CHAPTER 5

Linearization

We can hardly expect to make much headway in the global theory of dynamical systems on a smooth manifold X if we are completely ignorant of how systems behave locally (that is to say, near a point). We should like to attempt a local classification, but, bearing in mind our difficulties with linear systems, we do not expect complete success. However, regular (as opposed to fixed) points present no problems. As we shall see in the following section, there is essentially only one type, represented in the case of diffeomorphisms by a translation of **E**, the space on which X is modelled, and in the case of vector fields by any non-zero constant vector field on **E**. Thus, in both cases, interesting local behaviour occurs only at fixed points.

As we have seen in the previous chapter, linear systems exhibit a wide variety of behaviour near the fixed point zero. Moreover, if we introduce higher order terms, we admit further possibilities. For example, the vector field $v(x) = x^2$ on $\mathbf{R}$ has a "one-way zero" (see Figure 5.1), but no linear

FIGURE 5.1

vector field has an orbit structure looking like this. In order to prevent our classification programme from getting out of hand, we pick on a class of fixed points that occur often enough to be thought of as "the sort that one usually comes across". In the case of linear systems we were able to give this rather vague notion a precise topological meaning in terms of the space of all linear systems on a given Banach space **E**. When we generalize to smooth systems on a given manifold X we are still able to topologize the set of all systems, and to discuss open and dense subsets of the resulting space (see Chapter 7 and Appendix B below). Of course the space of all systems is infinite dimensional even when X is finite dimensional. Although the connection may seem tenuous at the moment, that is, in fact, one reason why we try to make our theory work for infinite dimensional $\mathbf{E}$ whenever possible.

Suppose that a given dynamical system ϕ has a fixed point p. If we are only interested in local structure, we may assume, after taking a chart at p, that $X = \mathbf{E}$ and $p = 0$. Suppose now that by altering ϕ slightly near 0 we always end up with another fixed point near 0, with a local phase portrait resembling that of ϕ at 0. In particular, then, the *linear approximation* of ϕ at 0 (if we can make sense of this term) must have a phase portrait near 0 resembling that of ϕ near 0, as also must all nearby linear systems. This rough and ready argument, together with the stability theorems of Chapter 4, suggests that we consider fixed points whose linear approximations are *hyperbolic systems* (automorphisms or vector fields as the case may be). Such points are said to be *hyperbolic fixed points*. We now give some precise definitions.

A fixed point p of a diffeomorphism f of X is *hyperbolic* if the tangent map $T_pf\colon T_pX \to T_pX$ is a hyperbolic linear automorphism. Corollary 4.27 has supplied us with a classification of hyperbolic linear automorphisms, and it turns out that this amounts to a local classification of their fixed points. We extend this to a classification of hyperbolic fixed points using the result due to Hartman that any hyperbolic fixed point p of f is topologically conjugate to the fixed point 0 of T_pf (see Chapter 2 for the definition of local topological conjugacy). This theorem does not require X to be finite dimensional.

(5.2) Exercise. (i) Prove that p is a hyperbolic fixed point of f if and only if it is a hyperbolic fixed point of f^{-1}.

(ii) Find an example of a diffeomorphism f having a fixed point p that is not topologically conjugate to the fixed point 0 of T_pf.

As usual, an analogous situation exists with flows. A fixed point p of a local integral $\phi\colon I \times U \to X$ of a vector field v on X (or equivalently, a zero p of v) is said to be *hyperbolic* (or *elementary*) if for some (and hence as we shall see any), $t \neq 0$ in I, p is a hyperbolic fixed point of ϕ^t. Let us examine the implications of this definition. Any admissible chart $\xi\colon U \to U'$ at p induces a vector field $w = (T\xi)v\xi^{-1}$ on U'. We may assume $\xi(p) = 0$. Now p is hyperbolic if and only if 0 is a hyperbolic fixed point of $\psi^t = \xi\phi^t\xi^{-1}$, and by Theorem 3.11 ψ is a local integral of w at 0. Let f be the principal part of w. By Theorem 3.32, $T(t) = D\psi^t(0) \in L(\mathbf{E})$ satisfies the differential equation $DT = Df(0)T$. Moreover $D\psi^0(0) = id$. By Proposition 4.33, $D\psi^t(0) = \exp(tDf(0))$, and thus, by Proposition 4.32, p is hyperbolic if and only if $Df(0) \in EL(\mathbf{E})$. Summing up:

(5.3) Proposition. *The point p is a hyperbolic zero of a vector field v if and only if, for any chart ξ at p, the differential at $\xi(p)$ of (the principal part of) the induced vector field $(T\xi)v\xi^{-1}$ is a hyperbolic vector field.* □

An equivalent, but rather more sophisticated, approach is to consider $Tv: TX \to T(TX)$, which is a section of the vector bundle $T\pi_X: T(TX) \to TX$, where $\pi_X: TX \to X$ is the tangent bundle of X. Now $T\pi_X$ may be identified with the tangent bundle on TX, $\pi_{TX}: T(TX) \to TX$ by the canonical involution (see Example A.38 of Appendix A) and Tv then becomes a vector field on TX. With this identification T_pv maps T_pX into $T(T_pX)$. Thus T_pv is a linear vector field on T_pX. We call this linear vector field the *Hessian* of v at p. Again, p is a hyperbolic zero of v if and only if the Hessian of v at p is a hyperbolic linear vector field.

(5.4) Exercise. Justify in detail the statements made in the preceding paragraph.

(5.5) Exercise. Let p be a critical point of a smooth function $g: R^n \to R$. Relate the Hessian of g at p in the classical sense to the Hessian of the gradient vector field $v = \nabla g$ (see Examples 3.3 and A.57) at p in the sense described above.

We prove in Corollary 5.26 below that if p is a hyperbolic fixed point of a vector field v then it is flow equivalent to the zero of the Hessian of v at p. Thus, using Exercise 4.44, we obtain a complete classification of hyperbolic fixed points of flows on real n-dimensional manifolds into $n+1$ classes, distinguished from one another by the dimension of the stable manifold (of the Hessian, let us say at this stage). Once again, the importance of the classification lies in the comparative ubiquity of the type of point under consideration (see Chapter 7).

(5.6) Exercise. Give an example of a smooth vector field v having a zero p that is not topologically equivalent to the zero of the Hessian of v at p.

We complete the chapter by discussing the structural stability of periodic orbits. In the case of a diffeomorphism f there is very little to be said beyond remarking that a periodic point of f is a fixed point of some positive power f^n. On the other hand, closed orbits of flows have a more interesting theory, and once again we are able to classify the types that "usually occur" in finite dimensions.

I. REGULAR POINTS

Recall that a point p of X is a *regular point* of a dynamical system on X if it is not a fixed point of the system. Equivalently, it is a regular point of a vector field v if $v(p) \neq 0_p$. The theorems that follow show that regular points are uninteresting from a local point of view. See Chapter 2 for the definitions of the various types of local equivalence in use.

(5.7) Theorem. *If p is a regular point of a diffeomorphism $f: X \to X$ and g is translation of the model space* **E** *of X by a non-zero vector x_0 then $f|p$ is topologically conjugate to $g|0$.*

Proof. Let $\xi: U \to U'$ be a chart at p with $\xi(p) = 0$. We may assume that $f(U) \cap U$ is empty, and that the diameter of U' is less than $\frac{1}{2}|x_0|$. Then ξ together with $(g|U')\xi(f|U)^{-1}$ is a (smooth) conjugacy from f at p to g at 0. □

Of course the above proof has little to do with the smoothness; it works in the context of homeomorphisms of Hausdorff topological spaces for any pair of regular points with homeomorphic neighbourhoods. Moreover, as we commented in Chapter 2, its simplicity could be taken as a criticism of our definition of local topological conjugacy. The analogous theorem for flows has a more secure status, and a correspondingly harder proof.

(5.8) Theorem. (*Rectification theorem*) *Let p be a regular point of a C^1 flow ϕ on X. Let x_0 be any non-zero vector of the model space* **E** *of X, and let ψ be the flow on* **E** *defined by $\psi(t, x) = x + tx_0$. Then $\phi|p$ is flow equivalent to $\psi|0$.*

Proof. By taking an admissible chart, we can assume that $p = 0 \in \mathbf{E}$ and that ϕ is the local integral of a vector field v on some open subset of **E**. Moreover we can assume after a linear conjugacy that $\mathbf{E} = \mathbf{R} \times \mathbf{F}$ for some Banach space **F** and that $v(0) = x_0 = (1, 0) \in \mathbf{R} \times \mathbf{F}$. Let h be the C^1 map defined on some neighbourhood of 0 in $\mathbf{E} = \mathbf{R} \times \mathbf{F}$ by

$$h(u, y) = \phi(u, (0, y))$$

(see Figure 5.9).

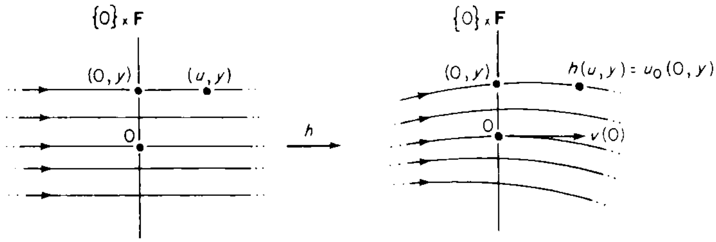


FIGURE 5.9

Then h preserves the action of **R**, since for all sufficiently small s and $t \in \mathbf{R}$ and $y \in \mathbf{F}$,

$$\begin{aligned} h\psi(t, (s, y)) &= h(s + t, y) \\ &= \phi(s + t, (0, y)) \\ &= \phi(t, \phi(s, (0, y))) \\ &= \phi(t, h(s, y)). \end{aligned}$$

Since h is the identity on $\{0\}\times \mathbf{F}$, and since

$$D_1 h(0) = D_1\phi(0,0) = v(0) = (1,0),$$

it is clear that $Dh(0): \mathbf{E} \to \mathbf{E}$ is the identity. Thus, by the inverse mapping theorem (see Exercise C.11 of Appendix C), h is a C^1 diffeomorphism between suitably restricted neighbourhoods of 0 in $\mathbf{E}$. Thus h is a (C^1) flow equivalence from ψ at 0 to ϕ at 0. □

(5.10) Remark. We emphasize that in both the above theorems the equivalence obtained is as smooth as the system under consideration.

II. HARTMAN'S THEOREM

Let T be a hyperbolic linear automorphism of $\mathbf{E}$ with skewness $a < 1$ (see Theorem 4.19) with respect to some norm $|\ |$ on $\mathbf{E}$. Basically, we are interested in showing the stability of T with respect to topological conjugacy under small smooth perturbations, or, more generally, under small Lipschitz perturbations. The central pillar of this theory is the theorem of Hartman [1], which we present below in a version strongly influenced by work of Moser [1] and Pugh [3].

It is immediate from the definition of hyperbolicity that the map $id - T$ is an automorphism. In fact, by the proof of Theorem 4.20 we have the further information that $|(id-T)^{-1}| \leq (1-a)^{-1}$. The first statement implies that T has a unique fixed point (namely the origin), and we start by proving that this property is stable under Lipschitz perturbations of T.

(5.11) Lemma. *Let $\eta: \mathbf{E} \to \mathbf{E}$ be Lipschitz with constant $\kappa < 1 - a$. Then $T + \eta$ has a unique fixed point.*

Proof. The fixed point set of $T + \eta$ is precisely that of $(id - T)^{-1}\eta$, which is a contraction. Thus the contraction mapping theorem (C.5 of Appendix C) gives the result. □

(5.12) Exercise. Prove that the fixed point of $T + \eta$ depends in a C^r fashion on the map η regarded as an element of the map space $C^r(\mathbf{E})$ (see Appendix B). (*Hint*: Consider the function from $\mathbf{E} \times C^r(\mathbf{E})$ to $\mathbf{E}$ sending (x, η) to $(id - T)^{-1}\eta(x)$.)

(5.13) Exercise. Let B be the closed ball in $\mathbf{E}$ with centre 0 and finite radius $b > 0$. Prove that if $\eta: B \to \mathbf{E}$ is Lipschitz with constant $\kappa < 1 - a$ and if $|\eta|_0 \leq b(1-a)$ then $T + \eta: B \to \mathbf{E}$ has a unique fixed point.

Recall (Appendix B) that $C^0(\mathbf{E})$ is the Banach space of all bounded continuous maps from $\mathbf{E}$ to $\mathbf{E}$, with the sup norm.

(5.14) Theorem. (*Hartman's linearization theorem*) *Let $\eta \in C^0(\mathbf{E})$ be Lipschitz with constant $\kappa < \min\{1-a, |T_s^{-1}|^{-1}\}$. Then $T+\eta$ is topologically conjugate to T.*

It is technically convenient to prove a more detailed statement, as follows:

(5.15) Theorem. *Let η, $\zeta \in C^0(\mathbf{E})$ be Lipschitz with constant κ less than* $\min\{1-a, |T_s^{-1}|^{-1}\}$. *Then there exists a unique $g \in C^0(\mathbf{E})$ such that*

$$(T+\eta)(id+g) = (id+g)(T+\zeta). \tag{5.16}$$

Moreover, $id+g$ is a homeomorphism, and is thus a topological conjugacy from $T+\zeta$ to $T+\eta$.

Proof. By the Lipschitz inverse mapping theorem (Exercise C.11 of Appendix C), the condition $\kappa < |T_s^{-1}|^{-1}$ implies that $T+\zeta$ is a homeomorphism. Thus we may write (5.16) as

$$(T+\eta)(id+g)(T+\zeta)^{-1} = (T+\zeta)(T+\zeta)^{-1} + g. \tag{5.17}$$

We define maps $\tilde{T}$ and $\tilde{\eta}$ of $C^0(\mathbf{E})$ into itself by

$$\tilde{T}(g) = Tg(T+\zeta)^{-1}$$

and

$$\tilde{\eta}(g) = \eta(id+g)(T+\zeta)^{-1} - \zeta(T+\zeta)^{-1}$$

so that (5.17) now reads $(\tilde{T}+\tilde{\eta})(g) = g$. Clearly $\tilde{\eta}$ is Lipschitz with constant κ. Equally clearly $\tilde{T}$ is a linear automorphism with inverse $g \mapsto T^{-1}g(T+\zeta)$. We also observe that $\tilde{T}$ is hyperbolic with skewness a, since it contracts and expands its invariant subspaces $C^0(\mathbf{E}, \mathbf{E}_s(T))$ and $C^0(\mathbf{E}, \mathbf{E}_u(T))$ in the correct proportions, and $C^0(\mathbf{E})$ is the direct sum of these subspaces. We now apply Lemma 5.11 and deduce the existence of a unique fixed point f of $\tilde{T}+\tilde{\eta}$. This gives the first part of the theorem.

Finally, let $g' \in C^0(\mathbf{E})$ be the unique map corresponding to g when η and ζ are interchanged in (5.15). Thus

$$(T+\eta)(id+g)(id+g') = (id+g)(T+\zeta)(id+g') = (id+g)(id+g')(T+\eta).$$

But also $(T+\eta)id = id(T+\eta)$. By the uniqueness in the first part of the theorem (applied with $\eta = \zeta$), $(id+g)(id+g') = id$. Similarly

$$(id+g')(id+g) = id.$$

Thus $id+g$ is a homeomorphism, since g and g' are continuous. □

(5.18) Exercise. Show that the hypothesis that g is bounded is necessary for uniqueness in Theorem 5.15. (*Hint*: Put $\mathbf{E} = \mathbf{R}$, and $\eta = \zeta = 0$.)

(5.19) Exercise. Obtain an alternative proof of Hartman's theorem by considering the map from $C^0(\mathbf{E})$ to $C^0(\mathbf{E})$ taking $g=(g_s, g_u)$ to

$$((T_s g_s + \eta_s(id+g) - \zeta_s)(T+\zeta)^{-1}, T_u^{-1}(\zeta_u + g_u(T+\zeta) - \eta_u(id+g)),$$

where $C^0(\mathbf{E}) = C^0(\mathbf{E}, \mathbf{E}_s(T)) \times C^0(\mathbf{E}, \mathbf{E}_u(T))$.

It is natural to ask how smooth the conjugacy $id+g$ is in Theorem 5.15, and tempting to suppose that it is just as smooth as the maps $T+\eta$ and $T+\zeta$ that it conjugates. However, a moment's thought shows that this cannot always be the case. If η, ζ and g are C^1, we may differentiate (5.16) at the unique fixed point p of $T+\zeta$, and deduce that the differential of $T+\zeta$ at p is similar to the differential of $T+\eta$ at its fixed point $p+g(p)$. Thus we have a necessary condition for smoothness of g that is quite a heavy restriction on η and ζ. To see this, put $\zeta=0$ and $\eta(0)=0$, and observe that, however small we make η and its derivatives, the automorphism $T+D\eta(0)$ will not usually be similar to T. Moreover, this similarity condition is not sufficient for C^1 conjugacy. We return to this and similar questions in the appendix to this chapter.

Our main application of Hartman's theorem is to compare a diffeomorphism near a hyperbolic fixed point with its linear approximation at the point. The result is a local one, whereas, in Hartman's theorem as stated above, maps are defined on the whole of $\mathbf{E}$. Thus the proof of the result consists of extending local maps to global ones.

(5.20) Corollary. *A hyperbolic fixed point p of a C^1-diffeomorphism $f: X \to X$ is topologically conjugate to the fixed point 0 of T_pf.*

Proof. We can assume, after conjugation with an admissible chart at p, that we are working in the model space $\mathbf{E}$. We may suppose that $p=0$ and that f is a C^1 diffeomorphism mapping some open neighbourhood U of 0 onto an open neighbourhood $f(U)$ of 0. We are given that the differential $Df(0)$ is hyperbolic. Let the skewness of $Df(0)$ be a with respect to some norm $|\ |$ on $\mathbf{E}$, and let κ be some positive number smaller than $\min\{1-a, |Df(0)_s^{-1}|^{-1}\}$. Since f is C^1, there is some closed ball B centre 0 radius b on which $|Df(x)-Df(0)| \leq \frac{1}{2}\kappa$. This implies that $f-Df(0)$ is Lipschitz with constant $\frac{1}{2}\kappa$ on B. Define $\eta: \mathbf{E} \to \mathbf{E}$ by

$$\eta(x) = \begin{cases} f(x) - Df(0)(x) & \text{if } |x| \leq b, \\ f\left(\dfrac{bx}{|x|}\right) - Df(0)\left(\dfrac{bx}{|x|}\right) & \text{if } |x| \geq b. \end{cases}$$

We assert that η is Lipschitz with constant κ. We must prove that, for x and x' in $\mathbf{E}$, $|\eta(x)-\eta(x')| \leq \kappa|x-x'|$, where (as we may suppose) $|x'| \leq |x|$. The case $|x| \leq b$ is trivial, and the case $b < |x'|$ reduces to the case $|x'| \leq b \leq |x|$, since multiplying the vectors by $b/|x'|$ leaves the left-hand side unaltered and decreases the right-hand side. Suppose now that $|x'| \leqslant b \leqslant |x|$. Then

$$\begin{aligned}
|\eta(x) - \eta(x')| &= \left|\eta\left(\frac{bx}{|x|}\right) - \eta(x')\right| \\
&\leqslant \tfrac{1}{2}\kappa \left|\frac{bx}{|x|} - x'\right| \\
&\leqslant \tfrac{1}{2}\kappa(|x - x'| + |x| - b) \\
&\leqslant \tfrac{1}{2}\kappa(|x - x'| + |x| - |x'|) \\
&\leqslant \kappa |x - x'|,
\end{aligned}$$

as required. Now η is also bounded. By Hartman's theorem there is a topological conjugacy h, say, from $Df(0)$ to $Df(0) + \eta$. Let V be a neighbourhood of 0 such that $h(V) \subset B$. Then, on V,

$$fh = (Df(0) + \eta)h = hDf(0).$$

Thus f is conjugate at 0 to $Df(0)$ at 0. □

(5.21) Corollary. *There are* $4n$ *topological conjugacy classes of hyperbolic fixed points that occur on n-dimensional manifolds.*

Proof. This follows immediately from Corollary 5.20 and the classification of hyperbolic linear homeomorphisms of $\mathbf{R}^n$ in Corollary 4.27. □

We call a fixed point p of a diffeomorphism $f: X \to X$ *structurally stable* if for each sufficiently small neighbourhood U of p in X there is a neighbourhood V of f in Diff X such that, for all $g \in V$, g has a unique fixed point q in U and q is topologically conjugate to p. Even although we have not discussed the topology of Diff X in this chapter, we have assembled all the facts that are needed to prove that fixed points are structurally stable if (and, in finite dimensions, only if) they are hyperbolic. Since we are discussing local phenomena, we can work in the model space $\mathbf{E}$. The basic essentials are contained in the following exercise.

(5.22) Exercise. Let $f: U \to f(U)$ be a diffeomorphism of open subsets of $\mathbf{E}$ with a fixed point at 0. Suppose that 0 is hyperbolic, and that $Df(0)$ has skewness a with respect to the norm $|\ |$ on $\mathbf{E}$. Choose κ, with $0 < \kappa < 1 - a$, so small that, for all $T \in L(\mathbf{E})$ with $|T - Df(0)| \leqslant \kappa$, T is hyperbolic and topologically conjugate to $Df(0)$. Let B be a closed ball in E with centre 0 and radius $b(<1)$ such that $|Df(x) - Df(0)| \leqslant \kappa/2$ for all $x \in B$. Use Exercise 5.13 to prove that, for all C^1 maps $g: U \to E$ with $|g - f|_1 \leqslant \kappa b/2$, g has a unique fixed point p in B. Show that $g|p$ is topologically conjugate to $f|0$, and hence that 0 is structurally stable under C^1-small perturbations of f.

Conversely, prove that if $\mathbf{E}$ is finite dimensional and if 0 is structurally stable under C^1-small perturbations of f then 0 is hyperbolic.

We have already discussed at some length in Chapter 4 the stability of linear automorphisms with respect to topological conjugacy under linear perturbations. As we commented there, it is possible to prove that hyperbolic automorphisms are stable in this sense (the result of Theorem 4.30) as a corollary of Hartman's theorem. This is just a matter of getting a local conjugacy at 0 between the original and the perturbed maps (one first doctors the perturbation to get it into $C^0(\mathbf{E})$) and then extending the local conjugacy to a global one, using the conjugacy relation much as in the proof of Theorem 4.24. We leave the details to the interested reader in the following pair of exercises.

(5.23) Exercise. Let $f: \mathbf{E} \to \mathbf{E}$ be Lipschitz with constant $\kappa > 0$. Given $\varepsilon > 0$, find a map $g \in C^0(\mathbf{E})$ such that g is Lipschitz with constant $\kappa + \varepsilon$ and $g = f$ on the unit ball in $\mathbf{E}$. (*Hint*: Assume $f(0) = 0$, and choose $N > 0$ with $1/N < \varepsilon/2\kappa$. Prove that the map $\rho: \mathbf{E} \to \mathbf{R}$ defined by

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1 \\ (N+1-|x|)/N & \text{if } 1 \leqslant |x| \leqslant N+1 \\ 0 & \text{if } N+1 \leqslant |x| \end{cases}$$

is Lipschitz with constant $1/N$. Deduce that $g = \rho . f$ is locally Lipschitz with constant $\kappa + \varepsilon$, and hence Lipschitz with constant $\kappa + \varepsilon$.)

(5.24) Exercise. Let $T \in HL(\mathbf{E})$ have skewness a with respect to the norm $|\ |$ on $\mathbf{E}$. Prove that any $T' \in L(\mathbf{E})$ with

$$|T' - T| < \min\{1 - a, |T_s^{-1}|^{-1}\}$$

is topologically conjugate to T.

III. HARTMAN'S THEOREM FOR FLOWS

Hartman's linearization theorem was independently discovered, in the flow context, by Grobman [1, 2]. We can readily deduce the global Banach space version for flows from its analogue for diffeomorphisms, as follows:

(5.25) Theorem. (*Hartman and Grobman*) *Let* $T \in EL(\mathbf{E})$. *For all Lipschitz maps* $\eta \in C^0(\mathbf{E})$ *with sufficiently small Lipschitz constant, there is a flow equivalence from* T *to* $T + \eta$.

Proof. Let $\phi: \mathbf{R} \times \mathbf{E} \to \mathbf{E}$ be the integral flow of T. Thus $\phi^1 (= \exp T)$ is a hyperbolic automorphism. Suppose that it has skewness $a < 1$ with respect to some equivalent norm $|\ |$ on $\mathbf{E}$. We recall a few necessary facts from Chapter 3. Firstly, the vector field $T+\eta$ has an integral flow, ψ say (Theorem 3.39). Next, the map $\psi^1-\phi^1$ is in $C^0(\mathbf{E})$ (Theorem 3.45). Finally, the Lipschitz constant of $\psi^1-\phi^1$ can be made arbitrarily small by decreasing the Lipschitz constant of η (Theorem 3.46). Thus, for the latter constant sufficiently small, we may apply Hartman's theorem to the hyperbolic automorphism ϕ^1 and the perturbation $\psi^1-\phi^1$, and obtain a topological conjugacy h from ϕ^1 to ψ^1 which (by Theorem 5.15) is unique subject to the condition $h-id\in C^0(\mathbf{E})$. We assert that $\psi^t h = h\phi^t$ for all $t\in\mathbf{R}$. Now, certainly,

$$\begin{aligned}\psi^1(\psi^t h\phi^{-t}) &= \psi^t\psi^1 h\phi^{-t}\\ &= \psi^t h\phi^1\phi^{-t}\\ &= (\psi^t h\phi^{-t})\phi^1.\end{aligned}$$

Moreover

$$\psi^t h\phi^{-t} - id = (\psi^t-\phi^t)h\phi^{-t}+\phi^t(h-id)\phi^{-t}$$

is C^0-bounded. Thus, by the above mentioned uniqueness of h, $h=\psi^t h\phi^{-t}$, and h is a flow equivalence from ϕ to ψ. □

Just as with diffeomorphisms, we may now deduce the theorem for hyperbolic fixed points of flows on manifolds. The proof, which we leave to the reader, is that of Corollary 5.20 with the obvious modifications.

(5.26) Corollary. *A hyperbolic zero p of a C^1 vector field on X is flow equivalent to the zero of its Hessian at p.* □

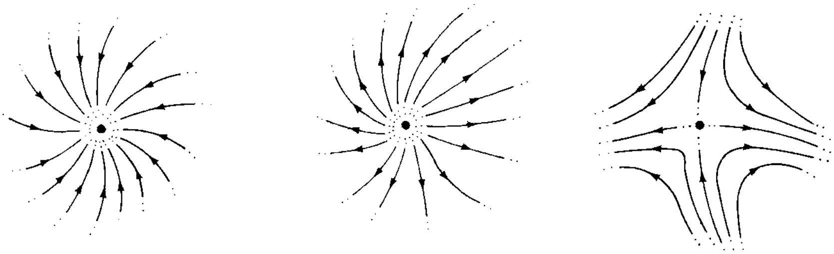
(5.27) Corollary. *Flow equivalence and orbit equivalence coincide for hyperbolic zeros of C^1 vector fields on an n dimensional manifold X. There are precisely $n+1$ equivalence classes of such points with respect to either relation.*

Proof. Immediate from Corollary 5.26 and Exercise 4.44. □

(5.28) Exercise. Show that the flow equivalence h of Theorem 5.25 is unique subject to the condition $h-id\in C^0(\mathbf{E})$.

(5.29) Exercise. Let v be a C^1 vector field on $\mathbf{E}$ with a zero at p. Prove that if p is hyperbolic then it is stable with respect to flow equivalence under C^1-small perturbations of v. Prove, conversely, that if $\mathbf{E}$ is finite dimensional and if p is stable with respect to topological equivalence under C^1-small perturbations of v then p is hyperbolic.

It is worth emphasizing at this stage that hyperbolicity of fixed points is not an invariant of topological equivalence or of flow equivalence. (We might well have made a similar remark earlier in the context of topological conjugacy.) That is to say, it is perfectly possible for a non-hyperbolic fixed point to be topologically equivalent, or even flow equivalent, to a hyperbolic fixed point. For example, the non-hyperbolic zero 0 of the vector field $v(x)=x^3$ on $\mathbf{R}$ is clearly flow equivalent to the hyperbolic zero 0 of the vector field $w(x)=x$. Suppose that a fixed point p of a flow ϕ is topologically equivalent to a hyperbolic fixed point q of a flow ψ. There are three possibilities: (a) $(T_q\psi^1)_s = T_q\psi^1$, (b) $(T_q\psi^1)_u = T_q\psi^1$, and (c) neither (a) nor (b). We call p (and also, of course, q) in case (a) a *sink*, in case (b) a *source* and in case (c) a *saddle point* (see Figure 5.30). We use the same terminology for fixed points of a diffeomorphism that are topologically conjugate to hyperbolic fixed points.



(*a*) sink (*b*) source (*c*) saddle

FIGURE 5.30

Any sink p has the property that any positive half orbit $\mathbf{R}^+ . x = \{t . x : t \geq 0\}$ starting at a point x near p stays near p and eventually ends at p, in the sense that $\omega(x)=p$. The point p is said to be *asymptotically stable in the sense of Liapunov*. In the language of differential equations, this type of stability is concerned with the way an individual solution of a system varies as the initial conditions are altered. This should not be confused with structural stability, which deals with the way the set of all solutions varies as we alter the system itself. We shall say some more about Liapunov stability in the appendix to this chapter, which also contains a section on the *index* of a fixed point. This last is an important integer valued invariant of local topological equivalence.

IV. HYPERBOLIC CLOSED ORBITS

Let p be a point on a closed orbit of a C^r flow ϕ $(r \geq 1)$ on a manifold X. Suppose that the orbit $\Gamma = \mathbf{R} . p$ of ϕ through p has period τ. Then $\phi^\tau : X \to X$ is a C^r diffeomorphism with a fixed point at p. The tangent map $T_p\phi^\tau$ is a linear homeomorphism of T_pX. It keeps the linear subspace $\langle v(x)\rangle$ of T_pX pointwise fixed, where $v(x)$ is the velocity of ϕ at x, because ϕ^τ keeps the orbit Γ pointwise fixed. We say that Γ is *hyperbolic* if T_pX has a $T_p\phi^\tau$-invariant splitting as a direct sum

$$T_pX = \langle v(p)\rangle \oplus F_p$$

where $T_p\phi^\tau|F_p$ is hyperbolic. This definition is independent of choice of p on Γ, for if $q = \phi(t, p)$ then $T_q\phi^\tau = (T_p\phi^t)\ (T_p\phi^\tau)\ (T_p\phi^t)^{-1}$, so $T_q\phi^\tau$ and $T_p\phi^\tau$ are linearly conjugate.

Thus hyperbolicity for a closed orbit means that the linear approximation to ϕ^τ is as hyperbolic as possible, bearing in mind that it cannot be hyperbolic in the direction of the orbit. We can get a clear geometrical picture of what this means in terms of *Poincaré maps.* Let Y be some open disc embedded as a submanifold of X through p, such that $\langle v(x)\rangle \oplus T_pY = T_pX$. We say that Y is *transverse* to the orbit at p, and call it a *cross section* to the flow there. Now $\phi(\tau, p) = p$. We assert that, for some small open neighbourhood U of p in Y, there is a unique continuous function $\rho: U \to \mathbf{R}$ such that $\rho(p) = \tau$ and $\phi(\rho(y), y) \in Y$. The function ρ is a *first return function* for Y. Any two such functions agree on the intersection of their domains; we could have phrased this in the language of germs (see Hirsch [1]). Intuitively $\rho(y)$ is the time that it takes a point starting at y to move along the orbit of the flow in the positive direction until it hits the section Y again. We call the point at which it does so $f(y)$ (see Figure 5.31). Thus $f: U \to Y$ is a map defined by

$$f(y) = \phi(\rho(y), y).$$

We call f a *Poincaré map* for Y. As with ρ, it is well defined up to domain.

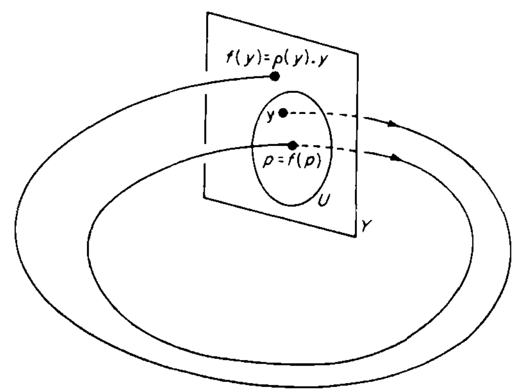


FIGURE 5.31

Note that these definitions do not require Γ to be hyperbolic. In fact, we now show that Γ is hyperbolic precisely when the Poincaré map for some section Y at p has a hyperbolic fixed point at p.

(5.32) Theorem. *For sufficiently small U, the first return function ρ is well defined and C^r, and the Poincaré map f is a well defined C^r diffeomorphism of U onto an open subset of Y. If T_pX has a $T_p\phi^\tau$-invariant splitting $\langle v(p)\rangle \oplus F_p$ then T_pf is linearly conjugate to $T_p\phi^\tau|F_p$. The orbit Γ is hyperbolic if and only if p is a hyperbolic fixed point of f.*

Proof. Notice that by Theorem 5.8 and Remark 5.10 we may identify p and some neighbourhood of it with the origin in the Banach space $\mathbf{E} = \mathbf{R} \times \mathbf{F}$ and some neighbourhood of it, with the flow given locally by

$$\phi(t, (s, y)) = (s + t, y).$$

This is because a C^r flow equivalence affects tangent maps at p only by a linear conjugacy. Under this identification the map ϕ^τ satisfies, near p,

$$\phi^\tau(s, y) = \phi^s(\phi^\tau(0, y)) = \phi^\tau(0, y) + (s, 0).$$

Thus if g and h are maps defined on some neighbourhood V of 0 in $\mathbf{F}$ by

$$\phi^\tau(0, y) = (g(y), h(y)) \in \mathbf{R} \times \mathbf{F}$$

then ϕ^τ is locally of the form

$$\phi^\tau(s, y) = (s + g(y), h(y)).$$

Notice that g and h are C^r and vanish at $y = 0$. Moreover if V is sufficiently small, h is a diffeomorphism of V onto an open subset of $\mathbf{F}$, by the inverse mapping theorem (Exercise C.11 of Appendix C).

We first prove the theorem for the particular cross section $\{0\} \times \mathbf{F}$ (or, rather, some neighbourhood of $p = (0, 0)$ in it). In this case, the first return function ρ is the map $(0, y) \mapsto \tau - g(y)$ and the Poincaré map f is $(0, y) \mapsto (0, h(y))$. If T_pX splits as in the statement of the theorem, we can suppose after a linear automorphism of $\mathbf{E}$ that $F_p = \{0\} \times \mathbf{F}$. Then the differential of ϕ^τ at $p = (0, 0)$ is given by

$$D\phi^\tau(p)(s, y) = (s, Dh(0)(y)), \tag{5.33}$$

and it is clear that $T_p\phi^\tau|F_p$ equals T_pf. It follows immediately that if Γ is hyperbolic then p is a hyperbolic fixed point of f. Suppose conversely that p is a hyperbolic fixed point of f. It is sufficient to prove that T_pX splits as in the statement of the theorem. Now the differential of ϕ^τ at p is given by

$$D\phi^\tau(p)(s, y) = (s + Dg(0)(y), Dh(0)(y)). \tag{5.34}$$

Since $Dh(0)$ is hyperbolic there is some $a>1$ for which iterates $|Dh(0)^n(y)|$ grow as a multiple of $a^{|n|}|y|$, either for n positive or, if y is on the stable manifold of $Dh(0)$, for n negative. Thus (s, y) is in the eigenspace of $D\phi^\tau(p)$ corresponding to 1 if and only if $y=0$. One easily checks directly that $D\phi^\tau(p)$ has the same resolvent as the map defined by (5.33), and so its spectrum is that of T_pf together with the number 1. By spectral theory (see Corollary 4.58) T_pX splits as required.

Finally consider an arbitrary cross section Y at p. Let π_1 and π_2 be the product projections of $\mathbf{R}\times\mathbf{F}$. Then π_2 maps some open neighbourhood of p in Y diffeomorphically onto an open neighbourhood of 0 in $\mathbf{F}$. A first return function for Y may be defined for sufficiently small y by the formula $y\mapsto\tau-g\pi_2(y)-\pi_1(y)+\pi_1(\pi_2|Y)^{-1}h\pi_2(y)$ and a Poincaré map by $y\mapsto(\pi_2|Y)^{-1}h\pi_2(y)$. Since the tangent map to the latter at p is linearly conjugate to $Dh(0)$, the proof is complete. □

(5.35) Remarks. The requirement in the statement of Theorem 5.32 that T_pX should split, is not so very inconvenient. One may, in general, speed up or slow down the flow ϕ near p along individual orbits so that the formula (5.34) becomes (5.33). The new flow has the same orbits as the old one and in addition has the splitting property (see the proof of Theorem 5.40 below).

When X is finite dimensional it is clear from (5.34) that

$$\det(D\phi^\tau(p)-\lambda(id))=(1-\lambda)\det(Dh(0)-\lambda(id)).$$

Thus, whether or not the splitting occurs, the eigenvalues of $T_p\phi^\tau$ are those of T_pf, *with the correct multiplicity*, together with an extra eigenvalue 1. The eigenvalues of T_pf are called the *characteristic multipliers* of Γ. Thus

(5.36) Corollary. *The orbit Γ is hyperbolic if and only if none of its characteristic multipliers lies on the unit circle in* $\mathbf{C}$. □

In order to analyse further the structure of a flow in the neighbourhood of a closed orbit, we recall the notion of the suspension of a diffeomorphism (Construction 1.23). The situation is now rather more complicated, as we wish to suspend diffeomorphisms which are only locally defined (in particular, Poincaré maps for cross sections of flows). Let Y be a manifold and let f be a C^r diffeomorphism ($r\geq 1$) of a connected open subset U of Y onto an open subset $f(U)$ of Y, with a fixed point p. Let I be the open interval $]-\varepsilon, 1+\varepsilon[$ for some small positive ε (certainly $\varepsilon<\frac{1}{2}$). Consider the quotient space of $\mathbf{R}\times Y$ under the equivalence relation $\sim$ given by $(s, y)\sim(s', y')$ if and only if $y\in U$, $y'=f(y)$ and $s=s'+1$. Let V be the quotient subset $I\times U/\sim$ (see Figure 5.37). Then V is a differentiable manifold, and the standard unit vector field in the direction of the first component of $\mathbf{R}\times Y$ induces a vector field on V. The suspension $\Sigma(f)$ of f is the local integral of

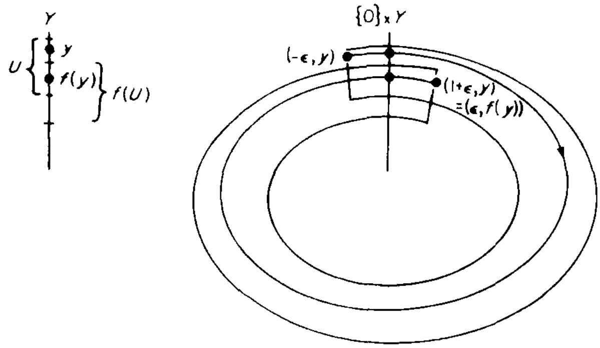


FIGURE 5.37

this vector field. Thus, if $[s, y]$ denotes the $\sim$ equivalence class of (s, y) and t is sufficiently small,

$$\Sigma(f)(t, [s, y]) = [s+t, y].$$

One identifies $y \in Y$ with $[0, y]$. Notice that $\Sigma(f)$ is C^r, and that the orbit through p is periodic of period 1. The set Y is a cross section of $\Sigma(f)$ with first return map the constant function with value 1, and f is a Poincaré map for Y.

Now let $f' : U' \to f'(U')$ be a diffeomorphism of open subsets of a manifold Y'. A *topological conjugacy from f to f'* is a homeomorphism $h: U \cup f(U) \to U' \cup f'(U')$ such that, for all $y \in U$,

$$hf(y) = f'h(y).$$

The following result considerably simplifies the problem of classifying suspensions.

(5.38) Proposition. *If two diffeomorphisms $f: U \to f(U)$ and $f' : U' \to f'(U')$ are topologically conjugate, then their suspensions are flow equivalent.*

Proof. Let h be a topological conjugacy from f to f'. Let $\Sigma(f)$ and $\Sigma(f')$ be the suspensions, local integrals on $V = (I \times U)/\sim$ and $V' = (I \times U')/\sim'$ respectively, where I is the interval $]-\varepsilon, 1+\varepsilon[$. Define a map $H: V \to V'$ by $H([s, y]) = [s, h(y)]'$ where $[\ ]'$ denotes the equivalence class with respect to $\sim'$. Provided that H is well defined (i.e. respects the identifications under $\sim$ and $\sim'$) it is clear that it is a flow equivalence from $\Sigma(f)$ to $\Sigma(f')$. But, for all $s \in]-\varepsilon, \varepsilon[$ and for all $y \in U$,

$$(s+1, h(y)) \sim' (s, f'h(y) = (s, hf(y)),$$

and so the representatives $(s+1, y)$ and $(s, f(y)$ of $[s+1, y]$ both lead to the same value for H. Thus H is well defined. □

(5.59) Remark. Topologically conjugate could be replaced by C^j conjugate above, in which case the flow equivalence H would be C^j $(1 \leqslant j \leqslant r)$.

The main connection between the flow near a closed orbit Γ of a C^r flow ϕ on a manifold X and the Poincaré map at a cross section Y of the flow at the point p of Γ is made by the following theorem:

(5.40) Theorem. *Let $f: U \to f(U)$ be a Poincaré map at p for the cross section Y. Then there is a C^r orbit preserving diffeomorphism h from some neighbourhood of the orbit $\mathbf{R}\,.\,p$ of the suspension $\Sigma(f)$ to some neighbourhood of the orbit Γ of ϕ such that $h(p) = p$.*

Proof. Let ε be as in the definition of $\Sigma(f)$ above, and let λ be a C^∞ real increasing function on the closed interval $[\varepsilon, 1-\varepsilon]$ such that $\lambda(\varepsilon) = 0$, $\lambda(1-\varepsilon) = 1$ and all derivatives of λ of order $\geqslant 1$ vanish at ε and $1-\varepsilon$. Let A be the maximum value of the first derivative λ' on $[\varepsilon, 1-\varepsilon]$. We may assume, by changing the time scale if necessary that Γ has period 1. We may also assume that, provided U is sufficiently small, the first return function ρ satisfies $\rho(y) > \max\{1 - 1/A, 2\varepsilon\}$ for all $y \in U \cup f(U)$. We define a C^r orbit preserving diffeomorphism h from the domain $V = (I \times U)/\sim$ of $\Sigma(f)$ to a neighbourhood of Γ in X by

$$h([s, y]) = \begin{cases} \phi(s, y) & \text{for } s \in \,]-\varepsilon, \varepsilon[,\ y \in U \\ \phi(s + (\rho(y) - 1)\lambda(s), y) & \text{for } s \in [\varepsilon, 1-\varepsilon],\ y \in U \\ \phi(s + \rho(y) - 1, y) & \text{for } s \in \,]1-\varepsilon, 1+\varepsilon[,\ y \in U. \end{cases}$$

Notice that, since for $s \in \,]1-\varepsilon, 1+\varepsilon[$

$$\phi(s + \rho(y) - 1, y) = \phi(s - 1, f(y)),$$

h is well defined on V. □

Now if Γ is hyperbolic, then any Poincaré map f has a hyperbolic fixed point at p. Since by Corollary 5.20 f is topologically conjugate at p to T_pf at 0, we deduce from Proposition 5.38 that $\Sigma(f)$ is topologically equivalent to the suspension $\Sigma(T_pf)$ on some neighbourhood of its unique closed orbit. Putting all this together, we have:

(5.41) Corollary. *If Γ is hyperbolic then the flow ϕ is topologically equivalent at Γ to $\Sigma(T_pf)$ at its unique closed orbit.* □

In the finite dimensional case, the classification of hyperbolic linear maps up to topological conjugacy in Corollary 5.21 yields immediately a classification of hyperbolic closed orbits:

(5.42) Corollary. *There are, up to topological equivalence, precisely* $4n$ *different hyperbolic closed orbits that can occur in a flow on an* $(n+1)$-*dimensional manifold* $(n \geqslant 1)$.

Proof. The different types are distinguished by the dimensions and orientability (or lack of it) of the pair of submanifolds whose α-set or ω-set is the closed orbit. These are the so-called *unstable* and *stable manifolds* of the closed orbit, and we shall discuss them further in the next chapter. □

(5.43) Exercise. Visualize the above types of hyperbolic closed orbit for $n = 1$ and for $n = 2$.

Hyperbolic closed orbits are *structurally stable*, in the sense that, if Γ is such an orbit of a C^1 vector field v on X, and if w is a vector field on X that is C^1-close to v (see Appendix B), then for some neighbourhood U of Γ in X, w has a unique closed orbit in U, and this closed orbit is topologically equivalent to Γ. To construct a proof of this result, use Theorem 3.45 to show that, for a given cross section Y, the Poincaré map of w is C^1-close to the Poincaré map of v and then use the structural stability of hyperbolic fixed points of diffeomorphisms (see Exercise 5.22).

Appendix 5

I. SMOOTH LINEARIZATION

Recall that in Hartman's theorem we altered a hyperbolic linear homeomorphism T by a perturbation η and found a topological conjugacy $h = id + g$ from T to $T + \eta$. We pointed out at the time that h is not necessarily C^1 even when the perturbation η is C^∞, since differentiating the conjugacy relation would place algebraic restrictions on the first derivatives of $T + \eta$. The question now arises as to whether further differentiation places further restrictions on higher derivatives, and whether, even if these algebraic restrictions are satisfied, the smoothness of η has any effect on that of h. It turns out that, in finite dimensions at any rate, further restrictions are the exception rather than the rule, and that positive results on smoothness can be obtained. We state here, without proof, the major theorem to this effect, due to Sternberg [1, 2], and add, also without proof, two relevant theorems of Hartman [1]. We also set an exercise to show that Hölder continuity of the perturbation implies Hölder continuity of the conjugacy. This seems to be the only completely general result in which a property is transferred from the perturbation to the conjugacy.

(5.44) Theorem. (*Sternberg's Theorem*) *Let* $T \in L(\mathbf{R}^n)$ *have eigenvalues* $\lambda_1, \ldots, \lambda_n$ *(possibly complex or repeated) satisfying*

$$\lambda_i \neq \lambda_1^{m_1} \ldots \lambda_n^{m_n}$$

for all $1 \leqslant i \leqslant n$ *and for all non-negative integers* $m_1, \ldots, m_n$ *with* $\sum_{j=1}^n m_j \geqslant 2$. *Let* $\eta : U \to \mathbf{R}^n$ *be a* C^s *map* $(s \geqslant 1)$ *defined on some neighbourhood* U *of* 0 *with* $\eta(0) = D\eta(0) = 0$. *Then* $(T + \eta)|0$ *is* C^r *conjugate to* $T|0$, *where, for given* T, *r depends only on s and tends to* ∞ *with s.* $\square$

In particular, if η is C^∞, the maps T and $T + \eta$ are C^∞ conjugate at 0. Notice that the eigenvalue condition implies that $T \in HL(\mathbf{R}^n)$. There is a vector field version of Sternberg's theorem, where the above multiplicative eigenvalue condition is replaced by the additive one

$$\lambda_i \neq m_1\lambda_1 + \cdots + m_n\lambda_n,$$

the conclusion being that, near 0, the vector field $T+\eta$ is induced (in the sense of Theorem 3.11) from T by some C^r diffeomorphism (provided s is sufficiently large) and that r tends to ∞ with s. See Nelson [1] for a good proof of this theorem and a more precise statement of how r depends on s.

(5.45) Theorem. (*Hartman*) *Let $T \in L(\mathbf{R}^n)$ be a contraction and $\eta : U \to \mathbf{R}^n$ be a C^1 map defined on some neighbourhood U of 0, with $\eta(0) = D\eta(0) = 0$. Then $(T+\eta)|0$ is C^1 conjugate to $T|0$.* □

(5.46) Theorem. (*Hartman*) *Let $T \in HL(\mathbf{R}^n)$, where $n = 1$ or 2, and let η be as in Theorem 5.45. Then $(T+h)|0$ is C^1 conjugate to $T|0$.* □

(5.47) Exercise. *A map $g: \mathbf{E} \to \mathbf{E}$ is Holder continuous with constant $\lambda\,(>0)$ and exponent $\alpha > 0$* if, for all $x, x' \in \mathbf{E}$, $|g(x) - g(x')| \leq \lambda|x - x'|^\alpha$. Prove that, for fixed λ and α, the subset $\mathscr{H}(\lambda, \alpha)$ of $C^0(\mathbf{E})$ consisting of all Hölder continuous maps with constant λ and exponent α is closed in $C^0(\mathbf{E})$. Let $T \in HL(\mathbf{E})$, and let $\eta \in C^0(\mathbf{E})$ be Lipschitz and Hölder continuous with exponent $\alpha < 1$. Prove that for any $\lambda > 0$ the map of $C^0(\mathbf{E})$ to itself defined in Exercise 5.19 (with $\zeta = 0$, for simplicity) takes $\mathscr{H}(\lambda, \alpha)$ into itself, provided that α and the Lipschitz and Hölder constants of η are sufficiently small. Deduce that in this case the map g of Theorem 5.15 is in $\mathscr{H}(\lambda, \alpha)$. (Note that, if we are only interested in the local behaviour of $T+\eta$, the Hölder condition on η is a consequence of the Lipschitz condition, since

$$|x - x'| < |x - x'|^\alpha \qquad \text{for} \qquad |x - x'| < 1.)$$

In the negative direction, we give some counter examples, which emphasize the value of the foregoing theorems.

(5.48) Exercise. Prove that two linear automorphisms of $\mathbf{R}$ are Lipschitz conjugate at 0 (i.e. with local conjugacy a Lipeomorphism) only if they are equal.

(5.49) Exercise. Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(x, y) = (a^2x + y^2, ay)$ where a is fixed and positive. Prove that f is not C^2 conjugate to $Df(0)$ at 0. (*Hint*: Differentiate the relation $hf = Df(0)h$ twice at 0.)

(5.50) Exercise. Let $T: \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(x, y, z) = (ax, acy, cz)$, where $a > 1 > c > a^{-1} > 0$ and let $\eta: \mathbf{R}^3 \to \mathbf{R}^3$ be defined, for some fixed positive ε, by $\eta(x, y, z) = (0, \varepsilon acxz, 0)$. Prove that there is no Lipschitz local conjugacy from T to $T+\eta$ at 0. (*Hint*: Show that, near 0, any local conjugacy h preserves the z-axis. Now suppose that h is Lipschitz. Prove that, near 0, h preserves the x-axis, and satisfies

$$c^{-n}h_2(x, 0, c^n z) - a^n h_2(a^{-n}x, 0, z) = n\varepsilon h_1(x, 0, c^n z)h_3(a^{-n}x, 0, z),$$

where $h = (h_1, h_2, h_3)$ and n is any positive integer. Obtain a contradiction by letting $n \to \infty$.

As we have seen the smoothness of the perturbation η is not always fully echoed by the smoothness of the conjugacy h from T to $T + \eta$. However, it is in other ways. For instance, the image under h of the stable manifold of T is a submanifold which is always as smooth as η: this is one of the main theorems of the next chapter. Another question that arises naturally is the dependence of h on η when both are regarded as points in map spaces. Here again the smoothness of η shows itself.

For example, let $\mathscr{B}^r$ denote the set of Lipschitz maps in $UC^r(\mathbf{E})$ (see Appendix B) with constant less than $\min\{1 - a, |T_s^{-1}|^{-1}\}$. We may drop the U of UC^r if $r = 0$ or if $\mathbf{E}$ is finite dimensional. With the notations of Theorem 5.15, we then have:

(5.51) Theorem. *For fixed ζ, the map θ sending $\eta \in \mathscr{B}^0$ to the corresponding $g \in C^0(\mathbf{E})$ is Lipschitz, and its restriction to $\mathscr{B}^r$ is C^r.*

Proof. We define a map $\chi : \mathscr{B}^0 \times C^0(\mathbf{E}) \to C^0(\mathbf{E})$ by

$$\chi(\eta, g) = (id - \tilde{T})^{-1}((\eta(id + g) - \zeta)(T + \zeta)^{-1}),$$

where $\tilde{T}$ is as in the proof of Theorem 5.15. Then χ is a uniform contraction on the second factor, and uniformly Lipschitz on the first factor with constant $(1 - a)^{-1}$. Thus, by Theorem C.7 of Appendix C, the fixed point map is Lipschitz, and this map is precisely θ. Now restrict χ to $\mathscr{B}^r \times C^0(\mathbf{E})$. The restricted map is C^r (see Theorem B.19 of Appendix B), and thus, by Theorem C.7 again, the restriction of θ to $\mathscr{B}^r$ is C^r. □

(5.52) Exercise. Investigate the dependence of the map g of Theorem 5.15 on the pair (η, ζ) together.

(5.53) Exercise. Investigate the dependence of $h - id$ on η in (the proof of) Theorem 5.25.

II. LIAPUNOV STABILITY

Let p be a fixed point of a flow ϕ on a topological space X. We say that p is *stable in the sense of Liapunov*† (*or Liapunov stable*) if, given any neigh-

† The results of this section date back to Liapunov's doctoral thesis, written at the end of the last century. See Liapunov [1] for a French translation.

bourhood U of p, there is some neighbourhood V of p such that for all $x \in V$, $\mathbf{R}^+ . x \subset U$, where $\mathbf{R}^+ . x = \{t . x : t \geq 0\}$. If this is not the case then p is *unstable*. We say that p is a *asymptotically stable* if it is Liapunov stable and, in addition, for some neighbourhood W of p, $x \in W$ implies $t . x \to p$ as $t \to \infty$. We may similarly talk of *Liapunov stable* and *asymptotically stable* zeros of a vector field on X (see Exercise 3.44).

(5.54) Example. The flow $t . x = x e^{-t}$ on $\mathbf{R}$ has an asymptotically stable fixed point at 0. The flow $t . z = z e^{it}$ on $\mathbf{R}^2 (= \mathbf{C})$ has a fixed point 0 that is stable but not asymptotically stable. The flow $t . (x, y) = (x e^t, y e^{-t})$ on $\mathbf{R}^2$ has an unstable fixed point at $(0, 0)$. Figure 5.54 illustrates a flow on S^2 with a

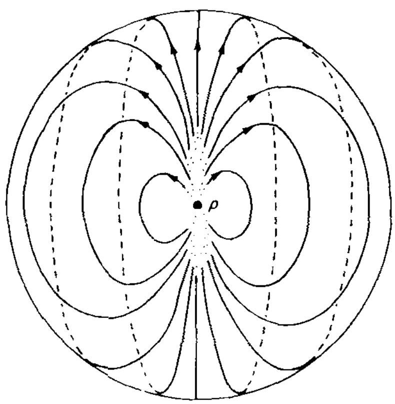


FIGURE 5.54

fixed point p that satisfies the additional condition for asymptotic stability without being Liapunov stable. It is the one-point compactification of the integral flow of a constant vector field on $\mathbf{R}^2$ (with p the point of compactification ∞). Of course one could start with $\mathbf{R}^n$ for any $n \geq 1$ and obtain a similar example.

Now let X be a *finite dimensional* smooth manifold. Let $f: X \to \mathbf{R}$ be a smooth function with an isolated critical point at p. Suppose that f has a (necessarily strict) maximum at p. Our experience of elementary several variable calculus leads us to expect that the level surfaces $\{x \in X : f(x) = \text{constant}\}$ of f near p should be, as in Figure 5.55, a sequence of smoothly embedded spheres of codimension 1 enclosing p. If ∇f is the gradient vector field of f (with respect to some Riemannian metric on X, see Examples 3.3 and A.57) then p is a zero of ∇f. Moreover the orbits of ∇f intersect the level surfaces of f orthogonally going inwards, the direction of increasing $f(x)$.

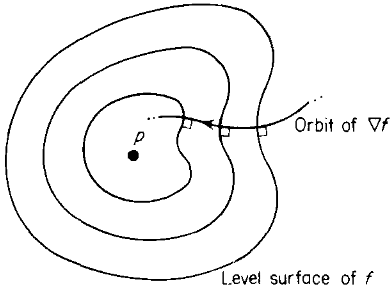


FIGURE 5.55

Thus p is asymptotically stable, as a zero of ∇f. (N.b. we shall shortly obtain a completely rigorous proof of this last statement.)

Suppose now that ϕ is the integral flow of a vector field v on X, and that, for all $x \neq p$ near p, $\langle \nabla f(x), v(x) \rangle > 0$. This condition says that $v(x)$ makes an acute angle with $\nabla f(x)$, and thus the orbit of v through x crosses the level surface at x in the same direction as the orbit of ∇f, namely inwards. Thus p is also asymptotically stable as a fixed point of ϕ (see Figure 5.56). If we

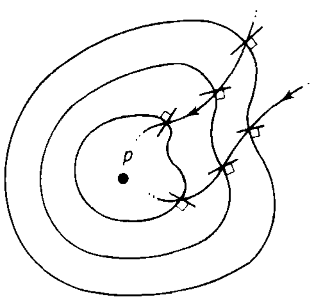


FIGURE 5.56

weaken the above inequality to $\geqslant$, we allow orbits of v to be tangential to the level surfaces. It is, of course, possible for an orbit to be tangential to a level surface at a point where it crosses over the surface. However it is difficult to see how an orbit can make much progress in an outwards direction if neither it *nor nearby orbits* may cross any level surface transversally outwards. We are led to suspect that, in this case, p is Liapunov stable as a fixed point of ϕ. This is, in fact, correct, but it is not easy to make the above approach rigorous. We leave it as a useful picture to have in mind, and start again.

Let $g: X \to \mathbf{R}$ be a smooth function†. Recall from the section on first integrals in Appendix 3 that the directional derivative $L_v g: X \to \mathbf{R}$ is the

† Actually, here, and in what follows, g need only be defined on some neighbourhood of p. However we do not wish to overcomplicate the notation.

continuous function defined by $Tg(v(x)) = (g(x), L_v g(x)) \in \mathbf{R}^2 = T(\mathbf{R})$. For all $t_0 \in \mathbf{R}$, $L_v g(t_0 . x)$ is the derivative with respect to t of $g(t . x)$ at t_0. With respect to any Riemannian metric $\langle\ ,\ \rangle$ on X, $L_v g(x) = \langle v(x), \nabla g(x)\rangle$. We say that g is *positive definite* at p if it has a strict local minimum at p and $g(p) = 0$. We define *positive semi-definite* by dropping the word "strict", and *negative definite* and *semi-definite* in the obvious way. We say that g is a *Liapunov function for v (or for ϕ)* at p if g is positive definite at p and $L_v g$ is negative semi-definite at p. It is *strong* if $L_v g$ is negative definite. Notice that a strong Liapunov function for v at p has an isolated minimum at p, since $L_v g$ vanishes at any critical point of g. Thus, modulo a constant, g takes the place of $-f$ in the above geometrical description; this change of sign is traditional, and has no significance.

(5.57) Theorem. (*Liapunov's Theorem*) *If there exists a Liapunov function g for ϕ at p then p is Liapunov stable as a fixed point of ϕ. If further g is strong then p is asymptotically stable.*

Proof. Let g be a Liapunov function for ϕ at p, and let U be any neighbourhood of p. We wish to show that for some neighbourhood V of p, $x \in V$ implies $\mathbf{R}^+ . x \subset U$. By taking a smaller neighbourhood if necessary we may assume that U is compact and that, for all $x \in U$, $g(x) > 0$ and $L_v g(x) \leq 0$. The frontier ∂U of U is compact and non-empty (provided $U \neq X$; if $U = X$ then, of course, we put $V = X$). Thus the infimum, m say, of $g|\partial U$ is strictly positive. By continuity of g, there exists a neighbourhood V of p on which $g(x)$ is strictly less than m, and we assert that this V has the above property. This is because if, for any $x \in V$, $\mathbf{R}^+ . x$ leaves U, it does so at some point $t_0 . x$ of $\partial U (t_0 > 0)$, but since $g(t . x)$ is non-increasing while $t . x$ remains in U it can never attain a suitable value $g(t_0 . x) \geq m$. To be more precise, $\phi_x^{-1}(\text{int}\ U)$ is open in $\mathbf{R}$, and hence is a union of open intervals. If $t_0 < \infty$ is the end point of the interval containing 0 then $g(t_0 . x)$ is in ∂U but, since $d(g(t . x))/dt \leq 0$ on $]0, t_0[$, $g(t_0 . x) \leq g(x) < m$, which contradicts $m = \inf\{g(y) \colon y \in \partial U\}$.

Now suppose that g is a strong Liapunov function for ϕ at p, and that U and V are neighbourhoods as above, with $L_v g(x) < 0$ for all $x \neq p$ in U. Then $g(t . x)$ decreases as t increases, and therefore tends to some limit $l \geq 0$ as $t \to \infty$. If $l > 0$, then, by continuity of g, $t . x$ never enters some open neighbourhood W of p. Thus, for all $t \geq 0$, $d(g(t . x))/dt \leq M < 0$, where M is the supremum of $L_v g$ on the compact set $U \backslash W$. However this inequality implies that $g(t . x) \to -\infty$ as $t \to \infty$, a contradiction. We deduce that $l = 0$. Now let $k = \inf\{g(y) \colon y \in U \backslash W\}$. Since k is strictly positive, $g(t . x) < k$ for all sufficiently large t, and hence $t . x$ is eventually in W. Since this last argument holds equally well for any open neighbourhood W of p, $t . x \to p$ as $t \to \infty$, and hence p is asymptotically stable. $\square$

The attractive feature of Liapunov's theorem is that one does not need to integrate the vector field (i.e. solve the differential equations) in order to apply it. For this reason, it is sometimes called *Liapunov's direct method.* For example, if v is the vector field $v(x, y) = (-2y^3, x^3 - y^4)$ on $\mathbf{R}^2$ and g is the positive definite function $g(x, y) = x^4 + 2y^4$, then it is a trivial observation that

$$L_v g(x, y) = 4x^3(-2y^3) + 8y^3(x^3 - y^4) = -8y^4$$

is negative semi-definite, and so $(0, 0)$ is a Liapunov stable zero of v. The converse to Liapunov's theorem also holds (see Antosiewicz's survey [1] for details), and so we know that a Liapunov function (resp. strong Liapunov function) exists at any Liapunov stable (resp. asymptotically stable) fixed point. The snag about applying Liapunov's theorem is that in any specific case it may be very hard to recognize whether such a function does exist, and to find one if it does.

It would be rather a tall order to prove that a given fixed point is unstable by showing that no Liapunov functions exist there. The following instability theorem also due to Liapunov is sometimes useful:

(5.58) Theorem. *If there exists a C^1 function $h: X \to \mathbf{R}$ with $h(p) = 0$ such that $L_v h$ is positive definite at p and h is strictly positive on a sequence of points converging to p, then p is unstable as a fixed point of ϕ.*

Proof. Let U be a compact neighbourhood of p such that $L_v h(x) > 0$ for all $x \neq p$ in U, and let V be any neighbourhood of p. Then V contains a point $x \neq p$ such that $h(x) > 0$. Suppose that $\mathbf{R}^+ . X \subset U$. Then $h(t . x)$ is strictly increasing with t, so by continuity of h, there is some open neighbourhood W of p that contains no points of $\mathbf{R}^+ . x$. Since $U \backslash W$ is compact, the infimum of $L_v h$ on $U \backslash W$ is strictly positive, and thus $h(t . x) \to \infty$ as $t \to \infty$. But h is bounded on $U \backslash W$, so we have a contradiction. □

(5.59) Exercise. (i) Use the function $g(x, y) = x^6 + 3y^2$ to prove that $(0, 0)$ is an asymptotically stable zero of the vector field $v(x, y) = (-3x^3 - y, x^5 - 2y^3)$ on $\mathbf{R}^2$. Prove that the vector field $v(x, y) = (3x^3 + y, x^5 - 2y^3)$ has an unstable zero at $(0, 0)$.

(ii) Prove that the function $f(x, y) = x^2 + 4y^2 + 2xy^2 + y^4$ is a Liapunov function for the vector field $v(x, y) = (-2xy^2, xy - 2y)$ at $(0, 0) \in \mathbf{R}^2$.

(5.60) Example. If a C^1 function $f: X \to \mathbf{R}$ has an isolated critical point at p, and this is a (necessarily strict) local maximum of f, then the gradient vector field ∇f (with respect to any Riemannian metric on X), has an asymptotically stable fixed point at p. For, let $g(x) = f(p) - f(x)$. Then g is positive definite at p, and $L_{\nabla f} g(x) = \langle -\nabla f(x), \nabla f(x) \rangle$ is negative unless $\nabla f(x) = 0$, which implies that x is a critical point of f. Thus $L_{\nabla f}g$ is negative definite at p, and so g is a strong Liapunov function for ∇f at p. Similarly if f has an isolated critical point at q and this is not a local maximum of f, then ∇f has an unstable fixed point at q by the instability theorem.

(5.61) Example. Recall (Example 3.49) that in conservative mechanics the Hamiltonian energy function $H = T + V$ is a first integral for the Hamiltonian vector field. Thus, rather trivially, H satisfies the condition that its directional derivative with respect to the vector field is negative semi-definite at any zero of the vector field. These zeros correspond to equilibrium positions of the system, and since the kinetic energy T is a positive definite function of the momenta, H is positive definite at an equilibrium position if and only if the potential energy V has a strict minimum there. For dissipative systems, H decreases along the orbits at points where the momenta are non-zero, and so the directional derivative is still negative semi-definite, and the above conclusion continues to hold.

Liapunov stability can be defined for orbits other than fixed points, and analogues of the above theorems can be obtained in this broader context. Equivalently one may discuss stability of a fixed point of a time dependent vector field, since stability of the solution γ of $x' = v(x)$ is equivalent to stability of the zero solution of $y' = v(y + \gamma(t)) - v\gamma(t)$. See, for example, Hale [1].

III. THE INDEX OF A FIXED POINT

Let v be a continuous vector field on an open subset V of $\mathbf{R}^2$, and let C be a simple closed curve in V. Suppose that v does not pass through any zero of v. Then we can associate with C an integer, called its *index with respect to v* which we may describe intuitively as follows. Consider a variable point x starting at some point $x_0 \in C$ and moving around C in the positive (anticlockwise) direction. The (anticlockwise) angle $\theta(x)$ that $v(x)$ makes with the positive x axis is only defined up to a multiple of 2π. However if we start, say, with $0 \leq \theta(x_0) < 2\pi$, we can choose a representative $\theta(x)$ that varies continuously with x. When we return to x_0 after one trip round C, $\theta(x)$ may not return to the original value $\theta(x_0)$: it takes up a value that differs from $\theta(x_0)$ by $2n\pi$, for some integer n. Thus $2n\pi$ is the *total angular variation* of the vector field around the curve C. The number n, which obviously does not depend upon the starting point x_0 and the speed with which x moves round C, is the index of C. In Figure 5.62, C has index 1 in (i)–(iii), but index 0 in (iv), index 2 in (v) and index -1 in (vi).

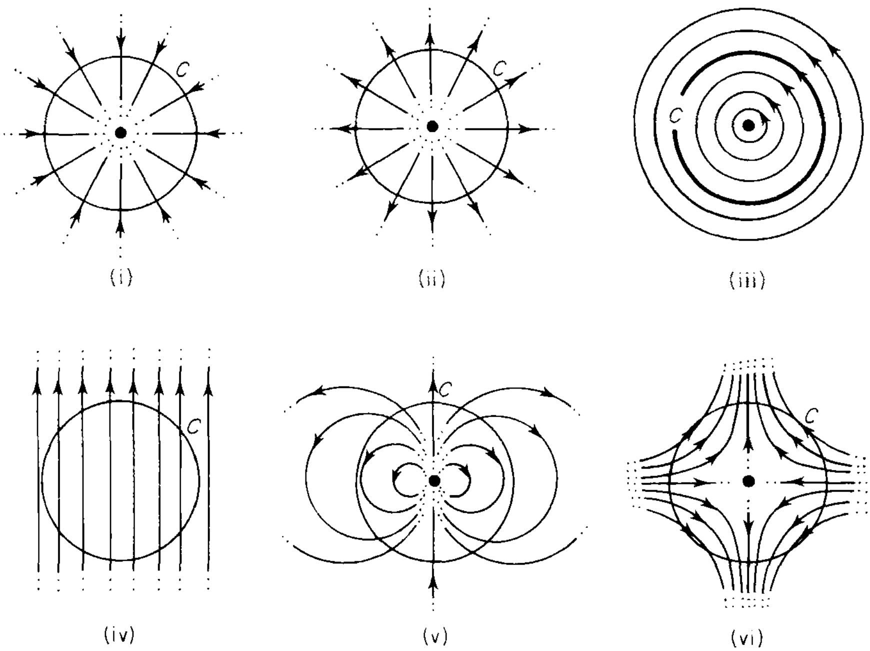


FIGURE 5.62

(5.63) Exercise. For any integer n, positive or negative, visualize a vector field and curve for which the index is n.

(5.64) Exercise. Let v be C^1 and let C be parametrized by the C^1 map $g:[0, \tau]\to \mathbf{R}^2$. Up to a multiple of π,

$$\theta g(t) = \tan^{-1}(v_2 g(t)/v_1 g(t)).$$

Prove that the index of C is

$$\int_0^\tau \langle vg(t), -iDv(g(t))(g'(t))\rangle\, dt/2\pi|vg(t)|^2$$

(identifying $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way). Use the formula to prove that the index of the unit circle with respect to $v(x, y) = (x, -y)$ is -1.

Suppose that C is deformed continuously into a curve C', through a family of curves none of which contains a zero of v. Then it is intuitively obvious that the total angular variation and hence the index, changes continuously, and, since the index is an integer, this can only mean that it remains constant. Similarly if the vector field v is deformed continuously into a new vector field v', then the index of C with respect to both these vector fields is the same provided that at no intermediate stage does the perturbed vector field have a zero on C. These two facts enable us to make some interesting observations. For example, C bounds a topological 2-dimensional ball B in $\mathbf{R}^2$. If B is in V and if B contains no zero of v, then C may be deformed until it lies in a small neighbourhood N of a given point x_0 of B. Since $v(x)$ is near $v(x_0)$ for $x \in N$, the total angular variation is now small, and hence zero, and hence the original curve C had index zero with respect to v (see Figure 5.65). Again, if

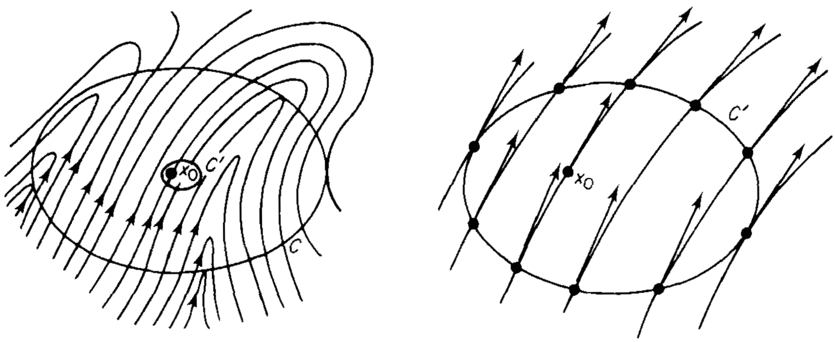


FIGURE 5.65

p is an isolated zero of v, and if C_1 and C_2 are circles with centre p, small enough to contain no other zeros of v and no points of $\mathbf{R}^2 \setminus V$, then C_1 has the same index as C_2, since we may deform the one radially to the other. Thus we may unambiguously define the *index of p with* respect to v to be the index of any sufficiently small circle with centre p. Moreover, by an argument familiar in contour integration and illustrated by Figure 5.66, the index of C is the sum of the indices of the zeros p_i in the interior domain B, provided that they are finite in number and that B is in V.

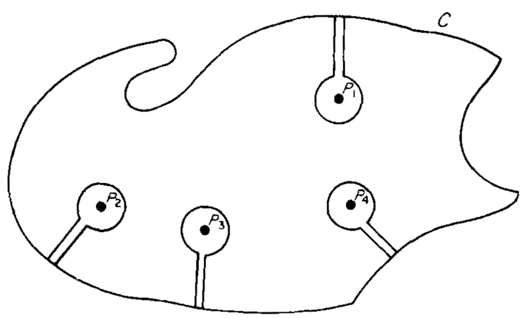


FIGURE 5.66

As an example of deforming the vector field, we may prove the *fundamental theorem of algebra*, which states that any complex polynomial $p(z)$ has a zero. For suppose that $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$, and let $r>1$ be large enough for the inequality $r^n>|a_1|r^{n-1}+\cdots+|a_n|$ to hold. Then $p_t(z)=z^n+t(a_1z^{n-1}+\cdots+a_n)$, for $0\leqslant t\leqslant 1$, defines a deformation from the vector field $z\mapsto p(z)$ on $\mathbf{C}=\mathbf{R}^2$ to the vector field $z\mapsto p_0(z)=z^n$, and p_t has no zeros on the circle $z=r$. Using the formula of Exercise 5.64, the index of the circle with respect to p_0 is

$$\int_0^{2\pi}\langle r^n\, e^{int}, -inr^{n-1}\, e^{i(n-1)t}ir\, e^{it}\rangle\, dt/2\pi r^{2n},$$

which simplifies to n. Thus the index of the circle is n with respect to p. Since, by our above remarks the index would be zero if p had no zeros inside the circle, we deduce that p has zeros.

It is possible to generalize the concept of index to higher dimensions. Let us think of the Jordan curve C as being the image of a continuous map $g:S^1\to V$ which preserves orientation (i.e. as x moves anticlockwise round S^1, $g(x)$ moves anticlockwise round C). Then the direction $vg(x)/|vg(x)|$ of the vector field at $g(x)\in C$ is a point $\theta(x)$ of S^1 (note that θ now has domain S^1 rather than C), so we have a continuous map $\theta:S^1\to S^1$. The index of C is the number of times the new map θ wraps S^1 round itself (in the anticlockwise direction). At first sight this new approach is even vaguer than the previous one, but it may be made completely precise using homology theory. The index is just the *degree*, $\deg\theta$, of the map θ. For the definition of degree, and for other bits of algebraic topology used in this section, we recommend the reader to Greenberg [1], Hu [2] and Maunder [1]. Now suppose that v is a continuous vector field on an open subset V of $\mathbf{R}^n$ and that $g:M\to V$ is a continuous (not necessarily injective) map of a compact (connected) oriented $(n-1)$-dimensional manifold, and that no zeros of v lie on $g(M)$. Then we may define a continuous map $\theta:M\to S^{n-1}$ by $\theta(x)=vg(x)/|vg(x)|$ and the *index of* g *with respect to* v, $\operatorname{ind}_v g$, to be $\deg\theta$. Equivalently, we proceed as follows. The map $vg:M\to\mathbf{R}^n\backslash\{0\}$ induces a homology group homomorphism $(vg)_*:H_{n-1}(M)\to H_{n-1}(\mathbf{R}^n\backslash\{0\})$. Both groups are isomorphic to $\mathbf{Z}$, and the given orientation of M and the standard orientation of $\mathbf{R}^n\backslash\{0\}$ give generators which we identify with $1\in\mathbf{Z}$. In this case, $\operatorname{ind}_v g$ is just $(vg)_*(1)$.

If g is homotopic to h by a homotopy that avoids the zero set Fix v of v, or if v is homotopic to w through vector fields v_t with no zeros on $g(M)$, then we obtain a homotopy from $vg:M\to\mathbf{R}^n\backslash\{0\}$ to vh or wg, as the case may be. Hence, $(vg)_*=(vh)_*=(wg)_*$, and:

(5.67) Proposition. $\operatorname{Ind}_v g=\operatorname{ind}_v h=\operatorname{ind}_w g$. □

In particular, since a constant map clearly has index 0 with respect to any vector field:

(5.68) Corollary. *If g is homotopic to a constant map in* $V\backslash\mathrm{Fix}\, v$ *then* $\mathrm{ind}_v\, g = 0$. □

(5.69) Exercise. Prove that if $w: V \to \mathbf{R}^n$ is a vector field that never takes the opposite direction to v at any point of $g(M)$ (i.e. $\langle wg(x), vg(x)\rangle > -|wg(x)| \cdot |vg(x)|$ for all $x \in M$) then $\mathrm{ind}_v\, g = \mathrm{ind}_w\, g$. Let $f\cdot \mathbf{R}^n \to \mathbf{R}^n$ be a continuous map taking the closed unit ball into itself. By putting $V = \mathbf{R}^n$, $M = S^{n-1}$, $v = id - f$, $w = id$ and $g =$ the inclusion, prove *Brouwer's theorem* that any continuous map of a closed ball into itself has a fixed point.

If g is a topological embedding of M in V, and $N = g(M)$, then $\mathbf{R}^n\backslash N$ has two connected components, one of which, D say, is bounded. If p is any point of D, and $g_*: H_{n-1}(M) \to H_{n-1}(\mathbf{R}^n\backslash\{p\}) \cong \mathbf{Z}$ is the induced map, then $g_*(1)$ is independent of p and takes the value ± 1. If it is $+1$ we say that g is *orientation preserving*, otherwise *orientation reversing*. If $g_1: M_1 \to V$ is another embedding of an oriented manifold M_1 with image N, then $gh = g_1$ defines a homeomorphism $h: M_1 \to M$, and, since the degree of the composite of two maps is the product of the degrees of the maps, *ind*$_v\, g_1 = (\mathrm{ind}_v\, g) \cdot (\deg h)$. Note that $\deg h = \pm 1$. It is $+1$ (i.e. h is orientation preserving) if and only if g and g_1 either both preserve or both reverse orientation. Thus we may define the *index of N with respect to v*, $\mathrm{ind}_v\, N$, to be $\mathrm{ind}_v\, g$ for any orientation preserving embedding $g: M \to V$ with image N. As in the $n = 2$ case, we may now define the *index of* an isolated zero p of v, $\mathrm{ind}_v\, p$, to be $\mathrm{ind}_v\, N$ for any sufficiently small $(n-1)$-sphere with centre p. Equivalently $\mathrm{ind}_v\, p = v_*(1)$, where $v_*: H_n(B, B\backslash\{p\}) \to H_n(\mathbf{R}^n, \mathbf{R}^n\backslash\{0\})$ is the induced homomorphism of relative homology groups, restricting v to some neighbourhood B of p that contains no other zeros of v. We make the same definition if p is a regular point of v. Since N is homologous to zero† in $N \cup D$, where D is the bounded component of $\mathbf{R}^n\backslash N$, we deduce:

(5.70) Proposition. *If $D \subset V$ and v has no zeros in D then* $\mathrm{ind}_v\, N = 0$. *If $D \subset V$ and v has only finitely many zeros $p_1, \ldots, p_r$ in D then* $\mathrm{ind}_v\, N = \sum_{i=1}^r \mathrm{ind}_v\, p_i$. □

(5.71) Corollary. *If p is a regular point of v then* $\mathrm{ind}_v\, p = 0$. □

(5.72) Exercise. Let $n = 2$, and let C be an integral curve of v. Prove that if C is a closed orbit then $\mathrm{ind}_v\, C = 1$. (*Hint*: Take $x \in C$ with minimal second coordinate x_2. Let τ be the period of C, and identify S^1 with $\mathbf{R}/\tau\mathbf{Z}$. Consider

† At least, this is clear if $N \cup D$ is polyhedral. More generally the proposition can be proved using Exercise H.2 of p. 361 of Spanier [1], together with Alexander duality (Theorem 16 of p. 296 of Spanier [1]).

the homotopy $G: S^1 \times [0, \tau] \to \mathbf{R}^2 \backslash \{0\}$ defined for all $t \in [0, \tau]$, by

$$G([t], u) = \begin{cases} \nu(v(x)) & \text{for } t = 0, \\ \nu((2t) . x - x) & \text{for } 0 < t < u/2, \\ \nu(u . x - x) & \text{for } t = u/2, 0 < u < \tau \\ -\nu(v(x)) & \text{for } t = u/2,\ u = \tau \\ \nu(u . x - (2t - u) . x) & \text{for } u/2 < t < u, \\ \nu(v(x)) & \text{for } t \geq u, \end{cases}$$

where $\nu : \mathbf{R}^2 \backslash \{0\} \to S^1$ is defined by $\nu(x) = x/|x|$. Prove that $[t] \to G([t], \tau)$ has degree 1.)

(5.73) Exercise. Let $V = \mathbf{R}^n$ and let v be a linear automorphism. Prove that $\text{ind}_v\ 0$ is 1 if $\det v > 0$ and -1 if $\det v < 0$. Thus $\text{ind}_v\ 0 = (-1)^m$, where m is the number of eigenvalues with negative real part, counting multiplicities.

We wish to prove that the index of a fixed point with respect to a vector field is a topological invariant. For this statement to make sense, we must be able to talk about orbits of the vector field, so we now assume that v integrates to give a partial flow on V (see Exercise 3.40). Actually there is no loss in assuming that the partial flow is a flow (see Exercise 3.44), so we make this assumption. We reformulate the definition of index in the context of flows. Let ϕ be any flow on V, let M be as above and let $g: M \to V$ be any continuous map such that $g(M)$ contains no fixed points of ϕ. Then $g(M)$ does not contain periodic points of arbitrarily small period, for if there were a sequence of periodic points in $g(M)$ whose periods converged to 0, then $g(M)$ would contain the limit of some subsequence, and this would necessarily be a fixed point. Let σ be any continuous positive valued function on M such that, for any $x \in M$ with $g(x) \in \text{Per}\ \phi$, $\sigma(x)$ is strictly less than the smallest positive period per $g(x)$ of $g(x)$. Let $\alpha: M \to \mathbf{R}^n \backslash \{0\}$ be given by $\alpha(x) = \sigma(x) . g(x) - g(x)$. We define the *index of g with respect to ϕ*, $\text{ind}_\phi\ g$ to be the integer $\alpha_*(1)$, where $\alpha_*: H_{n-1}(M) \to H_{n-1}(\mathbf{R}^n \backslash \{0\})$ is the induced homomorphism of homology. It is easy to check that the definition is independent of choice of the map σ. For, if τ is another map with the above properties, then so also is the map $x \to (1-u)\sigma(x) + u\tau(x)$ for all $u \in [0, 1]$. This gives us a homotopy from α to the map $\beta: M \to \mathbf{R}^n \backslash \{0\}$ defined by $\beta(x) = \tau(x) . g(x) - g(x)$, and hence $\beta_*(1) = \alpha_*(1)$. Notice that if ϕ is an integral flow of v, then $v(y) = \lim\ (t . y - y)/t$. Therefore, putting $\sigma(x) = t$, small and constant, gives us a map α such that $\text{ind}_\phi\ g = \alpha_*(1) = ((1/t)\alpha)_*(1)$, and, for all $u \in [0, 1]$, putting

$$\alpha_u(x) = \begin{cases} ((ut) . g(x) - g(x))/ut & \text{for } 0 < u \leq 1 \\ v(g(x)) & \text{for } u = 0 \end{cases}$$

gives us a homotopy from vg to $(1/t)\alpha$. Thus $\text{ind}_v\ g = \text{ind}_\phi\ g$ in this case.

Let ψ be a flow on an open subset W of $\mathbf{R}^n$. In proving our topological invariance result, we need to assume that the domains V and W in question are homeomorphic to the open n-ball.

(5.74) Theorem. *Let V be a topological n-ball and let $h: V \to W$ be a topological equivalence from ϕ to ψ. Then $\operatorname{ind}_\psi hg = \operatorname{ind}_\phi g$ if h is orientation preserving and $\operatorname{ind}_\psi hg = -\operatorname{ind}_\phi g$ if h is orientation reversing.*

Proof. If $f: V \to \mathbf{R}^n$ is a homeomorphism, it is a topological equivalence from ϕ to the induced flow $t . f(x) = f(t . x)$ on $\mathbf{R}^n$. Since hf^{-1} and f^{-1} are topological equivalences with domain $\mathbf{R}^n$, and h factorizes as the composite $(hf^{-1})(f)$, we may assume from the start that $V = \mathbf{R}^n$.

Let $\sigma: M \to \mathbf{R}$ be as in the definition of $\operatorname{ind}_\phi g$. By definition of topological equivalence, for each $y \in V$ there exists an increasing homeomorphism $\alpha_y: \mathbf{R} \to \mathbf{R}$ such that $h(t . y) = \alpha_y(t) . h(y)$ for all $t \in \mathbf{R}$. For all $x \in M$, put $\tau(x) = \alpha_{g(x)}(\sigma(x))$. Note that τ is positive valued. Moreover, since $\alpha_{g(x)}$ is bijective and $\sigma(x) < \operatorname{per} g(x)$ if $g(x) \in \operatorname{Per} \phi$, $\tau(x) < \operatorname{per} hg(x)$ if $hg(x) \in \operatorname{Per} \psi$.

We assert that τ is continuous. To see this, note that h maps the compact set $F = \{t . g(x) : x \in M, t \in [0, \sigma(x)]\}$ homeomorphically onto the set $G = \{t . hg(x) : x \in M, t \in [0, \tau(x)]\}$. Let $x_0 \in M$. Then, since $\sigma(x) . g(x)$ tends to $\sigma(x_0) . g(x_0)$ as $x \to x_0$ and h is continuous, $\tau(x) . hg(x) \to \tau(x_0) . hg(x_0)$ as $x \to x_0$. If τ is not continuous at x_0 then there exists some sequence $(x_r)_{r \geq 1}$ such that $x_r \to x_0$ as $r \to \infty$ and $\tau(x_r)$ is bounded away from $\tau(x_0)$. We may assume that either.

(i) $\tau(x_r) \to$ some limit $l < \tau(x_0)$, or

(ii) $\tau(x_r) > \tau(x_0)$ for all $r \geq 1$.

If (i) holds, then $\tau(x_r) . hg(x) \to l . hg(x_0)$ as $r \to \infty$, and thus $l . hg(x_0) = \tau(x_0) . hg(x_0)$, contrary to our above remark that $\tau(x) < \operatorname{per} hg(x)$. If (ii) holds, we choose t with $\tau(x_0) < t < \inf\{\tau(x_r): r \geq 1\}$ and $t < \operatorname{per} hg(x_0)$ if $hg(x_0) \in \operatorname{Per} \psi$. Then $(t . hg(x_r))_{r \geq 1}$ is a sequence in G converging to $t . hg(x_0)$. Since $t . hg(x_0)$ is not in G, this contradicts compactness of G. Thus τ is continuous at x_0.

The above properties of τ enable us to define $\operatorname{ind}_\psi hg$ as $\gamma_*(1)$, where $\gamma: M \to \mathbf{R}^n \backslash \{0\}$ is given by $\gamma(x) = \tau(x) . hg(x) - hg(x)$. Let $\alpha: M \to \mathbf{R}^n \backslash \{0\}$ be given by $\alpha(x) = \sigma(x) . g(x) - g(x)$, so that $\operatorname{ind}_\phi g = \alpha_*(1)$. For all $x \in M$, $\gamma(x) = h(\sigma(x) . g(x)) - hg(x)$. Let $\gamma_u: M \to \mathbf{R}^n \backslash \{0\}$ be defined by

$$\gamma_u(x) = h(\sigma(x) . g(x) - ug(x)) - h((1-u)g(x)).$$

This gives a homotopy from $\gamma_0 = \gamma$ to γ_1, where

$$\gamma_1(x) = h(\sigma(x) . g(x) - g(x)) - h(0) = h\alpha(x) - h(0),$$

and so $\text{ind}_\psi hg = (\gamma_1)_*(1)$. If h is orientation preserving, the homomorphism $h_*: H_{n-1}(\mathbf{R}^n\backslash\{0\}) \to H_{n-1}(W\backslash\{h(0)\})$ is just $id: \mathbf{Z} \to \mathbf{Z}$. So is the map back to $H_{n-1}(\mathbf{R}^n\backslash\{0\})$ induced from translation through $-h(0)$. Thus $\text{ind}_\psi \text{hg} = \alpha_*(1) = \text{ind}_\phi g$. Similarly, if h is orientation reversing $h_* = -id$, and $\text{ind}_\psi hg = -\alpha_*(1) = -\text{ind}_\phi g$. □

The corresponding result for the index of a map $g: M \to V$ with respect to a vector field is:

(5.75) Corollary. *Let V be a topological n-ball and let $h: V \to W$ be a topological equivalence from a C^1 vector field v on V to a C^1 vector field w on W. Then* $\text{ind}_w hg = \text{ind}_v h$ *if h orientation preserving, and* $\text{ind}_w hg = -\text{ind}_v h$ *if h is orientation reversing.* □

(5.76) Exercise. If V and v are as shown in Figure 5.76 (i), find a map $g: S^1 \to V$ and a diffeomorphism $h: V \to W$ for which the conclusion of Corollary 5.75 does not hold when $w = (Th)vh^{-1}$. If V and v are as shown in Figure 5.76 (ii), explain how to find, for given integers r, s with $r - s$ even, a map $g: S^1 \to V$ and a diffeomorphism $h: V \to W$ such that $\text{ind}_v g = r$ and $\text{ind}_w hg = s$, where $w = (Th)vh^{-1}$.

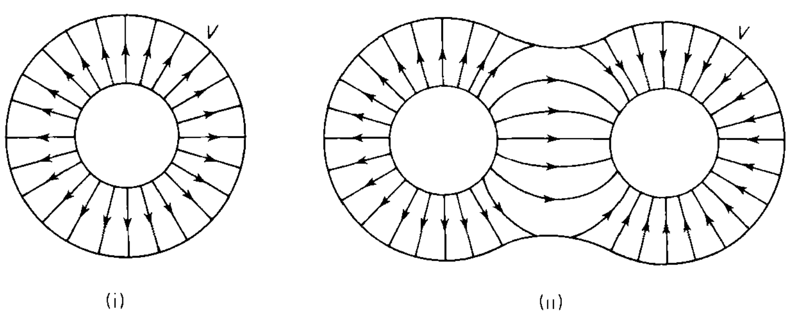


FIGURE 5.76

If p is an isolated fixed point of a flow ϕ on V then we may, as for vector fields, define $\text{ind}_\phi p$ to be $\text{ind}_\phi g$ where $g: S^{n-1} \to V$ is an orientation preserving embedding with image a small sphere with centre p. In this case the index is preserved by any topological equivalence h. We do not get a change of sign when h is orientation reversing, since, although $\text{ind}_\psi hg = -\text{ind}_\phi g$, it is also true that $\text{ind}_\psi h(p) = -\text{ind}_\psi hg$, since $hg: S^{n-1} \to W$ is orientation reversing. That is to say:

(5.77) Corollary. *If an isolated fixed point p of a flow ϕ on V is topologically equivalent to a fixed point q of a flow ψ on W then* $\text{ind}_\phi p = \text{ind}_\psi q$. □

Notice that if ϕ is the integral flow of the vector field v then $\text{ind}_\phi\, p = \text{ind}_v\, p$. Thus Corollary 5.77 also holds for vector fields.

We are now able to define the index $\text{ind}_\phi\, p$ of an isolated fixed point p of any flow ϕ (or C^1 vector field v) on any finite dimensional smooth manifold X by taking a chart $\xi: U \to U' \subset \mathbf{R}^n$ at p and defining $\text{ind}_\phi\, p$ to be the index of $\xi(p)$ with respect to the induced (partial) flow on U'. Since another chart η at p gives rise to a fixed point $\eta(p)$ that is topologically equivalent to $\xi(p)$, the integer $\text{ind}_\phi\, p$ does not depend on the choice of chart at p. Corollary 5.77 continues to hold when V and W are general finite dimensional manifolds. By Corollary 5.77, Exercise 5.73 and Corollary 5.20 we have:

(5.78) Proposition. *The index of a hyperbolic fixed point p of a flow ϕ is $(-1)^m$ where m is the dimension of the stable manifold at p.* □

If a flow ϕ on a finite dimensional manifold X has only finitely many fixed points, its *index sum* is $\sum \text{ind}_\phi\, p$ (summing over $p \in \text{Fix}\, \phi$). Similarly for vector fields. Thus, for example, the index sum of the north–south flow on S^n is 2 if n is even, and 0 if n is odd. If X is not compact then the index sum of ϕ may take any integer value, but if X is compact its topological structure puts stronger restrictions on the geometrical properties of the flows it can carry. In this case we have the celebrated theorem, due to Poincaré in dimension 2 and to Hopf in general, that the index sum is independent of the flow ϕ and depends only on the manifold X.

(5.79) Theorem. (*Poincaré–Hopf*) *The index sum of a flow ϕ (resp. C^0 vector field v) on a compact manifold X is independent of ϕ (resp. v), and equals the Euler characteristic of X.*

The proof of this theorem needs some transversality theory and it essentially contained in Hirsch [1]. Milnar [2] and Guillemin and Pollack [1] give nice versions of the proof for smooth v.

The whole of the above theory goes over perfectly well to diffeomorphisms $f: X \to X$, and even to continuous maps $f: X \to X$. If V is open in $\mathbf{R}^n$, $f: V \to \mathbf{R}^n$ is continuous, and $g: M^{n-1} \to V$ is a map of a compact oriented $(n-1)$-manifold whose image contains no points of Fix f, then the *index of g with respect to f*, $\text{ind}_f\, g$, is $\alpha_*(1)$, where $\alpha_*: H_{n-1}(M) \to H_{n-1}(\mathbf{R}^n \backslash \{0\})$ is induced by the map $\alpha(x) = fg(x) - g(x)$. So, in fact, $\text{ind}_f\, g = \text{ind}_v\, g$ where v is $f - id$ regarded as a vector field.

(5.80) Exercise. Prove that if $h: \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism then $\text{ind}_{hfh^{-1}}\, hg = \text{ind}_f\, g$ if h is orientation preserving, and $-\text{ind}_f\, g$ if not.

The theory now develops as for flows. The *index of an isolated fixed* point p of f is more commonly known as the *Lefschetz number*, $\text{Lef}_p\, (f)$, *of f at p*. Thus if p is an isolated fixed point of the flow ϕ then $\text{ind}_\phi\, p = \text{Lef}_p\, (\phi^t)$ for any $t > 0$ with $t < \text{per } q$ for all $q \in \text{Per } \phi \cap S$, where S is a small sphere with centre p. The index sum of a continuous map $f : X \to X$ is called the Lefschetz number, Lef (f), of f. For compact X, the analogue of the Poincaré–Hopf theorem is not that this number is independent of f, but rather that it is an invariant of homotopy. This actually generalizes the Poincaré–Hopf theorem, since, for any two flows ϕ and ψ on X, ϕ^t and ψ^t are homotopic to the identity and hence to each other.