Annotated Bibliography

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15 February 2019

1 An Elementary Introduction to Modern Convex Geometry [2]

This paper focuses on Geometry itself rather than on algorithms. This is necessary background for future work. It discusses the concentration of volume within basic shapes such as balls and hypercubes - for ex. most of the volume lies at the boundary of the set, and the volume of the unit ball becomes exponentially smaller than the volume of a unit cube (this seems fairly counterintuitive, but the largest distance between points in a ball is 2, the diameter, while for a cube, opposing corners are at distance $\sqrt{n}$ in dimension n).

This is a new field that I haven't been familiar with prior to doing research. I got a better sense of how one might approach problems in this area - general techniques others have used, useful theorems that I may need later on, etc. This is all about sets in high dimensions, and we can't actually picture anything that's going on, so this was useful to see how we can still reason about these objects even though we may not directly be able to see it.

2 Geometric Random Walks: A Survey [20]

This is a very useful introduction to random walks and the main techniques used up until now to solve the problem of sampling from convex sets. It discusses 3 different random walks: the ball walk, grid walk, and hit-and-run. Some of the tools needed to analyze these are briefly discussed. These are isoperimetry, and more specifically, a localization lemma that reduces certain n-dimensional integrals to 1-dimensional ones. The analysis of the convergence of Markov Chains (the random walks) to their stationary distribution boils down to bounding their conductance, which is where the isoperimetric inequalities discussed earlier come into play. Finally, it shows how to reduce convex optimization and volume computation from sampling.

Some of these topics are not likely to be helpful in solving the problem we're working, while some like isoperimetry may prove to be quite useful. Either way, it's important to be aware of the current approaches that others have taken.

3 Solving Convex Programs by Random Walks [5]

The paper presents a cutting plane method for the feasibility problem - determine if a set represented by an oracle is nonempty, which in turn can be used to solve convex programs. Specifically, at each iteration, we keep a set describing where the feasible region may be given the information we have received up until now from the oracle. We update this region at each step when we receive new information. In general, we want to ask questions such that we may decrease the size of the feasible region as much as possible.

Here, the feasible region is simply the intersection of half-spaces determined by the cutting planes used so far - other approaches keep a series of ellipses for ex. The main problem reduces to, given a convex set, sample points uniformly at random from its interior. The paper goes on to prove that for set in isotropic position, the average of $O(n)$ (and $O(\log^2 m)$ if this is a polytope) of these points will in expectation be close to the centroid of the body.

This is the basis for a lot of what we are doing in our current research. Instead of a random walk, we want to find a deterministic method to sample from the body of a polytope. The above results are useful since our main goal is to be able to deterministically approximate the centroid.

4 Computing the Volume is Difficult [3]

The paper proves a negative result about deterministic algorithms attempting to provide a lower and upper bound on the volume of a convex set. This result pertains to algorithms that only have access to the set through a separation oracle. The main result states that there is polynomial time algorithm with

$$\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq \left(c\frac{d}{\log d}\right)^d$$

proof idea: Running the algorithm on $K = B^d$ wil produce a sequence of points asked to the oracle. Consider the polytope formed by the convex hull of these points. Our algorithm cannot distinguish between these two sets. The same is true for the polar of this polytope, K^*. Finally, relate the volume of these by a theorem from Bourgain and Milman, $vol(K)vol(K^*) \geq c_0^d/d!$

Most importantly, the above only holds for algorithms using only an oracle's description of a set. On the other hand, our problem has an explicit description: the set $\{x \in \mathbb{R}^d : Ax \leq b\}$. Therefore, it may be possible to use this additional information to come up with an algorithm with good lower/upper bounds.

Bárány, I. and Z. Füredi (1987). "Computing the volume is difficult." Discrete & Computational Geometry 2(4): 319-326.

5 Shake-and-Bake [6]

The paper studies "shake-and-bake" algorithms which generates a set of points along the boundary of a polytope which converges to the uniform distribution over ∂K. The algorithms described are random walks where each point in the sequence is along the boundary of the polytope. Given a point on the boundary, it remains to choose a direction to move in to find the next point in the sequence. So that the markov chain converges to the uniform distribution, it turns out that this direction should not be chosen uniformly at random from a sphere centered at x, the current point, but rather from a cosine distribution. When thinking about the boundary, it's useful to view it as a thin strip having width ϵ around the boundary. ∂K is defined as the set of points as ϵ goes to 0.

Indeed, we may view the problem as follows. At point x, we choose direction v, and consider the intersection between the line $x + \theta v$ and this thin strip of width ϵ. The intersection is two line segments, of which we are interested in the one not containing v (this is supposed to represent the intersection of the ray with ∂K). Clearly, these segments are not all the same, depending on how oblique the facet we hit is. That is, if the facet is parallel to the one at x, then this segment has width ϵ. On the other extreme, if this facet is nearly perpendicular, then this segment is very long.

The deterministic algorithms we have in mind also find points along the boundary of the polytope, so it is useful to compare the convergence of our algorithm with that of the provably correct shake-and-bake ones. Does it converge to the same distribution (this is the main thing we would like to be true)? If so, it would be nice if it does so more quickly.

6 Isoperimetric Problems for Convex Bodies and a Localization Lemma [12]

The classical isoperimtric problem is to find a set with minimal surface are for a given volume. The solution to this problem, the sphere, has been known since the ancient Greeks. However, here we consider a different version of this problem: We want to find a surface which divides a convex body in two with smallest surface area relative to the resulting volume of the two halves. That is, we want to find the largest ψ (the cheeger constant) such that

$$vol_{n-1}(\partial S) \geq \psi \min\{vol(S), vol(K \setminus S)\}$$

for any $S \subset K$. The paper goes on to prove that

$$\psi \geq \frac{\ln 2}{M_1(K)}$$

where $M_1(K)$ is the average distance of a point to the center of gravity of the body K. If K is isotropic, it follows that $\psi \leq \frac{\ln 2}{\sqrt{n}}$

But what is the motivation for studying this? For example, the mixing time for the ball walk is $O^*(\frac{n^2}{\psi^2})$. For isotropic bodies, we have the upper bound $\psi \geq \frac{c}{\sqrt{n}}$. Thus, the algorithm itself is currently known to require $O(n^3)$ steps to converge. However, it is conjectured that ψ is lower bounded by a constant independent of the dimension. If this were true, it would imply that the algorithm actually requires $O(n^2)$ steps.

It was interesting to see the various ways to see the localization lemma being used throughout the paper to provide different proofs for various existing inequalities. At the moment, we're not sure how to prove anything, so this may or may not turn out to be useful, however, it's important to be aware of this technique either way.

7 The Kannan-Lovász-Simonovits Conjecture [18]

This paper discusses more of the implications of the conjecture (mentioned initially in the previous paper), as well as proves a better lower bound of

$$\psi \gtrsim \frac{1}{(Tr(A^2))^{1/4}}$$

which in the case of an isotropic body implies $\psi \gtrsim n^{-1/4}$, where A is the covariance matrix. Various other conjectures would be implied if this one turns out to be true. One example is the slicing conjecture, one of the main open questions in convex geometry: does any convex body of unit volume have a hyperplane section whose volume is at least a universal constant?

In standard localization, we repeatedly bisect space until we reach a single needle/segment in 0 or 1 dimensions for which the inequality is easy to prove. However, this needle may have high variance (i.e. if the needle is very long) and thus a much smaller value of ψ. As such, this needle would not provide any useful information about the isoperimetric constant of the original body. Instead, this paper takes a different approach of replacing the discrete bisection step with averaging over a series of inifinitesimal needles.

8 Hit and Run from a Corner [13]

This paper proves that the the hit-and-run random walk mixes in $O^*(n^3)$ time from any interior starting point - it has already been shown to be true if we begin from a warm start. The ball walk, on the other hand, will only have good mixing time using a warm start, that is, one can choose pathologic starting points where the walk may require exponential number of steps (if we start in a tight corner, for example). The main result needed to be proven is to bound the conductance. This is generally done in two steps:

1. Show that, given two points that are close, then their 1-step distributions (that is, the set of points they could be in after a single step) will have significant overlap.

2. Show that large subsets have large boundaries. This essentially amounts to bounding the cheeger constant of the stationary distribution.

This is currently the state of the art for sampling from convex sets given no additional assumptions. This algorithm moves through the polytope along randomly chosen lines, and utilizes a different distance function which better encapsulates this sort of movement. A few results are presented regarding this function. We deal with moving along lines as well (specifically from bonudary to boundary) so we may be able to apply some of these ideas to our current work.

9 Pseudorandomness [19]

This paper is a survey of pseudorandomness, which deals with generating numbers or objects through a deterministic method, yet appear to be random. This a 300+ page monograph on the topic, so I have not had a chance to go through it entirely. However, the last section is of most interest. It discusses computational indistinguishability. Essentially, we say that a variable looks random if there is no algorithm which can efficiently distinguish it from a truly random variable.

This is essentially what we are trying to do - deterministically create a sequence of points which appears to be random. However, for us, it is sufficient to be able to "fool" only certain types of algorithms. We're looking at the marginal distribution of the sequence of points, so it is enough to focus our attention on hyperplanes. But what does that actually mean? to fool a hyperplane? Consider dividing a polytope in two by separating the two sections by a hyperplane. If we chose points uniformly at random from the polytope, then in the limit, the ratio of the number of points on one section versus the other will equal the ratio of the actual volumes of the two sets. We would like this property to hold even for our deterministically created sequence of points.

Vadhan, S. P. (2012). "Pseudorandomness." Foundations and Trends in Theoretical Computer Science 7(1–3): 1-336.

10 Chaotic Billiards [8]

Billiards are simply models where particles move around in a closed space and bounce off the walls/boundary. This book studies various aspects of this movement, such as how the trajectory of the billiard disperses throughout the space and whether the entire space is uniformly covered as time goes on.

These are some of the same problems that we are interested in solving, however, there are a few distinctions: we are focused on billiards in higher dimension, while this book only considers 2D space. Secondly, the book deals mainly with arbitrary sets, not polygons, and not even necessarily convex sets, while our attention is solely on polytopes. Nonetheless, we may find there are some techniques here of interest. For instance, the cube is a special case of our problem, and the method we used to reason about them appeared in this text for the square (which could easily be extended to higher dimension).

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