Ergod. Th. & Dynam. Sys. (1985), **5**, 145–161
Printed in Great Britain

Invariant families of cones and Lyapunov exponents

MACIEJ WOJTKOWSKI
Department of Mathematics, University of Arizona, Tucson, AZ 85721, *USA*

(*Received* 8 *November* 1983)

Abstract. We show that in several cases preservation of cones leads to non-vanishing of (some) Lyapunov exponents. It gives simple and effective criteria for non-vanishing of the exponents, which is demonstrated on the example of the billiards studied by Bunimovich. It is also shown that geodesic flows on manifolds of non-positive sectional curvature can be treated from this point of view.

0. *Introduction*

Consider a compact manifold M^n with a Riemann metric and a diffeomorphism $\phi: M^n \to M^n$ preserving probability measure μ. According to the multiplicative ergodic theorem of Oseledec (see [**10**], [**11**] and [**13**]) for almost all $x \in M^n$ there is a basis $\{e_1, \ldots, e_n\}$ in $T_x M^n$ and real numbers called Lyapunov characteristic exponents

$$\chi_1(x) \leq \cdots \leq \chi_n(x)$$

such that

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \|D_x \phi^k(e_i)\| = \pm \chi_i(x), \qquad i = 1, \ldots, n.$$

If the measure μ is equivalent to Lebesgue measure then μ almost everywhere

$$\chi_1(x) + \cdots + \chi_n(x) = 0$$

(see [**3**]).

Presence of non-zero exponents implies positivity of the metric entropy of ϕ. If all Lyapunov exponents are non-zero almost everywhere then ergodic components of ϕ have positive measure and under additional assumptions ϕ is Bernoulli (see [**11**]).

Lyapunov exponents can be put in a more general framework. Consider a probability space (X, μ) and a measure-preserving transformation $\tau: X \to X$. Let $A: X \to \mathrm{GL}(n, \mathbb{R})$ be a measurable mapping to $n \times n$ matrices such that $\log^+ \|A(\cdot)\| \in L^1(X, \mu)$. Then for μ-almost all $x \in X$ there are subspaces $\{0\} = V_x^0 \subset V_x^1 \subset \cdots \subset V_x^n = \mathbb{R}^n$ and numbers $\chi_1(x) \leq \cdots \leq \chi_n(x)$ such that

$$\lim_{k \to +\infty} \frac{1}{k} \log \|A(\tau^{k-1}x) \cdots A(\tau x) A(x) v\| = \chi_i(x)$$

if $v \in V_x^i \setminus V_x^{i-1}$, $i = 1, \ldots, n$.

The classical Alexeev's method ([1]) of establishing non-vanishing of Lyapunov exponents can be described in the following way.

Let $\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^s$ and for a given $a > 0$ consider a cone

$$C_a = \{(u, t) \in \mathbb{R}^r \times \mathbb{R}^s \mid \|u\| \geq a\|t\|\}.$$

If, for almost all $x \in X$,

(i) $A(x)C_a \subset C_a$;

(ii) there is $\eta > 1$ such that if $v \in C_a$ then $\|A(x)v\| \geq \eta\|v\|$;

then

$$0 < \log \eta \leq \chi^{n-r+1} \leq \cdots \leq \chi^n.$$

This approach, although very simple, is quite efficient in the study of particular dynamical systems. It can be formulated in the setting of a diffeomorphism of a manifold. In this paper we will establish that in several cases the condition (i) alone leads to non-vanishing of (some) Lyapunov exponents. This idea initially appeared in [6].

The plan of the paper is as follows: In § 1 we study non-negative matrices and establish uniform exponential growth of the spectral radius of the product of non-negative matrices from a large family. This section may be of independent interest.

In § 2 we study formal properties of Lyapunov exponents and establish a criterion for positivity of the maximal Lyapunov exponent using the results of § 1.

In § 3 this criterion is applied to billiards studied by Bunimovich [4] which gives a particularly simple proof that for these billiards Lyapunov exponents are non-zero almost everywhere.

In § 4 and § 5 we show that the non-negative matrices of § 1 can be replaced by other families of matrices preserving some cone. In § 5 we study symplectic matrices and establish a criterion applicable to geodesic flows on manifolds of non-positive sectional curvature.

I wish to thank Prof. Anatole Katok for valuable discussion which led to the appearance of § 3.

1. *Non-negative matrices*

(1) *Notation.* By $0, \mathbb{1}$ we will denote the vectors in $\mathbb{R}^n$ or the $n \times n$ matrices with all entries equal to 0 and 1 respectively. I denotes the identity matrix. For two $n \times n$ matrices (or vectors in $\mathbb{R}^n$) A, B, $A \geq B$, $A > B$ means respectively that every entry of A is $\geq$, $>$ than the corresponding entry of B. If $A \geq B$ we say that A dominates B. Let

$$0_n = \{v \in \mathbb{R}^n \mid v \geq 0\}, \qquad 0_n^+ = \{v \in \mathbb{R}^n \mid v > 0\}.$$

Further, let

$$\mathcal{P} = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A \geq 0\}, \qquad \mathcal{P}^+ = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A > 0\},$$

$$S\mathcal{P} = \{A \in \mathcal{P} \mid |\det A| = 1\}, \qquad S\mathcal{P}^+ = S\mathcal{P} \cap \mathcal{P}^+.$$

For $A \in \mathcal{P}$, $A = (a_{ij})_{1 \le i,j \le n}$ let $\tilde{A} = (\tilde{a}_{ij})_{1 \le i,j \le n}$ be a zero-one matrix defined by

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0. \end{cases}$$

Let $\mathcal{P}_\varepsilon = \{A \in \mathrm{GL}(n, \mathbb{R}) \| A \ge \varepsilon \mathbb{1}\}$. Clearly we have $\mathcal{P}^+ = \bigcup_{\varepsilon > 0} \mathcal{P}_\varepsilon$. Let $\mathcal{F} \subset \mathcal{P}$ consist of all matrices A such that there are no permutation matrices P_1, P_2 for which $P_1 A P_2$ is a triangular matrix. Obviously $\mathcal{F} \supset \mathcal{P}^+$. Finally, $\mathcal{F}_\varepsilon = \{A \in \mathcal{F} | A \ge \varepsilon \tilde{A}\}$. Hence $\mathcal{F} = \bigcup_{\varepsilon > 0} \mathcal{F}_\varepsilon$. $S\mathcal{F} = \{A \in \mathcal{F} | |\det A| = 1\}$ and $S\mathcal{F}_\varepsilon = S\mathcal{F} \cap \mathcal{F}_\varepsilon$. By $r(A)$ we denote the spectral norm of A. $\|\cdot\|$ denotes the standard euclidean norm in $\mathbb{R}^n$ and the corresponding matrix norm. We will assume throughout this section that $n \ge 2$.

(2) Consider the function $F: 0_n \to \mathbb{R}$ defined by $F(v) = v_1 \cdot \ldots \cdot v_n$ where $v = (v_1, \ldots, v_n)$. $(F(v))^{1/n}$ is a homogeneous function of degree 1 and we will use it to measure the length of a vector in 0_n^+.

LEMMA 1.1. *If $v \in 0_n$ then $\|v\| \ge \sqrt{n}(F(v))^{1/n}$.*

LEMMA 1.2. *Let $P \in \mathcal{P}$ be a permutation matrix. Then for every $v \in 0_n$, $F(Pv) = F(v)$.*

LEMMA 1.3. *Let $D \in \mathcal{P}$ be a diagonal matrix. Then for every $v \in 0_n$, $F(Dv) = \det DF(v)$.*

For $A \in \mathcal{P}$ we introduce the 'norm' $\rho(A)$ defined by

$$\rho(A) = \inf_{v \in 0_n^+} \left(\frac{F(Av)}{F(v)} \right)^{1/n}.$$

LEMMA 1.4. *If $A, B \in \mathcal{P}$ then $\rho(AB) \ge \rho(A)\rho(B)$.*

Proof.

$$\rho(AB) = \inf_{v \in 0_n^+} \left(\frac{F(ABv)}{F(Bv)} \right)^{1/n} \left(\frac{F(Bv)}{F(v)} \right)^{1/n} \ge \rho(A)\rho(B). \qquad \square$$

LEMMA 1.5. *If $A \in \mathcal{P}$ then $\|A\| \ge \rho(A)$.*

Proof. By lemma 1.1,

$$\|A\| \ge \frac{\|A\mathbb{1}\|}{\|\mathbb{1}\|} \ge \left(\frac{F(A\mathbb{1})}{F(\mathbb{1})} \right)^{1/n} \ge \rho(A). \qquad \square$$

Combining lemmas 1.4 and 1.5 we get

PROPOSITION 1.1. *If $A \in \mathcal{P}$ then $r(A) \ge \rho(A)$.*

Proof. $\|A^k\| \ge \rho(A^k) \ge (\rho(A))^k$. Hence

$$r(A) = \lim_{k \to +\infty} \|A^k\|^{1/k} \ge \rho(A). \qquad \square$$

From lemmas 1.2 and 1.3 we get

LEMMA 1.6. *If P_1 and P_2 are permutation matrices and $A \in \mathcal{P}$, then $\rho(P_1 A P_2) = \rho(A)$.*

LEMMA 1.7. *If $A \in \mathcal{P}$ and $D \in \mathcal{P}$ is a diagonal matrix then*

$$\rho(DA) = \rho(AD) = (\det D)^{1/n} \rho(A).$$

We will now formulate a series of propositions giving estimates of $\rho(A)$ for different families of matrices.

PROPOSITION 1.2. *If* $A \in \mathcal{P}$ *then* $\rho(A) \geq |\det A|^{1/n}$.

PROPOSITION 1.3. *If* $A \in \mathcal{P}_\varepsilon$ *then* $\rho(A) \geq (|\det A| + n^n \varepsilon^n)^{1/n}$.

PROPOSITION 1.4. *If* $A \in \mathcal{F}_\varepsilon$ *then* $\rho(A) \geq (|\det A| + 4\varepsilon^n)^{1/n}$.

Proof of propositions 1.2, 1.3 *and* 1.4. Let $A \in \mathcal{P}$. In view of lemma 1.6, without loss of generality we can assume that $\det A > 0$. If $A = (a_{ij})_{1 \leq i,j \leq n}$ and $v \in 0_n^+$,

$$F(Av) = \left(\sum_h a_{1h(1)} a_{2h(2)} \cdots a_{nh(n)}\right) F(v) + \text{the other terms}$$

where the summation is over all permutations h and 'the other terms' are non-negative. This proves proposition 1.2.

Further, if $A \in \mathcal{F}_\varepsilon$,

$$\begin{aligned} F(Av) &= \det AF(v) + 2\left(\sum_{h \text{ odd}} a_{1h(1)} \cdots a_{nh(n)}\right) F(v) + \text{the other terms} \\ &\geq \det AF(v) + \varepsilon^n (F(\tilde{A}v) - \det \tilde{A} F(v)). \end{aligned}$$

The inequality was obtained by estimating all non-zero entries of A by ε.

For $A \in \mathcal{P}_\varepsilon$ we have $\tilde{A} = \mathbb{1}$ and $\det \tilde{A} = 0$. Moreover, by straight-forward calculation, $\rho(\mathbb{1}) = n$. Hence $F(\mathbb{1}v) \geq n^n F(v)$ and we get proposition 1.3.

To prove proposition 1.4 it is sufficient to prove that for $A \in \mathcal{F}$

$$\rho(\tilde{A}) \geq (|\det \tilde{A}| + 4)^{1/n}.$$

This will be done in lemmas 1.8 and 1.9. □

LEMMA 1.8. *If Z is a zero-one matrix such that Z dominates I (i.e. $Z \geq I$) and it does not dominate any other permutation matrix, then there is a permutation matrix P such that $P^{-1}ZP$ is triangular.*

Proof. By induction on the dimension of Z. For $n = 2$ there is nothing to prove. To obtain an inductive step we must prove that under the assumption above there is a column of Z with exactly one 1 (placed obviously on the main diagonal).

The characteristic polynomial of Z is

$$\det(Z - \lambda I) = (1 - \lambda)^n.$$

Hence the only eigenvalue of Z is equal to 1. But because $Z \geq 0$, then there must be an eigenvector $v \geq 0$, $v \neq 0$. We have thus $(Z - I)v = 0$. So if, for instance, $v_1 > 0$, then the first column of $Z - I$ is zero. □

LEMMA 1.9. *If* $A \in \mathcal{F}$ *then* $\rho(\tilde{A}) \geq (|\det \tilde{A}| + 4)^{1/n}$.

Proof. Since $\det A \neq 0$ then there is a permutation matrix P_1 such that $\tilde{A} \geq P_1$, i.e.

$$Z = P_1^{-1} \tilde{A} \geq I.$$

Since $A \in \mathcal{F}$ then in view of lemma 1.8, Z dominates at least one more permutation matrix. By lemma 1.6, $\rho(\tilde{A}) = \rho(Z)$. We will get the desired estimate for $\rho(Z)$ by

induction on the dimension n of Z, $n \geq 2$. For $n = 2$,

$$Z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and the estimate is obvious (it is also a special case of proposition 1.3). To obtain the inductive step, consider matrices Z_i, $i = 1, \ldots, n$ obtained from Z by changing all entries in the first row into zeros except the ith entry which we preserve (some Z_i may thus have zero first row). We have, for $v \in 0_n$,

$$F(Zv) = \sum_{i=1}^{n} F(Z_i v),$$

and

$$\det Z = \sum_{i=1}^{n} \det Z_i.$$

If some Z_{i_0} dominates at least two permutation matrices then by inductive assumption and proposition 1.2,

$$\begin{aligned} F(Zv) &\geq (|\det Z_{i_0}| + 4)F(v) + \sum_{i \neq i_0} |\det Z_i| F(v) \\ &\geq (|\det Z| + 4)F(v). \end{aligned}$$

Hence we are left with the case when all Z_i dominate at most one permutation matrix. Without loss of generality we can assume then that Z_1 and Z_2 dominate exactly one permutation matrix and Z_1 dominates I. Then it is clear that Z_2 can dominate only a cycle. If the length of the cycle is equal to $c \geq 2$ then by a straightforward computation,

$$F(\widetilde{Z_1 + Z_2} v) \geq 2^c F(v)$$

and

$$\det(\widetilde{Z_1 + Z_2}) = \begin{cases} 0 & \text{if } c \text{ is even} \\ 2 & \text{if } c \text{ is odd.} \end{cases}$$

Hence also in this case,

$$\begin{aligned} F(Zv) &= F(\widetilde{Z_1 + Z_2} v) + \sum_{i=3}^{n} F(Z_i v) \\ &\geq (|\det \widetilde{Z_1 + Z_2}| + 4)F(v) + \sum_{i=3}^{n} |\det Z_i| F(v) \\ &\geq (|\det Z| + 4)F(v). \end{aligned}$$

□

Propositions 1.1 and 1.4 together with lemma 1.4 yield:

COROLLARY 1.1. *If $A_1, \ldots, A_k \in S\mathcal{F}_\varepsilon$ then $r(A_1 \cdots A_k) \geq (1 + 4\varepsilon^n)^{k/n}$.*

We are also able to derive from the results above another characterization of the family of matrices $\mathcal{F}$ (and $S\mathcal{F}$):

COROLLARY 1.2. *$A \in \mathcal{F}$ if and only if $A \in \mathcal{P}$ and $\rho(A) > |\det A|^{1/n}$. In particular, $A \in S\mathcal{F}$ if and only if $A \in S\mathcal{P}$ and $\rho(A) > 1$.*

Proof. In view of proposition 1.4 and lemmas 1.6 and 1.7, we have only to prove that if $T \in \mathcal{P}$ is a triangular matrix with 1's on the main diagonal, then $\rho(T) = 1$.

To this end let $v = (1, \alpha, \ldots, \alpha)$, $\alpha > 0$. Then $v \in 0_n^+$ and $\lim_{\alpha \to 0} F(Tv)/F(v) = 1$. □

From corollary 1.2 and lemma 1.4 we get immediately:

COROLLARY 1.3. *If $A \in \mathcal{P}$ and $B \in \mathcal{F}$ then AB and BA belong to $\mathcal{F}$. In particular, if $A \in S\mathcal{P}$ and $B \in S\mathcal{F}$ then AB and BA belong to $S\mathcal{F}$.*

Remark. The privileged role played by the function F in the study of non-negative matrices is no accident. Consider the $n-1$-dimensional projective space $P\mathbb{R}^{n-1}$ and its subset $P0_n$ of lines in 0_n^+. Diagonal matrices act freely and transitively in $P0_n^+$ and they preserve the measure given by the $(n-1)$-form

$$\sum_{i=1}^{n} (-1)^{n+1} \frac{v_i \, dv_1 \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_n}{F(v)},$$

where $\widehat{\ }$ means that we omit the corresponding 1-form. A non-negative matrix takes $P0_n^+$ into itself and proposition 1.2 says that it contracts the measure.

2. *Lyapunov exponents*

Let $\tau : X \to X$ be a measurable transformation preserving a probability measure μ and let $A : X \to \mathrm{GL}(n, \mathbb{R})$ be a measurable map such that $\log^+ \|A(\cdot)\| \in L^1(X, \mu)$. We will call the pair (τ, A) a measurable cocycle.

For natural n, let $A^n : X \to \mathrm{GL}(n, \mathbb{R})$ be defined by

$$A^n(x) = A(\tau^{n-1}x) \cdots A(\tau x) A(x).$$

Thus for any natural n we have a measurable cocycle (τ^n, A^n) which we call the nth power of the cocycle (τ, A). For any $Y \subset X$, $\mu(Y) > 0$, we have the derived transformation $\tau_Y : Y \to Y$. It is defined in the following way: for $x \in Y$, let $k_Y(x) = \min\{n \geq 1 \mid \tau^n x \in Y\}$. Then

$$\tau_Y x = \tau^{k_Y(x)} x.$$

We have $k_Y \in L^1(Y, \mu)$ and the function is called the *return time.* Let $A_Y : Y \to \mathrm{GL}(n, \mathbb{R})$ be defined by

$$A_Y(x) = A^{k_Y(x)}(x).$$

The measurable cocycle (τ_Y, A_Y) will be called the *derived cocycle.* Given a measurable cocycle (τ, A) we define for $v \in \mathbb{R}^n$, $v \neq 0$,

$$\chi(x, v) = \chi(x, v; \tau, A) = \lim_{l \to +\infty} \frac{1}{l} \log \|A^l(x) v\|.$$

By Oseledec's multiplicative ergodic theorem (cf. [**10**], [**13**]) the limit above exists almost everywhere, i.e. $\chi(x, v)$ is defined for μ almost all $x \in X$. It is called the Lyapunov exponent. Moreover, $\chi(x, v) < +\infty$ (the value $-\infty$ is not excluded) and for a given $x \in X$, $\chi(x, v)$ has at most n distinct values. We have also

$$\chi_{\max}(x) = \max\{\chi(x, v) \mid v \in \mathbb{R}^n, v \neq 0\} = \lim_{l \to +\infty} \frac{1}{l} \log \|A^l(x)\|.$$

LEMMA 2.1. *For natural n, $x \in X$, $v \in \mathbb{R}^n$, $v \neq 0$, we have $\chi(x, v; \tau^n, A^n) = n\chi(x, v; \tau, A)$.*

LEMMA 2.2. *Let $Y \subset X$, $\mu(Y) > 0$. Then for $v \in \mathbb{R}^n$, $v \neq 0$ and almost all $x \in Y$,*

$$\chi(x, v; \tau_Y, A_Y) = t(x)\chi(x, v; \tau, A),$$

where $t(x) = \lim_{l \to +\infty} (1/l) \sum_{i=0}^{l-1} k_Y(\tau_Y^i x)$, i.e. $t(x)$ is the ergodic mean of the return time k_Y with respect to the derived transformation τ_Y.

Proof. Let $t_l(x) = \sum_{i=0}^{l-1} k_Y(\tau_Y^i x)$.

$$\chi(x, v; \tau_Y, A_Y) = \lim_{l \to +\infty} \frac{1}{l} \log \|(A_Y)^l v\| = \lim_{l \to +\infty} \frac{1}{l} \log \|A^{t_l(x)} v\|$$

$$= \chi(x, v; \tau, A) \lim_{l \to +\infty} (t_l(x)/l). \qquad \square$$

We will need the following classical lemma from ergodic theory.

LEMMA 2.3. *Let $f \in L^1(X, \mu)$ be positive almost everywhere. Then $f_+ = \lim_{l \to +\infty} (1/l) \sum_{i=1}^{l-1} f \circ \tau^i$ is (defined and) positive almost everywhere.*

Proof. We have $f_+ \geq 0$. Let $Y = \{x \in X \mid f_+(x) = 0\}$. Let us assume that $\mu(Y) > 0$. Considering f and τ restricted to Y we get by Birkoff's ergodic theorem, $\int_Y f_+(x)\, d\mu = \int_Y f(x)\, d\mu > 0$, which gives a contradiction. $\square$

THEOREM 2.1. *Let (τ, A) be a measurable cocycle with values in $S\mathcal{F}$, i.e. $A: X \to S\mathcal{F}$, then the maximal Lyapunov exponent is positive almost everywhere.*

Proof. According to corollary 1.2, for all x, $\rho(A(x)) > 1$ and so $\log \rho(A(x)) > 0$. Further, by lemmas 1.4 and 1.5,

$$\frac{1}{l} \log \|A^l(x)\| \geq \frac{1}{l} \log \rho(A^l(x)) \geq \frac{1}{l} \log \rho(A(\tau^{l-1}x)) \cdots \rho(A(\tau x))\rho(A(x))$$

$$= \frac{1}{l} \sum_{i=0}^{l-1} \log \rho(A(\tau^i x)).$$

So by lemma 2.3 we get our theorem. $\square$

In many cases it is useful to have the following generalization of theorem 2.1.

THEOREM 2.2. *Let (τ, A) be a measurable cocycle with values in $S\mathcal{P}$, (i.e. $A: X \to S\mathcal{P}$) such that for almost every $x \in X$ there is $N(x) \geq 1$ such that $A^{N(x)}(x) \in S\mathcal{F}$. Then the maximal Lyapunov exponent is positive almost everyhere.*

Proof. For natural n let $X_n = \{x \in X \mid N(x) = n\}$. We have $\bigcup_{n=1}^{+\infty} X_n = X \bmod 0$. Take X_n such that $\mu(X_n) > 0$. By corollary 1.3 the measurable cocycle

$$((\tau^n)_{X_n}, (A^n)_{X_n})$$

(i.e. the derived cocycle of the nth power of (τ, A)) has values in $S\mathcal{F}$. Hence by theorem 2.1 its maximal Lyapunov exponent is positive almost everywhere. But then by lemmas 2.1 and 2.2 we get that the maximal Lyapunov exponent of (τ, A) is positive almost everywhere in X_n. $\square$

Theorems 2.1 and 2.2 can be put in a setting of a diffeomorphism of a surface (with singularities). Let $\phi: M^2 \to M^2$ be a diffeomorphism of a two-dimensional manifold M^2 preserving a probability measure μ equivalent to an area element ν defined by some fixed Riemann metric on M^2 (i.e. $d\mu/d\nu = f$ where $f \in L^1(M^2, \nu)$). Suppose further that there is a measurable bundle of sectors $C(p) \subset T_pM^2$, $p \in M^2$ such that

$$D_p\phi(C(p)) \subset C(\phi(p)),$$

(the sectors $C(p)$ are defined and the invariance property holds only for almost every $p \in M^2$). Additionally we require that for almost all points $p \in M^2$ there is a natural number $n(p)$ such that $D_p\phi^{n(p)}(C(p))$ is contained strictly inside $C(\phi^{n(p)}(p))$, i.e. the boundary lines of $C(p)$ are mapped inside $C(\phi^{n(p)}(p))$. In such a situation we can choose a basis $e_1(p)$, $e_2(p)$ in almost every tangent plane T_pM^2 so that

$$C(p) = \{v \in T_pM^2 | v = a_1e_1(p) + a_2e_2(p),\ a_1a_2 \geq 0\},$$

and the area of the parallelogram spanned by $e_1(p)$, $e_2(p)$ is equal to $1/f(p)$. In these coordinates all differentials $D_p\phi$ are described by non-negative or non-positive matrices and theorem 2.2 applies except that we must take care of the relation between the norm $\|\cdot\|_1$ in T_pM^2 induced from $\mathbb{R}^2$ by our choice of basis and the norm $\|\cdot\|$ defined by the Riemann metric. Of course at each point $p \in M$ the two norms are equivalent

$$\beta(p)\|\cdot\|_1 \leq \|\cdot\| \leq \alpha(p)\|\cdot\|_1,$$

but generally α, β are only measurable functions.

By considering $X_d = \{p \in M^2 | \alpha(p) \leq d, \beta(p) \geq d^{-1}\}$, and the derived cocycle we have, in view of lemma 2.2, that its Lyapunov exponents are zero or non-zero together with those of $(\phi, D\phi)$ and, on the other hand, if we compute them in the norm $\|\cdot\|_1$ we get the same result. Hence by theorem 2.2 Lyapunov exponents of $(\phi, D\phi)$ are non-zero almost everywhere.

Having in mind the application to billiards it is important to note that we can allow singularities of ϕ without affecting the conclusion (non-vanishing of Lyapunov exponents).

The situation described above appears in several places, among them [**8**], [**12**]. We will describe in detail application of these ideas to billiards studied by Bunimovich [**4**].

3. *Billiards*

We start with some elementary differential geometry. Let $l(t)$, $|t| < \varepsilon$, be a smooth family of directed lines in the plane. We introduce the following infinitesimal characterization of the family along $l(0)$. We define the curvature of the family at a point $p \in l(0)$ to be the curvature with sign at p of the orthogonal section of our family (the orthogonal section is the curve that intersects all the lines of our family orthogonally). We choose the curvature to be negative if the acceleration vector of the section points in the direction of the lines and positive in the other case (see figure 1). If the orthogonal section fails to be a regular curve at p (if it is not defined) then we put $k = \infty$.

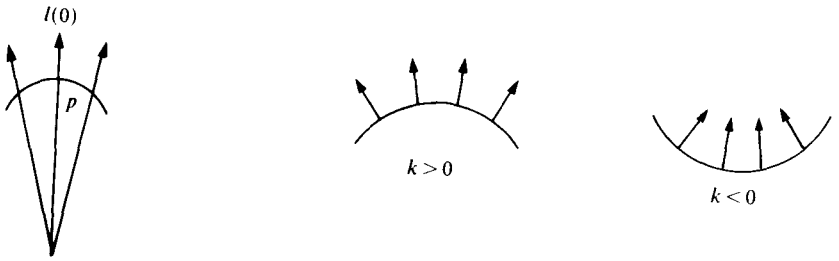

FIGURE 1

Let k_1, k_2 be curvatures of the family at points p_1, $p_2 \in l(0)$. Let l be the distance between p_1 and p_2 and the direction of $l(0)$ be from p_2 to p_1 (figure 2). Then

$$k_2 = \frac{k_1}{1 - lk_1}. \tag{1}$$

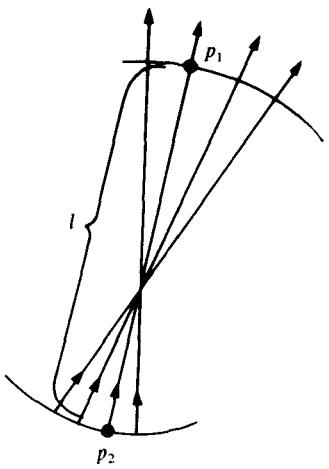

FIGURE 2

Let us now assume that the family of lines is reflected from some smooth curve. Then at the point of reflection we have two curvatures: k_b, the curvature of the family before reflection and k_a, the curvature of the family after reflection. We have

$$k_a = k_b + \frac{4}{d}, \tag{2}$$

where $|d|$ is the length of the segment of $l(0)$ inside the curvature disk of the curve at a point of reflection; d is positive if the reflection takes place on the side of the curve on which the curvature disk lies (focusing reflection) and negative in the opposite case (dispersing reflection) (figure 3). When the reflection takes place at a point at which the radius of the curvature is infinite (flat point) we have $d = \infty$ and $k_a = k_b$.

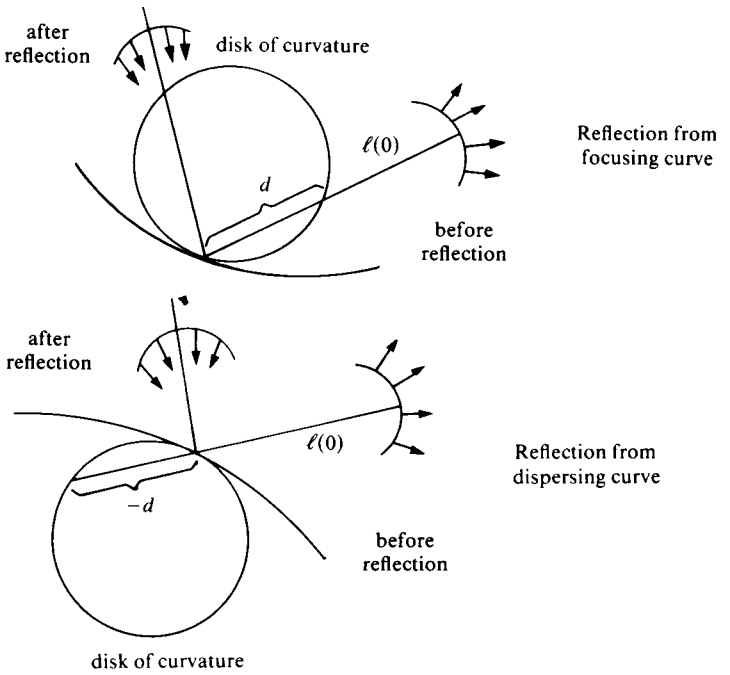

FIGURE 3

All the above statements can be proved by elementary geometrical considerations and are at least implicit in the literature of the subject (see Bunimovich's paper [4], which contains an extensive bibliography, or [7]).

Consider now a connected domain Q in the plane with piecewise smooth boundary ∂Q. We will distinguish between focusing pieces of ∂Q-the curvature disk lies on the side of Q, and dispersing pieces of ∂Q-the curvature disk lies on the other side or is infinite.

By a *billiard* in Q we mean a dynamical system resulting from the uniform motion of a point mass in Q with reflections at the boundary ∂Q according to the law 'the angle of incidence equals the angle of reflection'. The dynamics of the billiard can be reduced to the transformation T of S where S is the set of unit vectors attached at ∂Q and pointing outward. T is defined in the following way: For $v \in S$ draw a straight line through the point at which v is attached in the direction opposite to that of v up to the next point of intersection with ∂Q. Tv is the unit vector attached at this point and symmetric to our straight line with respect to the boundary ∂Q (figure 4). The definition of T follows an established tradition. Clearly S is a two-dimensional manifold with singularities. T preserves a smooth measure (cf. [4]). For almost all points in S, T and all its iterates are differentiable so it makes sense to speak about Lyapunov exponents of (T, DT). A tangent vector to S at v can be viewed as a parametrized family of directed lines $\ell(t)$, $|t| < \varepsilon$, where $\ell(0)$ has the direction of v and passes through the point at which v is attached. Families with the same curvature describe tangent vectors differing by a scalar factor so that

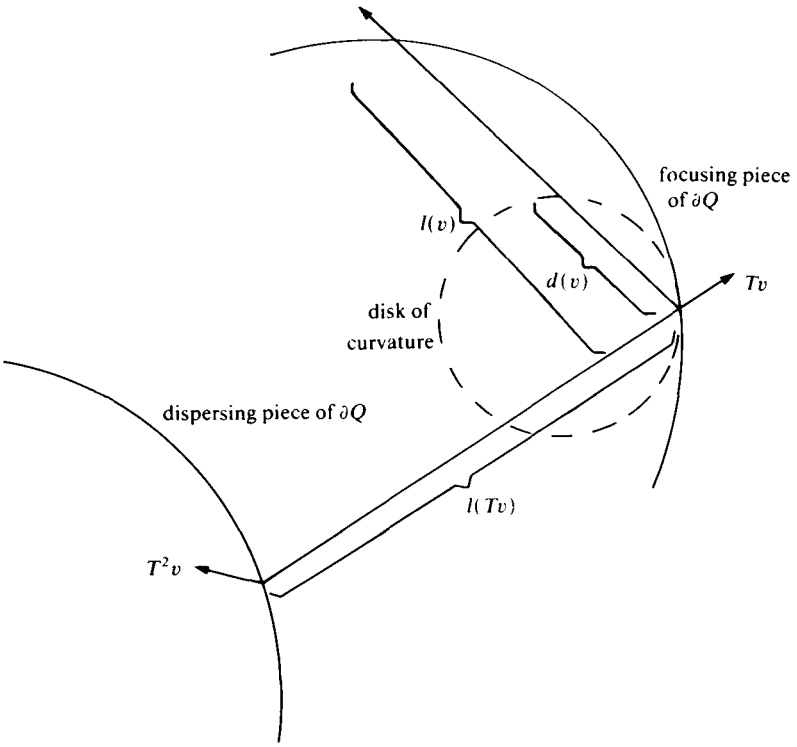

FIGURE 4

the curvature turns out to be the projective coordinate in the tangent plane of S. So, to detect a family of invariant cones for DT we have only to look at the evolution of curvature of a family of lines as it reflects from ∂Q for which we have formulae (1) and (2). Denote projectivization of D_vT by P_v. From (1) and (2) we get

$$P_v(k)=\frac{4}{d(v)}+\frac{k}{1-l(v)k}, \tag{3}$$

where $l(v)$ is the length of the segment of $\ell(0)$ between the point at which v is attached and the next point of intersection with ∂Q; $|d(v)|$ is the length of the segment of $\ell(0)$ inside the disk of curvature of ∂Q at the latter point; the sign of $d(v)$ is the same as in formula (2) (see figure 4).

We will now assume that Q satisfies Bunimovich conditions:

(a) At every point of a focusing piece of ∂Q, the disk of curvature lies entirely in Q (i.e. if $d(v)>0$ then $l(v)\geq d(v)$ in formula (3)).

(b) For almost all $v\in S$ there is $n(v)>0$ such that $d(T^{n(v)}v)>0$ and $l(T^{n(v)}v)>d(T^{n(v)}v)$ or $d(T^{n(v)}v)<0$.

Remark. The only focusing pieces of ∂Q that are allowed under condition (a) are arcs of circles. Indeed if a curve has variable curvature then except for critical points of the curvature the disk of curvature fails to lie locally on one side of the curve.

We will now define an invariant bundle of cones. For $v \in S$ we define a cone C_v in the tangent plane to S at v by a condition on the curvature k related to a tangent vector:

$$\frac{2}{l(v)} \leq k \leq +\infty \qquad \text{if } v \text{ is attached at a focusing piece of } \partial Q$$

$$-\infty \leq k \leq 0 \qquad \text{if } v \text{ is attached at a dispersing piece of } \partial Q.$$

The bundle of cones C_v is invariant under DT, i.e. $DT_v(C_v) \subset C_{Tv}$. Indeed let us check it, for example, in the situation when both v and Tv are attached at focusing pieces of ∂Q. We get by (3) that if $2/l(v) \leq k \leq +\infty$ then

$$\frac{2}{l(Tv)} \leq \frac{2}{d(v)} \leq \frac{4}{d(v)} - \frac{2}{l(v)} \leq P_v(k) \leq \frac{4}{d(v)} - \frac{1}{l(v)} < +\infty.$$

Hence by theorem 2.2 and the discussion at the end of § 2 we obtain immediately that the Lyapunov exponents are non-zero almost everywhere. The assumption of strict inclusion in theorem 2.2 is ensured by condition (b).

Pesin theory ([**11**]) does not apply formally to our billiards because of singularities of S and T, but the ideas behind it do apply (see [**9**]). So the properties of the Bunimovich billiards, such as the positivity of metric entropy, and the Bernoullian property for some power of T on each of the countably many ergodic components, can be obtained formally from non-vanishing of Lyapunov exponents. Ergodicity of such billiards (announced in [**4**]) lies beyond the scope of our approach.

4. *Other types of cones*

We will now consider matrices preserving other types of cones than the positive octant of § 1. Let $Q: \mathbb{R}^n \to \mathbb{R}$ be a non-degenerate quadratic form of the type $(1, n-1)$. Without loss of generality we can assume that

$$Q(v) = v_1^2 - v_2^2 - \cdots - v_n^2 = (Gv, v)$$

where

$$G = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix},$$

and $(\cdot, \cdot)$ denotes the standard scalar product. Linear transformations that preserve Q form the Lorentz group $O(1, n-1)$.

Consider the cone $C = \{v \in \mathbb{R}^n | Q(v) \geq 0\}$. As in § 1 we introduce the family of matrices

$$\mathcal{F} = \{A \in \mathrm{GL}(n, \mathbb{R}) | Q(Av) > 0 \text{ for } v \in C,\ v \neq 0\},$$

$$S\mathcal{F} = \{A \in \mathcal{F} | |\det A| = 1\}.$$

We will measure the length of vectors in C by $\sqrt{Q(v)}$.

LEMMA 4.1. *If $v \in C$ then $\|v\| \geq \sqrt{Q(v)}$.*

Again we put for $A \in \mathcal{F}$

$$\rho(A) = \inf_{v \in \operatorname{int} C} \sqrt{\frac{Q(Av)}{Q(v)}}.$$

Clearly the analogues of lemmas 1.4 and 1.5 hold so that we can conclude that $r(A) \geq \rho(A)$.

PROPOSITION 4.1. *If $A \in S\mathcal{F}$ then $\rho(A) > 1$.*

Proof. For $A \in S\mathcal{F}$ we have to compare the quadratic forms A^*Q, $(A^*Q)(v) = Q(Av)$, and Q on C. First we prove that there is a Lorentz transformation $L \in O(1, n-1)$ which diagonalizes A^*Q, i.e. L^*A^*Q is diagonal. For this purpose consider the family PC of lines contained in C, $PC \subset P\mathbb{R}^{n-1}$. PC is diffeomorphic to a closed $n-1$-dimensional disk.

For a linear subspace $V \subset \mathbb{R}^n$, let $V^\perp$ denote its orthogonal complement with respect to the quadratic form Q and V^Δ the orthogonal complement with respect to A^*Q.

LEMMA 4.2. *If $A \in \mathcal{F}$ then for $l \in PC$, $(l^\Delta)^\perp \in \operatorname{int} PC$.*

Proof. If $A \in \mathcal{F}$ then $A(PC) \subset \operatorname{int} PC$ or equivalently

$$(A(PC))^\perp \subset (\operatorname{int} PC)^\perp.$$

But $V \in (PC)^\perp$ iff $V \cap \operatorname{int} C = \varnothing$ so that if $A \in \mathcal{F}$ then

$$A^{-1}((PC)^\perp) \subset (\operatorname{int} PC)^\perp.$$

Now for $l \in PC$,

$$(l^\Delta)^\perp = (A^{-1}((Al)^\perp))^\perp. \qquad \square$$

So the mapping $PC \ni l \mapsto (l^\Delta)^\perp \in \operatorname{int} PC$ has a fixed point $l_0 \in \operatorname{int} PC$, i.e. $l_0^\Delta = l_0^\perp$. The Lorentz group acts transitively on $\operatorname{int} PC$ so that there is a Lorentz transformation L_1 that maps the v_1-axis onto l_0 and hence also the subspace $v_1 = 0$ onto l_0^Δ. We have as a result

$$Q(AL_1v) = \lambda_1 v_1^2 - (Hv', v'),$$

where H is some symmetric positive definite $(n-1) \times (n-1)$ matrix, $v' = (v_2, \ldots, v_n)$. There is an orthogonal matrix U such that

$$U^T H U = \Lambda' = \begin{pmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Clearly L_2, defined by $L_2 v = (v_1, Uv')$, is a Lorentz transformation. For $L = L_1 L_2$ we get

$$Q(ALv) = \lambda_1 v_1^2 - \lambda_2 v_2^2 - \cdots - \lambda_n v_n^2.$$

Since $|\det A| = 1$ we must have $\lambda_1 \cdot \ldots \cdot \lambda_n = 1$. But $Q(ALv) > 0$ if $Q(Lv) = Q(v) \geq 0$ so that $\lambda_1 > \max\{\lambda_2, \cdots, \lambda_n\}$. Thus $\lambda_1 > 1$ and there is $\eta > 1$ such that $\lambda_1 > \eta > \max\{\lambda_2, \cdots, \lambda_n\}$. Finally,

$$Q(ALv) - \eta Q(Lv) = (\lambda_1 - \eta) v_1^2 + (\eta - \lambda_2) v_2^2 + \cdots + (\eta - \lambda_n) v_n^2 \geq 0.$$

It follows immediately that $\rho(A) > \sqrt{\eta} > 1$. $\square$

Remark. Proposition 4.1 is false if we start with a quadratic form Q of type $(k, n-k)$, $2 \leq k \leq n-2$.

Using proposition 4.1 we can conclude in the way described in § 2 that for a measurable cocycle with values in $S\mathcal{F}$ the maximal Lyapunov exponent is positive almost everywhere. For $n=2$ we are back in the framework of § 1 and proposition 4.1 follows from corollary 1.2.

5. *Symplectic Matrices*

Consider the linear symplectic space $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic form $\omega(u, v) = (\mathcal{J}u, v)$ where

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbb{R}^n \times \mathbb{R}^n$ (or in $\mathbb{R}^n$).

We take a very special quadratic form Q on $\mathbb{R}^n \times \mathbb{R}^n$:

$$Q(v) = (x, y) \qquad \text{where } v = (x, y), \qquad x, y \in \mathbb{R}^n.$$

It is a quadratic form of the type (n, n). Let $C = \{v \in \mathbb{R}^{2n} | Q(v) \geq 0\}$. We will proceed along the same lines as in § 1 and § 4 but restrict our attention to symplectic matrices only. So let

$$\mathcal{F} = \{S \in \mathrm{Sp}\,(n, \mathbb{R}) | Q(Sv) > 0 \quad \text{if } v \in C, v \neq 0\}.$$

We have

LEMMA 5.1. *If $v \in C$ then $\|v\| \geq \sqrt{2}\sqrt{Q(v)}$.*

Again for $S \in \mathcal{F}$ we put $\rho(S) = \inf_{v \in \operatorname{int} C} \sqrt{Q(Sv)/Q(v)}$.

LEMMA 5.2. *If $S \in \mathcal{F}$ and $v = (x, x)$, $x \in \mathbb{R}^n$ then $\|Sv\| \geq \rho(S)\|v\|$.*

Proof. $\|Sv\| \geq \sqrt{2}\sqrt{Q(Sv)} \geq \rho(S)\sqrt{2Q(v)} = \rho(S)\|v\|$. □

PROPOSITION 5.1. *If $S \in \mathcal{F}$ then $\rho(S) > 1$.*

Proof. Note that for $A \in \mathrm{GL}(n, \mathbb{R})$,

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix}$$

is a symplectic matrix and it preserves Q. Let

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices. We claim that if $S \in \mathcal{F}$ then A is invertible. Indeed if there is $x \in \mathbb{R}^n$ such that $Ax = 0$ then for $u = (x, 0)$, $Q(u) = 0$ and $Q(Su) = 0$, thus contradicting the fact that S takes the cone C strictly into itself. So we can consider

$$S_1 = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix} S.$$

S_1 is symplectic, i.e. $S_1^T \mathcal{J} S_1 = \mathcal{J}$ and we get easily that

$$S_1 = \begin{pmatrix} I & R \\ P & I+PR \end{pmatrix},$$

where P and R are symmetric. S_1 clearly belongs to $\mathcal{F}$ so that for $u=(x,0)$, $x \in \mathbb{R}^n$, $Q(S_1 u) = (Px, x) > 0$ if $x \neq 0$, i.e. P is positive definite. Also R must be positive definite. Indeed if it is not then there is $y \in \mathbb{R}^n$ such that $(Ry, y) \leq 0$, $y \neq 0$. But then for $u = (-Ry, y) \in C$ we have $S_1(u) = (0, z)$, $z \in \mathbb{R}^n$, so that $Q(S_1 u) = 0$ which contradicts the fact that $S_1 \in \mathcal{F}$.

For $u = (x, y)$ we have

$$Q(S_1 u) = (x, y) + (Ry, y) + (P(x+Ry), x+Ry),$$

so that

$$Q(Su) = Q(S_1 u) \geq Q(u),$$

and if $Q(Su) = Q(u)$ then $u = 0$. We get that for $u \neq 0$, $Q(Su) > Q(u)$. Consider the ratio $\sqrt{Q(Su)/Q(u)}$ for $u \in \operatorname{int} C$, $\|u\| = 1$. As u approaches the boundary of C, $\|u\| = 1$, the ratio tends to infinity so that the infimum is attained at a point from $\operatorname{int} C$, i.e. $\rho(S) > 1$. □

As a byproduct of the above proof we get the following.

PROPOSITION 5.2. *A symplectic matrix*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathcal{F}$ iff $A^T C$ and BA^T are positive definite ($D^T B$ and CD^T are positive definite).

Note that $A^T C$ and BA^T (also $D^T B$ and CD^T) are symmetric because S is symplectic.

THEOREM 5.1. *If a measurable cocycle has values in $\mathcal{F}$ then all its Lyapunov exponents are different from zero.*

Proof. By lemma 5.2 and proposition 5.2, using the methods of § 2, we have exponential growth on the whole n-dimensional subspace $\{(x, y) \in \mathbb{R}^{2n} | x = y\}$. So the cocycle has n positive Lyapunov exponents. But if a cocycle has values in symplectic matrices then the Lyapunov exponents appear in pairs λ, $-\lambda$ ([**3**]). Hence we also have n negative exponents. □

There is an infinitesimal version of theorem 5.1. We will formulate it explicitly because it is related to non-vanishing of Lyapunov exponents for geodesic flows on manifolds of non-negative sectional curvature satisfying additional property (rank 1) (see [**2**] and [**5**]).

Let us consider a continuous flow ϕ^t: $M \to M$ on a compact manifold M preserving a probability measure μ and a continuous mapping $L: M \to \mathfrak{sp}(n, \mathbb{R})$, where $\mathfrak{sp}(n, \mathbb{R})$ is the Lie algebra of $\operatorname{Sp}(n, \mathbb{R})$. For $p \in M$ consider the linear differential equation

$$\frac{du}{dt} = L(\phi^t(p))u, \qquad u \in \mathbb{R}^{2n}. \tag{$*$}$$

General solutions of (*) have the form $u(t) = S^t u(0)$ where S^t is the matrix solution of $dS/dt = L(\phi^t(p))S$ such that $S^0 = I$.

Clearly all matrices S^t, $t \in \mathbb{R}$ are symplectic. We can define Lyapunov exponents by

$$\chi(p, u) = \lim_{t \to +\infty} \frac{1}{t} \log \|S^t u\| \qquad \text{where } p \in M,\ u \in \mathbb{R}^{2n}.$$

By Oseledec's multiplicative ergodic theorem we know that the limit exists μ-almost everywhere and for a given $p \in M$, $\chi(p, u)$ has at most $2n$ different values

$$\lambda_{-n} \leq \cdots \leq \lambda_{-1} \leq \cdots \leq \lambda_n,$$

and because S^t are symplectic then $\lambda_{-n} = -\lambda_n$ (for the last assertion see [3]).

Let us write $L(\phi^t(p))$ in the form

$$L(\phi^t(p)) = \begin{pmatrix} G(t) & N(t) \\ K(t) & -G^T(t) \end{pmatrix},$$

where K and N are symmetric matrices because $L \in \mathfrak{sp}(n, \mathbb{R})$. We will assume that both K and N are non-negative definite for all $t \in \mathbb{R}$. This condition is equivalent to $S^t(S^s)^{-1}C \subset C$ for all $t \geq s$. Indeed if $u(t) = (x(t), y(t))$ is the solution of (*) then

$$\frac{d}{dt} Q(u(t)) = \left(\frac{dx}{dt}, y\right) + \left(x, \frac{dy}{dt}\right) = (Gx + Ny, y) + (x, Kx - G^T y)$$
$$= (Kx(t), x(t)) + (Ny(t), y(t)) \geq 0.$$

Hence if $u \in C$ then $Q(u) \geq 0$ and for $t \geq s$,

$$Q(S^t(S^s)^{-1}u) \geq Q(S^s(S^s)^{-1}u) = Q(u) \geq 0,$$

i.e. $S^t(S^s)^{-1}C \subset C$. Vice versa if $S^t(S^s)^{-1}C \subset C$ for all $t \geq s$, then for $u = (x, y)$, $Q(u) = 0$ we have

$$\frac{d}{dt} Q(S^t(S^s)^{-1}u)|_{t=s} = (K(s)x, x) + (N(s)y, y) \geq 0.$$

Hence taking $u = (x, 0)$ and $u = (0, y)$ we get that both $K(s)$ and $N(s)$ are non-negative definite. We will say that the pair (ϕ^t, L) has 'rank 1' if additionally it satisfies the following condition:

For almost all $p \in M$ there are no solutions of (*) $u(t) = (x(t), y(t))$ such that $K(t)x(t) \equiv 0$ and $N(t)y(t) \equiv 0$ except for the trivial zero solution.

It is not difficult to see that the last condition can be replaced by:

For almost all $p \in M$ there is $t(p) > 0$ such that if $Q(u) \geq 0$, $u \neq 0$, then $Q(S^{t(p)}u) > 0$, i.e. $S^{t(p)}$ takes the cone C strictly into itself.

THEOREM 5.2. *If the pair (ϕ^t, L) as described above has 'rank 1' then all Lyapunov exponents are different from zero.*

Proof. The Lyapunov exponents for (ϕ^t, L) are the same as for the measurable cocycle (ϕ^1, S^t). Similarly to the proof of theorem 2.2, let $M_N = \{p \in M | t(p) \leq N\}$ for natural N. We can apply theorem 5.1 to the derived cocycle $((\phi^N)_{M_N}, (S^N)_{M_N})$ (note that $S^t \in \mathcal{F}$ if $t \geq t(p)$). But then also for (ϕ^t, L) the Lyapunov exponents are non-zero almost everywhere. □

Theorem 5.2 can be used to derive the fact that for geodesic flows on manifolds of non-negative sectional curvature with the rank 1 Riemann metric the Lyapunov exponents are non-zero (except for one corresponding to the direction of the flow) (see [**2**], [**5**]). In the case of the geodesic flow,

$$L=\begin{pmatrix} 0 & I \\ -K & 0 \end{pmatrix},$$

where $-K$ is non-negative definite. The rank 1 condition for the Riemann metric implies that for almost all points there are no solutions of $(*)$ $u(t)=(x(t),0)$ such that $K(t)x(t)\equiv 0$.

REFERENCES

[1] V. M. Alekseev. Quasi random dynamical systems. *Mat. USSR Sbornik* **7** (1969), 1-43.

[2] W. Ballmann & M. Brin. On the ergodicity of geodesic flows. *Ergod. Th. & Dynam. Sys.* **2** (1982), 311-315.

[3] G. Benettin, I. Galgani, A. Giorgilli & J.-M. Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: theory. *Meccanica* (1980), 9-20.

[4] L. A. Bunimovich. On the ergodic properties of nowhere dispersing billiards. *Commun. Math. Phys.* **65** (1979), 295-312.

[5] K. Burns. Hyperbolic behaviour of geodesic flows on manifolds with no focal points. *Ergod. Th. & Dynam. Sys.* **3** (1983), 1-12.

[6] E. Cornelis & M. Wojtkowski. A criterion for the positivity of the Lyapunov characteristic exponent. Preprint (1983).

[7] G. Gallavotti. Lectures on billiards. In *Lecture Notes in Physics*, **38**, Springer (1975).

[8] A. Katok. Bernoulli diffeomorphisms on surfaces. *Ann. Math.* **110**, (1979), 529-547.

[9] A. Katok & J.-M. Strelcyn. Invariant manifolds for smooth maps with singularities, I, II. Preprint (1981).

[10] V. I. Oseledec. The multiplicative ergodic theorem. The Lyapunov characteristic numbers of dynamical systems. *Trans. Mosc. Math. Soc.* **19** (1968), 197-231.

[11] Ya. B. Pesin. Lyapunov characteristic exponents and smooth ergodic theory. *Russ. Math. Surveys* **32** (1977), 55-114.

[12] F. Przytycki. Examples of conservative diffeomorphisms of the two-dimensional torus with coexistence of elliptic and stochastic behaviours. *Ergod. Th. & Dynam. Sys.* **2**, (1982), 439-463.

[13] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Publ. Math. IHES* **50** (1979), 27-58.