{G}^{(u)}_{i1} \cup \tilde{G}^{(u)}_{i2}$

a2) $\Sigma\, \alpha_m \leq \alpha$

a3) *For every* j, $1 \leq j \leq N_{mi}$, *there exists a* j', $1 \leq j' \leq N_{(m-i)i}$ *for which* $\tilde{G}^{(u)}_{mij}$ *is* s-*inserted into* $\tilde{G}^{(u)}_{(m-1)ij'}$ *and if* $\tilde{G}^{(u)}_{(m-1)ij'}$ *contains at least one parallelogram* $\tilde{G}^{(u)}_{mij}$ *then*

$$\sum_{\tilde{G}^{(u)}_{mij} \subset \tilde{G}^{(u)}_{(m-1)ij'}} \mu(\tilde{G}^{(u)}_{mij}) \geq (1-\alpha_m)\mu(\tilde{G}^{(u)}_{(m-1)ij'})$$

a4) *If* $\mathcal{F}^{(u)}_{(m-1)i}$ *is the union over* j' *of parallelograms* $\tilde{G}^{(u)}_{(m-1)ij'}$ *which do not contain any* $\tilde{G}^{(u)}_{mij}$ *then*

$$\mu\left(\bigcup_i \mathcal{F}^{(u)}_{(m-1)i}\right) \leq \delta\phi(\delta)\beta_m$$

and $\Sigma\, \beta_m \leq \beta$.

Similar statements 1)-3) *are true if* "s" *and* "u" *are interchanged everywhere. An important additional property is that for every parallelogram* $A_i \not\subset W^{(\delta)}$, *only one pair* $(\tilde{G}_{i1}^{(u)}, \tilde{G}_{i2}^{(u)})$, $(\tilde{G}_{i1}^{(s)}, \tilde{G}_{i2}^{(s)})$ *consists of two non-empty parallelograms. If* $\tilde{G}_{i2}^{(u)} (\tilde{G}_{i2}^{(s)})$ *is empty, then* $\tilde{G}_{i1}^{(u)} = A_i\ (\tilde{G}_{i1}^{(s)} = A_i)$.

A similar statement is true if in all the expressions the index "u" is substituted for the index "s" and vice versa.

Discussion. The geometrical situation described in the statement of the main theorem is as follows. Inside a covering element A_i we construct, for every m, a sequence of non-intersecting parallelograms $\tilde{G}_{mij}^{(u)}$. Each parallelogram $G_{mij}^{(u)}$ either belongs to the set $\mathfrak{F}_{mi}^{(u)}$, or contains inside itself parallelograms $\tilde{G}_{(m+1)ij'}^{(u)}$. In the latter case they fill almost the whole of the parallelogram $\tilde{G}_{mij}^{(u)}$.

The property a4) is extremely important. Roughly speaking, it shows that the parallelograms removed at the m-th step cannot be placed along the submanifolds of codimension one. In a sense, this allows us to restrict ourselves to triple collisions only.

Let

$$\tilde{G}_i^{(u)} = \bigcap_{m=0} \tilde{G}_{mi}^{(u)}, \quad \mathfrak{F}_i^{(u)} = \bigcup_{m=0}^{\infty} \ \bigcup_{G_{mij}^{(u)} \in \mathfrak{F}_{mi}^{(u)}} \tilde{G}_{mij}^{(u)}$$

and

$$\mathfrak{D}^{(u)} = \bigcup_{A_i :\, \mu(\mathfrak{F}_i^{(u)}) \le \sqrt{\phi(\delta)}\, \mu(A_i)} A_i \quad , \quad \mathfrak{D}_1^{(u)} = \bigcup_{A_i :\, \mu(\mathfrak{F}_i^{(u)}) > \sqrt{\phi(\delta)}\, \mu(A_i)} A_i$$

We shall demonstrate that

$$\mu(\mathfrak{D}_1^{(u)}) \le \text{const}\ \beta\delta\sqrt{\phi(\beta)}, \quad \mu(\mathfrak{D}^{(u)}) \ge \mu\left(\bigcup_{i=1}^{r} A_i\right)(1-\text{const}\ \beta\delta\sqrt{\phi(\delta)})\,.$$

We have

$$\mu(\mathfrak{D}_1^{(u)}) \le \sum_{A_i \subset \mathfrak{D}_1^{(u)}} \mu(A_i) \le \frac{1}{\sqrt{\phi(\delta)}} \sum_{i=1}^{p} \mu(\mathcal{F}^{(u)}) \le \frac{\text{const}}{\sqrt{\phi(\delta)}} \mu\left(\bigcup_{i=1}^{p} \mathcal{F}_i^{(u)}\right) =$$

$$= \frac{\text{const}}{\sqrt{\phi(\delta)}} \mu\left(\bigcup_{i=1}^{p} \bigcup_{m=0}^{\infty} \mathcal{F}_{im}^{(u)}\right) \le \frac{\text{const}}{\sqrt{\phi(\delta)}} \sum_{m=0}^{\infty} \mu\left(\bigcup_{i=1}^{p} \mathcal{F}_{im}^{(u)}\right) \le \text{const}\, \delta \sqrt{\phi(\delta)}\, \beta\,.$$

In the above expressions we make use of the fact that every point $x \in U_\lambda$ belongs to no more than const elements of the covering. The second inequality easily follows from the first.

We shall note now that for any i $\mu(A_i - \mathcal{F}_i^{(u)}) \ge (1 - \text{const}\, a)\mu(\tilde{G}_i^{(u)})$. Indeed, let for some m and with $\varepsilon_m < 1$ the inequality

$$\mu\left(A_i - \bigcup_{n=0}^{m-1} \mathcal{F}_{ni}^{(u)}\right) \ge \varepsilon_m\, \mu\left(\bigcup_{G_{mij}^{(u)} \not\subset \bigcup_{n=0}^{m-1} \mathcal{F}_{ni}^{(u)}} G_{mij}^{(u)}\right)$$

be true. Then due to a3) we have

$$\mu\left(A_i - \bigcup_{n=0}^{m} \mathcal{F}_{ni}^{(u)}\right) \ge \varepsilon_m\, \mu\left(\bigcup_{G_{mij} \not\subset \bigcup_{n=0}^{m} \mathcal{F}_{ni}^{(u)}} G_{mij}^{(u)}\right) \ge$$

$$\ge \varepsilon_m (1 - a_m)\, \mu\left(\bigcup_{G_{(m+1)ij} \subset G_{mij} \not\subset \bigcup_{n=0}^{m} \mathcal{F}_{ni}^{(u)}} G_{(m+1)ij}^{(u)}\right).$$

Multiplying these ineaualities with respect to all the m's, we shall obtain the statement required. From it follows, among other things, that for $A_i \subset \mathfrak{D}^{(u)}$, $\mu(\tilde{G}_i^{(u)}) \ge (1 - \text{const}\, a)(1 - \sqrt{\phi(\delta)})\mu(A_i)$.

Now we shall derive the basic corollary.

COROLLARY 1. *If the conditions of the theorem are valid then the neighborhood* U_λ *belongs to one ergodic component.*

Our reasoning will be based on the lemma proved below.

LEMMA. *Let, for all sufficiently small* δ's, *there be, for a covering* $\mathfrak{A}^{(\delta)}$, *a set* $\mathfrak{D}$ *which is a union of the covering elements* $\mu(\mathfrak{D}) \geq \mu(\overset{p}{\cup} A_i)(1-\delta\psi(\delta))$, *where* $\psi(\delta) \to 0$ *with* $\delta \to 0$. *Then there can be found a connected subset* $\mathfrak{E}$ *which is a union of the elements of the covering* $\mathfrak{A}^{(\delta)}$ *for which* $\mu(\mathfrak{E}) \geq \mu(\overset{p}{\cup} A_i)\cdot(1-\psi_1(\delta))$, *where* $\psi_1(\delta) \to 0$ *with* $\delta \to 0$.

Using this lemma, we shall prove the corollary. Let $\mathfrak{D}^{(u)}$ and $\mathfrak{D}^{(s)}$ be the sets constructed above (we are discussing the set $\mathfrak{D}^{(u)}$; for $\mathfrak{D}^{(s)}$ the index "s" must be changed to "u" and vice versa) and $\mathfrak{D} = \mathfrak{D}^{(u)} \cup \mathfrak{D}^{(s)} \cup W^{(s)}$. Then $\mu(\mathfrak{D}) \geq (1-\text{const}\,\delta\sqrt{\phi(\delta)})\mu(\overset{p}{\cup} A_i)$. With the help of the lemma we shall find a connected subset $\mathfrak{E} \subset \mathfrak{D}$ consisting of elements of the covering $\mathfrak{A}^{(\delta)}$ and we shall put $\mathfrak{E}' = \bigcup_i \tilde{G}_i^{(u)} \cup \bigcup_i \tilde{G}_i^{(s)}$. Then on the strength of what has been said above

$$\mu(\mathfrak{E}') \geq (1-\text{const}\,\alpha)\mu(\mathfrak{E}) \geq (1-\text{const}\,\alpha)(1-\psi_1(\delta))\mu\left(\bigcup_{i=1}^{p} A_i\right).$$

We shall now show that $\mathfrak{E}'$ belongs to one ergodic component.

First of all we shall note that $\tilde{G}_i^{(u)} \cap \tilde{G}_i^{(s)}$ consists mod 0 of a single ergodic component. The corresponding arguments go back to Hopf ([14]). Roughly speaking, the idea is as follows: the leaf $\gamma^{(s)}(x)$ of the parallelogram $\tilde{G}_i^{(u)}$ belongs to one ergodic component since for any continuous function f the time averages $f^+(y) = \lim \frac{1}{n}\sum_{k=0}^{n-1} f(T^k y)$ are constant on $\gamma^{(s)}(x)$ because for $y_1, y_2 \in \gamma^{(s)}(x)$ the distance between $T^k y_1$ and $T^k y_2$ tends to zero with $k \to \infty$. For the same reason $f^-(y) = \lim \frac{1}{n}\sum_{k=0}^{n-1} f(T^{-k} y)$ are constant on $\gamma^{(u)}(x)$. Consequently, due to the fact that $f^+ = f^-$ with probability one, for $\gamma^{(s)}(x) \cap \gamma^{(u)}(x) \neq \emptyset$ the union $\gamma^{(s)}(x) \cup \gamma^{(u)}(x)$ belongs to one ergodic component. And from this it follows that $\tilde{G}^{(u)} \cap \tilde{G}^{(s)}$ belongs to one ergodic component. If $\tilde{G}^{(u)}_{i2}(\tilde{G}^{(s)}_{i2})$ is empty then the same arguments show that $\tilde{G}^{(u)}_{i2}(\tilde{G}^{(s)}_{i2})$ belongs to the single ergodic component. The property of "absolute continuity" (cf. [4], [24], [21]) enables this reasoning to be made completely rigorous.

If $\mu(A_{i1} \cap A_{i2}) > 0$, then, for sufficiently small α $(\alpha < \text{const})$

$$(\alpha = \text{const})\ \mu(\tilde{G}^{(u)}_{i1} \cap \tilde{G}^{(s)}_{i1}) \cap (\tilde{G}^{(u)}_{i2} \cap \tilde{G}^{(s)}_{i2})) > 0\,.$$

But then $(\tilde{G}^{(u)}_{i1} \cap \tilde{G}^{(s)}_{i1}) \cup (\tilde{G}^{(u)}_{i2} \cap \tilde{G}^{(s)}_{i2})$ belongs to the single ergodic component. And from this, due to the connectedness of $\mathfrak{S}$, it follows directly that the whole of $\mathfrak{S}'$ belongs to one ergodic component.

Letting δ tend to infinity first and then α to zero, we obtain the statement of the corollary. Using arguments similar to those in [28], [8], we can now extend the statement to such points $x^{(0)}$ along the trajectory of which either one triple collision or one degenerate collision has occurred for which $(n(q), v) = 0$. The complement to the set of those $x^{(0)}$'s, where either there has been no multiple or degenerate collisions or there has been only one such collision, is a union of a countable number of manifolds of codimension 2. Therefore the set of such $x^{(0)}$'s itself is linearly connected. Let us consider in a more detailed way the condition $\mathcal{B}^{(s)}(x) > 0$, $\mathcal{B}^{(u)}(x) > 0$. As was explained in point 2 this condition is not fulfilled, on the one hand, on a union of a countable number of manifolds of codimension no less than 2. Therefore the removal of the union does not break the linear connectedness. There remains a set of such x's, along the trajectories of which, either for all sufficiently great t's or for all sufficiently small t's, the set of disks falls into two groups between which there is either no interaction at all, or the interaction is degenerate and is not accompanied by a change in the velocities of the disks. It can be demonstrated that the set does not break the linear connectedness of the complement. Thus, we find that the set of such $x^{(0)}$'s for which the statement of Corollary 1 is true has a probability one and is linearly connected. From this and from Corollary 1 follows the basic result: the transformation T and, consequently, the flow $\{S^t\}$ are ergodic.

If we make use of the results of [24], [21] again, we can demonstrate that the transformation T is a K-automorphism.

*The proof of the lemma.** We shall use induction with respect to the number of dimensions. Initially let $\dim U_{\lambda_0} = 2$. Then U_{λ_0} can be thought of as an ordinary square and $\mathfrak{A}^{(\delta)}$ as a covering formed by squares parallel to the side edges. We can therefore speak of vertical columns and horizontal rows going from one edge of the square to the opposite edge and which are formed by the squares of the covering $\mathfrak{A}^{(\delta)}$. It follows from the conditions of the lemma that a horizontal row $\Gamma^{(h)}$ can be found consisting of some elements of the covering $\mathfrak{A}^{(\delta)}$ and belonging as a whole to $\mathfrak{D}$. Then we shall take all the vertical columns $\Gamma_i^{(v)}$ consisting of the elements of the covering $\mathfrak{A}^{(\delta)}$ and belonging to $\mathfrak{D}$. The measure of the union of such columns is no less than $\mu(U_{\lambda_0})(1-\text{const}\,\psi(\delta))$. Let $\mathfrak{S} = \Gamma^{(h)} \cup \bigcup_i \Gamma_i^{(v)}$. It is evident that $\mathfrak{S}$ is connected and the statement of the lemma is fulfilled for it.

Let $\dim U_{\lambda_0} = p > 2$. Then we can speak of one-dimensional vertical columns and of (m-1)-dimensional horizontal layers. There will be found such a horizontal layer $\Gamma^{(h)}$, conforming to the conditions of the lemma, for which $\mu(\Gamma^{(h)} \cap D) \geq \mu(\Gamma^{(h)})(1-\text{const}\,\delta\psi(\delta))$. According to the assumption of the induction, a connected set $\mathfrak{S}^{(h)}$ of elements of the covering $\mathfrak{A}^{(\delta)}$, belonging to $\Gamma^{(h)} \cap \mathfrak{D}$, can be found for this layer for which $\mu(\mathfrak{S}^{(h)}) \geq \mu(\Gamma^{(h)})(1-\tilde{\psi}_1(\delta))$, $\tilde{\psi}_1(\delta) \to 0$ with $\delta \to 0$. We shall call the vertical one-dimensional column pure if it consists entirely of parallelograms belonging to $\mathfrak{D}$. We shall assume that $\mathfrak{S} = \mathfrak{S}^{(h)} \cup \bigcup_i \Gamma_i^{(v)}$, where

*I must thank G. A. Margulis for his useful remarks concerning the lemma owing to which it has become possible to reduce the proof significantly.

$\Gamma_i^{(v)}$ are pure vertical columns, one of the elements of the covering $\mathfrak{A}^{(\delta)}$ of which belongs to $\mathcal{E}^{(h)} \cap \Gamma^{(v)}$. It is evident that $\mathcal{E}$ is connected and the statement of the lemma is fulfilled for it. The lemma is proved.

4. *Construction*

We shall construct parallelograms G_{mij} in such a way that the parallelograms $T^{-m} G_{mij} = \tilde{G}_{mij}$ could fulfill all the requirements of the main theorem. The process of construction will be inductive in nature. We shall discuss in detail the case $r = 3$ and then mention the necessary changes that must be made for greater values of r's.

Let $G_{0i1} = A_i$ for $m = 0$. The construction can be performed in two steps. The first step is based on the lemma according to which in this case we can restrict ourselves to parallelograms, which over a fairly long period of time fall into smaller parts under the action of T no more than once. First of all we must point out that the transformation T is piecewise continuous and has a discontinuity on a countable number of manifolds of discontinuity $\Sigma_i(T)$. By $\Sigma_i(T^s)$ we shall denote submanifolds of discontinuity for $T^s, \tau(\Sigma_i(T^s)) = \max\limits_{x \in \Sigma_i(T^s)} (\tau(x) + \tau(Tx + \cdots + \tau(T^{s-1}x))$.

Here it is supposed naturally that $\Sigma_i(T^s)$ is not a submanifold of discontinuity for T^{s-1}.

Lemma 3. *There exists a function $F_1(\delta)$, $F_1(\delta) \to \infty$ for $\delta \to 0$ such that for a set $W^{(1)}$ of parallelograms A_1, intersecting with at least two manifolds of discontinuity $\Sigma_i(T^s)$ for which $\tau(\Sigma_i(T^s)) \leq F_1(\delta)$, the equality $\lim\limits_{\delta \to 0} \delta^{-1} \mu(W^{(1)}) = 0$ is fulfilled.*

The proof of that lemma is simple. For $r = 3, 4, 5$ it is possible to demonstrate, in addition, that $F(\delta) \sim$ const $\ln \delta^{-1}$. We shall begin by removing the parallelograms G_{0i1} belonging to $W^{(1)}$. Each of the remaining parallelograms G is characterized by the fact that there can be found no more than one value K_0 for which $T, T^2, \cdots, T^{k-1}$ are continuous on G and $T^{k+1}, \cdots, T^m$ are continuous on every connected component and $\tau(x) + \cdots + \tau(T^m x) \leq F_1(\delta)$. If such a k exists then $T^{-1}, T^{-2}, \cdots, T^{-m^r}$ are continuous on G for every m for which $\tau(T^{-m^r}x) + \tau(T^{-m^r+1}x) + \cdots + \tau(T^{-1}x) \leq F_1(\delta)$.

In any of the remaining parallelograms G_{0i1} we shall fix the leaf $\gamma_{0i1}^{(u)}$ of the partition $\xi^{(u)}$. We shall find for this leaf the number k mentioned above. Let us suppose that for $l < k-1$ a parallelogram G_{li1}, $T^{-l}G_{li1} \subset G_{0i1}$ with an expanding base $\gamma_{li1}^{(u)}$, $T^{-l}\gamma_{li1}^{(u)} \subset \gamma_{0i1}^{(u)}$ has already been constructed. By virtue of Lemmas 1 and 2, $T\gamma_{li1}^{(u)}$ shall be a convex submanifold. Let us assume that $\rho_{l+1} = \dfrac{C\delta^{3/2}}{1+t_{l+1}}$, where $C = C(x^{(0)})$ is a constant that will be given below, and $t_{l+1} =$
$= \min\limits_{x \in T^{-l}G_{li1}} (\tau(x) + \cdots + \tau(T^l x))$. We shall construct a set $\gamma_{(l+1)i1}^{(u)} =$
$= \{x \in T\gamma_{li1}^{(u)} : \operatorname{dist}(x, \partial(T\gamma_{li1}^{(u)})) \geq \rho_{l+1}\}$ and take the concave framing $\gamma^{(s)}(x)$ of the ellipsoid with the second fundamental form $\mathcal{B}^{(s)}(x)$ through every point $x = (q, v) \in \gamma_{(l+1)i1}^{(u)}$ in such a way that $\partial(T^{-1}\gamma^{(s)}(x)) \subset \partial G_{li1}$).* It follows from Lemmas 1 and 2 that $T^{-1}\gamma^{(s)}(x)$ is a concave submanifold.

LEMMA 4. *The constant* $C(x^{(0)})$ *can be chosen in such a way that* $\partial(T^{-1}\gamma^{(s)}(x)) \subset \partial^{(s)} G_{li1}$.

It follows from Lemma 4 that $\cup\gamma^{(s)}(x)$ is a parallelogram, where we take the union for $x \in \gamma_{(l+1)i1}^{(u)}$ as $G_{(l+1)i1}$.

There will exist, for $k \leq l \leq m$, two parallelograms G_{li1} and G_{li2}, and the construction procedure described above can be applied to either of them.

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*The ellipsoid and the construction procedure were described in detail at the beginning of point 3, when the neighborhood U was defined.

It remains to discuss the case $l = k-1$. The transformation T is discontinuous on $G_{k-1,i1}$. On decreasing $F_1(\delta)$ if necessary, (this is equivalent to decreasing δ with F_1 fixed), we can assume that T subdivides $G_{(k-1)i1}$ into two components, and on both of them T is continuous. Correspondingly, $\gamma^{(u)}_{(k-1)i1}$ is subdivided into two components $\hat{\gamma}^{(u)}_{(k-1)i1}$ and $\hat{\hat{\gamma}}^{(u)}_{(k-1)i1}$. We shall construct $T\hat{\gamma}^{(u)}_{(k-1)i1}$ and $T\hat{\hat{\gamma}}^{(u)}_{(k-1)i1}$, move back from the edge by a value of ρ_k and construct two parallelograms G_{ki1} and G_{ki2} in the same way as described above. We shall consider further only such parallelograms G_{kij} for which $\mu(T^{-1} G_{kij} \mid G_{(k-1)i1}) \geq \frac{\alpha}{8}$, $j = 1,2$. Let us put $G_{ij} = T^{-k} G_{kij}$, $j = 1,2$.

We shall now proceed to the second step. So far we have practically made no use of the fact that $r = 3$. Now we shall bear it in mind and, besides, we shall analyze the reflection structure.

Let us suppose that on the m-th step, parallelograms G_{mij} with expanding bases were constructed $\gamma^{(u)}_{mij}$. As will be seen the sizes of $\gamma^{(s)}(x)$ are the most important factor in the construction. In the absence of degeneration, they must decrease exponentially with respect to m. To evaluate the sizes of $\gamma^{(s)}(x)$ "in a linear approximation," it should be noted that for any $x \in G_{mij}$ and $l < m$ a natural mapping $\mathfrak{A}^m_l(y)$ of the space tangent to $\gamma^{(s)}(x)$ at a point $y \in \gamma^{(s)}(x)$ onto the space tangent to $T^{-m+l}\gamma^{(s)}(x)$ at a point $T^{-m+l}y$ is defined. This mapping can be described formally by an equation in variations. It is easy to see in our case that $\mathfrak{A}^m_{l-1}(y) = U_l^{-1}|\mathfrak{A}^m_l(y)(I+\tau(T^{-m+l-1}y)\mathcal{B}_{T^{-m+l-1}\gamma^{(s)}(y)}(T^{-m+l-1}y))U_l$. It follows from this that at typical points $\mathfrak{A}^m_0 \geq (1+\text{const})^m I$. Since $T^{-m}\gamma^{(s)}(x)$ must be of the order of δ we find that $\gamma^{(s)}(x)$ must be of the order of $\delta(1+\text{const})^{-m}$. A significantly weaker assumption will suffice for our construction procedures:

A simple case. For every point $y \in G_{mij}$ the operator $\mathfrak{A}^m_0 \geq m^8 I$, where I is the identity operator.

The boundary $\partial Q^{(2)}$ can be subdivided, for any $\varepsilon > 0$, into subsets of a diameter ε with a piecewise smooth boundary having a uniformly limited curvature and contained inside itself a sphere with a radius const ε. Such a subdivision of the boundary will be termed an ε-net.

We shall consider in a simple case the $\frac{\delta}{m^{1,2}}$-net on the boundary $\partial Q^{(2)}$. It subdivides $\gamma^{(u)}_{mij}$ into subsets. We shall call a subset internal when its boundary coincides with the boundary of a net element. When the contrary is the case, we shall call it a boundary subset. Inside an element of the $\frac{\delta}{m^{1,2}}$-net containing a boundary subset we shall construct the $\frac{\delta}{m^{2,4}}$-net and consider the intersection between the $\gamma^{(u)}_{mij}$ and the elements of this finer net. We shall subdivide them also into internal and boundary subsets. Leaving the internal elements aside we shall again use the boundary elements to construct the $\frac{\delta}{m^{3,6}}$-net. Then we shall make another such step and again we shall obtain the $\frac{\delta}{m^{4,8}}$-net. At this step we shall remove the boundary elements, take only the internal subsets and decompose them into connected components. We shall denote these components as $\gamma^{(u)}_{mijk}$, $G_{mijk} = \bigcup_{x \in \gamma^{(u)}_{mijk}} \gamma^{(s)}(x)$. It is obvious that G_{mijk} is a parallelogram.

Let $\Sigma^{(1)}_m$ be a $\frac{\delta}{m^{1,2}}$-neighborhood of discontinuity manifolds of the transformation T. We shall not consider the parallelograms G_{mijk} that intersect $\Sigma^{(1)}_m$. On the remaining parallelograms T is continuous. If $\gamma^{(u)}_{mijk} = \gamma^{(u)}_{mij} \cap G_{mijk}$ then $T\gamma^{(u)}_{mijk}$ is convex submanifold. We shall assume that $\hat{\gamma}^{(u)}_{mijk} = \{x \epsilon T\gamma_{mijk} : \operatorname{dist}(x, \partial(T\gamma_{mijk})) \geq \frac{\delta}{m^6}\}$. Through every point $x \epsilon \hat{\gamma}^{(u)}_{mijk}$ we shall construct the concave framing $\gamma^{(s)}(x)$ of the ellipsoid with the second fundamental form $\mathcal{B}^{(s)}(x)$ in such a way that $\partial(T^{-1}\gamma^{(s)}(x)) = T^{-1}(\partial\gamma^{(s)}(x)) \subset \partial G_{mij}$. Then it can be easily demonstrated that the set $\bigcup_{x \epsilon \gamma^{(u)}_{mijk}} \gamma^{(s)}(x)$ will be a parallelogram which we shall denote by $G_{(m+1)ij'}$.

Zero series. There can be found for at least one point $x \epsilon G_{m_0 ij}$ a vector e for which $\|\mathfrak{A}_0^{m_0} e\| \leq m_0^8 \|e\|$ but for $G_{(m_0-1)ij'} \supset T^{-1} G_{m_0 ij}$ this does not hold. In that case we shall call the m_0-th reflection an initial zero reflection. A series of reflections with numbers $m_0, m_0+1, \cdots, m$ will be termed a zero series if the above-mentioned inequality is fulfilled for all those reflections. If m_1 is such that for m_1 the inequality if fulfilled and for m_1+1 already it is not, then we shall call m_1 the final reflection of the zero series.

Beginning with the m_0-th reflection we shall watch the appearance inside the parallelograms G_{mij} of such points x for which in the tangent space of a point x there can be found a vector e such that $\|\mathfrak{B}^{(s)}(x)e\| \leq \frac{\ln^2 m}{m} \|e\|$. We must note that when the inverse inequality is fulfilled, for all $x \epsilon G_{mij}$ and all $1 < m$, $\mathfrak{A}_0^m \geq e^{\text{const} \ln^2 m} I$, that is, we are in the domain of the simple case.

The existence of the above-mentioned vector e means that in the direction corresponding to it the contraction and, consequently, the expansion operate for a long time almost linearly, that is, almost exactly as in the case of the reflection from flat walls. In other words, over a time period of the order $\frac{m}{\ln^2 m}$ the evolution of the layer $\gamma^{(u)}(x)$ is not accompanied by any distortion at all along the trajectory of the point x in the direction of the vector e and its shifts. For $r = 3$ such a degeneration can occur either when one of the disks does not interact with the other two or when there is a degeneration several times in succession, which is expressed by the fact that in the process of interaction the distortion of $\gamma^{(u)}(x)$ takes place mainly in one and the same direction. Construction of parallelograms is conducted with due regard to these circumstances. When a vector e appears for which $\|\mathfrak{B}^{(s)}(x)e\| \leq \frac{\ln^2 m}{m} \|e\|$ we can determine the nature of the degeneration on the basis of the expression for this vector. Let $\overline{m} \geq m_0$ be the first reflection for which $\|\mathfrak{B}^{(s)}(x)e\| \leq \frac{\ln^2 m}{m} \|e\|$. In particular, $\overline{m}$ can be the moment of the termination of the first stage.

For $m > \bar{m}$ we shall consider parallelograms $G_{mij'}$, $T^{-m+\bar{m}} G_{mij'} \subset G_{mij}$ in which a degeneration remains only as long as their relative measure is sufficiently great, that is, the ratio $\mu(G_{mij'})/\mu(G_{mij}) = \mu(T^{-m+\bar{m}} G_{mij'})\mu(G_{mij})$ is great. In this case $\gamma^{(u)}(x)\,(\gamma^{(s)}(x))$ will be expanded (contracted) linearly in the directions of degeneration. Due to the linear character of the contraction, the parallelogram parts removed near the boundary will have a relatively small measure and the property a3) will not be violated.

After the reflection which eliminates "almost flat" directions we shall consider the intersections between the discontinuity manifold for T and three degeneration manifolds corresponding to three successive degenerations. We shall consider the $\dfrac{\delta}{m^{1/2}\ln^2 m}$ neighborhood of a discontinuity manifold and the $1/m^4$ neighborhood of each of the degeneration manifolds and their intersection. The measure of such intersections decreases by the law $\dfrac{\text{const}\,\delta}{m^{1,25}\ln^2 m}$. At the m-th step we shall consider only those parallelograms G_{mij} for which the measure of intersection between G_{mij} and the union of the triple intersections described above does not exceed $\dfrac{1}{m^{1/4}}\,\mu(G_{mij})$. The parallelograms with the inverse inequality will not be considered at all. The measure of these parallelograms satisfies the estimate of the point a4) in the statement of the theorem. The basic property will hold for the remaining parallelograms: the parallelogram part, where degenerated "almost flat" directions appear, is used in the construction of subsequent parallelograms only if its conditional measure is sufficiently great, that is, there are degenerate directions for most in terms of the measure of the parallelogram points. But then we can again use considerations applicable in the linear case in reflections from smooth edges. If there are no "almost flat" directions, after $m^{3/8}$ reflections the layer $\gamma^{(s)}(x)$ will be contracted and will have size of the order of $\dfrac{\delta}{m^{1,125}}$. Besides, there exist a number of cases for which special considerations are necessary.

REMARKS.

1. The text of the last section cannot of course be considered as a complete proof. We hope, however, that the reader who understands the method described in point 3.3 will be able to reconstruct on his own all the missing details.

2. For an arbitrary r the proof is analogous. However, the list of degeneracies is much longer in this case and the analysis of the intersections is more complex. It is this fact that explains the complexity and the length of the complete proof which is not yet published in complete detail.

3. The entropy of the dynamical system $\{S^t\}$ corresponding to the hard-sphere gas can be determined in the following way. Let us consider the operator $\mathcal{B}^{(u)}(x)$ defined above, which is non-negative in our case, and suppose that $h(x) = \operatorname{tr} \mathcal{B}^{(u)}(x)$. Then it can be shown that the entropy of the dynamical system $h(\{S^t\}) = \int_u h(x)\, d\,\mu(x)$. This formula is a special case of the corresponding formula from [24].

4. N. S. Krylov adduces some arguments, according to which, for a sufficiently general type of interaction potential, the dynamical system, which appears when the particles are moving under the effect of this potential, possesses the property of exponential instability. These arguments cannot be considered as convincing and it is not clear whether the arguments are sufficient even to prove that the entropy is positive.

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