STRUCTURAL ANALYSIS

Volume - I

Third Edition

S S Bhavikatti

Emeritus Fellow

BVB College of Engineering and Technology, Hubli

Formerly Dean and Professor

NITK Suratkhal, SDMCET, Dharwad and

***Principal*, RYMEC, Bellary**

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PREFACE

Structural Analysis, also commonly referred to as *Theory of Structures* is an important subject for civil engineering students who are required to analyse and design structures.

This being a vast field, it is taught in at least two courses in almost all universities at the undergraduate level. Apart from these two courses, subjects like Matrix Method of Structural Analysis and Plastic Analysis are taught as papers in many postgraduate courses and electives in Structural Engineering.

I have covered this subject for undergraduate students in two books—Structural Analysis Vol. I and Structural Analysis Vol. 2.

This book, ***Structural Analysis Vol. 1***, covers determinate structures and gives an introduction to indeterminate structures, as is the conventional way of teaching this subject. The analysis of indeterminate structures by consistent deformation method is presented in two chapters. Though the 'Three Moment Equation' method is outdated and does not help students in studying advance methods of analysis, a chapter has been devoted to it, since many universities still have it in their syllabi.

I have explained the concepts and underlying theory in each chapter systematically. A highlight of the book is the numerous solved problems presented methodically. The set of unsolved problems, given under 'Exercises', are to test the students ability in solving them correctly.

The third edition includes the requirement of the latest syllabuses, Analysis of Pin-jointed Plane Frames as appendix. It discusses their stability and equilibrium, and methods of analysing force in truss members.

HIG-41
2nd Main,
Navanagar,
Hubli-580025

S S Bhavikatti

CONTENTS

INTRODUCTION 1

1.1 INTRODUCTION REMARKS

We have come across various structures in our day to day life ranging from simple ones like the curtain rods, and electric poles to more complex ones like multistoried buildings, shell roofs, bridges, dams, heavy machineries, automobiles, aeroplanes and ships. These structures are subjected to various loads like concentrated loads, uniformly distributed loads, uniformly varying loads, random loads, internal or external pressures and dynamic forces. The structure transfers its load to the supports and ultimately to the ground. Treating an entire structure as a single rigid body and finding the reactions from supports is the first step in analysing a structure which is covered under **Engineering Mechanics.**

While transferring the loads acting on the structure, the members of the structure are subjected to internal forces like axial forces, shearing forces, bending and torsional moments. **Structural Analysis** deals with analysing these internal forces in the members of the structures. The behaviour of the materials of the structures subjected to different types of internal forces is covered under **Strength of Materials. Structural Design** deals with sizing various members of the structure to resist the internal forces to which they are subjected in the course of their life cycle. However, the process of finding reactions, internal forces, behaviour of materials of structures to such forces and sizing of the members are so interconnected that it is difficult to separate them. Hence, a combination of topics in Engineering Mechanics, Strength of Materials, Structural Analysis and Structural Design is very common in various books and syllabi of engineering courses. The analysis of pin-jointed determinate plane frames has been covered in the book 'Engineering Mechanics' and the determination of bending moment and shear forces is determinate beams in the book 'Strength of Materials'.

In this book determination of deflections in beams and frames by various methods is dealt. Finding shear forces and bending moment due to moving loads is explained. The analysis of determinate structures like 3-hinged arches, cables and suspension bridges is explained. An introduction to the analysis of indeterminate structures by consistant determination method and three moment equations is presented.

1.2 IDEALISATIONS AND ASSUMPTIONS

The following idealisations and assumptions are made during analysis, under normal conditions.

1.2.1 Material Properties

Materials are assumed to be *homogeneous* and *isotropic*. Homogeneous material refers to the identical particles that exist throughout the material and isotropic refers to the physical properties of all the materials which are identical in all the directions in the particles of the materials.

Another assumption is *stress-strain* relation is *linear*, which means in case of metals, the analysis is carried out within the limit of proportionality and in case of materials like concrete, the stress-strain relation is approximated to a linear relation.

1.2.2 Boundary Conditions

The boundary conditions for structures are assumed to fall under the following idealised cases only:

i) Free end At the free end a structure can have linear or rotational displacement in any direction and hence no reaction is developed e.g. free end of a cantilever beam.

ii) Roller support At such end the member is free to move along the support and can rotate freely. Hence, there is no reaction along the support and the resisting moment is zero. In other words, there will be a reaction only in the direction normal to the support as shown in Fig. 1.1. The deflected shape of the beam at such end is also shown in Fig. 1.1.

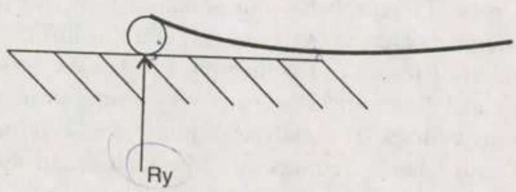


Figure 1.1 (a) *Roller end*

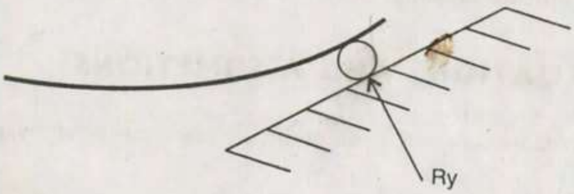


Figure 1.1 (b) *Roller end*

iii) Hinged ends At such ends it is assumed that the member cannot have linear motion in any direction but can rotate freely along the support point, i.e. the end is pinned. In such cases, the support can only develop the required resisting force in any direction and not the resisting moment. Since a force in any direction can be represented by its component forces, the reaction and deflected shape at the hinged end of a beam are as shown in Fig.1.2.

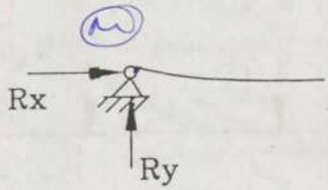


Figure 1.2 Hinged end

iv) Fixed ends Such ends cannot have any linear or rotational movement. At the fixed end, the support can develop not only the resisting force in any direction but also the resisting moment. The reaction and deflected shape at a fixed end in a typical beam are shown in Fig.1.3.

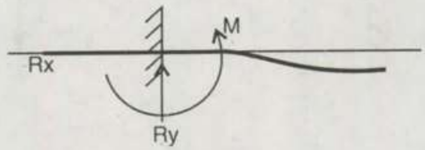


Figure 1.3 Fixed end

1.2.3 Small Deflections

Deflections are assumed to be small i.e., the changes in the shape of the structure due to loading are negligible. Hence, for all calculations, the changes in length of a member and the angle between any two members is neglected.

1.2.4 Loads

Concentrated load A heavy load distributed over a small area is assumed as a concentrated load acting at a point. For example, in the analysis of the beam shown in Fig.1.4, the weight of the beam and the load transferred by the secondary beams are idealised as concentrated loads. In Fig.1.5, the weight of a parapet wall at the free end of the balcony is a concentrated load.

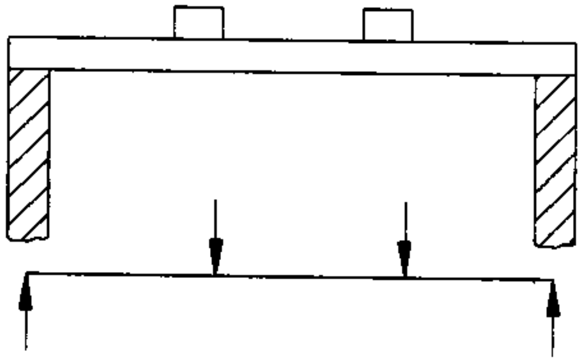


Figure 1.4

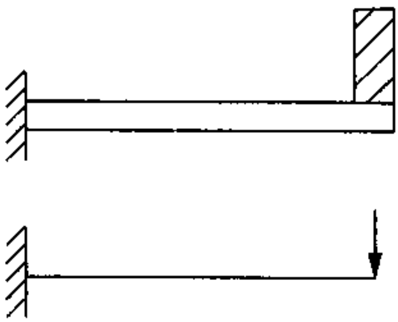


Figure 1.5

Uniformly distributed load Live load includes the weight of persons and any other moveable materials (like furnitures), which vary from time to time on the structure. For structural analysis, such loads are idealised as uniformly distributed load over an area. IS 875 gives live loads to be considered for the design of different types of structures.

1.2.6 Idealising the Structure

Structures are usually idealised as three dimensional. But without losing significant accuracy many structures are idealised as one or two dimensional which helps in simplifying the analysis considerably e.g. a beam is idealised as a one dimensional structure since it has considerable dimension in one direction compared to the dimensions in the other two (cross sectional) directions (Fig.1.6). A building frame is taken as a plane frame, neglecting the connection to adjacent frames by slabs and secondary beams (Fig.1.7). If the frames are equally stiff in both directions in the horizontal plane, they are analysed as space frames (three dimensional).

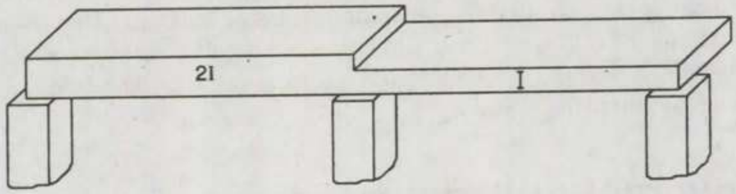


Figure 1.6 (a)

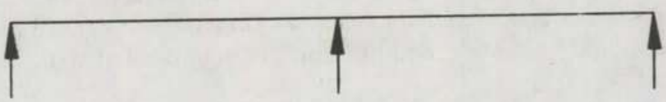


Figure 1.6 (b)

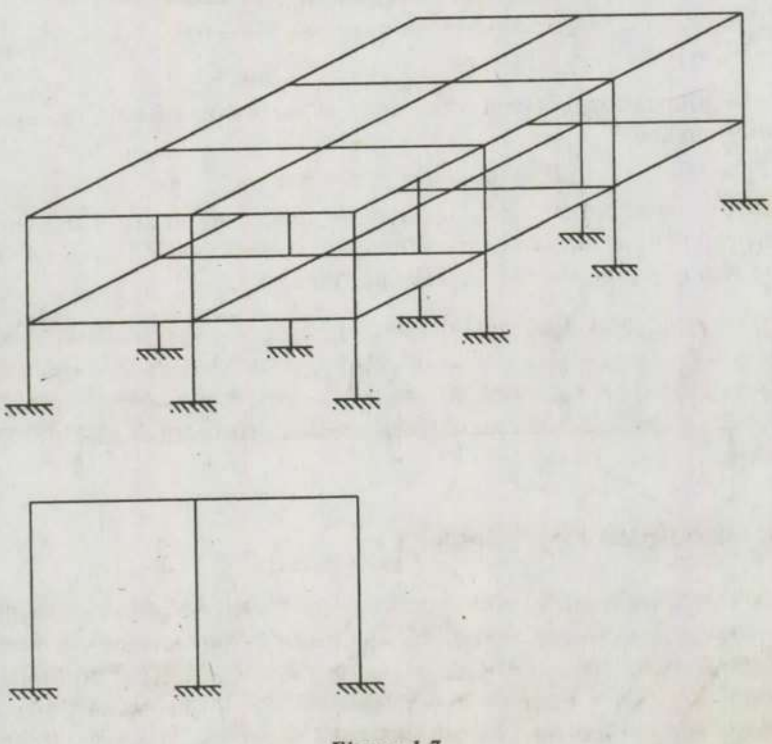


Figure 1.7

1.2.7 Law Of Superposition

The law of superposition holds good when the material is assumed to be perfectly elastic and obeys **Hooke's law** for the range of loads considered. This means that the structure can be analysed for different loads separately and the results be superposed to get the final results due to different combinations of loadings. This assumption also permits analysing the structure with convenient end conditions and suitable loadings separately and then combining the results suitably to get a solution for the required end conditions and loadings.

1.3 CONDITIONS OF EQUILIBRIUM

The basic tool in structural analysis is the usage of equilibrium equations which states that the structure or part of it remains in its stationary position. Hence, if the entire structure is considered, the reactions from the support and the loads on the structure should be in static equilibrium. The equations of static equilibrium are as follows:

i) the summation of all the forces along any axis is zero.

ii) the summation of all the moments about any axis is also zero.

The equations of static equilibrium are based on **Newton's law**. For a three dimensional system, the equations of equilibrium are as follows:

$$\Sigma F_x = 0.0 \qquad \Sigma F_y = 0.0 \qquad \text{and} \qquad \Sigma F_z = 0.0 \tag{1.1}$$

$$\Sigma M_x = 0.0 \qquad \Sigma M_y = 0.0 \qquad \text{and} \qquad \Sigma M_z = 0.0 \tag{1.2}$$

For a two dimensional system with x and y as the orthogonal axis, the equations of equilibrium are

$$\Sigma F_x = 0.0 \qquad \Sigma F_y = 0.0 \qquad \Sigma M = 0.0 \tag{1.3}$$

The above equilibrium conditions may be applied to a part of the structure also provided that, in such case, apart from the external loads, the reactive forces from the removed part are also considered. (Fig. 1.8)

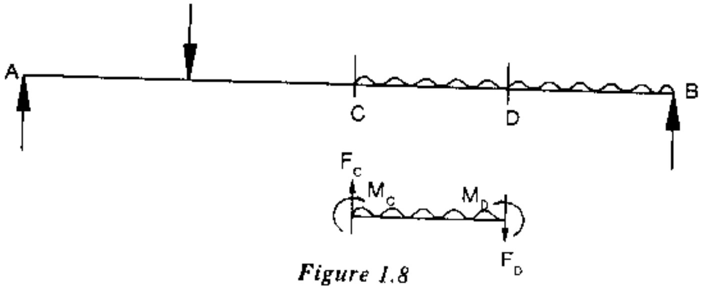


Figure 1.8

1.4 COMPATIBILITY CONDITIONS

Compatibility conditions means requirement of continuity, such as in joints where two or more members meet. The following two compatibility conditions are to be satisfied at any joint:

i) The members meeting at a joint will continue to meet at the same joint even after deformation takes place.
ii) At rigid-joints, the angle between any two members remains the same even after deformation takes place. The compatibility conditions will help in formulating additional equations.

1.5 STATICALLY DETERMINATE AND INDETERMINATE STRUCTURES

The structures are grouped into statically determinate and statically indeterminate structures. A structural system which can be analysed with the use of equations of statical equilibrium only is called as *statically determinate structure* e.g. beams or trusses with both ends simply supported, one end hinged and another on rollers, and the cantilever type. A structure which cannot be analysed with the use of equations of equilibrium only is called a *Statically Indeterminate Structures* e.g. fixed beams, continuous beams, propped cantilevers. To analyse indeterminate structures, apart from using equations of equilibrium one has to determine the various deformations and make use of compatibility conditions. Indeterminate structures are also called *Redundant Structures*.

1.6 SIMPLE AND COMPOUND SYSTEMS

A determinate structure is called a *simple structure,* if it is made up of a single system. If a system is to be split into a number of systems which may then be analysed with equations of equilibrium alone the system is termed as *compound systems.* Fig.1.9 shows a compound system which may be analysed by using equations of equilibrium only, after splitting it into two simple beams.

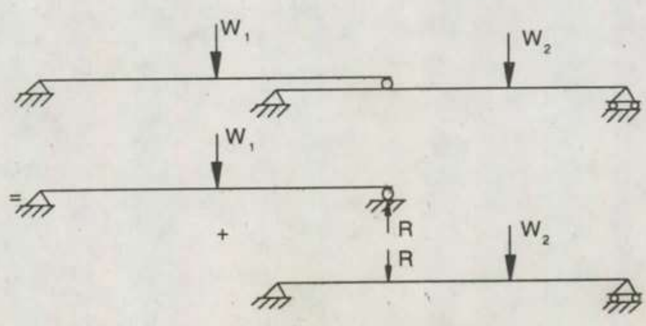


Figure 1.9 *Compound system*

1.7 LINEAR AND NON-LINEAR SYSTEMS

A system is called a *linear system* if its material has linear stress-strain relationship and a small deflection. In such cases, the law of superposition holds good. A system will be treated as a *non-linear system* if its material does not have linear stress-strain relationship or its deformation is so large that a change of geometry cannot be neglected in the analysis. If the non-linearity is due to stress-strain relationship, it is called *material non-linearity* and if the non-linearity is due to considerable changes in the geometry, it is called *geometric non-linearity*. In some cases both material non-linearity and geometric non-linearity need to be considered. Non-linear analysis is lengthy and repetative type with minor changes in each cycle. Hence it is ideally suited for computer aided analysis.

DEFLECTION OF DETERMINATE BEAMS USING MOMENT AREA AND CONJUGATE BEAM METHODS

2

2.1 INTRODUCTION

In any building, the deflection of beams should be kept below the permissible limit as specified by the code to

i. have aesthetic views,
ii. give psychological comfort to the user,
iii. safeguard the flooring materials, window frames, etc.

In a mechanical component, deflections play an important role in determining the performance of the machine. Hence, the deflections are to be limited to specified values. It is essential than an analyst knows how to calculate the accurate deflections in beams. Apart from these reasons, a structural analyst need to know the deflection of determinate beams for analysing indeterminate beams.

The following methods are available for finding the deflections of determinate beams.

i. Double Integration / Macaulay's Method
ii. Moment Area Method
iii. Conjugate Beam Method
iv. Strain Energy Method
v. Castigliano's Method
vi. Unit Load Method

The double integration method is explained in the code 'Strength of Materials'. In this chapter, the moment area method and the conjugate beam method are explained. The other three methods latter are explained in the latter chapters.

2.2 MOMENT AREA THEOREMS

The moment area method is based on the following two theorems.

Theorem 1 *The change in the slope between two points on a straight member under flexure is equal to the area of* $\frac{M}{EI}$ *diagram between those two points.*

where M is the Bending Moment
E is the Young's Modulus
and I is the Moment of Inertia.

Consider the beam AB (Fig. 2.1(a)). Let C and D be any two points on this beam. The $\frac{M}{EI}$ diagram is also shown in the figure. Fig. 2.1(b) shows the elastic curve of the beam after loading. According to this theorem, θ_{CD} which is the angle between the tangents at C and D is equal to the area of $\frac{M}{EI}$ diagram between C and D (shaded portion). Thus $\theta_{CD} = \int_C^D \frac{M}{EI} dx$ 2.1

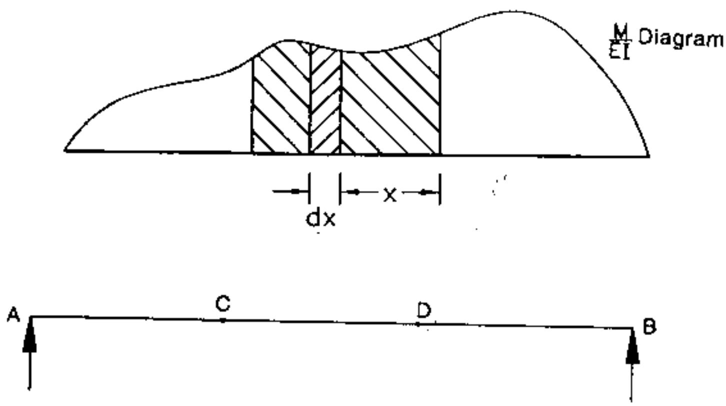


Figure 2.1 (a)

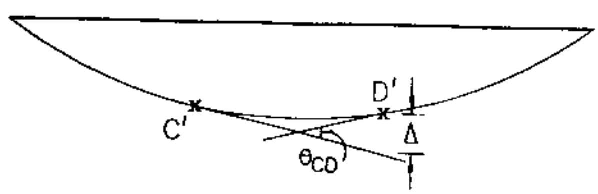


Figure 2.1 (b)

Theorem 2 *Deflection at a point in a beam in the direction perpendicular to its original straight line position measured from the tangent to the elastic curve at another point is given by the moment of* $\frac{M}{EI}$ *diagram about the point where the deflection is required.*

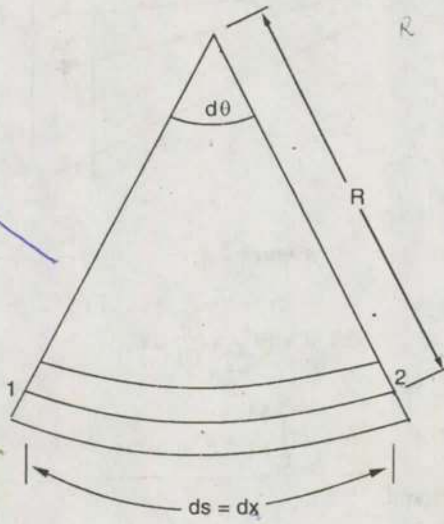
In Fig.2.1(b), Δ,the vertical (perpendicular to the horizontal position of AB) deflection at point D′ from the tangent to the elastic curve at C′ is given by the moment of $\frac{M}{EI}$ diagram between C and D about the point D. Thus,

$$\Delta = \int_C^D \frac{Mx}{EI} dx \qquad 2.2$$

2.3 DERIVATION OF MOMENT AREA THEOREMS

Fig.2.2 shows the elemental length 'dx' of Fig.2.1 to an enlarged scale. Let R be the radius of curvature. Then, from flexure formula

$$\frac{M}{I} = \frac{E}{R} \qquad 2.3$$



$R = \frac{1}{\rho} = \frac{M}{EI}$

$\boxed{S = r\theta}$

Figure 2.2

From Fig. 2.2,

$R d\theta = ds = dx$, since axial deformations are considered negligible

$$\therefore \qquad R = \frac{dx}{d\theta} \qquad 2.4$$

Substituting this value of R in equation 2.3, we get

$$\frac{M}{I} = \frac{E}{(dx/d\theta)}$$

or

$$d\theta = \frac{M}{EI}dx$$

$$\therefore \qquad \theta_{CD} = \int_C^D d\theta$$

$$= \int_C^D \frac{M}{EI}dx$$

Thus theorem 1 is proved.

Now consider Fig. 2.3 in which portion CD is blown up to an enlarged scale. Let the change of slope in elemental length dx be '$d\theta$.' Distance of elemental length from D is x (Ref. Fig. 2.1(a). Hence deflection.

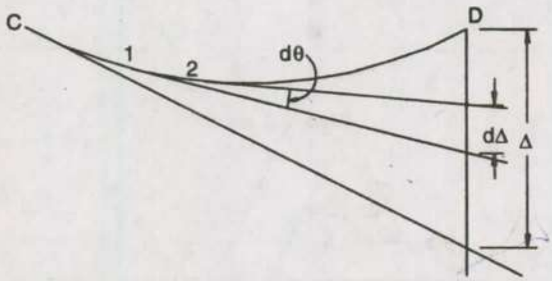


Figure 2.3

$$d\Delta = x d\theta = x \frac{M}{EI} dx$$

$$\therefore \qquad \Delta = \int_C^D \frac{M}{EI} x \, dx$$

Thus theorem 2 is proved.

2.4 SIGN CONVENTION IN THE MOMENT AREA METHOD APPLIED TO BEAMS

The following sign convention is used in applying moment area theorems to find the deflection of beams:

1. Sagging moment area is positive, which means that, the tangent at D makes an anticlockwise angle with tangent at C.
2. The moment of positive moment gives rise to positive deflection, which implies that the deflected position of a point (D) is above the tangent drawn at the other point (C).

Following are the illustrated problems to explain the wide application of moment area method. This method is advantageous, if the tangent at a particular point in the beam is along the axis itself e.g. beam with one end fixed or with a symmetric point.

slope

Example 2.1 Determine the rotation and deflection at the free end of the cantilever beam subjected to uniformly distributed load over an entire span as shown in Fig.2.4(a).

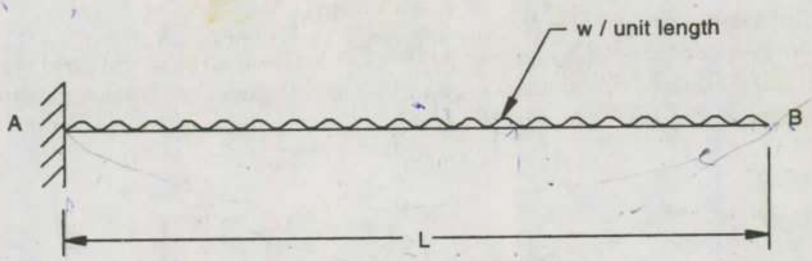


Figure 2.4 (a)

$2wL \times \frac{L}{2}$

$\frac{wL^2}{2}$

Solution

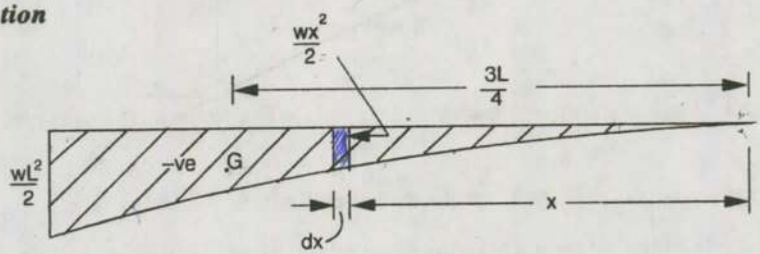


Figure 2.4 (b)

The bending moment diagram is shown in Fig.2.4(b). At any distance x from free end, bending moment is $-\frac{wx^2}{2}$.

Now,

$$\theta_{BA} = \theta_B - \theta_A = \theta_B \qquad \because \quad \theta_A = 0$$

∴ From the moment area theorem,

$$\theta_B = \int_0^L \frac{M}{EI}\,dx$$

$$= \int_0^L -\frac{wx^2}{2EI}\,dx$$

$$= -\frac{w}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{-wL^3}{6EI}$$

$$= \frac{wL^3}{6EI}\text{, clockwise with tangent at A}$$

Δ_B = deflection of B with respect to tangent at A
= vertical deflection, since tangent at A is horizontal.

From the second moment area theorem,

$$\Delta_B = \int_0^L \frac{M}{EI}\, x\, dx = \int_0^L -\frac{wx^3}{2EI}\, dx$$

$$= -\frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^L = \frac{-wL^4}{8EI}$$

$$= \frac{wL^4}{8EI}, \text{ downward}$$

Note: Area of such a parabolic curve is = $\frac{1}{3} \times L \times$ Ordinate at the end and its centre of gravity is at a distance $\frac{3L}{4}$ from the end where the value is zero.

Example 2.2 Find the rotation and deflection at the free end in the cantilever beam shown in Fig. 2.5(a).

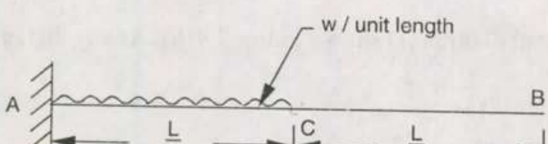


Figure 2.5 (a)

Solution

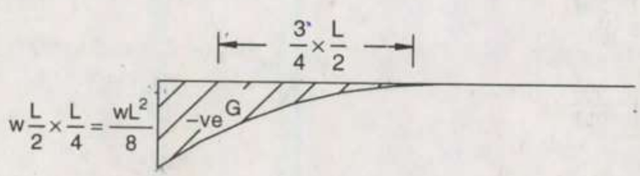


Figure 2.5 (b)

The bending moment diagram is a parabola as shown in Fig.2.5(b) with maximum ordinate as $\frac{wL}{2} \times \frac{L}{4} = \frac{wL^2}{8}$ Its centre of gravity from the point C is at a distance $\frac{3}{4} \times \frac{L}{2} = \frac{3}{8}L$. Area of the bending moment diagram is

$$= \frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{wL^3}{48}$$

and it is negative area since the bending moment is a hogging moment.

$$\theta_{BA} = \theta_B - \theta_A = \theta_B \qquad \because \quad \theta_A = 0$$

From the moment area theorem,

$$\theta_B = \text{Area of } \frac{M}{EI} \text{ diagram between A and B}$$

$$= \left[-\frac{wL^3}{48} \right] \times \frac{1}{EI}$$

$$= -\frac{wL^3}{48EI}$$

$$= \frac{wL^3}{48EI}, \text{ clockwise}$$

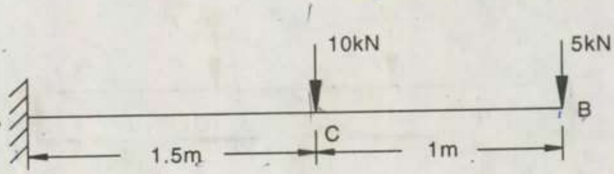
Since tangent at A is horizontal, vertical deflection at

$$B \quad = \text{Moment of } \frac{M}{EI} \text{ diagram about B}$$

$$= \left[-\frac{wL^3}{48EI} \right] \times \left(\frac{3}{8} L + \frac{L}{2} \right)$$

$$= -\frac{7wL^4}{384EI} = \frac{7wL^4}{384EI}, \text{ downward}$$

Example 2.3 Determine the slope and deflection at the free end of a cantilever beam as shown in Fig.2.6(a) by moment area method. (Take EI = 4000 kNm²).



15.16

Figure 2.6 (a)

Solution

The bending moment diagram for this beam is as shown in Fig.2.6(b).

$M_A = MA - 15 - \frac{5 \times 2.5}{12}$

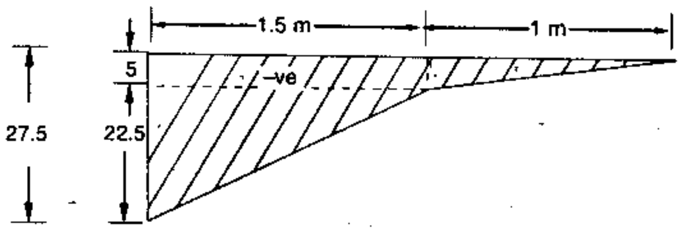


Figure 2.6 (b)

$$\theta_B = \frac{\text{Area of bending moment diagram}}{EI}$$

$$= -\frac{1}{EI}\left[\frac{1}{2}\times 1\times 5+5\times 1.5+\frac{1}{2}\times 22.5\times 1.5\right]$$

$$= -\frac{26.875}{EI} = -\frac{26.875}{4000}$$

$= -6.71875 \times 10^{-3}$ radians.

$= 6.71875 \times 10^{-3}$ radians, clockwise

$$\Delta_B = \text{Moment of } \frac{M}{EI} \text{ diagram about B}$$

$$= -\frac{1}{EI}\left[\frac{1}{2}\times 1\times 5\times\frac{2}{3}+5\times 1.5\times(1+0.75)+\frac{1}{2}\times 22.5\times 1.5\times(1+1)\right]$$

$$= -\frac{48.542}{EI} = -\frac{48.542}{4000}$$

$= -0.01213$m

$= -12.13$mm $= 12.13$mm, downward

Example 2.4 Determine the rotation at supports and deflection at mid-span and under the loads in the simply supported beam as shown in Fig. 2.7(a).

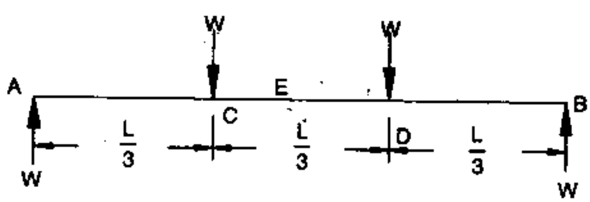


Figure 2.7 (a)

Solution

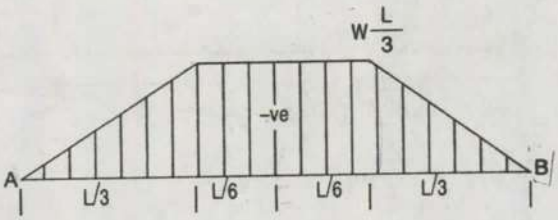


Figure 2.7 (b)

In this case, the bending moment diagram is as shown in Fig.2.7(b). Due to symmetry, the slope at E is zero. In other words, the tangent at E is horizontal. Hence, it is convenient to workout rotations and deflection with respect to the tangent at E.

$$\theta_A = \text{Area of } \frac{M}{EI} \text{ diagram between A and E}$$

$$= \frac{1}{EI}\left[\frac{1}{2}\times\frac{L}{3}\times\frac{WL}{3}+\frac{L}{6}\times\frac{WL}{3}\right]$$

$$= \frac{1}{EI} WL^2 \times \frac{2}{18} = \frac{WL^2}{9EI}$$

Deflection of E w.r.t. A = Deflection of A w.r.t. E

$$= \text{Moment of } \frac{M}{EI} \text{ diagram between AE about A}$$

$$= \frac{1}{EI}\left[\frac{1}{2}\times\frac{L}{3}\times\frac{WL}{3}\times\frac{2}{3}\times\frac{L}{3}+\frac{L}{6}\times\frac{WL}{3}\left(\frac{L}{3}+\frac{L}{12}\right)\right]$$

$$= \frac{WL^3}{EI}\left[\frac{1}{81}+\frac{1}{18}\times\frac{5}{12}\right]$$

$$= \frac{23}{648}\times\frac{WL^3}{EI}$$

$\left(\frac{L}{6}\right)\left(\frac{WL}{3}\right)[$

$\frac{2}{3} \times 1.5$

Deflection of C above E

$$= \text{Moment of } \frac{M}{EI} \text{ diagram between CE about C}$$

$$= \frac{1}{EI}\left(\frac{L}{6}\right)\times\frac{WL}{3}\times\frac{L}{12}$$

$$= \frac{WL^3}{216}\times\frac{1}{EI}$$

∴ Deflection of C below A

$$= \left[\frac{23}{648} - \frac{1}{216}\right] WL^3$$

$$= \frac{20}{648} WL^3 = \frac{5}{162} WL^3$$

Due to symmetry, deflection of D below B = $\frac{5}{162} WL^3$

Example 2.5 Determine the slope at A, deflection at C and mid-span E in the beam as shown in Fig. 2.8(a).

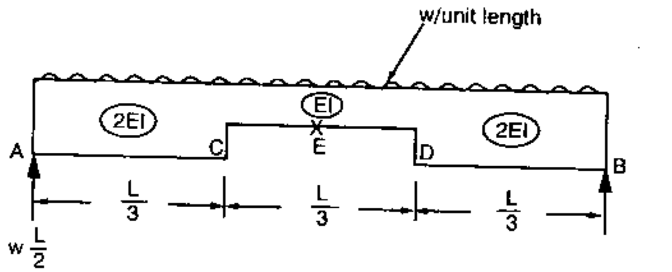


Figure 2.8 (a)

Solution

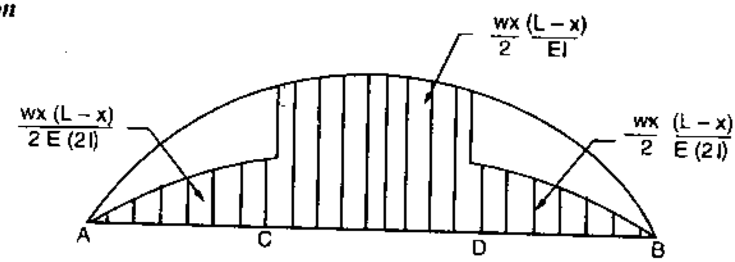


Figure 2.8 (b)

Due to symmetry, the slope at mid-span E is zero and tangent at this point is horizontal. Hence, slope at A w.r.t. the horizontal axis is the area of M/EI diagram between A and E and vertical deflection of E is the moment of M/EI diagram between A and E about E (i.e., we calculate upward deflection of A from tangent at E).

The bending moment at any point at a distance x from end A is given by

$$M_x = \frac{w}{2} (Lx - x^2)$$

Elemental length dx is first considered for finding the area of moment diagram at the required point and then it is integrated within the appropriate limits.

The $\frac{M}{EI}$ diagram for this problem is shown in Fig. 2.8(b).

θ_A = Area of $\frac{M}{EI}$ diagram between A and E

= Area of $\frac{M}{EI}$ diagram between A and C + Area of moment diagram between C and E.

$$= \int_0^{L/3} \frac{Mx}{2EI} + \int_{L/3}^{L/2} \frac{Mx}{EI} dx$$

$$= \int_0^{L/3} \frac{w}{4EI}(Lx - x^2)dx + \int_{L/3}^{L/2} \frac{w}{2EI}(Lx - x^2)dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_{L/3}^{L/2}$$

$$= \frac{wL^3}{4EI}\left[\frac{1}{18} - \frac{1}{81}\right] + \frac{wL^3}{2EI}\left[\frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{81}\right]$$

$$= \frac{wL^3}{4EI}\left[\frac{9-2}{162}\right] + \frac{wL^3}{2EI}\left[\frac{81-27-36+8}{648}\right]$$

$$= \frac{7}{648} \times \frac{wL^3}{EI} + \frac{13wL^3}{648EI}$$

$$= \frac{20wL^3}{648EI} = \frac{5wL^3}{162EI}$$

Note: $\frac{13wL^3}{648EI}$ is the slope at C since this is the area of $\frac{M}{EI}$ diagram between C and E

$\Delta_E = \Delta_A$ w.r.t. to horizontal axis at E,

= Moment of $\frac{M}{EI}$ diagram between A and E about A.

$$= \int_0^{L/3} \frac{1}{2EI} \frac{w}{2} (Lx - x^2)x\, dx + \int_{L/3}^{L/2} \frac{1}{EI} \frac{w}{2} (Lx - x^2)x\, dx$$

$$= \frac{w}{4EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/3} + \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_{L/3}^{L/2}$$

$$= \frac{wL^4}{4EI}\left[\frac{1}{81} - \frac{1}{81\times4}\right] + \frac{wL^4}{2EI}\left[\frac{1}{24} - \frac{1}{64} - \frac{1}{81} + \frac{1}{81\times4}\right]$$

$$= 0.010706 \ \frac{wL^4}{EI}$$

Deflection at C

At C, first deflection above the tangent at E will be found by moment area method and then it is subtracted from the deflection of E to get the deflection of C below the original axis.

Δ_C = Moment of $\frac{M}{EI}$ diagram between C and E about C.

$$\therefore \qquad \Delta_C = \int_{L/3}^{L/2} \frac{1}{EI} \frac{w}{2} (Lx - x^2)\left(x - \frac{L}{3}\right) dx$$

$$= \frac{w}{2EI} \int_{L/3}^{L/2} \left(Lx^2 - x^3 - \frac{L^2}{3}x + \frac{L}{3}x^2 \right) dx$$

$$= \frac{w}{2EI} \int_{L/3}^{L/2} \left(\frac{4L}{3}x^2 - x^3 - \frac{L^2}{3}x \right) dx$$

$$= \frac{w}{2EI} \left[\frac{4L}{3}\frac{x^3}{3} - \frac{x^4}{4} - \frac{L^2}{6}x^2 \right]_{L/3}^{L/2}$$

$$= \frac{wL^4}{2EI} \left[\frac{4}{9\times8} - \frac{1}{64} - \frac{1}{24} - \frac{4}{243} + \frac{1}{81\times4} + \frac{1}{6\times9} \right]$$

$$= 1.703961 \ 10^{-3} \ \frac{wL^4}{EI}$$

$$\therefore \qquad \Delta_C = \Delta_E - \Delta_C$$

$$= (0.010706 - 1.703961 \times 10^{-3}) \frac{wL^4}{EI}$$

$$= 0.00900205 \ \frac{wL^4}{EI}$$

Example 2.6 Determine the slope and deflections at the end of the beam shown in Fig.2.9(a). EI is constant throughout.

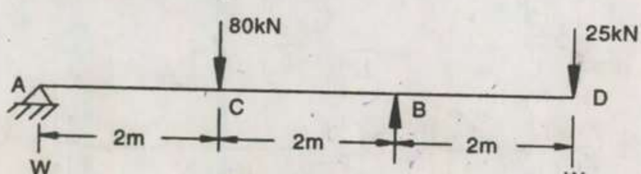


Figure 2.9 (a)

Solution

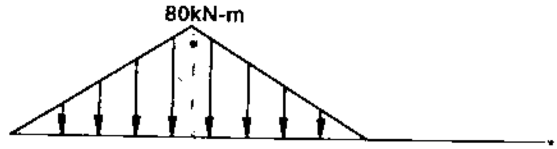


Figure 2.9 (b)

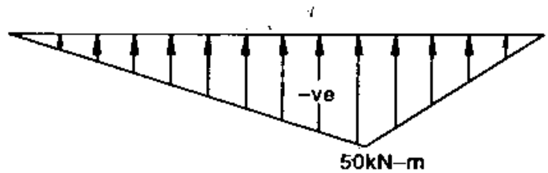


Figure 2.9 (c)

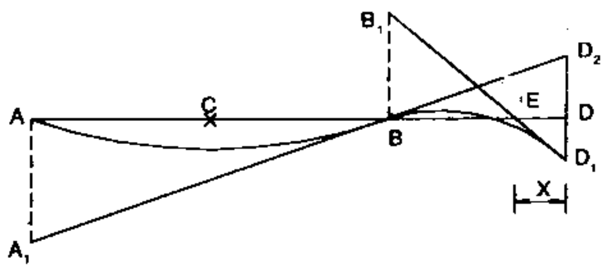


Figure 2.9 (d)

Bending moment diagram due to load at C and that due to the load at D are shown in Fig. 2.9(b) and Fig. 2.9(c) respectively. The deflected shape and the tangents at B and D are shown in Fig. 2.9 (d).

Now slope at B $= \dfrac{AA_1}{AB}$

where AA_1 = Deflection at A w.r.t. the tangent at B

= Moment $\dfrac{M}{EI}$ diagram between A and B about A,

$$= \frac{1}{EI}\left[\frac{1}{2}\times 80\times 4\times 2-\frac{1}{2}\times 50\times 4\times\frac{8}{3}\right]$$

$$= \frac{53.333}{EI}$$

$$\therefore \quad \theta_B = \frac{53.333}{EI}\times\frac{1}{4} = \frac{13.333}{EI}$$

By similar triangle principle,

$$DD_2 = \frac{AA_1}{2} = \frac{26.667}{EI}\text{, anticlockwise}$$

Now D_1D_2 = Deflection of point D w.r.t. the tangent at B

$= \text{Moment } \frac{M}{EI}$ diagram between B and D about D.

$$= -\frac{1}{EI} \times \frac{1}{2} \times 50 \times 2 \times \frac{4}{3} = -\frac{66.667}{EI}$$

$= \frac{66.667}{EI}$, downward as shown in figure.

∴ Vertical deflection at D

$$DD_1 = \frac{66.667}{EI} - \frac{26.667}{EI} = \frac{40}{EI}$$

To find slope at D:

BB_1 = Deflection of B w.r.t. the tangent at D.

$= \text{Moment } \frac{M}{EI}$ diagram between B and D about B.

$$= \frac{1}{EI}\left(-\frac{1}{2}\times 50\times 2\times \frac{2}{3}\right) = -\frac{33.333}{EI}$$

$= \frac{33.333}{EI}$, downward as shown in figure.

Let E be the intersection of B_1D_1 and BD. Let DE = x

Then, $$\frac{DE}{BE} = \frac{x}{2-x}$$

But $$\frac{DE}{BE} = \frac{DD_1}{BB_1}$$

∴ $$\frac{x}{2-x} = \frac{40}{33.333} = 1.2$$

$$x = (2-x)\,1.2$$

or $$x = 1.091$$

∴ Slope at $$D = \frac{DD_1}{x} = \frac{40}{EI} \times \frac{1}{1.091}$$

$$\theta_D = \frac{36.667}{EI}$$

Note: Thus moment area method can be applied to any type of determinate beam, but it can be used conveniently only if the tangent at one of the points in the beam is horizontal i.e., if there is a fixed end or a point of symmetry.

2.5 CONJUGATE BEAM THEOREMS

These theorems can be derived from moment area theorems and are very useful in solving a problem even if there is no point in the beam where the slope is zero.

Now consider the simply supported beam shown in Fig. 2.10(a). $\frac{M}{EI}$ diagram and deflected shapes for this beam are shown in Fig. 2.10(b) and 2.10(c) respectively.

$$\text{Now, } \theta_C = \theta_A - \text{Area of } \frac{M}{EI} \text{ diagram between A and C}$$

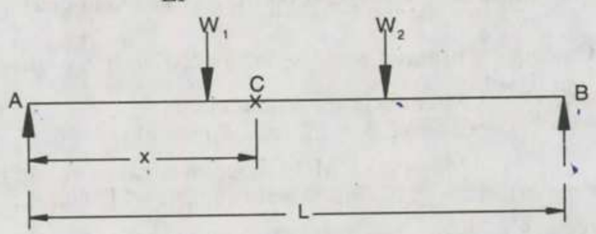


Figure 2.10 (a)

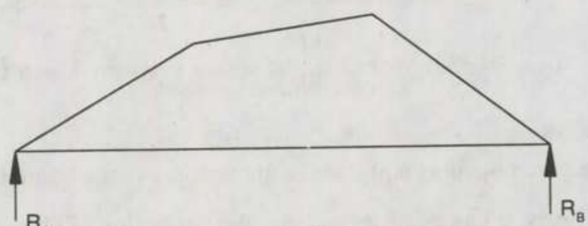


Figure 2.10 (b) $\frac{M}{EI}$ *Diagram*

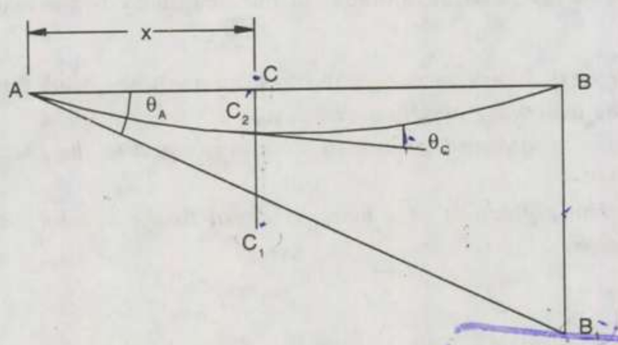


Figure 2.10 (c) Deflected shape

$$\theta_A = \frac{BB_1}{AB}$$

$$= \frac{1}{L}\text{ moment of area of }\frac{M}{EI}\text{ diagram between A and B about B.}$$

$$\theta_C = \frac{\text{Moment of M/EI diagram about B}}{L}$$

$$- \text{Area of }\frac{M}{EI}\text{ diagram between A and C} \qquad 2.5$$

Now deflection at C

$$= CC_2 = CC_1 - C_2C_1$$

$$= x_2\theta_A - 9\text{Deflection of C w.r.t. tangent at A)}$$

$$= \frac{x_C \times \text{Moment of (M / EI) diagram between A and B about B}}{L}$$

$$- \text{Area of }\frac{M}{EI}\text{ diagram between A and C about C} \qquad 2.6$$

Consider a beam at same span, loaded with $\frac{M}{EI}$ diagram.

$$\text{Then reaction at A, } R'_A = \frac{\text{Moment of the load about B}}{L}$$

$$= \frac{\text{Moment of M/EI diagram between A and B about B}}{L}$$

$$\text{Shear force at C} = R'_A - \text{load between A and C}$$

$$= \frac{\text{Moment of M/EI diagram between A and B about B}}{L}$$

$$- \text{Area of }\frac{M}{EI}\text{ diagram between A and C}$$

$\therefore$ θ_C in the given beam is equal to the shear force in the beam loaded with $\frac{M}{EI}$ diagram. Similarly, it can be observed that the deflection of C, given by equation 2.6, is equal to the bending moment in the imaginary beam loaded with $\frac{M}{EI}$ diagram.

The imaginary beam is called the conjugate beam, and from the above discussion the following two theorems result:

Theorem 1 *The rotation at a point in a beam is equal to the shear force in the conjugate beam.*

Theorem 2 *The deflection in a beam is equal to the bending moment in the conjugate beam.*

2.6 CONJUGATE BEAMS

The aforesaid statements are made considering simply supported beams. However, suitable changes may be made in the boundary conditions for different beams to see that the theorems stated above hold good. The *Conjugate Beam* can be defined as follows:

Conjugate beam is an imaginary beam of same span as the original beam loaded with $\frac{M}{EI}$ *diagram of the original beam, such that the shear force and bending moment at a section will represent the rotation and deflection at that section in the original beam.*

Table 2.1

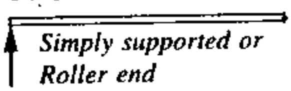
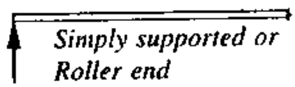
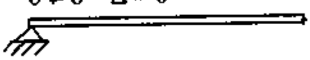
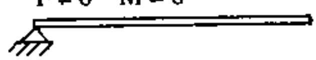
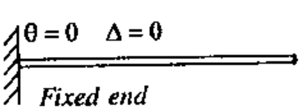
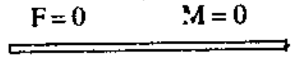
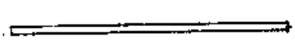
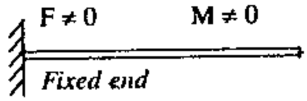
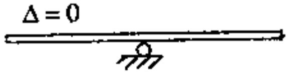
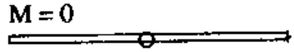
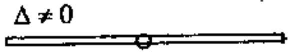
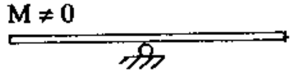
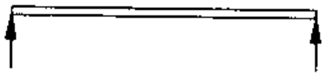
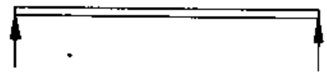
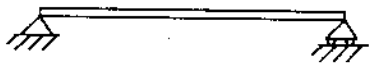
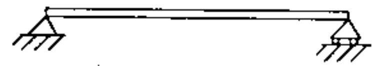
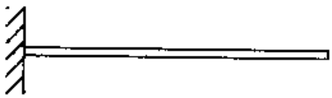
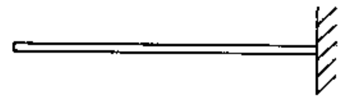
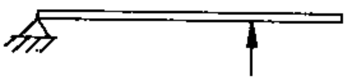
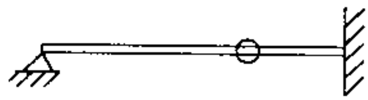
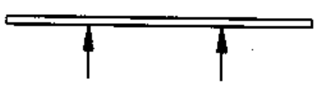
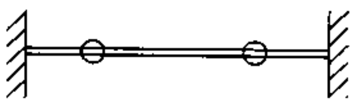
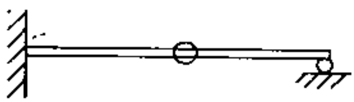
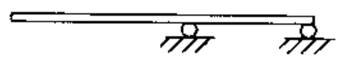
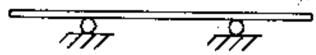
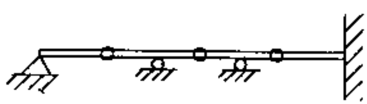
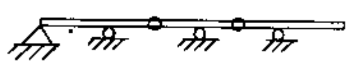
Sl. No.	*Original beams*	*Conjugate beams*
1	$\theta \neq 0 \quad \Delta = 0$  *Simply supported or Roller end*	$F \neq 0 \quad M = 0$  *Simply supported or Roller end*
2	$\theta \neq 0 \quad \Delta = 0$  *Hinged end*	$F \neq 0 \quad M = 0$  *Hinged end*
3	$\theta = 0 \quad \Delta = 0$  *Fixed end*	$F = 0 \quad M = 0$  *Free end*
4	$\theta \neq 0 \quad \Delta \neq 0$  *Free end*	$F \neq 0 \quad M \neq 0$  *Fixed end*
5	$\theta \neq 0$ and continuous $\Delta = 0$  *Interior support*	$F \neq 0$ and continuous $M = 0$  *Interior hinge*
6	$\theta \neq 0$ and discontinuous $\Delta \neq 0$  *Interior hinge*	$F \neq 0$ and discontinuous $M \neq 0$  *Interior support*

Table 2.2

Sl. No	*Original beams*	*Conjugate beams*
1		
2		
3		
4		
5		
6		
7		
8		

At fixed end, in the original beam, rotation and deflections are zero. Hence in the corresponding conjugate beam shear force and bending moment should be zero. This end condition exists at the free end. Hence wherever there is fixed end condition in the original beam, there will be free end condition in the conjugate beam. Various end conditions in the original beam and the corresponding end condition in the conjugate beam are shown in Table 2.1. The original beams and the corresponding conjugate beams are shown in Table 2.2

2.7 CONJUGATE BEAM METHOD

This method is very convenient to find deflections of various beams. The following illustration shows the application of this method to various determinate structures.

Sign convention Sagging moment is positive moment. Left side upward force or right side downward force (↑ ↓) gives positive shear. Hence, positive shear gives clockwise rotation and positive moment gives downward deflection.

Example 2.7 Determine θ_A, θ_B, θ_C and deflection Δ_C in the beam shown in Fig.2.11(a)

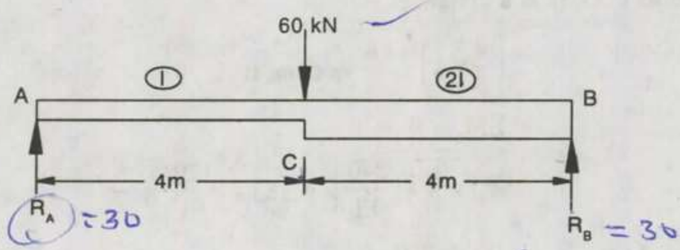


Figure 2.11 (a)

Solution

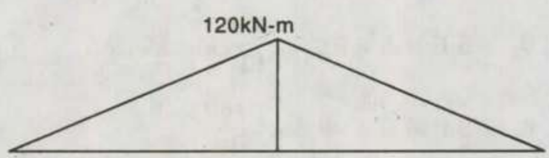


Figure 2.11 (b) B.M.D

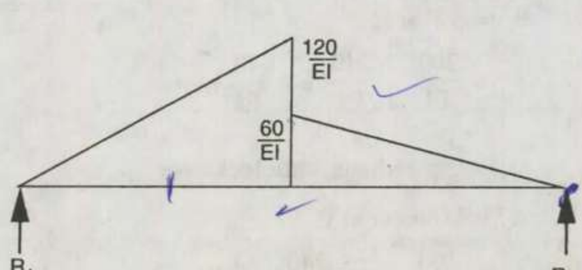


Figure 2.11(c) Conjugate beam

Bending moment diagram for the given beam is shown in Fig.2.11(b). Hence the conjugate beam with its loading $\left(\frac{M}{EI}\text{diagram}\right)$ is as shown in Fig.2.11(c).

Load between A and C is

$$\frac{1}{2}\times\frac{120}{EI}\times 4=\frac{240}{EI}$$

and its C.G. is at

$$\left(4+\frac{4}{3}\right)=5.333 \text{ m from B.}$$

Load between C and B is

$$\frac{1}{2}\times\frac{60}{EI}\times 4=\frac{120}{EI}$$

and its C.G. is at a distance

$$\left(\frac{2}{3}\times 4\right)=\frac{8}{3} \text{ m from B}$$

$$\therefore \quad \Sigma M_B = 0 \rightarrow$$

$$R_A'\times 8=\frac{240}{EI}\times 5.333+\frac{120}{EI}\times\frac{8}{3}$$

$$R_A'=\frac{200}{EI}\text{ units}$$

$$\therefore \quad R_B'=\frac{240}{EI}+\frac{120}{EI}-\frac{200}{EI}=\frac{160}{EI}\text{ units.}$$

$$\theta_A = \text{S.F at A} = R_B' = \frac{200}{EI}\text{, clockwise.}$$

$$\theta_B = \text{S.F at B} = -R_B' = -\frac{160}{EI}$$

$$=\frac{160}{EI}\text{ radians, anticlockwise}$$

$$\theta_C = \text{S.F at C}$$

$$=\frac{200}{EI}-\frac{240}{EI}=-\frac{40}{EI}\text{ units.}$$

$$=\frac{40}{EI}\text{ radians, anticlockwise}$$

$$\Delta_C = \text{Moment at C}$$

$$=\frac{200}{EI}\times 4-\frac{240}{EI}\times\frac{4}{3}$$

$$= \frac{480}{EI}, \text{ downward}$$

Example 2.8 Determine the rotations at A, B, C, E and deflections at C, D and E in the beam shown in Fig. 2.12(a).

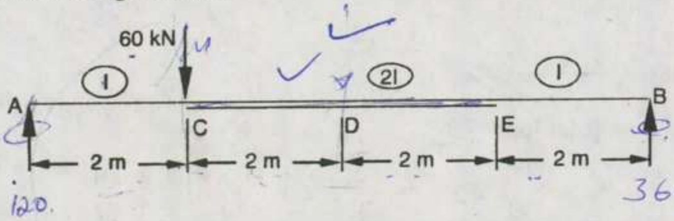


Figure 2.12 (a)

Solution

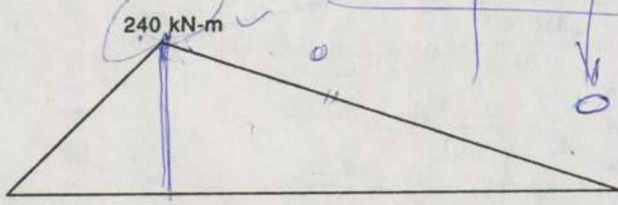


Figure 2.12 (b) B.M.D

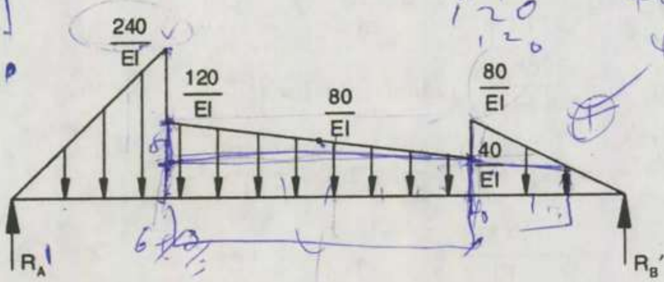


Figure 2.12 (c) Conjugate beam

Bending moment diagram for the given beam is shown in Fig. 2.12(b) and hence conjugate beam with its loading $\left(\frac{M}{EI}\text{diagram}\right)$ is as shown in Fig. 2.12(c).

The load in portion CE is split into a triangular load of maximum intensity $= \frac{120}{EI} - \frac{40}{EI} = \frac{80}{EI}$ and a udl of $\frac{40}{EI}$ per metre length.

$\therefore \quad \Sigma M_B = 0 \rightarrow$

$$R_A' \times 8 = \frac{1}{2} \times \frac{240}{EI} \times 2\left(6+\frac{2}{3}\right) + \frac{1}{2}\frac{80}{EI} \times 4\left(2+\frac{2}{3}\times 4\right)$$

$$+ \frac{40}{EI}\times 4 \times 4 + \frac{1}{2}\frac{80}{EI} \times 2 \times \frac{4}{3}$$

$\theta \leftarrow$ $R_A' = \dfrac{386.667}{EI}$

$\therefore \quad R_B' = \text{Total load} - R_A'$

$$= \frac{1}{2} \times \frac{240}{EI} \times 2 + \frac{1}{2}\frac{80}{EI} \times 4 + \frac{40 \times 4}{EI} + \frac{1}{2}\frac{80}{EI} \times 2 - \frac{386.667}{EI}$$

$$= \frac{253.333}{EI}$$

$\theta_A = F_A = R_A'$

$= \dfrac{386.667}{EI}$ radians, anticlockwise

$\theta_B = F_B' = -R_B' = -\dfrac{253.333}{EI}$

$= \dfrac{253.333}{EI}$ radians, anticlockwise

$\theta_C = F_C \quad = \dfrac{386.667}{EI} - \dfrac{1}{2} \times \dfrac{240}{EI} \times 2$

$= \dfrac{146.667}{EI}$, radians, clockwise

$\theta_E \quad = -R_B' + \dfrac{1}{2} \times \dfrac{80}{EI} \times 2 = -\dfrac{253.333}{EI} + \dfrac{80}{EI}$

$= -\dfrac{173.333}{EI}$

$= \dfrac{173.333}{EI}$ radians, clockwise

Δ_C = Moment at C in conjugate beam.

ans $= \dfrac{386.667}{EI} \times 2 - \dfrac{1}{2} \times \dfrac{240}{EI} \times 2 \times \dfrac{2}{3}$

$= \dfrac{613.333}{EI}$, downward

Δ_E = Moment at E

$= \dfrac{253.333}{EI} \times 2 - \dfrac{1}{2} \times \dfrac{80}{EI} \times 2 \times \dfrac{2}{3}$

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-eom

$$= \frac{453.333}{EI}, \text{ downward}$$

Δ_D = Moment at D

$$= \frac{253.333}{EI} \times 4 - \frac{1}{2} \times \frac{80}{EI} \times 2 \times \left(2 + \frac{2}{3}\right) - \frac{40}{EI} \times 2 \times 1 - \frac{1}{2} \times \left(\frac{80}{EI} - \frac{40}{EI}\right) \times 2 \times \frac{2}{3}$$

$$= \frac{693.333}{EI}, \text{ downward}$$

Example 2.9 Determine the deflection and slope at quarter point of simply supported beam of span L subjected to an udl w per unit length.

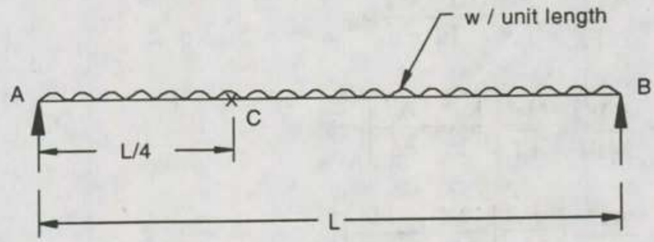


Figure 2.13 (a) Original beam

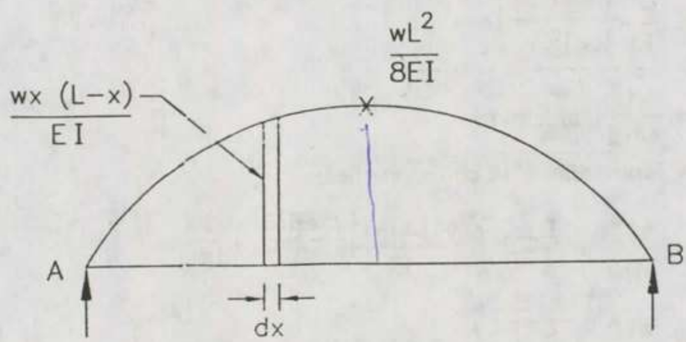


Figure 2.13 (b) Conjugate beam

Solution

Fig.2.13(a) and 2.13(b) show the original beam and its conjugate beam.

At any distance x from support A,

$$M = wx(L - x),$$

$\therefore$ Load on conjugate beam $= \frac{1}{EI} w(Lx - x^2)$

Total load on conjugate beam

$$= \text{Area of } \frac{M}{EI} \text{ diagram}$$

$= \frac{2}{3} \times$ Area of rectangle enclosing the parabolic $\frac{M}{EI}$ diagram

$$= \frac{2}{3} \times \frac{wL^2}{8EI} \times L = \frac{wL^3}{12EI}$$

$\therefore \quad R_A' = R_B' = \frac{1}{2}$ Total load

$$= \frac{wL^3}{24EI}$$

θ_C = S.F. at C in conjugate beam

$$= \frac{wL^3}{24EI} - \int_0^{L/4} \frac{w(Lx - x^2)}{EI} dx$$

$$= \frac{wL^3}{24EI} - \frac{w}{EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^{L/4}$$

$$= \frac{wL^3}{24EI}\left[\frac{1}{24} - \left(\frac{1}{32} - \frac{1}{64 \times 3}\right)\right]$$

$$= \frac{w^3}{EI}\left[\frac{8-6+1}{192}\right]$$

$$= \frac{wL^3}{64EI}, \text{ clockwise}$$

Δ_C = Moment at C in conjugate beam

$$= \frac{wL^3}{24EI} \times \frac{L}{4} - \int_0^{L/4} \frac{w(Lx - x^2)}{EI}\left(\frac{L}{4} - x\right) dx$$

$$= \frac{wL^3}{24EI} \times \frac{L}{4} - \frac{w}{4EI}\int (L^2x - 5Lx^2 + 4x^3)\, dx$$

$$= \frac{wL^3}{24EI} \times \frac{L}{4} - \frac{w}{4EI}\left[\frac{L^2x^2}{2} - \frac{5Lx^3}{3} + x^4\right]_0^{L/4}$$

$$= \frac{wL^4}{EI}\left[\frac{1}{96} - \frac{1}{4}\left(\frac{1}{32} - \frac{5}{3 \times 64} + \frac{1}{256}\right)\right]$$

$$= 8.138 \times 10\text{–}3\ \frac{wL^4}{EI}$$

Example 2.10 Determine the slope and deflections at B and C in the cantilever beam shown in Fig.2.14(a).

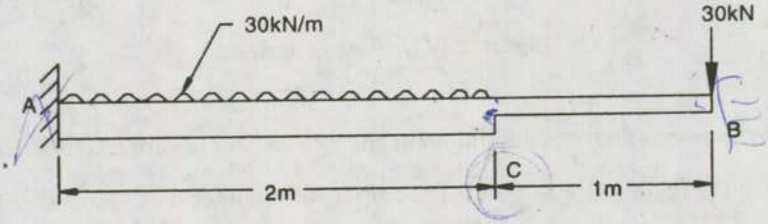


Figure 2.14 (a)

Solution

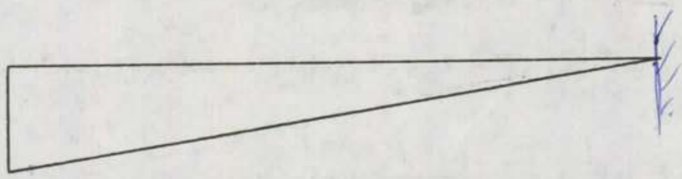


Figure 2.14 (b) B.M.D due to concentrated load

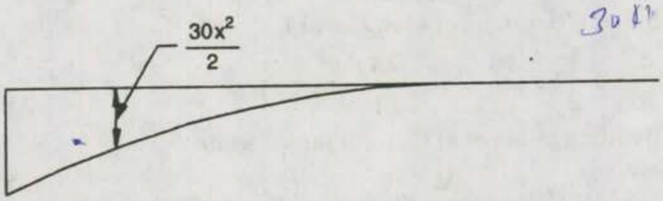


Figure 2.14 (c) B.M.D due to udl

θ_c = S.F

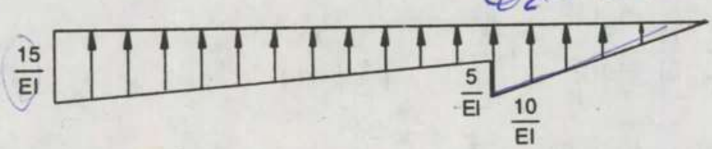


Figure 2.14 (d) $\frac{M}{EI}$ ***diagram due to concentrated load***

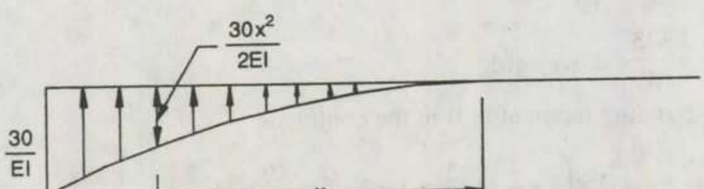


Figure 2.14 (e) $\frac{M}{EI}$ ***diagram due to concentrated load***

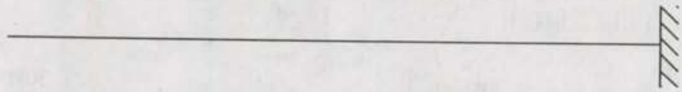


Figure 2.14 (f) Conjugate beam

Bending moment and $\frac{M}{EI}$ diagrams for the two load cases are drawn separately as shown in Fig.2.14 (b,c and d,e). The conjugate beam for the cantilever is shown in Fig.2.14 (f).

θ_C = S.F. in conjugate beam

$= \left(\frac{15}{EI}+\frac{5}{EI}\right)\frac{1}{2}\times 2 + \frac{1}{3}\times$ rectangle enclosing parabola, shown in Fig.2.14(e).

$= \frac{20}{EI} + \frac{1}{3} \times \frac{30}{EI} \times 2 = \frac{40}{EI}$, clockwise

θ_B = S.F. at B in conjugate beam

= S.F at C + Load between C and B

$= \frac{40}{EI} + \frac{1}{2} \times \frac{10}{EI} \times 1 = \frac{45}{EI}$, clockwise

Δ_C = Bending moment at C in conjugate beam

= B.M. at C due to $\frac{M}{EI}$ diagram in Fig. 2.14(d) and due to $\frac{M}{EI}$ diagram in Fig. 2.14(e).

$$= \left[\frac{1}{2}\left(\frac{15}{EI}-\frac{5}{EI}\right)2\times\frac{4}{3}+\frac{5}{EI}\times 2\times 1\right] + \int_0^2 \frac{30x^2}{2EI}\, x\, dx$$

$$= \frac{40}{3EI} + \frac{10}{EI} + \frac{15}{EI}\left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{40}{3EI} + \frac{10}{EI} + \frac{60}{EI}$$

$$= \frac{83.33}{EI}, \text{ downward}$$

Δ_B = Bending moment at B in the conjugate

$$\left[\frac{1}{2}\left(\frac{15}{EI}-\frac{5}{EI}\right)2\times\left(1+\frac{4}{3}\right)+\frac{5}{EI}\times 2\times 2+\frac{1}{2}\times\frac{10}{EI}\times 1\times\frac{2}{3}\right] + \int_0^2 \frac{30x^2}{2EI}\,(x+1)\, dx$$

$$= \frac{46.667}{EI} + \frac{15}{EI}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^2$$

$$= \frac{146.667}{EI}, \text{ downward.}$$

Example 2.11 Find the rotation and deflection at the free end in the overhanging beam shown in Fig.2.15(a).

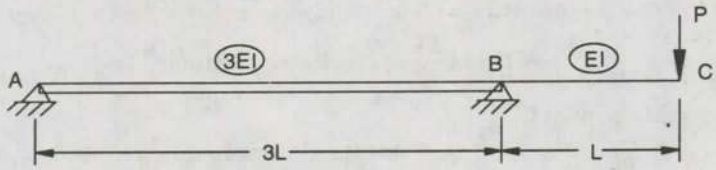


Figure 2.15 (a)

Solution

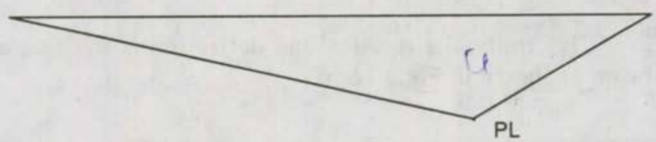


Figure 2.15 (b)

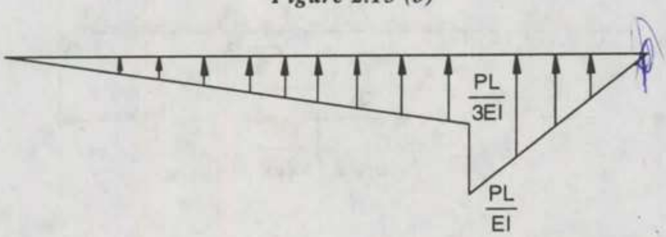


Figure 2.15 (c)

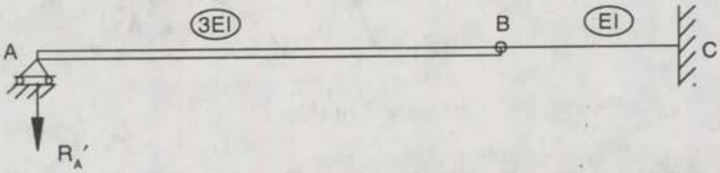


Figure 2.15 (d)

Fig.2.15(b) shows the bending moment diagram for the given beam. The load on conjugate beam is shown in Fig.2.15(c) and the conjugate beam in Fig.2.15(d).

$$= \Sigma M_B = 0 \rightarrow$$

$$R_A' \times 3L = \frac{1}{2} \times 3L \times \frac{PL}{3EI} \times L$$

$$R_A' = \frac{PL^2}{6EI}\text{, downward}$$

$$\theta_C = \text{S.F at C.}$$

$$= -\frac{PL^2}{6EI} + \frac{1}{2} \times 3L \times \frac{PL}{3EI} + \frac{1}{2} \times L \times \frac{PL}{EI}$$

$$= 0.833\frac{PL^2}{EI}\text{, clockwise rotation}$$

$$\Delta_C = \text{Moment at C}$$

$$= -\frac{PL^2}{6EI} \times 4L + \frac{1}{2} \times 3L \times \frac{PL}{3EI} \times 2L + \frac{1}{2} \times \frac{PL}{EI} \times L \times \frac{2L}{3}$$

$$= 0.667\frac{PL^3}{EI}\text{, downward.}$$

Example 2.12 Determine the rotation and deflection at the free end in the overhanging beam as shown in Fig.2.16(a).

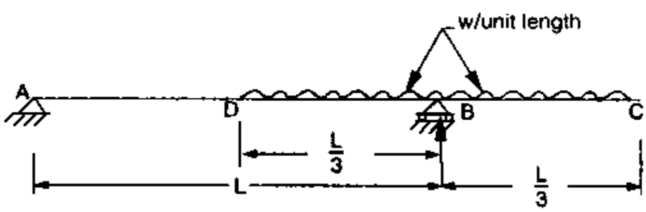


Figure 2.16 (a)

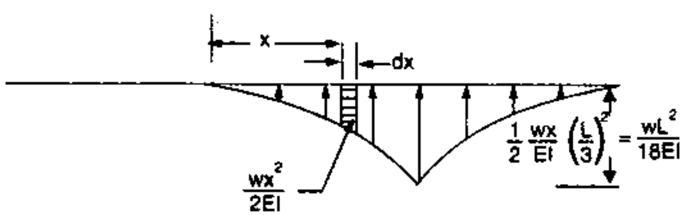


Figure 2.16 (b)

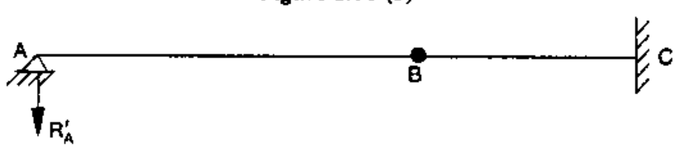


Figure 2.16 (c) *Conjugate beam*

Solution

In the given beam,

$$= \Sigma M_B = 0, \rightarrow R_A = 0$$

B.M.D. is a concave parabola.

$\frac{M}{EI}$ diagram and conjugate beams are as shown in Fig.2.16(b) and Fig.2.16(c), respectively.

Now the total load on conjugate beam

$$= \frac{1}{3} \times \frac{2L}{3} \times \frac{wL^2}{18EI} = \frac{wL^3}{81EI}$$

and its C.G. is at B.

$\therefore$ $\Sigma M_B = 0$ in conjugate beam gives,

$$R_A' \times L = \int_0^{L/3} \frac{wx^2}{2EI}\left(\frac{L}{3} - x\right) dx$$

$$= \frac{w}{2EI}\left[\frac{Lx^3}{3 \times 3} - \frac{x^4}{4}\right]_0^{L/3}$$

$$= \frac{wL^4}{2EI}\left(\frac{1}{9 \times 27} - \frac{1}{4 \times 81}\right)$$

$$R_A' = \frac{wL^3}{1944EI}$$

At the free end,

$$\theta_C = F_C = -R_A + \text{total load}$$

$$= \frac{1}{3} \times \frac{2L}{3} \times \frac{wL^2}{18EI} = \frac{wL^3}{81EI}$$

$$= -\frac{wL^3}{1944EI} + \frac{wL^3}{81EI} = \frac{wL^3}{1944EI}[-1 + 24]$$

$$= \frac{23}{1944EI} wL^3, \text{ clockwise}$$

$$\Delta_C = \text{Moment at C in the conjugate beam}$$

$$= -\frac{wL^3}{1944EI} \times \left(L + \frac{L}{3}\right) + \frac{wL^3}{81EI}\frac{L}{3}$$

$$= \frac{wL^4}{1944 \times 3EI}[-4 + 24]$$

$$= \frac{5}{1458EI} wL^4, \text{ downward}$$

Example 2.13 Determine the rotation at A and deflection at C in the overhanging beam shown in Fig.2.17(a).

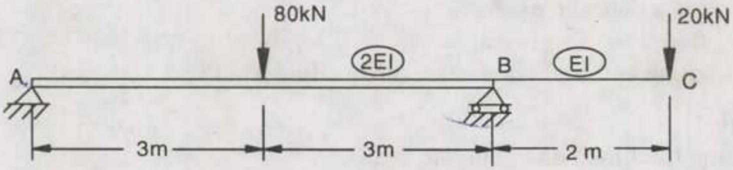


Figure 2.17 (a)

Solution

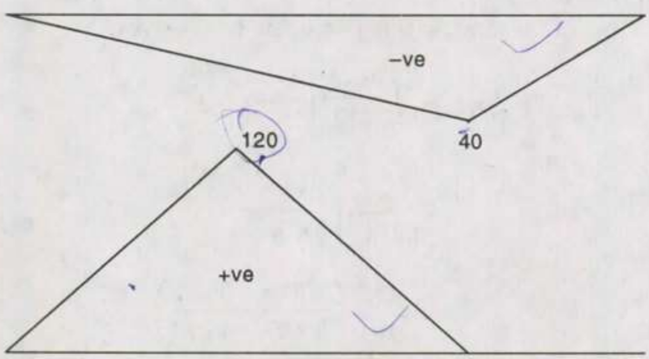


Figure 2.17 (b) B.M.D

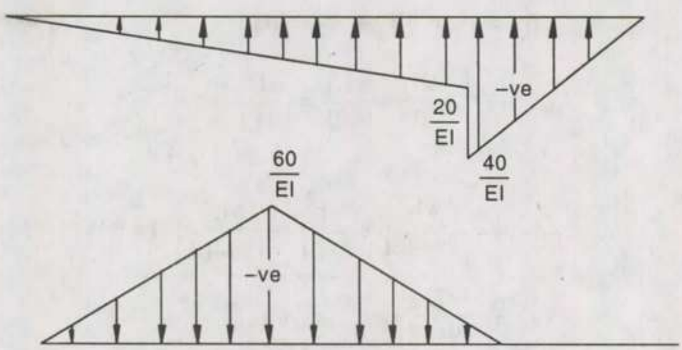


Figure 2.17 (c) $\frac{M}{EI}$ *diagram*

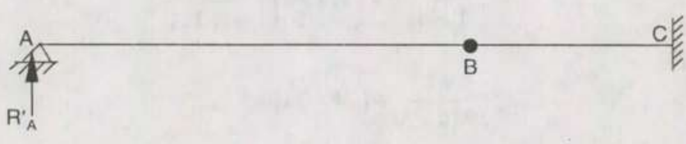


Figure 2.17 (d) *Conjugate beam*

Bending moment diagrams, $\frac{M}{EI}$ diagrams and the conjugate beams are shown in Fig.2.17(b,c and d) respectively.

Note : It will be convenient to handle B.M.D due to each load separately.

$$\Sigma M_B = 0,$$

$$R_A' \times 6 = \frac{1}{EI}\left[\frac{1}{2}\times\frac{20}{EI}\times 6\times 2+\frac{1}{2}\times\frac{60}{EI}\times 6\times 3\right]$$

$$R_A' = \frac{110}{EI}$$

$$\therefore \quad \theta_A = \frac{110}{EI}\text{, clockwise}$$

Deflection at C

= Moment at C in conjugate beam.

$$= \frac{110}{EI}\times 8 - \frac{1}{2}\times\frac{60}{EI}\times 6\times 5+\frac{1}{2}\times 6\times\frac{20}{EI}\times 4+\frac{1}{2}\times\frac{40}{EI}\times 2\times 1$$

$$= \frac{260}{EI}\text{, downward.}$$

EXERCISES

2.1 Determine the rotation and deflections at B and C in the cantilever beam shown in Fig.2.18 by moment area method.

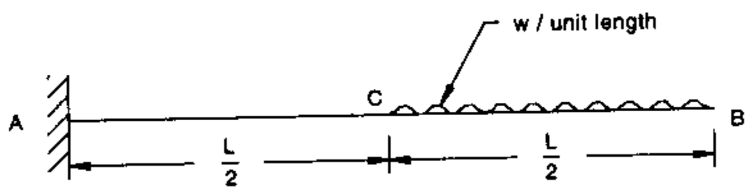


Figure 2.18

Ans : $\theta_B = \frac{7}{48}\frac{wL^3}{EI}; \theta_C = \frac{1}{8}\frac{wL^3}{EI}; \Delta_B = \frac{41}{384}\frac{wL^4}{EI}; \Delta_C = \frac{7}{192}\frac{wL^4}{EI}$

2.2 Determine the rotation at A and deflections under concentrated load at mid-span in the beam shown in Fig.2.19 by moment area method.

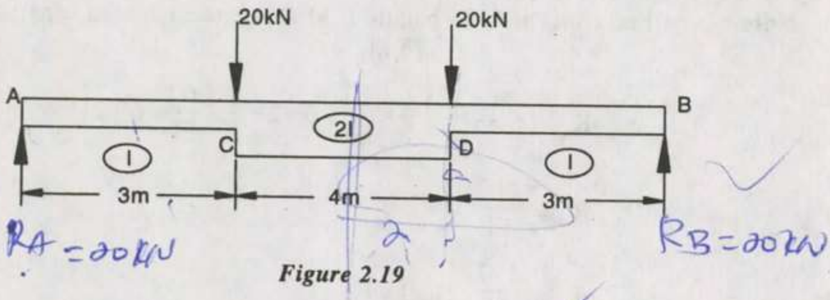


Figure 2.19

$$Ans: \theta_A = \frac{150}{EI};\ \Delta_C = \frac{360}{EI};\ \Delta_E = \frac{420}{EI}$$

2.3 Determine the deflection under the concentrated load and the maximum deflection in the beam as shown in Fig.2.20 using conjugate beam method.

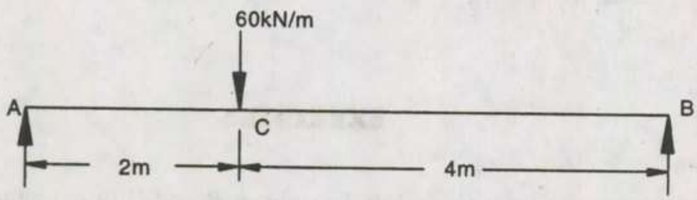


Figure 2.20

$$Ans: \Delta_C = \frac{213.333}{EI};\ \Delta_{max} = \frac{232.25}{EI}$$

2.4 Using conjugate beam method, determine the deflection and rotation at the free end in the beam as shown in Fig.2.21.

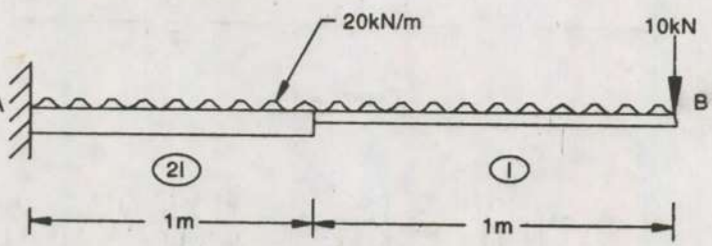


Figure 2.21

$$Ans: \theta_B = \frac{30.833}{EI},\ \Delta_B = \frac{36.25}{EI}$$

2.5 Determine the rotation at A and deflection at C in the overhanging beam shown in Fig.2.22.

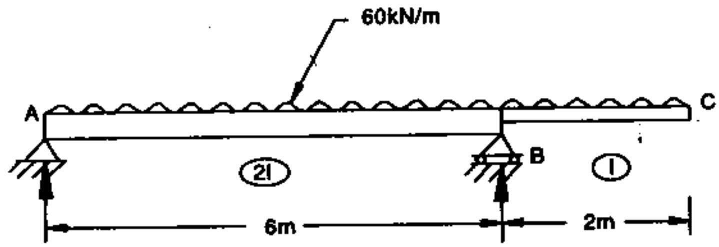


Figure 2.22

Ans : $\theta_A = \dfrac{210}{EI}$, $\Delta_C = \dfrac{420}{EI}$, upward

DEFLECTION OF BEAMS AND RIGID FRAMES BY ENERGY METHODS

3

3.1 INTRODUCTION

When an external load acts on a structure, the structure undergoes deformation by the external force and hence the work is done. To resist these external forces, the internal forces develop gradually from zero to their final value and the work is done. This internal work done is stored as energy in the structure and it helps the structure to spring back to the original shape and size, whenever the external loads are removed, provided the material of the structure is still within the elastic limit. *This internal work, which is stored as energy is due to the straining of the material and hence is called strain energy.*

When equilibrium is reached, as per the well known law of conservation of energy, the work done by the external forces must equal the strain energy stored. This concept of energy balance is utilized in structural analysis to develop a number of methods to find the deflections of structures. This chapter deals with the following methods for finding the deflections of beams and frames:

a. Strain energy /Real work method
b. Virtual work unit load method
c. Castigliano's method.

3.2 STRAIN ENERGY AND COMPLEMENTARY ENERGY

Consider the general structural system shown in Fig.3.1

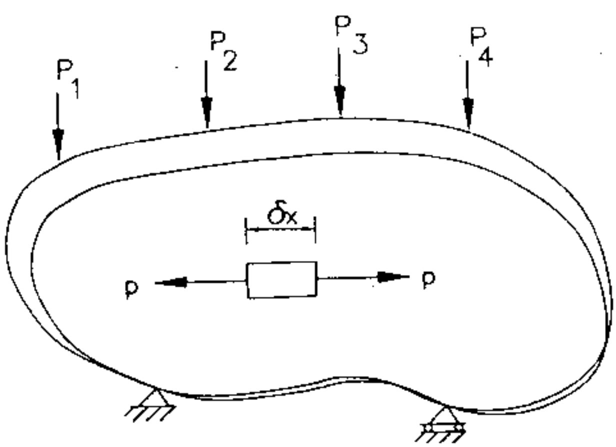


Figure 3.1

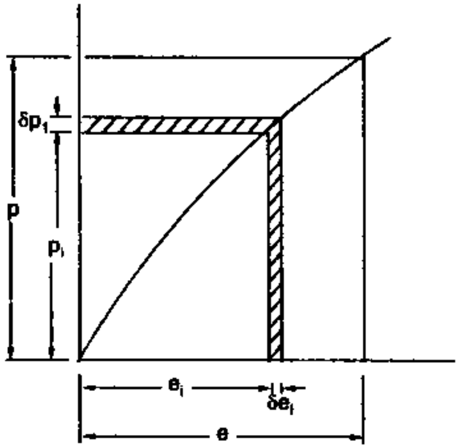


Figure 3.2

Let the cross-sectional area of the element shown in Fig. 3.1 be δa and its length δx. The stress in the element gradually increases from zero to its final value p as strain increases from zero to its final value 'e'. Let this stress-strain relation be as shown in Fig. 3.2.

Let stress be p_i, hence work done on the element, when strain δe_i takes place

$$= \text{Force} \times \text{Displacement}$$

$$= p_i\, \delta a\, \delta e_i\, \delta x$$

$$= p_i\, \delta e_i\, \delta v$$

$\delta a\, \delta x = \delta v$, where v is the volume of the element

∴ Strain energy stored in the element.

$$= \int_0^e p_i\, \delta e_i\, \delta v$$

$$= \text{Area under stress-strain curve} \times dv \qquad 3.1$$

If the stress–strain curve is linear, strain energy of the element,

$$= \frac{1}{2} pe\ dv,$$

∴ Strain energy stored in the structure

$$U = \int \frac{1}{2}\ p\ e\ dv,$$

$$= \int \frac{1}{2} \times \text{stress} \times \text{strain} \times dv \qquad 3.2$$

Fig. 3.3 shows load vs deformation relation. Let, during deformation the load acting be P_i and deformation be $\delta\Delta_i$. Then, work done by the load under consideration,

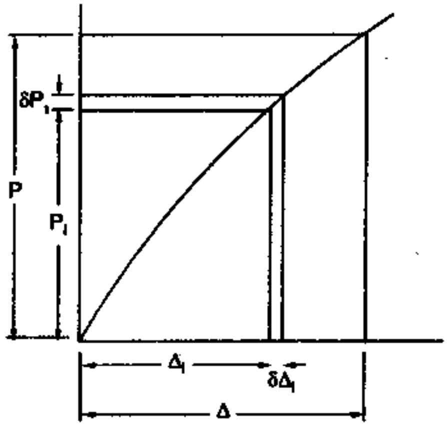


Figure 3.3

$$= \int P_i\ \delta\Delta i$$

= Area under the load deformation curve 3.3

$$= \frac{1}{2} \times P\Delta, \text{ in case of linear elasticity problems}$$

If there are 'n' number of loads, total work done by the external loads is the summation of the expression 3.3 for all the loads.

Hence, in case of a linear elasticity (linear stress-strain curve and material still within elastic limit) work done by external loads,

$$= \int_0^n \frac{1}{2} P\ \Delta$$

Complementary energy at any instant during deformation of the element is given by $e_i\ \delta p_i\ dv$. Hence, the complementary energy of the element when final deformation takes place,

$$\text{C.E.} = \int_0^{p_f} e_i \delta p_i\ dv$$

= Area above the stress-strain curve × dv

$$= \int \frac{1}{2} p\ e\ dv, \text{ in case of linear elasticity problems}$$

Thus complementary energy of the entire structure

$$U_c = \int \frac{1}{2}\ p\ e\ dv, \text{ in case of linear elasticity problems} \qquad 3.5$$

From equations 3.2 and 3.5 we conclude, in case of linear elasticity problems, strain energy and complementary energy are equal to each other.

Similarly, complementary work done is given by

$$= \int_0^{P_i} \Delta_i \, \delta P_i$$

In case of linear elasticity problems, work done

$$= \frac{1}{2} \Delta \, P$$

If there are 'n' number of loads, work done by external loads

$$= \sum \frac{1}{2} \Delta \, P$$

3.3 STRAIN ENERGY DUE TO BENDING

Consider the beam shown in Fig. 3.4 subject to pure bending. Let the area of the element be δa and its distance from the neutral axis be 'y'. From the flexure theorem the bending stress 'f' is given by

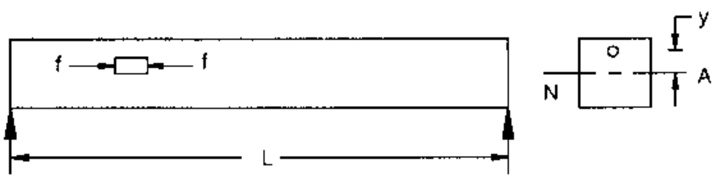


Figure 3.4

$$f = \frac{M}{I} y, \text{ where M is the bending moment}$$

I is the moment of inertia of the section.

$\therefore$ Strain, $e = \frac{f}{E} = \frac{M}{EI} y$, where E is Young's Modulus

Then, Strain Energy in the element

$$= \frac{1}{2} \times \text{Stress} \times \text{Strain} \times dv$$

$$= \frac{1}{2} \times \frac{M}{I} y \times \frac{M}{EI} y \times dv$$

$$= \frac{1}{2} \times \frac{M^2 y^2}{EI^2} dv$$

$\therefore$ Strain energy in the beam,

$$= \int \frac{1}{2} \times \frac{M^2 y^2}{EI^2} \, dv$$

$$= \int_0^L \int_0^A \frac{1}{2} \times \frac{M^2 y^2}{EI^2} \, \delta a \, dx, \text{ where A is the area of cross-section.}$$

$$= \int_0^L \frac{M^2}{2EI^2} \left(\int_0^A y^2 \, \delta a \right) dx$$

$$= \int_0^L \frac{M^2}{2EI^2} \, I \, dx, \text{ since } \int_0^A y^2 \, \delta a = I$$

$$U = \int_0^L \frac{M^2}{2EI} \, dx$$

Apart from bending strain energy, shear strain energy also takes place. Researches have proved that shear strain energy is very small compared to strain energy due to flexure. Hence, in this text shear strain energy is neglected and only strain energy due to pure bending is considered.

3.4 DEFLECTION BY STRAIN ENERGY METHOD

This method also called 'real work method', since work done by the actual loads are considered. From the law of conservation of energy, we can say,

Strain energy U = Real work done by loads

$$U = \sum_0^n \frac{1}{2} P\Delta \qquad 3.8$$

The equation 3.8 can be used to find deflection in beams and frames subject to bending stresses. Following are the illustrated problems using this method.

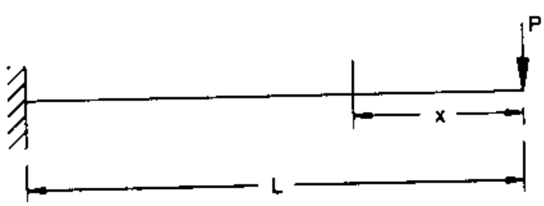


Figure 3.5

Example 3.1 Using strain energy method determine the deflection of the free end of a cantilever of length L subjected to a concentrated load P at the free end.

Solution

The bending moment at a distance x from the free end is,

$$M = Px$$

$$\therefore \qquad S.E = \int_0^L \frac{M^2}{2EI}dx$$

$$= \int_0^L \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{P^2L^3}{6EI}$$

Work done by the load = $\frac{1}{2}P\Delta$, where Δ is the deflection of the free end.

$\therefore$ From conservation of energy,

S.E = Work done by external loads

$$\frac{P^2L^3}{6EI} = \frac{1}{2}P\Delta$$

$$\text{or} \qquad \Delta = \frac{PL^3}{3EI}$$

Example 3.2 Determine the deflection under 60 kN load in the beam shown in Fig.3.6.

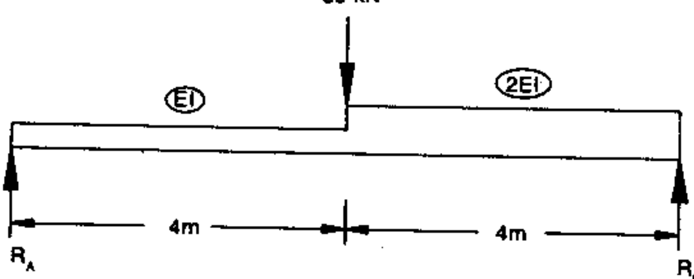


Figure 3.6

Solution

Reaction $R_A = R_B = 30$ kN

$\therefore$ Bending moment at any distance x from A or at a distance x from B

= 30x kN

$$\therefore \qquad S.E = \int_0^4 \frac{(30x)^2}{2EI}dx + \int_0^4 \frac{(30x)^2}{2\times 2EI}dx$$

$$= \frac{3}{4} \times \frac{900}{EI} \int_0^4 x^2\,dx$$

$$= \frac{3}{4} \times \frac{900}{EI} \left[\frac{x^3}{3}\right]_0^4 = \frac{3}{4} \times \frac{900}{EI} \times \frac{4^3}{3}$$

$$U = \frac{14400}{EI}$$

Work done by the load:

$$W_E = \frac{1}{2} \times P\Delta = \frac{1}{2} \times 60 \times \Delta$$

Equating strain energy of the beam to the work done by load; we get,

$$\frac{14400}{EI} = \frac{1}{2} \times 60 \times \Delta$$

$$\Delta = \frac{480}{EI}$$

Example 3.3 Determine the vertical deflection of point C in the frame shown in Fig.3.7. Given $E = 200\ kN/mm^2$ and $I = 30 \times 10^6\ mm^4$.

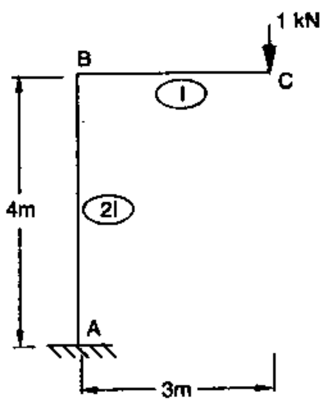


Figure 3.7 (a)

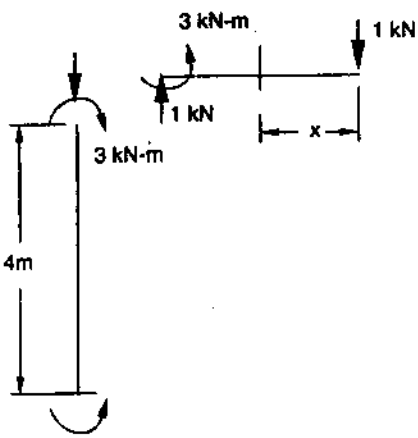


Figure 3.7 (b)

Solution

Free body diagram for various portion of the structure are shown in Fig.3.7(b) and the details of bending moment expressions for various portion of the structures are noted below :

Portion	*Origin*	*Limit*	*Expression*
BC	C	0 – 3	1x = x
AB	B	0 – 4	3

$$\therefore \qquad S.E = \int_0^3 \frac{(x)^2}{2EI}\,dx + \int_0^4 \frac{(3)^2}{2E\times 2I}\,dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{4EI}\left[9x\right]_0^4$$

$$= \frac{1}{6EI}\times 3^3 + \frac{1}{4EI}\times 9\times 4$$

$$= \frac{13.5}{EI}$$

$$\text{Work done} = \frac{1}{2}\times 1\times \Delta = \frac{\Delta}{2}$$

Equating work done to strain energy, we get

$$\frac{\Delta}{2} = \frac{13.5}{EI}$$

$$\Delta = \frac{27}{EI}$$

Noting the units used in strain energy (bending moment) i.e., kN and metres, EI should be used in kN–m^2

$$EI = 200\times 30\times 10^6\times 10^{-6} = 6000 \text{ kN–m}^2$$

$$\Delta = \frac{27}{6000} \text{ metres}$$

$$= 0.045 \text{ metres} = 4.5 \text{ mm}$$

Example 3.4 Determine the horizontal displacement of the roller end D of the portal frame shown in Fig.3.8, when P = 5kN. EI is 8000 kN–m² throughout.

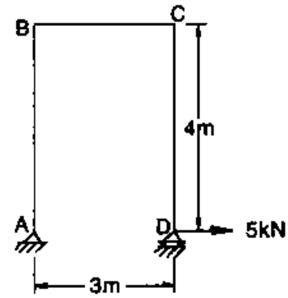


Figure 3.8 (a)

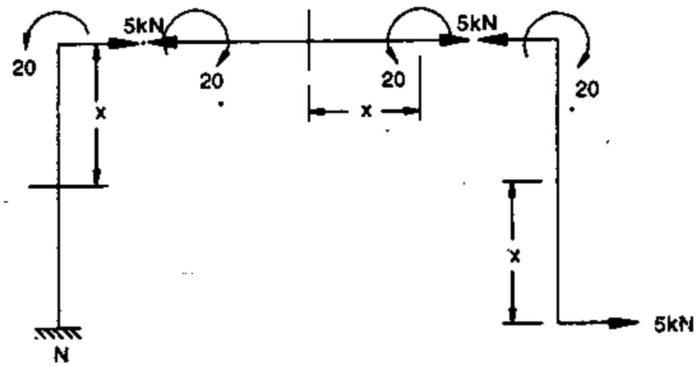


Figure 3.8 (b)

Solution

Looking at Fig.3.8(b), the bending moment expressions for various portions are noted below:

Portion	*CD*	*BC*	*AB*
Origin	D	C	B
Limit	0 – 4	0 – 3	0 – 4
Mx	5x	20	20 – 5x

$$\therefore \quad S.E = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$= \frac{1}{2EI}\left[\frac{25x^3}{3}\right]_0^4 + \frac{1}{2EI}\left[400x\right]_0^3 + \frac{1}{2EI}\left[400x - 200\frac{x^2}{2} + \frac{25x^3}{3}\right]_0^4$$

$$= \frac{266.67}{EI} + \frac{600}{EI} + \frac{1}{2EI}\left[1600 - 1600 + \frac{25\times 64}{3}\right]$$

$$= \frac{1133.33}{EI}$$

$$\text{Work done} = \frac{1}{2}\times P \times \Delta = \frac{1}{2}5\Delta = 2.5\Delta$$

Equating S.E to work done, we get,

$$2.5\Delta = \frac{1133.33}{EI}$$

$$\Delta = \frac{453.33}{EI} = \frac{453.33}{8000} = 0.0567 \text{ metres.}$$

$$= 56.7\text{mm}$$

Strain energy method can be conveniently used for finding deflection in structures only under the following situations :

1. The Structure is subjected to a single concentrated load.
2. Deflection required is at the loaded point and is in the direction of the load.

3.5 VIRTUAL WORK

Various results have shown that the strain energy / real work method is having limited use in finding displacement. The method based on virtual work principle proves to be the most versatile and powerful method. Before deriving the principle of virtual work for deformable bodies, meaning of virtual work is explained and principle of virtual displacement is derived.

Consider the body shown in Fig.3.9 subject to a set of real forces P_1, P_2, P_3, P_4,..... P_n. Let the body undergoes deformation due to some other forces as shown in Figure.

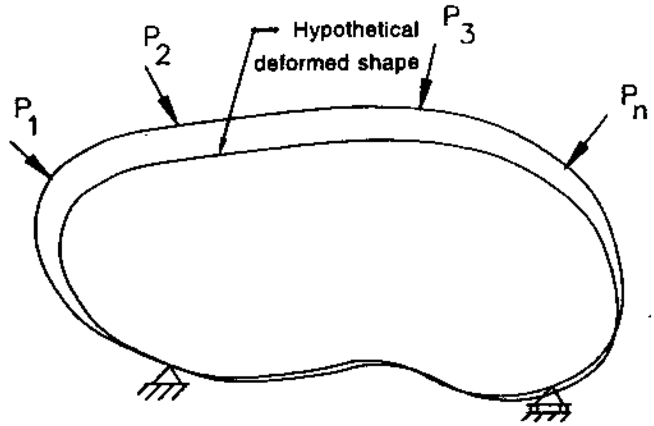


Figure 3.9

This hypothetical deformation is called virtual deformation and the work done by real forces due to virtual displacements is called virtual work. Virtual means the effect exists but actually it is not the fact. The virtual work may be defined as *the work done by real forces due to hypothetical displacements or the work done by hypothetical forces during real displacements.*

3.5.1 Bernoulli's Principle of Virtual Displacement

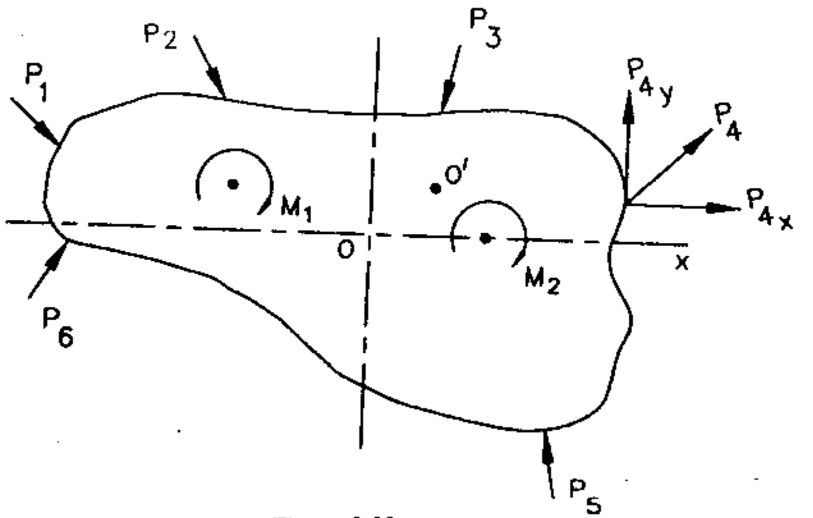


Figure 3.10

Consider the rigid body shown in Fig. 3.10 subject to P system of forces and M system of moments. Let P_x, P_y be the components of force in x and y directions. Since the body is in equilibrium,

$$\Sigma P_x = 0 \tag{a}$$

$$\Sigma P_y = 0 \tag{b}$$

$$\Sigma M + \Sigma P_x\, y + \Sigma P_y\, x = 0 \tag{c}$$

Suppose a virtual displacement OO' is given to the rigid body. Let the component of OO' in x-direction be δx and in y-direction be δy. Since it is a rigid body, all forces will have displacements 'δx' and 'δy' in x and y directions.

$$\begin{aligned}\therefore \quad \text{Virtual work done} &= \Sigma P_x\,\delta x + \Sigma P_y\,\delta y\\ &= \delta x\,\Sigma P_x + \delta y\,\Sigma P_y\\ &= 0\end{aligned}$$

Since $\Sigma P_x = \Sigma P_y = 0$, from equation (a) and (b)

$$\therefore \quad \text{Virtual work done} = 0 \tag{3.9}$$

Now consider the rotation of the rigid body by a virtual rotation '$d\theta$'. If a point Q(x,y) is at a distance 'r' from the origin, then, referring to Fig. 3.11,

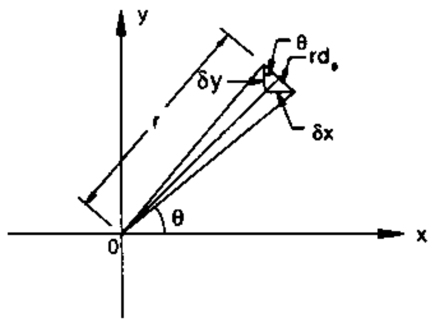


Figure 3.11

$$\begin{aligned}\delta x &= r d\theta \sin\theta\\ &= y d\theta\\ \text{and} \quad \delta y &= r d\theta \cos\theta\\ &= x d\theta\end{aligned}$$

$\therefore$ Virtual work done by real forces due to virtual displacements

$$\begin{aligned}&= \Sigma M\, d\theta + \Sigma P_x\,\delta x + \Sigma P_y\,\delta y\\ &= \Sigma M\, d\theta + \Sigma P_x\, y d\theta + \Sigma P_y\, x\, d\theta\\ &= d\theta\,(\Sigma M + \Sigma P_x\, y + \Sigma P_y\, x)\end{aligned}$$

But from equation (c), $\Sigma M + \Sigma P_x\, y + \Sigma P_y\, x = 0$

$$\therefore \quad \text{Virtual work} = 0 \tag{3.10}$$

From equation 3.9 and 3.10 we can conclude, '*If a rigid body is in equilibrium under a system of forces and / or moments, the virtual work done by this system of forces and / or moments during virtual displacement is zero.*' This principle was first observed by *Johann Bernaulli* in 1717. Hence, this is called Bernaulli's principle of virtual displacement.

3.5.2 Principle of Virtual Work for Deformable Bodies

Consider an elastic body shown in Fig.3.12 subjected to a system of forces. These forces causes real internal stresses and every element in the body is in equilibrium under the action of external forces and internal stresses.

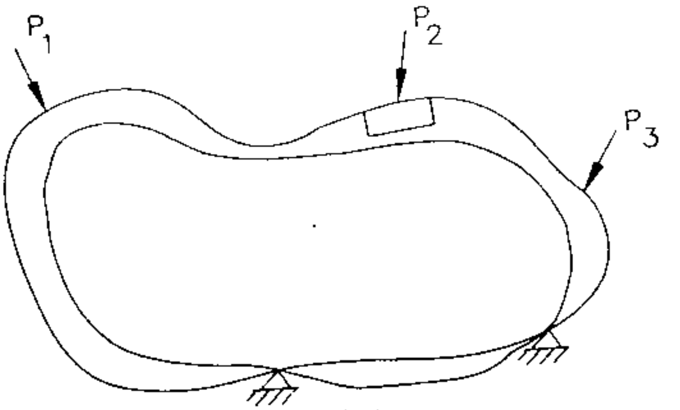


Figure 3.12

Now, imagine a virtual displacement, which means displacement due to some hypothetical force system. Let $d\omega_\theta$ be the virtual work of the external forces acting on an element. Since the body is elastic, there will be virtual displacement and also virtual deformation (strain). Hence, the external work applied to the element is dissipated in two forms:

1. The virtual work, dw_r, treating the element as a rigid body.
2. The virtual work of deformation of the element 'dw_i'. This may be called as virtual strain energy of the element also. Hence

$$d\omega_\theta = d\omega_i + d\omega_r$$

But from the principle of virtual displacement,

$$d\omega_r = 0$$

$$\therefore \quad d\omega_\theta = d\omega_i$$

Integration over entire body will give,

$$W_\theta = W_i \qquad 3.11$$

where W_θ is the total virtual work done by the force system and W_i represents the internal virtual strain energy of the entire body.

The equation 3.11 represents the principle of virtual work and may be stated as follows:

"If a deformable body in equilibrium under a system of forces is given virtual deformation, the virtual work done by the system of forces is equal to the internal virtual work done by the stresses due to that system of forces."

The above principle is free from the system causing the deformation. Hence it may be applied to all cases of deformations i.e., due to loads, temperature, settlement of support, lack of fit etc. This equation is valid for all elastic materials, irrespective of whether the system is linearly or non-linearly elastic.

3.6 UNIT LOAD METHOD

Consider the body shown in Fig. 3.13(a) which is subjected to forces P_1, P_2, P_3, P_4 ..., P_n applied gradually. Let the displacement under load at points be Δ_1, Δ_2, Δ_3, ...

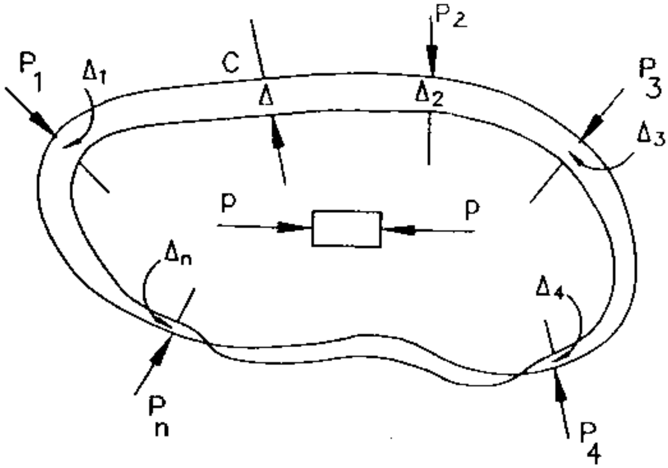


Figure 3.13 (a)

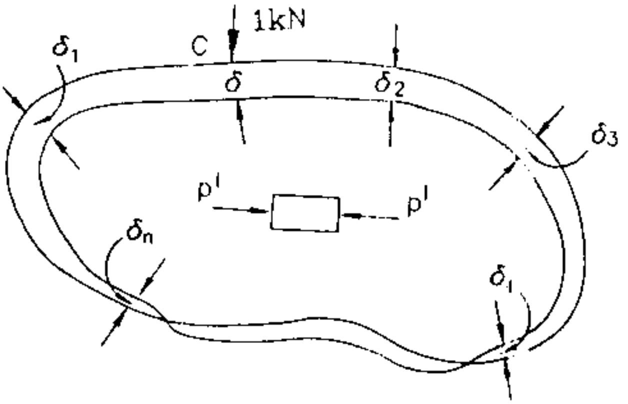


Figure 3.13 (b)

Δ_n, and at point C be Δ. Then,

$$\text{External work done} = \frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \frac{1}{2}\Delta_3 P_3 + \ldots + \frac{1}{2}\Delta_n P_n$$

and strain energy stored = $\int \frac{1}{2}$ pe dv

where p is stress and e is the strain in the element considered.

$$\therefore \qquad \frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \ldots + \frac{1}{2}\Delta_n P_n = \int \frac{1}{2} \text{pe dv} \qquad \text{(a)}$$

Now, consider the same body subjected to an unit load applied gradually at C when it is free of system of P forces. Let the displacements at 1, 2, 3, . . . n be δ_1, δ_2, δ_3, . . . δ_n respectively and the displacement at C be δ. Let the stress produced in the element be p' and the strain be e'. Then

External work done = $\frac{1}{2} \times 1 \times \delta$

Internal work done = $\int \frac{1}{2}$ p′ e′ dv

$$\therefore \qquad \frac{1}{2} \times 1 \times \delta = \int \frac{1}{2} p' e' \, dv \qquad \text{(b)}$$

Now, if P system of forces is applied to the body shown in Fig.3.13(b),

External work done = $\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \ldots + \frac{1}{2}\Delta_n P_n + 1 \times \Delta$

since, unit load is already acting,

Internal work done = $\int \frac{1}{2}$ p e dv + $\int$p′ e dv, since the stress p′ is acting throughout the deformation. Equating internal work to external work

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta P_2 + \ldots + \frac{1}{2}\Delta_n P_n + 1 \times \Delta = \int \frac{1}{2} p\, e\, dv + \int p' e\, dv \qquad \text{(c)}$$

Subtracting equation (a) from (c) we get,

$$1 \times \Delta = \int p' \, e \, dv$$

$$\Delta = \int p' \, e \, dv \qquad 3.12$$

where Δ = deflection at point where unit load is applied and is measured in the direction of unit load

p′ = stress in an element due to unit load

and e = strain in the element due to given load system.

The equation 3.12 is the basis for the unit load method.

3.7 THE UNIT LOAD METHOD-APPLICATION TO BEAM DEFLECTIONS

Consider the beam shown in Fig.3.14(a) subjected to a system of P forces.

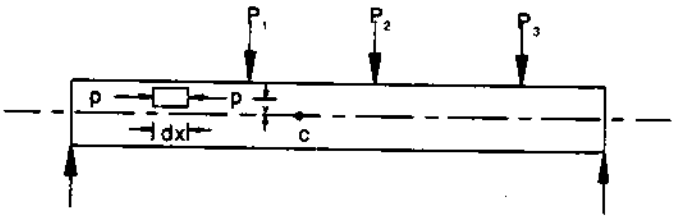


Figure 3.14 (a)

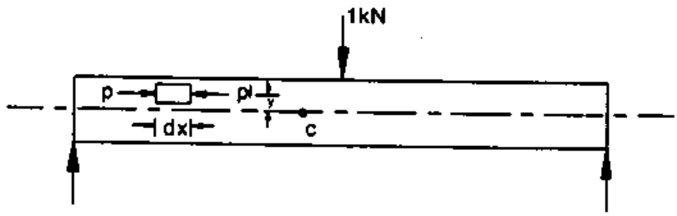


Figure 3.14 (b)

The stress in the element at distance y from neutral axis is

$$p = \frac{M}{I}y, \text{ where M is the moment acting at the section}$$

∴ Strain in the element due to given system of forces,

$$e = \frac{M}{EI}y$$

Let 'm' be the moment at the section where the element is considered, due to unit load acting at C. Then

$$\text{Stress } p' = \frac{my}{I}$$

∴ from equation (3.12),

$$\Delta = \int \frac{m}{I}y.\ \frac{M}{EI}y\ dv$$

$$\Delta = \int_0^L \frac{Mm}{EI^2}\left(\int_0^A y^2 dA\right)dx$$

$$= \int_0^L \frac{Mm}{EI^2} I dx \qquad \text{Since } \int_0^A y^2 dA = I$$

$$= \int_0^L \frac{Mm}{EI} dx \tag{3.13}$$

From the equation 3.13, deflection at any point C can be found. It needs bending moment due to a given load system and unit load acting at C. This procedure is applicable to rigid frames also, where only flexure effect is considered (i.e., in the analysis in which the effect of axial and shear forces are neglected). The procedure is illustrated with a set of problems below:

Example 3.5 Determine the deflection at free end of the overhanging beam shown in Fig. 3.15. Use unit load method.

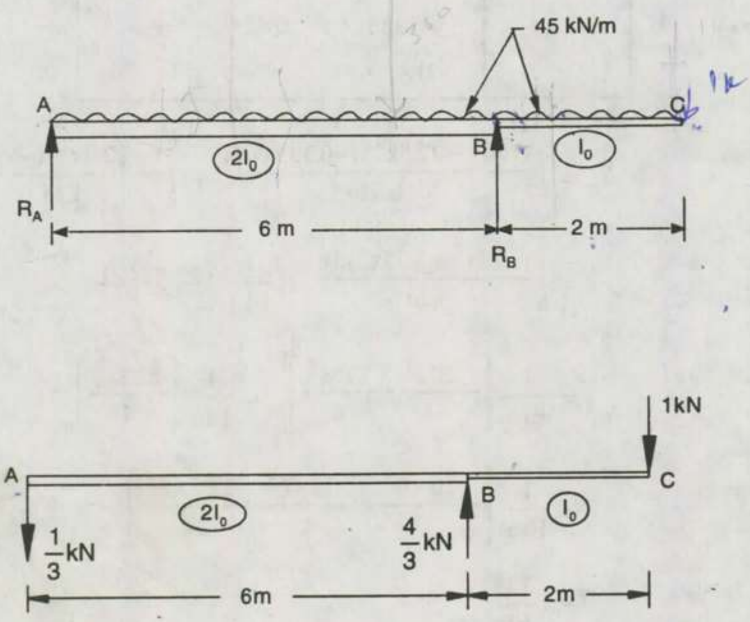


Figure 3.15

Solution

$\sum M_A = 0 \rightarrow$

$R_B \times 6 = 45 \times 8 \times 4$

$R_B = 240$ kN

$\sum v = 0 \rightarrow$

$\therefore \quad R_A = 45 \times 8 - 240 = 120$ kN

When unit load is acting at C (Fig.3.15(b))

$$R_B = \frac{1\times 8}{6} = 1.333 \text{ kN}$$

$$\therefore \qquad R_A = 0.333 \text{ kN} \downarrow$$

Taking sagging moment as positive and hogging moment as negative, expressions for moment in various portions are noted down as given below :

Portion	*AB*	*BC*
Origin	A	C
Limit	0 – 6	0 – 2
M	$120x - \frac{1}{2}\,45x^2$	$-\frac{1}{2}\,45x^2$
m	–0.333x	–x
I	2Io	Io

$$\Delta_C = \int_0^6 \frac{(120x - 22.5x^2)(-0.333x)dx}{E\,2Io} + \int_0^2 \frac{(-22.5x^2)(-x)dx}{EIo}$$

$$= \int_0^6 \frac{(-20x^2 + 3.75x^3)dx}{EIo} + \frac{1}{EIo}\int_0^2 22.5x^3\,dx$$

$$= \frac{1}{EIo}\left[-\frac{20x^3}{3} + \frac{3.75x^4}{4}\right]_0^6 + \frac{1}{EIo}\left[\frac{22.5x^4}{4}\right]_0^2$$

$$= \frac{1}{EIo}\left[-\frac{20\times 6^3}{3} + \frac{3.75\times 6^4}{4} + \frac{22.5\times 2^4}{4}\right]$$

$$= -\frac{135}{EIo}$$

$$= \frac{135}{EIo}, \text{ upward}$$

Example 3.6 Determine the deflection and rotation at the free end of the cantilever beam shown in Fig.3.16(a). Use unit load method. Given $E = 2 \times 10^5$ N/mm^2 and $I = 12 \times 10^6$mm^4.

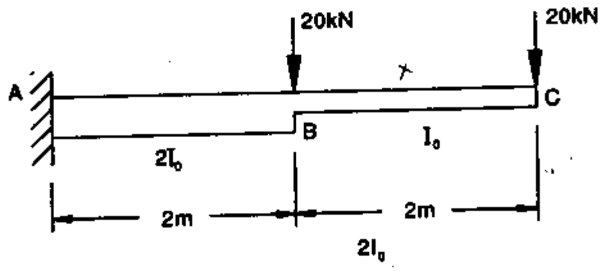


Figure 3.16 (a)

Solution

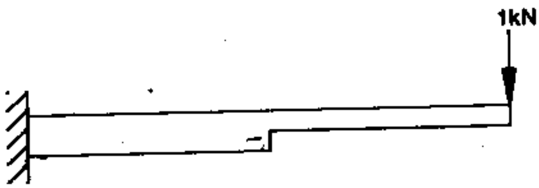


Figure 3.16 (b)

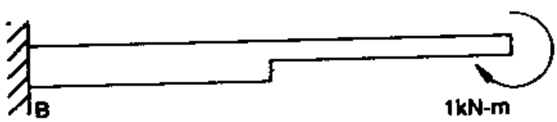


Figure 3.16 (c)

The bending moment expressions

a. M for given loads

b. m_1 for unit vertical load at C

c. m_2 for unit moment at C for various portions are tabulated below:

Portion	*CB*	*BA*
Origin	C	B
Limit	0 – 2	0 – 2
M	– 20x	– [20 (2 + x) + 20x]
m_1	– x	– (x + 2)
m_2	– 1	– 1
I	Io	2Io

$$\Delta = \int_0^L \frac{Mm_1}{EI}\,dx$$

$$= \int_0^2 \frac{(-20x)(-x)}{EIo}\,dx + \int_0^2 \frac{(20(2+x)+20x)(x+2)}{E2Io}\,dx$$

$$= \int_0^2 \frac{20x^2}{EIo}\,dx + \int_0^2 \frac{(40x+40)(x+2)}{2EIo}\,dx$$

$$= \left[\frac{20}{3}\frac{x^3}{EIo}\right]_0^2 + \frac{1}{2EIo}\left[\frac{40x^3}{3}+\frac{120x^2}{2}+80x\right]_0^2$$

$$= \frac{53.333}{EIo} + \frac{1}{EIo}[253.333]$$

$$= \frac{306.67}{EIo}$$

$$\theta_c = \int_0^L \frac{Mm_2}{EI} = \int_0^2 \frac{(20x)}{EIo}\,dx + \int_0^2 \frac{(40x+40)1}{E2Io}\,dx$$

$$= \left[\frac{20}{EIo}\frac{x^2}{2}\right]_0^2 + \frac{1}{2EIo}\left[\frac{40x^2}{2}+40x\right]_0^2$$

$$= \frac{40}{EIo} + \frac{160}{2EIo}$$

$$= \frac{120}{EIo}$$

Example 3.7 Determine the deflection at the free end of the overhanging beam shown in Fig.3.17(a). Assume uniform flexural rigidity.

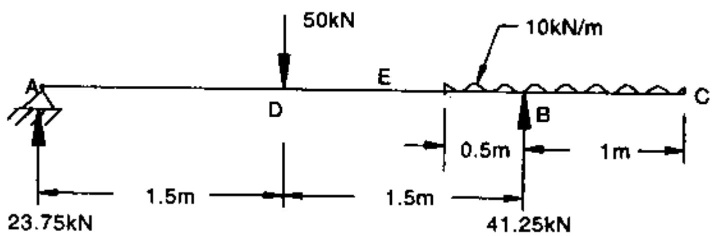


Figure 3.17 (a)

Solution

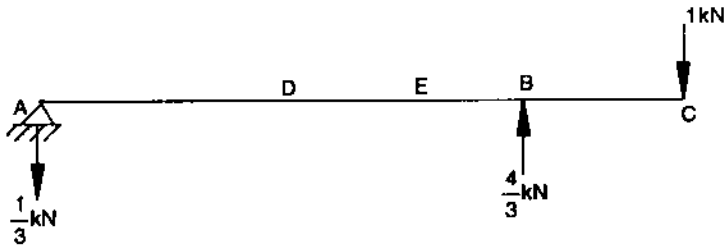


Figure 3.17 (b)

From given load,

$R_B \times 3 = 50 \times 1.5 + 10 \times 1.5 \times 3.25$

$\therefore R_B = 41.25$

$\therefore R_A = 50 + 1.5 \times 10 - 41.25 = 23.75$ kN

Taking sagging moment as positive and hogging moment as negative, the following table is prepared.

Portion	*AD*	*DE*	*EB*	*BC*
Origin	A	D	B	C
Limit	0 – 1.5	0 – 1	0 – 0.5	0 – 1
M	$23.75x$	$23.75(x + 1.5) - 50x$	$41.25x - \frac{10(x+1)^2}{2}$	$-\frac{10x^2}{2}$
m	$-\frac{1}{3}x$	$-\frac{1}{3}(x+1.5)$	$-(1+x)+\frac{4}{3}x$	$-1x$
EI	EI	EI	EI	EI

$$\Delta_c = \int_0^{1.5} \frac{23.75x}{EI}\frac{(-x)}{3}dx + \int_0^1 \frac{[23.75(x+1.5)-50x][-(1/3)(x+1.5)]}{EI}dx$$

$$+ \int_0^{0.5} [41.25x - 5(x+1)^2]\,[-1+\frac{x}{3}]\,dx + \int_0^1 \frac{5x^3}{EI}dx$$

$$= \int_0^{1.5}\left[-\frac{23.75}{3}\frac{x^2}{EI}\right]dx - \frac{1}{3}\int_0^1 \frac{[-26.25x^2 - 3.75x + 53.438]}{EI}dx$$

$$+ \int_0^{0.5} \frac{\left[\frac{-5}{3}x^3 - 15.417x^2 - 32.917x + 5\right]}{EI}dx + \int_0^1 \frac{5x^3}{EI}dx$$

$$EI\Delta_c = \left[-\frac{23.75x^3}{9}\right]_0^{1.5} + \frac{1}{3}\left[\frac{26.25x^3}{3} + 3.75\frac{x^2}{2} - 53.438x\right]_0^1$$

$$+ \left[-\frac{5x^4}{12} + 15.417\frac{x^3}{3} - 32.917\frac{x^2}{2} + 5x\right]_0^{0.5} + \left[\frac{5x^4}{4}\right]_0^1$$

$$= [-8.906 - 14.231 - 0.798 + 1.25]$$

$$= -22.885$$

$$\Delta_c = \frac{22.885}{EI}, \text{ upward.}$$

Example 3.8 Determine the vertical and the horizontal deflection at the free end of the bent shown in Fig. 3.18(a). Assume uniform flexural rigidity EI throughout.

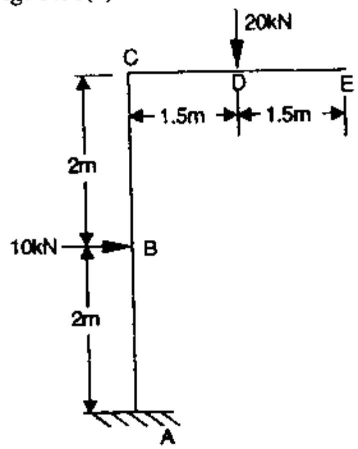


Figure 3.18 (a)

Solution

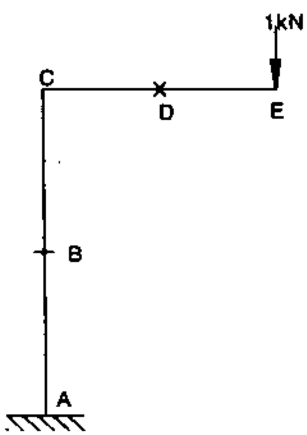


Figure 3.18 (b)

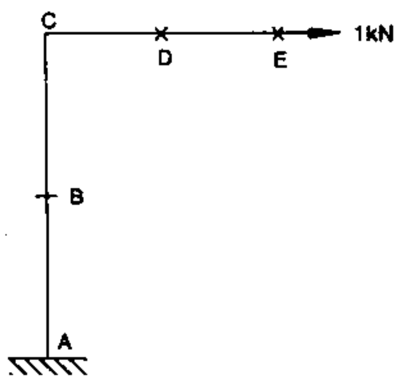


Figure 3.18 (c)

Expressions for M-moments due to given loads, m_1-moments due to unit vertical load at the free end and m_2-moments due to unit horizontal load at the free end are noted below in the tabular form.

Portion	*ED*	*DC*	*CB*	*BA*
Origin	E	D	C	B
Limit	0–1.5	0–1.5	0–2	0–2
M	0	–20x	–30	–30–10x
m_1	x	–(1.5 + x)	–3	–3
m_2	0	0	–x	–(x + 2)
EI	EI	EI	EI	EI

$$EI\Delta_{EV} = \int Mm_1\, dx$$

$$= 0 + \int_0^{1.5} 20x\,(1.5+x)\,dx + \int_0^2 90dx + \int_0^2 (90+30x)\,d$$

$$= \int_0^{1.5} (30x+20x^2)\,dx + \int_0^2 90x + \int_0^2 (90+30x)\,dx$$

$$= \left[\frac{30x^2}{2}+\frac{20x^3}{3}\right]_0^{1.5} + (90x]_0^2 + \left[90x+\frac{30x^2}{2}\right]_0^2$$

$$= 56.25 + 180 + 240$$

$$= 476{,}25$$

$$\therefore \quad \Delta_{EV} = \frac{476.25}{EI}$$

$$EI\Delta_{EH} = \int Mm_2\, dx$$

$$= 0 + 0 + \int_0^2 30x\,dx + \int_0^2 (30+10x)(x+2)\,dx$$

$$= \left[15x^2\right]_0^2 + \int_0^2 (10x^2+50x+60)\,dx$$

$$= 60 + \left[\frac{10x^3}{3}+50\frac{x^2}{2}+60x\right]_0^2$$

$$= 306.67$$

$$\Delta_{EH} = \frac{306.67}{EI}$$

Example 3.9 Determine the vertical deflections at A and C in the frame shown in Fig.3.19. Take E = 200 GPa, I = 150 × 10^4 mm^4.

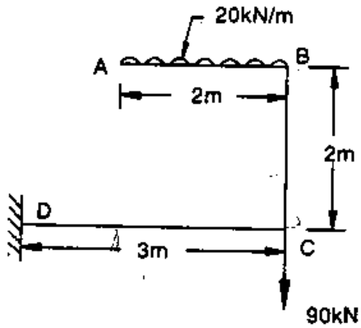


Figure 3.19 (a)

Solution

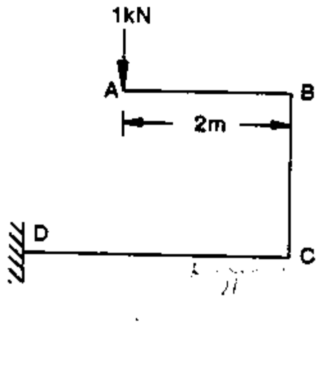


Figure 3.19 (b)

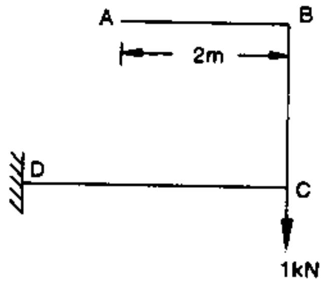


Figure 3.19 (c)

The bending moment expressions for M due to given load, m_1 due to unit vertical load at A and m_2 due to unit vertical load at C are tabulated first.

Portion	*AB*	*BC*	*CD*
Origin	A	B	C
Limit	0–2	0–2	0–3
M	$10x^2$	40	40–90x
m_1	x	2	2–x
m_2	0	0	–x
EI	EI	EI	EI

$$EI\Delta_A = \int_0^2 10x^2.x\,dx + \int_0^2 80dx + \int_0^3 (40-90x)(2-x)\,dx$$

$$= \left[\frac{10x^4}{4}\right]_0^2 + [80x]_0^2 + \int_0^3 (80-220x+90x^2)\,dx$$

$$= \frac{10(2^4)}{4} + 80(2) + \left[80x - 220\frac{x^2}{2} + \frac{90x^3}{3}\right]_0^3$$

$= 260$

Now, $E = 240\ GPa = 240 \times 10^9\ N/m^2$

$I = 150 \times 10^4\ mm^4 = 150 \times 10^4 \times 10^{-12}\ m^4$

$= 150 \times 10^{-8}\ m^4$

$$\therefore \quad \Delta = \frac{260}{240\times 10^9 \times 150 \times 10^{-8}} = 7.222 \times 10^{-4}\ m$$

$= 0.722$ mm

$EI\Delta_C = \int Mm_2\,dx$

$$= 0 + 0 + \int_0^3 (40-90x)(-x)\,dx$$

$$= \int_0^{0.5} (-40x+90x^2)\,dx$$

$$= \left[-20x^2 + 90\frac{x^3}{3}\right]_0^3$$

$= 630$

$$\Delta_c = \frac{630}{240\times 10^9 \times 150 \times 10^{-8}} = 1.75 \times 10^{-3}\ m$$

$= 1.75$ mm

Example 3.10 Determine the horizontal displacement and rotation at roller support in the frame shown in Fig.3.20(a). Flexural rigidity EI is constant throughout.

Solution

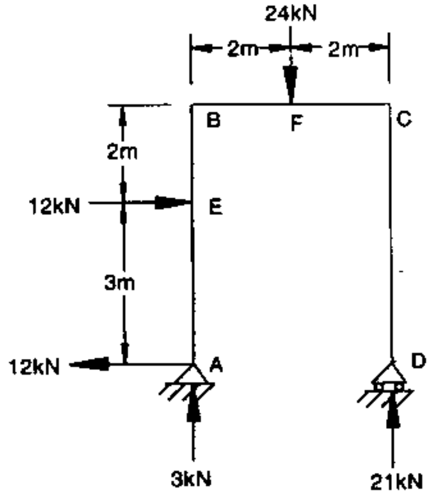


Figure 3.20 (a)

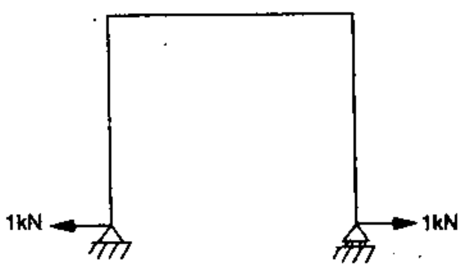


Figure 3.20 (b)

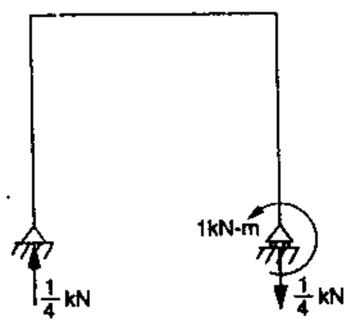


Figure 3.20 (c)

For given loading,

$$R_{DV} \times 4 = 24 \times 2 + 12 \times 3$$
$$R_{DV} = 21 \text{ kN}$$
$$R_{AV} = 24 - 21 = 3 \text{ kN and } R_{AH} = \overleftarrow{12\text{kN}}$$

Bending moment expressions for various portions in the frame are noted down for given loadings (M-values), for unit horizontal load at D (m_1-values) and for unit moment at D (m_2-values). All moment expressions are evaluated here taking forces from the end D.

Portion	AE	EB	BF	FC	CD
Origin	E	B	F	C	D
Limit	0–3	0–2	0–2	0–2	0–5
M	36–12x	21×4–24×2=36	21(x+2)–24x=42–3x	21x	0
m_1	3–x	5–x	5	5	x
m_2	0	0	$1-\frac{1}{4}(x+2)=\frac{1}{2}-\frac{x}{4}$	$1-\frac{x}{4}$	1
EI	EI	EI	EI	EI	EI

$$EI\Delta_{DV} = \int Mm_1 dx$$

$$= \int_0^3 (36-12x)(3-x)dx + \int_0^2 36(5-x)dx + \int_0^2 (42-3x)5dx + \int_0^2 21x(5)dx + 0$$

$$= \int_0^3 \left(108-72x+12x^2\right)dx + \int_0^2 (180-36x)\,dx + \int_0^2 (210-15x)\,dx + \int_0^2 105xdx$$

$$= \left[108x-36x^2+4x^3\right]_0^3 + \left[180x-18x^2\right]_0^2 + \left[210x-7.5x^2\right]_0^2 + \left[52.5x^2\right]_0^2$$

$$= 996$$

$$\Delta_{DH} = \frac{966}{EI}$$

$$EI\theta_D = \int Mm_2 dx$$

$$= 0+0+\int_0^2 (42-3x)\left(\frac{1}{2}-\frac{x}{4}\right)dx + \int_0^2 21x\left(1-\frac{x}{4}\right)dx + 0$$

$$= \int_0^2 \left(21-12x+\frac{3x^2}{4}\right)dx + \int_0^2 \left(21x-5.25x^2\right)dx$$

$$= \left[21x-6x^2+\frac{x^3}{4}\right]_0^2 + \left[\frac{21x^2}{2}-\frac{5.25x^3}{3}\right]_0^2$$

$$= 42-24+2+42-5.25\frac{8}{3}$$

$$= 48$$

$$\therefore \theta_D = \frac{48}{EI}$$

3.8 CASTIGLIANO'S THEOREMS

Castigliano published two important theorem in structural analysis (1879). The first theorem helps in determining deflection and the second one in determining redundant reaction component. In this article first theorem is explained.

First theorem *In a linearly elastic structure, partial derivative of the strain energy with respect to a load is equal to the deflection of the point where the* *load is acting, the deflection being measured in the direction of the load.*

The load may be a force or a moment. Mathematically, this theorem may be represented by,

$$\frac{dU}{dP_i} = \Delta_i, \frac{dU}{dM_j} = \theta_j \qquad 3.15$$

where U = total strain energy
P_i, M_j - loads
Δ_i, θ_j - deflections.

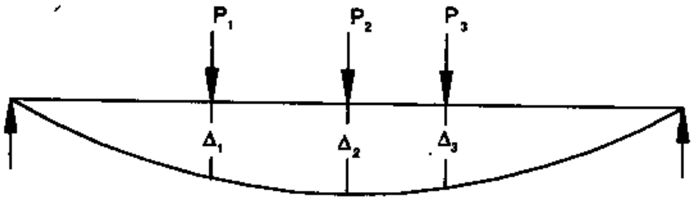


Figure 3.21 (a)

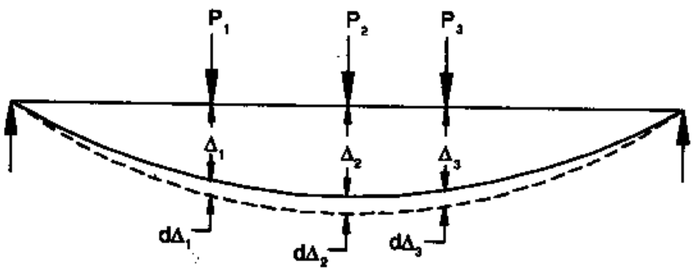


Figure 3.21 (b)

To prove this theorem, consider a simply supported beam shown in Fig. 3.21(a) on which loads P_1, P_2 and P_3 are applied gradually. Let the deflections under the loads P_1, P_2 and P_3 be Δ_1, Δ_2, Δ_3 respectively.

$$\therefore \qquad U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \qquad (a)$$

Let the additional load dP_1 be added after the loads P_1, P_2 and P_3 and applied and let the additional deflections be $d\Delta_1$, $d\Delta_2$, $d\Delta_3$. Then the additional strain energy stored dU is given by

$$dU = \frac{1}{2}dP_1\,d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \qquad (b)$$

$\therefore$ Total strain energy of the system is

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \qquad (c)$$

If $(P_1 + dP_1)$, P_2 and P_3 were to be applied simultaneously, strain energy stored is given by,

$$= \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1) + \frac{1}{2}P_2(\Delta_2 + d\Delta_2) + \frac{1}{2}P_3(\Delta_3 + d\Delta_3) \qquad \text{(d)}$$

Since the final strain energy in both the cases should be same,

Equation (c) = Equation (d)

i.e., $$\frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3 d\Delta_3 = \frac{1}{2}dP_1\Delta_1 \qquad \text{(e)}$$

But from eqn (b), $\frac{1}{2}(P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3) = \frac{1}{2}(dU - \frac{1}{2}dP_1 d\Delta_1)$. Neglecting small quantity of higher order from eqn (e), we get

$$\frac{1}{2}dU = \frac{1}{2}dP_1\Delta_1$$

or $$\frac{dU}{dp_1} = \Delta_1 \qquad 3.16$$

Similarly, if the moments are considered, it may be shown that

$$\frac{dU}{dM} = \theta \qquad 3.17$$

3.9 FINDING DEFLECTION USING CASTIGLIANO'S METHOD

If a load is acting at a point and is in the desired direction then that load is treated as a general load (P or M) and the general expressions for bending moment to cover the entire structure are noted down. If no such load is acting, a dummy load (P or M) is applied and then the bending moment expressions are noted. The strain energy for the entire structure is differentiated w.r.t. the load (P or M) to get the desired deflection. Substitute the value of that load as (P or M). If dummy load is used, naturally the value of that load is zero. Note that,

$$\frac{d}{dP}\int F(x,P)\,dx = \int \frac{d}{dP}[F(x,P)]\,dx$$

also. Whenever dummy load is used, the second form will be more useful. This method is illustrated with the following examples.

Example 3.11 A simply supported beam of span L, carries a concentrated load P at a distance 'a' from left hand side support as shown in Fig. 3.22. Using Castigliano's

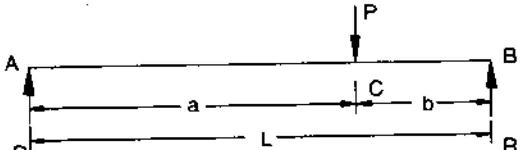


Figure 3.22

theorem determine the deflection under the load. Assume uniform flexural rigidity.

Solution

Reaction at A, $R_A = \dfrac{Pb}{L}$

and Reaction at B, $R_B = \dfrac{Pa}{L}$

Portion	*AC*	*CB*
Origin	A	B
Limit	0–a	0–b
M	$\dfrac{Pb}{L}x$	$\dfrac{Pa}{L}x$
EI	EI	EI

∴ S.E of the beam

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \frac{1}{2EI}\,dx + \int_0^b \left(\frac{Pa}{L}x\right)^2 \frac{1}{2EI}\,dx$$

$$= \left[\frac{P^2b^2}{L^2}\frac{1}{6EI}x^3\right]_0^a + \left[\frac{P^2a^2}{L^2}\frac{1}{6EI}x^3\right]_0^b$$

$$= \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2a^2b^2}{6EIL}, \text{ since } a+b=L$$

$$\therefore \qquad \Delta_c = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Example 3.12 Determine the vertical deflection at the free end and rotation at A in the overhanging beam shown in Fig.3.23(a). Assume constant EI. Use castigliano's method.

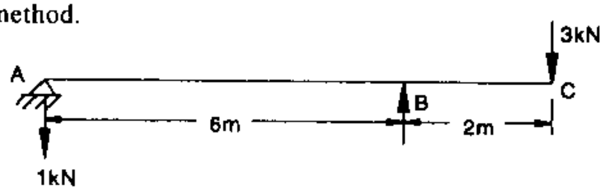


Figure 3.23 (a)

Solution

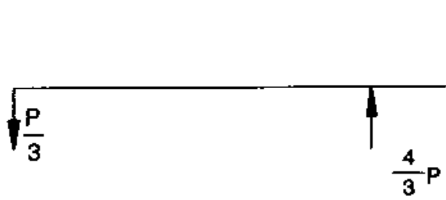


Figure 3.23 (b)

(1) Deflection at C: Taking 3kN force as P,

$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3}P \uparrow$$

$$\therefore \quad R_A = \frac{P}{3} \downarrow$$

Bending moment expressions are noted down in the tabular form.

Portion	*AB*	*BC*
Origin	A	C
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
EI	EI	EI

$$U = \int \frac{M^2}{2EI}dx$$

$$= \int_0^6 \frac{P^2x^2}{9}\frac{1}{2EI}dx + \int_0^2 \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3}\frac{P^2}{EI}$$

$$= \frac{5.333P^2}{EI}$$

$$\Delta_c = \frac{dU}{dp} = \frac{10.667P}{EI}$$

Substituting P = 3 kN, we get

$$\Delta_c = \frac{32}{EI}$$

Rotation at A : Apply a dummy moment M at A as shown in Fig. 3.23(c).

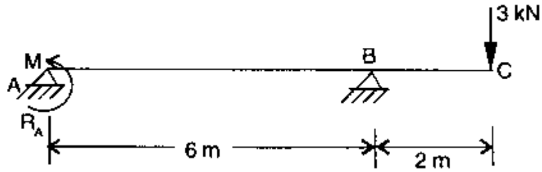


Figure 3.23 (c)

$$\Sigma M_B = 0$$

$$R_A = \frac{M-6}{6} = \frac{M}{6} - 1$$

Portion	*AB*	*BC*
Origin	A	C
Limit	0 – 6	0 – 2
M	$\left(\frac{M}{6}-1\right)x - M$	$-3x$

$$U = \int_0^6 \left[\left(\frac{M}{6}-1\right)x - M\right]_0^2 \frac{1}{2EI}\,dx + \int_0^2 \frac{(-3x)^2}{2EI}\,dx$$

$$\frac{dU}{dM} = \int_0^6 2\left[\left(\frac{M}{6}-1\right)x - M\right]\left(\frac{x}{6}-1\right)\frac{dx}{2EI} + 0$$

Since M is a dummy moment, its value is substituted as zero and then integrated.

$$\frac{dU}{dM} = \theta_A = \frac{1}{EI}\int_0^6 (-x)\left(\frac{x}{6}-1\right)dx$$

$$= \frac{1}{EI}\int_0^6 \left(-\frac{x^2}{6}+x\right)dx$$

$$= \frac{1}{EI}\left(-\frac{x^3}{18}+\frac{x^2}{2}\right)_0^6 dx$$

$$= \frac{6}{EI}$$

Note: First differentiate with respect to the dummy load, then substitute dummy load as zero and then integrate w.r.t. x.

Example 3.13 Determine the vertical and horizontal displacement at the free end D in the frame shown in Fig.3.24(a).

Take EI 12×10^{13} N-mm^2. Use Castigliano's theorem.

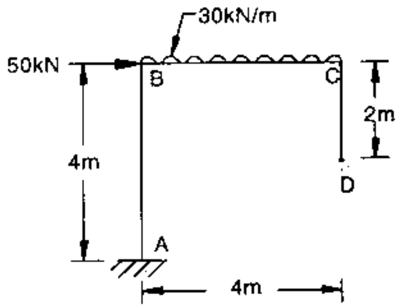


Figure 3.24 (a)

Solution

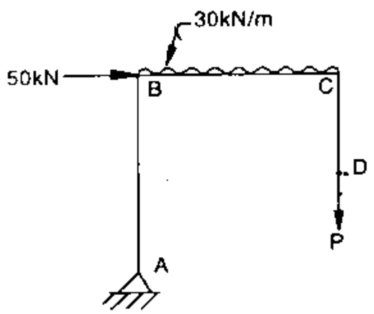


Figure 3.24 (b)

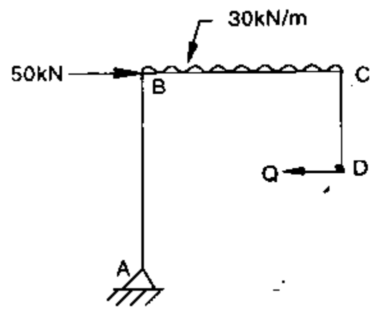


Figure 3.24 (c)

Vertical deflection

Since there is no load at D in vertical direction, a dummy load P is applied at D in vertical direction, in addition to given loads as shown in Fig.3.25(b) and the moment expressions are noted down.

Portion	*AB*	*BC*	*CD*
Origin	B	C	D
Limit	0–4	0–4	0–2
M	$-(4P + 240 + 50x)$	$-(Px + 15x^2)$	0
EI	EI	EI	EI

Strain energy $U = \int \frac{M^2}{2EI} dx$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI} dx + \int_0^4 \frac{(Px+15x^2)^2}{2EI} dx + 0$$

$$\therefore \qquad \Delta = \frac{\delta U}{\delta P} = \int_0^4 2\frac{(4P+240+50x]}{2EI}(4)\, dx + \int_0^4 2\frac{(Px+15x^2)x}{2EI} dx$$

Since P is dummy load, substitute P = 0

$$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI} dx + \int_0^4 \frac{15x^3}{EI} dx$$

$$= \frac{4}{EI}\left[240x+25x^2\right]_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4$$

$$= \frac{6400}{EI}$$

Now, $\quad EI = 12 \times 10^{13}$ N–mm^2

$\quad = 12 \times 10^4$ kN–m^2

$$\therefore \qquad \Delta_{DV} = \frac{6400}{12\times10^4} = 0.0533\text{m}$$

$$= 53.33$$

Horizontal deflection

Since there is no load in the horizontal direction at D, a dummy load is applied shown in Fig. 3.24(c) and the moment expressions are noted down.

Portion	*AB*	*BC*	*CD*
Origin	B	C	D
Limit	0–4	0–4	0–2
M	$-[Q(2-x)+240+50x)]$	$-(2Q+15x^2)$	Qx
EI	EI	EI	EI

$$U = \int_0^4 \frac{[Q(2-x)+240+50x]^2}{2EI} dx + \int_0^4 \frac{[(2Q+15x^2)]^2}{2EI} dx + \int_0^2 \frac{Q^2x^2}{2EI} dx$$

$$\Delta_{DH} = \frac{\delta U}{dQ} = \int_0^4 \frac{2[Q(2-x)+240+50x](2-x)}{2EI} dx + \int_0^4 \frac{2[2Q+15x^2)2}{2EI} dx + \int_0^2 \frac{2Qx^2}{2EI} dx$$

Substituting Q = 0

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI} dx + \int_0^4 \frac{30x^2}{EI} dx + 0$$

$$= \int_0^4 \frac{(480-140x-50x^2)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx$$

$$= \frac{1}{EI}\left[480x-70x^2-\frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$$

$$= \frac{373.33}{EI} = \frac{373.33}{12\times10^4} = 0.0031\text{m}$$

$$= 3.1\text{mm}$$

Example 3.14 A cantilever beam is in the form of a quarter of a circle in the vertical plane and is subjected to a vertical load P at its free end as shown in Fig.3.25(a). Find the vertical and horizontal displacements at the free end. Assume constant flexural rigidity.

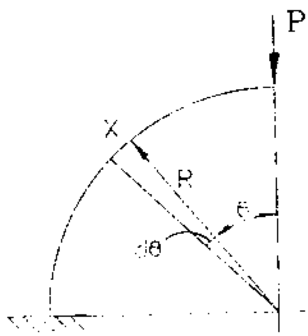


Figure 3.25 (a)

Solution

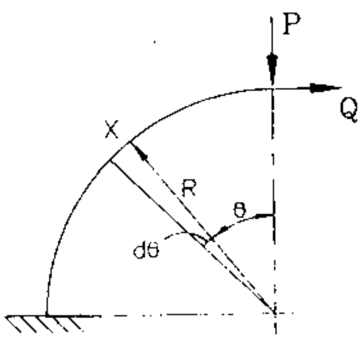


Figure 3.25 (b)

Vertical deflection of free end

Consider the section at x as shown in Fig.3.25(a).

$M = PR \sin\theta$

Strain energy in the elemental length 'R$d\theta$' is

$$= \frac{M^2}{2EI} R d\theta$$

$$= \frac{P^2R^2\sin^2\theta}{2EI} R\,d\theta$$

$$= \frac{P^2R^3}{2EI}\,\frac{1-\cos 2\theta}{2}\,d\theta$$

$$\therefore \qquad U = \int_0^{\pi/2} \frac{P^2R^3}{2EI}\,\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{P^2R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi P^2R^3}{8EI}$$

$$\Delta_V = \frac{\delta U}{dP} = \frac{\pi P R^3}{4EI}$$

Horizontal displacement

Since there is no horizontal force at the free end, apply a dummy horizontal force Q as shown in Fig.3.25(b).

Strain energy U is given by

$$U = \int_0^{\pi/2} \frac{[PR\sin\theta + QR(1-\cos\theta)]^2}{2EI} R d\theta$$

Since, $M = PR\sin\theta + QR(1-\cos\theta)$

Horizontal displacement

$$\Delta_H = \frac{\delta U}{\delta Q} = \int_0^{\pi/2} \frac{[PR\sin\theta + QR(1-\cos\theta)]}{EI}\,[R\,(1-\cos\theta)]\,Rd\theta$$

Now, Substituting $Q = 0$

$$\Delta_H = \frac{1}{EI}\int_0^{\pi/2} \frac{PR\sin\theta}{EI}[R\,(1-\cos\theta]\,Rd\,\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2} (\sin\theta - \sin\theta\cos\theta)\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{PR^3}{EI}\left(\cos\theta - \frac{\cos 2\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{PR^3}{EI}\left(0 + \frac{1}{4} - 1 + \frac{1}{4}\right)$$

$$= -\frac{PR^3}{2EI}$$

i.e., $\Delta_B \quad = \frac{PR^3}{2EI}$, towards support

3.10 OTHER USEFUL THEOREMS

3.10.1 Maxwell's Theorem of Reciprocal Deflection

Clerk Maxwell published this theorem in 1864. However, its usefulness was not appreciated till 1886 when Muller Breslau used it for the analysis of indeterminate structures. Maxwell's theorem of reciprocal deflections may be stated as "*Displacement at point A due to the load at point B is same as displacement of point B due to the same load acting at point A, the displacements being measured in the directions of the loads.*"

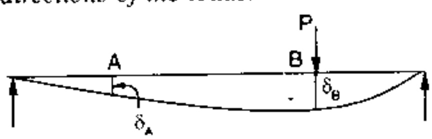


Figure 3.26 (a)

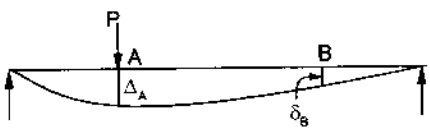


Figure 3.26 (b)

The reciprocal theorem is valid for linear as well as for rotational displacements. For example, in Fig. 3.26, δ_B is the displacement at B due to load P at A and δ_A is the displacement of A due to load P and B.

Then, according to this theorem,

$$\delta_A = \delta_B$$

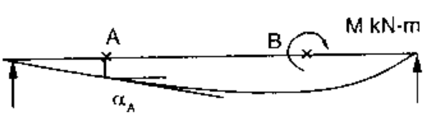


Figure 3.26 (a)

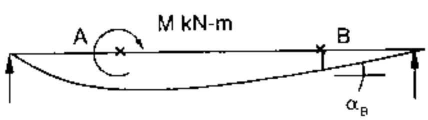


Figure 3.26 (b)

In Fig. 3.27, let α_A be the rotation at A due to the moment M at B and α_B be the rotation at B due to the moment M at A.

Then, according to this theorem,

$$\alpha_A = \alpha_B$$

Proof (a) Considering only linear displacements,

When load P is acting at B, let the displacement at A be δ_A and at B be Δ_B [Fig.3.26(a)].

$$\therefore \qquad \text{Work done} = \frac{1}{2} P\Delta_B \qquad \text{(a)}$$

When load P is acting at A, let the deflection at A be Δ_A and deflection at B be δ_B.

$$\text{Then} \qquad \text{work done} = \frac{1}{2} P\Delta_A \qquad \text{(b)}$$

Now imagine that load P is applied first at B and then at A. In this case,

$$\text{external work done} = \frac{1}{2} P\Delta_B + P\delta_B + \frac{1}{2} P\Delta_A \qquad \text{(c)}$$

If load P is applied first at A and then at B,

$$\text{work done} = \frac{1}{2} P\Delta_A + P\delta_A + \frac{1}{2} P\Delta_B \qquad \text{(d)}$$

Equations (c) and (d) represent work done when P is acting at both points A and B. Hence equating them we get,

$$\frac{1}{2} P\Delta_B + P\delta_B + \frac{1}{2} P\Delta_A = \frac{1}{2} P\Delta_A + P\delta_A + \frac{1}{2} P\Delta_B$$

$$\therefore \qquad \delta_B = \delta_A$$

Hence the theorem is proved. On the same way one can prove for moment and rotation cases also.

Maxwell's theorem can be stated even for the combination of linear and rotational cases also, with the requirement as the linear load (P) and rotational load (M) shall be the same or the load may be referred as unit load in the theorem. Thus, Maxwell theorem in general is *"Displacement at point A due to a unit load at point B is same as displacement at B due to the unit load at A, the displacements being measured in the directions of unit loads."*

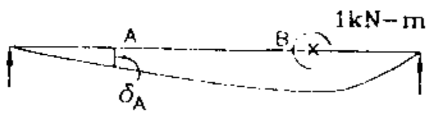


Figure 3.28 (a)

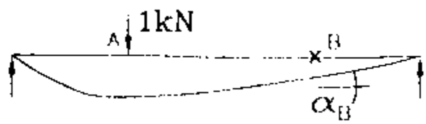


Figure 3.28 (b)

Referring Fig.3.29, $\delta_A = \alpha_B$

Proof Let, due to unit moment at B, the deflection at A be δ_A' and rotation at B be α_B'.

$\therefore$ Due to moment M at B deflection at A is $M\delta'_A$ and rotation at B is $M\,\delta'_B$. (refer Fig.3.29(a)).

$$\text{Work done} = \frac{1}{2} M\, M\, \alpha_B' = \frac{1}{2} M^2 \alpha_B' \qquad \text{(a)}$$

Due to unit load at A, let the deflection at A be δ_A' and rotation at B be α_B'. Hence deflection at A and rotation at B when load P is applied at A are [Fig.3.29(b)] $P\delta_A'$ and $P\alpha_B'$ respectively. Work done in this case

$$= \frac{1}{2} P.P\alpha_A' = \frac{1}{2} P^2 \delta_A' \qquad \text{(b)}$$

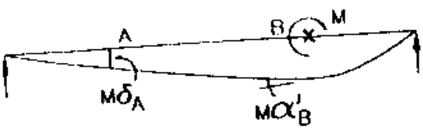


Figure 3.29 (a)

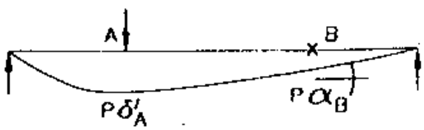


Figure 3.29 (b)

Now, consider the case when load M is applied before load P. Due to M, work done is $\frac{1}{2} M^2\alpha_B'$ and at this stage if additional load P is applied, there will be an additional displacement $P\delta_A'$ at A and $P\alpha_B'$ at B. Hence total work done

$$= \frac{1}{2} M^2 \alpha_B' + \frac{1}{2} P^2 \delta_A' + MP\alpha_B \qquad \text{(c)}$$

Similarly, first applying P before M,

$$\text{work done} = \frac{1}{2} PP\delta_A' + PM\,\delta_A + \frac{1}{2} MM\,\alpha'_B \qquad \text{(d)}$$

Equating (c) and (d) we get,

$$\frac{1}{2} M^2\alpha_B' + \frac{1}{2} P^2\,\delta_A' + MP\alpha_B = \frac{1}{2} P^2\delta_A + PM\delta_A + \frac{1}{2} M^2\alpha_B'$$

$$\alpha_B = \delta_A \qquad 3.19$$

Hence the theorem is proved.

EXERCISES

3.1 Using strain energy method, determine the vertical displacement at the free end of the cantilever beam shown in Fig.3.30. Given E = 200 kN/mm^2 and I = 20 × 10^{6}mm^4.

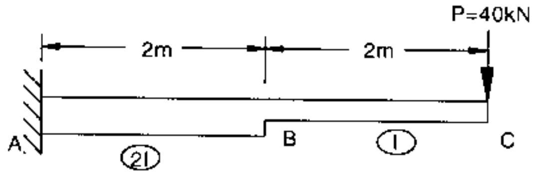


Figure 3.30

Ans : 12mm

3.2 Using unit load method, analyse exercise problem 3.1 and determine the displacement at point B and C.

Ans : Δ_B 3.333mm; Δ_C = 12mm

3.3 A simply supported beam of span 12 m shown in Fig. 3.31 is located with 40 kN loads at C, D and E. Determine the vertical deflection at D and rotation at E. Take EI = 20000 kN-m^2.

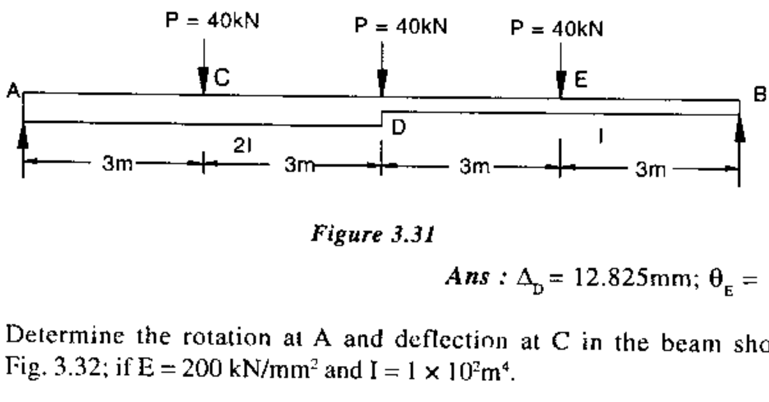


Figure 3.31

Ans : Δ_D = 12.825mm; θ_E = 1.332°

3.4 Determine the rotation at A and deflection at C in the beam shown in Fig. 3.32; if E = 200 kN/mm^2 and I = 1 × 10^{2}m^4.

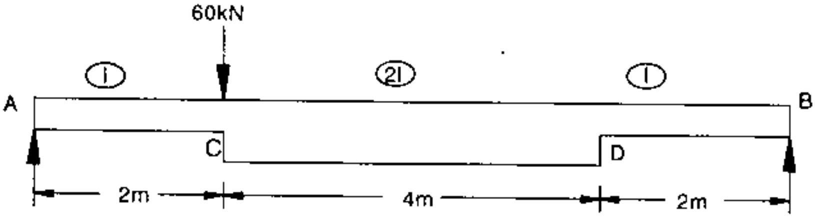


Figure 3.32

Ans : θ = 0.415°; Δ_C = 11.5mm

3.5 Determine the rotation at A and vertical deflection at the free end D in the overhanging beam shown in Fig.3.33, if $E=2\times10^5 N/mm^2$ and $I=1\times10^7 mm^4$

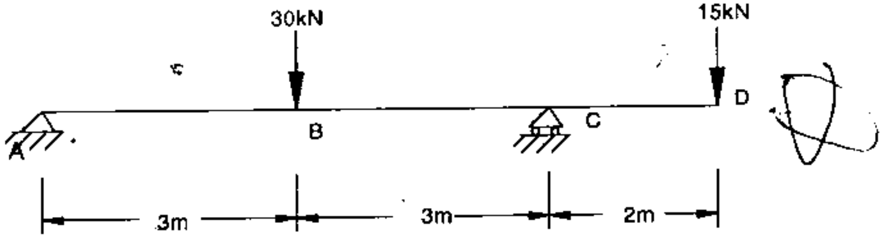


Figure 3.33

Ans : $\theta_A = 1.074$; $\Delta_D = 12.5$mm

3.6 Determine the vertical and horizontal deflection at the free end of the beam shown in Fig.3.34. Given E = 200 kN/mm² and I = 30 × 10⁷ N/mm².

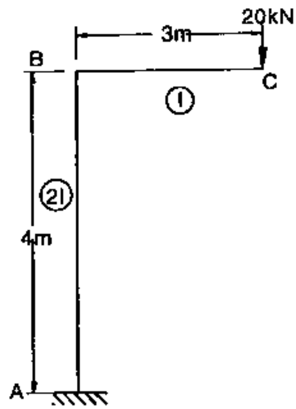


Figure 3.34

Ans : $\Delta_{VC} = 9$mm; $\Delta_{HC} = 4$mm

3.7 A 100mm diameter steel rod is bent to the shape as shown in Fig. 3.35 and is subjected to a vertical downward load of 500 N at the free end D. Determine the vertical downward deflection of end D. Take E = 200 kN/m².

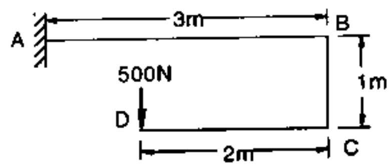


Figure 3.35

Ans : $\Delta_D = 69.60$mm

3.8 Determine the vertical and the horizontal displacements at the free end E in the frame shown in Fig.3.36. Given EI = 20000 kN-mm^2.

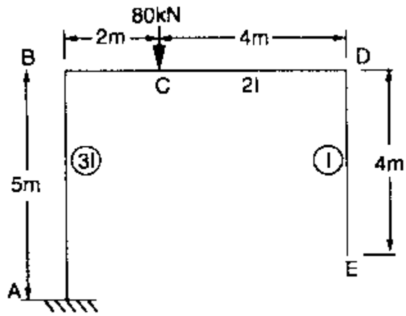


Figure 3.36

Ans : $\Delta_{VE} = 101.33$mm; $\Delta_{HE} = 36$mm

DEFLECTIONS OF PIN-JOINTED PLANE FRAMES

4

4.1 INTRODUCTION

The trusses supporting sloping roofs and bridge decks are the examples of pin-jointed plane frames. These frames use steel as their chief material. These slender steel members are connected at the ends by riveting or by welding to form a series of triangles. However, in the analysis it is assumed that the members are pin connected i.e., the ends are perfect hinges, also the loads act only at the joints. Hence there is no bending moment and shear forces in the members. All members are subjected to only direct forces i.e., tension or compression. In this chapter, the methods of finding the deflections of joints is discussed.

The following methods are used for finding the deflections of joints in pin connected trusses.

1. Unit Load Method
2. Castigliano's Method
3. Angle Weight Method
4. Joint Displacement Method
5. Williot-Mohr's Method
6. Stiffness Matrix Method

Unit load method and Castigliano's first theorem can be used for finding deflections of a single joint at a time. The angle weight method and the joint displacement methods are used to find the displacement of all joints at a time. Williot-Mohr's method is a graphical method for finding displacements of all joints at a time. Stiffness matrix method finds the displacement of all joints at a time and then finds the forces in the members. This method finds its application in computer-aided analysis.

The displacements of joints are required mainly for the following two purposes.

1. To check whether the maximum displacement is within the permissible limit.
2. To make use of it in the analysis of statically indeterminate structures.

In either case, the value required will be at limited joints. The displacement of all joints finds little use. Nowadays, the stiffness matrix method is finding its popularity over all other methods. This is mainly because of increased computer facility. The angle weight method, joint displacement method, Williot method and Castigliano's method have limited application. The Castigliano's method coincides with unit load method if the load is not acting at the point where displacement is required (in such case dummy load is to be applied). Hence, Castigliano's method is also not covered in this chapter. Stiffness matrix method is beyond the scope of this book, hence, only unit load method is explained in this chapter.

4.2 UNIT LOAD METHOD

Using energy equation, it has been proved in Chapter III (Art. 3.6) that deflection of a structure is given by the equation

$$\Delta = \int p'e \, dv \qquad 4.1$$

where Δ is the displacement at the point and in the direction of unit load applied

p' is the stress due to unit load

e is the strain due to the applied load

The above equation holds good irrespective of the type of structure, provided all types of stresses developed in the structure are accounted.

In case of pin-jointed frames, there is only one type of stress i.e. the direct stress. This stress may be different in different members but is constant at all points in a member. Hence,

$$\int p'edv = \sum p'e\, AL \qquad (a)$$

where $\sum$ is to cover all members

A is the cross-sectional area of member

L is the length of the member

p' is the stress due to unit load

$$\therefore \qquad p' = \frac{k}{A} \qquad (b)$$

where k is the force in the member due to unit load

e is the strain due to given load

$$\therefore \qquad e = \frac{P}{A} \times \frac{1}{E} = \frac{P}{AE} \qquad (c)$$

where P is force in the member due to given loadings. From equation (4.1) and equations a, b, c we get,

$$\Delta = \sum \frac{k}{A}\frac{P}{AE} AL$$

$$= \sum \frac{PkL}{AE} \qquad 4.2$$

or

$$\Delta = \sum k\,\delta L \qquad 4.3$$

Since $\delta L = \dfrac{PL}{AE}$ is the extension / shortening of members.

Equation 4.2 or 4.3 can be used to find the deflection of a joint at the point and in the direction of the unit load applied. The method needs analysis of the truss twice, once with the given loading to get 'P' terms and the second time with unit load to get 'k' terms. This method is illustrated with the following problems.

Example 4.1 Find the vertical deflection of the joint B in the truss loaded as shown in Fig.4.1(a). The cross sectional area of the members in mm are shown in brackets. Take E = 200 kN/mm².

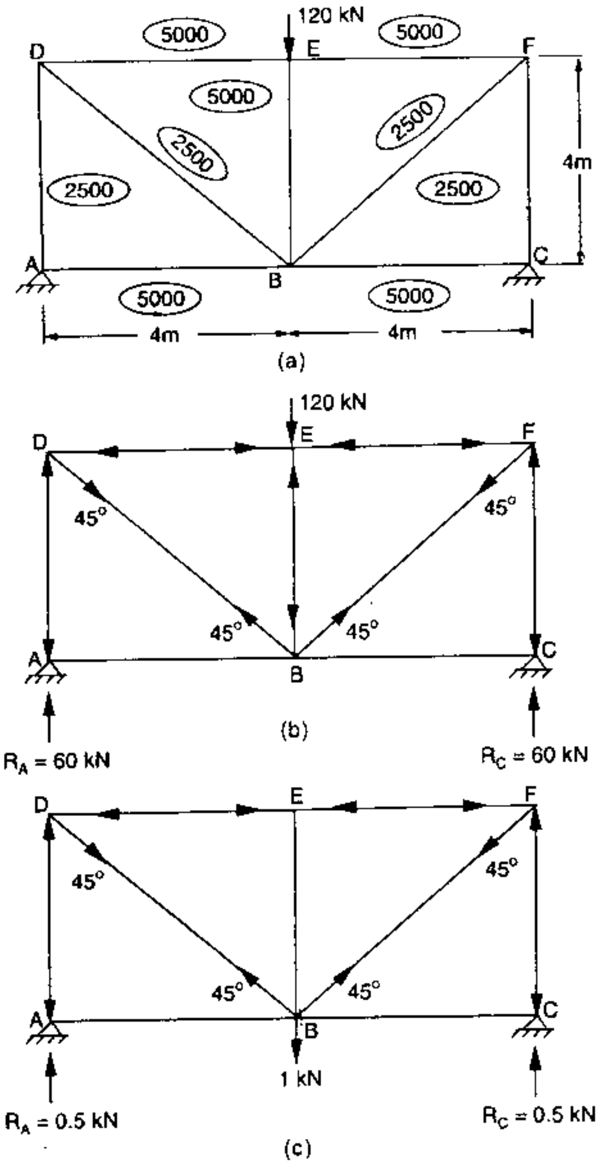


Figure 4.1

Solution

Forces due to given loading (P – forces)

Due to symmetry,	$R_A = R_C = \dfrac{120}{2} = 60$ kN
From joint A,	$P_{AD} = 60$ kN (compressive), $R_{AB} = 0$
Considering joint D,	$P_{BD} \sin 45° = 60$ or $P_{BD} = 60\sqrt{2}$ kN (tensile)
and	$P_{DE} = P_{BD} \cos 45° = 60$ kN (compressive)

From point E, $P_{EB} = 120$ kN (compressive)

The nature of the forces are marked on Fig. 4.1(b).

Note: Since forces marked in Table 4.1 are those acting on joints, the member force is tensile if o →———← o and compressive if o ←———→ o.

For other portion, symmetry is made use of and the magnitude and nature of forces are noted down in Table 4.1, taking tensile force as positive and compressive as negative.

Forces due to unit load: Unit load is applied at B in vertical downward direction as shown in Fig. 4.1(c). Then, $R_A = R_C = 0.5$ kN

From joint A, $F_{AD} = 0.5$ kN (compressive), $F_{AB} = 0$

From joint D, $F_{BD} \times \frac{1}{\sqrt{2}} = 0.5$ or $F_{BD} = 0.5\sqrt{2}$ (tensile)

$$F_{DE} = F_{BD} \times \frac{1}{\sqrt{2}} = 0.5 \text{ kN (compressive)}$$

From joint E, $F_{BE} = 0$

Using symmetry, forces for other portions can be written down and are tabulated in Table 4.1. Then, calculations in Table 4.1 are carried out to get vertical deflection of B as

$$\Delta = \sum \frac{PkL}{AE}$$

Table 4.1

Member	*Length in mm*	*Area in mm²*	*P-force in kN*	*k-force in kN*	$\frac{PkL}{A}$
AB	4000	5000	0	0	0
BC	4000	5000	0	0	0
DE	4000	5000	−60	−0.5	24
EF	4000	5000	−60	−0.5	24
AD	4000	2500	−60	−0.5	48
BE	4000	5000	−120	0	0
CF	4000	2500	−60	−0.5	48
BD	$4000\sqrt{2}$	2500	$60\sqrt{2}$	$0.5\sqrt{2}$	135.768
BF	$4000\sqrt{2}$	2500	$60\sqrt{2}$	$0.5\sqrt{2}$	135.768

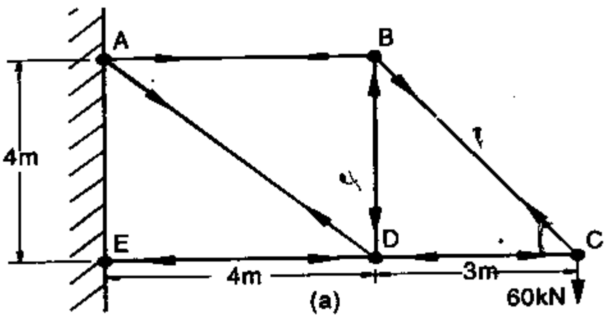
$$\sum \frac{PkL}{A} = 415.530$$

$$\therefore \quad \Delta = \Sigma\frac{PkL}{AE} = \frac{1}{E}\Sigma\frac{PkL}{A}$$

$$\therefore \quad = \frac{415.530}{E} = \frac{415.530}{200} = 2.078\text{mm.}$$

Note: All linear dimensions are in mm unit and forces in kN unit.

Example 4.2 Determine the vertical deflection of point D in the truss shown in Fig.4.2(a). The cross sectional areas of members AD and DE are 1500mm² while those of the other members are 1000mm². Take E = 200 kN/mm².



Solution

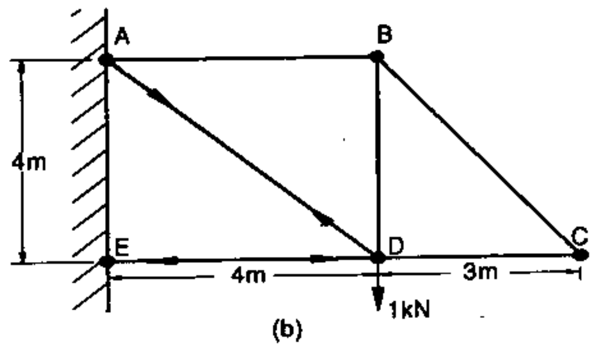


Figure 4.2

The truss is analysed by taking kN as unit of force and mm as unit for linear measurements.

P Forces:

From the equilibrium of joint C

$P_{CB} \times 0.8 = 60$ or $P_{CB} = 75$ kN (tensile)

$P_{CB} \times 0.6 = P_{CD}$ or $P_{CD} = 45$ kN (compressive)

At joint B

$P_{BD} = P_{CB} \times 0.8 = 75 \times 0.8 = 60$ kN (compressive)

$P_{BA} = P_{CB} \times 0.6 = 45$ kN (tensile)

At joint D

$$P_{DA} \times \frac{1}{\sqrt{2}} = P_{BD} = 60 \text{ kN}$$

$$P_{DA} = 60\sqrt{2} \text{ kN (tensile)}$$

$$P_{DE} = P_{DA} \times \frac{1}{\sqrt{2}} + P_{DC} = 60 + 45 = 105 \text{ kN (compressive)}$$

k - forces due to unit load at D:

Referring to Fig.4.2(b), from joint equilibriums C and B

$$P_{CB} = P_{CD} = P_{BA} = P_{BD} = 0$$

From joint D

$$P_{DA} \times \frac{1}{\sqrt{2}} = 1 \text{ or } P_{DA} = \sqrt{2} \text{ kN (tensile)}$$

and $$P_{DE} = P_{DA} \times \frac{1}{\sqrt{2}} = 1 \text{ kN (compressive)}$$

Further calculations are carried out in Table 4.2.

Table 4.2

Member	*Length*	*Area in mm*	*P-force in mm*	*K-force in kN*	$\frac{PkL}{A}$
AB	4000	1000	45	0	0
BC	5000	1000	75	0	0
CD	3000	1000	–45	0	0
DE	4000	1500	–105	–1	280
DB	4000	1000	–60	0	0
AD	$4000\sqrt{2}$	1500	$60\sqrt{2}$	$\sqrt{2}$	452.55

$$\sum \frac{PkL}{A} = 732.55$$

$$\text{Deflection at } D = \sum \frac{PkL}{AE}$$

$$= \frac{1}{E} \sum \frac{PkL}{A}$$

$$= \frac{1}{200} \times 732.55$$

$$\Delta_D = 3.663 \text{ mm}$$

Example 4.3 Figure 4.3(a) shows a pin-jointed truss loaded with a single load W = 100 kN. If the area of cross-section of all members shown in Figure 4.3(a) is 1000 mm^2, what is the vertical deflection of point C? Take E = 200 kN/mm^2 for all members.

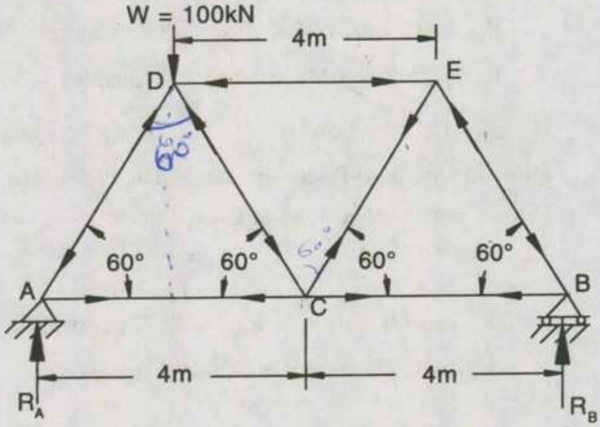


Figure 4.3(a)

Solution

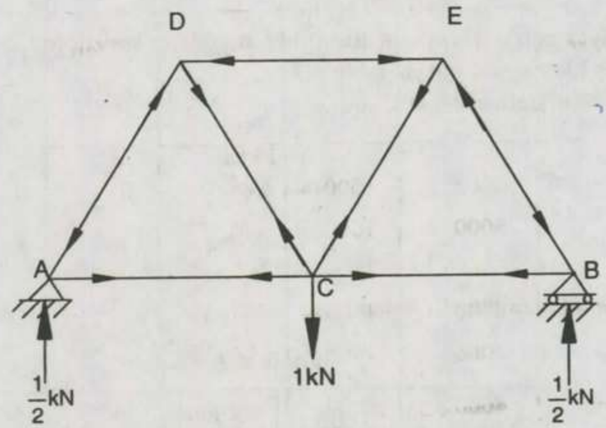


Figure 4.3(b)

P Forces:

Direction of these forces are marked in Fig.4.3(a)

$R_B \times 8 - 100 \times 2 = 0$ or $R_B = 25$ kN

$\therefore$ $R_A = 100 - 25 = 75$ kN

At joint A,

$$P_{AD} \sin 60° = 75 \quad \text{or } P_{AD} = 86.6 \text{ kN (comp)}$$

$$P_{AC} = P_{AD} \cos 60° = 43.3 \text{ kN (tensile)}$$

At joint D,

$$P_{DC}\sin 60^\circ + P_{AD}\sin 60^\circ - 100 = 0 \text{ or } P_{DC} = 28.87 \text{ kN (comp)}$$

$$P_{DE} = 86.6 \times \cos 60^\circ - 28.8 \cos 60^\circ = 28.87 \text{ kN (comp)}$$

At joint B,

$$P_{BE}\sin 60^\circ = 25 \quad \text{i.e., } P_{BE} = 28.87 \text{ (comp)}$$

$$P_{BC} = 28.87 \cos 60^\circ = 14.43 \text{ kN (tensile)}$$

At joint E,

$$P_{EC}\sin 60^\circ = 28.87 \sin 60^\circ \quad \therefore\ P_{EC} = 28.87 \text{ (tensile)}$$

Direction of these forces are marked in Fig. 4.3(b).

Due to unit vertical load at C – k forces:

$$R_A = R_B = 0.5 \text{ kN}$$

At joint A,

$$k_{AD}\sin 60^\circ = 0.5 \text{ or } k_{AD} = 0.577 \text{ (comp)}$$

$$k_{AC} = 0.577 \cos 60^\circ = 0.2886 \text{ (tensile)}$$

At joint D,

$$k_{DC}\sin 60^\circ = k_{AD}\sin 60^\circ = 0.577 \text{ (tensile)}$$

$$k_{DE} = 0.577 \times \cos 60^\circ + 0.577 \cos 60 = 0.577 \text{ (comp)}$$

Using symmetry, forces in the other members are noted down and furthe calculations are carried out in Table 4.3.

Now vertical deflection at C

$$= \sum \frac{PkL}{AE}$$

$$= \frac{L}{AE}\sum Pk$$

Since for all members L, A and E are same.

Table 4.3

Member	*P-force in kN*	*k-force in kN*	*Pk*
AD	–86.60	–0.577	49.97
DE	–28.87	–0.577	16.67
EB	–28.87	–0.577	16.67
BC	14.43	0.2886	4.16
AC	43.3	0.2886	12.50
DC	–28.87	0.577	–16.67
EC	28.87	0.577	16.67

$\Sigma Pk = 99.97$

$$L = 4000\text{mm}$$
$$A = 1000\text{mm}^2$$
and $$E = 200\ \text{kN/mm}^2$$

$\therefore$ $$\Delta_c = \frac{L}{AE}\sum Pk = \frac{4000}{1000 \times 200} \times 99.97$$

i.e., $$\Delta_c = 1.99\ \text{mm}$$

Example 4.4 Determine the horizontal displacement of roller support of the truss shown in Fig.4.4 (a). The cross-sectional areas of all top chord members are 6000 mm² and the other members have cross-sectional area = 3000 mm². Take E = 200 kN/mm².

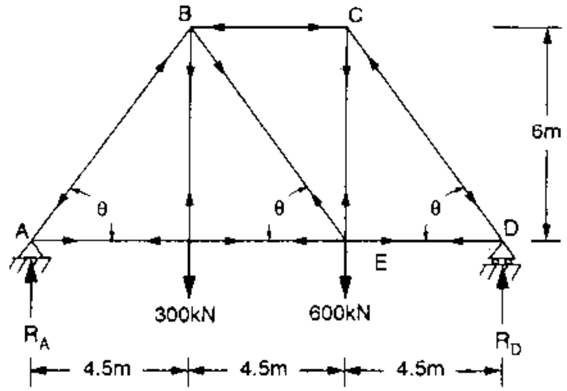


Figure 4.4 (a)

Solution

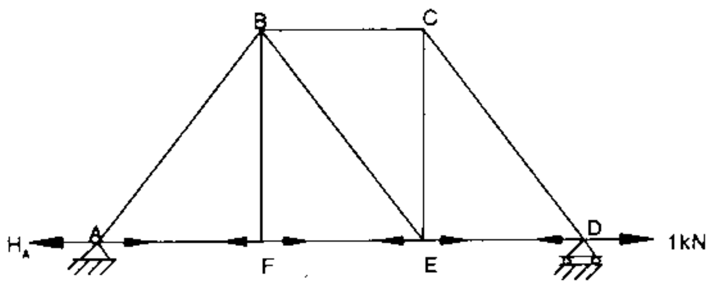


Figure 4.4 (b)

Forces in the members due to given loads (P–forces)

$$\sum M_A = 0 \rightarrow$$
$$R_D \times 13.5 - 300 \times 4.5 - 600 \times 9 = 0$$
$\therefore$ $$R_D = 500\ \text{KN} \quad \therefore R_A = 300 + 600 - 500 = 400\ \text{kN}$$

Noting that, for inclined members, sin θ = 0.8 and cos θ = 0.6, analysis is carried out.

At joint A,

$$P_{AB} = \frac{400}{0.8} = 500 \text{ kN (comp)}$$

$$P_{AF} = 500 \times 0.6 = 300 \text{ kN (tensile)}$$

At joint F,

$$P_{BF} = 300 \text{ kN (tensile)}$$

$$P_{EF} = P_{AF} = 300 \text{ kN (tensile)}$$

At joint B,

$$P_{BE} \times 0.8 = 500 \times 0.8 - 300 = 100$$

$$\therefore \quad P_{BE} = \frac{100}{0.8} = 125 \text{ kN (tensile)}$$

$$P_{BC} = 500 \times 0.6 + 125 \times 0.6 = 375 \text{ (comp)}$$

At joint D,

$$P_{DC} = \frac{500}{0.8} = 625 \text{ kN (comp)}$$

$$\therefore \quad P_{DE} = 625 \times 0.6 = 375 \text{ kN (tensile)}$$

At joint C,

$$P_{CE} = 625 \times 0.8 = 500 \text{ kN (tensile)}$$

The nature of forces are shown in Fig.4.4 (a) and are entered in Table 4.4.

Forces due to unit horizontal force at roller support D (k forces):

It can be easily seen that

$$k_{AF} = k_{FE} = k_{ED} = 1 \text{ kN tensile}$$

and all other member forces are zero. The nature of forces are shown in Fig.4.4 (b) and the values are entered in Table 4.4.

Table 4.4

Member	*Length in mm*	*Area in mm²*	*P– force in kN*	*k– force in kN*	$\frac{PkL}{A}$
AB	7500	6000	–500	0	0
BC	4500	6000	–375	0	0
CD	7500	6000	–625	0	0
DE	4500	3000	375	1	562.5
EF	4500	3000	300	1	450
FA	4500	3000	300	1	450
BF	6000	3000	300	0	0
CE	6000	3000	500	0	0
BE	7500	3000	125	0	0

$$\sum\frac{PkL}{A} = 1462.5$$

Horizontal movement of roller support,

$$\Delta_{HD} = \sum\frac{PkL}{A} = \frac{1}{E}\sum\frac{PkL}{A}$$

$$= \frac{1}{200} \times 1462.5$$

$$= 7.312\text{mm}$$

Example 4.5 A steel truss of span 15 m is loaded as shown in Fig.4.5. The cross-sectional area of each member is such that it is subjected to a stress of 100 N/mm². Find the vertical deflection of the joint C. Take E = 200 kN/mm².

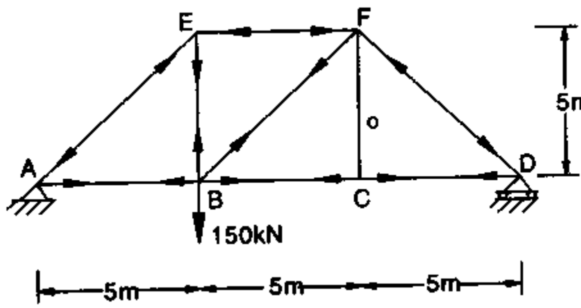


Figure 4.5 (a)

Solution

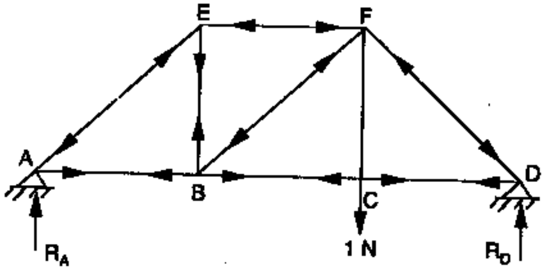


Figure 4.5 (a)

Forces due to given loading (P-forces) : It is given that all members are proportioned such that the stress $\frac{P}{A}$ = 100 N/mm^2. By the method of joint the direction of joint forces are marked in Fig. 4.5(a) and from this the nature of stresses are noted in Table 4.5 with tension as positive and compression as negative.

Forces due to unit vertical load at C (k-forces) : 1 N vertical load is applied at C as shown in Fig. 4.5(b).

$$R_D = \frac{1 \times 10}{15} = \frac{2}{3} \text{N}$$

$$\therefore \qquad R_A = \frac{1}{3} \text{N}$$

At joint A,

$$k_{AE} \sin 45^\circ = \frac{1}{3} \qquad \text{or} \qquad k_{AE} = \frac{1}{3}\sqrt{2} \text{ N (comp)}$$

$$k_{AB} = \frac{1}{3}\sqrt{2} \cos 45^\circ = \frac{1}{3} \text{ N (tension)}$$

At joint E,

$$k_{EB} = \frac{1}{3}\sqrt{2} \sin 45^\circ = \frac{1}{3} \text{ N (tensile)}$$

$$k_{EF} = \frac{1}{3}\sqrt{2} \cos 45^\circ = \frac{1}{3} \text{ N (comp)}$$

At joint D,

$$k_{DF} \sin 45^\circ = \frac{2}{3} \qquad \text{or} \qquad k_{DF} = \frac{2}{3}\sqrt{2} \text{ (comp)}$$

$$k_{DC} = \frac{2}{3}\sqrt{2}\ \cos 45° = \frac{2}{3}\ \text{N (tensile)}$$

At joint C,

$$k_{CF} = 1\text{N (tensile)}$$

$$k_{DC} = \frac{2}{3}\text{N (tensile)}$$

At joint B,

$$k_{BF}\cos 45° = \frac{1}{3} \qquad \text{or } k_{BF} = \frac{1}{3}\sqrt{2}\ \text{N}$$

These forces are noted and calculated in the following Table 4.5.

Table 4.5

Member	*Length in mm*	*P/A in N/mm²*	*k in N*	$\frac{PkL}{A}$
AE	$5000\sqrt{2}$	–100	$-\frac{1}{3}\sqrt{2}$	333333.33
EF	5000	–100	$-\frac{1}{3}$	166666.67
FD	$5000\sqrt{2}$	–100	$-\frac{2}{3}\sqrt{2}$	666666.67
DC	5000	100	$\frac{2}{3}$	333333.33
CB	5000	100	$\frac{2}{3}$	333333.33
BA	5000	100	$\frac{1}{3}$	166666.67
BE	5000	100	$\frac{1}{3}$	166666.67
CF	5000	0	1	0
BF	$5000\sqrt{2}$	100	$-\frac{1}{3}\sqrt{2}$	–333333.33

$$\sum\frac{PkL}{A} = 1833333.33$$

Vertical deflection of CE,

$$= \sum \frac{PkL}{AE} = \frac{1}{E}\sum \frac{PkL}{A}$$

$$= \frac{1833333.3}{200 \times 10^3}$$

$$= 9.167 \text{ mm}$$

Example 4.6 The members of the warren girder shown in Fig. 4.6(a) are so proportioned that all the members are stressed to 100 N/mm² when a vertical load of 60 kN is applied at L_1. If $E = 2 \times 10^5$ N/mm², find the vertical displacement of L_2.

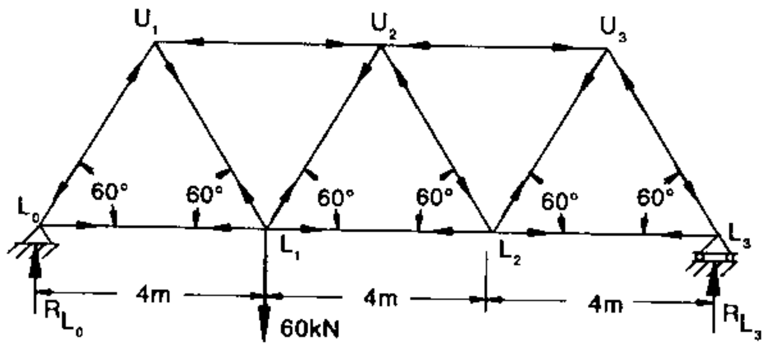


Figure 4.6 (a)

Solution

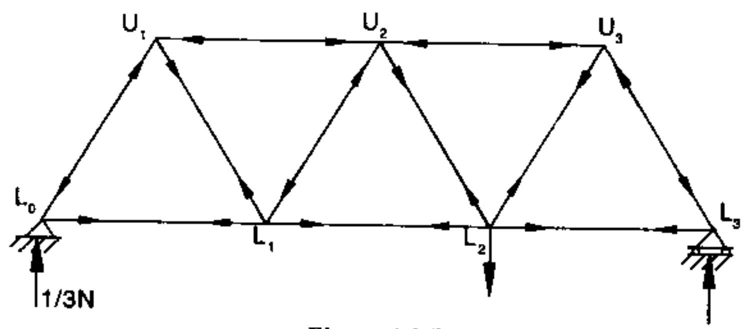


Figure 4.6 (b)

Forces due to given loads (P forces): Since the members have been so proportioned that stress $\frac{P}{A} = 100 \text{ N/mm}^2$ for all members in Fig. 4.6(a) forces are marked to identify nature and $\frac{P}{A}$. Column in Table 4.6 is filled up with tension as positive and compression as negative.

Forces due to unit vertical load at joint L (k forces) : Since vertical displacement at L_2 is required, a unit vertical load (1 N) is applied at L_2 and the forces developed are calculated.

Now,
$$R_{L_0} = \frac{1\times4}{12} = \frac{1}{3}N$$

$$R_{L_2} = 1 - \frac{1}{3} = \frac{2}{3}N$$

At joint L_0,

$$k_{L_0U_1} \sin 60^\circ = \frac{1}{3} \qquad \therefore\ k_{L_0U_1} = 0.3849N \text{ (comp)}$$

$$k_{L_0L_1} = 0.3849 \cos 60^\circ = 0.1925 \text{ N (tensile)}$$

At joint U_1,

$$k_{U_1L_1} = k_{U_1L_0} = 0.3849 \text{ N (tensile)}$$
$$k_{U_1U_2} = 0.3849\cos 60^\circ + 0.3849\cos 60^\circ = 0.3849 \text{ N (comp)}$$

At joint L_1,

$$k_{L_1U_2} = k_{U_1L_1} = 0.3849\ N\ (comp)$$
$$k_{L_1L_2} = k_{L_0L_1} + k_{U_1L_1}\cos 60^\circ + k_{L_1U_2}\cos 60^\circ$$
$$= 0.1925 + 0.3849\times0.5 + 0.3849\times0.5$$
$$= 0.5774\ N\ (tensile)$$

At joint L_3,

$$k_{L_3U_3} \sin 60^\circ = \frac{2}{3} \quad \text{or} \quad k_{L_3U_3} = 0.7698 \text{ N (comp)}$$

$$k_{L_2L_3} = 0.769 \cos 60^\circ = 0.3849 \text{ N (tensile)}$$

At joint U_3,

$$k_{U_3L_2} = k_{U_3L_3} = 0.7698 \text{ N (tensile)}$$

$$k_{U_3U_2} = 0.7698 \cos 60^\circ + 0.7698 \cos 60^\circ = 0.7698 \text{ N (comp)}$$

At joint U_2,

$$k_{L_2U_2} = k_{L_1U_2} = 0.3849 \text{ N (tensile)}$$

The values are tabulated and further calculations are carried out in Table 4.6..

From table,
$$\sum\frac{Pk}{A} = 384.91$$

$\therefore$ Vertical displacement of joint L_1

$$= \sum\frac{PkL}{AE} = \frac{L}{E}\sum\frac{Pk}{A}$$

$$= \frac{4000 \times 384.91}{2 \times 10^5}$$
$$= 7.698\text{mm}$$

Table 4.6

Member	*P/A in N/mm²*	*k in N*	$\frac{Pk}{A}$
L_0U_1	–100	–0.3849	38.49
U_1U_2	–100	–0.3849	38.49
U_2U_3	–100	–0.7698	76.98
U_3L_3	–100	–0.7698	76.98
L_0L_1	100	0.1925	19.25
L_1L_2	100	0.5774	57.74
L_2L_3	100	0.3849	38.49
L_1U_1	100	0.3849	38.49
L_1U_2	100	–0.3849	–38.49
L_2U_2	–100	0.3849	–38.49
L_2U_3	100	–0.7698	76.98

$$\sum \frac{Pk}{A} = 384.91$$

Example 4.7 The actual changes in length that takes place in the various members of the truss under a given load P are shown in Fig.4.7. Determine the deflection of joint C.

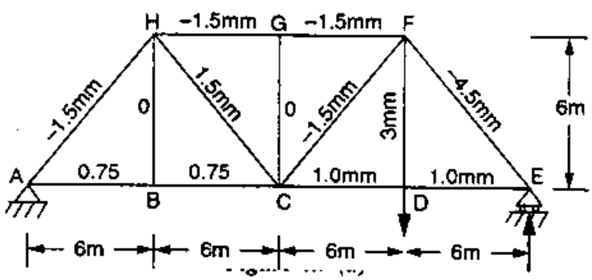


Figure 4.7 (a)

Solution

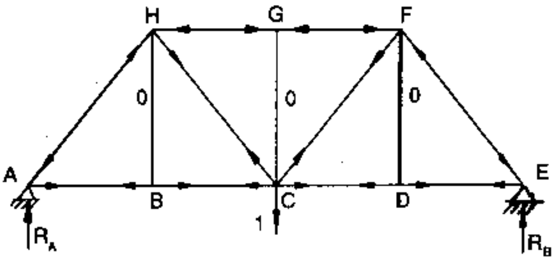


Figure 4.7 (b)

Now, to get vertical deflection at C, unit vertical force is to be applied at C and resulting forces (k–forces) are to be found. Then

$$\Delta_C = \sum \frac{Pk}{AE} = \Sigma\, \Delta L k$$

The change in length ΔL for all members due to the given loadings are already given.

Hence deflection Δ_C can be found.

k - forces :

Referring to Fig.4.7(b).

$R_A = R_E = 0.5$

At joint A,

$k_{AH} \sin 45° = 0.5$ $\quad$ or $k_{AH} = 0.5\sqrt{2}$ (comp)

$k_{AB} = 0.5\sqrt{2}\cos 45° = 0.5$ (tensile)

At joint B,

$k_{BH} = 0$

$k_{BC} = k_{AB} = 0.5$ (tensile)

At joint H,

$k_{HC} = k_{AH} = 0.5\sqrt{2}$ (tensile)

$k_{HG} = 0.5\sqrt{2}\sin 45° + 0.5\sqrt{2}\sin 45° = 1$ (comp)

At joint G,

$k_{GC} = 0$

Other values are entered in Table 4.7 by using symmetry and further calculations are carried out

Table 4.7

Member	$\Delta = \frac{Pk}{AE}$	*K in N*	*KΔL*
AB	0.75	0.5	0.375
BC	0.75	0.5	0.375
CD	1.0	0.5	0.50
DE	1.0	0.5	0.50
EF	–4.5	$-0.5\sqrt{2}$	3.182
FG	–1.5	–1.0	1.5
GH	–1.5	1.0	1.5
HA	–1.5	$-0.5\sqrt{2}$	1.061
HB	0	0	0
GC	0	0	0
FD	3.0	0	0
HC	1.5	$0.5\sqrt{2}$	1.061
FC	–1.5	$0.5\sqrt{2}$	–1.061

$\Sigma k\Delta L = 8.993$

∴ Deflection of joint

$$= \Sigma k\Delta L = 8.993 \text{ mm}$$

4.3 DEFLECTIONS DUE TO LACK OF FIT AND TEMPERATURE CHANGES

From virtual work principle, it has been proved that deflection of a joint can be obtained using the equation

$$\delta = \sum \frac{PkL}{AE} = \Sigma k\Delta L$$

where k is the unit load applied at the joint in the direction of which deflection is required.

The expression 'ΔL' i.e., changes in the length of members need not be only due to applied loads, but may be due to any other cause. Change in length may be due to selecting improper lengths while fabricating or may be due to temperature changes. Thus,

$$\Delta L = \delta L + \delta f + \delta t$$

where δL is the change in length due to given loadings

δf is the lack of fit

and δt is the changes due to temperature.

Example 4.8 Determine the vertical deflection of the joint H of the truss shown in Fig.4.8. Area of cross section of each of the member is 2000 mm^2. Take E = 200 kN/mm^2.

If the temperature of bottom chord members goes up by 20°C, what will be the additional deflection of joint H?. Given $\alpha = 12 \times 10^{-6}$ / °C

If the diagonal members are 10 mm too short before fabrication, what will be the deflection of joint due to lack of fit alone.

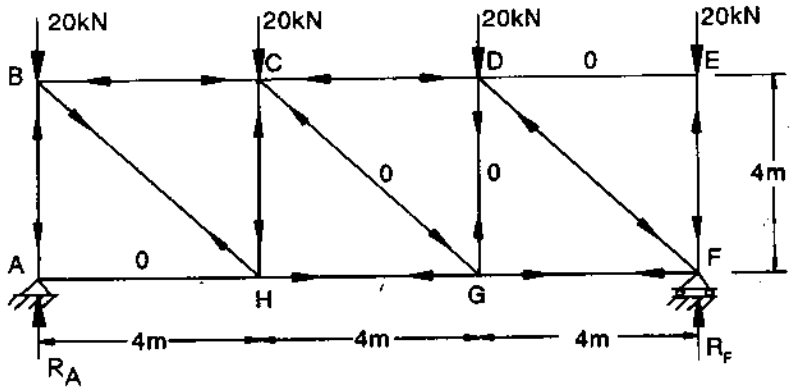


Figure 4.8 (a)

Solution

Forces in the members due to unit vertical load at H (k forces) (Ref.Fig.4.8b):

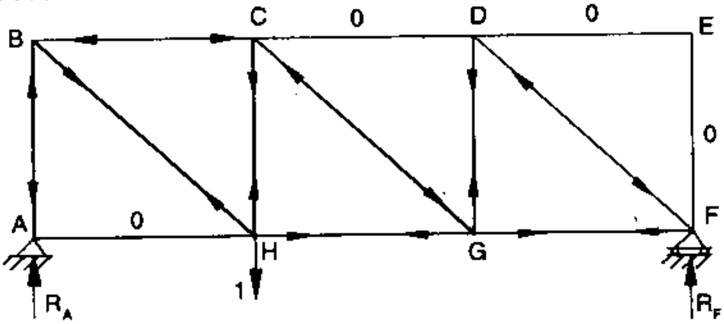


Figure 4.8 (b)

$$R_A = \frac{1 \times 8}{12} = \frac{2}{3} \quad \text{Hence } R_F = \frac{1}{3}$$

At joint A,

$$k_{AB} = \frac{2}{3} \text{ (comp) and } k_{AH} = 0$$

At joint B,

$$k_{BH} \cos 45° = k_{BA} \quad \text{or } k_{BH} = \frac{2}{3}\sqrt{2} \text{ (tensile)}$$

$$k_{BC} = \frac{2}{3}\sqrt{2} \times \cos 45° = \frac{2}{3} \text{ (comp)}$$

At joint H,

$$k_{HC} = 1 - \frac{2}{3}\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{1}{3} \text{ (tensile)}$$

$$k_{HG} = \frac{2}{3}\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{2}{3} \text{ (tensile)}$$

At joint C,

$$k_{CG} \times \frac{1}{\sqrt{2}} = \frac{2}{3} \text{ or } k_{CG} = \frac{2}{3}\sqrt{2} \text{ (comp)}$$

$$k_{CD} = \frac{2}{3} - \frac{2}{3}\sqrt{2} \times \frac{2}{3} = 0$$

At joint E,

$$k_{ED} = k_{EF} = 0$$

At joint F,

$$k_{FD} \times \frac{1}{\sqrt{2}} = \frac{1}{3} \text{ or } k_{FD} = \frac{1}{3}\sqrt{2} \text{ (comp)}$$

$$k_{FG} = \frac{1}{3}\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{1}{3} \text{ (tensile)}$$

At joint D,

$$k_{DG} = \frac{1}{3}\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{1}{3} \text{ (tensile)}$$

These forces are noted down in Table 4.8 with tension as positive and compression as negative.

P forces :

Due to symmetry, $$R_A = R_F = \frac{80}{2} = 40 \text{ kN}$$

At joint A,

$$P_{AB} = 40 \text{ KN (comp)}$$
$$P_{AH} = 0$$

At joint B,

$$P_{BH} \times \frac{1}{\sqrt{2}} = 40 - 20 = 20$$

$$P_{BH} = 20\sqrt{2} \text{ (tensile)}$$

$$P_{BC} = 20\sqrt{2} \times \frac{1}{\sqrt{2}} = 20 \text{ kN (comp)}$$

At joint H,

$$P_{HC} = 20\sqrt{2} \times \frac{1}{\sqrt{2}} = 20 \text{ kN (comp)}$$

$$P_{HG} = 20\sqrt{2} \times \frac{1}{\sqrt{2}} = 20 \text{ (tensile)}$$

At joint C,

$$P_{CG} \times \frac{1}{\sqrt{2}} = 20 - 20 = 0 \qquad \therefore P_{CG} = 0$$

$$P_{CD} = P_{BC} = 20 \text{ kN (comp)}$$

At joint G,

$$P_{GD} = 0$$
$$P_{GF} = 20 \text{ kN (tensile)}$$

At joint E,

$$P_{ED} = 0$$
$$P_{EF} = 20 \text{ (comp)}$$

At joint F,

$$P_{FD} \times \frac{1}{\sqrt{2}} = 40 - 20 = 20$$

or $$P_{FD} = 20\sqrt{2} \text{ (comp)}$$

These forces are entered in Table 4.8. Extension of various members, $\delta_L = \dfrac{PL}{AE}$ are calculated.

Table 4.8

Member	*k forces in kN*	*P in kN*	*L in mm*	$\delta L = \dfrac{PL}{AE}$	$k\delta L$	δt	$k\delta t$	δf	$k\delta f$
AB	–2/3	-40	4000	-0.4	0.2667	0	0	0	—
BC	- 2/3	–20	4000	–0.2	0.1333	0	0	0	—
CD	0	–20	4000	–0.2	0	0	0	0	—
DE	0	0	4000	0	0	0	0	0	–
EF	0	–20	4000	–0.2	0	0	0	0	—
FG	1/3	20	4000	0.2	0.0667	0.96	0.32	0	—
GH	2/3	20	4000	0.2	0.1333	0.96	0.32	0	—
HA	0	0	4000	0	0	0.96	0.32	0	—
HC	1/3	–20	4000	–0.2	–0.0667	0	0	0	—
GD	1/3	0	4000	0	0	0	0	0	—
BH	$\frac{2\sqrt{2}}{3}$	$20\sqrt{2}$	$4000\sqrt{2}$	0.4	0.3771	0	0	–10	–9.428
CG	$-\frac{2\sqrt{2}}{3}$	0	$4000\sqrt{2}$	0	0	0	0	–10	9.428
DF	$-\frac{1}{3}\sqrt{2}$	$-20\sqrt{2}$	$4000\sqrt{2}$	-0.4	0.1886	0	0	–10	4.714

$\Sigma k\delta L = 1.099$ $\Sigma k\delta t = 0.96$ $\delta f k = 4.174$

Deflection due to load alone

$$= \Sigma k\delta L$$
$$= 1.099\text{mm}$$

Deflection due to temperature alone

Now $\qquad \alpha = 12 \times 10^{-6}\ /\ ^\circ C$

$\qquad\qquad t = 20^\circ C$

L for bottom chord members = 4000 mm

For all bottom chord members

$$\delta t = \alpha t L = 12 \times 10^{-6} \times 20 \times 4000 = 0.96\text{mm}$$

For all other members, $\qquad \delta t = \alpha t L = 0$

Deflection due to temperature effect alone

$$= \Sigma k\delta t$$
$$= 0.96$$

Difference in actual length and required length 'δf' are entered in the Table 4.8 and $k\,\delta f$ values are tabulated. The deflection of joint H due to lack of fit alone

$$= \Sigma k\delta f$$
$$= 4.714\text{mm}$$

Example 4.9 A frame ABCD consists of two equilateral triangles and is hinged at A and is supported on rollers at D as shown in Fig.4.9. The sizes of the members are so proportioned that when a vertical load W = 80 kN acts at C, all the members will be stressed to 100 N/mm^2. But unfortunately the size of the member BD used is found to be 5 mm too long. What will be the vertical downward deflection of point C when load W = 80 kN is acting. Take E = 200 kN/ mm^2.

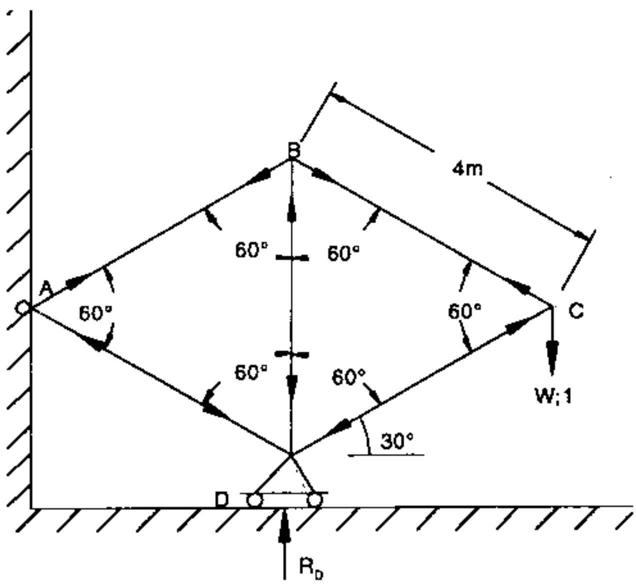


Figure 4.9

Solution

k-forces i.e., the forces in the members due to unit vertical load applied at C :

Taking moment about A,

$$R_D \times 4\sin 60° = 1 \times 2 \times 4 \sin 60°$$

or $\quad R_D = 2.0$

At joint C,

$$K_{CB} = K_{CD}$$

and $\quad K_{CB} \sin 30° + K_{CD} \sin 30° = 1$

or $\quad K_{CD} = 1$ (tensile)

$$K_{CD} = 1 \text{ (comp)}$$

At joint B,

$$k_{BA} = k_{BC} = 1 \text{ (tensile)}$$
$$k_{BD} = k_{BA} \cos 60° + k_{BC} \cos 60°$$
$$= 2 \times \cos 60° = 1 \text{ (comp)}$$

At joint D,

$$k_{DA} \sin 60° = k_{DC} \sin 60°$$
$$\therefore \quad k_{DA} = k_{DC} = 1 \text{ (comp)}$$

These forces are tabulated with tensile force as positive and compressive force as negative.

Extension due to given loading of W = 80 kN :

For all members $\frac{P}{A} = 100 \text{ N/mm}^2$

$L = 4000$ mm and $E = 200 \text{ kN/mm}^2 = 2 \times 10^5 \text{ N/nm}^2$

$$\therefore \quad \text{change in length} = \frac{PL}{AE} = \frac{100 \times 4000}{2 \times 10^5}$$
$$= 2\text{mm}$$

These values are tabulated.

Extensions due to erroneous fabrication δf : Only length of member BD is erroneous. It is 5 mm long These values are tabulated and total ΔL values are calculated.

Table 4.9

Member	*k-values*	*δL in mm*	*δf in mm*	*ΔL = δL+δF*	*kΔL*
AB	1	2	0	2	2
BC	1	2	0	2	2
CD	−1	−2	0	−2	2
DA	−1	−2	0	−2	2
BD	−1	−2	5	3	−3

$\Sigma k\Delta L = 5$

∴ Vertical deflection of joint C

$$\Sigma k\Delta L = 5\text{mm}$$

EXERCISES

1. Find the vertical deflection of the steel truss shown in Fig. 4.10 at the end C. Cross-sectional areas in mm^2 of all the members are shown in the Figure 4.10. Take $E = 200\ kN/mm^2$.

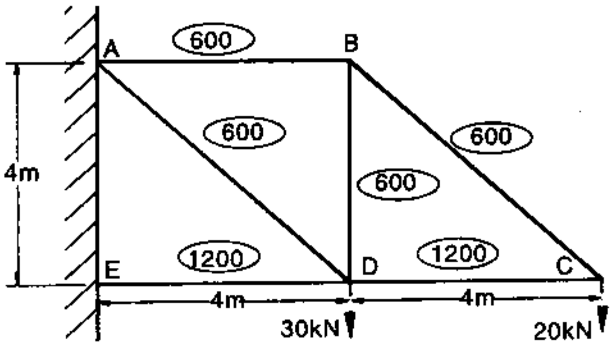


Figure 4.10

Ans : $\Delta_C = 10.60$ mm

2. Cross-sectional areas of all horizontal and vertical members of the frame shown in Fig.4.11 are $2000mm^2$ and those of inclined members are $3000mm^2$. Determine the vertical displacement of joint E. Take Young's modulus= $200\ kN/mm^2$.

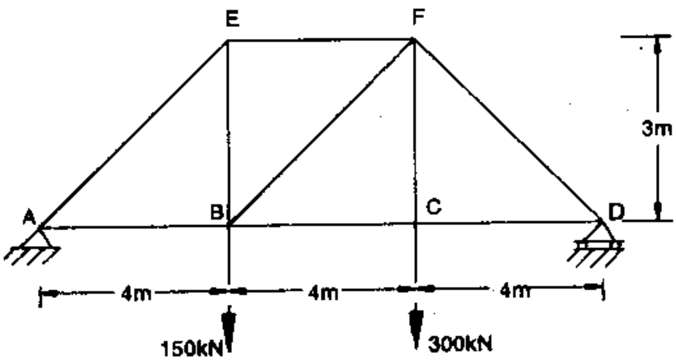


Figure 4.11

Ans : $\Delta_E = 10.83$ mm

3. Find the vertical deflection of joint D of the cantilever pin connected frame shown in Fig. 4.12. All members are having same cross-sectional area 500 mm^2 and Young's modulus E = 200 kN/mm^2. The inclined members are at 60° to the horizontal.

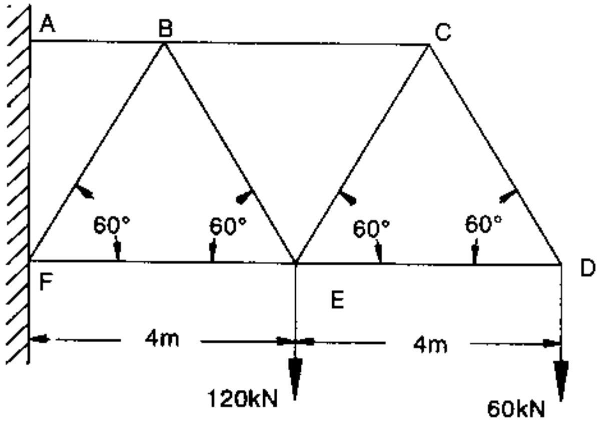


Figure 4.12

Ans : Δ_D = 5.44mm

4. Each bar of the truss shown in Fig.4.13 has a cross sectional area of 600mm^2 and E = 200 kN/mm^2. Calculate the horizontal deflection of joint C due to,
(a) Loading shown
(b) Temperature of member BC alone is increased by 15°C. Take coefficient of thermal expansion = 12×10^{-6}/ °C.
(c) Member AB being 8 mm too short.

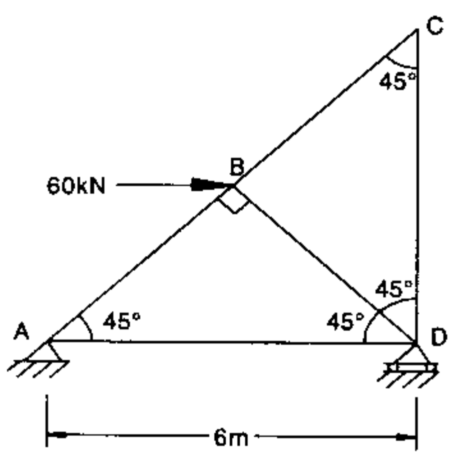


Figure 4.13

Ans : (a) 2.12 mm, (b) 1.08, (c) –11.313 mm

5. Determine the vertical displacement of joint E of truss shown in Fig.4.14 due to given loading and due to member BE and CE being 10 mm too long. Given, for all members cross-sectional area = 1000mm², Young's modulus = 200 kN/ mm².

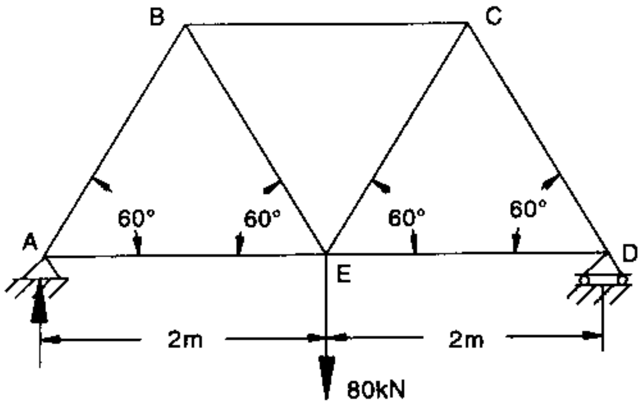


Figure 4.14

Ans : $\Delta_E = 12.746$ mm

6. Find the vertical displacement of joint E in the truss shown in Fig. 4.15, if the cross-sectional area of each vertical and horizontal member is 6000 mm² and that of each inclined member is 8000 mm². Take E = 200 kN/mm².

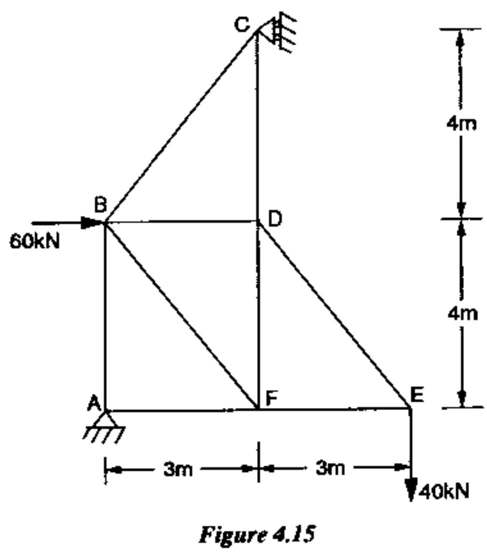


Figure 4.15

Ans : $\Delta_E = 1.098$ mm

MOVING LOADS AND INFLUENCE LINES FOR BEAMS

5

5.1 INTRODUCTION

The loads acting on a beam may be broadly classified as dead loads and live loads. Dead loads are those loads which do not change their position during the life of the beam e.g. self-weight of the beam, weight of the slab carried by the beam, finishing load etc. And loads which can change their position during the life of the beams are called live loads, e.g, weight of persons on slab/beam, weight of furnitures. etc. Bridge girders carry loads which roll over them from one end to other. The Indian Road Congress in its code of practice for road bridges, section II specifies different types of vehicular loads to be considered by giving wheel loads. Similarly, railway engine and train loadings are specified by railway bridge rules framed by the Ministry of Railways.

A designer analyses the beams for various positions of loads so as to get maximum shear force and bending moment values due to moving loads. To identify the positions of loads for maximum shear force and bending moment at specified sections, influence line diagrams can be used. In this chapter, the term *Influence Line Diagram* is explained and used for finding maximum values of shear force and bending moment values.

5.2 INFLUENCE LINE DIAGRAM

Influence line diagrams are drawn for various stress resultants like reaction, shear force, bending moment, torsion at specified points. Influence line diagram for a stress resultant is the one in which ordinate represent the value of the stress resultant for the position of unit load at the corresponding abscissa. For example if Fig. 5.1 represents influence line diagram for moment at section C in the beam AB, then the ordinate O represents the value of bending moment at C when a unit load is acting at section 1-1.

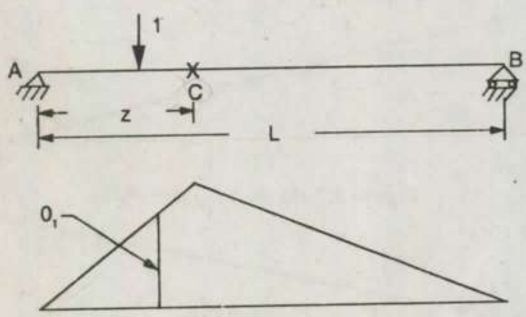


***Figure 5.1** ILD for moment at C*

Influence line diagrams are used to identify the position of moving loads for maximum value of stress resultants and for finding its maximum value.

5.3 INFLUENCE LINE DIAGRAMS FOR SIMPLY SUPPORTED BEAMS

In this article influence line diagrams for reactions at support A, support B and shear force and bending moment in a section at a distance z from the end A are drawn. Positive senses of shear force and bending moment are shown in Fig. 5.2.

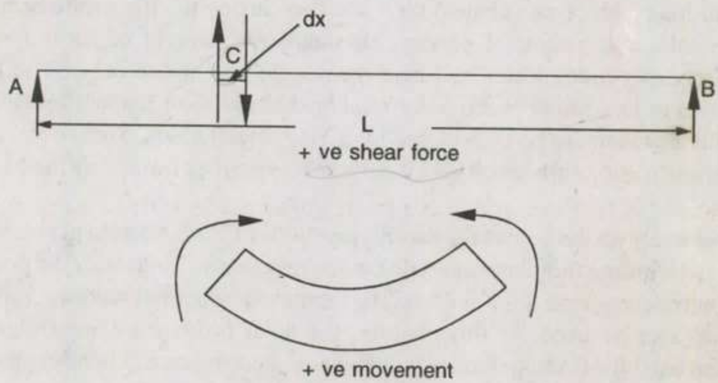


Figure 5.2

ILD for reaction R_A

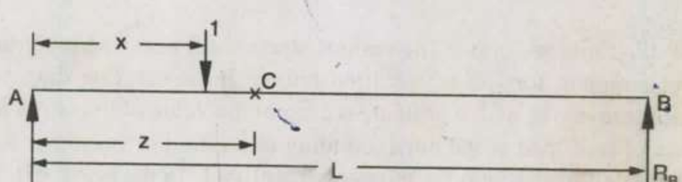


Figure 5.3 (a)

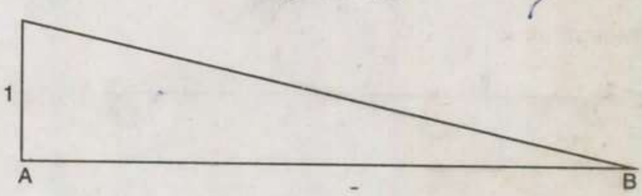


Figure 5.3 (b) *ILD for R_A*

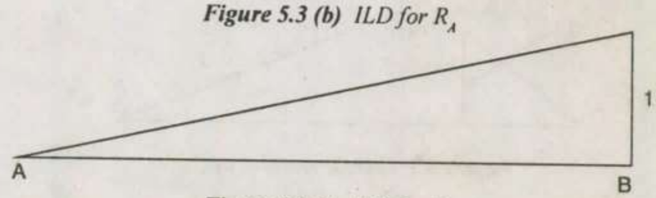


Figure 5.3 (c) *ILD for R_B*

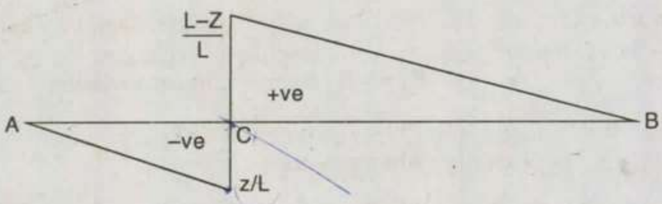


Figure 5.3 (d) ILD for F_C

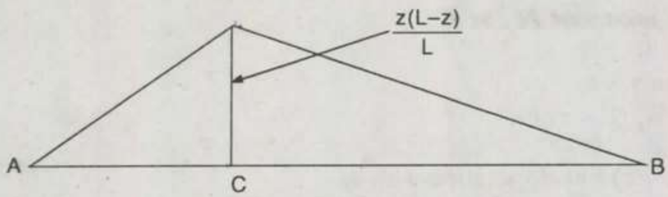


Figure 5.3 (e) ILD for M_C

Let the unit load be at a distance x from support A as shown in Fig.5.3(a).

Now $$R_A = \frac{1(L-x)}{L} = \left(1-\frac{x}{L}\right), \text{ linear variation with x}$$

when $x = 0$; $R_A = 1$
when $x = L$; $R_A = 0$

Hence ILD for R_A is as shown in Fig.5.3(b).

ILD for reaction R_B

Referring to Fig.5.3(a)

$$R_B = \frac{x}{L}, \text{ linear variation}$$

At $x = 0$; $R_B = 0$
At $x = L$; $R_B = 1$

Hence ILD for R_B is as shown in Fig.5.3(c).

ILD for shear force at C

Let C be the section at a distance z from A as shown in Fig.5.3(a).

(a) When $x < z$

$$F_C = -R_B = -\frac{x}{L}, \text{ linear variation,}$$

when $x = 0$; $F = 0$
when $x = z$; $F = z/L$

(b) When x > z

$$F_C = -R_A = \frac{L-x}{L}\text{, linear variation}$$

$$\text{when } x = z; \quad F_c = \frac{L-z}{L}$$

$$\text{when } x = L; \quad F_c = 0$$

∴ ILD for shear force at C is as shown in Fig.5.3(d).

ILD for moment M_c at C

(a) When x < z

$M_C = R_B (L - z)$

$\frac{X}{L}$ (L – z) linear variation with x,

$$\text{when } x = 0; \quad M = 0$$

$$\text{when } x = z; \quad M_C = \left(\frac{z(L-z)}{L}\right)$$

(b) When x > z

$$M_C = R_A\, z = \frac{L-x}{L} z\text{; linear variation with x}$$

$$\text{When } x = z; \quad M_C = \frac{L-z}{L} z$$

$$\text{When } x = L; \quad M_C = 0$$

Hence ILD for moment at C is as shown in Fig.5.3(e).

5.4 INFLUENCE LINE DIAGRAMS FOR CANTILEVER BEAMS

Consider a cantilever beam of span L as shown in Fig.5.4(a). Influence line diagrams for shear force and bending moments at fixed end A and at section C are to be determined. Let a unit load act at a distance x from the free end B.

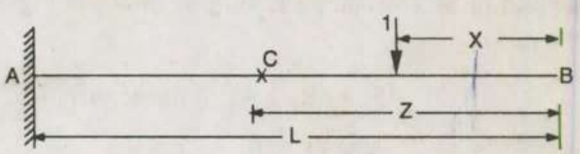


Figure 5.4 (a)

ILD for F_A

$$F_A = 1, \text{ constant}$$

Hence, ILD for F_A is as shown in Fig. 5.4 (b).

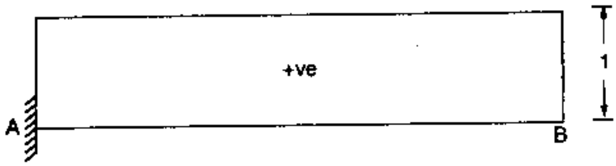


Figure 5.4 (b) *ILD for F_A*

ILD for M

$M_A = -(L - x)$; Linear variation,

when $x = 0 \quad M_A = -L$

when $x = L \quad M_A = 0$

This ILD for M_A is shown in Fig. 5.4(c).

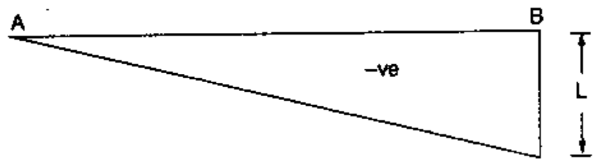


Figure 5.4 (c) *ILD for M_A*

ILD for F_C

When $x < z \qquad F_C = 1$ constant

when $x > z \qquad F_C = 0$

ILD for F_C is as shown in Fig. 5.4 (d).

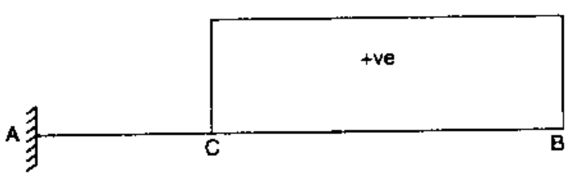


Figure 5.4 (d) *ILD for F_C*

ILD for M_c

For $x \leq z$ $\quad$ $M_C = -1\ (z-x)$, Linear variation

when $x = 0$; $\quad M_C = -z$

when $x = z$; $\quad M_C = 0$

For $x > z$ $\quad$ $M_C = 0$, constant

Hence, ILD for M_C is as shown in Fig.5.4(e).

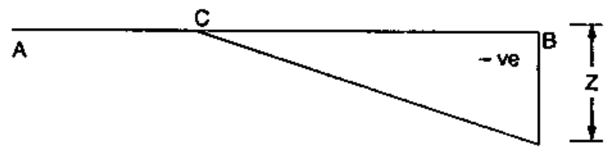


Figure 5.4 (e) ILD for M_C

5.5 INFLUENCE LINE DIAGRAMS FOR OVERHANGING BEAMS

A typical overhanging beam is shown in Fig. 5.5(a) for which ILD for the following are drawn:

1. Reaction at support A
2. Reaction at support B
3. Shear force at section D
4. Bending moment at section D
5. Shear force at section E and
6. Bending moment at section E

Fig.5.5.(b), (c) and (d) show positions of unit loads and origin for measuring their distance x.

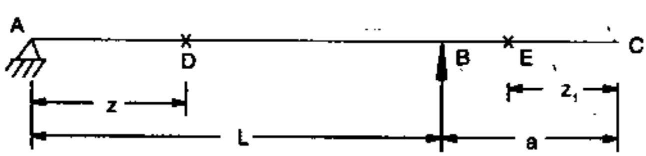


Figure 5.5 (a)

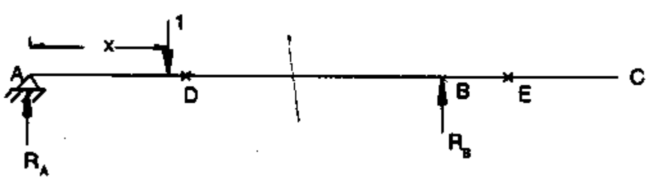


Figure 5.5 (b)

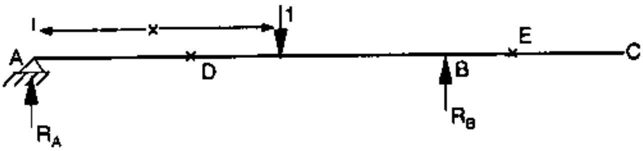


Figure 5.5 (c)

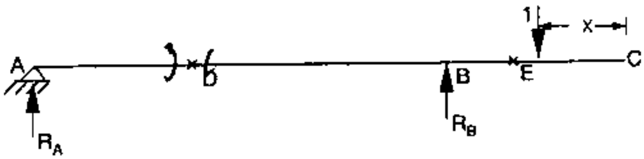


Figure 5.5 (d)

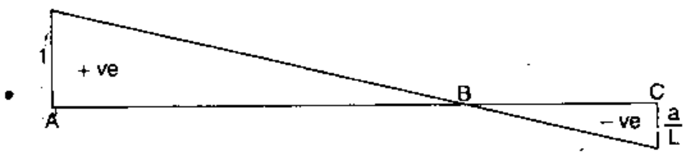


Figure 5.5 (e) ILD for R_A

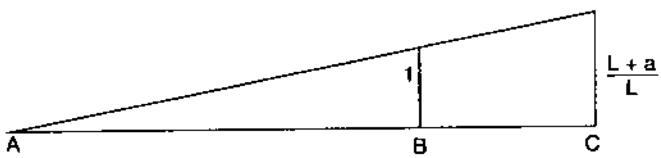


Figure 5.5 (f) ILD for R_B

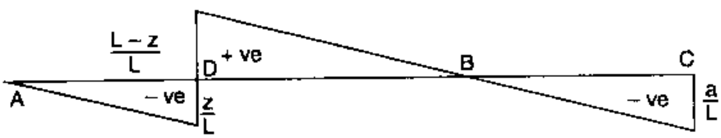


Figure 5.5 (g) ILD for F_D

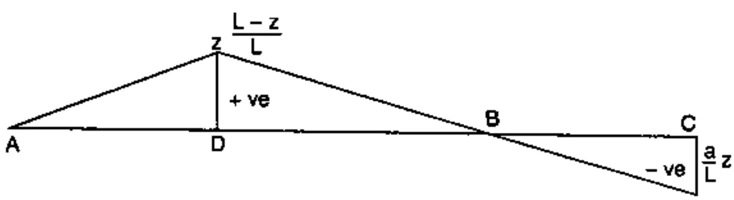


Figure 5.5 (h) ILD for M_D

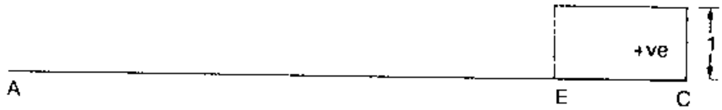


Figure 5.5 (i) ILD for F_L

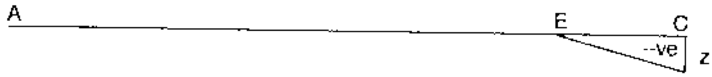


Figure 5.5 (j) ILD for M_l

ILD for R_A

When unit load is in portion AB (Fig.5.5(b)).

$$R_A = \frac{L-x}{L}, \text{ linear variation,}$$

when $x = 0$; $R_A = 1$

when $x = L$; $R_A = 0$

When unit load is in portion BC (Fig.5.5(d)).

$$R_A = -\frac{a-x}{L}, \text{ i.e. } R_A \text{ is downward linear variation}$$

when $x = 0$; $R_A = -a/L$

when $x = a$; $R_A = 0$

ILD for R_A is as shown in Fig. 5.5(e).

ILD for R_B

When unit load is in portion AB (Fig.5.5(b)).

$$R_B = \frac{x}{L}, \text{ linear variation,}$$

at $x = 0$; $R_B = 0$

at $x = L$; $R_B = 1$

When the unit load is in portion BC (Fig.5.5(d)).

$$R_B = \frac{L+a-x}{L}, \text{ linear variation,}$$

at $x = 0$; $R_B = \frac{L+a}{L}$

at $x = a$; $R_B = 1$

ILD for R_B is as shown in Fig.5.5(f).

ILD for F_D

When unit load is in portion AD

$$F_D = -R_B = -\frac{x}{L}, \text{ linear variation,}$$

at $x = 0$; $\quad F_D = 0$

at $x = z$; $\quad F_D = -\frac{z}{L}$

For portion DB (refer Fig.5.5(c)).

$$F_D = R_A = \frac{L-x}{L}, \text{ linear variation,}$$

when $x = z$; $\quad F_D = \frac{L-z}{L}$

when $x = L$; $\quad F_D = 0$

When unit load is in portion BC (refer Fig.5.5(d)).

$$F_D = R_A = -\frac{a-x}{L}$$

when $x = 0$; $\quad F_D = -\frac{a}{L}$

when $x = a$; $\quad F_D = 0$

Hence ILD for F_D is as shown in Fig.5.5(g).

ILD for M_D

When unit load is in portion AD (Fig. 5.5 (b)).

$$M_D = R_B(L-z) = \frac{x}{L}(L-z), \text{ linear variation,}$$

at $x = 0$; $M_D = 0$

at $x = z$; $M_D = \frac{z(L-z)}{L}$

When unit load is in portion DB (Fig. 5.5(c)).

$$M_D = R_A z = \frac{L-x}{L} z, \text{ linear variation,}$$

at $x = z$; $M_D = \frac{L-z}{L} z$

at $x = L$; $M_D = 0$

When unit load is in portion BC (Fig.5.5(d)).

$$M_D = R_A z = -\frac{a-x}{L} z, \text{ linear variation}$$

$$\text{at } x = 0;\ M_D = -\frac{a}{L}z$$

$$\text{at } x = a;\ M_D = 0$$

ILD for M_D is as shown in Fig.5.5(h).

ILD for F_E

Referring to Fig.5.5(b), when unit load is in portion AB

$$F_E = 0$$

When the load is in portion BE,

$$F_E = 0$$

when the load is in portion EC Fig.5.5(d).

$$F_E = 1, \text{ constant}$$

∴ ILD for F_E is as shown in Fig.5.5(i).

ILD for M_E

When the load is in portion AB and BE

$$M_E = 0$$

When the load is in portion EC (Fig.5.5(d)).

$$M_E = -1\,(z_1 - x), \text{ linear variation}$$

$$\text{at } x = 0; \quad M = -z_1$$

$$\text{at } x = z; \quad M_E = 0$$

ILD for M_E is as shown in Fig.5.5(j).

5.6 INFLUENCE LINE DIAGRAM FOR DOUBLE OVERHANGING BEAM

The typical double overhanging beam is shown in Fig.5.6. Influence line diagrams for R_A, R_B, shear and bending moment at E are to be found. So far, the detailed procedure for finding influence line diagrams was explained, now a short cut procedure is presented. Note that influence line diagram for all stress results discussed so far are varying linearly.

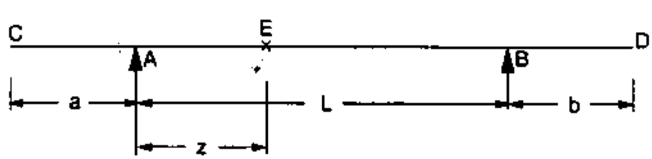


Figure 5.6 (a)

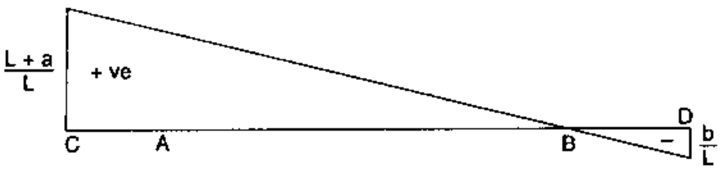


Figure 5.6 (b) ILD for R_A

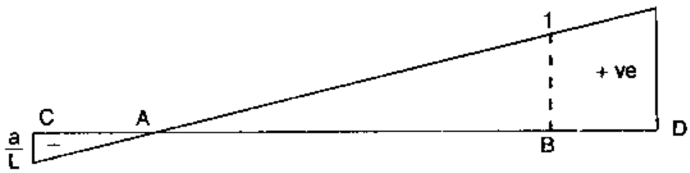


Figure 5.6 (c) ILD for R_B

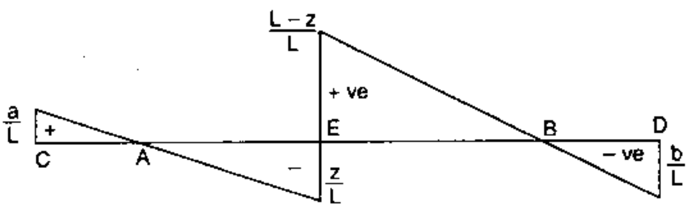


Figure 5.6 (e) ILD for F_E

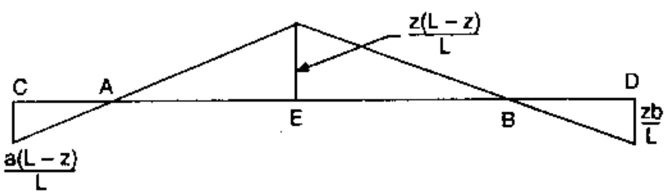


Figure 5.6 (e) ILD for M_E

ILD for R_A

When unit load is at C, $R_A = \dfrac{L + a}{L}$

Load at A, $R_A = 1$

Load at B, $R_A = 0$

and load at D, $R_A = -\dfrac{b}{L}$

The variation is linear. Hence, ILD for R_A is as shown in Fig. 5.6(b).

ILD for R_B

When load is at C, $R_B = -\dfrac{a}{L}$

load at A, $R_B = 0$

load at B, $R_B = 1$

load at D, $R_B = \dfrac{L+b}{L}$

ILD for R_B is shown in Fig.5.6(c).

ILD for F_E

When load is at C, $F_E = -R_B = \dfrac{a}{L}$

load at A, $F_E = -R_B = 0$

load at E, just to the left of E, $F_E = -R_B = -\dfrac{z}{L}$

load at E, just to the right of E, $F_E = R_A = \dfrac{L-z}{L}$

load at B, $F_E = R_A = 0$

load at D, $F_E = R_A = -\dfrac{b}{L}$

ILD for shear force at E, is as shown in Fig.5.6(d).

ILD for M_E

When load is at C, $M_E = R_B\,(L-z) = -\dfrac{a}{L}\,(L-z)$

load at A, $M_E = R_B\,(L-z) = 0$

load at E, $M_E = R_B\,(L-z) = \dfrac{z}{L}\,(L-z)$

load at B, $M_E = R_A z = 0$

load at D, $M_E = R_A z = -\dfrac{b}{L}\,z$

Variation is linear throughout.

∴ ILD for bending moment at E is as shown in Fig.5.6(e).

5.7 INFLUENCE LINE DIAGRAM IN A BALANCED CANTILEVER BEAM

A typical balanced cantilever beam is shown in Fig.5.7(a) and (b). It is to be noted that points C and D are subjected only to shear and not to moment. Influence line diagrams are drawn for reaction at A, moment at G and moment at B.

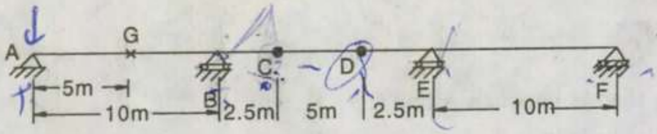


Figure 5.7 (a)

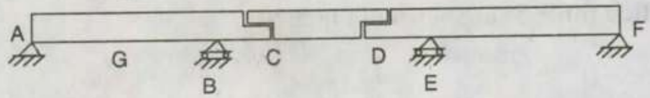


Figure 5.7 (b)

wing

ILD for R_A

When unit load is at A, $R_A = 1$

load at B, $R_A = 0$

load at C, $R_A = -\frac{2.5}{10} = -0.25$

load at D, $R_A = 0$

when the load is in the portion D to F, $R_A = 0$

ILD is as shown in Fig.5.7(c).

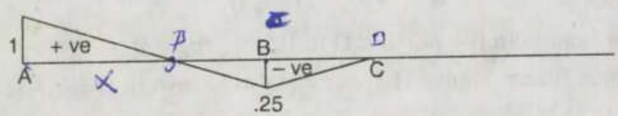


Figure 5.7 (c)

ILD for R_B

When the load is at A, $R_B = 0$

load at B, $R_B = 1$

load at C, $R_B = \frac{1 \times 12.5}{10.0} = 1.25$

load at D, $R_B = 0$

when the load is in the portion D to F, $R_B = 0$

Variation is linear throughout.

Hence ILD for the R_B is as shown in Fig.5.7(d).

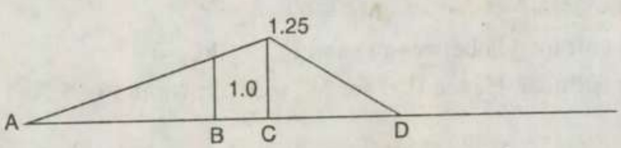


Figure 5.7 (d)

ILD for F_C

When the load is between A and C, $F_C = 0$

when the load is at C, $F_C = 1$

load at D, $F_C = 0$

when the load is between D to F, $F_C = 0$

Hence ILD for F_C is as shown in Fig.5.7(e).

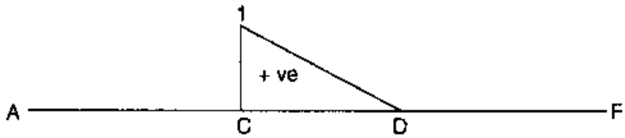


Figure 5.7 (e)

ILD for F_G

When the load is at A, $F_G = -R_B = 0$

load is at just to the left of G, $F_G = -R_B = -0.5$

load is at just to the right of G, $F_G = R_A = 0.5$

load at B, $F_G = R_A = 0$

load at C, $F_G = R_A = -\ 0.25$

load at D, $F_G = 0$

when the load is in the portion D to F, $F_G = 0$

Variation is linear. Hence ILD for F_G is as shown in Fig.5.7(f).

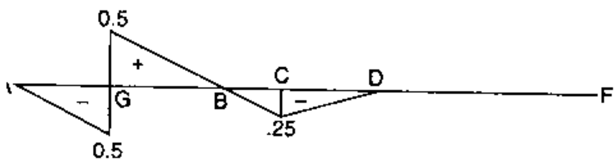


Figure 5.7 (f) ILD for F_G

ILD for M_G

When the load is at A, $M_G = R_B \times 5 = 0$

load at G, $M_G = R_B \times 5 = 0.5 \times 5 = 2.5$

load at B, $M_G = R_A \times 5 = 0 \times 5 = 0$

load at C, $M_G = R_A \times 5 = -\ 0.25 \times 5 = -1.25$

load at D, $M_G = 0$

when the unit load is between D and F, $M_G = 0$

Variation is linear. Hence ILD for M_G is as shown in Fig.5.7(g)

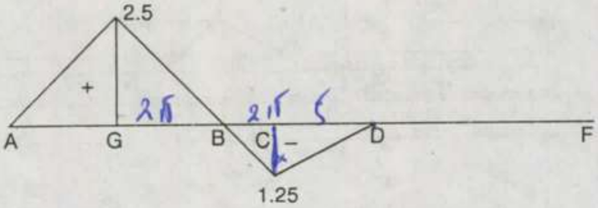


Figure 5.7 (g)

ILD for M_B

When the unit load is between A and B

$$M_B = R_B \times 0 = 0$$

When the load is at C, $M_B = R_A \times 10 = 0.25 \times 10 = -2.5$

when the load is in portion D to F, $M_B = 0$

ILD for M_B is as shown is Fig. 5.7(h).

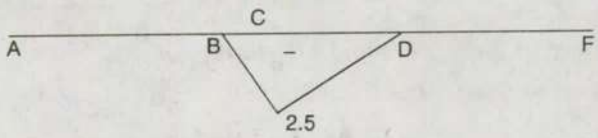


Figure 5.7 (h) ILD for M_B

5.8 INFLUENCE LINE DIAGRAMS FOR GIRDERS SUPPORTING FLOOR BEAMS

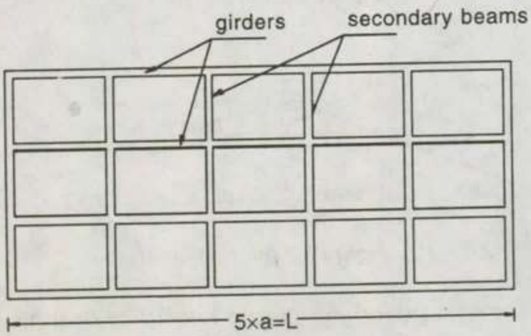


Figure 5.8 (a)

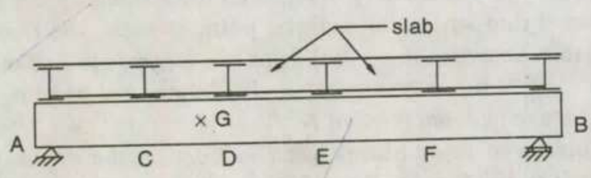


Figure 5.8 (b)

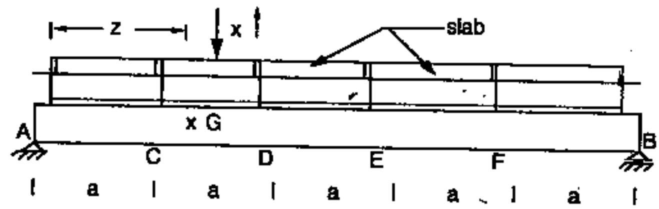


Figure 5.8 (c)

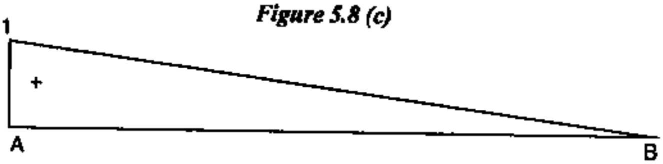


Figure 5.8 (d) ILD for R_A

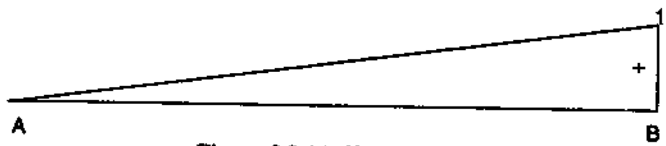


Figure 5.8 (e) ILDS for R_B

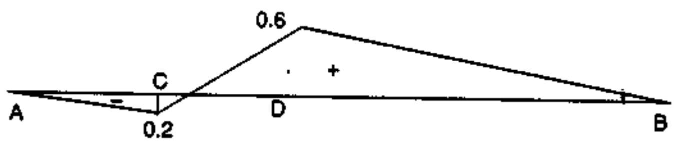


Figure 5.8 (f) ILD for F_G

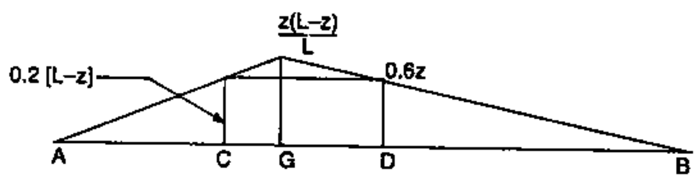


Figure 5.8 (g) ILD for M_G

Figure 5.8 (a)–(g) For girders supporting floor beams

A typical floor beam girder system is shown in Fig.5.8. In this system, floor slabs transfer the loads to the secondary beams and the secondary beams transfer the loads to girders. From structural analysis point of view, the floor slab may be seen as a cut slab as shown in Fig.5.8(c). Now it is required to find out the influence line diagrams for reaction at A, reaction at B, shear and bending moment at G which is at a distance z from the end A

The introduction of floor beams has no effect on the reactions R_A and R_B. Thus influence line diagram for R_A and R_B is similar to that of simply supported beams. Fig.5.8(d) and 5.8(e) shows the ILD for R_A and R_B respectively.

ILD for F_G

When the unit load is in the portion AC, shear force at G is – R

∴ when the load is at A, $F_G = -R_B = 0$

load at C, $F_G = -R_B = -\frac{a}{L} = -\frac{a}{5a} = -0.2$

when the load is in portion DB, $F_G = R_A$

∴ load at B, $F_G = 0$

load at D, $F_G = \frac{3a}{L} = \frac{3a}{5a} = 0.6$

When the load is in portion CD, say at x from D,

$F_G = R_A - R_C$, where R_C is load transferred at C.

$$= \frac{1(3a+x)}{L} - \frac{1x}{a}$$

$$= \frac{3a+x}{5a} - \frac{x}{a}$$

$$= 0.6 - \frac{4x}{5a} = 0.6 - 0.8\frac{x}{a}$$

It varies linearly with x

when load is at C, where x =a, $F_G = -0.2$

when load is at D, where x = 0, $F_G = 0.6$

∴ ILD for F_G is as shown Fig.5.8(f).

ILD for M_G

When the load is in the portion AC

$$M_G = R_B(L-z)$$

∴ when load is at A, $M_G = 0$

load at C, $M_G = \frac{a}{L}(L-z)$

when the unit load is in portion DB

$$M_G = R_A z$$

∴ When load is at D, $M_G = \frac{3a}{L} \times 1 \times z = 0.6z$, since L = 5a.

When the unit lad is at B, $M_G = 0 \times 2 = 0$

When the unit load is in portion CD

Let $M_G = R_A z - R_C(z-a)$

$$= \frac{1(3a+x)z}{L} - \frac{x}{a}(z-a)$$

$$= \frac{3az + xz - 5xz + 5ax}{5a} \text{ since } L = 5a$$

It is varying linearly.

$$\text{At } x = a, M_G = \frac{-az + 5a^2}{L} = \frac{a}{L}(5a - z)$$

$$= 0.2\,(L-z), \text{ since } 5a = L$$

$$\text{At } x = 0, M_G = R_A z = \frac{3az}{L} = 0.6\,z$$

Hence ILD for M_G is as shown in Fig.5.8(g).

5.9 USE OF INFLUENCE LINE DIAGRAMS

Using influence diagrams for shear force and bending moment at a section, SF and BM at that section for any given loading can be found easily.

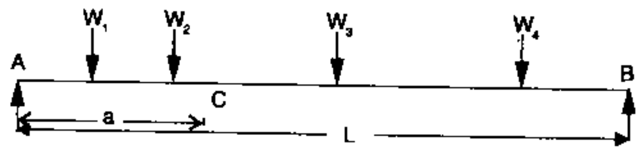


Figure 5.9 (a)

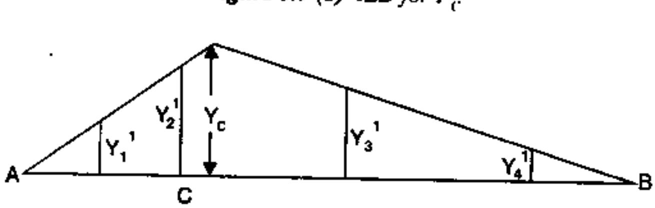


Figure 5.9 (b) ILD for F_C

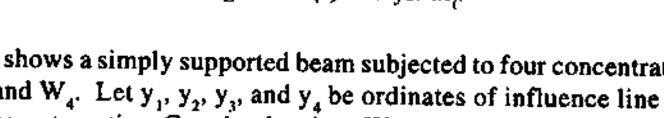


Figure 5.9 (c) ILD for M_C

Fig.5.9(a) shows a simply supported beam subjected to four concentrated loads W_1, W_2, W_3 and W_4. Let y_1, y_2, y_3, and y_4 be ordinates of influence line diagram for shear force at section C at load points W_1, W_2, W_3, and W_4 respectively. (Refer Fig.5.9(b)). Then shear force at C due to given loadings,

$$F_C = W_1 y_1 + W_2 y_2 + W_3 y_3 + W_4 y_4$$

Similarly, if y_1', y_2', y_3' and y_4' are the ordinates of ILD for M_C (Fig.5.9(c)). Then

$$M_C = W_1 y_1' + W_2 y_2' + W_3 y_3' + W_4 y_4'$$

Fig. 5.10(a) shows a simply supported beam subjected to uniformly distributed load w/unit length. If an elemental length dx is considered referring to Fig.5.10(b),

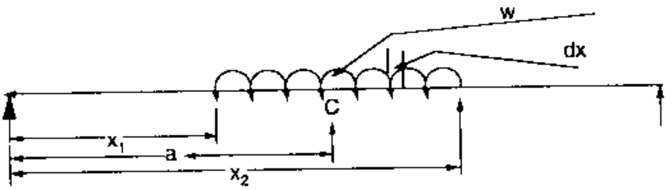


Figure 5.10 (a)

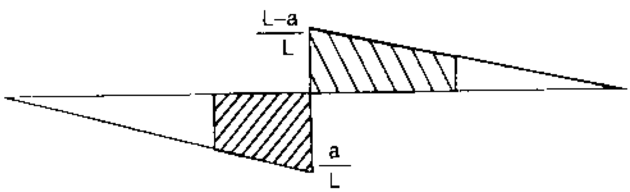


Figure 5.10 (b) *ILD for F_c*

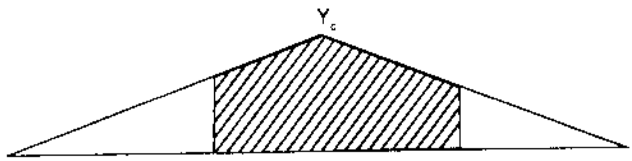


Figure 5.10 (c) *ILD for M_c*

S.F. at C due to elemental load w dx

$$= y\ w\ dx$$

$\therefore$ S.F. at C due to udl from x_1 to x_2

$$= \int_{x_1}^{x_2} y\ w\ dx$$

$$= w \times \text{Area of shaded portion in ILD for } F_C$$

Similarly, bending moment at C due to uniformly distributed load is equal to the intensity of load times the area under the load in ILD for M_C (fig.5.10(c)).

Example 5.1 Using influence line diagrams determine the shear force and bending moment at section C in the simply supported beam shown in Fig.5.11(a).

Solution

S.F at C

Influence line diagram for shear force at C is as shown in Fig.5.11(b).

$$\therefore\ F_C = -40 \times \frac{1}{7} - 10 \times \frac{1}{2}\left[\frac{1}{7} + \frac{2}{7}\right] \times 2 + \frac{10 \times 1}{2}\left[\frac{5}{7} + \frac{4}{7}\right] 2 + 60 \times \frac{3}{7} + 80 \times \frac{2}{7}$$

$$= 51.43 \text{ kN}$$

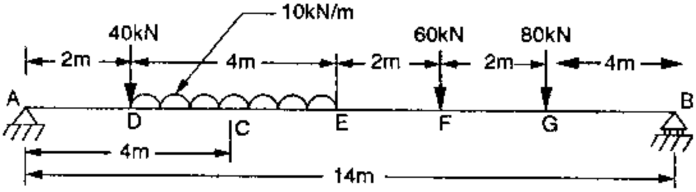


Figure 5.11 (a)

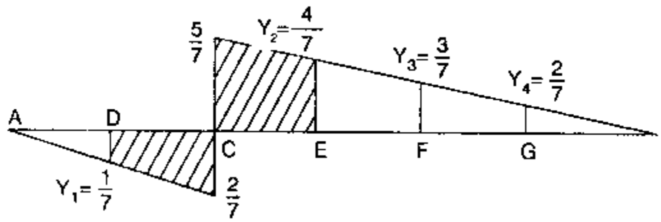


Figure 5.11 (b) ILD for S.E. at C

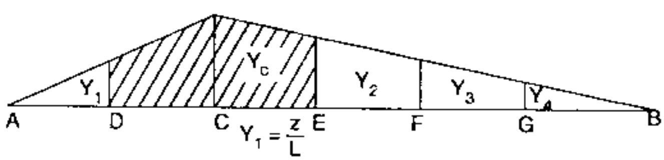


Figure 5.11 (c) ILD for B.M at C

ILD for bending moment at C is as shown in Fig. 5.11(c).

Maximum ordinate, $y_C = \dfrac{4(14-4)}{14} = \dfrac{20}{7}$

$$\therefore \qquad y_1 = \frac{10}{7},\ y_2 = \frac{8}{10}\times\frac{20}{7} = \frac{16}{7}$$

$$y_3 = \frac{6}{10}\times\frac{20}{7} = \frac{12}{7}$$

$$y_4 = \frac{4}{10}\times\frac{20}{7} = \frac{8}{7}$$

$$\therefore \qquad M_C = 40\times\frac{10}{7} + \frac{10\times1}{2}\left[\frac{10}{7}+\frac{20}{7}+\frac{20}{7}+\frac{16}{7}\right]2 + 60\times\frac{12}{7} + 80\times\frac{8}{7}$$

$$= 345.71 \text{ kNm}$$

5.10 MAXIMUM S.F AND B.M VALUES DUE TO MOVING LOADS

A designer finds the position of moving loads for which shear force and bending moment values are maximum. Influence line diagrams are used for this purpose and to find the values of maximum shear force and bending moment values. In this article, the following types of loads on a simply supported girder are considered.

1. Single point load
2. Uniformly distributed load longer than the span
3. Uniformly distributed load shorter than the span
4. A train of point loads.

5.10.1 Beam Subjected to a Single Point Load

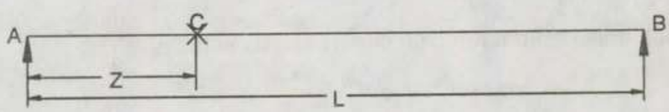


Figure 5.12 (a)

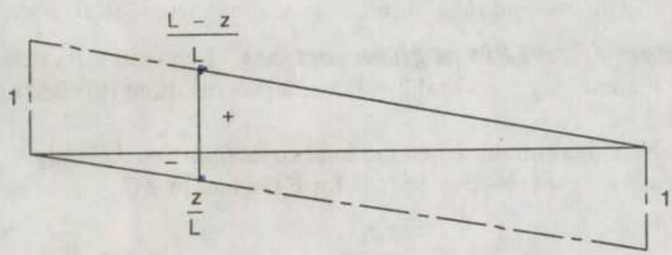


Figure 5.12 (b) ILD for F_c

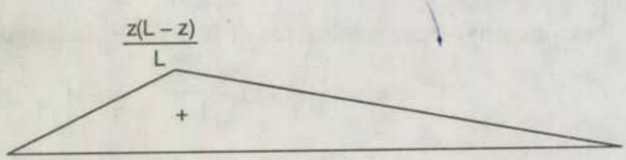


Figure 5.12 (c) ILD for M_c

(a) Maximum S.F and B.M at a given point Let w be the moving load and the values of maximum S.F and B.M required be at C, which is at a distance z from A (Fig.5.12(a)). ILD for S.F and B.M are as shown in Fig.5.12(b) and 5.12(c) respectively.

From Fig.5.12(b), it is clear that maximum negative S.F occurs when the load is just to the left of section C and its value is $= w\ \dfrac{z}{L}$.

Similarly, maximum positive S.F. occurs when the load is just to the right of the section and its value is $= W \dfrac{L-z}{L}$.

From ILD for moment M_C, it is clear that maximum bending moment will occur when the load is on the section itself and its value is $= \dfrac{Wz\,(L-z)}{L}$.

(b) Absolute maximum values any where in the beam For achieving this, ordinate of ILD should be maximum. Negative S.F ordinate is maximum when $z = L$ and is equal to 1.

$\therefore$ Absolute maximum negative S.F = w and it occurs when the load is at $z = L$, i.e., at B.

For positive S.F, ILD ordinate has maximum value of 1, when $z = 0$; i.e., at A.
Absolute maximum S.F = w.

Ordinate of ILD for moment is maximum when $z = \dfrac{L}{2}$. Hence when a load is at midspan, absolute maximum moment occurs and its value is $\dfrac{WL}{4}$.

5.10.2 Uniformly Distributed Load Longer Than the Span

Let a uniformly distributed load of intensity w move from left to right.

(a) Maximum SF and BM at given sections Load intensity times the area of ILD over loaded length gives the value of stress resultant (SF/BM). Referring again to Fig.5.12.

Negative SF is maximum, when the load covers portion AC only.
Max. negative F_C = w × Area of ILD for F_C in length AC

$$= w\,\frac{1}{2}\,z\,\frac{z}{L} = \frac{wz^2}{2L}$$

Positive SF is maximum when the uniformly distributed load occupies the portion CB only and

$$\text{Max. positive } F_C = w \times \text{Area of ILD for } F_C \text{ in length CB}$$

$$= w\,\frac{1}{2}\,(L-z)\,\frac{L-z}{L}$$

$$= \frac{w(L-Z)^2}{2L}$$

From Fig.5.12(c), it is clear that maximum moment at C will be, when the udl covers entire span,

$$M_{\text{i-c}}\ \max = w \times \text{Area of ILD for } M_C$$

$$= w\,\frac{1}{2}L\,\frac{z(L-z)}{L}$$

$$= \frac{wz\,(L-z)}{2}$$

(b) Absolute maximum values any where in the beam Negative SF is maximum when $z = L$ i.e. at B when the load occupies entire span AB

$$\text{Absolute max. S.F} = w\,\frac{1}{2}\,1\,L = \frac{1}{2}\,wL$$

Similarly, maximum positive S.F occurs when $z = 0$; i.e., at A when the load occupies entire span AB

$$\text{Max. +ve S.F} = w\frac{1}{2}\,1\,L = \frac{wL}{2}$$

Maximum moment at any section

$$= \frac{1}{2}\,\frac{wz(L-z)}{L}\,L = \frac{1}{2}\,wz\,(L-z)$$

This is maximum, when $z = \frac{L}{2}$ i.e., at mid-span.

$$\therefore \text{ Absolute max. moment } = \frac{1}{2}\,w\,\frac{L}{2}\,(L-L/2) = \frac{wL^2}{8}\text{, at mid-span.}$$

5.10.3 Uniformly Distributed Load Smaller Than the Span

Let the length of uniformly distributed load w/unit length be d. Let it move from left to right over beam AB of span L. We are considering the case when, d<L. Now, positions of this load for maximum shear force and bending moment at section C (Ref.Fig.5.13) are to be determined.

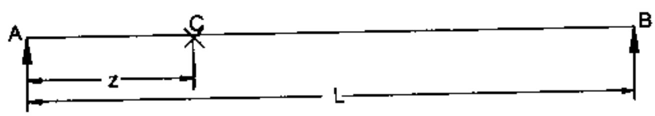


Figure 5.13 (a)

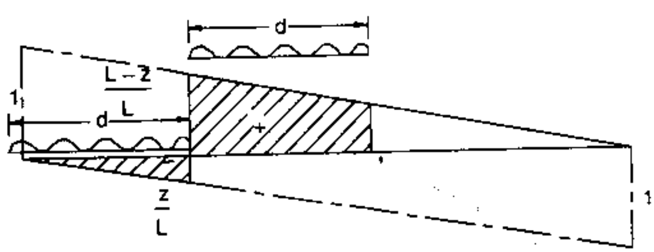


Figure 5.13 (b) ILD for F_C

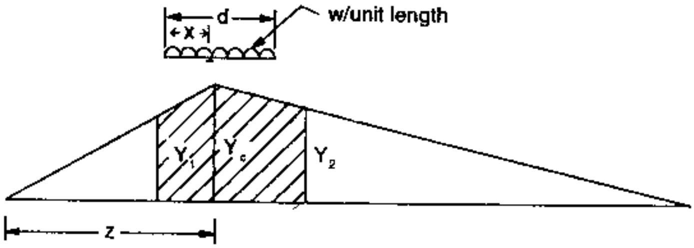


Figure 5.13 (c) ILD for M_C

From ILD for shear force at C, it is clear that maximum shear force will develop when the head of the load reaches the section.

For maximum positive shear force the tail of the udl should reach the section.

Maximum bending moment will develop at C when the load is partly to the left of the section and partly to the right of section. Let the position of the section be as shown in Fig.5.13(c). Referring to this figure,

$$M_c = w\frac{x(y_1 + y_c)}{2} + w\,(d-x)\,\frac{(y_c + y_2)}{2}$$

For M_c to be maximum,

$$\frac{dM_c}{dx} = 0 = \frac{w(y_1 + y_c)}{2} - \frac{w(y_c + y_2)}{2}$$

i.e., $y_1 = y_2$

Thus moment at C will be maximum when the ordinates of ILD for M_c at head and tail of the udl are equal.

Now $\quad y_1 = y_2$

i.e., $\quad \dfrac{(z-x)}{z}y_c = \dfrac{(L-z)-(d-x)}{L-z}y_c$

$\therefore \quad (z-x)(L-z) = z(L-z-d+x)$

$$Lz - z^2 - Lx + xz = Lz - z^2 - dz + xz$$

i.e., $Lx = dz$

or $\quad \dfrac{x}{d} = \dfrac{z}{L}$

i.e. bending moment at a section is maximum when the load is so placed that the section divides the load in the same ratio as it divides the span.

Once the position of moving load is identified for maximum values the required values can be easily found.

Position for absolute maximum moment Obviously for this y_c should be maximum.

Now, $\quad y_c = \dfrac{z(L-z)}{L}$

For y_c to be maximum,

$$\frac{dy_c}{dz} = 0 = L - 2z$$

or

$$z = \frac{L}{2}$$

i.e., Absolute maximum moment occurs at mid-span. The position of the load is to be such that the section divides the load in the same ratio as it divides the span which means that absolute maximum moment C.G of the load will be at the mid-span.

Example 5.2 A simply supported beam has a span of 15 m. Uniformly distributed load of 40 kN/m and 5 m long crosses the girder from left to right. Draw the influence line diagram for shear force and bending moment at a section 6 m from left end. Use these diagrams to calculate the maximum shear force and bending moment at this section.

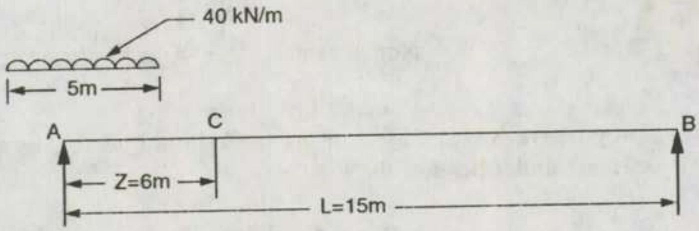


Figure 5.14 (a)

Solution

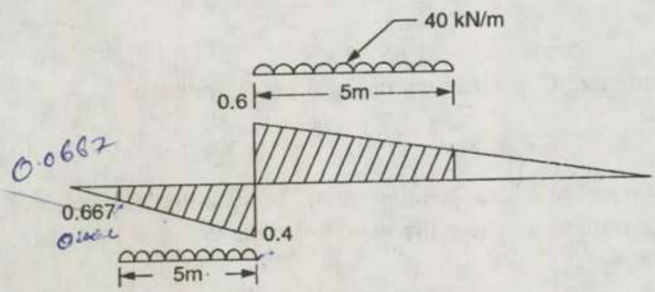


Figure 5.14 (b) ILD for F_c

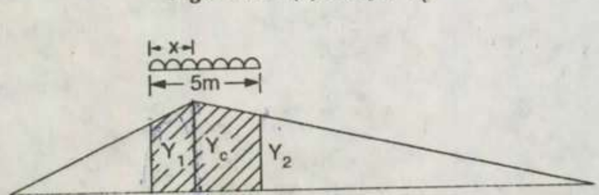


Figure 5.14 (c) ILD for M_c

The beam is shown in Fig.5.14. For point 'C', which is at z = 6m from A, ILD for shear force F_c and bending moment M_c are to be found.

ILD for F_c : ILD ordinate at just to the left of C is

$$= -\frac{z}{L} = -\frac{6}{15} = -0.4$$

ILD ordinate at just to right of C

$$= \frac{L-z}{L} = \frac{15-6}{15} = 0.6$$

ICD for F_C is as shown in Fig. 5.14(b).

At C, negative S.F is maximum when the head of load touches C. At this time, tail of the udl is at a distance of 1 m from A as shown in Fig. 5.14(B). Ordinate under tail end of load

$$= \frac{1}{6} \times 0.4 = 0.0667$$

$$\therefore \qquad \text{Negative max. } F_C = 40 \times \left[\frac{0.0667 + 0.4}{2}\right] \times 5$$

$$= 46.667 \text{ kN}$$

For maximum positive S.F at C, tail of the load should be at C as shown in Fig.5.14(b). Ordinate under head of the load

$$= \frac{4}{9} \times 0.6 = 0.267$$

$$\therefore \qquad \text{Max. positive SF} = 40 \times \text{Area of ILD under the load}$$

$$= 40 \times \frac{0.6 + 0.267}{2} \times 5$$

$$= 86.67 \text{ kN}$$

ILD for moment at C is as shown in Fig.5.14(c) in which

$$\frac{z(L-z)}{L} = \frac{6 \times 9}{15} = 3.6$$

For maximum moment, load position should be such that the section divides the load in the same ratio as it divides the span. Referring to Fig. 5.14(c).

$$\frac{x}{5-x} = \frac{6}{9}$$

or $\qquad 9x = 30 - 6x$

or $\qquad x = 2\text{m}$

$$y_1 = \frac{6-2}{6} \quad y_c = \frac{4}{6} \times 3.6 = 2.4$$

y_2 will be same as y_1

Max. moment = w × Area of ILD for M_c under the loaded length

$$= 40\left[\frac{2.4 + 3.6}{2} \times 2 + \frac{3.6 + 2.4}{2} \times 3\right]$$

$$= 600 \text{ kN-m}$$

5.10.4 A Train of Concentrated Loads

A train of concentrated loads moving over a simply supported beam from left to right is shown in Fig. 5.15(a).

It is required to find

a. Maximum shear force at C
b. Maximum bending moment at C
c. Absolute maximum shear force in the beam
d. Absolute maximum bending moment in the beam.

(a) Maximum shear force at C Influence line diagram for shear force at C is shown in Fig. 5.15(b). As soon as w_1 enters the span negative shear force develops at C. It increases as the load moves on. Some more loads may enter the span and hence the rate of increase in S.F goes up. This will continue till the load w_1 reaches the section C. As soon as w_1 cross section C, it contributes to positive shear, thus reducing the negative shear. Hence there will be a drop in shear force value. Further movement causes more increase in shear force till the second load reaches C. There is a second peak value and a sudden drop, when the second load crosses. Thus, shear force will have a peak value whenever a load is on the section. Highest value among these peak values is to be selected. By two or three trial values, it is possible to get maximum negative shear force value. It is to be noted that for maximum negative shear force, most of the loads are to the left of the section.

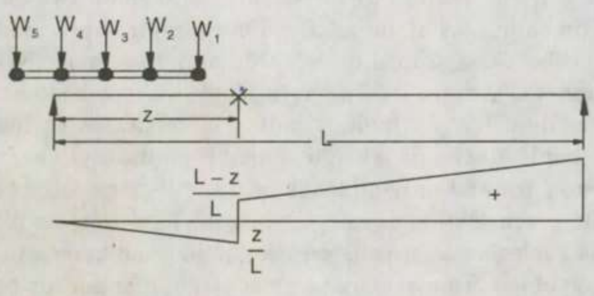


Figure 5.15 ILD for S.F at C

Similarly, for maximum positive shear force, there are peak values whenever a load comes on the section and the maximum value is obtained when most of the loads are to the right of the section.

(b) Maximum bending moment Let R_1 be the resultant of the loads on the left of the section and R_2 be resultant of the loads on the right of the section.

Distance between R_1 and R_2 be d and R_1 be at a distance x from C.

Let ordinate of ILD for moment at C be y_1 under R_1 and y_2 under R_2 and maximum ordinate at C be y_c (refer Fig.5.16).

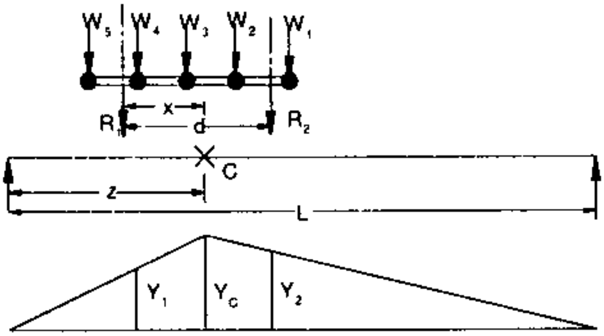


Figure 5.16

$$M_c = R_1 y_1 + R_2 y_2$$

$$= R_1 \frac{z-x}{z} y_c + R_2 \frac{(L-z)-(d-x)}{L-z} y_c$$

For M_c to be maximum

$$\frac{dM_c}{dx} = -\frac{R_1 y_c}{z} + R_2 \frac{y_c}{L-z} = 0$$

$$\frac{R_1}{z} = \frac{R_2}{L-z}$$

i.e., the average load on the left side portion of the beam is same as the average load on the right side portion of the beam. But, seldom we get exactly equal average load on both sides of the section. For example, when load w_i is to the left of the section, the average load on left side may be heavier. When it just rolls over the section, the average load on right-hand side may become heavier. Hence the above condition for maximum bending moment can be interpreted as the bending moment is maximum when that load is on the section.

Thus, due to a train of moving loads on a simply supported beam, maximum moment at the given section develops when the load w_i is on the section where the load w_i is such that as it rolls on the section and comes to the other side, heavier portion of the beam becomes lighter and lighter portion becomes heavier.

In case of some load entering and some leaving the span, the change of portion heavier becoming lighter and lighter portion becoming heavier may happen under more than one particular load. All such cases are to be considered to identify which position gives maximum moment at the section.

(c) Absolute maximum shear force At any section, influence line ordinate for negative shear is $\frac{z}{L}$ and for positive shear it is $\frac{L-z}{L}$. Hence when $z = 0$ i.e., at support A, ILD ordinate for +ve shear force is maximum (=1) and when z = L. i.e., at support B, ILD ordinate for negative shear force is maximum (= 1). ILD for shear force at support sections A and B are as shown in Fig.5.17.

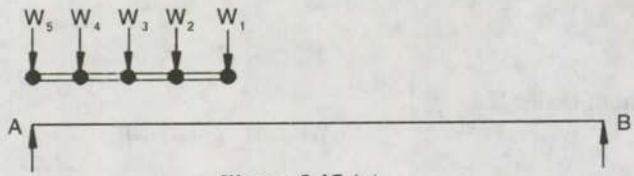


Figure 5.17 (a)

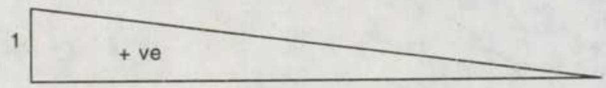


Figure 5.17 (b) *ILD for S.F at A*

Figure 5.17 (c) *ILD for S.F at B*

Obviously, maximum shear force occurs when one of the load is on support A. When the load starts moving from left to right, contribution of leading loads to shear force at A decreases but more number of loads may come on the beam and they will contribute to additional shear. However, no general conclusions can be drawn to say whether increase due to additional load is more or decrease due to the reduced contriubtion from leading loads is more. It needs a few trials to arrive at conclusions. However, it can be definitely said that one of the loads should be on the support A to get absolute maximum positive S.F.

Similarly, to get absolute maximum negative shear force, one of the loads should be on support B (just to the left of the section) and a few trials may be required to get absolute maximum negative shear force which occurs at support B.

(d) Maximum moment under a load Let a train of concentrated loads w_1, w_2, w_3 .... move on a simply supported beam AB from left to right as shown in Fig.5.18. Now the condition for moment to be maximum under wheel load w_2 is required. Let R be the resultant of all loads. Let its distance from w_2 be 'd' and from support A be x (refer Fig.5.18).

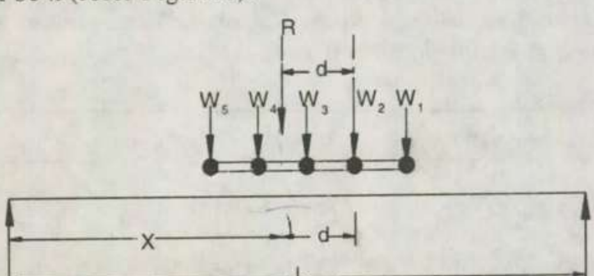


Figure 5.18

Now, $$R_A = \frac{R(L-x)}{L}$$

∴ Moment under load W_2

$$M = R_A\,(x+d) - Rd$$

$$= R\left[\frac{L-x}{L}\right](x+d) - Rd.$$

$$= \frac{R}{L}\,[Lx - x^2 + Ld - xd] - Rd$$

For the moment to be maximum

$$\frac{dM}{dx} = 0 = \frac{R}{L}\,[L - 2x - d]$$

or $$x = \frac{L}{2} - \frac{d}{2} \qquad \text{(i)}$$

Distance of W_2 from A

$$= x+d = \frac{L}{2} - \frac{d}{2} + d$$

$$= \frac{L}{2} + \frac{d}{2} \qquad \text{(ii)}$$

From (i) and (ii) we can conclude, for moment to be maximum under any particular load, the load and the resultant should be equidistant from the mid-span.

(e) Absolute maximum bending moment Influence line diagram ordinate for bending moment is maximum at the centre of span. Hence, bending moment will be maximum near the centre of the span when heavier loads are near the centre. Since the maximum moment always occurs under a wheel load, it can be concluded that *absolute maximum moment will occur under one of the loads when the resultant of all the loads and the load under consideration are equidistant from the centre of the beam.* The maximum moment under possible loads can be evaluated and the maximum of these selected as absolute maximum.

Example 5.3 Four point loads 8, 15, 15 and 10 kN have centre to centre spacing of 2 m between consecutive loads and they traverse a girder of 30 m span from left to right with 10 kN load lending. Calculate the maximum bending moment and shear force at 8 m from the left support.

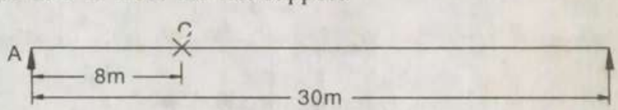


Figure 5.19 (a)

Solution

The beam is shown in Fig.5.19. ILD for shear force at 8m from left support is shown in Fig.5.19 along with possible load position for maximum negative shear force. Maximum negative S.F at C.

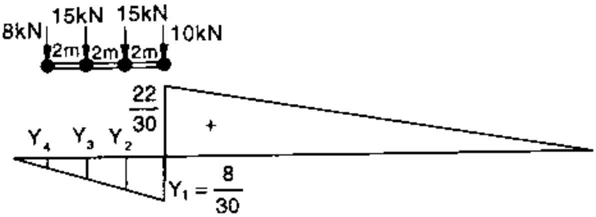


Figure 5.19 (b)

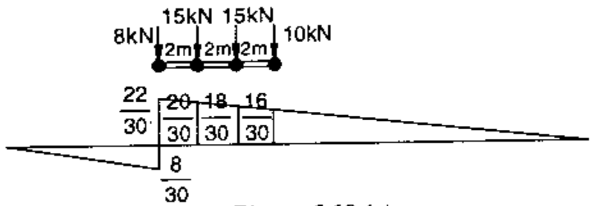


Figure 5.19 (c)

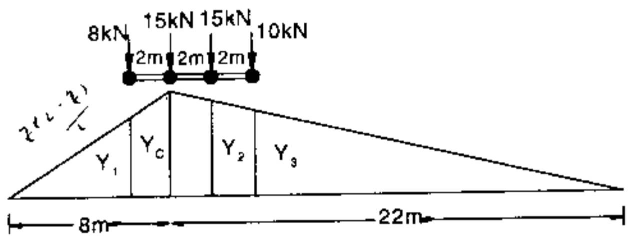


Figure 5.19 (d)

$= 10y_1 + 15y_2 + 15y_3 + 8y_4$

$= 10 \times \frac{8}{30} + 15 \times \frac{6}{30} + 15 \times \frac{4}{30} + 8 \times \frac{2}{30}$

$= 8.2$ kN

For maximum +ve SF at C, load position is as shown in Fig.5.19(c).

SF at C

$= 10 \times \frac{16}{30} + 15 \times \frac{18}{30} + 15 \times \frac{20}{30} + 8 \times \frac{22}{30}$

$= 30.2$ kN

Check for another positon i.e., when $w_3 = 15$ kN load is on the section.

$$\text{S.F at C} = 10 \times \frac{18}{30} + 15 \times \frac{20}{30} + 15 \times \frac{22}{30} - 8 \times \frac{6}{30}$$

$= 25.4$ kN

$\therefore$ Maximum +ve shear force is = 30.2 kN

ILD for bending moment at C is as shown in Fig.5.19(d). The maximum ordinate,

$$y_c = \frac{z(L-z)}{L} = \frac{8(30-8)}{30} = 5.867$$

To find the load position for maximum moment, average load on portion AC and CB are to be found as loads cross section C one after another.

Table 5.1

Load crossing	*Average load* AC W_{1av}	*Average load* BC W_{2av}	*Remarks*
10 kN	$\frac{38}{8}$	$\frac{10}{22}$	$W_{1av} > W_{2av}$
15 kN	$\frac{23}{8}$	$\frac{25}{22}$	$W_{1av} > W_{2av}$
15kN	$\frac{8}{8}$	$\frac{40}{22}$	$W_{1av} < W_{2av}$

Hence load positon for maximum moment at C is when second 15 kN load is on C. Referring to Fig.5.19(d),

$$\text{Max. } M_c = 8y_1 + 15y_c + 15y_2 + 10Y_3$$

$$= 8\,\frac{6}{8}\,y_c + 15y_c + 15\,\frac{20}{22}\,y_c + 10\,\frac{18}{22}\,y_c$$

$$= 251.21 \text{ kN-m, since } y_c = 5.867$$

Example 5.4 The simply supported beam shown in Fig.5.20 is subjected to a set of four concentrated loads which move from left to right. Determine,

a. Absolute maximum shear
b. Absolute maximum moment in the beam.

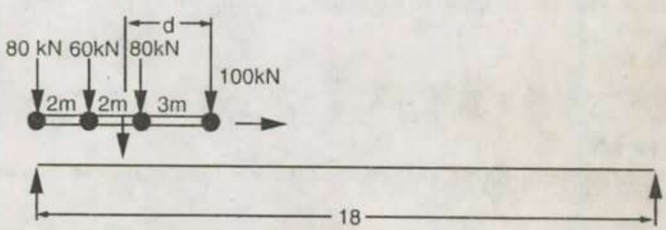


Figure 5.20 (a)

Solution

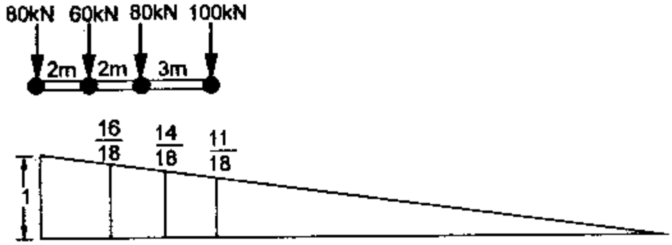


Figure 5.20 (b)

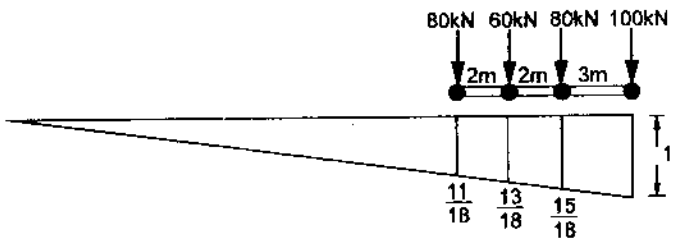


Figure 5.20 (c)

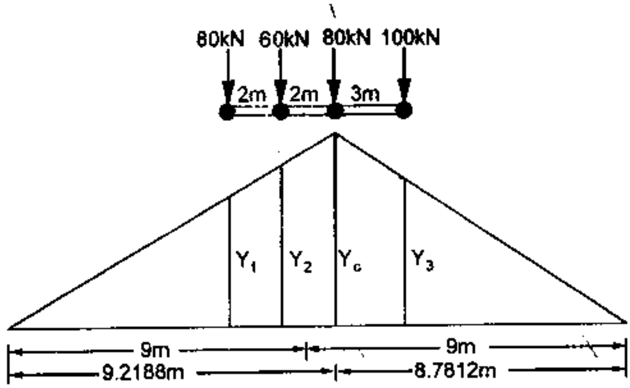


Figure 5.20 (d)

For absolute maximum – ve shear, the load position is as shown in Fig. 5.20(b).

$$\text{Absolute maximum – ve shear} = 80 \times 1 + 60 \times \frac{16}{18} + 80 \times \frac{14}{18} + 100 \times \frac{11}{18}$$

$$= 256.67 \text{ kN}$$

For absolute maximum + ve shear, the load position is as shown in Fig. 5.20c.

$$\text{Absolute max. + ve shear} = 80 \times \frac{11}{18} + 60 \times \frac{13}{18} + 80 \times \frac{15}{18} + 100$$

$$= 258.889 \text{ kN}$$

Note: No additional trials are made since end loads are not light. For finding absolute maximum moment, first C.G. of loads is to be found and the position of loads to be determined.

Taking moment about leading load 100 kN

$$a = \frac{80 \times 3 + 60 \times 5 + 80 \times 7}{100 + 80 + 60 + 80} = 3.4375\text{m}$$

i.e., resultant is very close to leading 80 kN load.

Hence maximum moment is likely to occur under 80 kN leading load. The distance between this load and the resultant is d = 3.4375 – 3 = 0.4375 m.

For maximum bending moment this load should be at

$$= \frac{L}{2} + \frac{d}{2} = \frac{18}{2} + \frac{0.4375}{2}$$

$$= 9.2188\text{m}$$

ILD ordinate for a section at 9.2188 is

$$y_c = \frac{9.2188 \times (18 - 9.2188)}{18}$$

$$= 4.497$$

ILD for this case is shown in Fig.5.20(d).

Absolute max. B.M

$$= 80y_1 + 60y_2 + 80y_c + 100y_3$$

$$= \left[80 \times \frac{5.2188}{9.2188} + 60 \times \frac{7.2188}{9.2188} + 80 + 100 \times \frac{5.7812}{8.7812}\right] 4.497,$$

since $y_c = 4.497$

$$= 1070.77 \text{ kN-m}$$

Note: Additional trial to check the moment under 60 kN load is not made since :ntre of gravity is very close to the load which is heavier than 60 kN load.

Example 5.5 A train of concentrated loads shown in Fig. 5.21 moves from left to right on a simply supported girder of span 16m. Determine the absolute maximum shear force and bending moment developed in the beam.

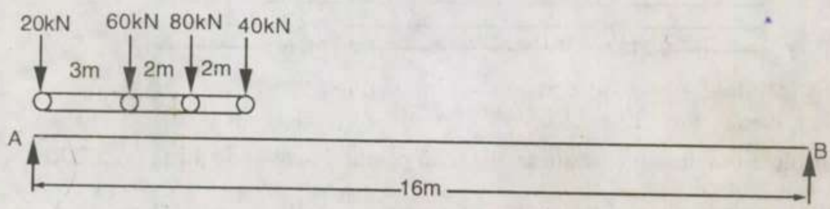


Figure 5.21 (a)

Solution

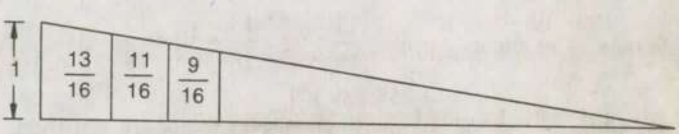


Figure 5.21 (b) ILD for F_A

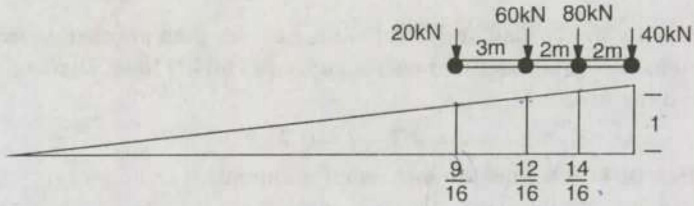


Figure 5.21 (c) ILD for F_B

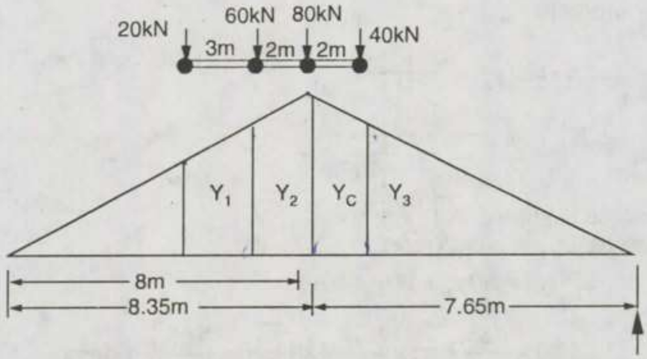


Figure 5.21 (d) ILD for M at 8.35 from A

ILD for shear force at A is as shown in Fig. 5.21(b). When 20 kN load is just on A,

$$\text{Positive S.F at A} = 20 \times 1 + 60 \times \frac{13}{16} + 80 \times \frac{11}{16} + 40 \times \frac{9}{16}$$

$$= 146.25 \text{ kN}$$

Since 20 kN load is lighter one more trial with 60 kN load at A is made. For this position

$$\text{S.F at A} = 60 \times 1 + 80 \times \frac{14}{16} + 40 \times \frac{12}{16} = 160 \text{ kN}$$

∴ Maximum positive S.F occurs at A and is equal to 160 kN.

Maximum negative shear force occurs at B when leading load is on B (refer Fig. 5.21(c)).

∴ Maximum negative S.F

$$= 40 \times 1 + 80 \times \frac{14}{16} + 60 \times \frac{12}{16} + 20 \times \frac{9}{16}$$

$$= 166.25 \text{ kN}$$

∴ Absolute maximum shear force = 166.25 kN

For finding position for absolute maximum moment, position of C.G of load system is to be located.

Distance of C.G for the leading load of 40 kN

$$a = \frac{80 \times 2 + 60 \times 4 + 20 \times 7}{40 + 80 + 60 + 20} = 2.7\text{m}$$

It is nearer to 80 kN load and this load is heavier than another nearer load of 60 kN. Hence, maximum moment will occur under 80 kN load. Distance between this load and the resultant,

$$d = 2.7 - 2 = 0.7$$

∴ Position of 80 kN load for maximum moment

$$= \frac{16}{2} + \frac{0.7}{2} = 8.35 \text{ m from A}$$

For the section at 8.35 m from A, ILD
Ordinate for moment

$$y_c = \frac{z(L-z)}{L}$$

$$= \frac{8.35(16-8.35)}{16}$$

$$= 3.9923$$

Absolute maximum moment (refer Fig.5.21(d)).

$$= 20y_1 + 60y_2 + 80y_c + 40y_3$$

$$= \left[20\times\frac{3.35}{8.35} + 60\times\frac{6.35}{8.35} + 80 + 40\times\frac{5.65}{7.65}\right] 3.9923$$

$$= 651.524 \text{ kN–m}$$

Example 5.6 A train of 5 wheel loads as shown in Fig.5.22(a) crosses a simply supported beam of span 24 m from left to right. Calculate the maximum positive and negative shear force values at the centre of the span and the absolute maximum bending moment anywhere in the span.

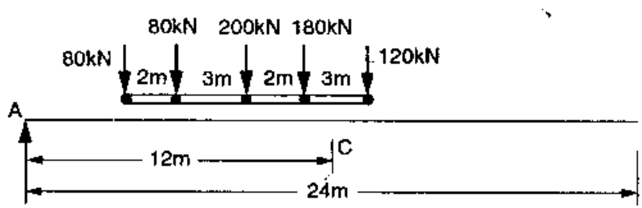


Figure 5.22 (a)

Solution

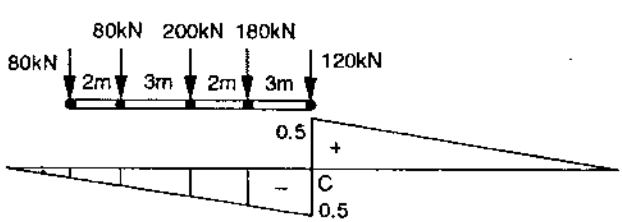


Figure 5.22 (b) ILD for S.F at mid-span with loading for max. –ve S.F

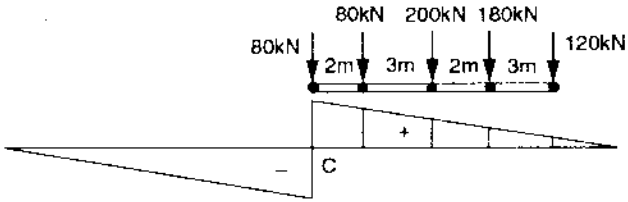


Figure 5.22 (c) ILD for S.F at mid-span with loading for max. +ve S.F

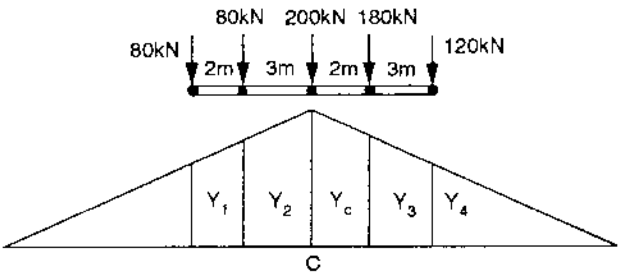


Figure 5.22 (d)

The ILD for shear force at centre of span is as shown in Fig. 5.22(b).

For maximum negative shear force the leading load should be on the section.

The maximum ILD ordinate for shear force at C

$$= \frac{12}{24} = 0.5$$

∴ Maximum negative S.F at mid-span

$$= 120 \times 0.5 + \left[180 \times \frac{9}{12} + 200 \times \frac{7}{12} + 80 \times \frac{4}{12} + 80 \times \frac{2}{12}\right] \times 0.5$$

$$= 205.833 \text{ kN}$$

For maximum shear force at C, the trailing load should be at C (refer Fig. 5.22(c)).
Maximum shear force at D

$$= 80 \times 0.5 + \left[80 \times \frac{10}{12} + 200 \times \frac{7}{12} + 180 \times \frac{5}{12} + 120 \times \frac{2}{12}\right] \times 0.5$$

$$= 179.167 \text{ kN}$$

When leading 80 kN load is on C,
positive S.F at C

$$= \left[-80 \times \frac{10}{12} + 80 \times 1 + 200 \frac{9}{12} + 180 \times \frac{7}{12} + 120 \times \frac{4}{12}\right] 0.5$$

$$= 154.167 \text{ kN}$$

Hence maximum shear force at mid-span C = 179.167 kN

Let C.G of loads from leading wheel load be x.

Then $$x = \frac{180\times3+200\times5+80\times8\times80\times10}{120+180+200+80+80}$$
$$= 4.5152 \text{ m}$$

This is nearer to 200 kN load. Hence maximum moment is likely to occur under this load.

Its distance from CG

$$= 5 - 4.5152 = 0.4848\,\text{m}$$

∴ Its position from end A

$$= 12 - \frac{0.4848}{2} = 11.7576\,\text{m}$$

∴ ILD ordinate under this load is

$$y_c = \frac{z(L-z)}{L} = \frac{11.7576(24-11.7576)}{24}$$
$$= 5.9975$$

Referring to Fig. 5.22(d).

∴ Absolute max $BM = 80y_1 + 80y_2 + 200y_c + 180y_3 + 120y_4$

$$= \left[80\times\frac{6.7576}{11.7576} + 80\times\frac{8.7576}{11.7576} + 200 + 180\times\frac{10.2424}{12.2424} + 120\times\frac{7.2424}{12.2424}\right]5.99$$

Since $y_c = 5.99$

$= 3157.64$ kN-m

Example 5.7 The system of concentrated loads shown in Fig. 5.23 rolls from left to right on the girder of span 15 m, 40 kN load leading. For a section 4 m from left support, determine

i. Maximum bending moment
ii. Maximum shear force

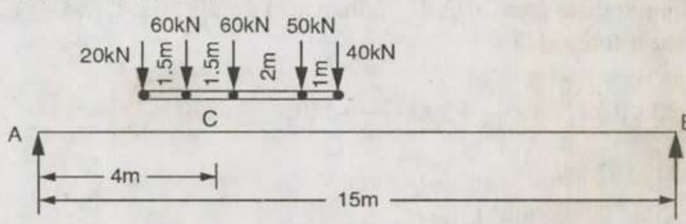


Figure 5.23 (a)

Solution

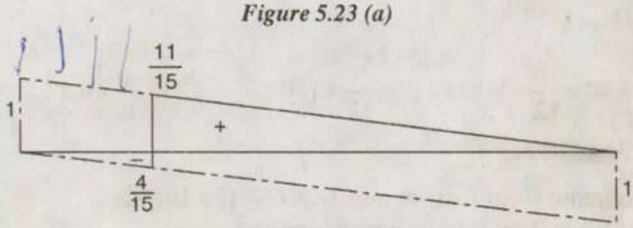


Figure 5.23 (b) ILD for S.F at C

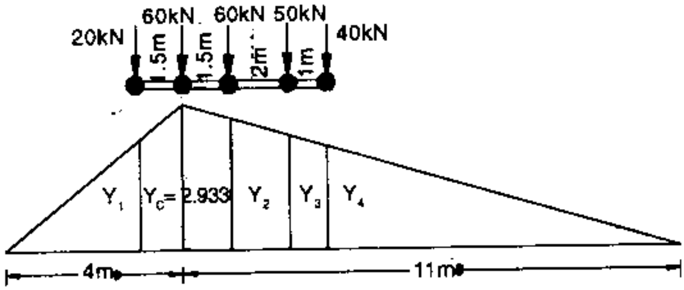


Figure 5.23 (c) ILD for moment at C

The ILD for shear force at the section C, 4m from A is shown in Fig.5.23(b).

The maximum ordinate for negative S.F at C

$$= \frac{4}{15}$$

while the maximum ordinate for the positive S.F at C

$$= \frac{11}{15}$$

For maximum negative shear force most of the loads will be on AC and one of them on C. When 40 kN load is on C

$$\text{negative S.F at C} = 40 \times \frac{4}{15} + 50 \times \frac{3}{15} + 60 \times \frac{1}{15}$$

$$= 24.667 \text{ kN}$$

When 50 kN load reaches the section,

$$\therefore \quad \text{negative S.F at C} = \left[-40 \times \frac{10}{15} + 50 \times \frac{4}{15} + 60 \times \frac{2}{15} + 60 \times \frac{5}{15}\right]$$

$$= -3.333 \text{ kN}$$

∴ Maximum negative S.F is when 40 kN load is on the section and its value is 24.667 kN.

For maximum positive shear force at C, let the 20 kN load be on C.

Then max. positive S.F at C

$$= \left[20 \times \frac{11}{15} + 60 \times \frac{9.5}{15} + 60 \times \frac{8}{15} + 50 \times \frac{6}{15} + 40 \times \frac{5}{15}\right]$$

$$= 118 \text{ kN}$$

Since 20 kN is a light load, let us try the case, when trailing 60 kN load is on C,

$$\text{S.F at C} = \left[-20 \times \frac{2.5}{15} + 60 \times \frac{11}{15} + 60 \times \frac{9.5}{15} + 50 \times \frac{7.5}{15} + 40 \times \frac{6.5}{15}\right]$$

$$= 121 \text{ kN}$$

∴ Maximum positive S.F occurs when trailing 60 kN load is on the section and its value

$$= 121 \text{kN}$$

Maximum bending moment :

The ILD for moment at section C is as shown in Fig.5.22(c).

Maximum ordinate

$$y_c = z\frac{(L-z)}{L} = \frac{4(15-4)}{15}$$

$$= 2.933$$

Maximum moment at C occurs when a particular load rolls over it, changing lighter portion to heavier portion and heavier portion to lighter portion. When different loads roll over C, the average loading on AC and CB are found as given in the Table below.

Table 5.2

Load rolling on C	*Average load on AC (w_{1av})*	*Average load on CB (w_{1av})*	*Remarks*
40	$\frac{190}{4}$	$\frac{40}{11}$	$w_{1a_v} > w_{2a_v}$
50	$\frac{140}{4}$	$\frac{90}{11}$	$w_{1a_v} > w_{2a_v}$
First 60	$\frac{80}{4}$	$\frac{150}{11}$	$w_{1a_v} > w_{2a_v}$
Second 60	$\frac{20}{4}$	$\frac{210}{11}$	$w_{1a_v} < w_{2a_v}$

∴ Maximum moment occurs at C when the second 60 kN load is on C as shown in Fig. 5.23(c).

∴ Maximum moment at C

$$= 20y_1 + 60y_c + 60y_2 + 50y_3 + 40y_4$$

$$= \left[20\frac{2.5}{4} + 60 + 60\frac{9.5}{11} + 50\frac{7.5}{11} + 40\frac{6.5}{11}\right]y_c$$

$$= 533.94 \text{ kN-m} \qquad \text{Since } y_c = 2.933$$

Example 5.8 Draw the influence line diagram for bending moment at a point 10 m distant from the left hand abutment on a bridge girder of span 25m and find the maximum bending moment at that point due to a series of wheel loads 100 kN, 200 kN, 200 kN, 200 kN, 200 kN at centre to centre distance of 4m, 2.5m, 2.5m and 2.5m. The loads can cross in either direction, 100 kN load leading in each case.

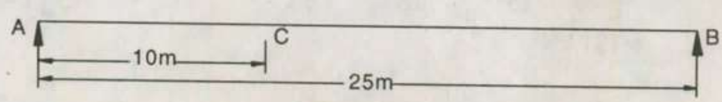


Figure 5.24 (a)

Solution

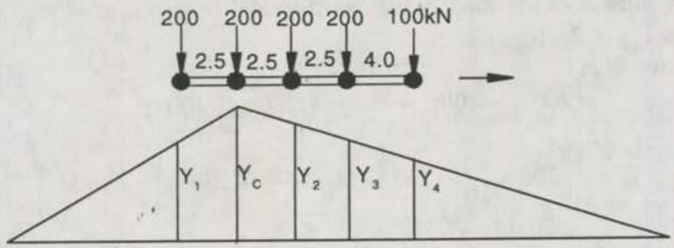


Figure 5.24 (b)

25

15 : e

13/5/

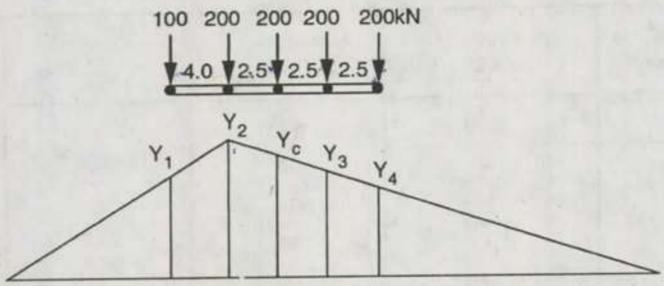


Figure 5.24 (c)

The influence line diagram for moment at C, 10m from the support A is as shown in Fig.5.24(b)

where, $$y_c = \frac{z(L-z)}{L} = \frac{10(25-10)}{25} = 6$$

To find load position for maximum moment at C, we have to find average load on portion AC and CB and identify the load which when crosses makes the lighter portion heavier and heavier portion lighter.

When the train of load moves from left to right :

Table 5.3

Load rolling on C	*Average load on AC (w_{1av})*	*Average load on CB (w_{2av})*	*Remarks*
100	$\frac{800}{10}$	$\frac{100}{15}$	$w_{1av} > w_{2av}$
First 200	$\frac{600}{10}$	$\frac{300}{15}$	$w_{1av} > w_{2av}$
Second 200	$\frac{400}{10}$	$\frac{500}{15}$	$w_{1av} > w_{2av}$
Third 200	$\frac{200}{10}$	$\frac{700}{15}$	$w_{1av} < w_{2av}$

∴ When third 200 kN load is just on the section, moment is maximum at C. Referring to Fig. 5.24(b).

Max. moment at C,

$$= 200y_1 + 200y_c + 200y_2 + 200y_3 + 100y_4$$

$$= \left[200 \times \frac{7.5}{10} + 200 + 200 \times \frac{12.5}{15} + 200 \times \frac{10}{15} + 100 \times \frac{6}{15}\right] y_c$$

$$= 4140 \text{ kN-m} \qquad \text{Since, } y_c = 6$$

When load moves from right to left (refer Fig. 5.24(c)).

Table 5.4

Load crossing	*Average load on AC w_{1av}*	*Average load on CB w_{2av}*	*Remarks*
100	$\frac{100}{10}$	$\frac{800}{15}$	$w_{1av} < w_{2av}$
First 200	$\frac{300}{10}$	$\frac{600}{15}$	$W_{1av} < w_{2av}$
Second 200	$\frac{500}{10}$	$\frac{400}{15}$	$w_{1av} > w_{2av}$

∴ Moment is maximum when second 200 kN load is on the section. Maximum moment in this case

$$= 100\, y_1 + 200y_2 + 200y_c + 200y_3 + 200y_4$$

$$= \left[100\frac{3.5}{10} + 200\frac{7.5}{10} + 200 + 200\frac{12.5}{15} + 200\frac{10}{15}\right] y_c$$

$$= 4110 \text{ kN-m} \qquad y_c = 6$$

Example 5.9 Determine maximum shear force and maximum bending moment at quarter span from left end when a uniformly distributed load longer than the span of intensity 20 kN/m, accompanied by a 100 kN concentrated load crosses the span of 12m. Use influence lines. The concentrated load can occupy any position.

Solution

The beam is shown in Fig.5.25(a). ILD for shear force at section C, 3m from left support is shown in Fig.5.25(b). For the same section ILD for moment is shown in Fig.5.25(c).

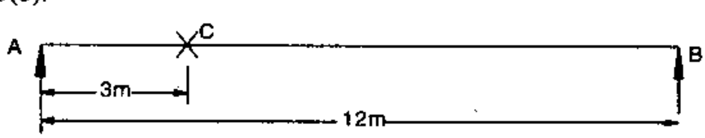


Figure 5.25 (a)

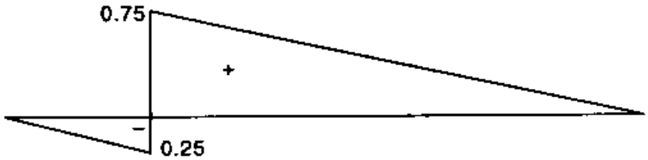


Figure 5.25 (b)

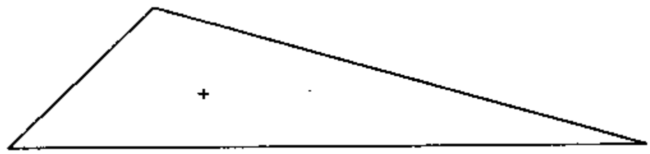


Figure 5.25 (c)

To get max. negative shear force at C, uniformly distributed load should occupy only portion of AC and concentrated load should be at the section.

$$\text{Max. positive SF at C} = \frac{1}{2} \times 0.25 \times 3 \times 20 + 100 \times 0.25$$

$$= 32.5\text{kN}$$

For maximum positive shear force at C, udl should occupy only poriton CB and concentrated load should be at C, then .

$$\text{Max. positive SF at C} = \frac{1}{2} \times 0.75 \times 9 \times 20 + 100 \times 0.75$$

$$= 142.5\text{kN}$$

For maximum moment at C, udl should occupy the entire span and the concentrated load should be on the section.

$$\text{Max. moment at C} = \frac{1}{2} \times y_c \times 12 \times 20 + 100 \times y_c = 200\, y_c\ \text{kN}$$

$$\text{But } y_c = \frac{2(L-z)}{L} = \frac{3(12-3)}{12} = 2.25$$

Max. moment at C = 220 × 2.25 = 495 kN-m

EXERCISES

5.1 Uniformly distributed load of intensity 30 kN/m crosses a simply supported beam of span 60 m from left to right. The length of udl is 15m. Find the value of maximum bending moment at a seciton 20m from left end. Find also the absolute value of maximum bending moment and shear force in the beam.

Ans : M_{20} = 5250kN-m; Abs.SF = 393.75kN; Abs M = 5906.25

5.2 A train of wheel loads as shown in Fig.5.26 crosses a girder of 25m span with 120 kN load leading.Determine the value of
i.Maximum bending moment at the section 8m from the left end of the girder.
ii.Absolute maximum bending moment on the girder.
Fig. 5.26

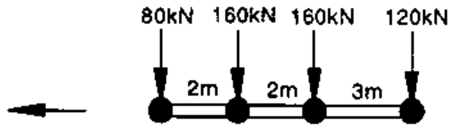


Figure 5.26

Ans : M_8 = 2425.6kN-m; Abs.M = 2751.46kN-m

5.3 The following system of concentrated loads roll from left to right on a girder of span 16m, with 60 kN load leading. Determine the absolute maximum moment and the maximum moment at 4m from the left support.

Ans : Abs.M = 620.14kN-m; M_4 = 465kN-m

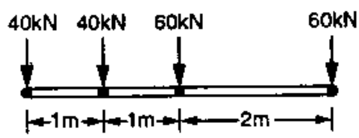


Figure 5.27

Ans : Abs. M = 620.14 kN-m, M_4 = 465 kN-m

5.4 In the problem given in 5.3, determine the absolute maximum shear force and the shear force at 4 m from the left support.

Ans : Abs max. +ve S.F = 155 kN
Abs max. –ve S.F = 160 kN
Max. –ve S.F at 4 m from left support = 25 kN
Max. +ve S.F at 4 m from left support = 110 kN

INFLUENCE LINES FOR BRIDGE TRUSSES 6

6.1 INTRODUCTION

In bridges, girder can be used only for small spans. For large spans, trussed bridges are used. There are two types of trussed bridges.

1. Through type and
2. Deck type

Typical through type and deck type bridges are shown in Fig.6.1 and 6.2 respectively.

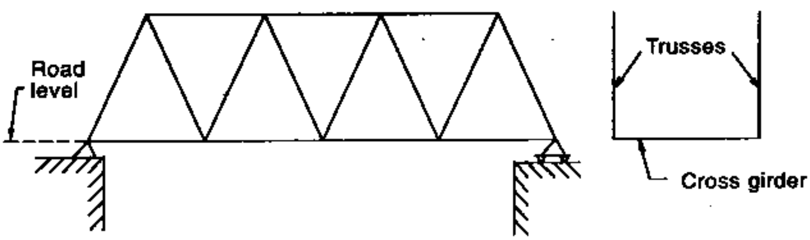


Figure 6.1 A typical through type trussed bridge

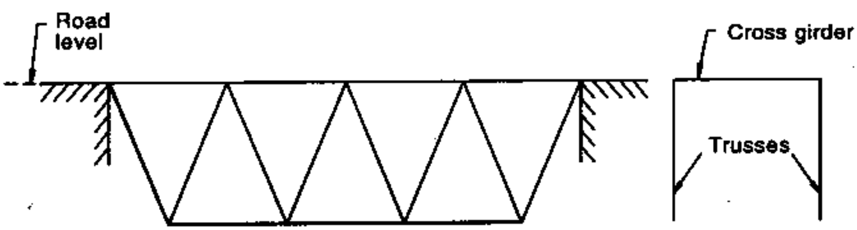


Figure 6.2 A typical deck type trussed bridge

Bridge truss consists of two trusses connected by cross girders at every panel points. In case of through type bridges, girders are at the lower joints and road or rail tracks are laid over the cross girders, and allows the vehicles, to move through the bridge. In case of deck type bridges, cross girders connect top chord panel points and road or rails are laid over these cross girders. In both types of trusses, end portals are provided.

Cross girders transfer the loads to the panel joints. In this analysis, the joints are assumed to be pin-connected. Hence the members of the bridge trusses are subjected to axial forces only. Now our interest is to find the influence line diagrams so that for various moving loads, design forces in the members can be found.

6.2 METHOD OF FINDING INFLUENCE LINE DIAGRAMS

Let the reactions at left and right side end of the bridge truss be R_1 and R_2 respectively. The influence line diagrams for these reactions are similar to those for the end reactions in beams (Ref. Fig. 6.3). To find the influence line diagrams for the forces in the members, method of section is used.

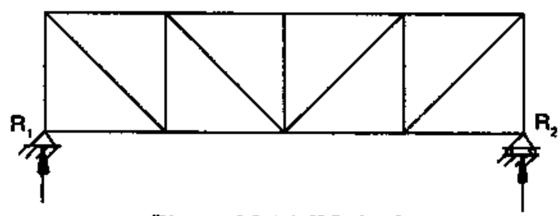


Figure 6.3 (a) ILD for R_1

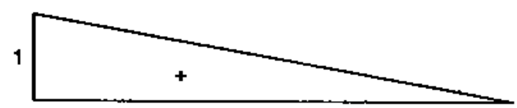


Figure 6.3 (b) ILD for R_1

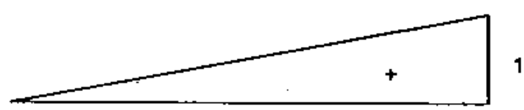


Figure 6.3 (c) ILD for R_2

It gives the simple relation with end reactions, when unit load is not on the panel. When the unit load is on the panel, the load first gets transferred to panel joints through the cross girders. Hence the force varies linearly, when the load is within the panel of the member under consideration. This method is illustrated with the examples given below. Influence line diagram is drawn for all the members in only one problem and in the rest it is done for a selected typical members only.

Example 6.1 Determine the influence line diagrams for forces in half the portion of the through type symmetric truss shown in Fig.6.4(a).

Solution

The load is moving over the bottom chord. ILD for the reaction at the left end support R_1 varies linearly from unit value at the left end support to zero at the other support. Similarly, ILD for the reaction R_2 at the right end support varies linearly from unit value at the right end support to zero at left end support (Fig.6.4(b) and Fig.6.4(c)).

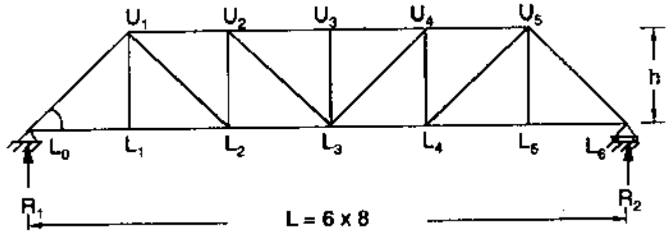


Figure 6.4 (a)

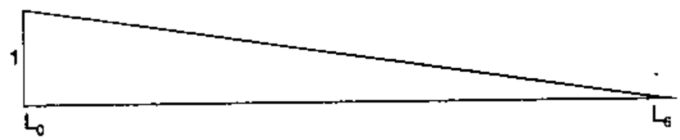


Figure 6.4 (b) ILD for R_1

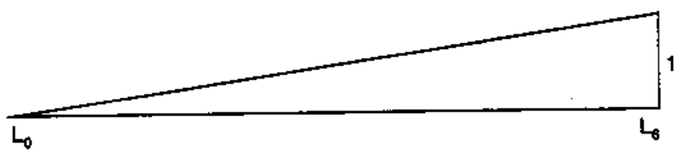


Figure 6.4 (c) ILD for R_2

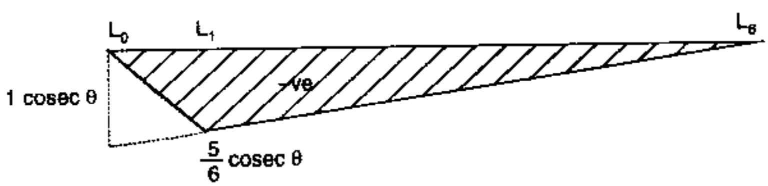


Figure 6.4 (d) ILD for $P_{L_0U_1}$

ILD for top chord members

ILD for force in L_0U_1 When the unit load is between L_1 and L_6, at joint L_0, $\Sigma v = 0$ gives

$$F_{L_0U_1} \sin\theta = R_1$$

$$F_{L_0U_1} = R_1 \operatorname{cosec}\theta \qquad \text{(compressive)}$$

$\therefore$ When unit load is at L_1 $\qquad F_{L_0U_1} = \frac{5}{6}\operatorname{cosec}\theta$

when unit load is at L_6, $F_{L_0U_1} = 0$ and between L_1 to L_6 it varies linearly. When the load is in the portion L_0 L_1, the unit load partly gets transferred to joint L_0 and partly to L_1. The load transferred at L_0, directly goes to the support without giving rise to axial forces in the members. When the load is exactly on L_0, no force develops in the members L_0U_1. Hence ILD for force in L_0U_1 linearly varies from zero at L_0 to $\frac{5}{6}$ cosec θ at L_1 as shown in Fig.6.5(d).

ILD for force in U_1 U_2

Consider the section 1 – 1 shown in Fig.6.5(a). When the load is in portion L_0 L_2, taking moment about L_2,we get

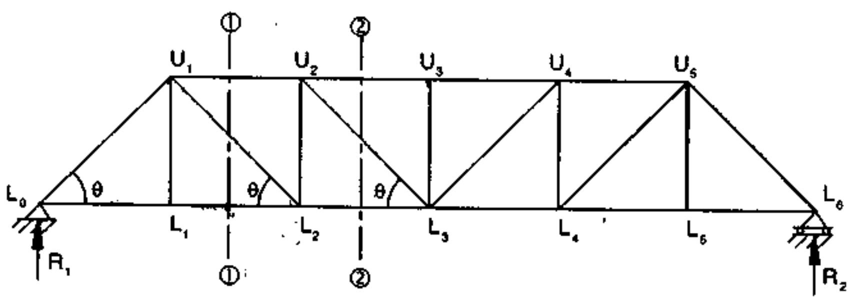


Figure 6.5(a)

$$P_{U_1U_2} = \frac{R_2 \times 4a}{h} \text{ (comp)}$$

$\therefore$ When load is at L_2, $\qquad P_{U_1U_2} = \frac{2}{6} \times \frac{4a}{h} = \frac{4a}{3h}$

and $\qquad P_{U_1U_2} = 0$ when the load is at L_0

When load is in portion L_2 to L_6 , taking moment about L_2 we get,

$$P_{U_1U_2} \times h = R_1\ 2a$$

$$P_{U_1U_2} = \frac{R_1 2a}{h} \text{ (comp)}$$

When load is at L_2, $\qquad R_1 = \frac{4}{6}$

$$\therefore \qquad P_{U_1U_2} = \frac{4}{6} \times \frac{2a}{h} = \frac{4a}{3h} \text{ (comp)}$$

and when load is at L_6, $\qquad R_1 = 0 \qquad \therefore \; P_{U_1U_2} = 0$

$\therefore$ ILD for $P_{U_1U_2}$ is as shown in Fig. 6.5(b).

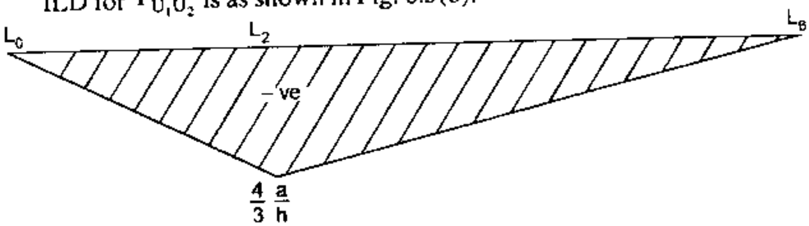


Figure 6.5(b) *ILD for* $P_{U_1U_2}$

Thus, it is similar to ILD for the moment at L_2 divided by height. Similarly, ILD for U_2U_3 may be found by taking section at (2)-(2) and it can be shown that mamimum ordinate will be, when the load is at L_3 and is equal to

$$= R_2 \frac{3a}{h}$$

$$= \frac{1}{2}\frac{3a}{h}$$

ILD for force in member U_2U_3 is shown in Fig. 6.5(c).

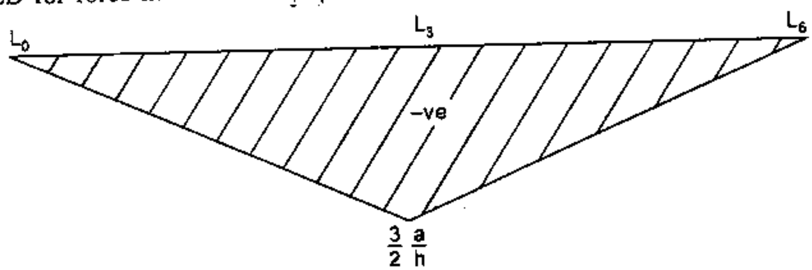


Figure 6.5(c) *ILD for* $P_{U_2U_3}$

This ILD is $\frac{1}{h}$ times ILD for moment at L_3.

ILD for bottom chord members

For L_0L_1

Considering joint L_0

$$\Sigma v = 0$$

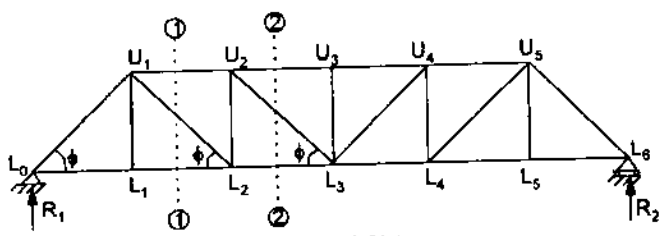


Figure 6.6(a)

$$P_{L_0U_1} \sin\theta = R_1 \text{ or } P_{L_0U_1} = \frac{R_1}{\sin\theta}$$

$$\Sigma H = 0 \rightarrow P_{L_0L_1} = P_{L_0U_1} \cos\theta = \frac{R_1 \cos\theta}{\sin\theta} = R_1 \cot\theta \text{ (tensile)}$$

Hence ILD for force in member $L_0 L_1$ is similar to that of ILD for $L_0 U_1$ but cosec θ replaced by cot θ is as shown in Fig.6.6(b).

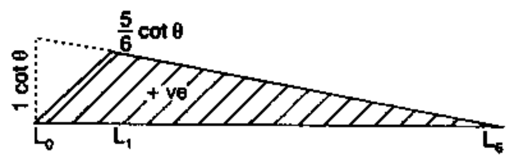


Figure 6.6 (b) *ILD for* $P_{L_0L_1}$

ILD for $P_{L_1L_2}$

Consider section 1-1, when the load is in portion $L_0 L_1$

ΣM about $U_1 = 0$, gives

$$P_{L_1L_2} \times h = R_2\, 5a$$

$$\therefore \qquad P_{L_1L_2} = R_2 \frac{5a}{h} \text{ (tensile)}$$

when load is at L_0, $\quad R_2 = 0 \quad \therefore P_{L_1L_2} = 0$

when load is at L_1, $\quad R_2 = \frac{1}{6} \quad \therefore P_{L_1L_2} = \frac{5a}{6h}$

when the load is in portion L_2 to L_6, moment equilibrium at U_1 gives,

$$P_{L_1L_2} = R_1 \frac{a}{h} \text{ (tensile)}$$

when load is at L_2 $\quad R_1 = \frac{4}{6} \quad \therefore P_{L_1L_2} = \frac{4}{6} \times \frac{a}{h} = \frac{4a}{6h}$

when load is at L_6, $\quad R_1 = 0 \quad P_{L_1L_2} = 0$

when the load is between $L_1 L_2$, it varies linearly. Thus, ILD for $L_1 L_2$ is same as that for $L_0 L_1$, since $\frac{a}{h} = \cot\theta$. Obviously, the equilibrium condition at joint L_2 shows that the two should be the same. This ILD is shown in Fig.6.6(c).

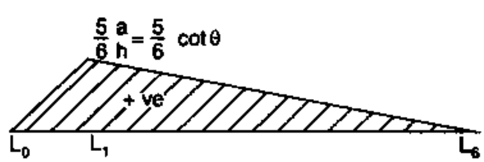


Figure 6.6 (c) *ILD for* $P_{L_1L_2}$

This is $\frac{a}{h}$ times ILD for the moment at U_1.

Similarly, ILD for $L_2 L_3$ can be found by considering section 2 – 2 and taking moment about U_2. The maximum ILD ordinate in this case is at L_2 and is

$$= R_B \frac{4a}{h} = \frac{2}{6} \times \frac{4a}{h} = \frac{4}{3} \cdot \frac{a}{h}$$

This is shown in Fig.6.6(d).

Thus, it is $\frac{a}{h}$ times ILD for moment at U_2.

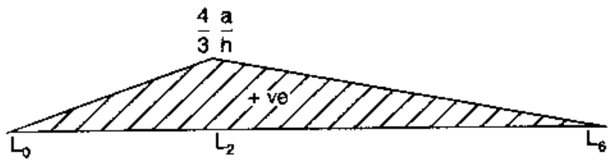


Figure 6.6 (d) ILD for $P_{L_2L_3}$

ILD for diagonal members Consider section 1-1, shown in Fig.6.7(a), considering the equilibrium of right side portion, when the load is in portion $L_0 L_1$

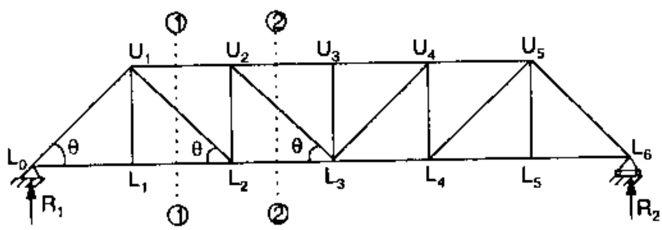


Figure 6.7 (a)

$$P_{U_1L_2} \sin\theta = R_2$$

$$P_{U_1L_2} = R_2 \operatorname{cosec}\theta \text{ (comp)}$$

when load is at L_0, $\quad R_2 = 0, \quad P_{U_1L_2} = 0$

when load is at L_1, $\quad R_2 = \frac{1}{6} \quad P_{U_1L_2} = \frac{1}{6} \operatorname{cosec}\theta$ (comp)

when load is in portion L_2 to L_6, considering left side portion,

$$\Sigma v = 0 \rightarrow P_{U_1L_2} \sin\theta = R_1$$

when load is at L_2, $\quad R_1 = \frac{4}{6}$

$$\therefore \quad P_{U_1L_2} = \frac{4}{6} \operatorname{cosec}\theta \text{ (tensile)}$$

when load is at L_6, $R_1 = 0$ $\therefore$ $P_{U_1L_2} = 0$

when the load is between L_1 and L_2 it varies linearly. Hence ILD is as shown in Fig.6.7(b).

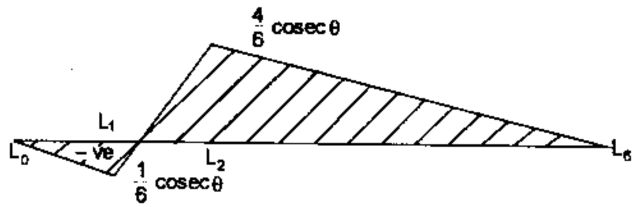


Figure 6.7(b) *ILD for* $P_{U_1L_2}$

To find ILD for $P_{U_2L_3}$, consider section 2 – 2 (Fig.6.7) and equilibrium of vertical forces. Ordinate at L_2 will be $\frac{2}{6}$cosec θ (negative) and at L_3 will be $\frac{3}{6}$ cosecθ (positive). Hence ILD is as shown in Fig.6.7(c).

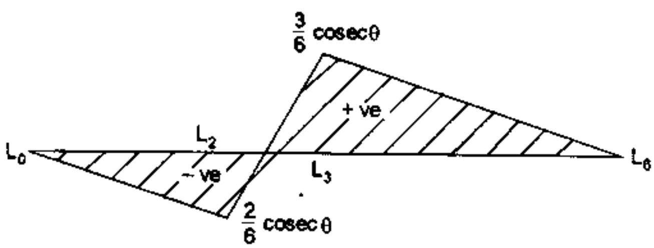


Figure 6.7(c) *ILD for* $P_{U_2L_3}$

ILD for vertical members

ILd for $P_{U_1L_1}$:

Considering the equilibrium of joint L_1 (Fig.6.8(a)), we get

$$P_{U_1L_1} = \text{Load at Joint } L_1$$

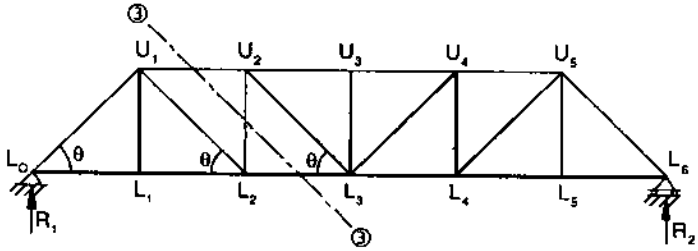


Figure 6.8(a)

The load at joint L_1 due to unit load moving, increases from 0 to 1 as the load moves from L_0 to L_1 and then reduces from 1 to 0 as the load moves from L_1 to L_2. When unit load is in portion L_2 to L_6, load at $L_1 = 0$. Hence ILD for $P_{U_1L_1}$ is as shown in Fig.6.8(b).

Figure 6.8(b) *ILD for $P_{U_1L_1}$*

ILD for $P_{U_2L_2}$

Consider section 3-3, when the load is between L_0 and L_2, considering right-hand portion

$$P_{U_2L_2} = R_2 \text{ (tensile)}$$

when load is at L_0, $\qquad P_{U_2L_2} = 0$

when load is at L_2, $\qquad P_{U_2L_2} = \frac{2}{6}$

when the load is between L_3 and L_6, considering left-hand side portion,

$$P_{U_2L_2} = R_1 \text{ (comp)}$$

$$= \frac{3}{6}\text{, when load is at } L_3$$

$$= 0\text{, when load is at } L_6$$

Between L_2 and L_3, it varies linearly. Hence ILD for $P_{U_2L_2}$ is as shown in Fig.6.8 (c).

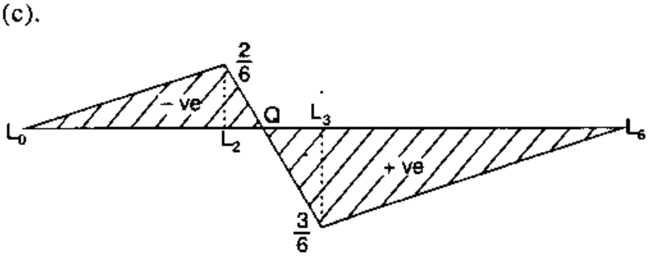


Figure 6.8(c) *ILD for $P_{U_2L_2}$*

ILD for $P_{U_3L_3}$

Since load moves on bottom chord, equilibrium of joint U_3 shows that, $P_{U_3L_3}$ is always zero.

Figure 6.8 (d) *ILD for $P_{U_3L_3}$*

Example 6.2 Determine the maximum forces in the members $U_2 L_2$, $U_3 L_3$ and $U_2 U_3$ of the truss shown in Fig.6.9, when an uniformly distributed load of 60 kN/m, longer than the span, moves from left to right on the top chord.

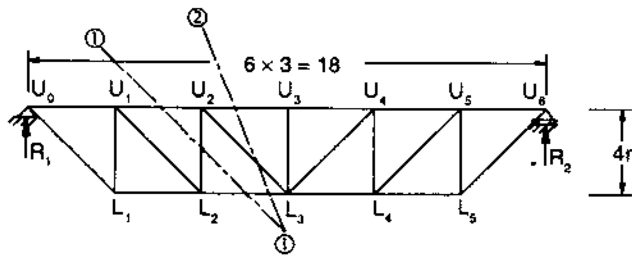


Figure 6.9 (a)

Solution

ILD ordinate for reaction at U_0 i.e., R_1 varies from one to zero linearly from U_0 to U_6. Similarly, ILD ordinate for R_2 varies from zero to unity from U_0 to U_6.

ILD for $P_{U_2L_2}$

Consider section 1–1. When load is in the portion $U_0 U_1$, considering right side portion,

$$P_{U_2L_2} = R_2 \quad \text{(tensile)}$$

i.e., when load is at U_0, $\quad P_{U_2L_2} = 0$

when load is at U_1, $\quad P_{U_2L_2} = \frac{1}{6} \quad$ (tensile)

when load is in the portion $U_2 U_6$, considering left side portion;

$$P_{U_2L_2} = R_1 \quad \text{(comp)}$$

$$= \frac{4}{6}, \text{ when load is at } U_2$$

$$= 0, \text{ when load is at } U_6$$

Between U_1 and U_2, it varies linearly. Hence ILD for $P_{U_2L_2}$ is as shown in Fig.6.9(b).

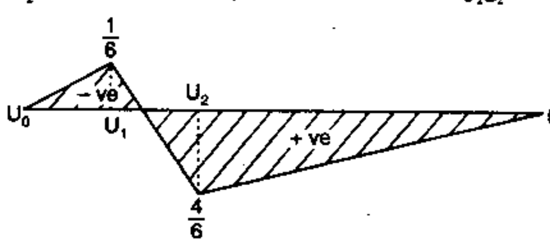


Figure 6.9(b) *ILD for* $P_{U_2L_2}$

Maximum compressive force develops in $U_2 L_2$ when the load covers U_0 to Q and maximum tensile force develops, when udl covers portion Q to U_6. By proportionating, we get

$$\frac{U_1Q}{QU_2} = \frac{(1/6)}{(4/6)},$$

$$\therefore \quad \frac{U_1Q}{U_1U_2} = \frac{(1/6)}{(1/6)+(4/6)} = \frac{1}{5} = 0.2$$

or $\quad U_1Q = 0.2 \times U_1U_2 = 0.2 \times 3 = 0.6$

$\therefore$ Max. compressive force in $U_2 L_2$

$$= \frac{1}{2} \times \frac{1}{6} (3 + 0.6) \times 60$$

$$= 18 \text{ kN}$$

and Max. tensile for

$$= \frac{1}{2} \times \frac{4}{6} (18 - 3.6) \times 60$$

$$= 288 \text{ kN}$$

ILD for $P_{U_3L_3}$

Considering equilibrium of forces at U_3, we get,

$$P_{U_3L_3} = \text{Load at } U_3 \text{(comp)}$$

When unit load is between U_0 U_2 and U_4 U_6 there is no load at U_3. It increases from zero to unity as unit load moves from U_2 to U_3 to U_3 and then it reduces to zero as unit load reaches U_4. Hence ILD for $P_{U_3L_3}$ is as shown in Fig.6.9(c).

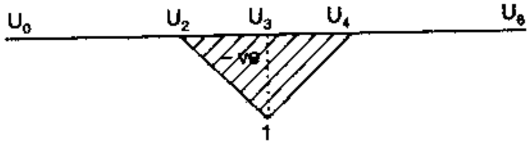


Figure 6.9 (c) *ILD for $P_{U_3L_3}$*

$\therefore$ Max. comp. force due to udl occurs when the udl covers U_2 to U_4

$\therefore$ Max. $P_{U_3L_3} = \frac{1}{2} \times 6 \times 1 \times 60 = 180$ kN (comp)

ILD for $P_{U_2U_3}$

Consider section 2-2, for any position of unit load,

$$P_{U_2U_3} \times 4 = M_{L_3}$$

$$P_{U_2U_3} = \frac{1}{4} \times M_{L_3} \text{ (comp)}$$

Hence ILD for $P_{U_2U_3}$ is $\frac{1}{4} \times$ ILD for moment at L_3

Max. ordinate occurs at L_3 and is

$$= \frac{1}{4} \times \frac{z(L-z)}{L}$$

$$= \frac{1}{4} \times \frac{9(18-9)}{18} = \frac{9}{8}$$

ILD is as shown in Fig. 6.9(d).

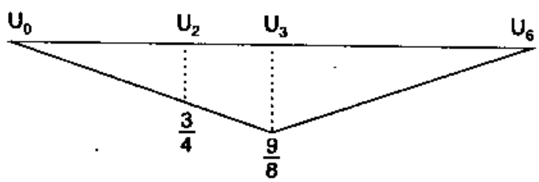


Figure 6.9 (d) *ILD for* $P_{U_2U_3}$

Max. force occurs when the load covers the entire span.

Max. force in $U_2\,U_3 = \frac{1}{2} \times \frac{9}{8} \times 18 \times 60 = 607.5$ kN.

Example 6.3 Draw the influence line diagram for forces in the members $U_3\,L_4$, $U_3\,U_4$ and $U_3\,L_3$ of the frame shown in Fig.6.10 and find the maximum forces developed, when uniformly distributed load of intensity 40 kN/m, longer than the span moves from left to right on bottom chord.

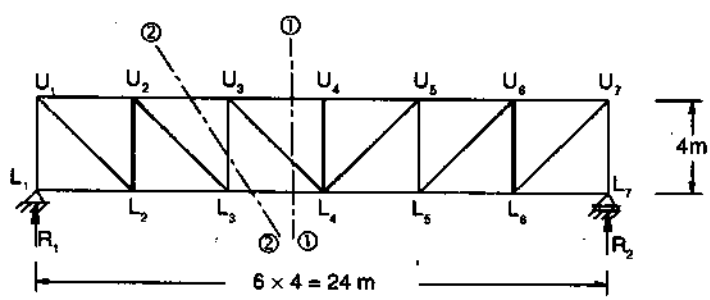


Figure 6.10 (a)

Solution

Let the reactions at the left end support and right end supports be R_1 and R_2 respectively. Take the section 1-1 as shown in the Figure.

ILD for $P_{U_3L_4}$

When unit load is between L_1 and L_3 considering right-hand side portion of truss,

$$\Sigma v = 0 \rightarrow$$

$$P_{U_3L_4} \times \sin 45° = R_2$$

$$P_{U_3L_4} = \frac{R_2}{\sin 45^\circ} = \sqrt{2}\ R_2 \text{ (comp), which varies linearly.}$$

$$= 0, \text{ when load is at } L_1$$

$$= \sqrt{2} \times \frac{2}{6} = \frac{\sqrt{2}}{3}, \text{ when load is at } L_3$$

When unit load is in portion L_4 to L_7, considering the equilibrium of left-hand side portion of truss,

$$\Sigma v = 0 \rightarrow$$

$$P_{U_3L_4} \times \sin 45° = R_1$$

$$P_{U_3L_4} = \sqrt{2}\ R_1 \text{ (tensile), which varies linearly}$$

$$= 0, \text{ when load is at } L_7$$

$$\sqrt{2} \times \frac{3}{6} = \frac{1}{\sqrt{2}}, \text{ when load is at } L_4$$

Between L_3 and L_4, it varies linearly. Hence ILD for is $P_{U_3L_4}$ is as shown in Fig.6.10(b). Let it intersect $L_3 L_4$ at Q. Then

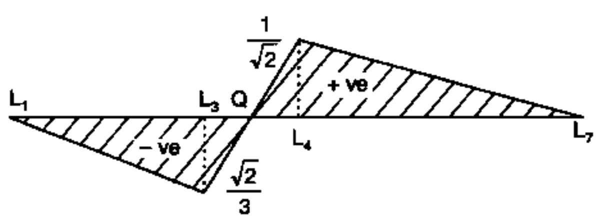


Figure 6.10 (b) ILD for $P_{U_3L_4}$

$$\frac{L_3Q}{QL_4} = \frac{\sqrt{2}/3}{1/\sqrt{2}}$$

$$\frac{L_3Q}{L_3L_4} = \frac{\sqrt{2}/3}{\sqrt{2}/3 + 1/\sqrt{2}} = \frac{1/3}{1/3 + 1/2} = 0.4$$

$$\therefore \quad L_3Q = 0.4 \times 4 = 1.6 \text{ m}$$

∴ Max. compressive force occurs when udl covers the portion $L_1 Q$,

$$= \frac{1}{2} \times \frac{\sqrt{2}}{3} \times L_1Q \times 40$$

$$= \frac{\sqrt{2}}{2 \times 3} \times 9.6 \times 40$$

$$= 90.51 \text{ kN.}$$

Maximum tensile force in the member $U_3 L_4$ will occur when 40 kN/m udl cover the portion QL_7 and its value is

$$= \frac{1}{2} \times QL_7 \times \frac{1}{\sqrt{2}} \times 40$$

$$= \frac{1}{2} \times (12 + 2.4) \times \frac{1}{\sqrt{2}} \times 40$$

$$= 203.65 \text{ kN}$$

ILD for $U_3 U_4$

Considering section 1-1 (Fig.6.10) and moment equilibrium about L_4, it can be observed that

$$P_{U_3U_4} \times 4 = M_{L_4}$$

$$P_{U_3U_4} = \frac{1}{4} M_{L_4} \text{ (compressive)}$$

∴ ILD for $P_{U_3U_4}$ is similar to that of moment at L_4. It is as shown in Fig.6.10(c) with maximum ordinate.

$$= \frac{1}{4}\frac{z(L-z)}{L} = \frac{1}{4}\frac{12(24-12)}{24} = 1.5$$

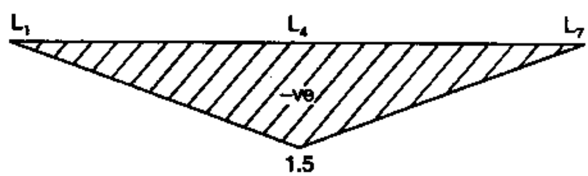


Figure 6.10 (c) *ILD for $P_{U_3U_4}$*

Force in U_3U_4 is maximum when load covers the entire span and its value is

$$= \frac{1}{2} \times 24 \times 1.5 \times 40$$

$$= 720 \text{ kN-m (comp)}$$

ILD for $U_3 L_3$

Consider section 2 – 2, when the unit load is in portion L_1 L_3, considering right-hand side portion,

$$\Sigma v = 0 \rightarrow 0$$

$$P_{U_3L_3} = R_2 \text{(tensile), varies linearly}$$

$$= 0, \text{ when load is at } L_1$$

$$= \frac{2}{6}, \text{ when load is at } L_3$$

when the unit load is in portion L_4 L_7, considering left-hand side portion,

$$\Sigma v = 0 \rightarrow$$

$$P_{U_3L_3} = R_1 \text{ (comp), varies linearly}$$

$$= 0, \text{ when load is at } L_7$$

$$= \frac{3}{6} \text{ when load is at } L_4$$

Between L_3 L_4, it varies linearly

∴ ILD is as shown in Fig.6.10(d).

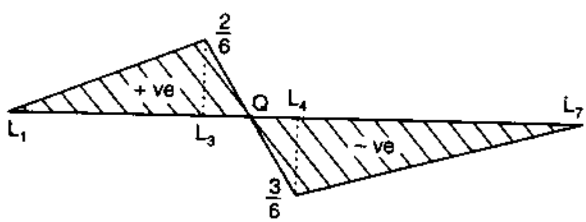


***Figure 6.10 (d)** ILD for $P_{U_3L_3}$*

$$\frac{L_3Q}{QL_4} = \frac{(2/6)}{(3/6)}$$

$$\frac{L_3Q}{L_3L_4} = \frac{(2/6)}{(2/6)+(3/6)} = 0.4$$

∴ $L_3Q = 1.6$ m

Maximum tensile force in U_3 L_3 due to 40 kN/m udl occurs when load covers the portion L_1 Q only and is

$$= \frac{1}{2} \times \frac{2}{6} \times L_1Q \times 40$$

$$= \frac{1}{2} \times \frac{2}{6} \times (8 + 1.6) \times 40$$

$$= 64 \text{ kN}$$

Maximum compressive force occurs when the load is in portion QL_7 and is

$$= \frac{1}{2} \times \frac{3}{6} \times QL_7 \times 40$$

$$= \frac{1}{2} \times \frac{3}{6} \times (12 + 2.4) \times 40$$

$$= 144 \text{ kN}$$

Example 6.4 Determine the maximum forces in the members U_2U_3, L_3U_3 and L_3L_4 of the bridge truss shown in Fig.6.11(a), if uniformly distributed load of 60 kN/m, longer than the span, traverses along the bottom chord members.

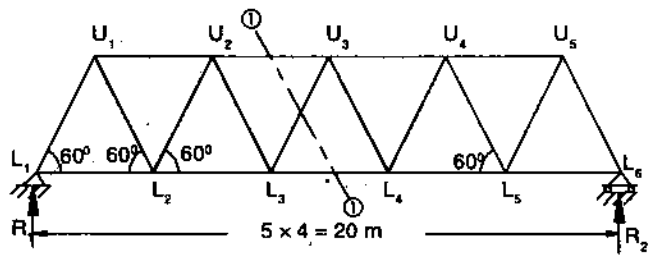


Figure 6.11 (a)

Solution Consider section (1) – (1):

ILD for $P_{U_2U_3}$

Moment equilibrium condition about L_3 gives

$$P_{U_2U_3} \times 4 \sin 60° = M_{L_3}$$

$$P_{U_2U_3} = \frac{1}{4\sin 60°} M_{L_3} \text{ (comp)}$$

$$\therefore \quad \text{ILD for } P_{U_2U_3} = \frac{1}{4\sin 60°} \times \text{ILD for moment at } L_3$$

$$\text{Max. ordinate} = \frac{1}{4\sin 60°} \frac{z(L-z)}{L} \text{ where } z = L_1L_3 = 8\text{m}$$

$$= \frac{1}{4\sin 60°} \frac{8(20-8)}{20}$$

$$= 1.3856$$

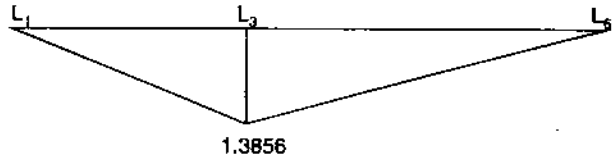


Figure 6.11 (b) ILD for $P_{U_2U_3}$

Hence ILD for $P_{U_2U_3}$ is as shown in Fig.6.11(b).

∴ Maximum force in U_2 U_3 will be, when the udl of 60 kN/m occupies the entire span and is

$$= \frac{1}{2} \times 1.3856 \times 20 \times 60$$

$$= 831.36 \text{ kN (comp)}$$

ILD for $P_{L_3U_3}$

Let R_1 be the reaction at left support and R_2 at the right support. When load is between L_1 and L_3, considering right-hand side portion of truss,

$$\Sigma v = 0 \rightarrow$$

$$P_{L_3U_3} \sin 60° = R_2$$

$$P_{L_3U_3} = \frac{1}{\sin 60°} R_2, \text{ (tensile), varies linearly,}$$

$= 0$ when load is at L_1,

$$= \frac{1}{\sin 60°} \times \frac{2}{5} \text{ (tensile), when load is at } L_3$$

When load is between L_4 and L_6

$$P_{L_3U_3} \sin 60° = R_1$$

$$P_{L_3U_3} = \frac{1}{\sin 60°} R_1 \text{ (comp), varies linearly}$$

$= 0$, when load is at L_6

$$= \frac{1}{\sin 60°} \times \frac{2}{5} \text{ (comp), when load is at } L_4$$

It varies linearly between L_3 and L_4.

∴ ILD is as shown in Fig.6.11(c).

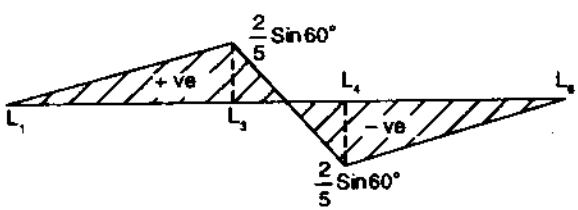


Figure 6.11 (c) ILD for $P_{L_3U_3}$

Q is at mid-span of L_3 and L_4, since ordinate at L_3 = ordinate at L_4. Hence maximum tensile force and compressive force are equal and occur when moving load covers half the span.

$$= \frac{1}{2}\,\frac{2}{5\sin 60^\circ} \times 10 \times 60$$
$$= 138.564 \text{ kN}.$$

ILD for $L_3 L_4$

Consider 1-1 (Fig.6.11): For any unit load position,

$$P_{L_3L_4} \times 4\sin 60^\circ = M_{U_3}$$

$$P_{L_3L_4} = \frac{1}{4\sin 60^\circ} M_{U_3} \text{ (tensile)}$$

$\therefore$ ILD of $P_{L_3L_4}$ is $\frac{1}{4\sin 60^\circ}$ of ILD for moment at U_3

$\therefore$ Max. ordinate is at U_3 and is

$$= \frac{1}{4\sin 60^\circ} \times \frac{10(20-10)}{20}$$

$$= \frac{5}{4\sin 60^\circ} = 1.4434$$

However, since load is on the lower chord member, linear variation starts from L_3 to L_4. ILD is as shown in Fig.6.11(d).

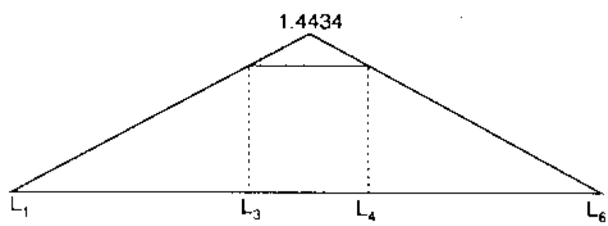


Figure 6.11 (d) *ILD for $P_{L_3L_4}$*

Ordinate at L_3 and L_4 = $1.4434 \times \frac{8}{10} = 1.15472$

Maximum force in member L_3L_4 occur when the load covers entire span and is

$$\left[\frac{1}{2} \times 1.1572 \times 8 + 1.1572 \times 4 + \frac{1}{2} \times 1.1572 \times 8\right] \times 60$$

$$= 831.40 \text{ kN}$$

Example 6.5 For a unit load moving from left to right on the truss shown in Fig. 6.12(a), draw the influence line diagram for the forces in the members, L_1L_2, U_1L_2, U_2U_3 and U_2L_2.

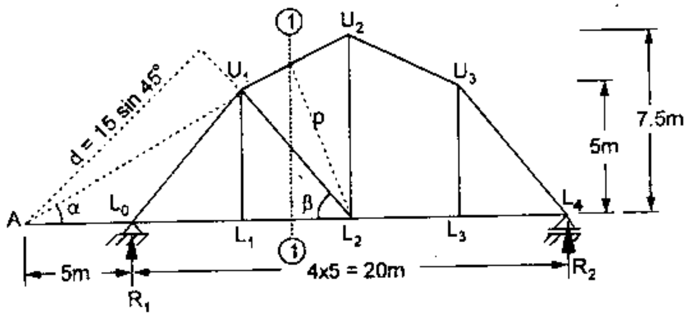


Figure 6.12 (a)

Solution

Let the continuation of U_2U_1 meet horizontal at bottom chord level, at A.

Now,
$$\frac{AL_2}{AL_1} = \frac{7.5}{5} = 1.5$$

Denoting AL_0 by x, we have

$$\frac{x+10}{x+5} = 1.5$$

or
$$x + 10 = 1.5x + 7.5$$

or
$$x = 5$$

∴
$$\alpha = \tan^{-1}\frac{7.5}{15} = 26.565°$$

Let 'p' be the perpendicular distance of U_1U_2 from the joint L_2

$$\begin{aligned} p &= AL_2 \sin\alpha \\ &= 15 \sin\alpha \\ &= 6.7082 \text{ m} \end{aligned}$$

ILD for $P_{L_1L_2}$

Consider section 1–1 as shown in Fig. 6.12(a).

Moment equilibrium condition of parts of the truss gives,

$$P_{L_1L_2} \times 5 = M_{U_1}$$

$$P_{L_1L_2} \times \frac{1}{5} M_{U_1} \text{ (tensile)}$$

Hence ILD for $P_{L_1L_2}$ is similar to that for the moment at U_1. Hence it is a triangle with maximum ordinate equal to

$$= \frac{1}{5}\frac{z(L-z)}{L} = \frac{1}{5}\frac{5(20-5)}{20} = 0.75$$

This is shown in Fig. 6.12(b).

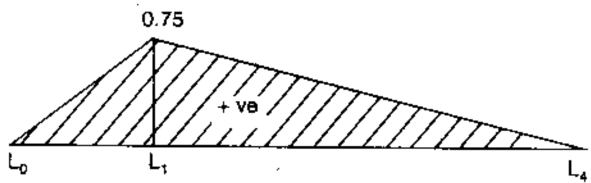


Figure 6.12 (b) ILD for $P_{L_1L_2}$

ILD for $P_{U_1U_2}$

Consider section 1–1: for any position of load, moment equilibrium of parts of the truss about L_2 shows

$$P_{U_1U_2}\ p = M_{L_2}$$

$$P_{U_1U_2} = \frac{1}{p} M_{L_2}$$

$$= \frac{1}{6.7082} M_{L_2} \text{ (compressive)}$$

Thus ILD ordinates $P_{U_1U_2}$ are $\frac{1}{6.7082}$ × ordinates of M_{L_2}.

ILD of $P_{U_1U_2}$ is a triangle with maximum ordinate at L_2 which

$$= \frac{1}{6.7082} \frac{z(L-x)}{L}$$

$$= \frac{1}{6.7082} \frac{10(20-10)}{20}$$

$$= 0.7454$$

∴ LLD for $P_{U_1U_2}$ is as shown in Fig. 6.12(c).

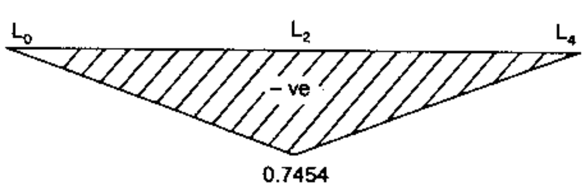


Figure 6.12 (c) ILD for $P_{U_1U_2}$

ILD for $P_{U_2L_2}$

Joint equilibrium at U_2

$$\Sigma H = 0 \rightarrow P_{U_1U_2} = P_{U_2U_3}$$

$$\Sigma v = 0 \rightarrow P_{U_2L_2} = 2P_{U_1U_2} \sin\alpha$$

$$= 2 \times \sin 26.565° \; P_{U_1U_2}$$

$$= 0.8944 \; P_{U_1U_2} \text{ (tensile)}$$

$\therefore$ ILD for $P_{U_2L_2}$ is similar to ILD for $P_{U_1U_2}$ but maximum ordinate is

$$0.8944 \times 0.7454$$

$$= 0.6667$$

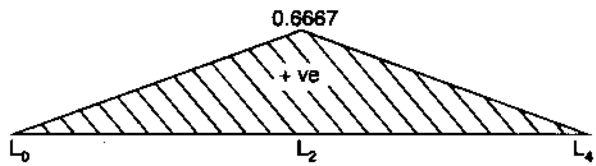


Figure 6.12 (d) *ILD for $P_{U_2L_2}$*

ILD for $P_{U_1L_2}$

Now $\tan \beta = \frac{5}{5} = 1$

$\therefore$ $\beta = 45°$

Taking section 1–1 and considering right part of truss, when load is between L_0L_1 , $\Sigma M_A = 0$ gives

$$P_{U_1L_2} \; 15 \sin \beta = 25 \; R_2$$

$$P_{U_1L_2} = \frac{25R_2}{15 \sin 45°} = 2.3570 \; R_2 \text{ (compressive)}$$

When load is at L_0, $R_2 = 0$; $\therefore$ $P_{U_1L_2} = 0$

When load is at L_1, $R_2 = 0.25$; $\therefore$ $P_{U_1L_2} = 2.3570 \times 0.25$
$= 0.5893$

When load is in the portion L_2L_4, considering the left portion and taking $\Sigma M_A = 0$, we get

$$P_{U_1L_2} \times 15 \sin \beta = R_1 \times 5$$

$$P_{U_1L_2} = \frac{5}{15 \sin 45°} R_1 \qquad \text{Since } \beta = 45°$$

$$= 0.4714 \; R_1 \text{ (comp)}$$

when load is at L_2, $R_1 = 0.5$; $\therefore$ $P_{U_1L_2} = 0.2357$,

when load is at L_4, $R_1 = 0$; $\therefore$ $P_{U_1L_2} = 0$

ILD is as shown in Fig.6.12(e).

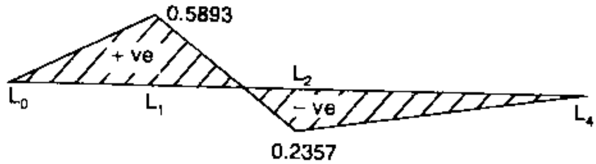


Figure 6.12 (d) *ILD for $P_{U_1U_2}$*

Example 6.6 Draw the influence line diagrams for the forces in the members U_3U_5, U_3M_4, U_3L_3, M_4L_4, M_4L_3, M_4L_5, L_3L_4, and L_4L_5 of the compound truss shown in Fig.6.13(a).

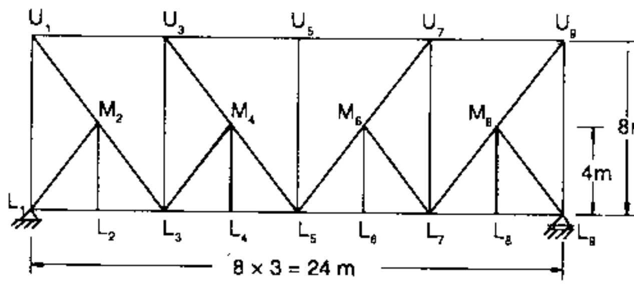


Figure 6.13 (a)

Solution

This is a through type truss in which the load moves on bottom chord. Hence bottom chord is made stronger. For analysis, this type of truss may be split into two — primary trusses of four panels each of length 6m and four secondary trusses each of panel 2 × 3 = 6 m.

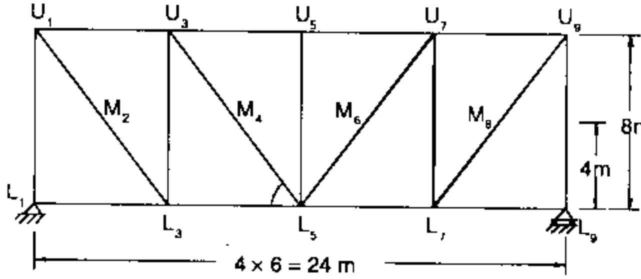


Figure 6.13 (b) *Primary Trusses*

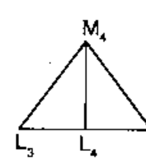


Figure 6.13 (c) *Secondary Trusses*

Fig.6.13(b), shows the primary truss and Fig.6.13(c) shows a secondary truss. Influence line diagrams for forces in the members common to both primary and secondary trusses will be obtained by combining ILD for the forces in the member of primary and of the secondary trusses.

ILD for force in $U_3 U_5$

Taking section (1)–(1) in primary truss (Fig.6.14a) for any position of unit

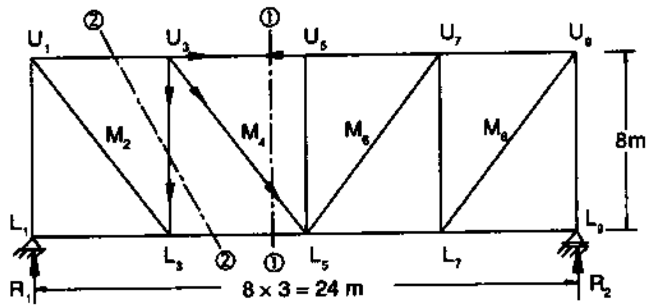


Figure 6.14 (a)

load,

$$P_{U_3U_5} \times 8 = M_{L_5}, \text{(comp)}$$

$$\therefore \qquad P_{U_3U_5} = \frac{1}{8} M_{L_5}, \text{(comp)}$$

$\therefore$ ILD is a triangle with maximum ordinate under L_5

$$= \frac{1}{8}\frac{z(L-z)}{L}$$

$$= \frac{1}{8}\frac{12(24-12)}{24}$$

$$= 0.75$$

This is shown in Fig.6.14(b).

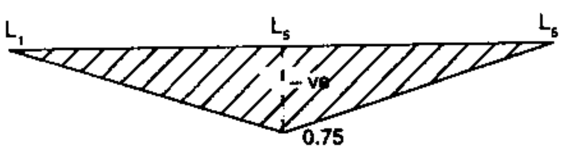


Figure 6.14 (b) *ILD for $P_{U_3U_5}$*

ILD for force in $U_3 M_4$

Let R_1 be the reaction of left support and R_2 be the reaction of right support. When load is between $L_1 L_3$, considering right side part of truss, we get

$$P_{U_3M_4} \sin\theta = R_2$$

$$P_{U_3M_4} = R_2 \operatorname{cosec}\theta$$

$$P_{U_3M_4} = 1.25\, R_2 \text{ (comp), since } \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{0.8}$$

when load is at L_1, $R_2 = 0$ $\quad\therefore P_{U_3M_4} = 0$

when load is at L_3, $R_2 = 0.25$ $\quad\therefore P_{U_3M_4} = 1.25 \times 0.25 = 0.3125$

when load is in portion L_5 to L_9,

$$P_{U_3M_4} \sin\theta = R_1$$

$$P_{U_3M_4} = R_1 \operatorname{cosec}\theta \text{ (tensile)}$$

when load is at L_5, $R_1 = 0.5$, $\quad\therefore P_{U_3M_4} = 1.25 \times 0.5 = 0.625$

when load is at L_9, $R_1 = 0$ $\quad\therefore P_{U_3M_4} = 0$

Between L_3 and L_5, $P_{U_3M_4}$ varies linearly. ILD for $P_{U_3M_4}$ is as shown in Fig. 6.14(c).

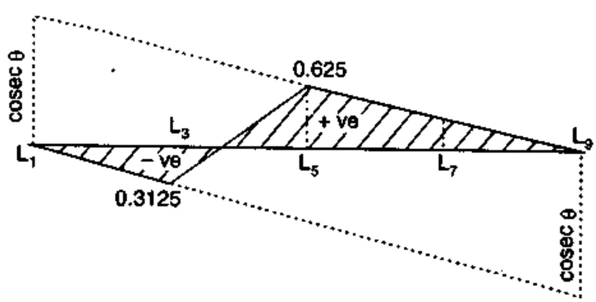


Figure. 6.14 (c) *ILD for* $P_{U_3M_4}$

ILD for $P_{U_3L_3}$

Consider section 2-2 :

When unit load is in portion L_1L_3, considering the equilibrium of vertical forces in right side part of the truss, we get

$$P_{U_3M_4} = R_2 \text{ (tensile)}$$
$$= 0, \text{ when load is at } L_1$$
$$= 0.25, \text{ when load is at } L_3$$

When the unit load is in portion L_5L_9, considering equilibrium of vertical forces in left side part, we get

$$P_{U_3M_4} = R_1 \text{ (compressive)},$$
$$= 0.5, \text{ when load is at } L_5$$
$$= 0, \text{ when load is at } L_9$$

For load position between L_3 L_5, ILD varies linearly.

$\therefore$ ILD for $P_{U_3L_3}$ is as shown in Fig. 6.14(d).

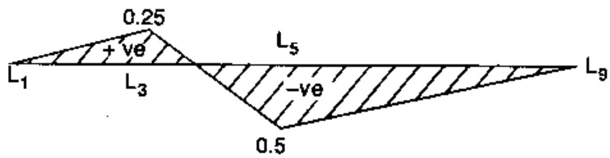


Figure. 6.14 (d) ILD for $P_{U_3L_3}$

ILD for force in M_4L_4

This member is seen only in secondary truss. Hence considering secondary truss (Fig. 6.13 (c)), $P_{M_4L_4}$ = load at L_4 (tensile)

Load at L_4 is zero when load is between L_1 L_3 and L_5 L_9. When unit load moves from L_3 to L_5, it increases linearly from zero at L_3 to unit value at L_4 and then decreases to zero at L_5. Hence ILD for $P_{M_4L_4}$ is as shown in Fig. 6.14(e).

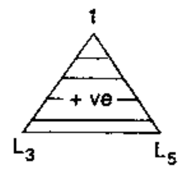


Figure. 6.14 (d) ILD for $P_{M_4L_4}$

ILD for force $P_{M_4L_5}$

In primary truss, ILD for $P_{M_4L_5}$ is same as that for $P_{U_3M_4}$. This is shown in Fig. 6.15 (b). In secondary truss taking section 1-1, Fig. 6.15(c) then considering the equilibrium

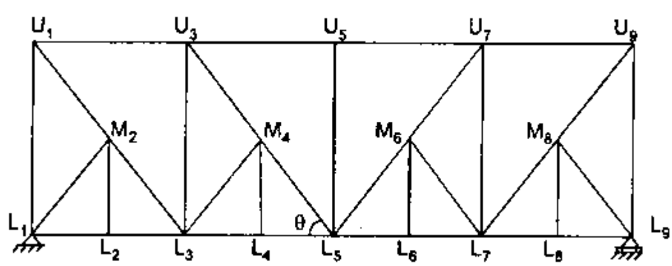


Figure 6.15 (a)

of parts show that $P_{M_4L_5}\, r = M_{L_4}$ where r is $\perp$r distance of M_4 L_5 from L_4 and M_{L_4} is moment about L_4

$$P_{M_4L_5} = \frac{1}{r} M_{L_4} \text{ (comp)}$$

Now $\qquad r = L_4 L_5 \sin\theta = 3\sin\theta$

In secondary truss, ILD for moment at L_4 is a triangle with maximum ordinate at L_4 as

$$\frac{3(6-3)}{6} = 15$$

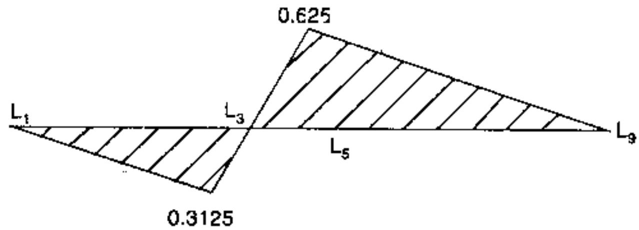


Figure. 6.15(b) ILD for $P_{M_4L_5}$ in primary truss

$\therefore$ ILD for $P_{M_4L_5}$ in secondary truss is a triangle with maximum ordinate at L_4 as

$$= \frac{1}{3 \sin \theta} \times 15$$

$$= 0.5 \ \text{cosec}\, \theta = 0.625$$

This is shown in Fig. 6.15(c).

Since the member M_4L_5 is in both primary and secondary trusses, its final ILD is the sum of ILD shown in Fig. 6.15 (b) and 6.15(c). This is shown in Fig. 6.15(d).

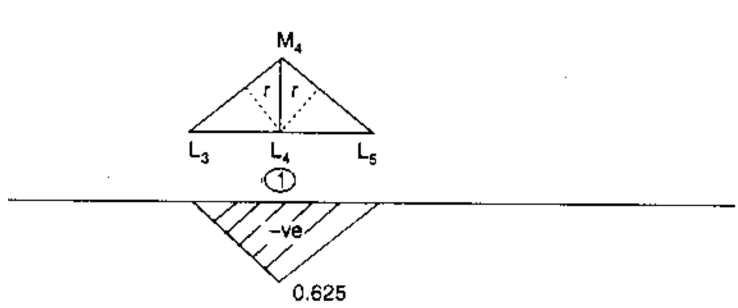


Figure. 6.15(c) ILD for $P_{M_4L_5}$ in secondary truss

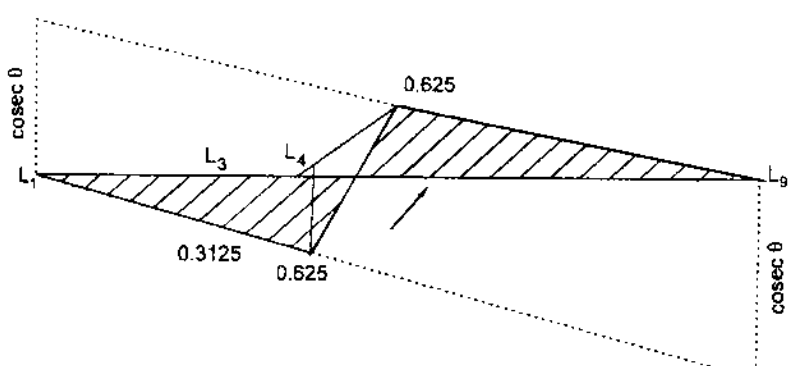


Figure. 6.15(d) ILD for $P_{M_4L_5}$ in compound truss

ILD for $P_{L_3M_4}$

This member is present only in secondary truss. By method of section in secondary truss, we can observe that,

$$P_{M_3L_4} \times r = M_{L_4}$$

where $\qquad r = 3 \sin\theta = 2.4$

$\therefore$ ILD for $P_{M_3L_4}$ is similar to that of $P_{M_3L_5}$ in secondary truss (Ref.Fig.6.15(c)).

ILD for $P_{L_3L_4}$

This member is present both in primary and secondary truss. Considering section 1–1 in primary truss (Fig.6.16.a), we find

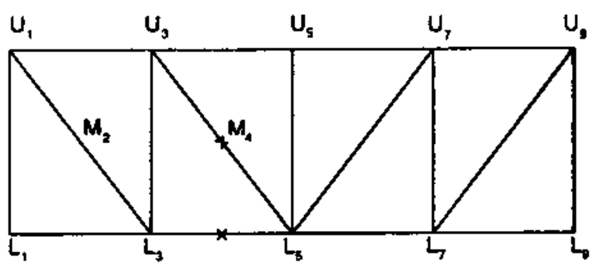


Figure 6.16 (a) Primary Truss

$$P_{L_3L_4} \times 8 = M_{U_3}$$

$$P_{L_3L_4} = \frac{1}{8} M_{U_3} \text{ . (tensile)}$$

Moment at M_{U_3} is having ILD as a triangle with maximum ordinate at U_3

$$= \frac{z(L-z)}{L} = \frac{6(24-6)}{24} = 4.5$$

$\therefore$ In primary truss ILD for $P_{L_3L_4}$ is a triangle with maximum ordinate under

$U_3 = \frac{1}{8} \times 4.5 = 0.5625$ as shown in Fig.6.16(b).

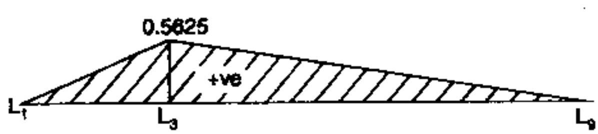


***Figure. 6.16 (b)** ILD for $P_{L_3L_4}$ in primary truss*

In secondary truss, $P_{L_3L_4}$ is given by

$$P_{L_3L_4} \times 4 = \text{moment about } M_4$$

$$P_{L_3L_4} = \frac{1}{4} \times \text{Moment about } M_4 \text{ (tensile)}$$

Moment about M_4 in secondary truss has a triangular ILD with maximum ordinate

$$= \frac{3(6-3)}{6} = 1.5 \text{ at mid-span}$$

∴ In secondary truss, ILD for $P_{L_3L_4}$ is a triangle with maximum ordinate,

$$= \frac{1}{4} \times 1.5 = 0.375 \text{ at mid-span.}$$

This is shown in Fig.6.16(c).

Figure 6.16 (c) *ILD for $P_{L_3L_4}$ in secondary truss*

Combination of ILD for $P_{L_3L_4}$ in primary (Fig.6.16.b) and that in secondary (Fig.6.16(c)) gives ILD for $P_{L_3L_4}$ in the compound truss. This is shown in Fig.6.16(d).

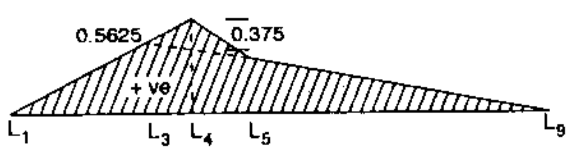


Figure 6.16 (d) *ILD for $P_{L_3L_4}$ in compound truss*

Since $P_{L_4L_5} = P_{L_3L_4}$ ILD for $P_{L_4L_5}$ is same as that for $P_{L_3L_4}$.

EXERCISES

6.1 Draw the influence line diagrams for the forces in the members $U_1 U_2$, $U_1 L_2$ and $U_1 L_2$ of the truss shown in Fig.6.17. Determine the maximum forces in these members when uniformly distributed load of intensity 8 kN/m and length 5m traverses the span.

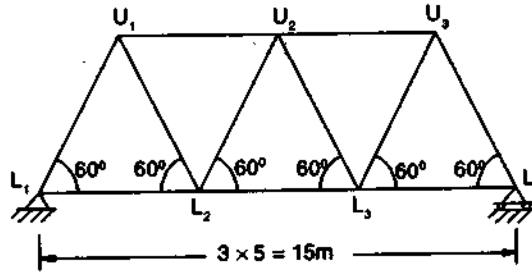


Figure 6.17

Ans : $P_{U_1U_2} = P_{U_1L_2} = P_{U_1L_1} = 25.66$ kN

6.2 Determine the maximum forces in the members 1,2,3 and 4 of the truss shown in Fig.6.18 when uniformly distributed load of 30 kN/m longer than the span traverses along the girder.

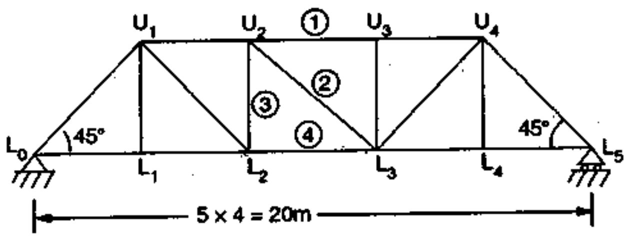


Figure 6.18

Ans : $P_1 = -360$ kN, $P_2 = \pm 84.853$ kN

$P_3 = 360$ kN, $P_4 = \pm 60$ kN

6.3 Find the maximum forces developed in the members BC, BH and HG of the truss shown in Fig. 6.19. when a load of 30 kN/m longer than the span moves from left to right.

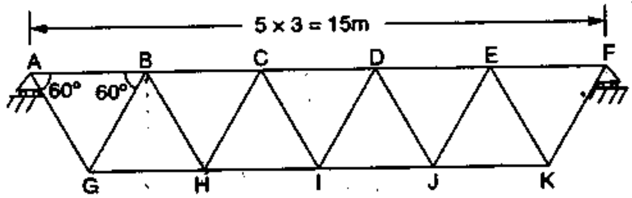


Figure 6.19

Ans : $P_{BC} = 318.19$ kN, $P_{BH} = 12.59$ kN (comp) and

116.91 kN (tensile), $P_{HG} = 207.846$ kN

THREE HINGED ARCHES 7

7.1 INTRODUCTION

Beams transfer the applied load to end supports by bending and shear action. In this process, either one or two points at a particular section is subjected to maximum stress. The material in most of the portion is under stress and hence under-utilised. The horizontal distance from one support to another is called the *span*. For larger spans, beams are very uneconomical and many a time the self-weight of beams contributes to the stress in such large proportions that it is difficult to design beams for larger spans. For large spans like bridges arches are provided instead of beams (Fig.7.1). Arches are nothing but curved beams (usually in the vertical plane) that transfer loads to their plane.

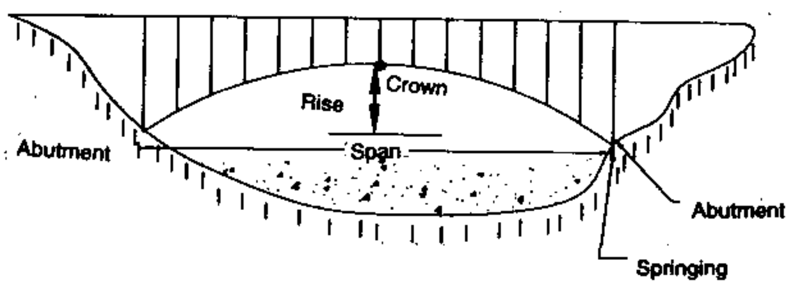


Figure 7.1 Typical arch bridge

Arches transfer loads to abutments at springing points. Hinges may be provided at these points. The topmost point is called the *crown* which sometimes has a hinge. The height of the crown above the support level is called *rise*.

Because of the curved nature of arches, they give rise to horizontal forces. Abutments are designed for horizontal forces also. Any section in the arch will be subjected to normal thrust, radial shear and bending moment (Fig.7.2). However, the bending moment is considerably less compared to a beam of the same span. Thus, loads get transferred partly by axial compression and partly by flexure. In axial

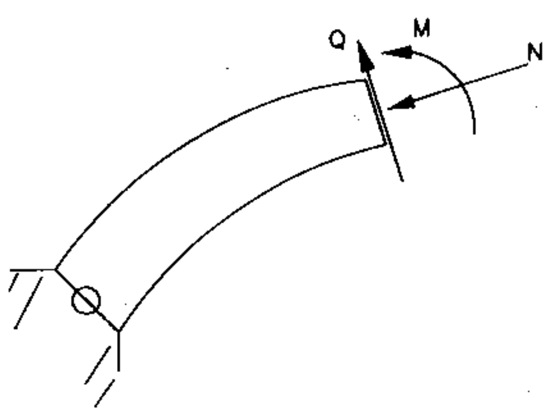


Figure 7.2 Internal forces in an arch

compression, each and every particle of the cross-section of the structure is subjected to stress equally. Hence the material is utilised fully. Reduction in the bending moment results in smaller sections for the arch compared to the section required for the beams to transfer the same load. There are three types of arches depending upon the number of hinges provided.

1. Three hinged arch (Fig.7.3(a))
2. Two hinged arch (Fig.7.3(b))
3. Hingeless arch or fixed arch (Fig.7.3(c))

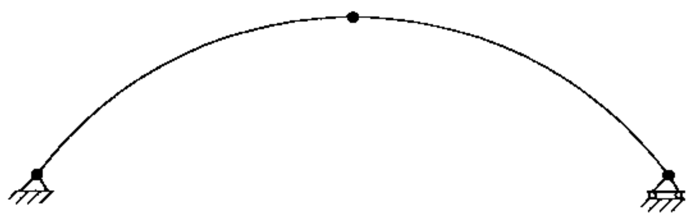


Figure 7.3 (a) Three hinged arch

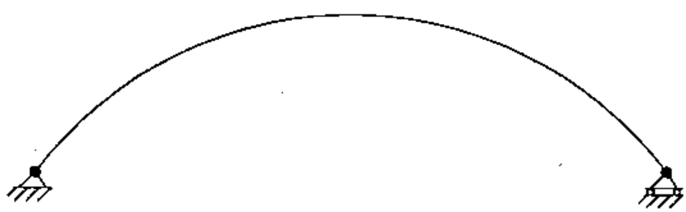


Figure 7.3 (b) Two hinged arch

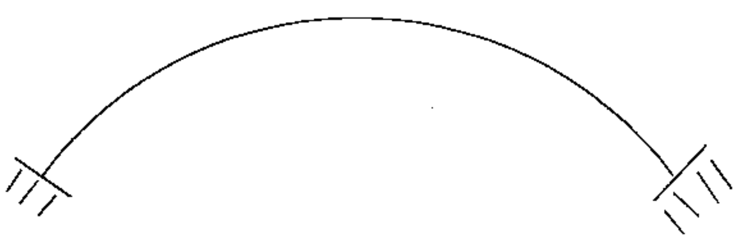


Figure 7.3 (c) Hingeless / Fixed arch

Figure 7.3 Types of arches

7.2 THREE HINGED ARCHES

Three hinged arch is a determinate structure whereas two hinged and fixed arches are indeterminate structures.

7.2.1 Types of Three Hinged Arches

Three hinged arch as may have different shapes. Commonly used shapes are

a. Circular
b. Parabolic

From the property of a circle, the radius R of the circular arch of span 'L' and rise 'h' may be found as

$$\frac{L}{2}\times\frac{L}{2} = h(2R - h) \qquad 7.1$$

or

$$R = \frac{L^2}{8h} + \frac{h}{2} \qquad 7.2$$

Taking origin at supports A, the coordinates of any point D on the arch (Refer Fig. 7.4) may be defined as

$$x = \left[\frac{L}{2} - R\sin\theta\right]$$

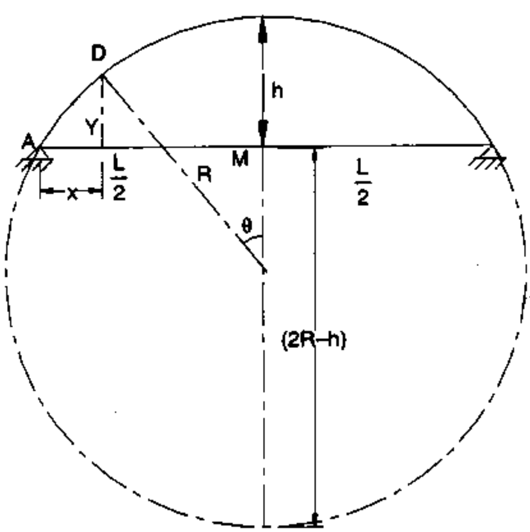


Figure 7.4

$$y = R\cos\theta - (R - h)$$
$$= h - R(1 - \cos\theta)$$

In the case of a parabolic arch, taking the springing point as the origin (refer Fig. 7.5) its equation is given by

$$y = \frac{4hx}{L^2}(L - x)$$

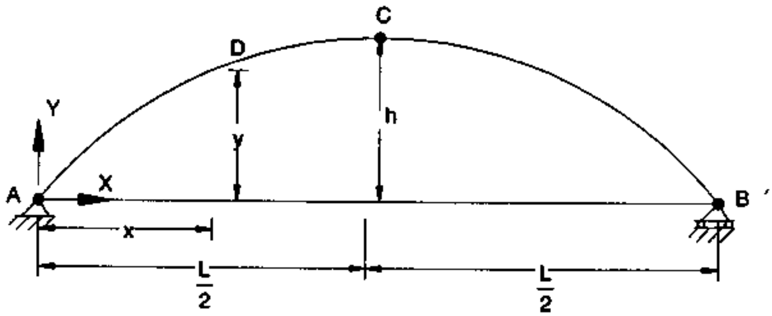


Figure 7.5

If the crown is taken as the origin, the equation of parabolic curve (refer Fig.7.6) is

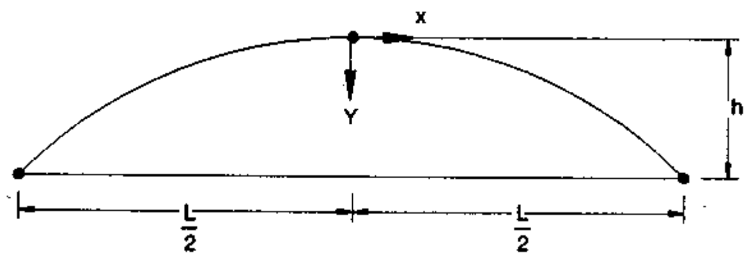


Figure 7.6

$$\frac{x^2}{y} = a, \text{ where 'a' is a constant.}$$

If the springing points are at the same level (Fig.7.6), then

at $x = \frac{L}{2}$, $y = h$,

$$\frac{L^2}{4h} = a$$

Hence the equation is $$\frac{x^2}{y} = \frac{L^2}{4h}$$

If the springing points are not at the same level (Fig.7.7): Let h_1 and h_2 be the depth of the abutments from the crown and let L be the span. Then,

$$\frac{x^2}{y} = \text{constant}$$

$$\frac{x}{\sqrt{y}} = \text{constant}$$

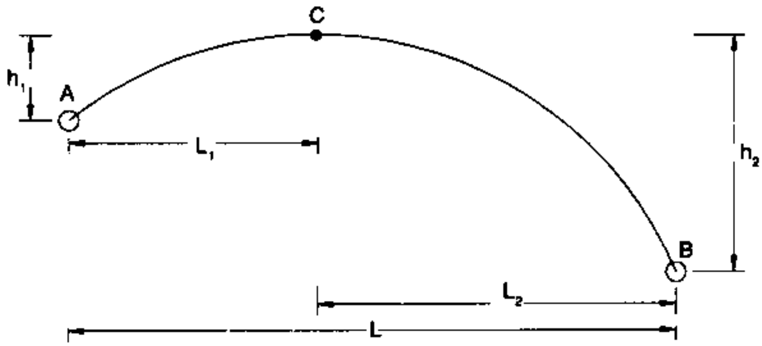


Figure 7.7

Applying this equation to points A and B, we get

$$\text{constant} = \frac{L_1}{\sqrt{h_1}} = \frac{L_2}{\sqrt{h_2}} = \frac{L_1 + L_2}{\sqrt{h_1} + \sqrt{h_2}}$$

$$= \frac{L}{\sqrt{h_1} + \sqrt{h_2}}$$

$$\therefore \qquad L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}}$$

and

$$L_2 = \frac{L\sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}}$$

7.3 ANALYSIS FOR STATIC LOADS

Consider a three hinged arch subjected to loads as shown in Fig. 7.8. Since the nds are hinged there will be two reaction components at each end namely vertical ıd horizontal. Hence, totally there are four reaction components namely, V_A, H_A, $_B$, and H_B.

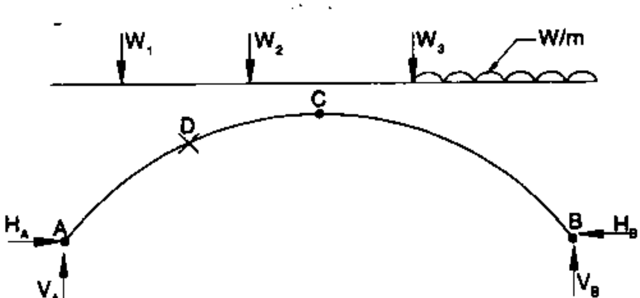


Figure 7.8

For any plane structure there are three independent equations of equilibrium which can be used conveniently.

$$\left.\begin{array}{l}\sum H = 0 \\ \sum V = 0 \\ M_A \text{ or } M_B = 0\end{array}\right\} \quad 7.9$$

In this case, the fourth equation is also available i.e.,

$$M_c = 0, \text{ since C is a hinge}$$

If no horizontal load is acting, which is the usual case, equation 7.9(a) gives $H_A = H_B$, say H. In such case, the following three equations are used.

$$\left.\begin{array}{lr}\sum V = 0 & \text{(a)} \\ M_A \text{ or } M_B = 0 & \text{(b)} \\ \text{and} \quad M_C = 0 & \text{(c)}\end{array}\right\} \quad 7.10$$

Since the loads tend to spread the arch, the horizontal thrust is in the inward direction as shown in the figure.

Now consider a section at D,

Let V be the vertical shear

Q, the radial shear

and N, the normal thrust.

All these forces are shown in their positive senses in Fig. 7.9. Let the normal to the section make an angle θ with the horizontal.

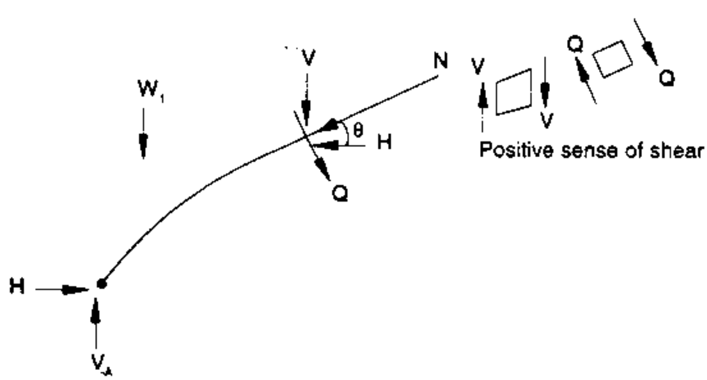


Figure 7.9

Then,

$$N = V \sin\theta + H\cos\theta \quad 7.11$$

and

$$Q = V\sin\theta - H\cos\theta \quad 7.12$$

The moment at D can be obtained by considering all the forces including the reaction on any one part of the arch. Sagging moment M is taken as positive moment in this text.

Example 7.1 A three hinged circular arch hinged at the springing and crown points has a span of 40m and a central rise of 8m. It carries a uniformly distributed load 20 kN/m over the left half of the span together with a concentrated load of 100 kN at the right quarter span point. Find the reactions at the supports, normal thrust and shear at a section 10m from the left support.

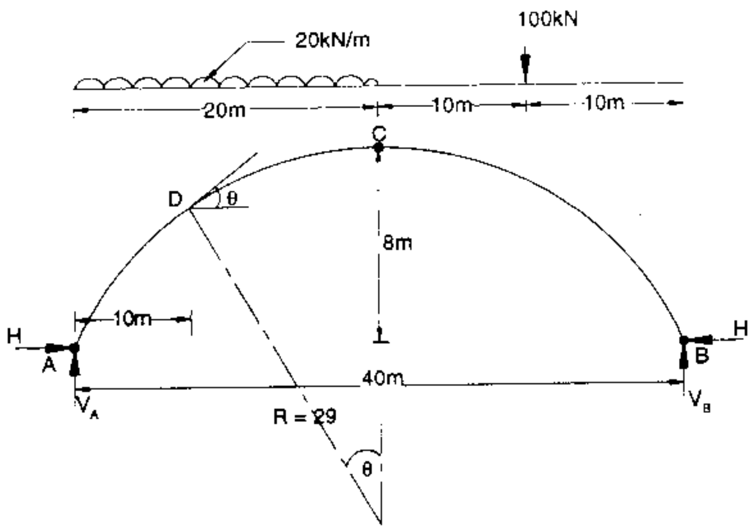


Figure 7.10

Solution

The arch is shown in Fig. 7.10

$$\Sigma M_B \to 0$$

$$V_A \times 40 - 20 \times 20 \times 30 - 100 \times 10 = 0$$

$$V_A = 325 \text{ kN}$$

$$\Sigma V = 0$$

$$V_A + V_B = 20 \times 20 + 100$$

$$\therefore \quad V_B = 500 - V_A = 500 - 325$$

$$= 175 \text{ kN}$$

Since C is hinged,

$$M_C = 0$$

$$= V_B \times 20 - 100 \times 10 - H \times 8 = 0$$

$$= 175 \times 20 - 100 \times 10 - H \times 8 = 0$$

or $$H = 312.5 \text{ kN}$$

Let D be the point 10m from the left support where the normal thrust and shear are to be found. Now from the property of circles,

$$h(2R - h) = \frac{L}{2} \times \frac{L}{2}$$

$$8(2R - 8) = \frac{40}{2} \times \frac{40}{2} = 400$$

$\therefore$ $$R = 29 \text{ m}$$

$\therefore$ $$\text{Slope at } D = \theta = \sin^{-1}\frac{10}{R} = \sin^{-1}\frac{10}{29}$$

$\therefore$ $$\theta = 20.171°$$

Vertical shear at $$D = V = V_A - 20 \times 10$$

$$= 325 - 200 = 125\text{kN}$$

$\therefore$ $$N = V \sin\theta + H \cos\theta$$

$$= 125 \sin 29.171° + 312.5 \cos 20.171°$$

$$= 336.437 \text{ kN}$$

$$Q = V \cos\theta - H \sin\theta$$

$$= 125 \cos 20.171° - 312.5 \sin 20.171°$$

$$= 9.575\text{kN}$$

Example 7.2 A circular arch of span 25m with a central rise 5m is hinged at the crown and springing. It carries a point load of 100kN at 6m from the left support. Calculate

i. the reactions at the supports
ii. the reactions at crown
iii. moment at 5m from the left support.

Solution

The arch with loading on it is shown in Fig.7.11(a). Taking the moment about B we get

$$V_A \times 25 = 100 \times (25 - 6)$$

$\therefore$ $$V_A = 76 \text{ kN}$$

$\therefore$ $$V_B = 100 - 76 = 24 \text{ kN}$$

Considering moment about C, we get

$$0 = 24 \times 12.5 - H \times 5$$

$$H = 60 \text{ kN}$$

Considering the equilibrium of the left half of the arch, the reactions at crown (refer Fig.7.11(b)) are

$$H = \overleftarrow{60} \text{ kN and } V_C = 24 \text{ kN} \uparrow$$

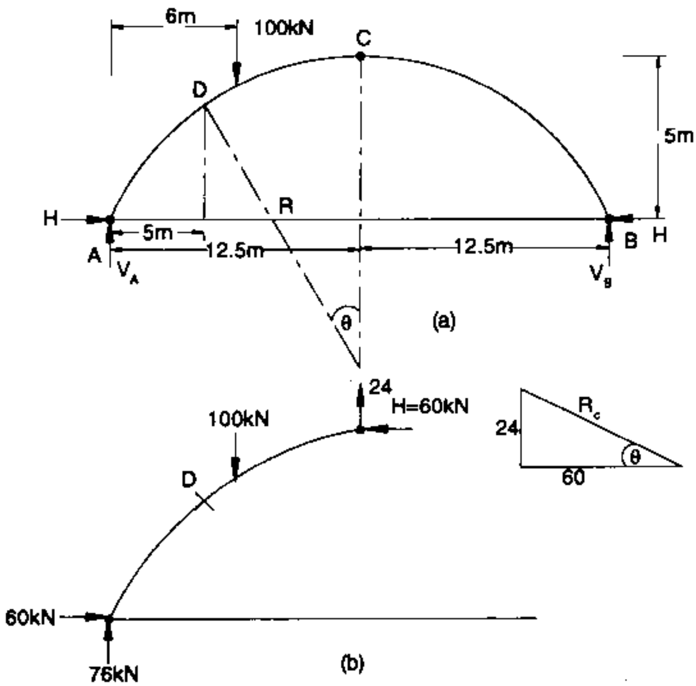


Figure 7.11

Moment at 5m from the left support :

From the property of circles, we get

$$h\ (2R - h) = \frac{L}{2} \times \frac{L}{2}$$

$$5\ (2R - 5) = \frac{25}{2} \times \frac{25}{2}$$

$$R = 18.125\text{m}$$

Referring to Fig. 7.11(a), we find

$$R\ \sin\theta = 12.5 - 5 = 7.5$$

i. e.,

$$\sin\theta = \frac{7.5}{18.125} = 0.4138$$

$$\theta = 24.443°$$

$$y_D = h - R(1 - \cos\theta)$$

$$= 5 - 18.125\ (1 - \cos 24.443°)$$

$$= 3.375$$

$\therefore$

$$M_D = V_A \times 5 - Hy_D$$

$$= 76 \times 5 - 60 \times 3.375$$

$$= 177.5 \text{ kN–m}$$

Example 7.3 A three hinged semi-circular arch of radius R carries a uniformly distributed load of intensity w/unit length over its entire horizontal span. Determine the reactions of supports and maximum bending moment in the arch.

Solution

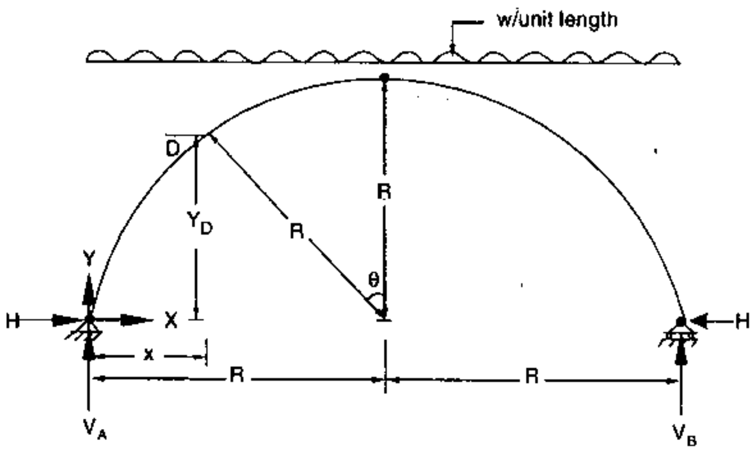


Figure 7.12

The arch is shown in Fig. 7.12. Since it is semi-circular, the span = 2R. Due to symmetry,

$$V_A = V_B = \frac{w \times 2R}{2} = wR$$

Taking moment about the crown point C, we get

$$H \times R - V_A \times R + wR \times \frac{R}{2} = 0$$

$$HR - wRR + \frac{wR^2}{2} = 0 \text{ or } H = \frac{wR}{2}$$

But, $x = R - R \sin\theta$ and $y = R\cos\theta$,

$$M_x = V_A\, x - Hy - \frac{wx^2}{2}$$

$$\therefore \quad M_x = wR\,(R - R\sin\theta) - \frac{wR}{2} R\cos\theta - \frac{w}{2}(R - R\sin\theta)^2$$

$$= \frac{wR^2}{2}[2(1-\sin\theta) - \cos\theta - (1-\sin\theta)^2]$$

$$= \frac{wR^2}{2}(2 - 2\sin\theta - \cos\theta - 1 + 2\sin\theta - \sin^2\theta)$$

$$= \frac{wR^2}{2}(1 - \cos\theta - \sin^2\theta)$$

For M_x to be maximum

$$\frac{dM_x}{d\theta} = 0 = \frac{wR^2}{2}(\sin\theta - 2\sin\theta\cos\theta)$$

$\therefore$ $$\sin\theta\ (1-2\cos\theta) = 0$$

$\theta = 0$ gives the crown point where moment is zero (minimum)

$1 - 2\cos\theta = 0$ Should give maximum point

i.e., $$\cos\theta = 0.5 \text{ or } \theta = 60°$$

$\therefore$ x for maximum moment point

$$= R\ (1 - \sin 60°)$$

$$M_{max} = \frac{wR^2}{2}\ (1 - \cos 60° - \sin^2 60°)$$

$$= -\frac{wR^2}{8}$$

Example 7.4 A three hinged parabolic arch hinged at the supports and at the crown has a span of 24m and a central rise of 4m. It carries a concentrated load of 50kN at 18m from left support and a uniformly distributed load of 30kN/m over the left half portion. Determine the moment, thrust and radial shear at a section 6m from the left support.

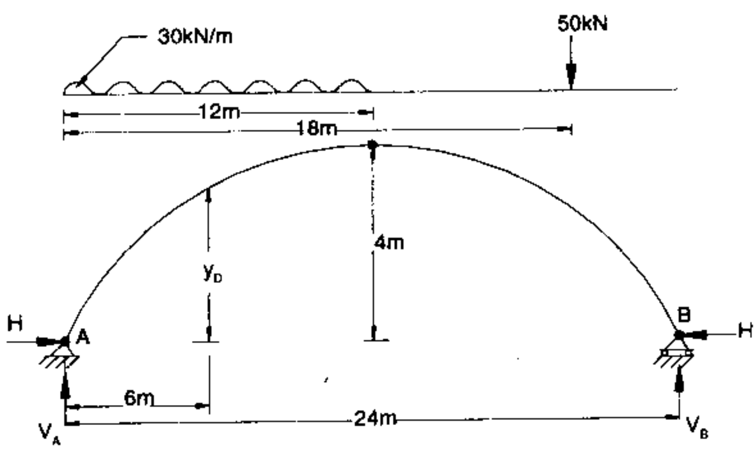


Figure 7.13

Solution

The arch is shown in Fig.7.13. Taking moment about B, we get

$$V_A \times 24 - 30 \times 12 \times 18 - 50 \times 6 = 0$$

$$V_A = 282.50\text{kN}$$

$\therefore$ $$V_B = 30 \times 12 + 50 - 282.50 = 127.5\text{kN}$$

Taking moment about crown C,

$$V_B \times 12 - H \times 4 - 50 \times 6 = 0$$

$$127.5 \times 12 - H \times 4 + 50 \times 6 = 0$$

or $$H = 307.5\text{kN}$$

At 6m from the left support,

$$M = V_A \times 6 - Hy_D - 30 \times \frac{6^2}{2}$$

In the parabolic arch,

$$y = \frac{4hx(L-x)}{L^2}$$

$\therefore$ at x = 6m

$$y_D = \frac{4 \times 4 \times 6(24-6)}{24^2} = 3\text{m}$$

$$M = 282.5 \times 6 - 307.5 \times 3 - 30 \times \frac{6^2}{2}$$

$$= 232.5\text{kN–m}$$

Vertical shear at D,

$$V = V_A - 30 \times 6$$

$$= 282.5 - 30 \times 6 = 102.5\text{kN}$$

Curve is given by

$$y = \frac{4hx(L-x)}{L^2}$$

$$\therefore \quad \frac{dy}{dx} = \tan\theta = \frac{4h(L-2x)}{L^2}$$

$\therefore$ at x = 6m,

$$\tan\theta = \frac{4 \times 4(24 - 2 \times 6)}{24 \times 24}$$

$$\therefore \quad \theta = 18.435°$$

$$\therefore \quad N = V\sin\theta + H\cos\theta$$

$$= 102.5 \sin 18.435° + 307.5 \cos 18.435°$$

$$= 324.133\text{kN}$$

and radial shear

$$Q = V\cos\theta - H\sin\theta$$

$$= 102.5 \cos 18.435° - 307.5 \sin 18.435°$$

$$= 0$$

Example 7.5 A symmetric three hinged parabolic arch has a span of 30m and a central rise of 6m. The arch carries a distributed load which varies uniformly from 40 kN/m at each abutment to zero at mid-span. Determine

a. the horizontal thrust at the abutments
b. maximum positive bending moment in the arch.

Solution

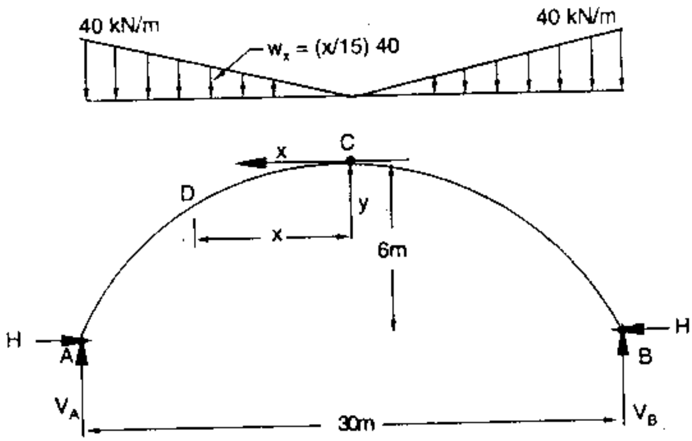


Figure 7.14 (a)

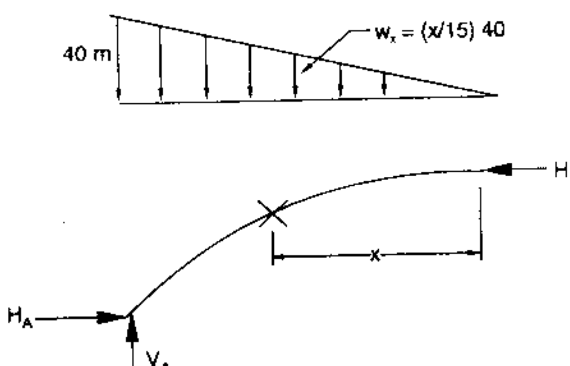


Figure 7.14 (b)

The arch is shown in the Fig. 7.14 (a). Due to symmetry

$$V_A = V_B = \frac{1}{2} \times \text{total load}$$

$$= \frac{1}{2} \times 2\left(\frac{1}{2} \times 15 \times 40\right)$$

$$= 300\text{kN}$$

$$\Sigma M_c = 0 \rightarrow$$

$$V_A \times 15 - H \times 6 - \frac{1}{2} \times 15 \times 40 \times 10 = 0$$

$$H = 250 \text{ kN}$$

Taking origin at crown point C, the equation of parabola is given by

$$\frac{x^2}{y} = \text{constant} = a$$

at x = 15 and y = 6

$$a = \frac{15^2}{6} = 37.5$$

or $$x^2 = 37.5y \quad \text{or} \quad y = \frac{x^2}{37.5}$$

Reaction at crown is the horizontal force H. Here, due to symmetry, there is no vertical shear (refer. Fig. 7.14(b)).

$$\therefore \quad M_x = Hy - \frac{1}{2} x\, W_x \frac{2x}{3}$$

$$= 250 \frac{x^2}{3.75} - \frac{1}{2} x \frac{x}{15} 40 \frac{2x}{3}$$

$$= \frac{250}{37.5} x^2 - \frac{40x^3}{45}$$

For maximum bending moment,

$$\frac{dMx}{dx} = 0 = \frac{250}{37.5} 2x - \frac{40}{45} 3x^2$$

$$\therefore \quad x\left(\frac{500}{37.5} - \frac{40}{15}x\right) = 0$$

The above equation has two solutions: $x = 0$ and $\frac{500}{37.5} - \frac{40}{15}x = 0$

x = 0 gives the crown point, where moment is minimum i.e. zero. Hence the point of maximum moment is given by

$$x = \frac{500}{37.5} \times \frac{15}{40} = 5$$

$$M_{max} = \frac{250x^2}{37.5} - \frac{40x^3}{45}$$

$$= \frac{250 \times 25}{37.5} - \frac{40 \times 125}{45} = 55.555 \text{ kN-m}$$

at x = 5m from crown i.e. at 10 m from abutment.

Example 7.6 Show that the parabolic shape is a funicular shape for a three hinged arch subjected to a uniformly distributed load over to its entire span.

Solution

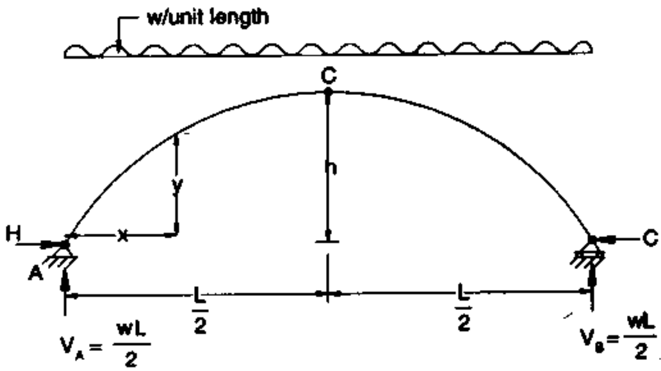


Figure 7.15

Let the span of the arch be 'L' and rise be 'h' as shown in Fig.7.15. Due to symmetry,

$$V_A = V_B = \frac{1}{2} \times \text{total load}$$

$$= \frac{1}{2} w.L = \frac{wL}{2}$$

Taking moment about C, we get

$$0 = V_A \frac{L}{2} - Hh - w\frac{L}{2}\frac{L}{4}$$

$$= \frac{wL}{2}\frac{L}{2} - Hh - \frac{wL^2}{8}$$

or $$H = \frac{wL^2}{8}$$

At any section distance x from A,

$$M = V_A x - Hy - \frac{wL^2}{2}$$

But in a parabolic arch, $$y = \frac{4hx(L-x)}{L^2}$$

$\therefore$ $$M = \frac{wL}{2} x - \frac{wL^2}{8} \frac{4hx(L-x)}{L^2} - \frac{wL^2}{2}$$

$$= \frac{wLx}{2} - \frac{w}{2} x(L-x) - \frac{wL^2}{2} = 0$$

Thus, for a parabolic arch is subjected to a uniformly distributed load over its entire span, the bending moment at any section is zero. Hence, the parabolic shape is a funicular shape for a three hinged arch subjected to uniformly distributed load over entire span.

Example 7.7 A three hinged parabolic arch having supports at different levels shown in Fig.7.16 carries a uniformly distributed load of intensity 30kN/m over the portion left of the crown. Determine the horizontal thrust developed. Find also the bending moment, normal thrust and radial shear force developed at a section 15m from the left support.

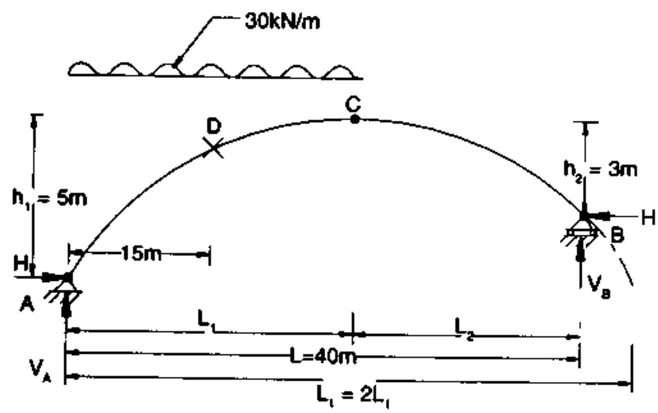


Figure 7.16

Solution

Taking C as the origin the equation of the parabola is

$$\frac{x^2}{y} = a, \text{ where 'a' is a constant.} \qquad ...(1)$$

Let the horizontal distance between A and C be L_1 and that of C and D be L_2. Then,

$$\frac{L_1^2}{5} = a = \frac{L_2^2}{3}$$

$$\frac{L_1}{\sqrt{5}} = \frac{L_2}{\sqrt{3}} = \frac{L_1 + L_2}{\sqrt{5}+\sqrt{3}} = \frac{L}{\sqrt{5}+\sqrt{3}}$$

$$\therefore \quad L_1 = \frac{L\sqrt{5}}{\sqrt{5}+\sqrt{3}} = \frac{40\sqrt{5}}{\sqrt{5}+\sqrt{3}} = 22.54\text{m}$$

$$\therefore \quad L_2 = 40 - 22.54 = 17.46\text{m}$$

$\Sigma M_c = 0 \rightarrow$

$V_B \times 17.46 = H \times 3$

$H = 5.82\ V_B$ ...(2)

$$\Sigma M_A = 0 \rightarrow V_B \times 40 + H \times 2 = 30 \times 22.54 \times \frac{22.54}{2}$$

$$= 7620.774 \quad ...(3)$$

From eqns. (2) and (3), we get

$V_B \times 40 - 5.82\ V_B \times 2 = 7620.774$

$V_B = 147.58$

$H = 5.82 \times 147.58 = 858.92$

and $V_A = 30L_1 - V_B$

$= 30 \times 22.5 - 147.58 = 528.02$ kN

The portion left of C may be treated as a parabola of span

$L = 2 \times 22.54 = 45.08$m

∴ Eqn. of parabola is

$$y = \frac{4h_1x\,(L'-x)}{L'^2} = \frac{(4)(5)(x)(45.08-x)}{45.08^2}$$

At $x = 15$m, $y = 4.44$m

∴ M at this section,

$$= V_A \times 15 - H \times 4.44 - 30 \times 15\ \frac{15}{2}$$

$$= 528.62 \times 15 - 858.92 \times 4.44 - 30\ \frac{15 \times 15}{2}$$

$$= 740 \text{ kN-m}$$

$$\frac{dy}{dx} = \tan\theta = \frac{4h(L'-2x)}{L'^2}$$

At $x = 15$m

$$\tan\theta = \frac{4 \times 5 \times (45.08 - 2 \times 15)}{45.08^2}$$

$\theta = 8.44°$

$V = 528.62 - 30 \times 15 = 78.62$ kN

$N = V\sin\theta + H\cos\theta$

$= 78.62 \sin 8.44° - 858.92 \sin 8.44°$

$= 861.25$ kN

$Q = V\cos\theta - H\sin\theta$

$= 78.62 \cos 8.44° - 858.92 \sin 8.44°$

$= -48.30$ kN

7.4 BENDING MOMENT DIAGRAMS

In the arch, at any section D (x, y), the bending moment may be looked as a sum of the moment in an equivalent beam minus the ordinate time the horizontal thrust. Thus

M = Beam moment – Hy

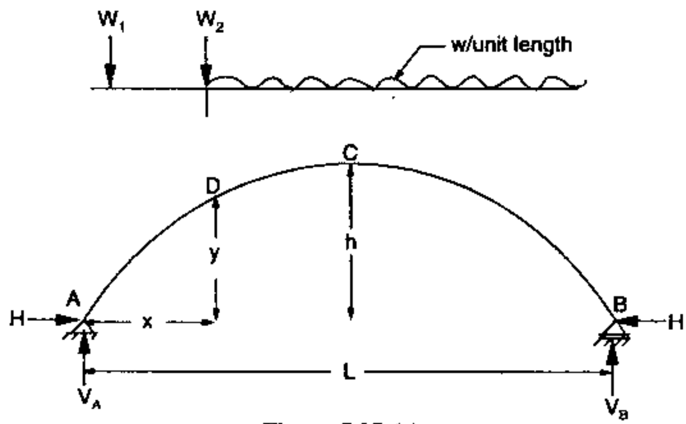


Figure 7.17 (a)

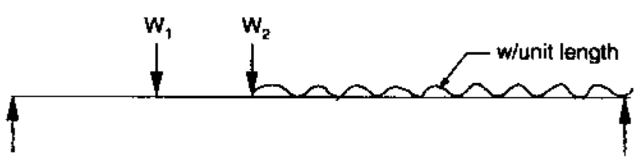


Figure 7.17 (b)

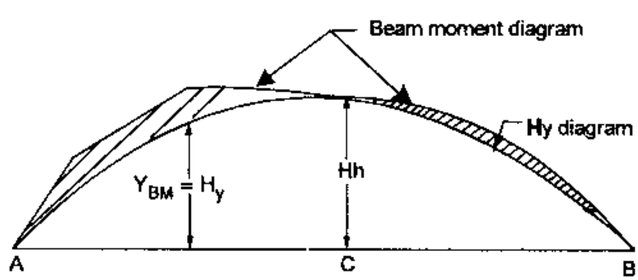


Figure 7.17 (c)

Hence, first bending moment diagram for the equivalent beam may be drawn. Then to subtract Hy moment, we note that the moment at the central hinge is zero. Hence, the curve of the arch is drawn with the ordinate at the crown point equal to the ordinate of the bending moment diagram as shown in Fig. 7.17(c).

Example 7.7 A symmetric three hinged parabolic arch of span 36m and rise 6m is subjected to a concentrated load of 120kN at a point 12m from left support. Draw the bending moment diagram for the arch.

Solution

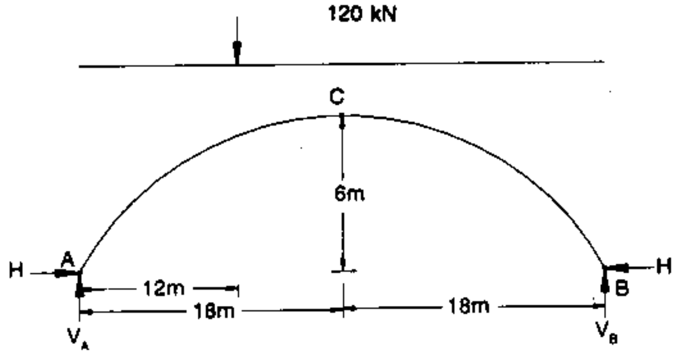


Figure 7.18 (a)

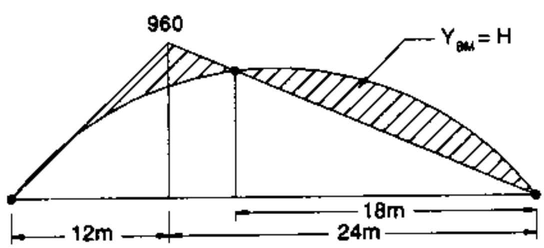


Figure 7.18 (b)

The arch is shown in Fig.7.18(a).

$$\Sigma M_B = 0 \rightarrow$$
$$V_A \times 36 - 120(36 - 12) = 0$$
$$V_A = 80 \text{ kN}$$
$$\therefore \quad V_B = 120 - 80 = 40 \text{ kN}$$

Beam moment diagram is a triangle with maximum ordinate at the load point, its ordinate being

$$= \frac{120 \times 12\,(36-12)}{36} = 960\text{kN-m}$$

This is drawn first (Fig. 7.18b). Now, at mid-span the net bending moment is zero. The ordinate of the beam moment diagram at mid-span is

$$= \frac{960 \times 18}{24} = 720\text{kN-m}$$

Since $M_c = 0$ in the arch, $Hh = 720$

or
$$H = \frac{720}{h} = \frac{720}{6} \text{ kN} = 120 \text{ kN}$$

A parabola is drawn with its central ordinate equal to 720kN-m as shown in Fig.7.18(b). The equation of this parabola is

$$y_{BM} = Hy = H\frac{4hx(L-x)}{L^2}$$

Example 7.8 A three hinged symmetric parabolic arch of span 60 m and rise 12m is subjected to a concentrated load of 40kN acting at 10m from its left support and uniformly distributed load of intensity 10kN/m acting over its entire right half portion. Draw the bending moment diagram.

Solution The arch with its loading is shown in Fig.7.19(a). The equivalent beam for beam calculations is shown in Fig.7.19(b).

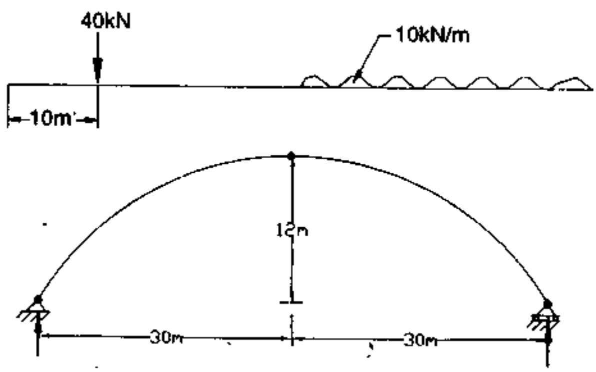


Figure 7.19 (a)

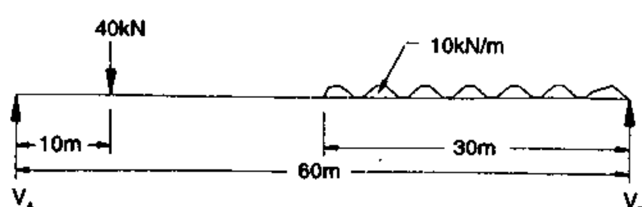


Figure 7.19 (b) Equivalent beam

To calculate the beam moment,

$$V_A = \frac{40(60-10)+10\times30\times15}{60}$$
$$= 108.33\text{kN}$$
$$V_B = 40 + 10 \times 30 - 108.33$$
$$= 231.67\text{kN}$$

$$M_D = 108.33 \times 10 = 1080.33$$
$$M_C = 108.33 \times 30 - 40 \times 20$$
$$= 2450\text{kN–m}$$

In the portion AC, variation of BMD is linear, and in BC, it is parabolic as shown in Fig.7.19(c).

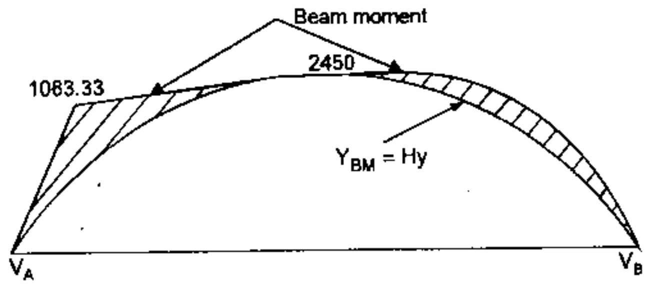


Figure 7.19 (c) Bending moment diagram

At the centre of the span, the bending moment in the arch should be zero.

$$H \times 12 = 2450$$
$$H = 204.167\text{kN}$$

Hence a parabola with y_{MB} = Hy is drawn over this diagram. The difference diagram is the bending moment diagrams for the arch. At the central hinge, the two diagrams cross since the bending moment at the hinge is zero.

7.5 INFLUENCE LINE DIAGRAMS

Consider the three hinged arch of span L and rise h shown in Fig.7.20. The influence line diagrams for the following are discussed in this section.

1. For horizontal thrust H
2. Moment at section D
3. Normal thrust at D
4. Radial shear at D

where D is a point distance 'z' from the left support A.

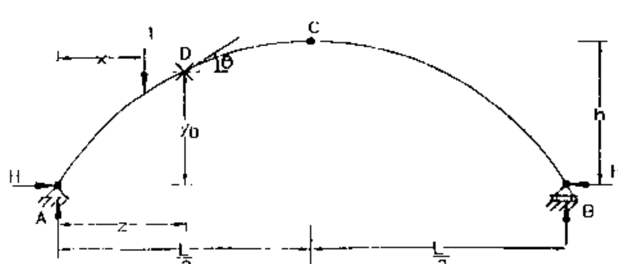


Figure 7.20 (a)

Let 'x' be the distance of the unit load from the support A.

ILD for H

Taking the moment about A, we get

$$V_B = \frac{x}{L}$$

$$\therefore \qquad V_A = \frac{L-x}{L}$$

When the load is in portion AC taking the moment about hinge C, we get

$$Hh = V_B \frac{L}{2} = \frac{x}{L}\frac{L}{2}$$

$$\text{or} \qquad H = \frac{x}{2h}\text{, linear variation}$$

When $\quad x = 0; \qquad H = 0$

When $\quad x = \frac{L}{2}; \qquad H = \frac{L}{4h}$

When the unit load is in portion CB, considering the left half portion, we get

$$Hh = V_A \frac{L}{2} = \frac{L-x}{L}\frac{L}{2} = \frac{L-x}{2}$$

$$\therefore \qquad H = \frac{L-x}{2h}\text{, linear variation}$$

When $\quad x = \frac{L}{2}, \qquad H = \frac{L}{4h}$

When $\quad x = L \qquad H = 0$

Hence, ILD for H is a triangle with its maximum ordinate equal to $\frac{L}{4h}$ at hinge C as shown in Fig.7.20(b).

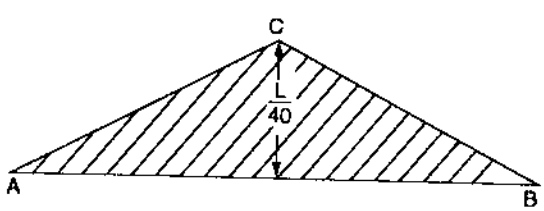


Figure 7.20 (b) *ILD for H*

ILD for moment at D

Bending moment at any given section in the arch

$$= \text{Beam moment} - Hy$$

where, beam moment means, the moment in an equivalent beam. Thus,

$$M_D = \text{Beam moment at D} - Hy_D$$

Hence ILD for M_D in the arch will be drawn as the difference diagram of beam moment diagram and the 'Hy_D' moment diagram. We know that ILD for beam moment at D is a triangle with maximum ordinate $\frac{z(L-z)}{L}$ at D as shown in Fig.7.21(c). Since ILD for H is a triangle with maximum ordinate of $\frac{L}{4h}$ at hinge C, Hy_D diagram is a triangle with $\frac{L}{4h}\,y_D$ as the maximum ordinate at C. This is to be subtracted from the beam moment diagram. Hence, this triangle is drawn on the same side as beam moment diagram (refer. Fig. 7.21(c)) and the difference diagram is marked as the ILD for the bending moment at D.

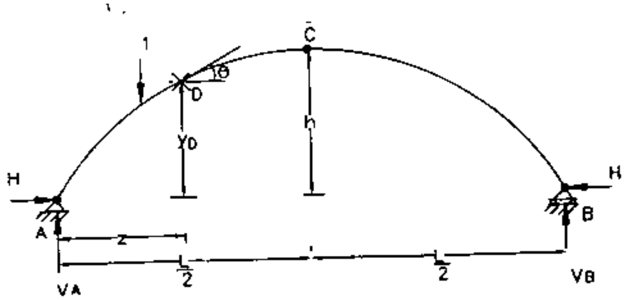


Figure 7.21 (a)

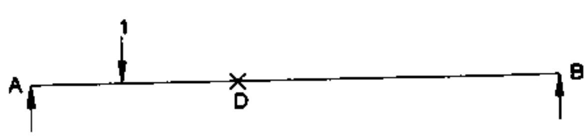


Figure 7.21 (b) Equilateral beam

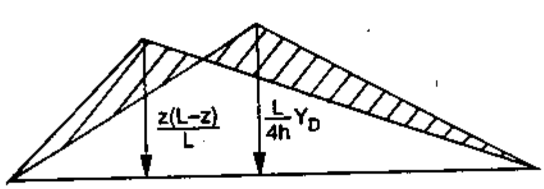


Figure 7.21 (c) ILD for M_D

In the case of a parabolic arch, we know

$$y_D = \frac{4hz(L-z)}{L^2}$$

Hence the maximum ordinate for Hy_D term

$$= \frac{L}{4h} \times \frac{4hz(L-z)}{L^2}$$

$$= \frac{z(L-z)}{L}$$

which is the same as that of the beam moment ordinate at D.

ILD for normal thrust at D (N_D)

As discussed in section 7.3, the normal thrust at section D is given by

$$N = V \sin \theta + H \cos \theta$$

where θ is the slope of the arch with the horizontal (refer.Fig.7.22) and V is the vertical shear. Now ILD for V sin θ and H cos θ will be drawn so as to get a diagram for the normal thrust.

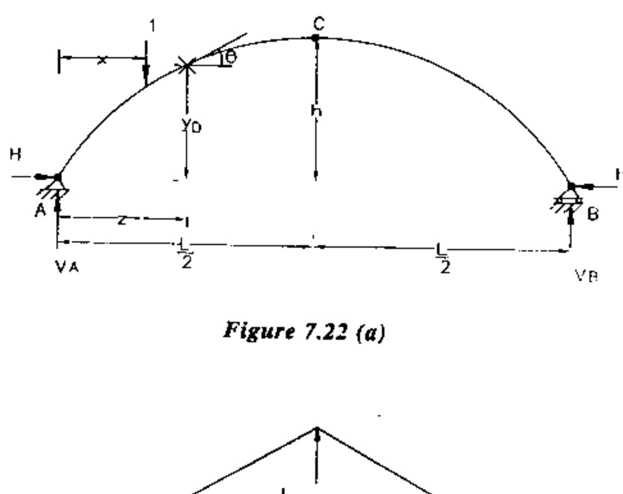


Figure 7.22 (a)

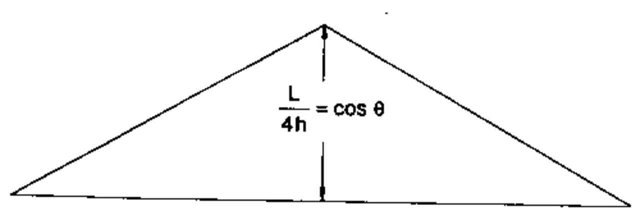


Figure 7.22 (b) ILD for H cosθ

H cos θ diagram is a triangle similar to ILD for the horizontal thrust but multiplied by cos θ as shown in Fig. 7.21(b). Since V is the vertical shear at D, which is obviously the same as that in an equivalent beam, V sin θ diagram is shown in Fig. 7.22(c). Now, to get ILD for N_D, the diagrams are drawn in such a way that

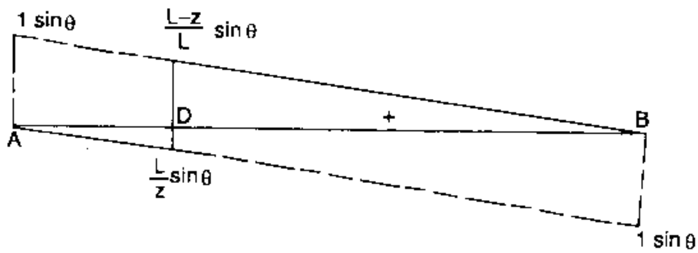


Figure 7.22 (c) *ICD for H sin θ*

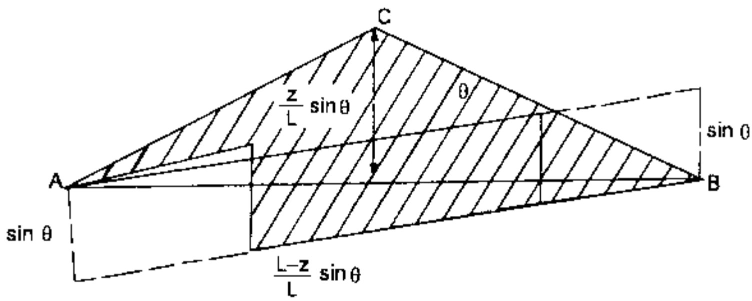


Figure 7.22 (d) *ILD for $N_D = V \sin\theta + H \cos\theta$*

the addition is obtained by drawing ILD for M cos θ on one side and drawing ILD for V sin θ on the opposite side as shown in in Fig. 7.22(d).

ILD for radial shear (Q_D)

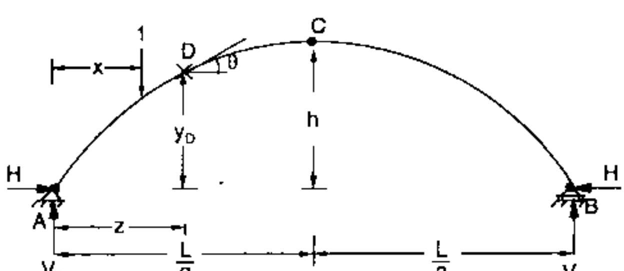


Figure 7.23 (a)

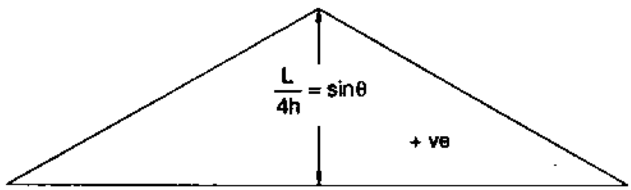


Figure 7.23 (b) H sinθ

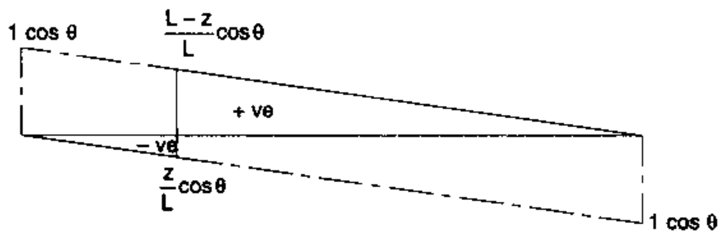


Figure 7.23 (c) V cosθ

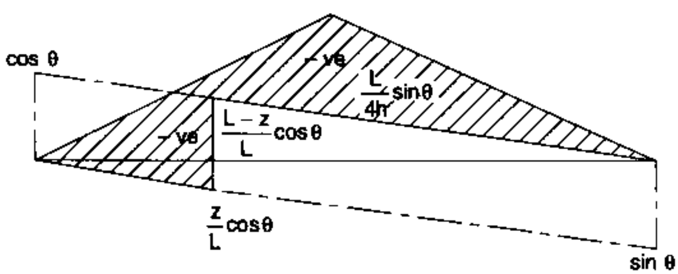


Figure 7.23 (d) ILD for $Q_D = V\cos\theta - H\sin\theta$

As seen in section 7.3,

$$Q_D = V\cos - H\sin\theta$$

First, the V cos θ diagram is drawn. This is similar to the ILD for SF in a beam but multiplied by the constant cos θ (refer.Fig.7.23(c)). Then H sin θ diagram is drawn so that the difference diagram is available. This difference diagram is the ILD for Q_D and is shown hatched in the Fig. 7.23(d).

Note: To get the sum of two diagrams one diagram should be drawn reversed and to get the difference, they should be drawn on the same side.

7.6 MAXIMUM BENDING MOMENT DIAGRAMS

Consider a three hinged symmetric parabolic arch of span L and rise h shown in Fig.7.24(a). Now our interest is to draw the maximum bending moment diagram when the following loads move over the arch.

1. Concentrated load
2. Uniformly distributed load.

7.6.1 When a Concentrated Load Moves

The influence line diagram for the bending moment at section D, which is at a distance z from the left hinge A is shown in Fig. 7. 24(b). Let W be the concentrated load moving from left to right. Now, the moment at D when the concentrated load is at distance x from the left support is equal to w times the ordinate of ILD for M_D. From the Fig.7.24(b), it is obvious that the ordinate is positive maximum at D and is negative maximum at mid-span.

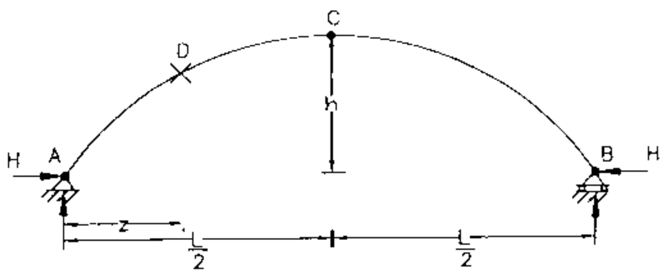


Figure 7.24 (a)

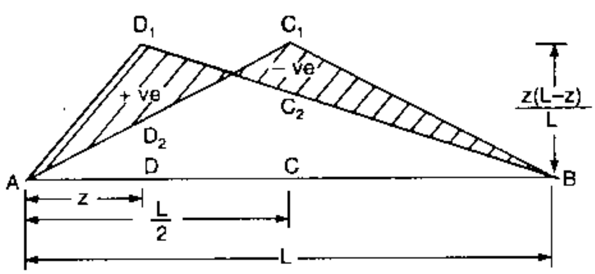


Figure 7.24 (b) ILD for M_D

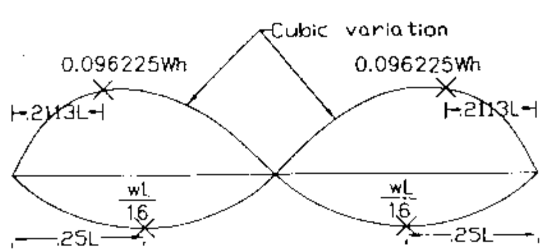


Figure 7.24 (c) Maximum moment diagram for concentrated load

$\therefore$ Max. positive moment at D,

$$M = W \times \text{ordinate of ILD at D}$$

$$= W\left[\frac{z(L-z)}{L} - \frac{z}{L/2}\frac{z(L-z)}{L}\right]$$

$$M = \frac{Wz(L-z)}{L}\left(1-\frac{2z}{L}\right)$$

$$= \frac{Wz\,(L-z)\,(L-2z)}{L^2}, \text{ cubic variation}$$

This equation holds good for the section from $z = 0$ to $\frac{L}{2}$

Now, at $z = 0$, $M = 0$

at $z = \frac{L}{2}$; $M = 0$

For maximum moment,

$$\frac{dM}{dz} = 0 = \frac{W}{L^2}\left((L-z)(L-2z) - z(L-2z) - 2z(L-z)\right)$$

$$= \frac{W}{L^2}\,(L^2 - 3Lz + 2z^2 - Lz + 2z^2 - 2zL + 2z^2)$$

$\therefore$ $$0 = 6z^2 - 6Lz + L^2$$

$\therefore$ $$z = \frac{6L\sqrt{36L^2 - 4 \times 6 \times L^2}}{12}$$

$$= \frac{6L \pm 2L\sqrt{3}}{12} = \frac{L}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$$

$= 0.2113L$, since the above solution holds good only from 0 to $\frac{L}{2}$.

$\therefore$ $$M_{Max} = \frac{W \times 0.2113L}{L^2}(L - 0.2113L)(L - 2 \times 0.2113L)$$

$$= 0.096225\ WL$$

Hence, maximum positive bending moment diagram for the various positions of the moving load is as shown in Fig. 7.24(c) for the left half of the arch. Obviously, a similar diagram exists for the right half of the section as shown in Fig. 7.24(c).

From the ILD for moment at D Fig. 7.24(a) it is clear that maximum negative moment occurs when the load is at mid-span and its value is given by

$$M = W \times \text{ordinate of ILD at C}$$

$$= W\,(\overline{CC_1} - CC_2)$$

$$= W\left[\frac{z(L-z)}{L} - \frac{L/2}{L-z}\frac{z(L-z)}{L}\right]$$

$$= W\left[\frac{z(L-z)}{L} - \frac{z}{2}\right]$$

$$= \frac{W}{2L}\,[2(Lz - z^2) - Lz]$$

$$= \frac{W}{2L}\,(Lz - 2z^2)$$

For M to be maximum

$$\frac{dM}{dz} = 0$$

i.e., $\quad L - 4z = 0$

or $\quad z = \frac{L}{4}$

Hence maximum negative bending moment occurs at quarter point and its value

$$= \frac{W}{2L}\left[L\frac{L}{4} - 2\left(\frac{L}{4}\right)^2\right]$$

$$= \frac{WL}{16}$$

The influence line diagram for maximum negative bending moment is also shown in Fig.7.24(c).

7.6.2 When Uniformly Distributed Load Moves

Let the salient points in the ILD for the moment be named as shown in Fig.7.25(a). In this diagram,

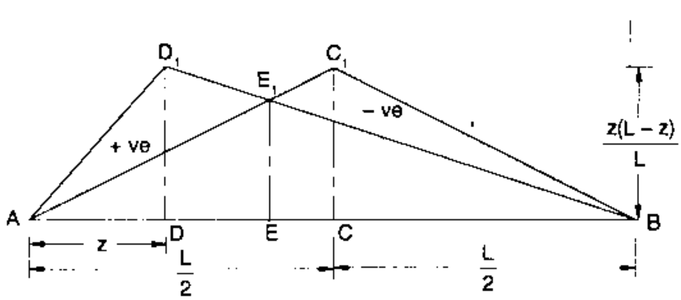


Figure 7.25 (a) ILD for moment at D

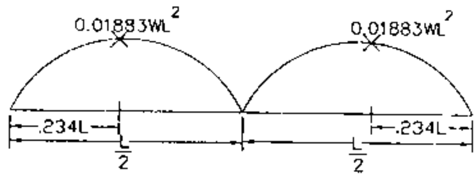


Figure 7.25 (b) Maximum moment diagram

$$DD_1 = CC_1 = \frac{z(L-z)}{L}$$

To locate E, EE_1 is calculated. From the triangles ACC_1 and BDD_1;

$$\frac{AE}{AC} CC_1 = EE_1 = \frac{BE}{BD} DD_1 \tag{ii}$$

$\therefore$ $$\frac{AE}{(L/2)} = \frac{BE}{L-z}, \text{ since } DD_1 = CC_1$$

$$\frac{AE}{BE} = \frac{L}{2(L-z)}$$

$\therefore$ $$\frac{AE}{AE+BE} = \frac{L}{L+2(L-z)}$$

i.e., $$\frac{AE}{L} = \frac{L}{3L-2z}$$

$$AE = \frac{L^2}{3L-2z} \qquad \text{...(iii)}$$

From (ii), $$EE_1 = \frac{CC_1 \times AE}{AC}$$

$$= \frac{z(L-z)}{L} \times \frac{2}{L} \times \frac{L^2}{3L-2z}$$

$$= \frac{2z(L-z)}{3L-2z} \qquad \text{...(iv)}$$

Now, from the influence line diagram, it is clear that when a load occupies the portion AE, the positive moment will be maximum and its value will be

$$= W \times \text{Area of } AE_1 D_1$$

$$= W(\text{Area of } \Delta AD_1B - \text{Area of } \Delta AE_1B)$$

$$= W \frac{1}{2} L (DD_1 - EE_1) \qquad \text{...(v)}$$

$$= \frac{WL}{2}\left[\frac{z(L-z)}{L} - \frac{2z(L-z)}{3L-2z}\right]$$

$$= \frac{WL}{2} z(L-z)\left[\frac{1}{L} - \frac{2}{3L-2z}\right]$$

$$= \frac{WL}{2} z\,(L-z)\ \frac{3L-2z-2L}{L(3L-2z)}$$

$$= \frac{W}{2}\ \frac{z(L-z)(L-2z)}{(3L-2z)}$$

$$= \frac{W}{2}\frac{z(L^2-3Lz+2z^2)}{3L-2z}$$

$$= \frac{W}{2}\frac{(L^2z-3Lz^2+2z^3)}{3L-2z} \qquad \text{...(vi)}$$

For positive M to be maximum,

$$\frac{dM}{dz} = 0$$

$$= \frac{(3L-2z)(L^2-6Lz+6z^2)-(L^2z-3Lz^2+2z^3(-2)}{(3L-2z)^2}$$

$$0 = 3L^3 - 18L^2z + 18Lz^2 - 2L^2z + 12Lz^2 - 12z^3 + 2L^2z - 6Lz^2 + 4z^3$$

$$= -8z^3 + 24Lz^2 - 18L^2z - 3L^3$$

or $$8z^3 - 24Lz^2 + 18L^2z - 3L^3 = 0$$

$z = nL$, we get

$$8n^3 - 24n^2 + 18n - 3 = 0$$

By trial and error method, the equation can be solved which gives

$$n = 0.234$$

$\therefore$ Maximum positive bending moment will be at $z = 0.234L$ and the corresponding moment (from eqn. (vi)) is equal to $0.01883\ WL^2$.

Maximum bending moment diagram can be plotted from eqn. (vi). This is shown in Fig. 7.25(b).

It is obvious that negative maximum moment is same as that of positive moment since $\Delta AC_1B = \Delta AD_1B$ and by subtracting the common area ΔAE_1B and multiplying with the load intensity, we get positive and negative moments. Hence the maximum negative moment diagram is the same as that of maximum positive moment diagram.

Example 7.9 A symmetric three hinged parabolic arch of span 20m and rise 3m is subjected to a 300kN concentrated load moving from left to right. Determine the maximum positive and negative bending moments at 5m from the left support. Determine the absolute maximum bending moment also.

Solution

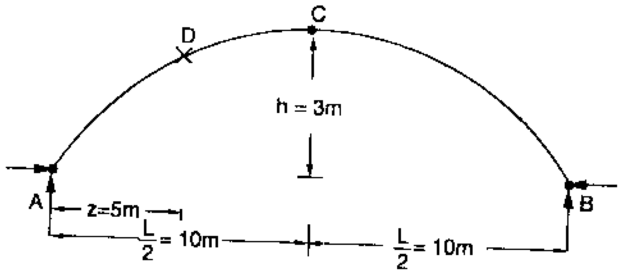


Figure 7.26 (a)

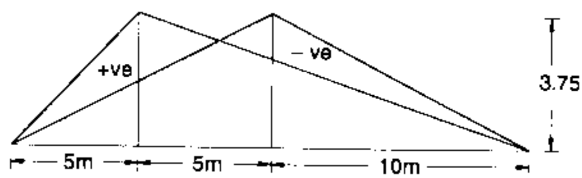


Figure 7.26 (b)

Referring to Fig.7.26,

$$L = 20\text{m}, h = 3\text{m}, z = 5\text{m}$$

ILD ordinates are

$$= \frac{z(L-5z)}{L} = \frac{5(20-5)}{20} = 3.75$$

Maximum bending moment occurs when the load is on the section and its value is given by

$$= \text{Load} \times \text{ILD ordinate}$$

$$= 300\left[3.75 - \frac{5}{10} \times 3.75\right]$$

$$= 562.5\text{kN–m}$$

Maximum negative moment occurs when the load is on the central hinge and its value

$$= W \times \text{ordinate at centre}$$

$$= 300\left[3.75 - \frac{10}{15} \times 3.75\right]$$

$$= 375\text{kN–m}$$

Absolute maximum positive moment occurs at section

$$= 0.2113 \times L$$

$$= 0.2113 \times 20$$

$$= 4.226 \text{ m, from either support}$$

∴ Absolute maximum positive moment = 0.096225 WL

= 0.096225 × 300 × 20

= 577.35kN-m

Absolute maximum negative moment occurs when the load is at the quarter span and its value $= \frac{WL}{16} = \frac{300 \times 20}{16}$

= 375kN-m.

Example 7.10 A three hinged symmetric parabolic arch has a span of 20m and a central rise of 4m. It is loaded with a uniformly distributed load of 30 kN/m for 8m length from the left support. Draw influence line diagram for the bending moment at a section 6m from the left support and then determine the bending moment at the section.

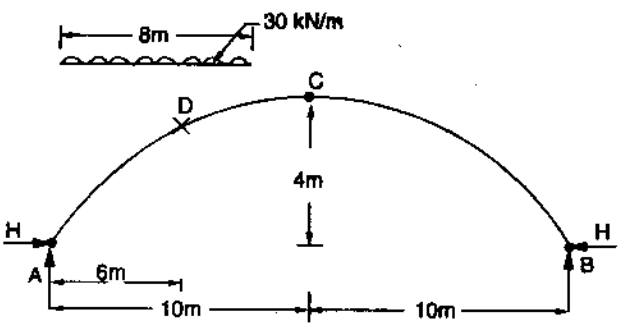


Figure 7.27 (a)

Solution Referring to Fig. 7.27(b).

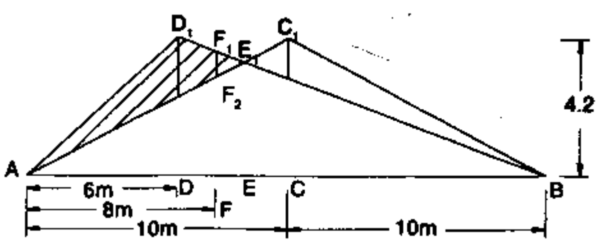


Figure 7.27 (b)

AB = 20; AC = 10m, AD = 6m

Fig. 7.27(b) is the ILD for the moment at D. Since it is parabolic arch

$$DD_1 = CC_1 = \frac{z(L-z)}{L} = \frac{6(20-6)}{20} = 4.2$$

From triangles ACC_1 and BDD_1, EE_1 may be found as

$$\frac{AE}{AC}\, CC_1 = EE_1 = \frac{BE}{BD}\, DD_1$$

$$\frac{AE}{10} \times 4.2 = \frac{BE}{14} \times 4.2$$

$$\frac{AE}{BE} = \frac{10}{14}$$

or $$\frac{AE}{AE+EB} = \frac{10}{10+14}$$

i.e., $$\frac{AE}{20} = \frac{10}{24}$$

$$AE = \frac{10 \times 20}{24} = 8.333$$

From Δ BDD_1,

$$FF_1 = \frac{BF}{BD} \times DD_1 = \frac{20-8}{20-6} \times 4.2 = \frac{12}{14} \times 4.2 = 3.6$$

From Δ ACC_1.

$$F_1F_2 = \frac{AF}{AC} \times 4.2 = \frac{8}{10} \times 4.2 = 3.36$$

$\therefore$ $$FF_2 = FF_1 - FF_2$$
$$= 3.6 - 3.36 = 0.24$$

The bending moment at section D,

$$= AW \times \text{Area of ILD under the load}$$
$$= W \times \text{hatched area (in Fig.7.27(b).}$$
$$= 30\,(\Delta\, AD_1B - \Delta\, AE_1B - \Delta\, F_1E_1F_2)$$

$$= 30\left(\frac{1}{2} \times 20 \times DD_1 - \frac{1}{2} \times 20 \times EE_1 - \frac{1}{2}(AE - AF) \times F_1F_2\right)$$

$$= 30\left(\frac{1}{2} \times 20 \times 4.2 - \frac{1}{2} \times 20 \times 3.6 - \frac{1}{2}(8.333 - 8) \times 0.24\right)$$

$$= 178.8\text{kN-m}$$

Example 7.11 A three hinged symmetric parabolic arch has a span of 30m and a rise of 6m. It is subjected to a moving load longer than the span of intensity 40 kN/m. Determine the maximum moment at a section 10 m from the left support. What is the absolute maximum moment and where will it occur?

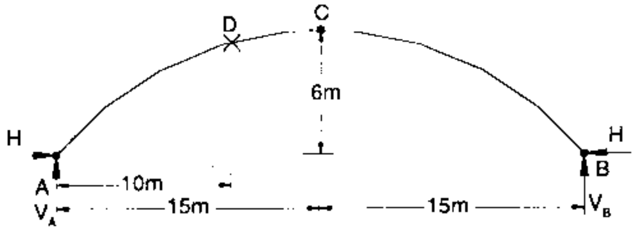


Figure 7.28 (a)

Solution

Referring to Fig. 7.28(b), L = 30 m; h = 6m; AD = 10m

ILD for the moment at D is as shown in Fig. 7.28(b).

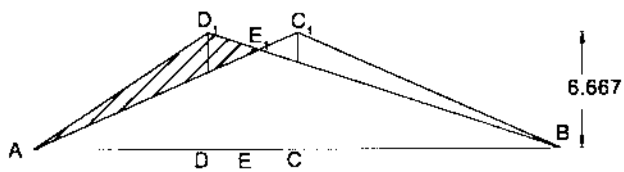


Figure 7.28 (b)

To locate the intersection point E in the ILD for M_D, we find EE_1 from Δ ACC_1 and Δ ADD_1

$$\frac{AE}{AC} \times 6.667 = EE_1 = \frac{BE}{BD} \times 6.667$$

$$\therefore \qquad \frac{AE}{15} = \frac{BE}{20}$$

$$\frac{AE}{BE} = \frac{15}{20} = \frac{3}{4}$$

$$\frac{AE}{AE+BE} = \frac{3}{3+4}$$

$$\frac{AE}{AB} = \frac{3}{7}$$

or $$AE = \frac{3}{7} \times AB = \frac{3}{7} \times 30 = 12.857\text{m}$$

$$\therefore \qquad EE_1 = \frac{AE}{AC} \times 6.667 = \frac{12.857}{15} \times 6.667$$

$$= 5.714$$

At D, the moment is maximum when the load occupies portion AE and is equal to W × area $AD_1 E_1$

$$M_{max\ D} = 40\ (\Delta\ AD_1B - \Delta\ AE_1B)$$

$$= 40\left(\frac{1}{2}\times 30\times DD_1 - \frac{1}{2}\times 30EE_1\right)$$

$$40\times\frac{1}{2}\times 30\ (6.667 - 5.714)$$

$$= 517.80\text{kN-m}$$

Absolute maximum moment occurs at

$$z = 0.234\times L = 0.234\times 30$$

$$= 7.02\text{m from either support}$$

and its value is

$$= 0.01883\times WL^2$$

$$= 0.01883\times 40\times 30^2$$

$$= 677.88\text{ kN-m, from either support}$$

Example 7.12 A three hinged symmetric circular arch has a span of 60m and a rise of 12m. A point load of 80 kN rolls over the arch from left to right. Find the maximum positive and negative bending moment at a section 24m from the left springing.

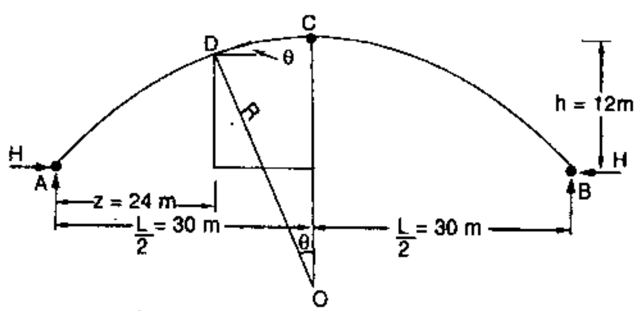


Figure 7.29 (a)

Solution

Referring to Fig.7.29(a)

$$h(2R - h) = \frac{L}{2}\times\frac{L}{2}$$

$$12(2R - 12) = 30\times 30$$

$$R = 43.5\text{m}$$

Now, $\quad R\sin\theta = 30 - 24 = 6$

$$\sin\theta = \frac{6}{R} = \frac{6}{43.5}$$

$\therefore \quad \theta = 7.928°$

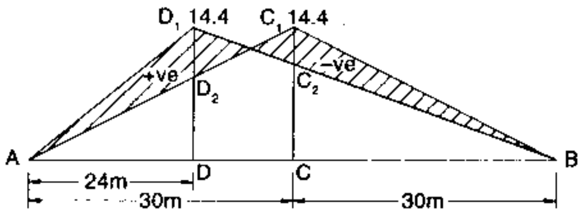


Figure 7.29 (b)

∴ Ordinate at D in the ILD diagram (Fig.7.29(b)),

$$\frac{z(L-z)}{L} = \frac{24(60-24)}{60} = 14.4$$

CC_1 in Fig 7.29 (b),

$$= \frac{L}{4h}\, y_D, \text{ where } y_D = \text{ordinate of D (refer.Fig.7.29(a))}$$

Now,

$$\begin{aligned} Y_D &= R\cos\theta - (R - h) \\ &= 43.5\,[\cos\theta - 1] + 12 \\ &= 11.584 \text{ m} \end{aligned}$$

$$\therefore \quad CC_1 = \frac{L}{4h}\, y_D = \frac{60}{4\times 12} \times 11.584$$

$$= 14.48$$

Maximum positive bending moment occurs when the load is on the section itself and is

$$M_{max} = W \times D_1D_2$$

Now, ordinate $D_1\,D_2$

$$= 14.4 - DD_2$$

$$= 14.4 - \frac{24}{30} \times 14.48$$

$$= 2.816$$

∴ Positive $M_{max} = 80 \times 2.816 = 225.28$ kN-m

Note: In case of circular arch, $DD_1 \neq CC_1$

∴ Maximum negative moment occurs at D when the load is at the crown. Now referring to Fig. 7.29(b).

$$C_1C_2 = 14.48 - \frac{BC}{BD} \times 14.4$$

$$= 14.48 - \frac{30}{36} \times 14.4$$

$$= 2.48$$

∴ Maximum negative bending moment at D

$$= W \times C_1C_2$$

$$= 80 \times 2.48 = 198.4 \text{ kN-m}$$

Example 7.13 A three hinged symmetric parabolic arch has a span of 16m and a rise of 3m. It is subjected to two rolling loads of magnitudes 40kN and 80kN separated by a distance of 2m. The load moves from left to right. Determine the maximum positive moment and negative moment at a section 4m from the left support.

Solution

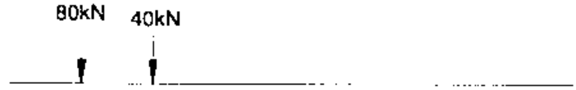


Figure 7.30 (a)

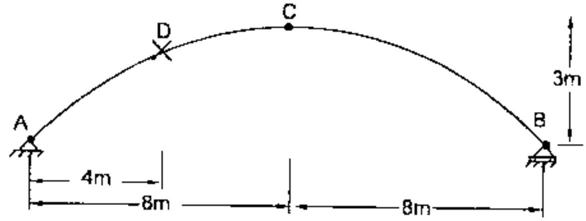


Figure 7.30 (b)

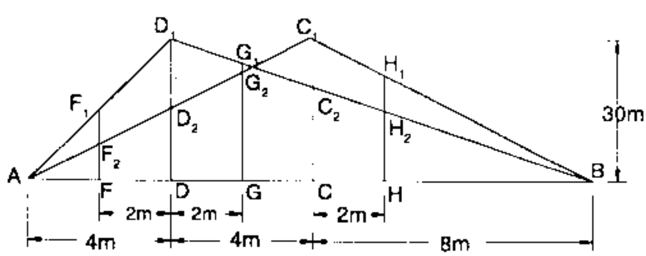


Figure 7.30 (c)

Referring to Fig.7.30

ordinate $$DD_1 = CC_1 = \frac{z(L-z)}{L}$$

$$= \frac{4(16-4)}{16} = 3$$

Positive moment at D is maximum when one of the rolling load is on it. When 40kN load is on D,

$$\begin{aligned} M_D &= 40 \times D_1D_2 + 80 \times F_1F_2 \\ &= 40\,[3 - DD_2] + 80\,[FF_1 - FF_2] \\ &= 40\left(3 - \frac{4}{8}\times 3\right) + 80\left(3\times\frac{2}{4} - \frac{2}{8}\times 3\right) \\ &= 120\text{kN-m} \end{aligned}$$

When 80kN load is on D,

$$M_D = 40\,[G_1G_2] + 80 \times D_1D_2$$
$$= 40\,[GG_1 - GG_2] + 80\,[3 - DD_2]$$
$$= 40\left(\frac{10}{12}\times 3 - \frac{6}{8}\times 3\right) + 80\left(3 - \frac{4}{8}\times 3\right)$$
$$= 130 \text{ kN-m} > 120\text{kN-m}$$

∴ Maximum positive moment is 130kN-m.

From the influence line diagram, it is clear that the maximum negative moment occurs when 80kN load is at crown C.

$$\text{Max. negative moment} = 80\,C_1C_2 + 40\,H_1H_2$$
$$= 80\,[3 - CC_2] + 40\,[HH_1 - HH_2]$$
$$= 80\left(3 - \frac{8}{12}\times 3\right) + 40\left(\frac{6}{8}\times 3 - \frac{6}{12}\times 3\right)$$
$$= 110\text{kN–m}$$

EXERCISES

7.1 A three hinged symmetric circular arch has a span of 36m and a rise of 6m. Determine the bending moment, normal thrust and radial shear at 9m from the left support, if the arch is subjected to a uniformly distributed load of intensity 30kN/m over left half portion and a concentrated load of 60 kN at 27m from the left springing.

Ans : M = 434.329kN-m; N = 474.124kN; Q = –14.369kN

7.2 If the above arch has a parabolic shape, what will be its bending moment, normal thrust and radial shear values.

Ans : M = 540.0 kN-m; N = 474.342 kN, Q = 0

7.3 Draw the bending moment diagram for the arch referred to in problem 7.2.

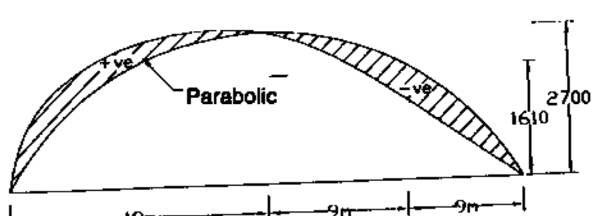


Figure 7.31

Ans : See Fig. 7.31

7.4 A three hinged symmetric parabolic arch of span 40m and rise 5m is subjected to rolling loads. Determine the maximum moment at a section 10m from the left support due to

a. a concentrated load of 40kN rolling from left to right

b. a uniformly distributed load of intensity 20 kN/m longer than the span rolling from left to right.

Ans : (a) + ve M_{max} = 150 kN-m, –ve M_{max} = 100 kN-m,
(b) +ve or – ve M_{max} = 600 kN-m.

7.5 For the problem given in 7.4 determine the absolute maximum bending moment and the section where it occurs.

Ans : (a) 153.96 kN-m at 8.452m from either end
(b) 602.56 kN-m at 9.36m from either end.

CABLES AND SUSPENSION BRIDGES

8

8.1 INTRODUCTION

Cables are used as temporary guys during the erection and as permanent guys for supporting masts and towers. The use of towers to support large tents like that of circus tents is well known to all. Cables are also used in the suspension bridges.

A suspension bridge consists of two cables with the number of suspenders (hangers) which support the roadway. Fig.8.1 shows a typical suspension bridges in which the cable is supported over towers. To reduce the bending moment in

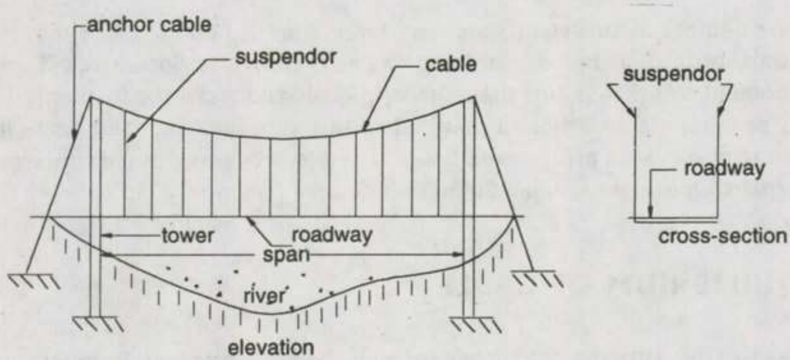


Figure 8.1

the towers anchor cables are provided. The central sag or dip of the cable varies from $\frac{1}{10}$ th to $\frac{1}{15}$ th of span. The cables will be having either guided pulley support or roller support as shown in Fig.8.2. In case of pedestrian suspension bridges, suspenders support the roadway directly. For heavy traffic and large spans stiffening girders are provided to support the roadway (Fig.8.3). Laksman Jhula at Rishikesh and Howra bridge are popular examples of suspension bridges.

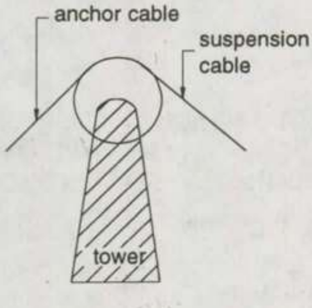


Figure 8.2 (a) Guided pulley support

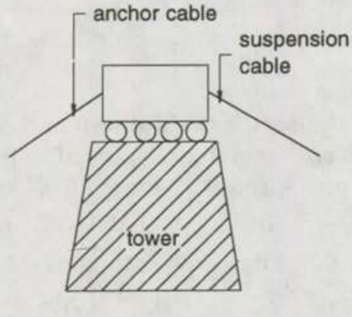


Figure 8.2 (b) Roller pulley support

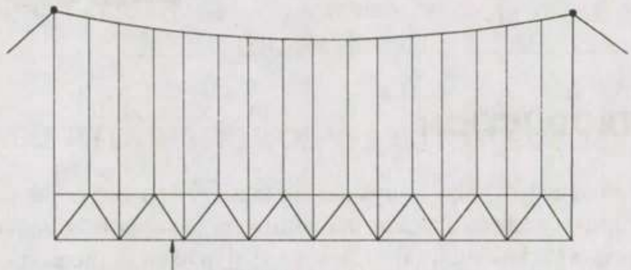


Figure 8.3

Since the number of suspenders are very large, the load on the cable may be taken as uniformly distributed. Cables being very flexible, do not resist any bending moment and they adjust their shape to loads and resist the load only by tension. Since steel is an efficient material in resisting tension, steel finds its application in suspension bridges and hence they are very economical for larger spans. Suspension bridges of span 200m to 300m are commonly built.

8.2 EQUILIBRIUM OF CABLE

A cable is a flexible structure which cannot resist bending moment. It deflects so that the bending moment is zero at any point which is achieved by developing horizontal thrust at the support and thus developing appropriate deflection.

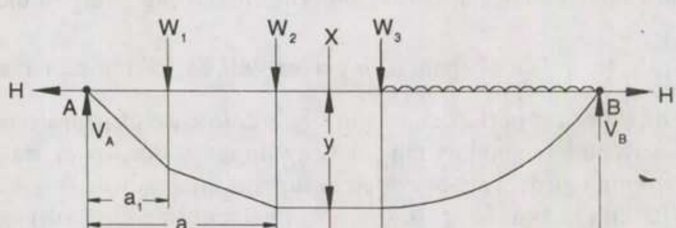


Figure 8.4

Consider the cable shown in Fig.8.4 which is subjected to various loads. Let the horizontal force developed be H and let V_A and V_B be the vertical reactions at supports A and B. At section X–X, let the deflection be y. Then

$$M_x = V_A x - W_1 (x - a_1) - W_2 (x - a_2) - Hy$$

Since the cable is flexible, $M_x = 0$

$$\therefore \quad Hy = V_A x - W_1 (x - a_1) - W_2 (x - a_2)$$

$$= \text{Beam moment}$$

Considering any segment of cable and using the above equation along with usual equations, a loaded cable can be analysed.

8.3 CABLE SUBJECTED TO CONCENTRATED LOADS

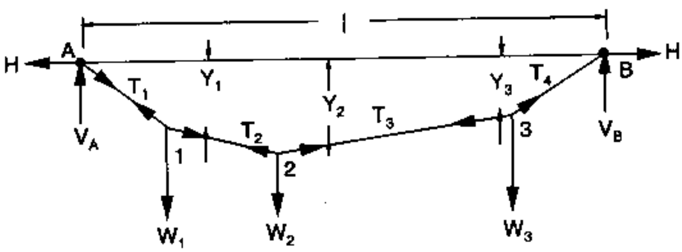


Figure 8.5

Consider the cable of length L spanning over a horizontal gap ℓ subjected to the concentrated loads as shown in Fig.8.5. Let V_A and V_B be the vertical reactions at supports and let H be the horizontal reactions at supports. The equilibrium condition is

$$Hy = M_{beam} \text{ or } y = \frac{M_{beam}}{H} \qquad \text{...8.1}$$

Hence the deflected shape is similar to the beam moment diagram. If M_1, M_2 and M_3 are the beam moments at load points 1, 2, and 3, the deflections y_1, y_2 and y_3 are given by

$$y_1 = \frac{M_1}{H} \; ; y_2 = \frac{M_2}{H} \; ; y_3 = \frac{M_3}{H}$$

Hence if the horizontal thrust is known or position of cable at any one point is known, the deflections at all points can be calculated and the shape is found. The actual length of the cable is the sum of lengths of each segments.

After finding deflections, slope of the various segments can be found and using the equations of equilibrium of load points 1, 2 and 3, forces in the various segments of cable can be found.

Example 8.1 A light cable is supported at two points 20m apart which are at the same level. The cable supports three concentrated loads as shown in Fig.8.6. The deflection at first point is found to be 0.8m. Determine the tension in the different segments and total length of the cable.

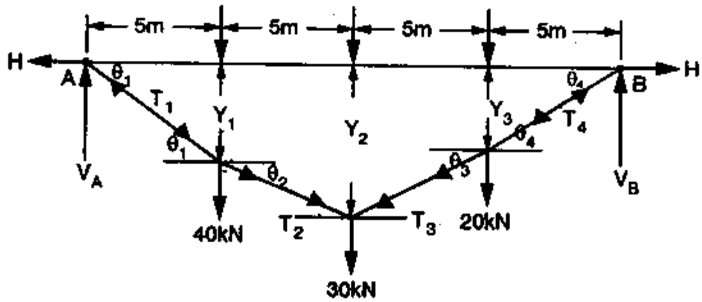


Figure 8.6

Solution

Taking moment about A, we get

$$M_A = 0 = 40 \times 5 + 30 \times 10 + 20 \times 15 - V_B \times 20$$

or $$V_B = 40 \text{ kN}$$

$\therefore$ $$V_A = (40 + 30 + 20) - 40$$
$$= 50 \text{ kN}$$

The beam moments at point 1, 2 and 3 are

$$M_{1\,beam} = 50 \times 5 = 250 \text{ kN-m}$$
$$M_{2\,beam} = 50 \times 10 - 40 \times 5 = 300 \text{ kN-m}$$
$$M_{3\,beam} = V_B \times 5 = 40 \times 5 = 200 \text{ kN-m}$$

Let H be the horizontal reaction. Since the cable is flexible, bending moment at all points is zero. Let y_1, y_2 and y_3 be the deflections of points 1, 2 and 3 respectively. Equating bending moments at point 1, 2 and 3 to zero we get,

$$Hy_1 = M_{1\,beam} = 250 \text{ kN-m}$$
$$Hy_2 = M_{2\,beam} = 300 \text{ kN-m}$$

and $$Hy_3 = M_{3\,beam} = 200 \text{ kN-m}$$

Since $y_1 = 0.8$m, from (i) we get

$$H = \frac{250}{0.8} = 312.5 \text{ kN}$$

$\therefore$ $$y_2 = \frac{300}{H} = 0.96 \text{ m}$$

and $$y_3 = \frac{200}{H} = 0.64 \text{ m}$$

Hence deflected shape is as shown in Fig.8.6.

Let the inclination of segments A–1, 1–2, 2–3 and 3–B be θ_1, θ_2, θ_3 and θ_4 respectively.

Then $$\theta_1 = \tan^{-1}\frac{y_1}{5} = \tan^{-1}\frac{0.8}{5} = 9.09^\circ$$

$$\theta_2 = \tan^{-1}\frac{y_1 - y_2}{5} = \tan^{-1}\frac{0.16}{5} = 1.833^\circ$$

$$\theta_3 = \tan^{-1}\frac{y_2 - y_3}{5} = \tan^{-1}\frac{0.32}{5} = 3.662°$$

$$\theta_4 = \tan^{-1}\frac{y_3}{5} = 7.294°$$

Let the tension in the segment A–1, 1–2, 2–3 and 3–4 be T_1, T_2, T_3 and T_4 respectively. Applying Lami's theorem at points 1 and 3, we get the values of T_1, T_2, T_3 and T_4

$$\frac{T_1}{\sin(90-\theta_2)} = \frac{T_2}{\sin(90+\theta_1)} = \frac{40}{\sin(180-\theta_1+\theta_2)}$$

$$T_1 = 40 \times \frac{\sin(90-1.833)}{\sin(180-9.09+1.833)} = 316.49\text{kN}$$

$$T_2 = 40 \times \frac{\sin(90+9.09)}{\sin(180-9.09+1.833)} = 312.679 \text{ kN}$$

From the consideration of equilibrium at point 3,

$$\frac{T_3}{\sin(90+\theta_4)} = \frac{T_4}{\sin(90-\theta_3)} = \frac{20}{\sin(180+\theta_3-\theta_4)}$$

$$T_3 = 313.162\text{kN}$$
$$T_4 = 315.072\text{kN}$$

Length of cable = Sum of length at each segment

$= 5 \sec\theta_1 + 5 \sec\theta_2 + 5 \sec\theta_3 + 5 \sec\theta_4$

$= 20.117$ m

Example 8.2 **A light cable 18 m long, is supported at two ends at the same level. The supports are 16 m apart. The cable supports 120 N load dividing the distance into two equal parts. Find the shape of the cable and tension in the cable.**

Solution

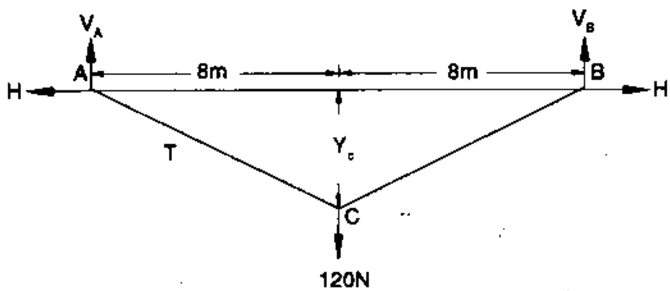


Figure 8.7

Referring to Fig.8.7, let y_c be the deflection at the centre of the cable. Due to symmetry

$$V_A = V_B = \frac{120}{2} = 60 \text{ N}$$

Now length of the cable

$$L = \left[\left(\frac{\ell}{2}\right)^2 + {y_c}^2\right]^{1/2} \times 2$$

$$18 = 2\left(8^2 + y_c^2\right)^{1/2}$$

$$81 = 8^2 + y_c^2$$

$$y_c = \sqrt{81-64} = 4.123 \text{ m}$$

Taking moment about C, we get

$$0 = Hy_c - V_A\frac{\ell}{2}$$

$$= H \times 4.123 - 60 \times 8$$

$$H = 116.42 \text{ N}$$

Tension in the cable

$$= \sqrt{V_A^2 + H^2}$$

$$= \sqrt{60^2 + 116.42^2}$$

$$= 130.97 \text{ N}$$

8.4 CABLE SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

Let a cable of length L be supported at points A and B which are at a horizontal distance l and are at the same level. (Fig.8.8(a)). The cable is subjected to a uniformly distributed load w/unit horizontal length.

Let h be the central deflection,

$$V_A = V_B = \frac{w\ell}{2}$$

Taking moment about central point and noting bending moment is zero at all points in the cable, we get,

$$Hh - \frac{w\ell}{2}\frac{\ell}{2} - \frac{w\ell}{2} \times \frac{\ell}{4} = 0$$

or

$$H = \frac{w\ell^2}{8h} \qquad \text{...8.2}$$

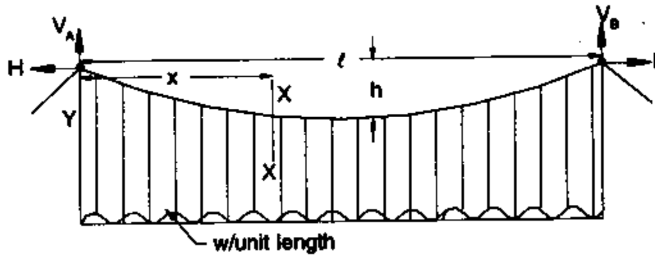


Figure 8.8 (a)

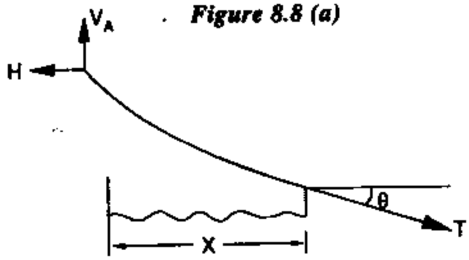


Figure 8.8 (b)

If 'V' is the shear at any section X–X distance x from A, then [refer. Fig. 8.8(b)].

$$T = \sqrt{V^2 + H^2}$$

$$V_{max} = \frac{w\ell}{2} \text{ at support}$$

$$\therefore \qquad T_{max} = \sqrt{\left(\frac{w\ell}{2}\right)^2 + \left(\frac{w\ell^2}{8h}\right)^2} \qquad ...8.3$$

$$V_{max} = \frac{w\ell}{2}\sqrt{1 + \frac{\ell^2}{16h^2}} \qquad ...8.4$$

$$V_{min} = 0, \text{ at centre} \qquad ...8.5$$

Hence $$T_{min} = \sqrt{0 + H^2} = H \qquad ...8.6$$

At any point, since cable cannot resist shear,

$$V = T \sin\theta$$

Now to find the shape of the cable, consider the portion on left side of section X–X. Let θ be the slope. Then

$$\Sigma H = 0 \rightarrow T\cos\theta = H$$

$$\Sigma V = 0 \rightarrow T\sin\theta = V_A - wx$$

$$= \frac{w\ell}{2} - wx$$

$\therefore$
$$\tan\theta = \left[\frac{w\ell}{2} - wx\right]\frac{1}{H}$$

i.e.,
$$\frac{dy}{dx} = \left[\frac{w\ell}{2} - wx\right]\frac{1}{H}$$

$\therefore$
$$y = \left[\frac{w\ell}{2}x - \frac{wx^2}{2}\right]\frac{1}{H}$$

$\therefore$
$$= \frac{wx(\ell - x)}{2H}$$

Substituting the value of $H = \frac{w\ell^2}{8h}$, we get

$$y = \frac{wx(\ell - x)}{2} \times \frac{8h}{w\ell^2}$$

$$y = \frac{4hx(\ell - x)}{\ell^2} \qquad \text{...8.7}$$

which is a parabola. Thus the shape of the cable is a parabola.
To find the length of the cable in any curve

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\cong 1 + \frac{1}{2}\left[\frac{4h\ (\ell - 2x)}{\ell^2}\right]^2$$

$\therefore$ Length of the cable is given by

$$L = \int_0^\ell ds = \int_0^\ell \left[1 + \frac{1}{2}\frac{16h^2}{\ell^4}(\ell^2 - 4\ell x + 4x^2)\right]dx$$

$$= \left[\ x + \frac{8h^2}{\ell^4}(\ell^2 x - \frac{4\ell x^2}{2} + \frac{4x^3}{3})\right]_0^\ell$$

$$= \ell + \frac{8h^2}{\ell^4}\ \ell^3\left[1 - 2 + \frac{4}{3}\right]$$

i. e.,
$$L = \ell + \frac{8h^2}{3\ell}$$

8.5 CABLE WITH ENDS AT DIFFERENT LEVELS

Consider the cable shown in Fig.8.9 which is supported at A and B. A is h_1 metres above the lowest point C and B is h_2 metres above C. This is subjected to a uniformly distributed load w per unit horizontal length over entire span ℓ. Let the horizontal distance between AC be ℓ_1, and that between CB be ℓ_2. Let the horizontal reactions at supports be H.

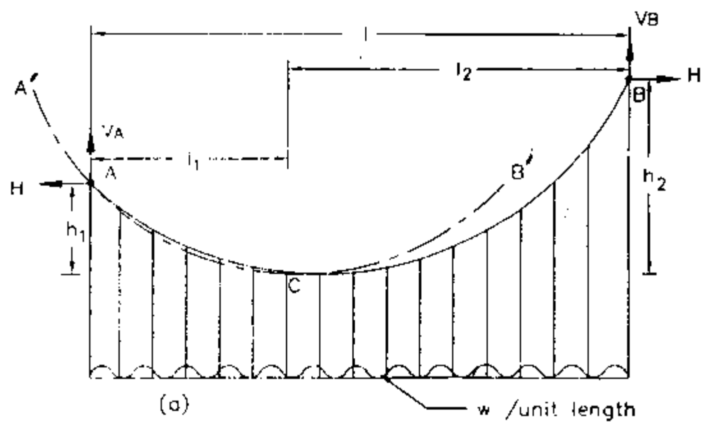


Figure 8.9 (a)

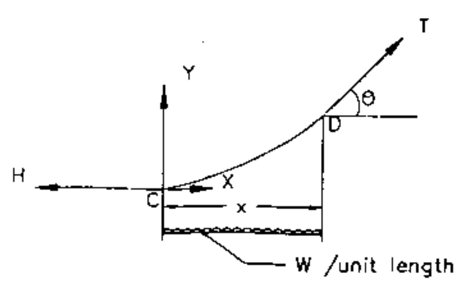


Figure 8.9 (b)

A cable can take only axial force i.e. it cannot take bending moment and shear force. Hence, at lowest point C, the axial force = H and there is no force in vertical direction.

Let D be a point where slope is θ. Taking the lowest point C as origin and considering the equilibrium of portion CD, we get

$$T \cos \theta = H$$

and $$T \sin \theta = wx$$

$$\therefore \qquad \tan\theta = \frac{wx}{H}$$

i e.,
$$\frac{dy}{dx} = \frac{wx}{H}$$

$\therefore$ $y = w\dfrac{x^2}{2H} + C_1$, where C_1 is a constant of integration

At C,
$$x = y = 0$$

$$\therefore \qquad 0 = 0 + C_1 \quad \text{or} \quad C_1 = 0$$

$$\therefore \qquad y = w\frac{x^2}{2H} \qquad \text{...8.9}$$

This is the equation of a parabola. Hence the cable is having a parabolic shape. Applying eqn. 8.9 to points A and B, we get

$$h_1 = \frac{w\ell_1^2}{2H} \quad \text{and} \quad h_2 = \frac{w\ell_2^2}{2H}$$

$$\therefore \qquad \frac{h_1}{h_2} = \frac{\ell_1^2}{\ell_2^2}$$

or
$$\frac{\sqrt{h_1}}{\sqrt{h_2}} = \frac{\ell_1}{\ell_2}$$

$$\therefore \qquad \frac{\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} = \frac{\ell_1}{\ell_1+\ell_2} = \frac{\ell_1}{\ell}$$

$$\left.\begin{aligned} \therefore \qquad \ell_1 &= \ell\,\frac{\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \\ \text{and} \qquad \ell_2 &= \ell\,\frac{\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}} \end{aligned}\right\} 8.10$$

To find H, calculate moment at C. If it is calculated from left-hand side

$$V_A\ell_1 - Hh_1 - \frac{w\ell_1^2}{2} = 0$$

or
$$V_A = \frac{w\ell_1}{2} + \frac{h_1}{\ell_1}H$$

If calculated from right-hand side ...(i)

$$V_B\ell_2 - Hh_2 - \frac{w\ell_2^2}{2} = 0$$

or $$V_B = \frac{w\ell_2}{2} + \frac{h_2}{\ell_2}H$$

Adding eqn. (i) and (ii) we get ...(ii)

$$V_A + V_B = \frac{w}{2}(\ell_1 + \ell_2) + \left[\frac{h_1}{\ell_1} + \frac{h_2}{\ell_2}\right]H$$

$$= \frac{w}{2}\ell + \left[\frac{h_1}{\ell_1} + \frac{h_2}{\ell_2}\right]H$$

But $V_A + V_B = w\ell$, the total downward load.

$\therefore$ $$w\ell = \frac{w}{2}\ell + \left[\frac{h_1}{\ell_1} + \frac{h_2}{\ell_2}\right]H$$

or $$\frac{w\ell}{2} = \left[\frac{h_1}{\ell_1} + \frac{h_2}{\ell_2}\right]H$$

Substituting the values of ℓ_1 and ℓ_2 from equation 8.10 we get

$$H = \frac{w\ell}{2\left[\frac{h_1\left(\sqrt{h_1}+\sqrt{h_2}\right)}{\ell\sqrt{h_1}} + \frac{h_2\left(\sqrt{h_1}+\sqrt{h_2}\right)}{\ell\sqrt{h_2}}\right]}$$

$$H = \frac{w\ell^2}{2\left[\sqrt{h_1}\left(\sqrt{h_1}+\sqrt{h_2}\right)+\sqrt{h_2}\left(\sqrt{h_1}+\sqrt{h_2}\right)\right]}$$

$$H = \frac{w\ell^2}{2\left(\sqrt{h_1}+\sqrt{h_2}\right)^2} \qquad \ldots 8.11$$

From equations (i) and (ii), V_A and V_B can be found
To find the length of the cable (refer Fig. 8.9(a)),

Length of cable ACB $= \frac{1}{2}$ × sum of length of ACB′ and A′CB

$$= \frac{1}{2} \times \left[2\ell_1 + \frac{8}{3}\frac{h_1^2}{2\ell_1} + 2\ell_2 + \frac{8}{3}\frac{h_2^2}{2\ell_2}\right]$$

$$= \ell_1 + \ell_2 + \frac{2}{3}\frac{h_1^2}{\ell_1} + \frac{2}{3}\frac{h_2^2}{\ell_2}$$

i.e.,
$$L = \ell + \frac{2}{3}\frac{h_1^2}{\ell_1} + \frac{2}{3}\frac{h_2^2}{\ell_2} \qquad ...8.12$$

8.6 FORCES ON ANCHOR CABLES AND TOWERS

The forces on anchor cables and towers depend upon the type of support given to cables. As mentioned in Art 8.1 there are two types of supports to cable-guided pulley and roller support.

8.6.1 Guided Pulley Support

Let the inclination of main cable to horizontal be θ and that of anchor cable be α (Fig.8.10). Assuming pulley as frictionless, tension in anchor cable is taken as same as tension in the main cable. Let this tension be T.

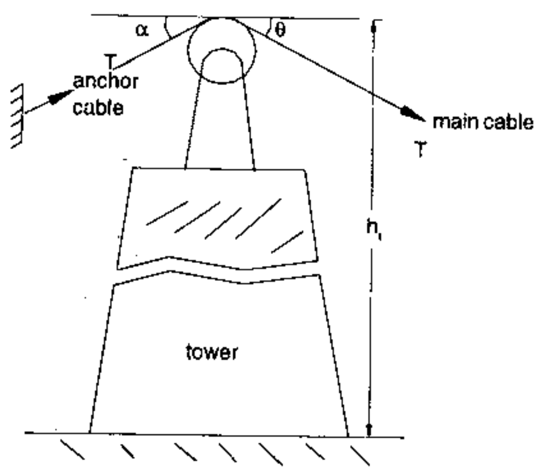


Figure 8.10

$\therefore$ Vertical load transmitted to tower

$$= T\sin\theta + T\sin\alpha$$
$$= T(\sin\theta + \sin\alpha) \qquad ...8.13$$

Horizontal load transmitted to tower

$$= T\cos\theta - T\cos\alpha$$
$$= T(\cos\theta - \cos\alpha) \qquad ...8.14$$

Bending moment on the tower

$$= \text{Horizontal force on tower} \times \text{Height of tower}$$
$$= T(\cos\theta - \cos\alpha)\, h_1 \qquad ...8.15$$

8.6.2 Roller Support

In this case, the suspension cable and the anchor cables are connected to a saddle resting on a tower (Fig.8.11). In this arrangement, the two cables need not have the same tension. Let T be the tension in main cable and T_1 in the anchor cable.

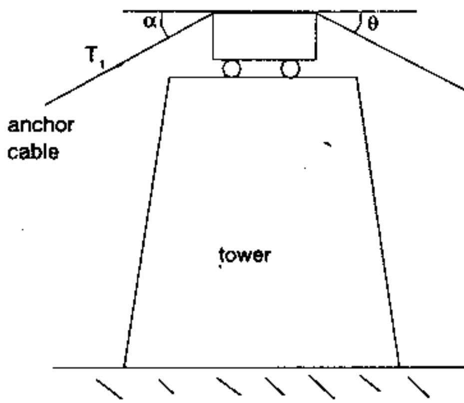


Figure 8.11

Assuming saddle to have frictionless rollers,

$$T_1 \cos\alpha = T\cos\theta$$

$$\therefore \qquad T_1 = T\,\frac{\cos\theta}{\cos\alpha} \qquad \text{...8.16}$$

Since saddle is having frictionless rollers, there is no horizontal force and hence no bending moment on tower.

$$\therefore \qquad \text{Vertical force on the tower} = T_1 \sin\alpha + T\sin\theta \qquad \text{...8.17}$$

Example 8.3 A bridge cable is suspended from towers 80m apart and carries a load of 30 kN/m on the entire span. If the maximum sag is 8m, calculate the maximum tension in the cable. If the cable is supported by saddles which are stayed by wires inclined at 30° to the horizontal, determine the forces acting on the towers. If the same inclination of back stay passes over pulley, determine the forces on the towers.

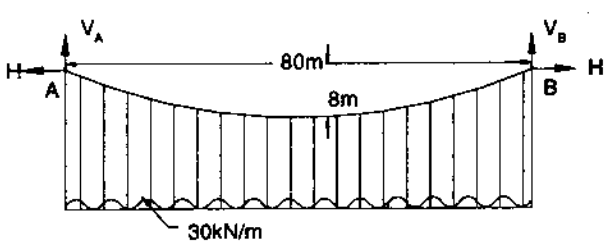


Figure 8.12 (a)

Solution

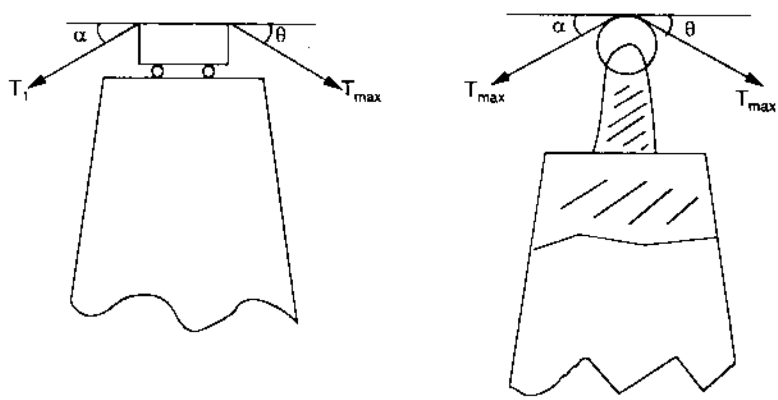


Figure 8.12 (b)

Referring to Fig. 8.12(a),

$$V_A = \frac{w\ell}{2} = \frac{30 \times 80}{2} = 1200 \text{ kN}$$

Taking moment about central point C

$$H \times 8 - \frac{w\ell}{2} \times \frac{\ell}{2} + \frac{w\ell}{2} \times \frac{\ell}{4} = 0$$

or $$H = \frac{w\ell^2}{64} = \frac{30 \times 80^2}{64} = 3000 \text{ kN}$$

Maximum tension occurs at support

$$\therefore \quad T_{max} = \sqrt{V^2 + H^2} = \sqrt{1200^2 + 3000^2} \text{ kN}$$

$$= 3231.1 \text{ kN}$$

$$H = T_{max} \cos\theta$$

$$\theta = \cos^{-1}\frac{H}{T} = \cos^{-1}\frac{3000}{3231.1}$$

$$= 21.80°$$

(i) If the cable is supported by saddle (referring to Fig.8.12(b)), the anchor cable tension T_1 is given by

$$T_1 \cos\alpha = T_{max} \cos\theta$$

$$T_1 \times \cos 30° = 3231.80 \times \cos 21.80°$$

$$T_1 = 3464.1 \text{ kN}$$

There is no horizontal force on the tower. The vertical force on the tower is given by,

$$= T_1 \sin\alpha + T_{max} \sin\theta$$

$$= 3464.1 \sin 30° + 3231.1 \sin 21.80°$$

$$= 2931.98 \text{ kN}$$

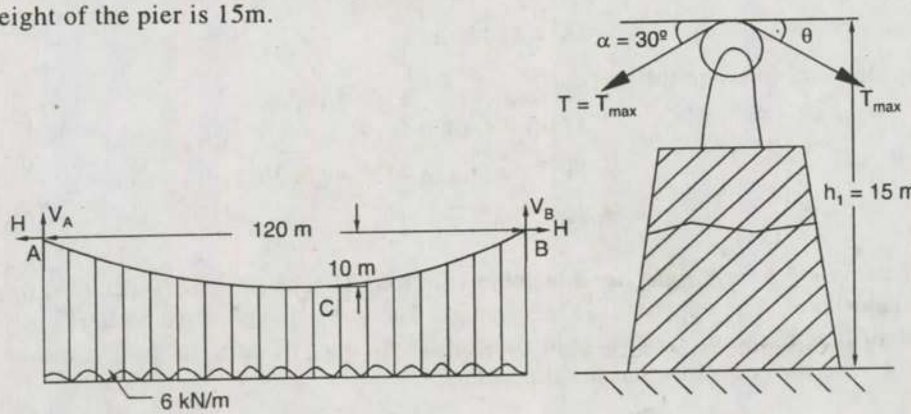
(ii) If the cable is on pulley, referring to Fig.8.12(c), the vertical force on tower is given by

$$T_{max}\,(\sin\alpha + \sin\theta)$$
$$= 3231.1\,(\sin 30^\circ + \sin 21.80^\circ)$$
$$= 2815.48 \text{ kN}$$

Horizontal force on the tower

$$= T_{max}\,(\cos\theta - \cos\alpha)$$
$$= 3231.1\,(\cos 21.8^\circ - \cos 30^\circ)$$
$$= 201.82 \text{ kN}$$

Example 8.4 A cable of span 120m and dip 10m carries a load of 6 kN/m of horizontal span. Find the maximum tension in the cable and the inclination of the cable at the support. Find the forces transmitted to the supporting pier if the cable passes over smooth pulleys on top of the pier. The anchor cable is at 30° to the horizontal. Determine the maximum bending moment for the pier if the height of the pier is 15m.



(a) *(b)*

Figure 8.13

Solution

Referring to Fig. 8.13 (a),
due to symmetry,

$$V_A + V_B = \frac{w\ell}{2} = \frac{6\times120}{2} = 360\text{kN}$$

Taking moment about central point C,

$$H\times h - \frac{w\ell^2}{2}\times\frac{\ell}{2} - \frac{w\ell}{2}\frac{\ell}{4}$$

$$\therefore \quad H = \frac{w\ell^2}{8h} = \frac{6\times120\times120}{8\times10} = 1080\text{kN}$$

$$T_{max} = \sqrt{V^2 + H^2} = \sqrt{360^2 + 1080^2}$$

$$= 1138.42 \text{ kN}$$

$$\cos\theta = \frac{H}{T_{max}} = \frac{1080}{1138.42}$$

$$\theta = 18.435°$$

Refering to Fig. 8. 12(b), horizontal force transferred to pier

$$= T_{max} (\cos 18.435° - \cos 30°)$$

$$= 1138.42 (\cos 18.435° - \cos 30°)$$

$$= 94.099 \text{ kN}$$

Max. bending moment in the pier

$$= H h_1$$

$$= 94.099 \times 15$$

$$= 1411.49 \text{ kN-m}$$

Vertical force on the pier

$$= T (\sin\theta + \sin\alpha)$$

$$= 1138.42 (\sin 18.435° + \sin 30°)$$

$$= 929.21 \text{ kN}$$

Example 8.5 A light flexible cable 18m long is supported at two ends at the same level. The supports are 16m apart. The cable is subjected to uniformly distributed load of 1 kN/m of horizontal length over its entire span. Determine the reactions developed at the support.

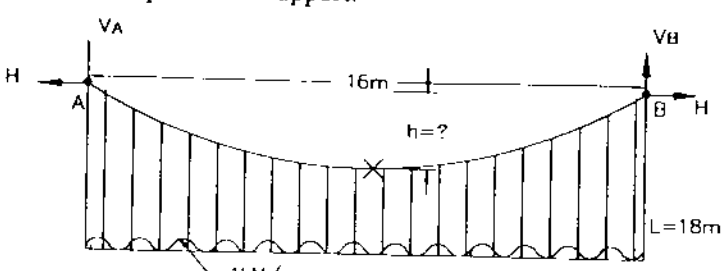


Figure 8.14

Solution

The cable is shown in Fig.8.14. The length of the cable L is given by

$$L = \ell + \frac{8}{3}\frac{h^2}{\ell}$$

Where ℓ is span and h is central dip.

Applying this, we get

$$18 = 16 + \frac{8}{3}\frac{h^2}{16}$$

or $$h = 3.464 \text{ m}$$

Let H be the horizontal force and V_A be the vertical reaction at A. Then

$$V_A = \frac{w\ell}{2} = \frac{1 \times 16}{2} = 8 \text{ kN}$$

$$H \times 3.464 = \frac{w\ell^2}{8} = \frac{1 \times 16^2}{8}$$

$$H = 9.237 \text{ kN}$$

$$T_{max} = \sqrt{V^2 + H^2}$$

$$= \sqrt{8^2 + (9.237)^2}$$

$$= 12.220 \text{ kN}$$

Its inclination to horizontal is given by

$$T_{max} \cos\theta = H$$

$$\theta = \cos^{-1}\frac{H}{T_{max}} = \cos^{-1}\frac{9.237}{12.220} = 40.898°$$

Example 8.6 A cable is suspended from the points A and B which are 80m apart horizontally and are at different levels, the point A being 5m vertically higher than the point B and the lowest point in the cable is 10m below A. The cable is subjected to a uniformly distributed load of 30kN/m over the horizontal span. Determine the horizontal and vertical reactions at each end and also the maximum tension in the cable.

Solution

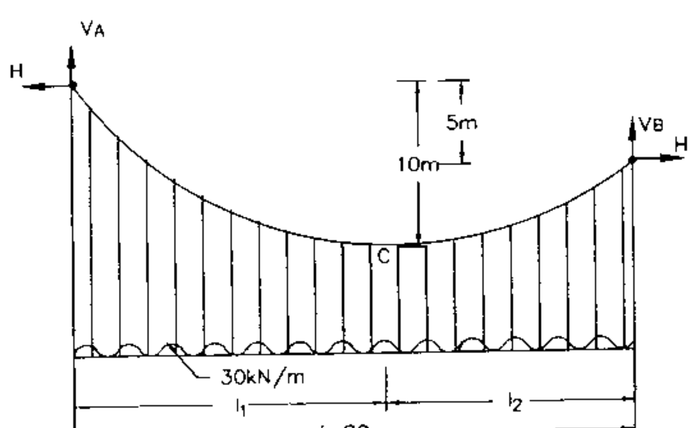


Figure 8.15 (a)

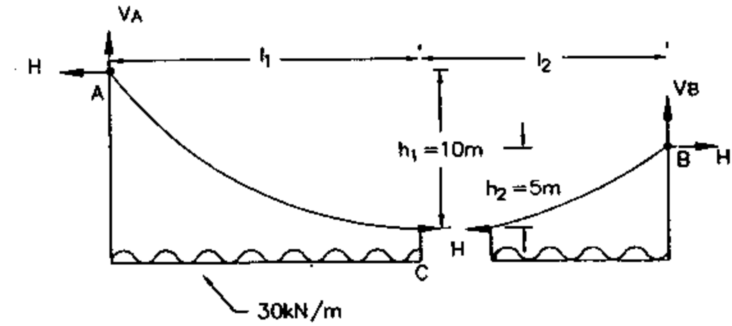


Figure 8.15 (b)

The cable is shown in the Fig.8.15(a). Let V_A and V_B be the vertical reactions at supports A and B respectively and H be the horizontal reaction. Let C be the lowest point.

At the lowest point, tensile force is horizontal and is obviously equal to H. Let horizontal distance between A and C be ℓ_1, and that between C and B be ℓ_2.

Referring to Fig.8.15(b), the moment equilibrium condition about A and B is given by

$$H \times 10 - \frac{w\ell_1^2}{2} = 0; H \times 5 - \frac{w\ell_2^2}{2} = 0$$

Hence, $$\frac{w\ell_1^2}{2} = \frac{w\ell_2^2}{2} 2 = 10H$$

$\therefore$ $$\ell_1^2 = 2\ell_2^2$$

or $$\ell_1 = \sqrt{2}\ell_2$$

But $\ell_1 + \ell_2 = 80$ m (given)

$\therefore$ $$\ell_2\left(\sqrt{2} + 1\right) = 80$$

$$\ell_2 = 33.137 \text{ m}$$

Hence $$\ell_1 = 80 - 33.137 = 46.863 \text{ m}$$

$\therefore$ $$H = \frac{w\ell_1^2}{20} = \frac{30 \times 46.863^2}{20}$$

$$= 3294.2 \text{ kN}$$

Considering the equilibrium of vertical forces on AC, we get

$$V_A = w\ell_1 = 30 \times 46.863 = 1405.89 \text{ kN}$$

Considering the equilibrium of CB, we get

$$V_B = w\ell_2 = 30 \times 33.137 = 994.11 \text{ kN}$$

Since $V_A > V_B$, maximum tension ocurs at support A,

$$\therefore \quad T_{max} = \sqrt{V_A^2 + H^2} = \sqrt{1405.89^2 + 3294.2^2}$$

$$= 3581.66 \text{ kN}$$

Example 8.7 A cable of span 100 m has its ends at heights 8m and 15m above the lowest point of the cable. It carries a uniformly distributed load of 10 kN/m per unit horizontal run of the span. Determine the horizontal and vertical reactions at the support. What is the length of the cable ?

Solution

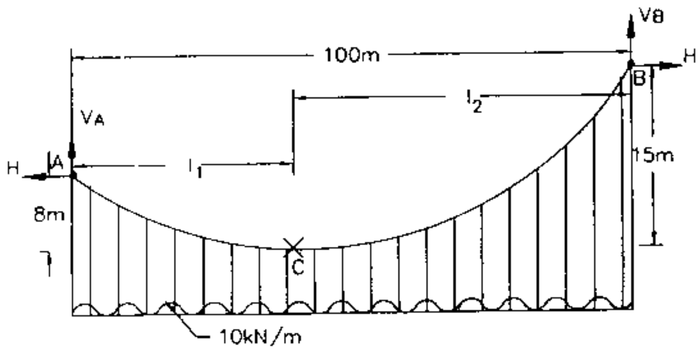


Figure 8.16 (a)

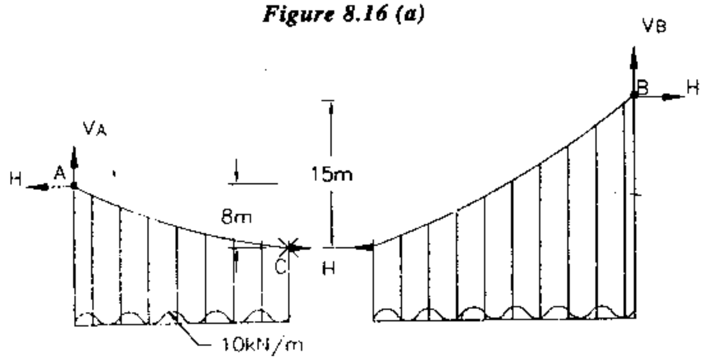


Figure 8.16 (b)

Referring to Fig.8.16(a)

$$h_1 = 8 \text{ m}; \; h_2 = 15 \text{ m} \quad \ell = 100 \text{ m}$$

Moment about A and B when calculated from mid-section gives

$$H \times 8 - 10 \times \ell_1 \frac{\ell_1}{2} = 0; \quad H \times 15 - 10 \; \ell_2 \frac{\ell_2}{2} = 0$$

$$8H = 5\ell_1^{\,2} \, ; \, 15H = 5\ell_2^{\,2}$$

$$\therefore \quad \frac{\ell_1^{\,2}}{\ell_2^{\,2}} = \frac{8}{15}$$

$L_1 =$

or
$$\frac{\ell_1}{\ell_2} = \frac{\sqrt{8}}{\sqrt{15}} = 0.730$$

$$\frac{\ell_1}{\ell_1 + \ell_2} = \frac{0.730}{1.730}$$

$$\ell_1 = \frac{0.73}{1.73}(\ell_1 + \ell_2) = \frac{0.73}{1.73} \times 100$$
$$= 42.214 \text{ m}$$

$\therefore$
$$\ell_2 = 100 - 42.214 = 57.786 \text{ m}$$

$$V_A = w\ell_1 = 10 \times 42.214 = 422.14 \text{ kN}$$

$$8H = 10\,\frac{\ell_1^{\,2}}{2}$$

$$H = \frac{10 \times 42.214^2}{8 \times 2} = 1113.76 \text{ kN}$$

$$V_B = w\ell_2 = 10 \times 57.786 = 577.86 \text{ kN}$$

Length of the cable

$$L = \ell + \frac{2}{3}\frac{h_1^{\,2}}{\ell_1} + \frac{2}{3}\frac{h_2^{\,2}}{\ell_1}$$

$$= 100 + \frac{2}{3} \times \frac{8^2}{42.214} + \frac{2}{3} \times \frac{15^2}{57.786}$$
$$= 103.607 \text{ m}$$

8.7 EFFECT OF TEMPERATURE ON CABLE

Due to rise in temperature, the length of cable increases. Since the supports are rigid, increase in length of the cable, increases the dip of the cable and hence reduces the horizontal thrust.

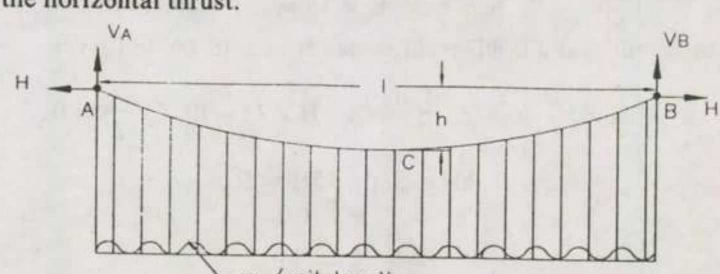


Figure 8.17

Referring to Fig. 8.17, the length of the cable is given by

$$L = \ell + \frac{8}{3}\frac{h^2}{\ell}$$

$$\therefore \quad \frac{dL}{dh} = \frac{16}{3}\frac{h}{\ell}$$

or

$$\delta L = \frac{16}{3}\frac{h}{\ell}\,\delta h \qquad \text{(i)}$$

If α is the coefficient of thermal expansion and t is the rise in temperature, then, increase in length is given by

$$\delta L = L\,\alpha\,t$$

$$= \alpha\,t\left[\ell + \frac{8h^2}{3\ell}\right]$$

Since $\alpha t \dfrac{8h^2}{3l}$ is a small quantity, it may be neglected. Then

$$\delta L = \ell\,\alpha\,t \qquad \text{(ii)}$$

From eqn. (i) and (ii) we get

$$\frac{16}{3}\frac{h}{\ell}\,\delta h = \ell\,\alpha\,t$$

or

$$\delta h = \frac{3}{16}\frac{\ell^2 \alpha t}{h} \qquad 8.18$$

The above expression gives increase in dip due to the rise in temperature.

Taking moment about the lowest point C, we get

$Hh = M_c$, where M_c is the beam moment at C

$$H = \frac{M_c}{h}$$

$$\frac{dH}{dh} = -\frac{M_c}{h^2} = -\left(\frac{M_c}{h}\right)\frac{1}{h}$$

$$= -\frac{H}{h}$$

$$\therefore \quad \frac{\delta H}{H} = -\frac{1}{h}\delta h$$

Substituting the value of δh from eqn. 8.18, we get

$$\frac{\delta H}{H} = -\frac{3}{16} \times \frac{\ell^2}{h^2}\alpha\,t \qquad 8.19$$

Thus, increase in temperature reduces the horizontal thrust.

Example 8.8 A cable of span 80m and dip 6m is subjected to a rise in temperature of 20° C. If the coefficient of thermal expansion of the cable material is $12 \times 10^{-6}/°C$, determine the increase in the dip of the cable. What are the changes in reactions and maximum tension, if the cable carries a load of 15 kN/m?

Solution

Referring to Fig. 8.17.

$$\ell = 80\text{m},\ h = 6\text{m}$$

$$t = 20° \text{ and } \alpha = 12 \times 10^{-6}/°C$$

$$\therefore \quad \delta h = \frac{3\ell^2 \alpha t}{16h}$$

$$= \frac{3 \times 80^2 \times 12 \times 10^{-6} \times 20}{16 \times 6}$$

$$= 0.048\text{m}$$

If cable carries a udl 15kN/m, then

$$Hh = \frac{w\ell^2}{8}$$

$$H = \frac{w\ell^2}{8h} = \frac{15 \times 80^2}{8 \times 6}$$

$$= 2000\text{kN}$$

From eqn. 8.19,

$$\frac{\delta H}{H} = -\frac{3}{16} \times \frac{\ell^2}{h^2} \alpha t$$

$$\frac{\delta H}{2000} = -\frac{3}{16} \times \frac{80^2}{6^2} \times 12 \times 10^{-6} \times 20$$

$$\delta H = 16 \text{ kN}$$

$$\therefore \quad H = 2000 + 16 = 2016 \text{ kN}$$

$$V_A = \frac{w\ell^2}{2} = \frac{15 \times 80}{2} = 600 \text{ kN}$$

Due to change in temperature, there will not be any change in vertical reactions. Before rise in temperature,

$$T = \sqrt{2000^2 + 600^2} = 2088.06 \text{ kN}$$

After rise in temperature

$$T = \sqrt{(2016)^2 + 600^2}$$

$$= 2103.392 \text{ kN}$$

Change in maximum tension

$$= 2103.392 - 2088.06$$
$$= 15.332 \text{ kN}$$

8.8 SUSPENSION BRIDGE WITH THREE HINGED STIFFENING GIRDER

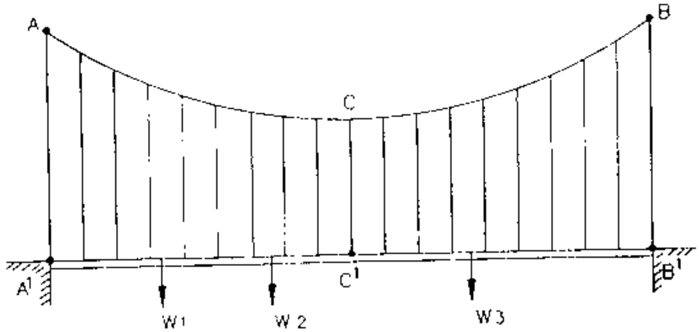


Figure 8.18 (a)

Consider the suspension cable stiffened with a three hinged girder as shown in Fig.8.18(a). The girder can be a heavy beam or a truss which has three hinges– two at the ends and one at the centre. The cable and the girder are connected by a number of hangers/suspenders. Since the number of suspenders are very large, the load on cable or girder, due to the forces in the suspenders, may be taken as uniformly distributed load. Let it be W_e per unit horizontal length (Fig.8.18b).

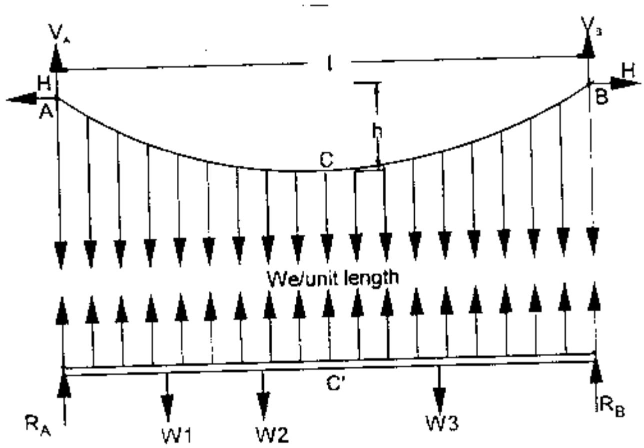


Figure 8.18 (b)

Considering the moment at the central hinge of girder C′, W_e, the unifromly distributed load exerted by suspenders can be determined. Then the beam may be analysed for the given load along with W_e. Note that due to W_e alone, there is a bending moment of $\frac{W_e x}{8}(\ell - x)$ at section x-x and a maximum moment i.e., $\frac{W_e \ell^2}{8}$ at C. This moment is a hogging moment. The shear force at that section due to W_e is $-W_e\left(\frac{\ell}{2} - x\right)$. The cable can be analysed for the uniformly distributed load W_c. The analysis is illustrated with the problems below.

Example 8.9 A three hinged stiffening girder of a suspension bridge of span 100m is subjected to two point loads of 200 kN and 300 kN at the distances of 25m and 50m from the left end. Find the shear force and bending moment for the girder at a distance 30m from the left end. The supporting cable has a central dip of 10m. Find also the maximum tension and its slope in the cable.

Solution

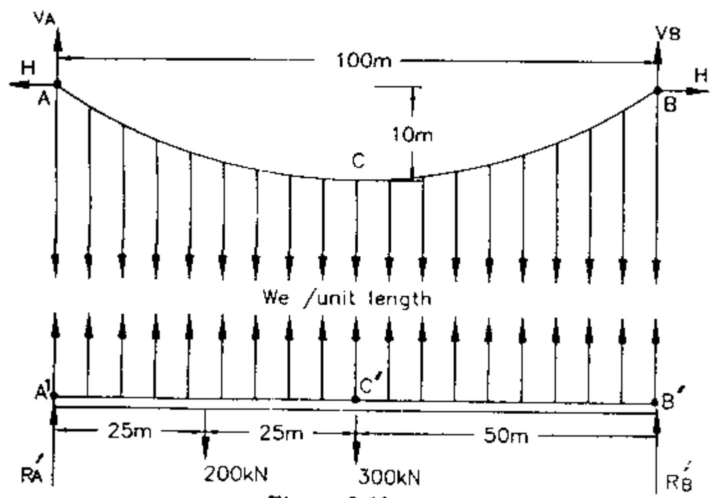


Figure 8.19

Let the suspenders exert a uniformly distributed load of W_c/unit horizontal length as shown in Fig. 8.19.

Reactions at A and B due to a given loading only be denoted as R'_A and R'_B, respectively. Then considering the girder

$$\Sigma M_A' = 0$$

$$R_B' \times 100 - 200 \times 25 - 300 \times 50 = 0$$

$$R_B' = 200$$

$$\Sigma M_{c'} = 0$$

Bending moment due to given loading + Bending moment due to $W_e = 0$

$$\therefore \quad R_B' \frac{\ell}{2} - \frac{W_e \ell^2}{8} = 0$$

$$200 \times \frac{100}{2} = \frac{W_e \times 100^2}{8}$$

$$W_e = 8\text{kN/m}$$

$$R_A' = 300 + 200 - R_B'$$

$$= 500 - 200$$

$$= 300\text{kN}$$

At x = 30m, from left end,

S.F = S.F due to given loading + S.F due to W_e

$$S.F = R_A' - 200 \;\; W_e\left(\frac{100}{2} - 30\right)$$

$$= 300 - 200 - 8(50 - 30)$$

$$= -60\text{kN}$$

$$= 60\text{kN}\downarrow$$

M = Moment due to given loading + Moment due to W_e

$$= 300 \times 30 - 200 \times 5 - 8 \times \frac{30(100-30)}{2}$$

$$= -400 \text{ kN-m}$$

$$= 400 \text{ kN-m hogging}$$

For the analysis of cable

$$V_A = W_e \frac{\ell}{2} = 8 \times \frac{100}{2} = 400 \text{ kN}$$

Taking moment about C, we get

$$H \times h = \frac{W_e\ell}{2}\frac{\ell}{2} - W_e\frac{\ell}{2}\;\frac{\ell}{4}\frac{W_e\ell^2}{8}$$

$$H \times 10 = \frac{W_e\ell^2}{8} = \frac{8 \times 100^2}{8} = 10000 \text{ kN}$$

or $$H = 1000 \text{ kN}$$

$$T_{max} = \sqrt{V_A^2 + H^2}$$

$$T_{max} = \sqrt{400^2 + 1000^2} = 1077.033 \text{ kN}$$

Its slope to horizontal is given by

$$T_{max}\cos\theta = H$$

$$\theta = \cos^{-1}\frac{1000}{1077.033} = 21.80°$$

Example 8.10 A suspension bridge of 120m span has two three hinged stiffening girders supported by two cables having a central dip of 12m. The roadway has a width of 6m. The dead load on the bridge is $5kN/m^2$ while the live load is $10kN/m^2$ which acts on the left half of the span. Determine the shear force and bending moment in the girder at 30m from the left end. Find also the maximum tension in the cable for this position of live load.

Solution

Width of bridge $= 6m$

Dead load on the bridge $= 5kN/m^2$

$\therefore$ Dead load on each girder $= 5 \times \frac{6}{2} = 15kN/m$

Live load on the bridge $= 10kN/m^2$

$\therefore$ Live load on each girder $= 10 \times \frac{6}{2} = 30kN/m$

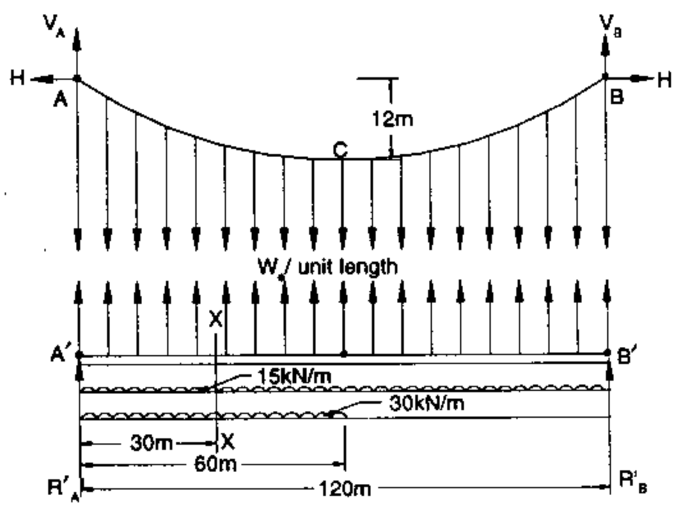


Figure 8.20

These loads are shown in Fig. 6.20. Let suspenders expert a load of intensity W_e per unit length. Let R_A' and R_B' be the reactions at A′ and B′ due to given loadings only.

Taking moment about A′ we get

$$R_B' \times 120 - 15 \times 120 \times 60 - 30 \times 60 \times 30 = 0$$

$$R_B' = 1350kN$$

$$\therefore \quad R_A' = 15 \times 120 + 30 \times 60 - R_B'$$

$$= 1800 + 1800 - 1350$$

$$= 2250 \text{ kN}$$

$$\Sigma M_C = 0$$

Moment due to given loading + Moment due to $W_e = 0$

$$R_B' \times 60 - 15 \times 60 \times 30 - W_e \times \frac{120^2}{8} = 0$$

$$1350 \times 60 - 15 \times 60 \times 30 = W_e \times \frac{120^2}{8}$$

$$W_e = 30\text{kN/m}$$

$\therefore$ At 30m from left support,

$$S.F = S.F \text{ due to given loading} + S.F \text{ due to } W_e$$

$$S.F = R_A' - 15 \times 30 - 30 \times 30 - W_e\left(\frac{120}{2} - 30\right)$$

$$= 2250 - 450 - 900 - 30(60-30)$$

$$= 0\text{kN}$$

Bending moment M is given by

M = Moment due to given loading + moment due to W_e

$$M = R_A' \times 30 - 15 \times 30 \times 15 - 30 \times 30 \times 15 - \frac{W_e}{2}\, 30\,(120 - 30)$$

$$= 2250 \times 30 - 15 \times 30 \times 15 - 30 \times 30 \times 15 - \frac{30}{2} \times 30 \times 90$$

$$= 6750\text{kN-m}$$

Considering the equilibrium of the cable

$$V_A = \frac{W_e \ell}{2} = \frac{30 \times 120}{2} = 1800 \text{ kN}$$

$$M_C = 0$$

$$H \times 12 = W_e \frac{\ell^2}{8} = 30 \times \frac{120^2}{8}$$

$$H = 4500 \text{ kN}$$

$$T_{max} = \sqrt{V_A^2 + H^2}$$

$$= \sqrt{1800^2 + 4500^2}$$

$$= 4846.65 \text{ kN}$$

Its inclination to horizontal is given by

$$T_{max} \cos\theta = H$$

$$4846.65 \cos\theta = 4500$$

$$\therefore \quad \theta = 21.80°$$

8.9 INFLUENCE LINES AND ROLLING LOADS

Suspension bridges are subjected to a number of vehicles rolling over them. Hence the analysis of suspension bridges subject to rolling loads is essential. As seen in chapter V, the best approach for the analysis of structures subjected to rolling loads is to construct influence lines for the stress resultants required. In this article, we will construct influence line diagrams for the following.

i) Equivalent cable load W_e
ii) Bending moment M in the girder
iii) Shear force F in the girder
iv) Horizontal force in the cable.

8.9.1 ILD for W_e

Referring to Fig. 8.21 (a), let the unit load be at a distance x from the end A′. Reaction at B′ due to this load is given by

$$R_B' = \frac{x}{\ell}$$

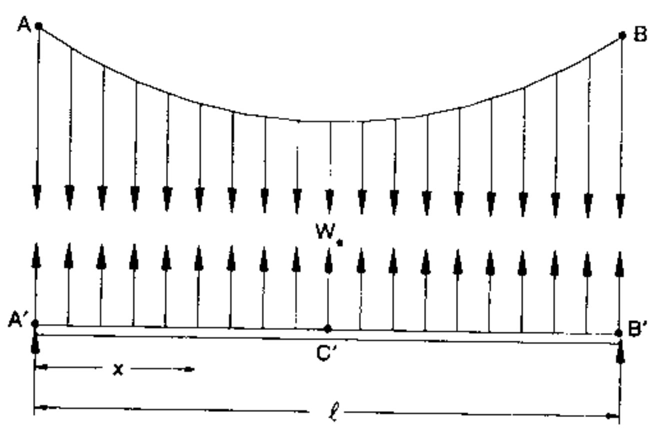


Figure 8.21

Let W_e be the equivalent load exerted by suspenders. Let the load be in the portion A′C′

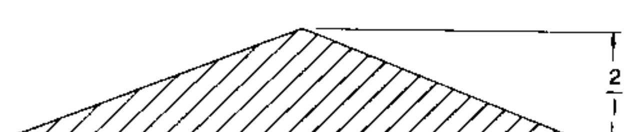


Figure 8.21 (a) ILD for W_e

$$\Sigma M'_C = 0$$

Moment due to unit load + Moment due to W_e = 0

$$\frac{x}{\ell}\,\frac{\ell}{2} - W_e\,\frac{\ell^2}{8} = 0$$

or $$W_e = \frac{4x}{\ell^2}\text{, linear variation} \qquad 8.20$$

At $x = 0$, $\quad W_e = 0$

At $x = \frac{\ell}{2}$ $\quad W_e = \frac{2}{\ell}$

Obviously, when unit load moves from C′ to B′, it reduces linearly from $\frac{2}{\ell}$ to 0. Hence ILD for We is as shown Fig. 8.21(b).

8.9.2 ILD for M

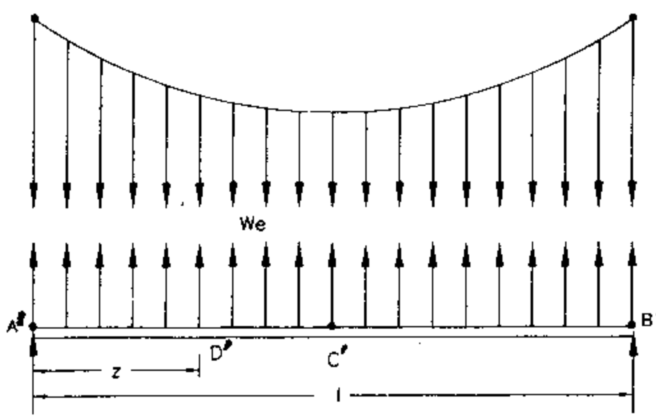


Figure 8.22 (a)

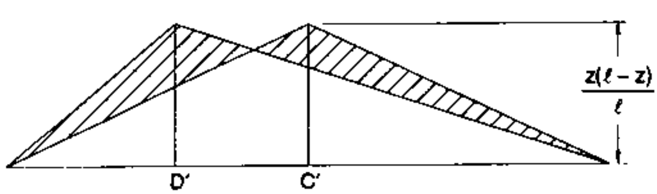


Figure 8.22 (b) ILD for M

Referring to Fig. 8.22(a), bending moment at D′, distance z from support A′ is required. When unit load is in portion A′D′ at a distance x from A′, the bending moment at this section is given by

M = Moment due to given unit load + Moment due to W_c

∴ ILD for M = ILD for moment due to unit load + ILD for moment due to W_e.

ILD for moment due to the unit load on the girder is a triangle with maximum ordinate $\frac{z(\ell - z)}{\ell}$ at section z distance from A′.

Moment due to W_e

$$= -W_c \frac{z(\ell - z)}{\ell}$$

From eqn. 8.20, $W_c = \frac{4x}{\ell^2}$

∴ Moment due to W_c at z

$$= -\frac{4x}{\ell^2}\frac{z(\ell - z)}{2}$$

$$= -\frac{2x}{\ell^2} z\,(\ell - z) \qquad 8.21$$

It varies linearly with x

At x = 0, M due to $W_c = 0$

At $x = \frac{\ell}{2}$, M due to $W_c = \frac{z(\ell - z)}{\ell}$

Thus, ILD for moment due to W_e is also a triangle with maximum ordinate of $\frac{z(\ell - z)}{\ell}$. This is drawn on the same side as ILD for moment due to given unit load, since the difference diagram is required. Fig. 8.22(b) shows ILD for M at section distance z from A′.

8.9.3 ILD for S.F

Shear force in the girder at D′,

Distance z from A′ = Shear force in the girder due to given unit load + Shear force due to W_e.

Hence ILD for S.F is drawn by appropriately superposing ILD for shear force due to W_e.

S.F due to W_e

$$= -W_e\left(\frac{\ell}{2} - z\right)$$

$$= -\frac{4x}{\ell^2}\left(\frac{\ell}{2} - z\right), \text{ since } W_e = \frac{4x}{\ell^2} \text{ from eqn. 8.20}$$

Linear variation with x.

when $x = 0$; $\quad$ S.F = 0

$x = \frac{\ell}{2}$; $\quad$ $\text{S.F} = -4\,\frac{\ell}{2}\,\frac{1}{\ell^2}\left(\frac{\ell}{2} - z\right)$

$$= \left(\frac{\ell - 2z}{\ell}\right)$$

Hence it is a triangle with maximum ordinate $\left(\frac{\ell - 2z}{\ell}\right)$ at centre. Since the difference of the diagrams is required, these are drawn on the same side and the difference diagram is hatched. Fig. 8.23 shows the ILD at D′ for the cases (i) when $z < \frac{\ell}{4}$ (ii) when $z = \frac{\ell}{4}$ and (iii) when $z > \frac{\ell}{4}$

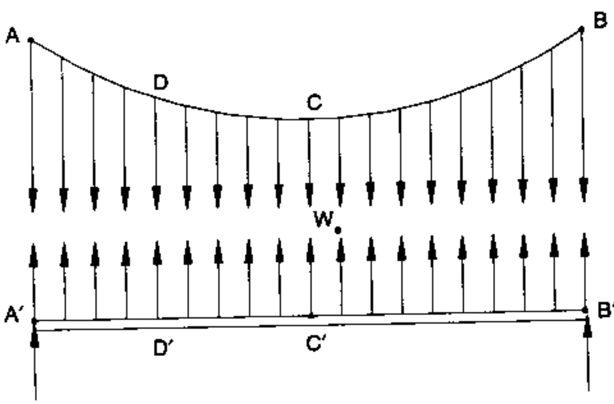


Figure 8.23 (a)

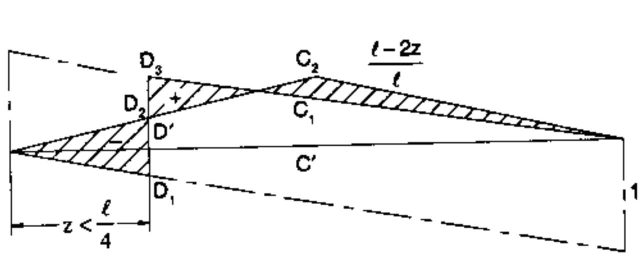


Figure 8.23 (b) ILD for F when $z < \frac{\ell}{4}$

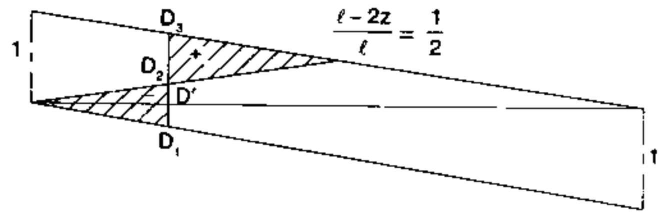


Figure 8.23 (c) ILD for F when $z = \frac{\ell}{4}$

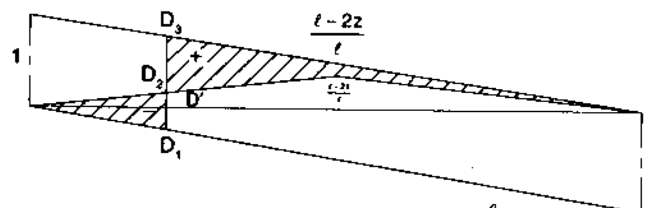


Figure 8.23 (d) ILD for F when $z > \frac{\ell}{4}$

8.9.4 ILD for H

Consider the equilibrium of cable when unit load is at section x-x distance x from A′. Let W_e be the equivalent udl acting on the cable. Taking moment about C, we get,

$$Hh - V_A\frac{\ell}{2} + W_e\frac{\ell}{2}\frac{\ell}{4} = 0$$

$$Hh - \frac{W_e\ell}{2}\frac{\ell}{2} + W_e\frac{\ell^2}{8} = 0$$

$$H = \frac{W_e\ell^2}{8h}$$

From equation 8.20, $$W_e = \frac{4x}{\ell^2}$$

$$\therefore \qquad H = \frac{4x\ \ell^2}{8h\,\ell^2} = \frac{4x}{8h}\text{, linear variation}$$

$$\text{At } x = 0,\ H = 0$$

$$\text{At } x = \frac{\ell}{2},\ H = \frac{4(\ell/2)}{8h} = \frac{\ell}{4h}$$

∴ ILD for H is as shown in Fig.8.24.

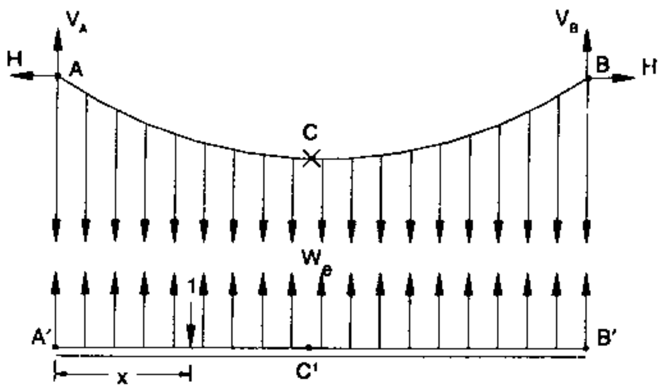


Figure 8.24 (a)

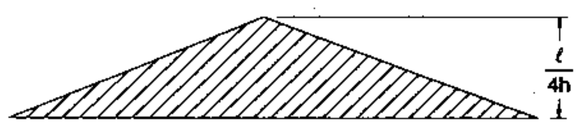


Figure 8.24 (b) ILD for H

Example 8.11 A suspension cable of 90m span and 9m dip is stiffened by a three hinged girder. The dead load is 10 kN/m. Determine the maximum tension in the cable and maximum bending moment at the section 30m from left support in the girder due to the concentrated load of 100 kN crossing the girder, assuming that the dead load is carried by the cable without stressing the girder.

Solution

Due to dead load which is entirely carried by cables, horizontal thrust H, is given by

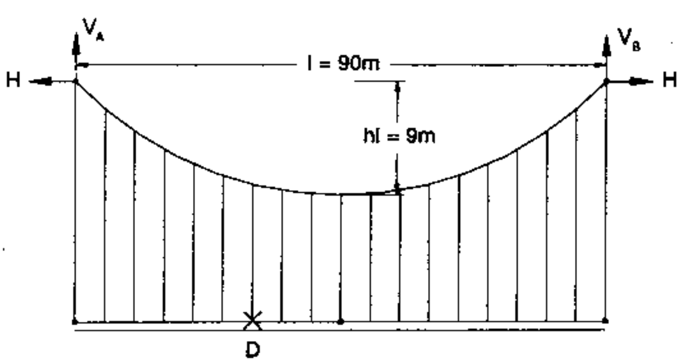


Figure 8.25 (a)

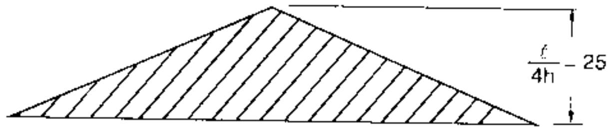


Figure 8.25(b) ILD for H

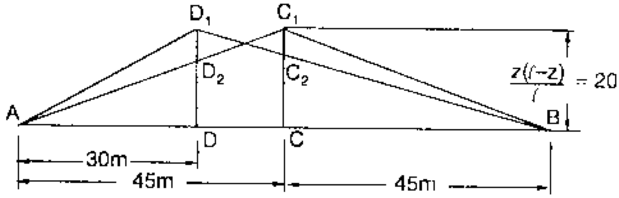


Figure 8.25(c) ILD for M_D

$$H_1 \times h = \frac{W\ell^2}{8}$$

$$H_1 = \frac{10\times 90^2}{8\times 9} = 1125 \text{ kN}$$

Corresponding vertical reaction $V_{1A} = \frac{W\ell}{2}$

$$= \frac{10\times 90}{2}$$

$$= 450 \text{ kN}$$

ILD for horizontal force H is as shown in Fig. 8.25(b).

Maximum ordinate $= \frac{\ell}{4h} = \frac{90}{4\times 9}$

$$= 2.5$$

∴ Horizontal thrust H_2, due to live load is maximum when live load is at centre

∴ $$H_2 = 100 \times 2.5 = 250 \text{ kN}$$

Corresponding vertical reactions at A $= V_{2A} = \frac{100}{2} = 50$ kN

∴ Max. horizontal thrust $= H_1 + H_2$

$= 1125 + 250 = 1375$ kN

Vertical reaction $= V_A = V_{1A} + V_{2A} = 450 + 50 = 500$ kN

$$T_{max} = \sqrt{V_A^2 + H^2}$$

$$= \sqrt{1375^2 + 500^2}$$

$$= 1463.09 \text{ kN}$$

Fig. 8.25 (c) shows ILD for bending moment in which ordinates

$$DD_1 = CC_1 = \frac{z(\ell - z)}{\ell}$$

$$= \frac{30(90-30)}{90} = 20$$

Maximum positive moment occurs when the load is on the section

$$M_{max} = W \times \text{ordinates } D_1D_2$$
$$= 100\,[DD_1 - DD_2]$$
$$= 100\left(20 - 20\frac{30}{45}\right)$$
$$= 666.67 \text{ kN-m}$$

Maximum negative moment occurs when the moving load is at mid-span section

$$= W \times C_1C_2 = 100\,(CC_1 - CC_2)$$

$$= 100\left[20 - \frac{45}{60} \times 20\right] = 100 \times 5$$

$$= 500 \text{ kN–m}$$

Note: At any section, moment due to uniformly distributed load covering entire span (similar to dead load) is zero since in ILD positive area is equal to negative area

Example 8.12 A suspension bridge, 100m span, has two three hinged stiffening girders supported by two cables with a central dip of 18m. If four point loads of 200kN each are placed along the central lines of the roadway at 30m, 33m 36m 39m from the left hand hinge, find the shear force and bending moment in girder at section 45m from left end. Calculate the maximum tension in the cable.

Solution

Here, $\ell = 100$m, $h = 18$m

The loads are placed centrally along the bridge. Hence half the loads come to each cable-girder system. The loads coming on a system are of 100kN intensity. Influence Line diagram for moment at 45 m from left support is as shown in Fig. 8.26(a).

The maximum ordinates are

$$= \frac{z(\ell - z)}{\ell} = \frac{45(100-45)}{100} = 24.75$$

Ordinate O_1 under 100 kN load at 39m from left end

$$= \frac{39}{45} \times 24.75 - \frac{39}{50} \times 24.75$$

$$= 39 \times 24.75\left[\frac{1}{45} - \frac{1}{50}\right]$$

$$= 2.145$$

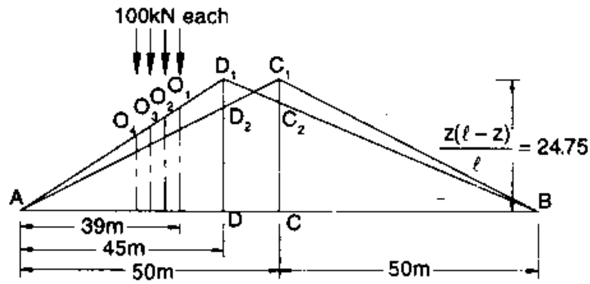


Figure 8.26 (a) ILD for M_b

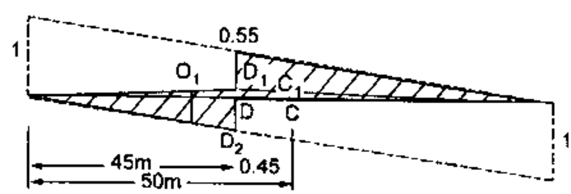


Figure 8.26 (d) ILD for F_D

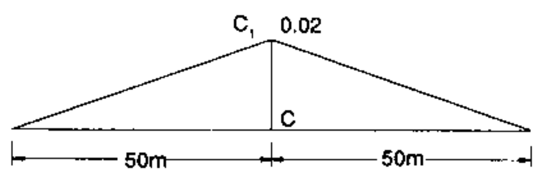


Figure 8.26 (c) ILD for We

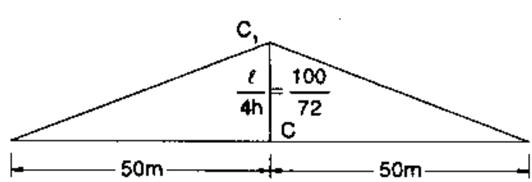


Figure 8.26 (d) ILD for H

Similarly,

$$O_2 = \frac{36}{39} \times 2.145 = 1.98$$

$$O_3 = \frac{33}{39} \times 2.145 = 1.815$$

$$O_4 = \frac{30}{39} \times 2.145 = 1.65$$

$$\therefore \quad M = 100 \times 2.145 + 100 \times 1.98 + 100 \times 1.815 + 100 \times 1.65$$
$$= 759 \text{ kN} - \text{m}$$

Influence line diagram for shear force at 45m from left support is shown in Fig. 8.26 (b).

The ordinates of ILD diagram for given unit load on a beam are

$$DD_2 = \frac{z}{\ell} = \frac{45}{100} = 0.45$$

and
$$DD_1 = \frac{\ell - z}{\ell} = \frac{55}{100} = 0.55$$

Central ordinate of ILD diagram accounting for the effect of W_e

$$CC_1 = \frac{\ell - 2z}{\ell} = \frac{100 - 2 \times 45}{100} = 0.1$$

$\therefore$ Ordinate at 39 m from left support is

$$O_1 = 0.45 \times \frac{39}{45} + 0.1 \times \frac{39}{50}$$
$$= 39\left[\frac{0.45}{45} + \frac{0.1}{50}\right]$$
$$= 39 \times 0.012$$
$$= 0.468$$

Similarly,

$$O_2 = \frac{36}{39} \times 0.468 = 0.432$$

$$O_3 = \frac{33}{39} \times 0.468 = 0.396$$

$$O_4 = \frac{30}{39} \times 0.468 = 0.36$$

$$\therefore \quad F = 100 \times O_1 + 100 \times O_2 + 100 \times O_3 + 100 \times O_4$$
$$= 100[0.468 + 0.432 + 0.396 + 0.36]$$
$$= 165.6 \text{ kN}$$

The ILD diagram for uniformly distributed load for cable tension W_e and horizontal thrust H are as shown in Fig. 8.26 (c) and 8.26 (d).

Maximum ordinate of ILD for W_e

$$= \frac{2}{\ell} = \frac{2}{100} = 0.02$$

∴ Ordinate of ILD for W_e at 39m

$$O_1 = \frac{39}{50} \times 0.02$$

$$O_2 = \frac{36}{50} \times 0.02$$

$$O_3 = \frac{33}{50} \times 0.02$$

$$O_4 = \frac{30}{50} \times 0.02$$

$$\therefore \quad W_e = 100 \times O_1 + 100 \times O_2 + 100 \times O_3 + 100 \times O_4$$

$$= 100\left(\frac{39+36+33+30}{50}\right) \times 0.02$$

$$= 5.52 \text{ kN/m}$$

∴ Vertical reaction at A,

$$V_A = W_e \frac{\ell}{2} = 5.52 \times \frac{100}{2} = 276.0 \text{ kN}$$

Referring to fig 8.26 (d), Maximum ordinate of ILD for H

$$= \frac{\ell}{4h} = \frac{100}{4 \times 18} = \frac{100}{72}$$

$$\therefore \quad O_1 = \frac{39}{50} \times \frac{100}{72} = 39 \times \frac{1}{36}$$

$$O_2 = \frac{36}{50} \times \frac{100}{72} = 36 \times \frac{1}{36} = 1$$

$$O_3 = 33 \times \frac{1}{36}$$

$$O_4 = 30 \times \frac{1}{36}$$

$$\therefore \quad H = 100 \times O_1 + 100 \times O_2 + 100 \times O_3 + 100 \times O_4$$

$$= 100\left(\frac{39}{36} + \frac{36}{36} + \frac{33}{36} + \frac{30}{36}\right)$$

$$= 383.33 \text{ kN}$$

$$T_{max} = \sqrt{V_A^2 + H^2}$$

$$= \sqrt{276^2 + 383.33^2}$$

$$= 472.356 \text{ kN}$$

Example 8.13 A suspension bridge has a cable of span 100m and dip of 10m. The cable is stiffened by a 3 hinged stiffening girder. Sketch the influence line diagram for bending moment at quarter span of girder. Determine the maximum moment at this section when a uniformly distributed load longer than the span of intensity 10kN/m traverses the span.

Solution

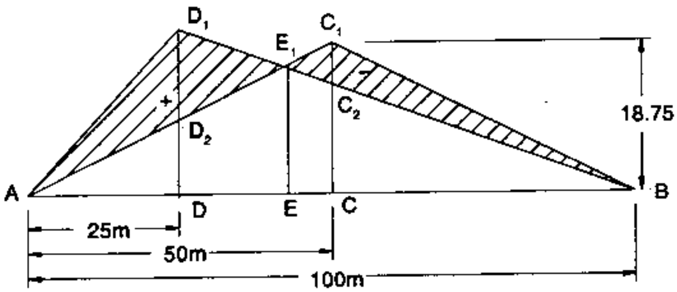


Figure 8.27

Figure 8.27 shows influence line diagram for bending moment at quarter span D. Max. ordinates in this diagram are,

$$= \frac{z(\ell - z)}{\ell} = \frac{25(100-25)}{100} = 18.75$$

Maximum moment occurs when the uniformly distributed load occupies portion AE (refer Fig.8.27). To locate E, we write expression for EE_1 from triangles ACB and ADB.

$$\frac{AE}{AC} \times CC_1 = EE_1 = \frac{BE}{BC} \times DD_1$$

$$\frac{AE}{50} \times 18.75 = \frac{BE}{75} \times 18.75$$

$$\frac{AE}{BE} = \frac{50}{75}$$

$$\frac{AE}{AE+BE} = \frac{50}{50+75}$$

or $$AE = \frac{50}{125} \times AB = \frac{50}{125} \times 100 = 40m$$

$\therefore$ $$EE_1 = \frac{40}{50} \times 18.75 = 15$$

$\therefore$ Maximum + ve moment M = Intensity of load × area of ΔAD_1E_1

$$= 10(\Delta AD_1B - \Delta AE_1B)$$

$$= 10\left(\frac{1}{2}\times 100\times DD_1 - \frac{1}{2}\times 100\times EE_1\right)$$

$$= 10\times\frac{1}{2}\times 100\,(DD_1 - EE_1)$$

$$= 10\times\frac{1}{2}\times 100\,(18.75 - 15)$$

$$= 1875 \text{ kN-m}$$

Maximum –ve moment

$$= 10\,(\Delta BC_1E_1 - \Delta AE_1B) = 1875 \text{ kN-m}$$

8.10 MAXIMUM MOMENT DIAGRAMS

Influence line diagram for bending moment at any section in three hinged stiffening girder of suspension bridge is the same as that for a parabolic arch. Both are having two triangles with maximum values at the section and the other at centre of span of magnitude $\frac{z(\ell - z)}{\ell}$ where z is the distance of the section from the support. Hence maximum bending moment diagrams for the rolling loads will be exactly same as that for three hinged parabolic arch. These expressions can be derived as explained in the Art. 7.6. These diagrams are shown in Fig. 8.28(c) and Fig. 8.28(d) for a concentrated load and for uniformly distributed loads respectively.

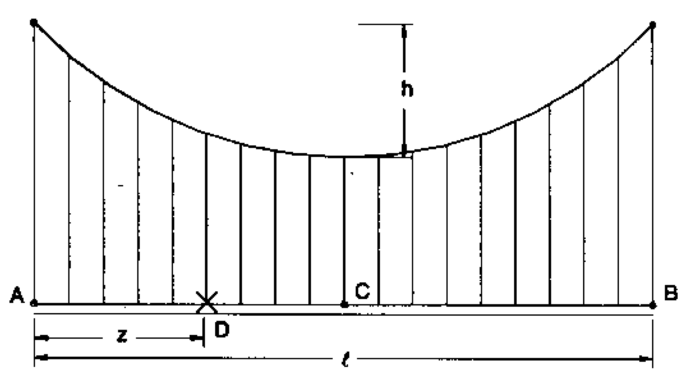


Figure 8.28 (a)

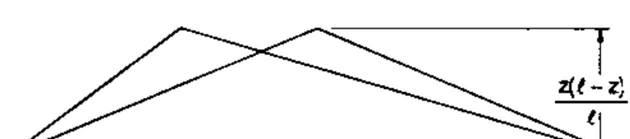


Figure 8.28 (b) ILD for M_D

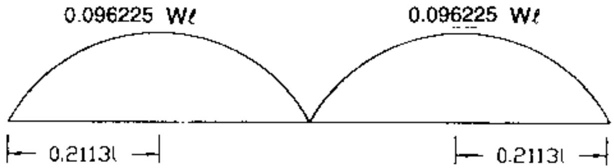


Figure 8.28 (c) BMD for maximum moment due to concentrated load

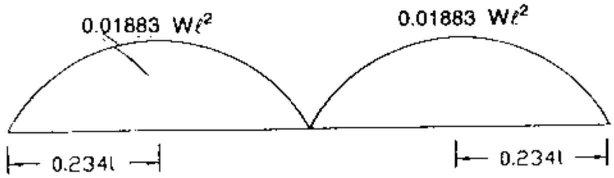


Figure 8.28 (d) BMD for maximum moment due to uniformly distributed load

8.11 MAXIMUM SHEAR FORCE DIAGRAM

Influence line diagram for shear force is as shown in Fig.8.23. From the figure it is clear that

(i) Maximum negative shear force occurs when the load is just to the left of the section,

(ii) maximum positive shear force occurs when the load is just to the right of the section. However when the section is very close to the support positive maximum shear may occur when the load is at mid-span.

(I) Diagram for maximum negative shear force due to a concentrated load W

Referring Fig. 8.23(b),

$$F_{max} = W \times D_1D_2$$
$$= W\,[D_1D' + D'D_2]$$
$$= W\left(\frac{z}{\ell} + \frac{z}{(\ell/2)}\frac{(\ell-2z)}{\ell}\right)$$
$$= \frac{Wz}{\ell^2}\,[\ell + 2(\ell - 2z)]$$
$$= \frac{Wz}{\ell^2}\,(3\ell - 4z)$$

$\therefore$ Maximum negative S.F occurs when $\frac{dF_{max}}{dz} = 0$

i.e., $3\ell - 8z = 0$

or $z = \frac{3}{8}\ell$

when $z = 0$, $F_{-max} = 0$

$z = \frac{\ell}{4}$, $F_{-max} = -\frac{W}{2}$

$z = \frac{3}{8}\ell$ $F_{-max} = -\frac{9}{16}W$

This is absolute maximum.

At $z = \frac{\ell}{2}$, $F_{-max} = -\frac{W}{2}$

∴ Maximum negative force diagram is as shown in Fig. 8.29.

(ii) Diagram for maximum positive shear force

Referring to Fig. 8.23, we find positive maximum shear force occurs when the load is just to the right of the section.

$$F_{+max} = W \times D_2D_3$$
$$= W \times (D'D_3 - D'D_2)$$
$$= W\left(\frac{\ell - z}{\ell} - \frac{z}{(\ell - 2)}\frac{\ell - 2z}{\ell}\right)$$
$$= \frac{W}{\ell^2}\left[\ell^2 - \ell z - 2\ell z + 4z^2\right]$$
$$= \frac{W}{\ell^2}\left[\ell^2 - 3\ell z + 4z^2\right]$$

∴ At $z = 0$, $F_{+max} = W$

At $z = \frac{\ell}{4}$, $F_{+max} = \frac{W}{\ell^2}\left(\ell^2 - \frac{3}{4}\ell^2 + 4\frac{\ell^2}{16}\right) = \frac{W}{2}$

At $z = \frac{3\ell}{8}$, $F_{+max} = \frac{W}{\ell^2}\left(\ell^2 - 3\ell\frac{3\ell}{8} + 4\frac{9\ell^2}{64}\right) = \frac{7}{16}W$

At $z = \frac{\ell}{2}$, $F_{+max} = \frac{W}{\ell^2}\left(\ell^2 - 3\ell\frac{\ell}{2} + 4\frac{\ell^2}{4}\right) = \frac{W}{2}$

Maximum positive S.F diagram is also shown in Fig. 8.29.

It is possible to show that maximum positive shear force never occurs,when the load is at centre. When the load is at centre, referring to Fig. 8.23(a).

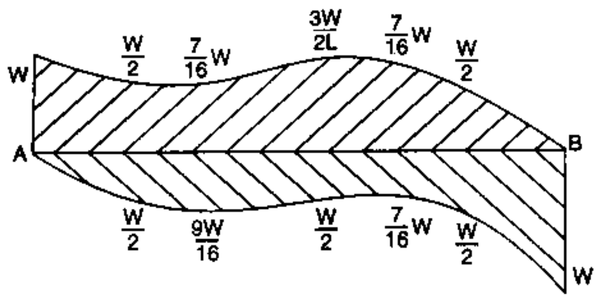


***Figure 8.29** Maximum positive and negative shear force diagram*

$$
\begin{aligned}
F_{+max} &= W \times C' C_2 \\
&= W \times \ (C' C_2 - C' C_1) \\
&= W\left(\frac{\ell - 2z}{\ell} - \frac{1}{2}\right) \\
&= \frac{W}{2\ell}\ (2\ell\ - 4z\ -\ \ell\ ) \\
&= \frac{W}{2\ell}\ (\ell\ - 4z)
\end{aligned}
$$

If this is more than maximum shear force found earlier, then,

$$\frac{W}{2\ell}(\ell - 4z) > \frac{W}{\ell^2}\left(\ell^2 - 3\ell z + 4z^2\right)$$

i.e., $$\ell^2 - 4\ell z > 2\ell^2 - 6\ell z + 8z^2$$

or $$8z^2 - 2\ell z + \ell^2 < 0$$

$$z = \frac{2\ell \pm \sqrt{4\ell^2 - 32\ell^2}}{16}$$

Hence there is no real point. Thus maximum positive shear force always occurs when the concentrated load is just to the right of the section.

Example 8.14 A suspension cable of 40m span and 4m dip is stiffened by a three hinged girder. The dead load is 10 kN/m. Determine the maximum tension in the cable and maximum bending moment in the girder due to a concentrated load of 100 kN crossing the girder.

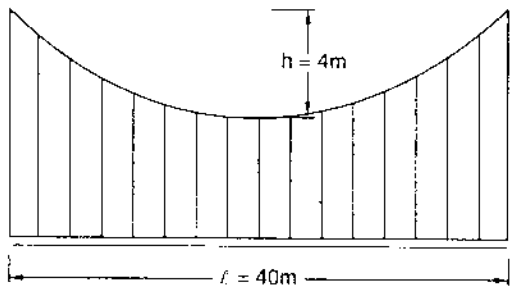


Figure 8.30 (a)

Solution

Referring to Fig. 8.30(a).

$$\ell = 40\text{m}; \quad h = 4\text{m}$$

ILD for horizontal thrust is a triangle with a maximum ordinate of,

$$\frac{\ell}{4h} = \frac{40}{4 \times 4} = 2.5 \text{ as shown in Fig. 8.30(b).}$$

∴ Horizontal thrust due to dead load,

$$= W \times \text{area of ILD triangle}$$

$$= 40 \times \frac{1}{2} \times 2.5 \times 10 = 500 \text{ kN}$$

Due to live load, horizontal thrust is maximum when the live load is at the centre of the span. Hence maximum horizontal thrust due to live load = 100 × 2.5 = 250 kN. ,

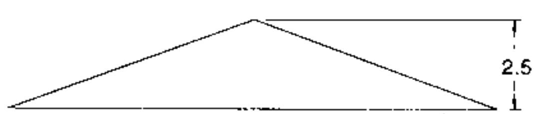


Figure 8.30 (b) ILD for H

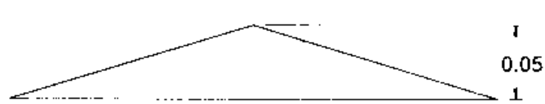


Figure 8.30 (c) ILD for We

∴ Maximum horizontal thrust in this case,

$$= 500 + 250 = 750 \text{ kN.}$$

Fig.8.30(c) shows ILD for tensile force from suspenders. The maximum ordinate in this case is at the centre of span and is,

$$= \frac{2}{\ell} = \frac{2}{40} = 0.05$$

W_e due to dead load,

$$= 10 \times \frac{1}{2} \times 40 \times 0.05 = 10 \text{ kN/m}$$

and due to live load at the centre of the span,

$$= 100 \times 0.05 = 5 \text{ kN/m},$$

$$\therefore \quad \text{Total } W_e = 10 + 5 = 15 \text{ kN/m},$$

$$V_A = W_e \frac{\ell}{2} = 15 \times \frac{40}{2} = 300 \text{ kN}$$

$$\therefore \quad T_{max} = \sqrt{V_A^2 + H^2}$$

$$= \sqrt{300^2 + 750^2}$$

$$= 807.775 \text{ kN}$$

Due to live load, moment is maximum at,

$$0.211 \times \ell = 0.211 \times 40 = 8.44 \text{ m}$$

from either support and its value is

$$= 0.096225 \text{ W } \ell$$

$$= 0.096225 \times 100 \times 40,$$

$$= 384.9 \text{ kN–m}$$

Corresponding moment due to dead load may be found by using ILD shown in Fig.8.30(d).

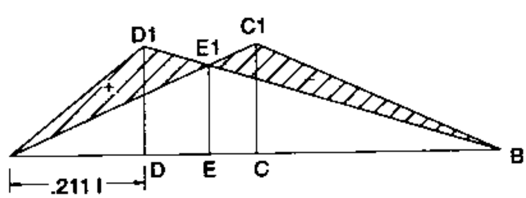


Figure 8.30 (d) ILD for M

M due to DL
$$= (\Delta AD_1E_1 - \Delta BC_1E_1)\,W$$
$$= [(\Delta AD_1B_1 - \Delta AE_1B) - (\Delta AC_1B - \Delta AE_1B)]\,W$$
$$= (\Delta AD_1B - \Delta AC_1B)\,W$$
$$= 0 \qquad (\text{since } DD_1 = CC_1)$$

Hence due to uniformly distributed dead load, bending moment in the stiffening girder is zero.

$$\therefore \quad M_{max} = M_{max} \text{ due to live load only}$$

$$= 304.9 \text{ kN-m}$$

Note: In suspension bridges bending moment due to uniformly distributed load over the entire span is zero.

8.12 SUSPENSION CABLE WITH TWO HINGED STIFFENING GIRDER

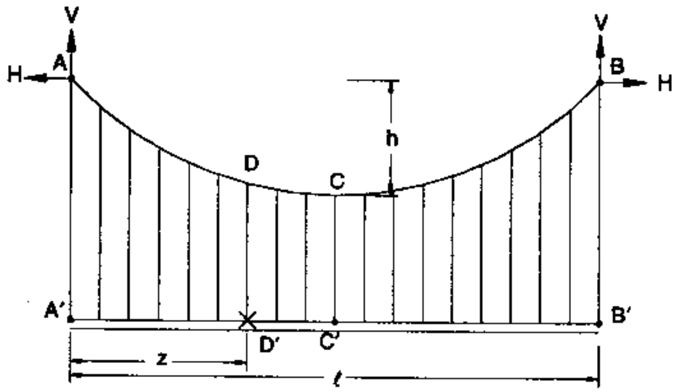


Figure 8.31 (a)

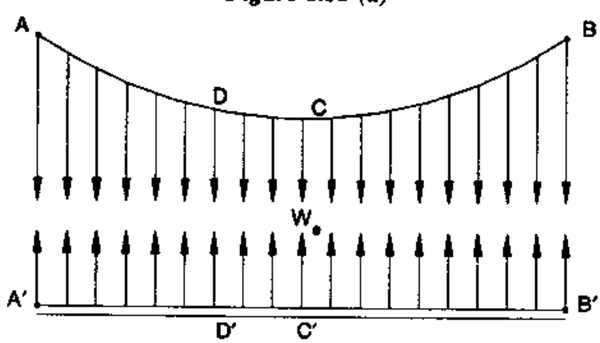


Figure 8.31 (b)

In this case, as the name suggest the stiffening girders are having two hinges. They are provided at the end supports as shown in Fig. 8.31. This is a statically indeterminate structure. However, the analysis is made simple by assuming that the girder is very stiff and hence, whatever is the load on the girder the stiffeners are equally stressed, i.e., Equivalent udl due to forces in suspenders is equal to the average load on the stiffening girder. Thus

$$W_e = \frac{W}{\ell}$$

where W = Total load on the girder

W_e = Equivalent uniformly distributed load

ℓ = Horizontal span.

Horizontal reaction,

$$H = \frac{W_e \ell^2}{8h}$$

Vertical reaction at the ends is given by

$$V = \frac{W_e \ell}{2}$$

$$T_{max} = \sqrt{V_A^2 + H^2}$$

For girder,

$$M_D' = \text{Beam moment} + \text{Moment due to } W_e$$

$$= \text{Beam moment} - W_e \frac{z(\ell - z)}{2}$$

$$F_D' = \text{Beam shear} + \text{Shear due to } W_e$$

$$= \text{Beam shear} - W_e \left(\frac{\ell}{2} - z\right)$$

Hence influence line diagrams for W_e, H, V, M and M_D are as shown in Fig.8.32.

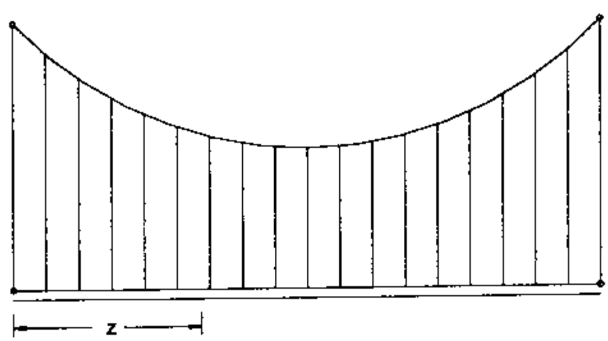


Figure 8.32 (a)

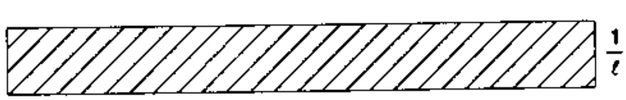


Figure 8.32 (b) ILD for We

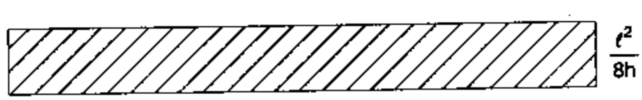


Figure 8.32 (c) ILD for H

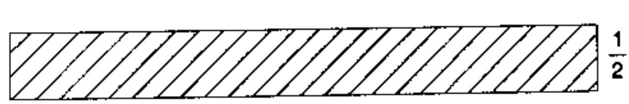


Figure 8.32 (d) ILD for V

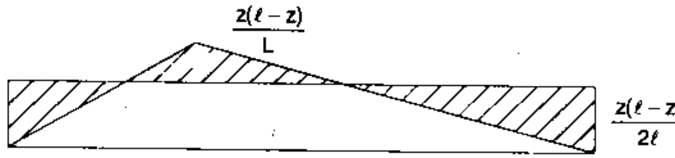


Figure 8.32 (e) ILD for M_D'

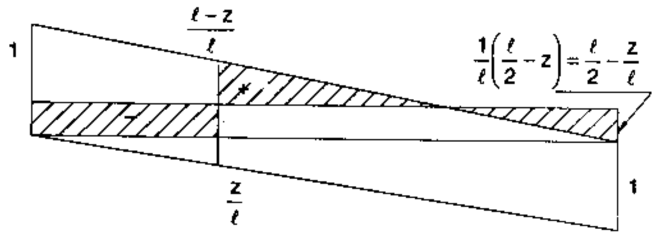


Figure 8.32 (f) ILD for F_D'

Example 8.15 A suspension bridge of span 80m and width 6m is having two cables stiffened with two, two hinged girders. The central dip of cables is 8m., the dead load on the bridge is 5 kN/m^2 and the live load is 10 kN/m which covers the left half of the span. Determine the shear force and bending moment at 20m from the left end. Find also the maximum tension in the cable.

Solution

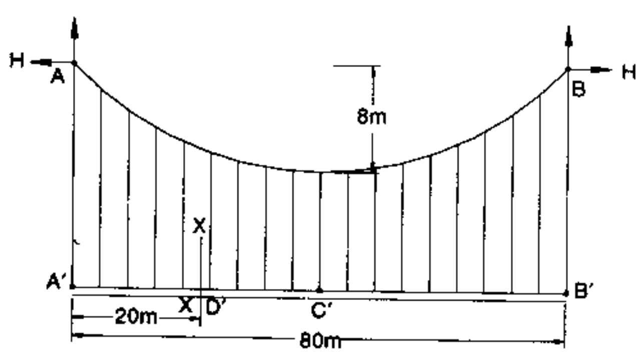


Figure 8.33 (a)

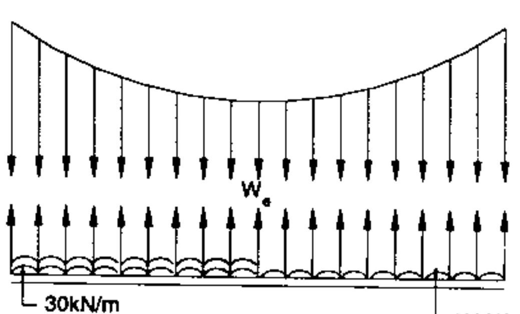


Figure 8.33 (b)

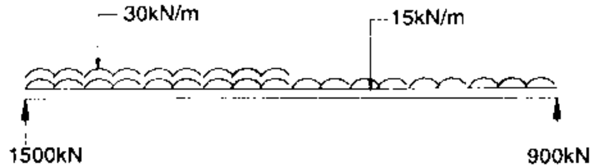


Figure 8.33 (c) Loading for beam shear and beam moment

Referring to Fig.8.33.

$$\ell = 80 \text{ m}, \quad h = 8\text{m}$$

Dead load per girder

$$= 5 \times \frac{6}{2} = 15 \text{ kN/m on entire span}$$

$$\text{Live load} = 10 \times \frac{6}{2} = 30 \text{ kN/m on left half portion}$$

∴ Total load on a girder

$$= 15 \times 80 + 30 \times \frac{80}{2}$$

$$= 2400 \text{ kN}$$

∴ Equivalent udl from cables

$$W_e = \frac{\text{Total load}}{\text{Span}}$$

$$= \frac{2400}{80} = 30 \text{ kN/m}$$

$$H = \frac{W_e \ell^2}{8h} = \frac{30 \times 80 \times 80}{8 \times 8} = 3000 \text{ kN}$$

$$V = W_e \times \frac{\ell}{2} = 30 \times \frac{80}{2} = 1200 \text{ kN}$$

$$\therefore \quad T_{max} = \sqrt{V_A^2 + H^2}$$

$$= \sqrt{3000^2 + 1200^2}$$

$$= 3231.10 \text{ kN}$$

Now consider the analysis of girder, which is subjected to a given load and to upward W_e. At D′ which is 20m from left support.

$$F_D = \text{Beam shear} + \text{shear due to } W_e$$

Due to given dead load and live load

$$\text{Reaction at A}' = \frac{\text{Moment about B}'}{80}$$

$$= \frac{15\times80\times40+30\times40(80-20)}{80}$$

$$= 1500 \text{ kN}$$

∴ Beam shear at D′

$$= 1500 - 15\times20 - 30\times20$$
$$= 600 \text{ kN}$$

∴ shear at D′

$$= 600 - W_e\left(\frac{\ell}{2} - z\right)$$

$$= 600 - 30\left(\frac{80}{2} - 20\right) = 0$$

M_D = Beam moment + Moment due to W_e

$$= \text{Beam moment } - W_e\,\frac{z(\ell - z)}{2}$$

$$= 1500\times20 - 15\times20\times10$$

$$- 30\times20\times10 - 30\times20\,\frac{(80-20)}{2}$$

$$= 3000\text{kN-m}$$

EXERCISES

8.1 A suspension cable has a span of 160m and central dip of 16m. It carries a load of 5kN/m of horizontal length. Calculate the maximum and minimum tension in the cable. Find the horizontal and vertical forces in each pier under the following conditions,

(i) If the cable passes over frictionless rollers on the top of the pier.

(ii) If the cable is firmly clamped to saddles on frictionless rollers on the top of piers.

In each case, the back stay is inclined at 30° with the horizontal.

Ans : T_{max} = 1077.33kN, T_{min} = 1000kN-m,

(i) V = 938.517kN; H = 67.262kN

(ii) V = 977.350kN, H=0;

8.2 A cable of span ℓ has its ends at the height of h_1 and h_2 above the lowest point of cable. It carries a uniformly distributed load of W per unit run of the span. Show that the horizontal reaction at each end is given by

$$H = \frac{W\ell^2}{2\left(\sqrt{h_1}+\sqrt{h_2}\right)^2}$$

If $W = 10$ kN/m, $\ell = 100$m, $h_1 = 8$m and $h_2 = 15$m, determine the horizontal and vertical reactions at each support.

8.3 The cables of a suspension bridge have a span of 100 m and a central dip of 10 m. Each cable is stiffened by a girder hinged at two ends and at mid-span. The dead load is a uniformly distributed load of 10 kN/m for the entire span. The live load is uniformly distributed load of length 20 m and of intensity 30 kN/m. Determine the maximum tension in the cable when the live load is placed on the left half of the span with the leading edge at the central hinge. Determine the bending moment and shear force in the girder at 25 m from left support.

Ans : $T_{max} = 2638.731$kN, $M_{max} = -37.5$kN-m, $F = 60$kN

8.4 A cable is suspended from two towers A and B at 120m apart. The supports are at the same level. The cenral dip is 10m. The cable is stiffened by three hinged girder hinged at the supports and at mid-span. Draw the influence line diagram for bending moment at the left quarter span section of the girder and find the maximum +ve and –ve bending moment at that section due to the passing of a concentrated load of intensity 50kN on the girders.

Ans : $M_{max} = 562.5$kN, $-M_{max} = 375$kN-m

8.5 A suspension cable, 100m span and 15m dip is stiffened with a three hinged girder. If a concentrated load of intensity 100kN crosses the span, determine the maximum tension in the cable. Also determine the greatest bending moment and shear force in the stiffening girder. State the position of the load in the above cases.

Ans : $T_{max} = 174.005$kN; $M_{max} = 962.25$kN-m at 21.13m distance and $F_{max} = 56.25$kN at 37.5m distance

8.6 A suspension cable, 100m span and 12m dip is stiffened with a two hinged girder. The girder carries a dead load of 10kN/m over entire span and a concentrated load of 800kN at 40m from left support. Determine the maximum tension in the cable and shear force and bending moment at a section 30m from left support.

Ans : $T_{max} = 2079.814$kN, $M = 6000$kN-m; $F = 320$kN

INTRODUCTION TO ANALYSIS OF INDETERMINATE STRUCTURES

9

9.1 INTRODUCTION

The structures which cannot be analysed using only the equations of statics (equilibrium equations) are called statically indeterminate structures. For example, consider the propped cantilever shown in Fig. 9.1(a). There are only three equations of equilibrium whereas the unknown reaction components are four in number as shown in Fig. 9.1(b). Thus, in this case, the equations of equilibrium are not sufficient for the analysis of provided cantiliver. Hence this is an indeterminate structure. Indeterminate structures are also called *Hyper Static* structures. Analysis of such structures is possible by imposing the compatibility conditions of deformations. In this chapter, the term *Degree of Indeterminacy* is defined and guidelines are given to find it. Various methods of analysing indeterminate structures are also discussed.

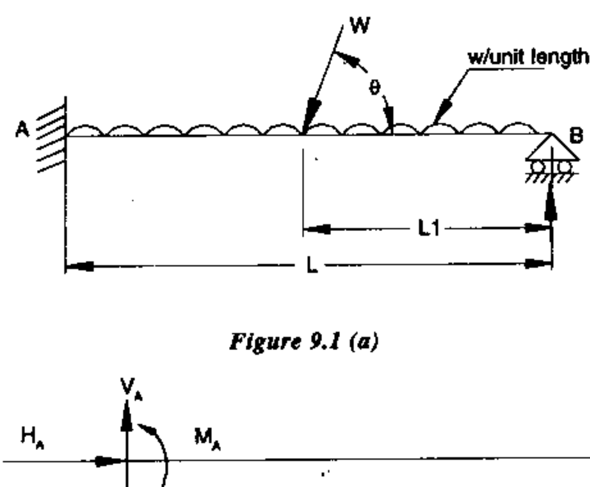


Figure 9.1 (a)

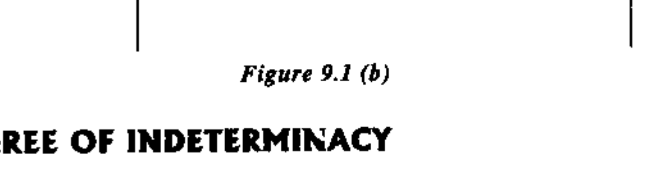


Figure 9.1 (b)

9.2 DEGREE OF INDETERMINACY

In the propped cantilever shown in Fig.9.1, the number of unknowns are four while the number of independent static equilibrium equations are three. Hence we need one additional equation to analyse the beam. The degree of indeterminancy of this propped cantilever is said to be one. Thus, the degree of indeterminacy may be defined as additional equations required to determine the unknowns and it may be represented mathematically as

Degree of indeterminacy

= No.of unknowns – No.of independant static equilibrium equations 9.1

9.3 DETERMINING DEGREE OF INDETERMINACY

In this article, the method of determining the degree of indeterminacy of the following structures is presented for

a. Beams
b. Rigidly-jointed frames
c. Pin-jointed frames.

9.3.1 Beams

To find the number of unknowns in the beams, the following guidelines may be used. Unknown reaction components at

a. roller support — 1
b. hinged support — 2
c. fixed support — 3

Hence total number of unknowns are found.

In the entire beam structure, there are only three independent equations. However, if the internal hinges are provided, there will be an additional equilibrium equation (moment at hinge= 0) for each hinge. Using these guidelines, the number of independent static equilibrium equations are determined. Then, using eqn. 9.1, the degree of indeterminacy is found.

The figures 9.2 (a) to (f) shows the degree of indeterminacy for various beams.

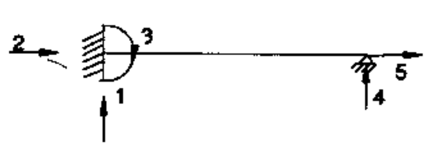


Figure 9.2 (a) $N = 5 - 3 = 2$

Figure 9.2 (b) $N = 6 - 3 = 3$

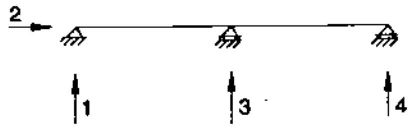


Figure 9.2 (c) $N = 4 - 3 = 1$

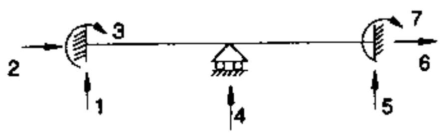


Figure 9.2 (d) $N = 7 - 3 = 4$

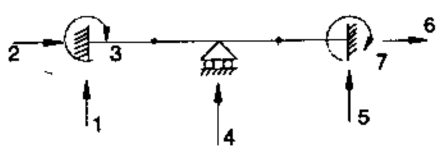


Figure 9.2 (e) $N = 7 - (3+2) = 2$

9.3.2 Rigid-Jointed Frames

(a) Plane Frames In case of single storey frames, the reaction components and independant static equilibrium equation may be found using the guidelines given for the beams. Hence the degree of indeterminacy may be found. This is illustrated with cases shown in Fig.9.3.

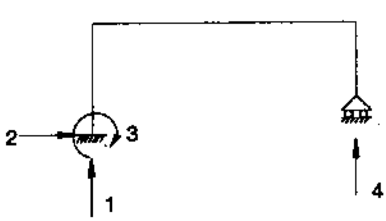


Figure 9.3 (a) $N = 4 - 3 = 1$

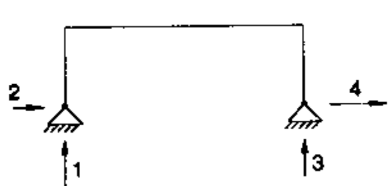


Figure 9.3 (b) $4 - 3 = 1$

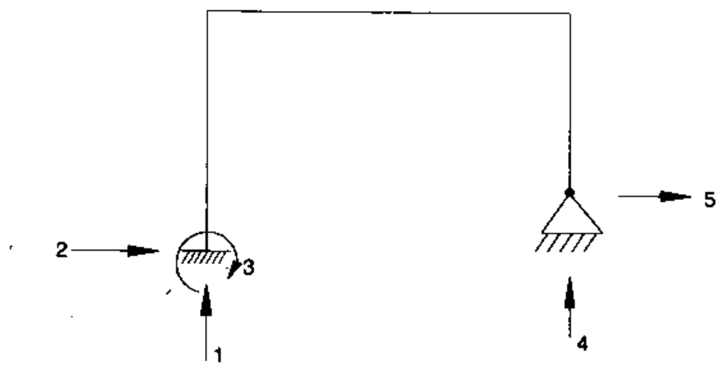


Figure 9.3 (c) N = 5 − 3 = 2

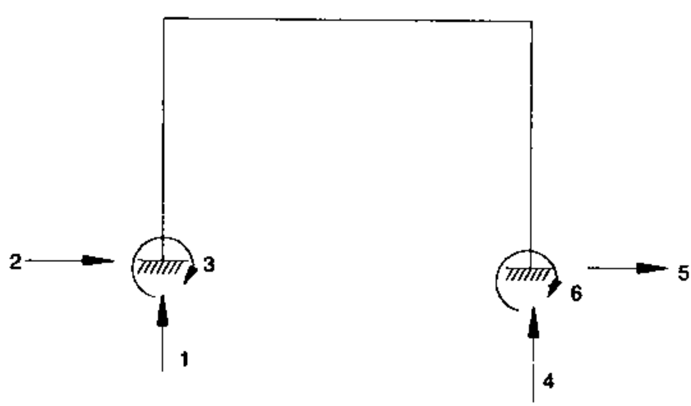


Figure 9.3 (d) N = 6 − 3 = 3

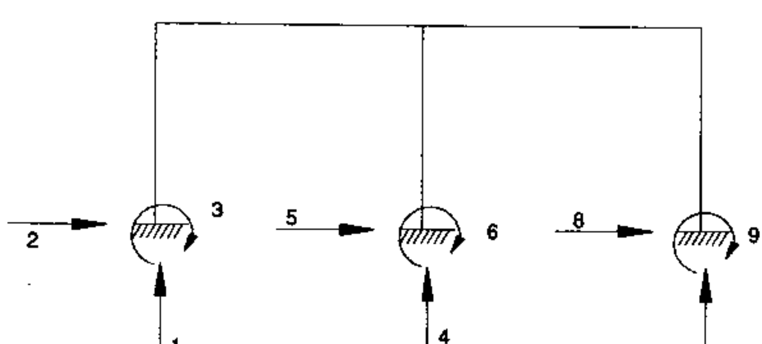


Figure 9.3 (e) N = 9 − 3 = 6

However, in case of multistoreyed frames, there will be internal indeterminacies also. For example, consider the two storeyed frames shown in Fig.9.4. Even if all the reaction components are found some how, it is not possible to find the bending moment and shear forces at any point in the second storey beams and columns.

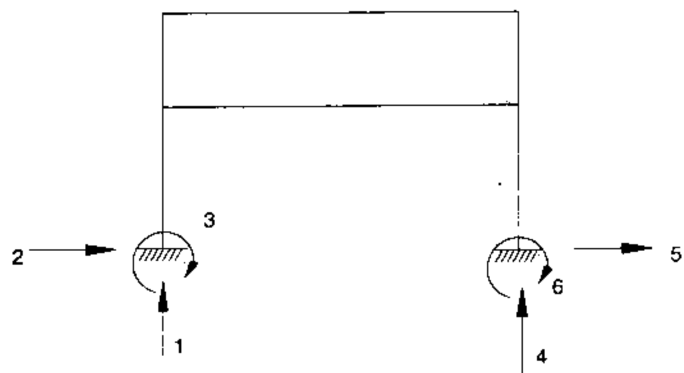


Figure 9.4

Hence the degree of indeterminacy is more than three $(6 - 3 = 3)$. To find the total degree of indeterminacy in such frames, imagine several cuts in the frames, such that if the forces at every cut and reaction components are known, it is possible to find the bending moment and shear force at any point by using equations of static equilibrium. In such a cut frame, the number of unknowns is equal to number of reaction components plus the three unknowns at every cut. The number of static equilibrium equations available are three for each portion. Hence the total number of independent static equilibrium equations are known. Using the equation 9.1, the degree of indeterminancy is found. Fig.9.5 illustrates this the method for two storeyed single bay frame. From this example, it is clear that, any number of cuts may be made to get in determinate portions.

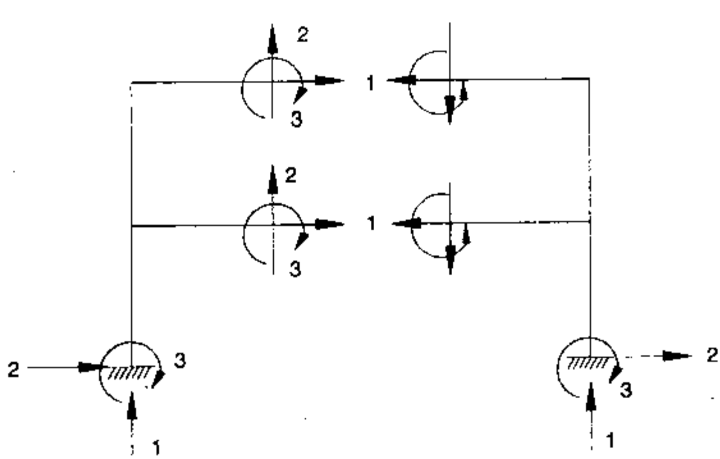


Figure 9.5 (a) $N = 3 \times 4 - 3 \times 2 = 6$

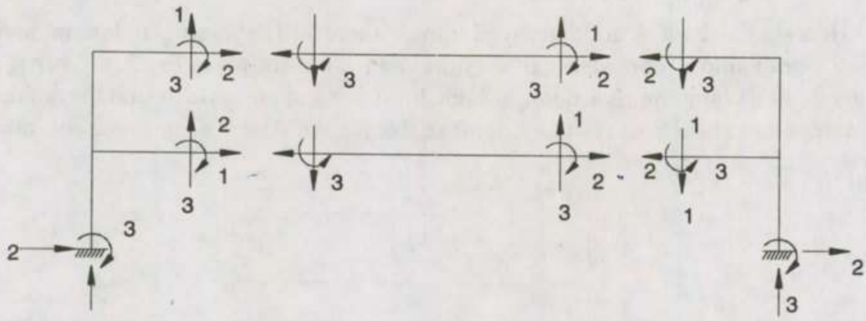


Figure 9.5 (b) $N = 3 \times 6 - 3 \times 4 = 6$

Fig. 9.6 illustrates the problem of finding the degree of indeterminacy in a multistorey frame.

In this, number of reaction components = 8

Number of cuts to make determinate portions = 6

∴ Number of unknowns at cuts = 3 × 6 = 18

∴ Total number of unknowns = 8 + 18 = 26.

Number of determinate portions = 3

∴ Number of independant equilibrium equations = 3 × 3 = 9

∴ Degree of indeterminacy N = 26 – 9 = 17

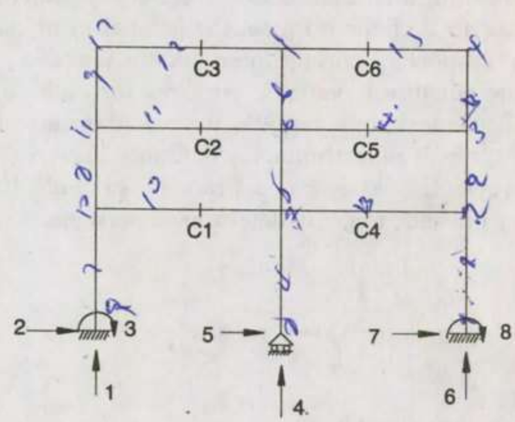


Figure 9.6 N = (8 + 3 × 6) – 3 × 3 = 17

Space frames Procedure to be followed in these cases is the same as that for the multi-storey frames. It is to be noted that, at any point there are six component of forces (three forces + three moments) in the space frames. The reaction components at supports are at:

a. Roller end — 1
b. Hinged end — 3
c. Fixed end — 6

26 - 9
17

These are shown in Fig.9.7

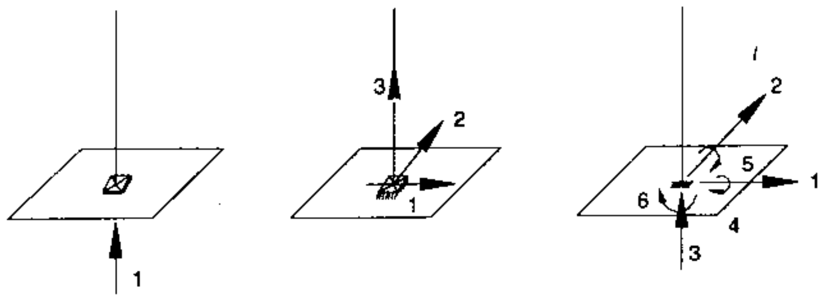


Figure 9.7

Now, consider the rigid-jointed space frame shown in Fig.9.8.

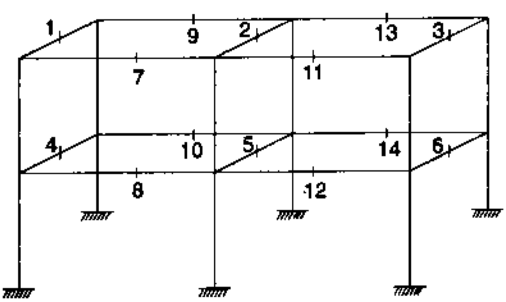


Figure 9.8 $N = 6 \times 6 + 6 \times 14 - 6 \times 6 = 84$

No. of reaction components = $6 \times 6 = 36$
No. of cuts made to get determinate portions = 14
No. of unknown at cuts = $14 \times 6 = 84$
No. of portions obtained = 6
No. of independent equations = $6 \times 6 = 36$
$\therefore$ Degree of indeterminacy = $36 + 84 - 36 = 84$

9.3.3 Pin-Jointed Frames

In these cases, first reaction components are identified. There are only three independent equilibrium equations. The difference in the number of unknown reaction components and the number of independent equilibrium equations give the external indeterminacy. For example, in the truss shown in Fig. 9.9.

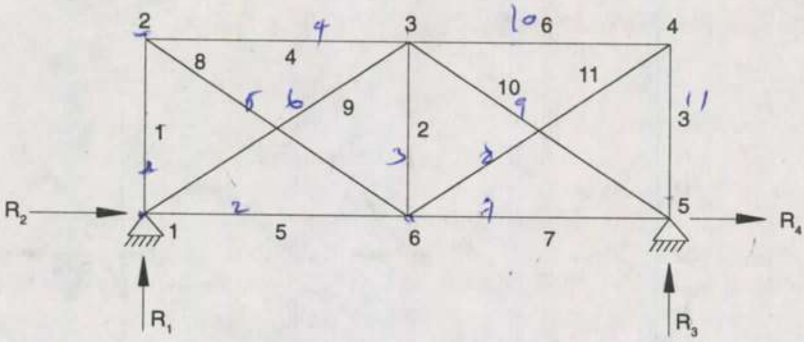


Figure 9.9

Degree of external indeterminacy

$$n_e = 4 - 3 = 1$$

Just by finding the reaction components, forces in all the members of this truss cannot be found. This truss is over-stiff i.e., it contains the number of members more than required for the stable frames. The number of excess members provided than the required stable configuration constitutes the degree of internal indeterminaney. The number of members required for a stable configuration is given by

$$= 2j - 3$$

where j is the number of joints.

∴ the number of unknown forces in the members is equal to the number of members. Let the number of members be m. Then, a degree of internal indeterminacy

$$n_i = m - (2j - 3) \qquad 9.2$$

Total degree of indeterminacy = degree of external indeterminacy + degree of internal indeterminacy

$$n = n_e + n_i \qquad 9.2$$

For the problem shown in Fig.9.9,

$$n_e = 4 - 3 = 1$$

$$n_i = 11 - (6 \times 2 - 3) = 2$$

$$\therefore \quad n = n_e + n_i = 3$$

It may be noted that the equation 9.2 is a necessary condition but not a sufficient condition. It is possible to satisfy the above condition by making a portion over-stiff and a portion deficient.

9.4 METHODS OF ANALYSIS

Presently, there are several methods available for the analysis of indeterminate structures. The first method to develop was *Consistent Deformation Method.* In this, redundant forces are identified. By removing restraint in the direction of redundant forces, released structure (which is a determinate structure) is obtained. In this released structure, displacements are obtained in the direction of the redundant forces. Then the displacement due to each redundant forces are obtained and the conditions of displacement compatibility are imposed to get additional equations. Solution for these equations gives the values of redundant forces. Then the released structure subjected to these known forces give the forces and moments in the structure. This method is suitable when the number of unknowns are one or two. When the number of unknowns become more, it is a lengthy method.

Next method to develop was *Slope Deflection Method.* In this case, basic unknowns are displacements. The equations are formed in terms of these unknown displacements. Then the equations are solved to get the unknown displacements. This method is also suitable when the unknown displacements are very few.

Solution of simultaneous equations was considered as a time consuming process. Hence there was a pause in its popularity.

In 1930, Hardy Cross popularised *Moment Distribution Method* which avoids the solutions of simultaneous equations. It involves an iterative procedure which converges satisfactorily in four to five iterations. In this method also, basic unknowns are displacements but, instead of finding them directly as in the case of slope deflection method, individual joints are allowed to rotate in successive steps until the compatibility condition is satisfied. This method became popular since the problems with higher degrees of indeterminancy could be handled in the days when the solution of simultaneous equations was considered to be a tough job. Gasper Kani of Germany gave another distribution procedure in which instead of distributing entire moment in successive steps, only the rotation contributions are distributed. This is again a method in which basic unknowns are displacements which are not found directly but they are allowed to change in successive steps till the compatibility is obtained. Even today, it is a popular method with design engineers who are not yet exposed to computer methods for the analvsis.

In 1930, Prof. Hardy Cross presented *Column Analogy Method* for the analysis of redundant beams and frames. It makes use of analogy between the equations in the beams/frames analysis and the analysis for eccentrically loaded columns. The problem with the degree of indeterminacy upto a maximum of only three are suitable. The method is ideally suided for the analysis of beams, portal frames and arches with fixed ends.

Now a days, solution of simultaneous equations of any order are possible with the computers. Hence, again the attention has gone to earlier methods of consistent deformation and slope deflection methods. These methods are systematically developed in the matrix forms and are called *Matrix Method of Structural Analyses.* Consistent deformation method has given rise to *Flexibility (force) Matrix Method* and the slope deflection method has given rise to *Stiffness (displacements) Matrix*

Method. Once a program is developed, the user has to give various data to get the solution. With the popularity of computers, these methods are also becoming popular. With the fast growing technology in computers now a days, the analysis of frames of any number of storeys and bays is possible.

Finite Element Analysis is the latest method for the analysis. With this, not only the structures with prismatic members but also the structures with two or three dimensional members can be analysed.

In this book consistent deformation method is explained for the analysis of pin-jointed frames, beams and rigid-jointed frames.

ANALYSIS OF PIN-CONNECTED INDETERMINATE FRAMES BY CONSISTENT DEFORMATION METHOD

10

10.1 FRAMES WITH EXTERNAL INDETERMINACY

Consider the truss shown in Fig. 10.1 which has hinged supports at both ends. As discussed earlier, this truss is a structure having one degree of external indeterminacy. One of the reaction component is identified as the redundant force, say horizontal force R at B. Then, restraint in that direction is to be removed and the horizontal force R is treated as an additional unknown force acting on that structure. This type of structure is shown in Fig. 10.2. The condition for consistency is that the truss of Fig. 10.2 should have zero horizontal displacement to represent the truss shown in Fig. 10.1. Since the truss in Fig. 10.2 is a determinate truss, its displacement can be evaluated and the consistency condition can be imposed on it. For finding the displacements, any one method discussed in Chapter IV may be used. Since the unit load method is ideally suited for finding the displacements in pin-connected frames (trusses) this method is employed.

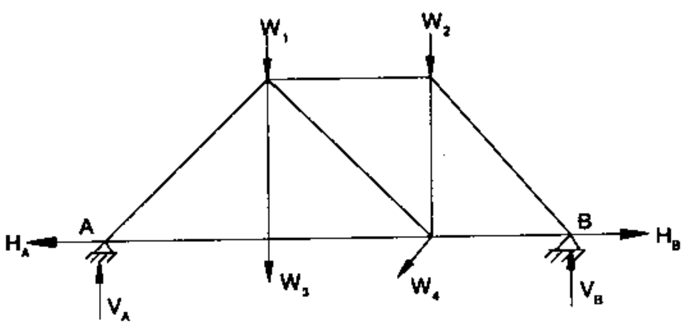


Figure 10.1

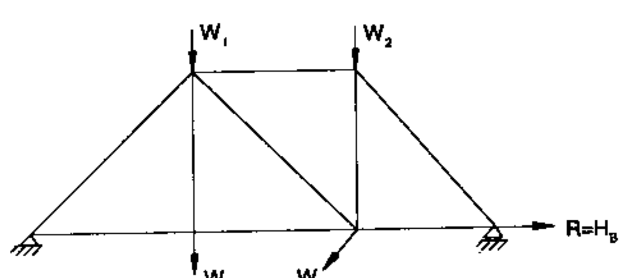


Figure 10.2

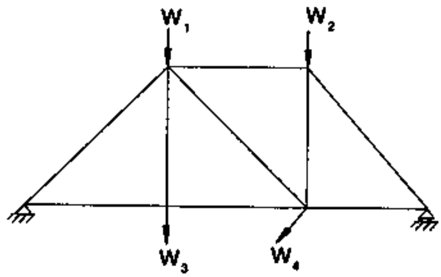


Figure 10.3 (a)

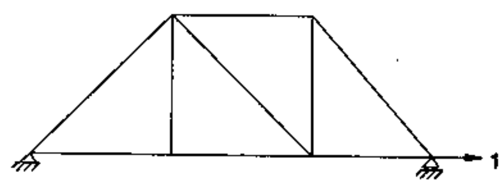


Figure 10.3 (b)

Now, the total displacement in the truss shown in Fig.10.2 may be split into two parts: One due to the given loadings and the other due to the redundant force R. Referring to Fig.10.3, let the forces in the members of the truss, due to given loadings be P_i and that due to unit load at B in the direction of R be k_i. Then, according to the unit load method, the horizontal displacement of B due to the given loadings is given by

$$\frac{\sum PkL}{AE} \qquad 10.1$$

Due to the unit load in the direction of R, let the force in the i^{th} member be k_i. Then, force in the i^{th} member due to force R will be equal to Rk_i.

∴ Horizontal displacement due to R (according to eqn. 4.1)

$$= \sum Rk\frac{kL}{AE}$$

$$= R\sum\frac{k^2L}{AE} \qquad 10.2$$

∴ The displacement of B,

$$= \sum\frac{PkL}{AE} + R\sum\frac{k^2L}{AE}$$

But, according to the consistency condition, the horizontal displacement is equal to zero.

∴
$$\sum\frac{PkL}{AE} + R\sum\frac{k^2L}{AE} = 0$$

or $$R = -\frac{\sum \frac{PkL}{AE}}{\sum \frac{k^2L}{AE}} \qquad 10.3$$

Thus, the analysis involves finding the forces in the determinate structure due to given loads and due to the unit load in the direction of the redundant force. Equation 10.3 is used to determine the value of the redundant force. Once the redundant force is found, the forces in the members can be assembled as

$$P_1 + Rk_1,\ P_2 + Rk_2,\ \ldots,\ P_4 + Rk_4 \qquad 10.4$$

The above procedure may be followed for frames having an internal redundancy problem. In this case, the force in one of the members may be treated as the redundant force.

Example 10.1 Determine the force in the members of the truss shown in Fig.10.4(a). The cross-sectional area of vertical and horizontal members is 4000mm^2 and that of the diagonals is 6000mm^2.

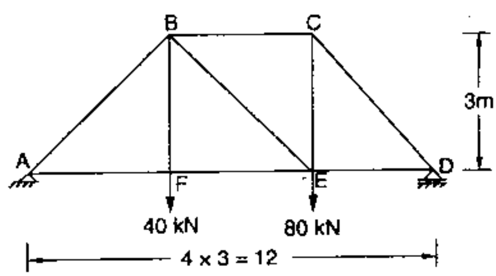


Figure 10.4 (a)

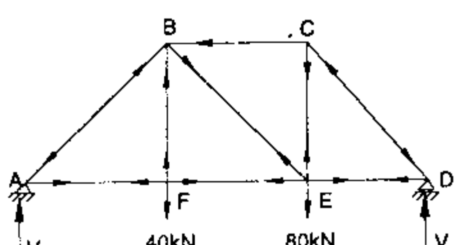


Figure 10.4 (b) *P-forces*

Solution

The horizontal reaction at D is taken as a redundant force 'R'. Hence, restraint in this direction is released and basic determinate structure is obtained as shown in Fig.10.4(b). Now the forces in this structure due to the given loadings (P-forces) and the unit force in the direction of R (k-forces) are to be found.

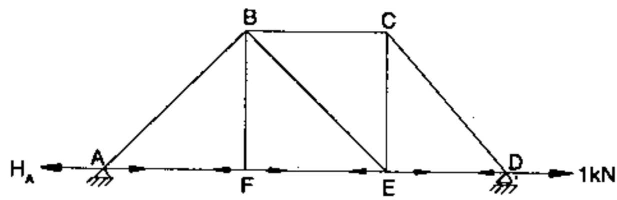


Figure 10.4 (c) K-forces

P-forces:

$\Sigma M_D = \rightarrow$

$$V_A \times 12 = 40 \times 8 + 80 \times 4$$

$$V_A = 53.33\text{kN}$$

$$\therefore \quad V_D = 40 + 80 - 53.33 = 66.67\text{kN}$$

The inclined members make an angle 'θ' with the horizontal where

$$\sin\theta = \frac{3}{5} = 0.6 \text{ ; and } \cos\theta = 0.8$$

At Joint A,

$$P_{AB} \sin\theta = 53.33 \text{ or } P_{AB} = \frac{53.33}{0.6} = 88.88 \text{ kN (comp)}$$

$$P_{AF} = P_{AB} \cos\theta = 88.8 \times 0.8 = 71.10 \text{ kN (tensile)}$$

At Joint F,

$\Sigma V = 0 \rightarrow \quad P_{FB} = 40$ kN (tensile)

$\Sigma H = 0 \rightarrow \quad P_{FE} = 71.10$ kN (tensile)

At Joint B,

$\Sigma V = 0 \rightarrow \quad P_{BE} \sin\theta = 88.89 \sin\theta - 40$

$$P_{BE} = \frac{13.33}{0.6} = 22.22\text{kN (tensile)}$$

$\Sigma H = 0 \rightarrow \quad P_{BC} = 88.89 \times \cos\theta + 22.22 \cos\theta$

$$= 88.89 \text{ kN (comp)}$$

At Joint D,

$\Sigma V = 0 \rightarrow \quad P_{DC} \sin\theta = V_D = 66.67$

$\therefore \quad P_{DC} = 111.12$ kN (comp)

$\Sigma H = 0 \rightarrow \quad P_{DE} = 111.12 \cos\theta = 88.89$ kN (tensile)

At Joint C,

$\Sigma V = 0 \rightarrow \quad P_{CE} = P_{CD} \sin\theta = 111.12 \times 0.6$

$$= 66.67 \text{ kN (tensile)}$$

The above values are tabulated in Table 10.1.

k-forces:

The reaction at joint A is a horizontal force of 1 kN.

$$\Sigma V = 0 \text{ at A and D gives } P_{AB} = P_{DC} = 0$$

Then, the equilibrium of joints C and B gives the force in other members (except bottom chord) as zero.

The horizontal equilibrium equations at A, F, E and D show that all of the bottom chord members are subjected to unit tensile force. These 'k' forces and the corresponding calculations are also shown in Table 10.1.

Now, the horizontal deflection of joint D is given by

$$\sum \frac{PkL}{AE} + R\sum \frac{k^2L}{AE} = 0$$

or

$$R = -\frac{\sum \frac{PkL}{AE}}{\sum \frac{k^2L}{AE}} = -\frac{\sum \frac{PkL}{A}}{\sum \frac{k^2L}{A}}, \text{ since E is consta}$$

Substituting values from the table, we get the value of R

$$= -\frac{231.11}{3.0} = -77.04$$

$\therefore$ Final forces in the members are given by $S = P + Rk$ as shown in the table.

Table 10.1

Member	*Length (mm)*	*Area (mm^2)*	*P-forces (kN)*	*k-forces (kN)*	$\frac{PkL}{A}$	$\frac{k^2L}{A}$	*S*
AB	5000	6000	– 88.89	0.0	0.0	0.0	– 88.89
BC	4000	4000	– 88.89	0.0	0	0.0	– 88.89
CD	5000	6000	– 111.12	0.0	0.0	0.0	–111.12
AF	4000	4000	71.11	1.0	71.11	1.0	– 5.93
FE	4000	4000	71.11	1.0	71.11	1.0	– 5.93
ED	4000	4000	88.89	1.0	88.89	1.0	11.85
BF	3000	4000	40.00	0.0	0.0	0.0	40.0
CE	3000	4000	66.67	0.0	0.0	0.0	66.67
BE	5000	6000	22.22	0.0	0.0	0.0	22.22
				Σ	231.11	3.0	

Example 10.2 Analyse the truss shown in Fig. 10.5(a) by consistent deformation method. Assume that the cross-sectional area of all members are same.

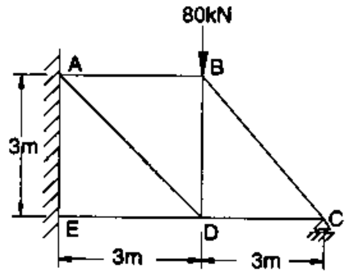


Figure 10.5 (a)

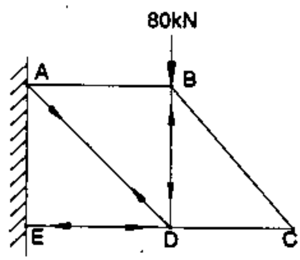


Figure 10.5 (b) *P-forces*

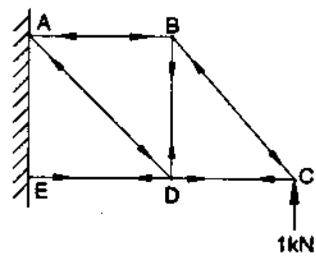


Figure 10.5 (c) *K-forces*

Solution

The vertical reaction at C is treated as a redundant force. Hence the basic determinate structure with the given loading is as shown in Fig.10.5(b) and the unit force applied for computing vertical displacement is as shown in Fig.10.5(c).

P-forces

The inclined members of the truss make an angle of 45° with the horizontal. The equilibrium of joint C is given (refer to Fig.10.5(b)) as follows

$$P_{CB} = P_{CD} = 0$$

At Joint B,

$$\Sigma v = 0 \quad \rightarrow \quad P_{BD} = 80 \text{ kN (comp)}$$

$$\Sigma H = 0 \quad \rightarrow \quad P_{BA} = 0$$

At Joint D,

$$\Sigma V = 0 \rightarrow P_{DA} \sin 45° = 80$$

$$P_{DA} = 80\sqrt{2} \text{ kN (tensile)}$$

$$\Sigma H = 0 \rightarrow P_{DE} = 80\sqrt{2} \times \frac{1}{\sqrt{2}} = 80 \text{ kN (comp)}$$

K-forces: Referring to Fig.10.5.(C)

At Joint C,

$$P_{CB} \sin 45° = 1$$

$$P_{CB} = \sqrt{2} \text{ kN (comp)}$$

$$P_{CD} = P_{CB} \cos 45° = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1 \text{ kN (tensile)}$$

At Joint B,

$$\Sigma V = 0 \rightarrow P_{BD} = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1 \text{ kN (tensile)}$$

$$\Sigma H = 0 \rightarrow P_{BA} = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1 \text{ kN (comp)}$$

At Joint D,

$$P_{DA} \sin 45° = 1 \quad \text{or } P_{DA} = \sqrt{2} \text{ (comp)}$$

$$P_{DE} = \sqrt{2} \cos 45° = 1 \text{ (tensile)}$$

Table 10.2 presents the above values with tension in positive and compression in negative. The values $\sum \frac{PkL}{AE}$ and $\sum \frac{k^2L}{AE}$ are calculated as follows.

Table 10.2

Member	*Length (mm)*	*P-forces (kN)*	*k-forces (kN)*	*PkL*	*k²L*	*s = P + Rk*
AB	3000	0.0	–1	0	3000	–40
BC	3000√2	0.0	–√2	0	6000√2	–56.569
CD	3000	0.0	1	0	3000	–40
DE	3000	–80	1	–240000	3000	–40
BD	3000	–80	1	–240000	3000	–40
AD	3000√2	–80√2	–√2	–480000√2	6000√2	56.569
			Σ	–1158822.5	28970.56	

$$R = \frac{-\sum \frac{PkL}{AE}}{\sum \frac{k^2L}{AE}} = \frac{-\sum PkL}{\sum k^2L}, \text{ since AE is same for all the members}$$

$$= \frac{11588822.5}{28970.56} = 40.00$$

∴ The final force S for the various members are as given in Table 10.2.

Example 10.3 Find the forces in the members of the truss shown in Fig.10.6(a). The cross-sectional area and Young's Modulus of all the members are the same.

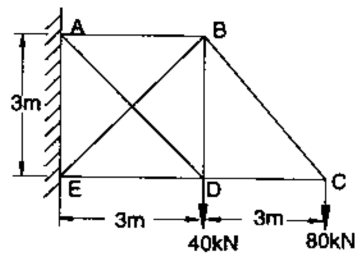


Figure 10.6 (a)

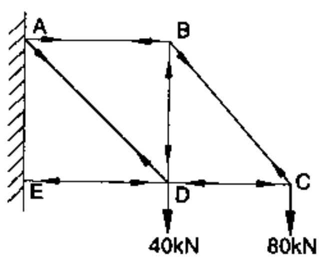


Figure 10.6 (b) P-forces

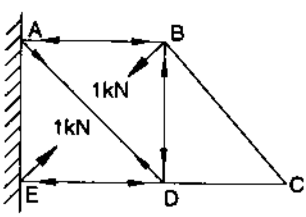


Figure 10.5 (c) K-forces

Solution

In this truss, there is internal indeterminacy of one degree. Force in the member BE is taken as the redundant force R. The basic determinate structure with the given loading is shown in Fig.10.6(b) and with the unit load in the direction of the redundant force is shown in Fig.10.6(c). The displacement in this direction should be evaluated to impose the consistency condition.

P-forces:

The Inclined members make 45° with the horizontal.

At Joint C,

$\Sigma V = 0 \rightarrow P_{BC} \sin 45° = 80$

$\therefore \quad P_{BC} = 80\sqrt{2}$ kN (tensile)

$\Sigma H = 0 \rightarrow \quad P_{CD} = 80\sqrt{2} \cos 45° = 80$ kN (comp)

At Joint B,

$\Sigma V = 0 \rightarrow P_{BD} = 80\sqrt{2} \sin 45° = 80$ kN (comp)

$\Sigma H = 0 \rightarrow P_{BA} = 80\sqrt{2} \cos 45° = 80$ kN (tensile)

At Joint D,

$\Sigma V = 0 \rightarrow P_{DA} \sin 45° = 40 + 80$

$\therefore \quad P_{DA} = 120\sqrt{2}$ kN (tensile)

$\Sigma H = 0 \rightarrow \quad P_{DE} = 120\sqrt{2} \cos 45° + P_{DC}$

$= 120 + 80 = 200$ kN (comp)

K–forces:

From the equilibrium at joint C, we get $P_{CB} = P_{CD} = 0$

At Joint B,

$$\Sigma V = 0 \rightarrow \quad P_{BD} = 1 \sin 45° = \frac{1}{\sqrt{2}} \text{ (comp)}$$

$$\Sigma H = 0 \rightarrow \quad P_{BA} = \frac{1}{\sqrt{2}} \text{ (comp)}$$

At Joint D,

$$\Sigma V = \rightarrow \quad P_{DA} \sin 45° = \frac{1}{\sqrt{2}} \quad \text{or } P_{DA} = 1 \text{ kN (tensile)}$$

$$\Sigma H = \rightarrow \quad P_{DE} = 1 \times \cos 45° = \frac{1}{\sqrt{2}} \text{ kN (comp)}$$

Table 10.3 is prepared with tensile forces in positive and compressive forces in negative.

Table 10.3

Member	*Length (mm)*	*p-forces (kN)*	*k-forces (kN)*	*PkL*	*k²L*	*S=P+Rk (kN)*
AB	3000	80	$-\frac{1}{\sqrt{2}}$	$-\frac{240000}{\sqrt{2}}$	1500	142.31
BC	$3000\sqrt{2}$	$80\sqrt{2}$	0.0	0.0	0.0	$80\sqrt{2}$
CD	3000	–80	0.0	0.0	0.0	–80.000
DE	3000	–200	$-\frac{1}{\sqrt{2}}$	$\frac{600000}{\sqrt{2}}$	1500	–137.69
BD	3000	–80	$-\frac{1}{\sqrt{2}}$	$\frac{240000}{\sqrt{2}}$	1500	–17.69
AD	$3000\sqrt{2}$	$120\sqrt{2}$	1	720000	$3000\sqrt{2}$	81.58
BE*	$3000\sqrt{2}$	...	1	...	$3000\sqrt{2}$	–88.12
			Σ	1144264.07	12985.28	

Note: While calculating displacement due to force R, the extension of the redundant member should also be considered.

The total displacement of the truss in the direction of BE = 0

i.e.,
$$\sum \frac{PkL}{AE} + R \sum \frac{k^2L}{AE} = 0$$

or
$$R = -\frac{\sum \frac{PkL}{AE}}{\sum \frac{k^2L}{AE}} = \frac{-\Sigma PkL}{\Sigma k^2 L},$$

Since AE is the same for all the members

$$R = -\frac{1144264.07}{12985.20} = -88.12 \text{ kN}$$

The final forces S are tabulated.

Example 10.4 The frame ABCDEF shown in Fig.10.7(a) has a regular hexagon shape and is subjected to 60 kN vertical downward loads at A and D. All the members are of the same material and have the same cross-sectional area. Determine the forces in all the members.

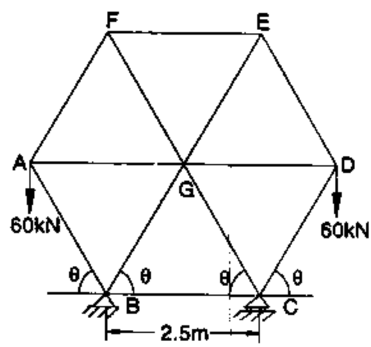


Figure 10.7 (a) *Even frame*

Solution

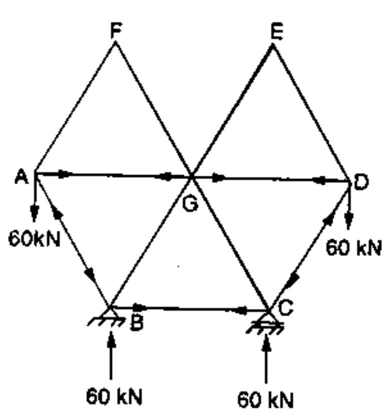


Figure 10.7 (b) *P-forces*

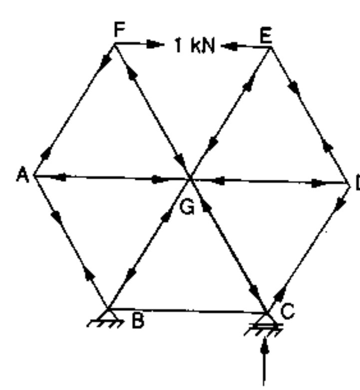


Figure 10.7 (c) *k-forces*

Since the shape of the frame is a regular hexagon, all inclined members make an angle of 60° with the horizontal. Let the force in the member EF be taken as the redundant force R. The basic determinate structure is obtained by dropping this member as shown in Fig. 10.7(b). Forces in this frame due to the given loading (p-forces) and the unit load in the direction of the redundant forces (k-forces) are calculated as follows.

P-forces:

Equilibrium of joints F and E, give

$$P_{FA} = P_{FG} = P_{EG} = P_{ED} = 0$$

At Joint A,

$$\Sigma V = 0 \rightarrow \quad P_{AB} \sin 60° = 60$$

$$\text{or} \quad P_{AB} = 69.28 \text{ kN (comp)}$$

$$\Sigma H = 0 \rightarrow \quad P_{AG} = 69.28 \cos 60° = 34.64 \text{ kN (tensile)}$$

At Joint B,

Due to symmetry, the reaction at B is 60 kN in the vertical upward direction.

$$\Sigma V = 0 \rightarrow \quad P_{BG} \sin 60° + 69.28 \sin 60° = 60$$

$$P_{BG} = 0$$

$$\Sigma H = 0 \rightarrow \quad P_{BC} = 69.28 \cos 60° = 34.64 \text{ kN (comp)}$$

Making use of symmetry, the forces in the other members are noted as shown in Table 10.4.

k-forces:

At Joint F,

$$\Sigma V = 0 \rightarrow \quad P_{FA} = P_{FG}$$

$$\Sigma H = 0 \rightarrow \quad P_{FA} \cos 60° + P_{FG} \cos 60° = 1$$

$$\text{or} \quad P_{FA} = 1 \text{ kN (tensile)}$$

$$\therefore \quad P_{FG} = 1 \text{ kN (comp)}$$

At Joint A,

$$\Sigma V = 0 \rightarrow \quad P_{AB} \sin 60° = 1 \sin 60°$$

$$\text{or} \quad P_{AB} = 1 \text{ kN (tensile)}$$

$$\Sigma H = 0 \rightarrow \quad P_{EG} = 1 \cos 60° + 1 \cos 60° = 1 \text{ kN (comp)}$$

At Joint B,

Vertical reaction at B = 0

$$\Sigma V = 0 \rightarrow \quad P_{BG} = 1 \text{ kN (comp)}$$

$$\Sigma H = 0 \rightarrow \quad P_{BC}\ 1 \text{ kN (tensile)}$$

Using symmetry, the forces in the other members are noted down (see Table 10.4) with a positive sign for tensile force and a negative sign for compressive force. Now, the force R in the redundant member EF is given by

$$R = -\frac{\sum \frac{PkL}{AE}}{\sum \frac{k^2L}{AE}} = \frac{\sum Pk}{\sum k^2},$$

since A, E and L are same for all the members.

$$R = \frac{-(-242.48)}{12} = 20.21 \text{ kN}$$

Forces in all the members are calculated and tabulated under S in Table 10.4.

Table 10.4

Member	*P-force (kN)*	*k-force (kN)*	*Pk*	K^2	*S=P+Rk (kN)*
AB	–69.28	1.0	–69.28	1.0	–49.07
BC	–34.64	1.0	34.64	1.0	–14.43
CD	–69.28	1.0	–69.28	1.0	–49.07
DE	0.0	1.0	0.0	1.0	20.21
EF	0.0	1.0	0.0	1.0	20.21
FA	0.0	1.0	0.0	1.0	20.21
AG	34.64	–1.0	0.0	1.0	14.43
BG	0.0	–1.0	0.0	1.0	–20.21
CG	0.0	–1.0	0.0	1.0	–20.21
DG	34.64	–1.0	–34.64	1.0	–14.43
EG	0.0	–1.0	0.0	1.0	–20.21
FG	0.0	–1.0	0.0	1.0	–20.21
		Σ	–242.48	12.0	

Example 10.5 Three wires AO, BO and CO support a load of 40 kN as shown in Fig.10.8a. The cross-sectional areas of all the wires is the same. Determine the forces in all the wires.

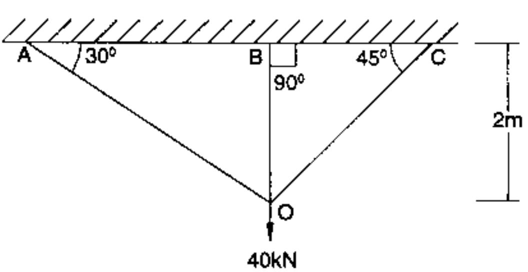


Figure 10.8 (a)

Solution

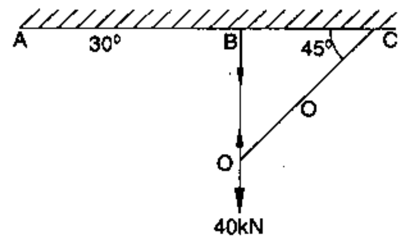


Figure 10.8 (b) P-forces

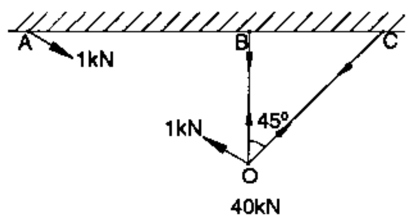


Figure 10.8 (c) k-forces

Force in member AO is treated as the redundant force R. The basic determinate structure with the given loadings is shown in Fig.10.8(b) and with unit force in the direction of the redundant force in Fig.10.8(c).

Now,

P-forces :

At Joint O,

$$\Sigma H = 0 \rightarrow \ P_{CO} = 0$$
$$\Sigma V = 0 \rightarrow \ P_{BO} = 40 \text{ kN (tensile)}$$

k–forces :

At Joint O,

$$\Sigma H = 0 \rightarrow \quad P_{CO} \sin 45° = 1 \sin 60°$$
$$P_{CO} = 1.22 \text{ kN (tensile)}$$
$$\Sigma V = 0 \rightarrow \quad P_{BO} = 1 \cos 60 + \dots \cos 60° + 1.22 \cos 45°$$
$$= 1.363 \text{ kN (comp)}$$

These values are entered in Table 10.5 and further calculations are carried out.

Table 10.5

Member	*Length in mm*	*P forces in kN*	*k forces in kN*	*PkL*	*K^2L*	*S=P+Rk*
AO	4000	...	1	0	4000	9.14
BO	2000	40	−1.363	−109040	3715.54	27.54
CO	2828.43	0	1.22	0	4209.84	11.15
				−109040	11925.38	

$$\therefore\ R = -\frac{\dfrac{\Sigma PkL}{AE}}{\dfrac{\Sigma k^2L}{AE}} = -\frac{\Sigma PkL}{\Sigma k^2L}\text{, since AE is the same for all members}$$

$$= -\frac{-109040}{11925.38} = 9.143\text{kN}$$

The values of final forces are tabulated in Table 10.5.

Example 10.6 Analyse the pin-connected plane frame shown in Fig.10.9(a). The cross-sectional area of each member is 2000mm². Take E equal to 200 kN/mm².

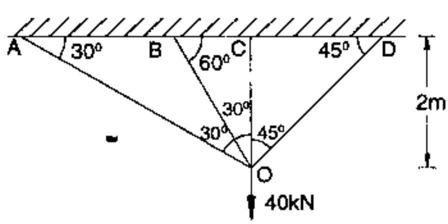


Figure 10.9 (a)

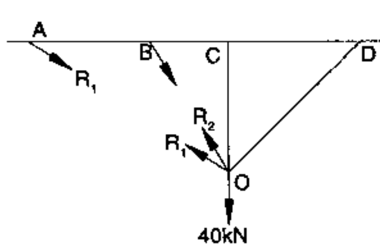


Figure 10.9 (b)

Solution

The degree of indeterminacy of the pin-jointed plane frame is two. Hence, the forces in the two members should be treated as redundant. The basic determinate structure with redundant forces is shown in Fig.10.9(b).

The forces in the basic determinate structure due to given loading (P-forces) are

$$P_{DO} = 0$$

$$P_{CO} = 40 \text{ kN (tensile)}$$

The forces due to unit load in the direction of R_1 (k_1 forces) are

$$P_{DO} \sin 45° = 1 \sin 60°$$

$$P_{DO} = 1.22 \text{ kN (tensile)}$$

$$P_{CO} = 1.22 \cos 45° + 1 \cos 60° = 1.363 \text{ kN (comp)}$$

The forces in the determinate structure due to unit forces in the direction of R_2 (k_2 forces) are given by

$$\Sigma H = 0 \rightarrow P_{DO} \sin 45° = 1 \sin 30°$$

$$P_{DO} = 0.707 \text{ (tensile)}$$

$$\Sigma V = 0 \rightarrow P_{CO} = 0.707 \cos 45° + 1 \cos 30°$$

$$P_{CO} = 1.366 \text{ (comp)}$$

The values of the forces P, k_1 and k_2 are entered in Table 10.6.

Table 10.6

Member	*Length (mm)*	*P (kN)*	*k_1 (kN)*	*k_2 (kN)*	*Pk_1L*	*Pk_2L*	*k_1^2L*	*k_2^2L*	*k_1k_2L*	*S*
AD	4000.00	—	1	—	—	—	4000	—	0	2.673
BO	2309.40	—	—	1.000	—	—	—	2309.40	0	12.521
CO	2000.00	40	−1.363	−1.366	−109040	−109280	3715.54	3731.91	3723.72	19.253
DO	2828.43	0	1.22	0.707	0	0	4209.84	1413.79	2439.63	12.113
				Σ	−109040	−109280	11925.38	7455.10	6163.35	

Now, the displacement in the direction of OA due to

i. P forces : $\sum \frac{Pk_1L}{AE}$

ii. R_1 forces : $\sum \frac{(R_1k_1)k_1L}{AE} = \sum \frac{R_1k_1^2L}{AE}$

iii. R_2 forces : $\sum \frac{(R_2k_2)k_1L}{AE} = \sum \frac{R_2k_1k_2L}{AE}$

∴ According to the consistency condition,

$$\sum \frac{Pk_1L}{AE} + \sum \frac{R_1k_1^2L}{AE} + \sum \frac{R_2k_1k_2L}{AE} = 0$$

Since AE is the same for all members

$$\sum Pk_1L + \sum R_1K_1^2L + \sum R_2k_1k_2L = 0$$

$$-109040 + 11925.388\ R_1 + 6163.35\ R_2 = 0$$

$$R_1 + 0.5168\ R_2 = 9.1435$$

Similarly, the consistency condition in the direction of OB gives

$$\sum Pk_2L + \sum R_1k_1k_2L + \sum R_2k_2^2L = 0$$

$$-109820 + 6163.35\ R_1 + 7455.1\ R_2 = 0$$

or $$R_1 + 1.2096\ R_2 = 17.818$$

From equations (a) and (b), we get

$$0.6928\quad R_2 = 8.6745$$

$$R_2 = 12.521\ \text{kN}$$

$$R_1 = 2.673$$

Now the forces in the members can be calculated using the expression,

$$S = P + k_1R_1 + k_2R_2$$

whose values are tabulated in the last column of Table 10.6.

Example 10.7 Find the forces in all the members of the truss shown in Fig.10.10(a). The cross-sectional areas of all the members are the same.

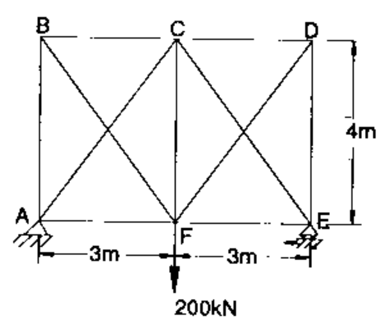


Figure 10.10 (a)

Solution

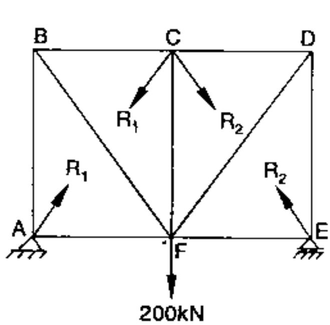


Figure 10.10 (b)

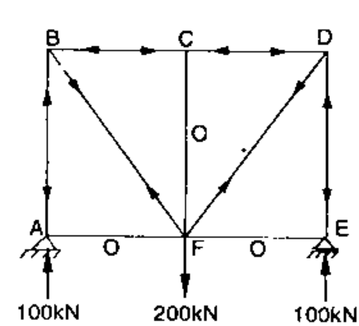


Figure 10.10 (c)

The truss has two degrees of internal indeterminacy. The basic determinate structure is obtained by removing the diagonals AC and EC as shown in the figure. The forces in AC and EC are taken as redundant forces R_1 and R_2 Ref.Fig.10.10(b)

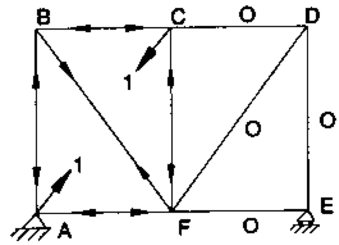


Figure 10.10 (b)

Analysis of truss for the given loads (P forces) :

Due to symmetry, the vertical reactions at the supports are equal to 100kN.

At Joint A,

$$\Sigma V = 0 \rightarrow \quad P_{AB} = 100 \text{ kN (comp)}$$
$$\Sigma H = 0 \rightarrow \quad P_{AF} = 0$$

At Joint B,

$$\Sigma V = 0 \rightarrow \quad P_{BF} \times 0.8 = 100$$
$$\text{or} \quad P_{BF} = 125 \text{ kN (tensile)}$$
$$\Sigma H = 0 \rightarrow \quad P_{BC} = 125 \times 0.6 = 75 \text{ kN (comp)}$$

Using symmetry, forces in the other half can be written down. Forces in the members due to unit force in the direction of R_1 (k_1 forces) can be calcualted as follows.

Reaction at supports = 0

∴ Forces in the members ED, EF, DF and DC = 0

At Joint A,

$$\Sigma V = 0 \rightarrow \quad P_{AB} = 1 \times 0.8 = 0.8 \text{ (comp)}$$
$$\Sigma H = 0 \rightarrow \quad P_{AF} = 1 \times 0.6 = 0.6 \text{ (comp)}$$

At Joint B,

$$\Sigma V = 0 \rightarrow \quad P_{BF} \times 0.8 = 0.8 \text{ or } P_{BF} = 1 \text{ (tensile)}$$
$$\Sigma H = 0 \rightarrow \quad P_{BC} = 1 \times 0.6 = 0.6 \text{ (comp)}$$

At Joint C,

$$\Sigma V = 0 \rightarrow \quad P_{CF} = 1 \times 0.8 = 0.8 \text{ (comp)}$$

The forces in the members due to unit force in the direction of R_2(k_2- forces) is calculated below.

Using the symmetry with k_1-forces, these values are tabulated in Table 10.7. The consistency requirement in the direction R_1 is

$$\sum \frac{Pk_1L}{AE} + \sum \frac{R_1k_1k_1L}{AE} + \sum \frac{R_2k_2k_1L}{AE} = 0$$

$1080000 + R_1 \times 17280 + R_2 \times 2560 = 0$, since A and E are the same

$$421.875 + 6.75\, R_1 + R_2 = 0 \qquad \text{(a)}$$

Consistency required in the direction of the R_2 is

$$\sum \frac{Pk_2L}{AE} + \sum \frac{R_1k_1k_2L}{AE} + \sum \frac{R_2k_2k_2L}{AE} = 0$$

Since A and E are the same for all the members, we get

$$1080000 + R_1 \times 2560 + R_2 \times 17280 = 0$$

$$62.5 + 0.1481\ R_1 + R_2 = 0 \qquad \text{....(b)}$$

Solving for R_1 from equation (a) and (b), we get

$$R_1 = -54.435 \text{ kN}$$

From equation (a), $\quad R_2 = -54.435$ kN

The member forces are calculated using the expression

$$S = P + R_1\, k_1 + R_2\, k_2$$

and are tabulated.

Table 10.7

Member	*Length in mm*	*P in kN*	*k in kN*	*k_2 in kN*	*Pk L*	*Pk_2 L*	*$k^2_1 L$*	*$k^2_2 L$*	*k_1k_2L*	*S*
AB	4000	−100	−0.8	0	320000	0	2560	0	0	−56.452
BC	3000	−75	−0.6	0	135000	0	1080	0	0	−42.399
CD	3000	−75	0	−0.6	0	135000	0	1080	0	−42.399
DE	4000	−100	0	−0.8	0	320000	0	2560	0	−56.452
EF	3000	0	0	−0.6	0	0	0	1080	0	32.661
AF	3000	0	−0.6	0	0	0	1080	0	0	32.661
AC	5000	—	1.0	—	0	0	5000	0	0	−54.435
BF	5000	125	1.0	0	625000	0	5000	0	0	70.565
EC	5000	—	—	1.0	0	0	0	5000	0	−54.435
FD	5000	125	0	1.0	0	625000	0	5000	0	7.565
FC	4000	0	−0.8	−0.8	0	0	2560	2560	2560	87096
				Σ	1080000	1080000	17280	17280	2560	

10.2 STRESSES DUE TO ERRORS IN LENGTH OF A MEMBER

At the time of fabricating the redundant frames, if a member is found to be slightly shorter or longer (lacks in fit) the member is forced in position. This causes forces in the member as well as in all other members. To see the method of finding such forces, let us consider the redundant frame ABCD shown in Fig. 10.11(a), in which the member AC is short by $\delta\ell$. When this member is forced into position, it is subjected to tensile force. Let the force developed in AC be X and P_i be the force developed in the i[th] member of the determinate structure shown in Fig. 10.11(b). Due to this force X, the member AB expands and the joints A and C move close. Hence, by consistency condition, Movement of joints of determinate structure + Expansion of member AC = $\delta \ell$

$$\sum \frac{PkL}{AE} + \frac{XL'}{A'E} = \delta \ell$$

where summation Σ is for the members of the determinate structure (frame without AC),

k – forces in members due to unit force in the direction x (Fig.10.11.c)

L′ and A′ are the length and cross-sectional area of member AC, respectively.

Now,

$$P_i = k_i X$$

$$\therefore \qquad \sum \frac{PkL}{AE} = \sum \frac{k^2 XL}{A'E}$$

Force in member AC due to unit force in the direction x-forces

$$= K' = 1$$

we can write,

$$= \frac{XL'}{A'E} \text{ as } \frac{k^2 XL'}{A'E}$$

$\therefore$ Equation (i) becomes

$$\sum \frac{k^2 XL}{AE} + \frac{k^2 XL'}{A'E} = \delta \ell \qquad \text{(ii)}$$

where summation is over all the members of the determinate structure.

Equation (ii) may be looked as

$$\sum \frac{k^2 XL}{AE} = \delta \ell \qquad 10.4$$

where summation is over all the members of the redundant truss.

From equation (iii)

$$X \sum \frac{k^2 L}{AE} = \delta \ell$$

Forces developed in the other members may be obtained as

$$P_i = k_i X \qquad 10.5$$

Example 10.8 While fabricating the frame ABCD shown in Fig.10.11(a), AC was last number to be fitted. At this stage, it was found that the member AC was 1mm short of the required length. Determine the forces developed, when AC was forced into its position, given that the :

cross-sectional area of the diagonal members = 2000mm^2
cross-sectional area of the other members = 1000mm^2 and the
Young's modulus E = 200 kN/mm^2.

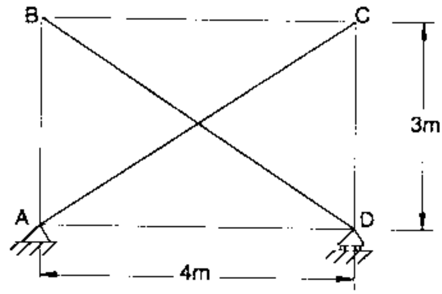


Figure 10.11 (a)

Solution

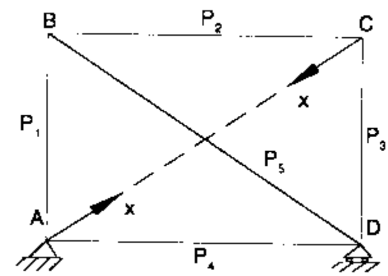


Figure 10.11 (b)

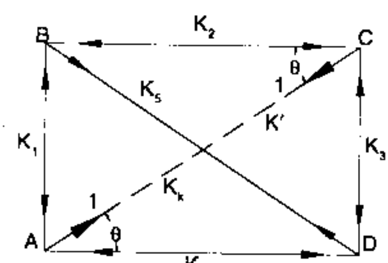


Figure 10.11 (c)

Since AC is short, it will be subjected to tensile force when forced into position. Let X be the force in AC. This force will exert forces X on joints A and C as shown in Fig.10.11(b). Due to 1kN force in the direction of X forces, the forces (k) developed are to be found.

Reactions at supports A and D are zero.

Consider the joint C,

$$K_{CB} = 1 \cos \theta = 0.8\text{kN (comp)}$$

and $$k_{CD} = 1 \sin \theta = 0.6\text{kN (comp)}$$

At Joint B,

$$k_{BD} \cos\theta = 0.8$$

$$k_{BD} = \frac{0.8}{0.8} = 1\text{kN} \quad \text{(tensile)}$$

$$k_{BA} = k_{BD} \sin\theta = 1 \times 0.6 = 0.6\text{kN (comp)}$$

At Joint A,

$$k_{AD} = 1 \cos\theta = 0.8\text{kN} \quad \text{(comp)}$$

These forces are tabulated in Table 10.8.

Table 10.8

Member	*L*	*A*	*k*	$\frac{k^2L}{A}$	$P = kX$
AB	3000	1000	–0.6	1.08	–9.772
BC	4000	1000	–0.8	2.56	–13.029
CD	3000	1000	–0.6	1.08	–9.772
AD	4000	1000	–0.8	2.56	–13.029
BD	5000	2000	1.0	2.50	16.287
AC	5000	2000	1.0	2.50	16.287
			Σ	12.28	

$$\therefore \qquad X = \frac{\delta\ell}{\sum \frac{k^2L}{AE}} = \frac{1}{\frac{1}{E}\times 12.28} = \frac{200}{12.28}$$

$$= 16.287\text{kN}$$

∴ The forces developed in the various members, due to lack of fit = P = kX. These are shown in Table 10.8.

Example 10.9 The member BE was the last to be fitted in the truss ABCDEF shown in Fig.10.12(a). While fitting, it was observed that the member was 1mm longer than the required length. Find the forces developed in all the members of

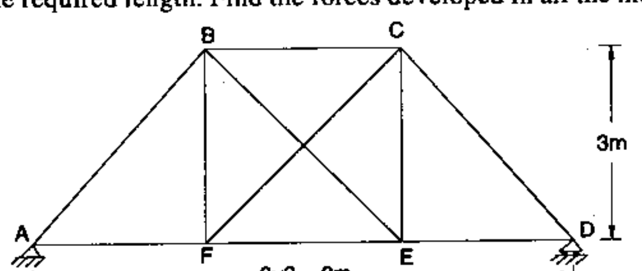


Figure 10.12 (a)

the truss due to forcing the member BE in to position . The following particulars are given : cross-sectional area of all the members is 4000mm^2 and the Young's Modulus of the material used is 200 kN/mm^2.

Solution

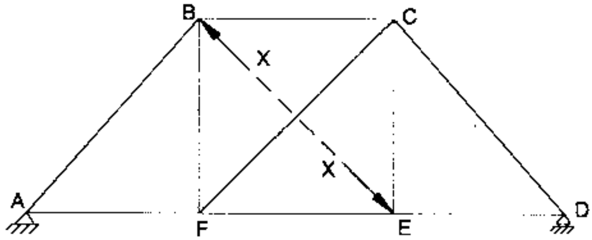


Figure 10.12 (b) P - forces

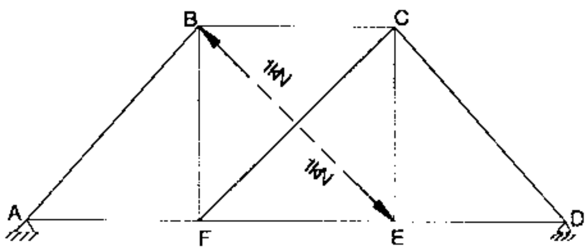


Figure 10.12 (c) K - forces

When fitted in, the member BE will be subjected to compressive forces, because it is longer than the required length. The compressive force X developed in the member acts on the joints B and E as shown in Fig.10.12(b). Let the forces developed in the ith member due to this the force (fitting) be P_i. To find these forces, we have to first find the k forces which are due to unit forces applied in the direction of the X forces (Fig.10.12(c)) in the determinate beam obtained after removing member BE.

K-forces :

No reaction develops at A and D.

$$\therefore \qquad K_{AB} = K_{AF} = K_{DC} = 0$$

At Joint B,

$$\Sigma H = 0 \rightarrow \quad P_{BC} = \frac{1}{\sqrt{2}} \text{ (tensile)}$$

$$\Sigma V = 0 \rightarrow \quad P_{BF} = \frac{1}{\sqrt{2}} \text{ (tensile)}$$

At Joint F,

$$\Sigma V = 0 \rightarrow \quad P_{FC} = 1 \text{ (comp)}$$

$$\Sigma H = 0 \rightarrow \quad P_{FC} = \frac{1}{\sqrt{2}} \text{ (tensile)}$$

At Joint C,

$$\Sigma V = 0 \rightarrow \quad P_{FC} = \frac{1}{\sqrt{2}} \text{ (tensile)}$$

With the above value Table 10.9 is prepared.

Table 10.9

Member	*L*	*k*	k^2L	$p = kX$
AB	$3000\sqrt{2}$	0	0	0
BC	3000	$\frac{1}{\sqrt{2}}$	1500	43.564
CD	$3000\sqrt{2}$	0	0	0
AF	3000	0	0	0
FE	3000	$\frac{1}{\sqrt{2}}$	1500	43.564
ED	3000	0	0	0
BF	3000	$\frac{1}{\sqrt{2}}$	0	0
CE	3000	$\frac{1}{\sqrt{2}}$	1500	43.564
BE	$3000\sqrt{2}$	–1	$3000\sqrt{2}$	61.608
FC	$3000\sqrt{2}$	–1	$3000\sqrt{2}$	61.608
		Σ	1298581	

Note: Tension is positive, while compression is negative.

$$\therefore \quad X = \frac{\delta\ell}{\sum \frac{k^2 L}{AE}} = \frac{\delta\ell}{\frac{1}{AE} \times \sum k^2 L} = \frac{\delta\ell}{\sum k^2 L} AE$$

$$= \frac{1}{12985.288} \times 4000 \times 200$$

$$= 61.608$$

∴ The forces developed in the i^{th} member is given by

$P_i = k_i X$. These values are calculated for all the members and shown in Table 10.9.

10.3 TEMPERATURE STRESSES

Changes in temperature cause a change in the length of a member. In redundant frames, a change in length of any member gives rise to forces in all the other members. Consider the redundant frame shown in Fig.10.13(a). Let the

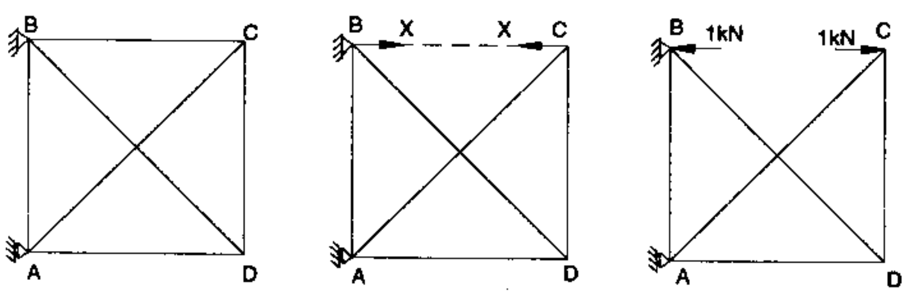


Figure 10.13

temperature of member BC rise by T. Then the free expansion of the member BC is given by $\delta = L\,\alpha\,T$, where L is the length of the member and α the co-efficient of thermal expansion. But free expansion is not possible. Hence, compressive force X develops in the member.

This causes the movement of joints B and C in the truss without member BC. The consistency condition demands a shortening of BC and the movement of joints B and C

$$= \delta = L\,\alpha\,T$$

$$\frac{XL'}{A'E} + \sum \frac{PkL}{AE} = \delta = L\,\alpha\,T$$

Where L' and A' are the length and cross-sectional area of BC respectively, k represents the forces in the truss without the member BC, due to unit load in BC and summation is over all the members of the truss without BC and P the forces induced in the members.

Now $$P_i = k_i X \qquad \text{(ii)}$$

$$\therefore \qquad \sum \frac{PkL}{AE} = \sum \frac{k^2 LX}{AE} \qquad \text{(iii)}$$

Now, the force in member BC due to unit force in the X-direction is unity.

From equations (i) (ii) and (iii) we get

$$k' = 1$$

$$\therefore \qquad \frac{XL'}{A'E} = \frac{k'^2 L' X}{A'E} \qquad \text{(iv)}$$

From equations (i), and (iii), we get

$$\frac{k'^2 L'}{A'E}\, x + \sum \frac{k^2 L}{AE}\; L\,\alpha T$$

where summation is over all the members except member BC. Equation (v) may be written as

$$X = \sum \frac{k^2 L}{AE} = L\,\alpha T$$

where summation is over all the members of redundant truss. Hence

$$X = \frac{L\alpha T}{\sum \frac{k^2 L}{AE}}$$

After finding the value of X , forces in all the other members can be found out using equation (ii)

$$P_i = k_i x \qquad \text{(ii)}$$

Note : Increase in temperature causes compression in the member.

Example 10.7 Find the forces developed in all the members of the truss shown

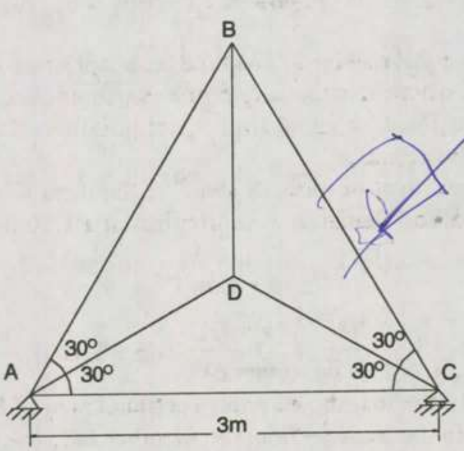


Figure 10.14(a)

in Fig.10.14(a), if the temperature of member AC goes up by 20°C. Take the co-efficient of thermal expansion $\alpha = 12 \times 10^{-6}$ /°C. Cross-sectional area of all the members is 2500 mm² and Young's Modulus is 200 kN/mm².

Solution

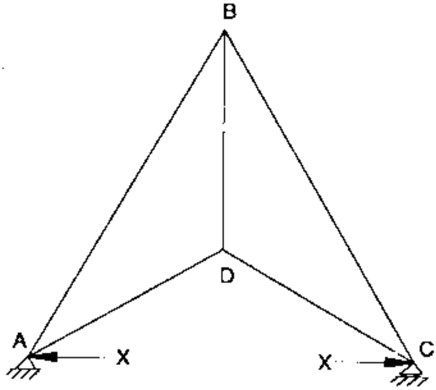


Figure 10.14 (b)

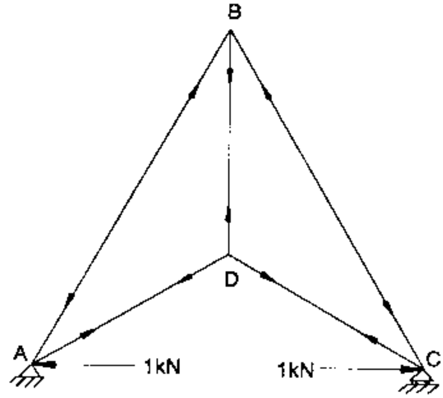


Figure 10.14 (c)

Free expansion of AC $L\alpha T$ $= 3000 \times 12 \times 10^{-6} \times 20$

$= 0.72$ mm

As free expansion is prevented, compressive force develops in the member AC. Let this force developed be X. It develops joint forces X as shown in Fig.10.14(b). Now, we need forces developed (k forces) due to unit loads acting at joints A and C in the determinate truss obtained after removing member AC as shown in Fig. 10.14(c).

In this case, reactions at A and C = 0

At Joint A,

$$\Sigma V = 0 \quad \rightarrow \quad K_{AD} \sin 30° = k_{AB} \sin 60°$$

i.e.,

$$k_{AD} \frac{1}{2} = k_{AB} \frac{\sqrt{3}}{2}$$

or

$$k_{AD} = \sqrt{3}\; k_{AB}$$

$$\Sigma H = 0 \quad \rightarrow \quad k_{AD} \cos 30° - k_{AB} \cos 60° = 1$$

From eqns.(i) and (ii), we get

$$\sqrt{3}\; k_{AB} \cos 30° - k_{AB} \cos 60° = 1$$

$$k_{AB} = 1 \text{ (comp)}$$

From (i), $k_{AD} = \sqrt{3} \times 1 = \sqrt{3}$ (tensile)

Similarly from the equilibrium conditions at joint C,

we get, $k_{CB} = 1$ (comp), $k_{CD} = \sqrt{3}$ (tensile)

At Joint B,

$\Sigma V = 0 \rightarrow$ $k_{BD} = 1 \times \cos 30° + 1 \times \cos 30°$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3} \text{ (tensile)}$$

These values are tabulated as shown in Table 10.10. Then Σk^2L is calculated.

Table 10.10

Member	*L*	*k*	*k^2L*	*P=kX*
AB	3000	−1	3000	− 14.641
BC	3000	−1	3000	− 14.641
AC	3000	−1	3000	− 14.641
AD	1732.05	$\sqrt{3}$	5196.15	25.359
BD	1732.05	$\sqrt{3}$	5196.15	25.359
CD	1732.05	$\sqrt{3}$	5196.05	25.359
		Σ	24588.45	

Now, free expansion

$$L\alpha = 300 \times 12 \times 10^{-6} \times 20$$
$$= 0.72$$

$$X = \frac{L\alpha T}{\sum \frac{k^2L}{AE}} = \frac{L\alpha T}{\frac{1}{AE} \times \Sigma k^2 L}$$

$$= L\alpha T \frac{AE}{\Sigma k^2 L}$$

$$= 0.72 \times \frac{2500 \times 200}{24588.45}$$

$$= 14.641 \text{ kN}$$

Forces in the other members are calculated using the formula P = kT. These values are listed in the last column of Table 10.10.

EXERCISES

10.1 Determine the forces in all the members of the truss shown in Fig.10.15. Take the cross-sectional areas for AD, BE and BC as 3000mm² and for all other members as 2000mm². The Young's Modulus = 200 kN/mm².

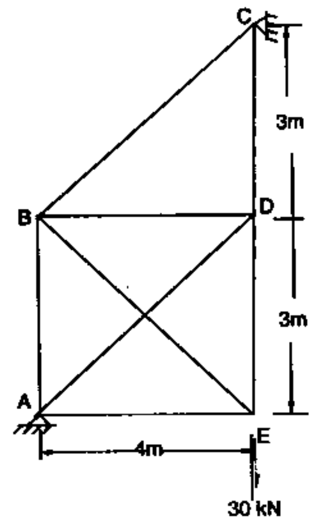


Figure 10.15

Ans : $F_{AB} = -22.505$kN $F_{AE} = -10.006$kN; $F_{AD} = -12.492$kN
$F_{BC} = -25$kN; $F_{BD} = 9.994$kN; $F_{BE} = 12.500$kN
$F_{CD} = 15$kN; $F_{DE} = 22.495$kN

10.2 The pin-jointed truss shown in Fig.10.16 is hinged at supports A and D and is subjected to a horizontal force of 60 kN at C. Determine the forces developed in all the members of the truss. Given cross-sectional areas of all members = 6000mm², Young's Modulus 200 kN/mm².

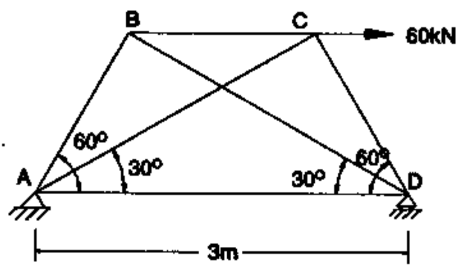


Figure 10.16

Ans : $F_{BA} = 11.37$kN; $F_{AD} = 0$; $F_{AC} = 32.26$kN;
$F_{BC} = 22.74$kN; $F_{BD} = -19.70$kN; $F_{CD} = -18.63$kN

10.3 In the truss ABCDEF shown in Fig.10.17 the member AC was fitted at the end. While fitting AC it was found that it was 0.5mm short. Find the forces developed in all the members of the truss due to forced fitting of AC. Assume cross -sectional area for all members to be 3000mm^2 and Young's Modulus to be 200kN/mm^2.

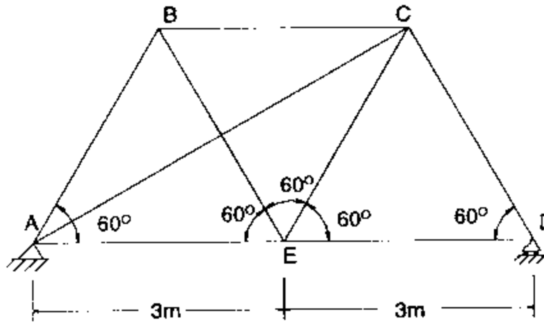


Figure 10.17

Ans : F_{BA} = – 16.99kN; F_{AC} = 29.42kN; F_{AE} = –16.99kN
F_{BC} = – 16.99kN; F_{BE} = 16.99kN; F_{CD} = 0
F_{CE} = – 16.99kN; F_{DE} = 0

10.4 Determine the forces in the members of the truss ABCDEF shown in Fig.10.18 due to rise in temperature of member BD by 30°C. Given: Cross-sectional area of all members = 5000 mm^2, Young's Modulus=200 kN/mm^2 and co-efficient of thermal expansion = 12×10^{-6}/°C.

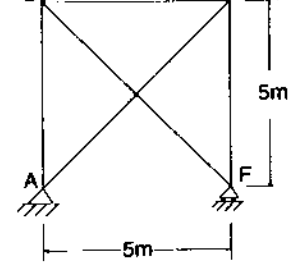


Figure 10.18

Ans : $F_{BC} = F_{BE} = F_{CD} = F_{DE}$ = – 74.56kN; $F_{BD} = F_{CE}$ = 105.44kN

ANALYSIS OF INDETERMINATE BEAMS AND RIGID FRAMES BY CONSISTENT DEFORMATION METHOD

11

11.1 INTRODUCTION

Analysis of propped cantilevers, fixed beams, continuous beams and single bay–single storey frames are discussed in this chapter. To find the deflection of basic determinate beams and frames, direct expressions, the conjugate beam method, the energy method or virtual work methods are used.

11.2 PROPPED CANTILEVERS

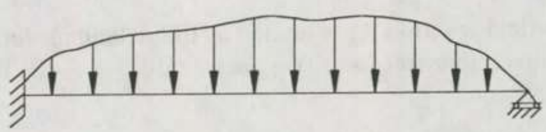


Figure 11.1 (a)

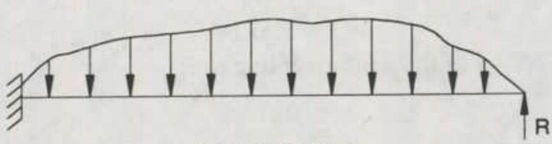


Figure 11.1 (b)

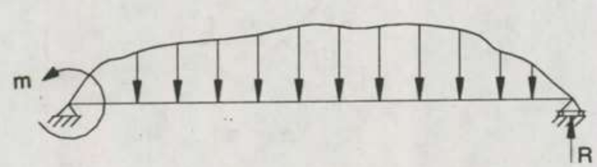


Figure 11.1 (c)

Propped cantilevers have only one degree of indeterminacy. If the reaction at the propped end is treated as a redundant force, the resulting basic determinate structure is a cantilever (refer Fig.11.1). If we treat the fixed end moment as the redundant force, the resulting basic determinate structure is a simply supported beam. Any one of the above two basic determinate structure can be used for the analysis. This is illustrated below with a number of examples.

Example 11.1 Determine the reaction components for the propped cantilever subject to uniformly distributed load as shown in Fig.11.2(a).

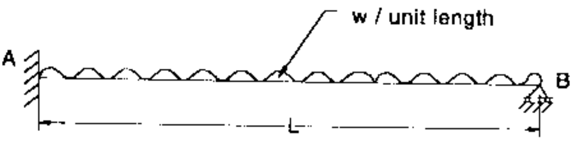


Figure 11.2 (a)

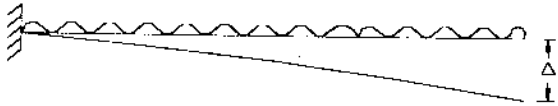


Figure 11.2 (b)

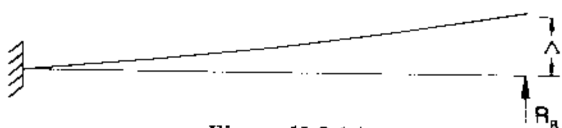


Figure 11.2 (c)

Solution Vertical reaction R_B is treated as the redundant force. By removing the restraint to this redundant force, we get a cantilever as the basic determinate structure.

Due to the given loading, the downward deflection at B is given by

$$\Delta = \frac{wL^4}{8EI} \tag{a}$$

Upward deflection of the cantilever due to R_B

$$= \frac{R_B L^3}{3EI} \tag{b}$$

Consistency condition requires the following equation to be true

$$\frac{wL^4}{8EI} = R_B \frac{L^3}{3EI}$$

$$\therefore \quad R_B = \frac{3}{8} wL$$

$$\therefore \quad R_A = wL - \frac{3}{8} wL = \frac{5}{8} wL$$

$$M_A = R_B L - wL.\frac{L}{2}$$

$$= \frac{3}{8} wL^2 - \frac{wL^2}{2} = -\frac{wL^2}{8}$$

Example 11.2 Determine the reaction components in the propped cantilever shown in Fig.11.3(a). EI is constant throughout.

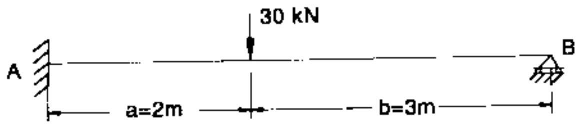


Figure 11.3 (a)

Solution

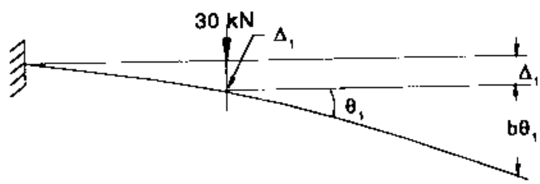


Figure 11.3 (b)

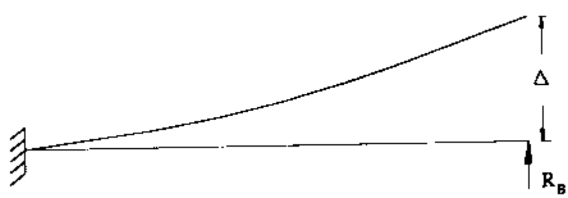


Figure 11.3 (c)

Reaction R_B at B is taken as the redundant force and hence the resulting basic determinate structure is a cantilever.

In a cantilever, deflection and rotations under the load P acting at distance 'a' from the fixed end are

$$\frac{Pa^3}{3EI} \text{ and } \frac{Pa^2}{2EI}$$

Hence in this case (refer Fig.11.2(b)),

$$\Delta_1 = \frac{30 \times 2^3}{3EI} = \frac{80}{EI} \text{ and } \theta_1 = \frac{30 \times 2^2}{2EI} = \frac{60}{EI}$$

∴ Deflection of B,

$$\Delta = \frac{80}{EI} + \frac{60}{EI} \times 3 = \frac{260}{EI}, \text{ downward}$$

Deflection at B due to R_B

$$= \frac{R_B \times L^3}{3EI} = \frac{R_B \times 5^3}{3EI}, \text{ upward}$$

From the consistency requirements,

$$\frac{260}{EI} = \frac{R_B \times 5^3}{3EI}$$

$\therefore$ $R_B = 6.24 kN$

$\therefore$ $R_A = 30 - 6.24 = 23.76$

$\therefore$ $M_A = R_B \times 5 - 30 \times 2 = 6.24 \times 5 - 60$

$= - 28.8$ kN–m

$= 20.8$ kN–m (hogging)

Example 11.3 : Determine the reaction components in the beam shown in Fig.11.4(a). EI is constant throughout.

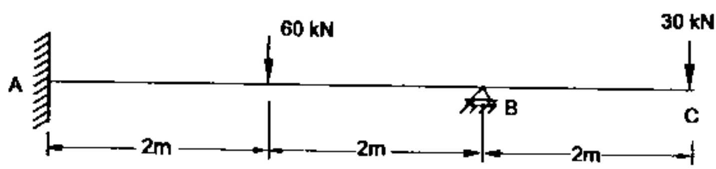


Figure 11.4 (a)

Solution

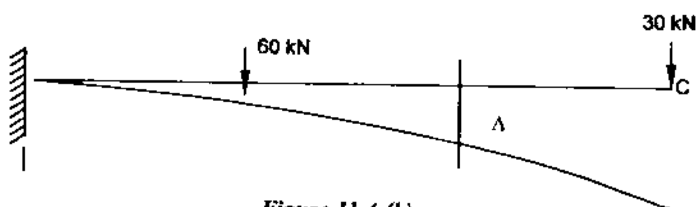


Figure 11.4 (b)

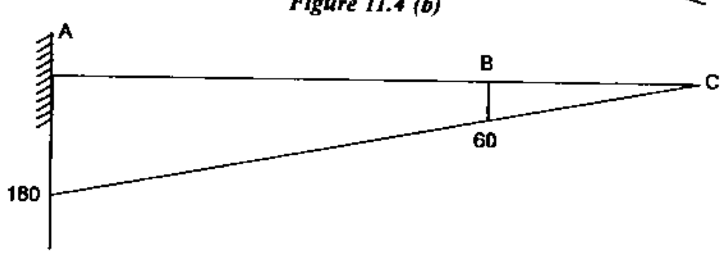


Figure 11.4 (c)

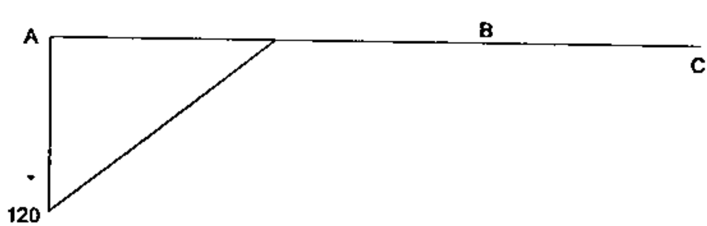


Figure 11.4 (d)

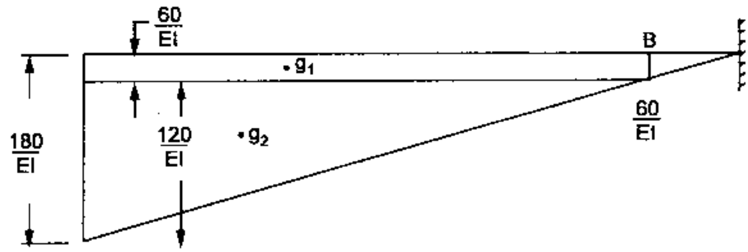


Figure 11.4 (e)

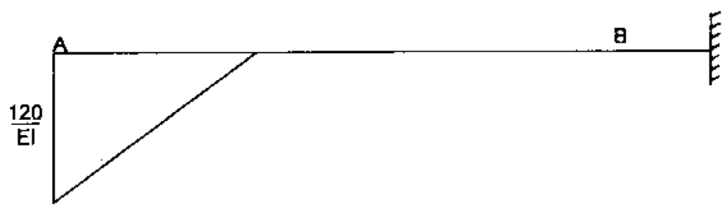


Figure 11.4 (f)

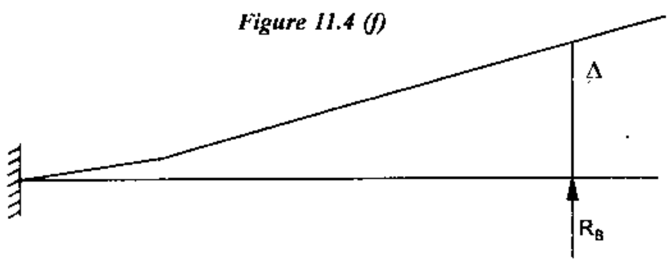


Figure 11.4 (g)

Taking the vertical reaction R_B as the redundant force, we get the basic determinate structure as the cantilever AC (Fig.11.4b). To obtain the downward deflection at B in the cantilever AC, we use conjugate beam method. Fig.11.4(c) and Fig.11.4(d) shows the bending moment diagrams on the determinate structure due to 30 kN and 60 kN loads, respectively. Fig.11.4(e) and Fig.11.4(f) show the conjugate beam with $\frac{M}{EI}$ diagrams.

Now,

Deflection at B = Bending moment at B in conjugate beam

$$= \left(\frac{60}{EI}\times 4\times 2+\frac{1}{2}\times\frac{120}{EI}\times 4\times\frac{8}{3}\right)+\frac{1}{2}\times\frac{120}{EI}\times 2\left(2+\frac{4}{3}\right)$$

$$= \frac{1}{EI}(480+640+400)$$

$$= \frac{1520}{EI} \qquad \text{(a)}$$

Vertical upward deflection in cantilever AC due to R_B (Fig.11.4 (g) is given by

$$\frac{R_B \times 4^3}{3EI} \qquad \text{(b)}$$

For consistency, the deflection at B should be zero. Hence equating (a) and (b), we get

$$\frac{R_B \times 4^3}{3EI} = \frac{1520}{EI}$$

$$R_B = 71.25 \text{ kN}$$

$\therefore$ $$V_A = 60 + 30 - 71.25 = 18.75 \text{ kN}$$

and $$M_A = 30 \times 6 - 71.25 \times 4 + 60 \times 2$$

$$= 15 \text{ kN-m}$$

Example 11.4 The cantilever shown in Fig.11.5 has a rigid support 10mm below the free end. Analyse this beam for the loading shown. EI = 150000kN-m^2

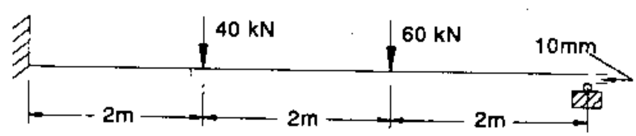


Figure 11.5 (a)

Solution

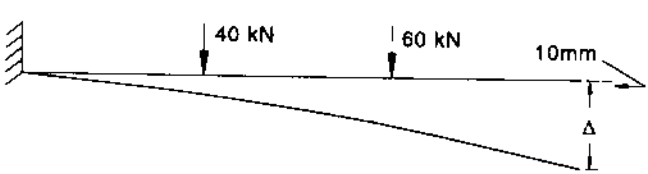


Figure 11.5 (b)

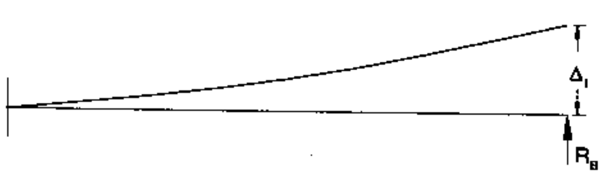


Figure 11.5 (c)

If there is no support 10mm below the free end, the deflection of the cantilever

Δ = Deflection due to 40 kN load + Deflection due to 60 kN load

$$= \left(\frac{40\times2^3}{3EI}+\frac{40\times2^2}{2EI}\times4\right)+\left(\frac{60\times4^3}{3EI}+\frac{60\times4^2}{2EI}\times2\right)$$

$$= \frac{2666.667}{EI}=\frac{2666.67}{150000}$$

$$= 0.0177778\text{m, downward}$$

Let R_B be the reaction at end B. Then, the upward deflection due to R_B is given by

$$\Delta_1 = \frac{R_B\times6^3}{3EI}=\frac{R_B\times6^3}{3\times150000}$$

From the consistency condition,

$$\Delta_1 = \Delta - 0.010$$

$$\therefore \quad \frac{R_B\times6^3}{3\times150000} = 0.0177778 - 0.010$$

$$= 0.0077778$$

$$R_B = 16.20\text{kN}$$

$$\therefore \quad V_A = 40 + 60 - 16.20 = 83.80 \text{ kN}$$

$$M_A = 16.20\times6 - 60\times4 - 40\times2$$

$$= -222.8 \text{ kN–m}$$

$$= 222.8 \text{ kN–m, hogging}$$

Example 11.5 The cantilever beam shown in Fig. 11.6(a) of span 4m is supported by a 2m long, 3mm diameter wire CB. Determine the force developed in the wire due to loading shown in the figure, if the flexural rigidity of the beam EI = 5000 kN-m² and the Young's Modulus of the wire = 200 kN/mm².

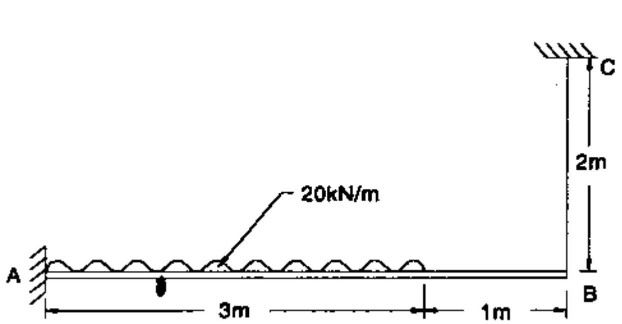


Figure 11.6 (a)

Solution

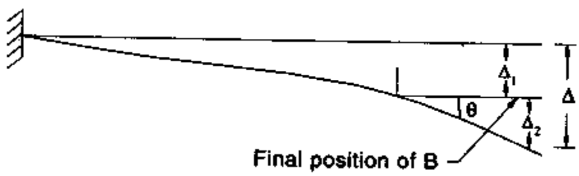


Figure 11.6 (b)

In the absence of the wire AC, let deflection of B in the cantilever AB be Δ. Then

$$\Delta = \frac{20\times 3^4}{8EI} + \frac{20\times 3^3}{6EI}\times 1$$

$$= \frac{292.5}{EI} = 0.0585\text{m}$$

Let the force developed in the wire be P kN. Then, its extension

$$\Delta_1 = \frac{PL}{AE} = \frac{P\times 2000}{\frac{\pi}{4}\times 3^2\times 200} = 1.41471 \text{ P m}$$

$$= 0.00141471 \text{ P m}$$

Due to force P, the upward deflection of the cantilever

$$\Delta_2 = \frac{P\times 4^3}{3EI} = \frac{P\times 64}{3\times 5000} = 4.26666\times 10^{-3}\text{m}$$

From the consistency condition,

$$\Delta = \Delta_1 + \Delta_2$$

$$0.0585 = 0.00141471 \text{ P} + 4.26666\times 10^{-3}$$

$$\therefore \quad \text{P} = 38.335 \text{ kN}$$

Example 11.6 The beam AB of span 3m rests over another beam BC of span 2m as shown in Fig.11.7(a). If a 60 kN load acts on beam AB at 2m from A, find the reactions at the supports A and C given that the flexural rigidity of beam AB is twice that of beam BC.

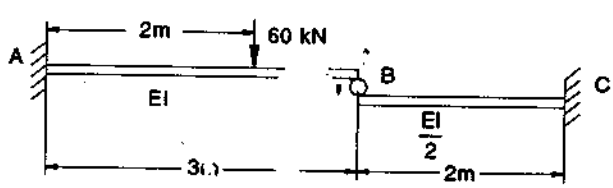


Figure 11.7 (a)

Solution

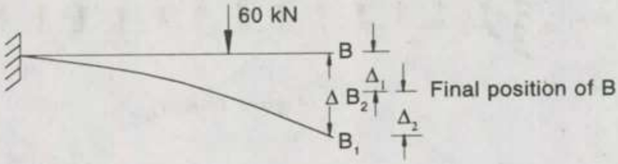


Figure 11.7 (b)

If the beam AB is not resting on BC, it will deflect by BB_1 at end B as shown in Fig.11.7b.

$$BB_1 = \Delta = \frac{60\times2^3}{3EI} + \frac{60\times2^2}{2EI}\times1$$

$$= \frac{160}{EI} + \frac{120}{EI} = \frac{280}{EI}$$

Let the reaction at B be P. This force pushes the beam AB up by B_1B_2 to bring it to the final position.

$$B_1B_2 = \Delta_2 = \frac{P\times3^3}{3EI} = \frac{9P}{EI}$$

The reactive force P on beam BC will bring down B by

$$\Delta_1 = \frac{P\times2^3}{3EI_{BC}} = \frac{8P}{3EI/2} = \frac{5.3333P}{EI}$$

From the consistency condition of deformation,

$$\Delta = \Delta_1 + \Delta_2$$

$$\frac{280}{EI} = \frac{9P}{EI} + \frac{5.3333P}{EI} = \frac{14.3333P}{EI}$$

$$\therefore \qquad P = \frac{280}{14.33333} = 19.54 \text{ kN}$$

$$V_A = 60 - 19.54 = 40.46 \text{ kN}$$

$M_A = 19.54\times3 - 60\times2 = -61.38$ kN–m $= 61.38$ kN–m, Hogging

$V_C = 19.54$ kN, $M_C = 19.54\times2 = 39.08$ kN–m, Hogging

11.3 FIXED BEAMS

Consider the fixed beam shown in Fig.11.8(a). Let the end moments developed be M_{FAB} and M_{FBA}. To find these end moments, the consistent deformation method may be used. Taking a simply supported beam as a basic determinate structure, the redundant forces are M_{FAB}.

MFAB
MFBA → redundant

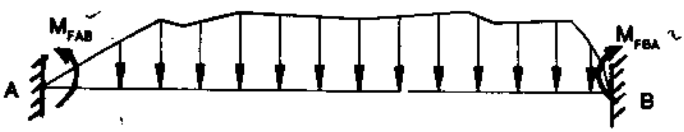


Figure 11.8 (a)

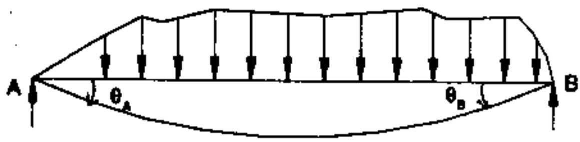


Figure 11.8 (b)

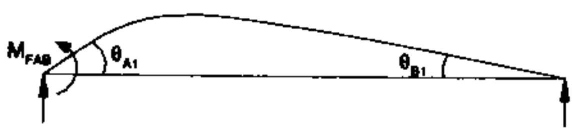


Figure 11.8 (c)

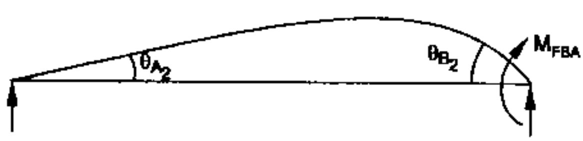


Figure 11.8 (d)

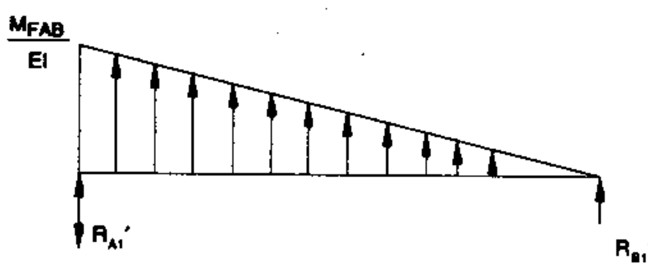


Figure 11.8 (e)

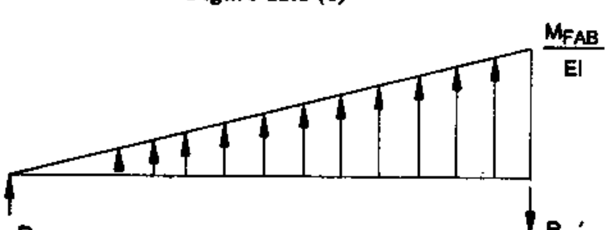


Figure 11.8 (f)

Let θ_A and θ_B be the rotations of A and B due to the given loading
θ_{A1} and θ_{B1} be ,, ,, ,, M_{FAB}
θ_{A2} and θ_{B2} be ,, ,, ,, M_{FBA}, for a simply supported beam
Then, the beam consistency condition needs,

$$\theta_A = \theta_{A1} + \theta_{A2}$$

and
$$\theta_B = \theta_{B1} + \theta_{B2}$$

From the conjugate beam method (refer Fig. 11.8 (e))
Now

$$R_{A1}' = \theta_{A1} = \frac{M_{FAB}L}{3EI} \text{ and } R_{B1}' = \theta_{B1} = \frac{M_{FBA}L}{6EI}$$

$$R_{A2}' = \theta_{A2} = \frac{M_{FBA}L}{6EI} \text{ and } R_{B2}' = \theta_{B2} = \frac{M_{FBA}L}{3EI}$$

Hence,
$$\theta_A = \frac{M_{FBA}L}{3EI} + \frac{M_{FBA}L}{6EI} = \frac{L}{6EI}(2M_{FAB} + M_{FBC}) \tag{11.1}$$

and
$$\theta_B = \frac{L}{6EI}(M_{FAB} + 2M_{FBA}) \tag{11.2}$$

For the given loading, the rotations θ_A and θ_B in the simply supported beam may be found and equations 11.1 and 11.2 be formed. Solutions to the two equations give the desired fixed end moments M_A and M_B. The procedure is illustrated below with four standard examples 11.7 to 11.10. Using these standard examples, the fixed end moments for any problem can be found. This is illustrated in examples 11.11 to 11.14.

Example 11.7 Using the consistent deformation method, find the fixed end moments developed in the fixed beam shown in Fig. 11.9(a). Draw the bending moment and shear force diagrams.

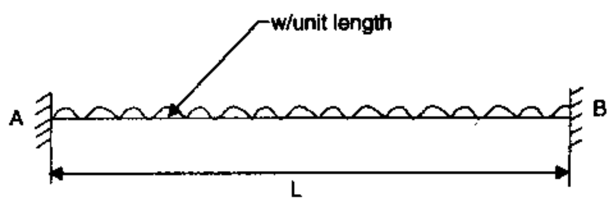


Figure 11.9 (a)

Solution

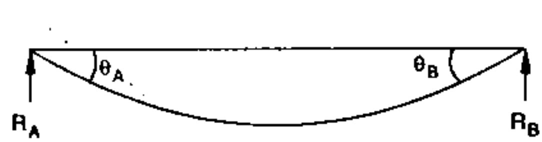


Figure 11.9 (b)

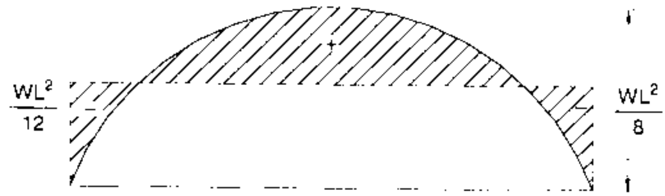


Figure 11.9 (c)

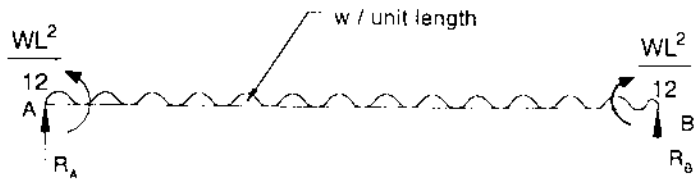


Figure 11.9 (d)

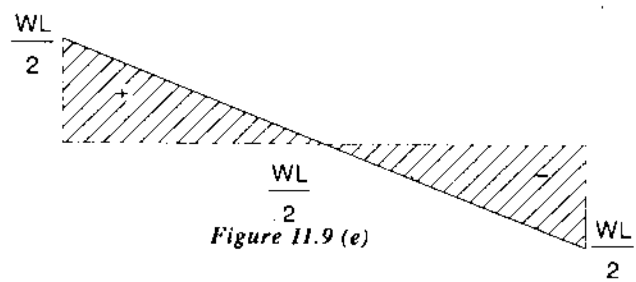


Figure 11.9 (e)

Due to symmetry rotations $\theta_A = \theta_B$ (refer Fig. 11.9(b)). Let M_{FAB} and M_{FBA} be the fixed end moments at A and B respectively. Due to symmetry they are also equal. Hence rotation at A, due to M_{FAB} and M_{FBA} is given by

$$\theta_A = M_{FAB}\frac{L}{3EI} + \frac{M_{FBA}}{6EI} \quad \text{(as equation 11.1)}$$

$$= \frac{M_{FBA}L}{6EI}(2+1) \quad \text{since } M_{FBA} = M_{FBA}$$

$$= \frac{M_{FBA}L}{2EI}$$

Due to the uniformly distributed load, rotation at B

$$= \frac{WL^3}{24EI}$$

Hence for consistency,

$$\frac{M_{FBA}L}{2EI} = \frac{WL^3}{24EI}$$

$$\therefore \qquad M_{FAB} = \frac{WL^2}{12} = M_{FBA}$$

The end moments provide tension at the top. The free moment is a parabola with maximum value of $\frac{wL^2}{8}$ at centre which gives tension at the bottom. Hence the difference of these two diagrams gives the bending moment diagram as shown in Fig.(11.9(c)). To find the reaction, consider the free body diagram of fixed beam as shown in Fig.11.9(d). Taking the moment about B, we get

$$R_A \times L - wL \times \frac{L}{2} - \frac{WL^2}{12} + \frac{WL^2}{12} = 0$$

$$\therefore \qquad R_A = \frac{WL}{2}$$

$$R_B = WL - R_A = \frac{WL}{2}$$

Hence, shear force diagram is as shown in Fig.11.9(e).

Example 11.8 A fixed beam of span L is subjected to a concentrated load W at a distance 'a' from end A as shown in Fig.11.10(a). Determine the end moments developed.

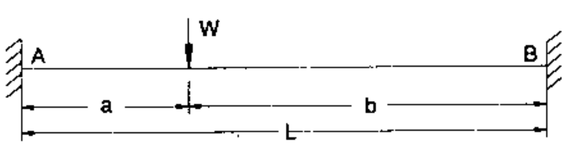


Figure 11.10 (a)

Solution

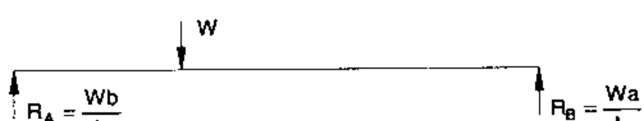


Figure 11.10 (b)

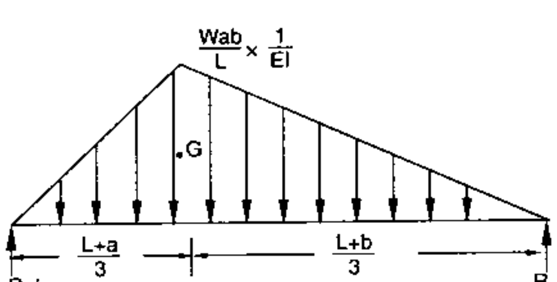


Figure 11.10 (c)

Taking a simply supported beam as the basic determinate structure, the maximum bending moment is a triangle with peak value under load as $\frac{wab}{L}$. The conjugate beam and loading on it are as shown in Fig.11.10(c). The centroid of the triangle is at a distance $\frac{L+a}{3}$ from A and $\frac{L+b}{3}$ from B. The total load on the conjugate beam is $\frac{1}{2}\;\frac{Wab}{LEI} \times L = \frac{Wab}{2EI}$

$$\therefore \qquad \theta_A = R_A' = \frac{\sum M_B}{L} = \frac{Wab}{2EI}\;\frac{L+b}{3}\;\frac{1}{L}$$

$$= \frac{Wab}{6EI}\;\frac{L+b}{L} \qquad \text{(a)}$$

Similarly,

$$\theta_B = R_B' = \frac{\sum M_A}{L} = \frac{Wab}{6EI}\;\frac{L+a}{L} \qquad \text{(b)}$$

Let M_{FAB} and M_{FBA} be the fixed end moments.

Then from equation 11.1 and 11.2,

$$\theta_A = \frac{L}{6EI}\,(2M_{FAB} + M_{FAB})$$

and

$$\theta_B = \frac{L}{6EI}\,(M_{FAB} + 2M_{FBA})$$

Substituting the values of θ_A from (a) (which is the consistency condition now) we get

$$\frac{Wab}{6EI}\;\frac{L+b}{L} = \frac{L}{6EI}\,(2M_{FAB} + M_{FBA})$$

i.e.,

$$Wab\,\frac{L+b}{L^2} = 2M_{FAB} + M_{FBA} \qquad \text{(c)}$$

From eqn.(b), we get

$$\frac{Wab}{6EI}\;\frac{L+a}{L} = \frac{L}{6EI}\,(M_{FAB} + 2M_{FBA})$$

i.e.,

$$Wab\,\frac{L+a}{L^2} = M_{FAB} + 2M_{FBA} \qquad \text{(d)}$$

Subtracting (d) from twice (c), we get

$$\frac{Wab}{L^2}\,(2L + 2b - L - a) = 3M_{FAB}$$

$$\frac{Wab}{L^2}\ (L + 2b - a) = 3M_{FAB}$$

Since L = a + b, we get

$$\frac{Wab}{L^2}\ (3b) = 3M_{FAB}$$

or

$$M_{FAB} = \frac{Wab^2}{L^2}$$

Substituting it in eqn. (c), we get

$$Wab\ \frac{L+b}{L^2} = \frac{2Wab^2}{L^2} + M_{FBA}$$

$$M_{FBA} = \frac{Wab}{L^2}(L + b - 2b)$$

$$= \frac{Wa^2b}{L^2},\ \text{since } L = a + b$$

Thus,
$$M_{FAB} = \frac{Wab^2}{L^2} \qquad 11.4$$

and
$$M_{FBA} = \frac{Wa^2b}{L^2} \qquad 11.5$$

Particular case,

If $a = b = \frac{L}{2}$,

$$M_{FAB} = M_{FBA} = \frac{\frac{WL}{2}\frac{L^2}{4}}{L^2} = \frac{WL}{8}$$

Example 11.9 To a fixed beam, an external moment M is applied at a distance 'a' from the left-hand support as shown in Fig.11.11(a). Determine the fixed end moments developed.

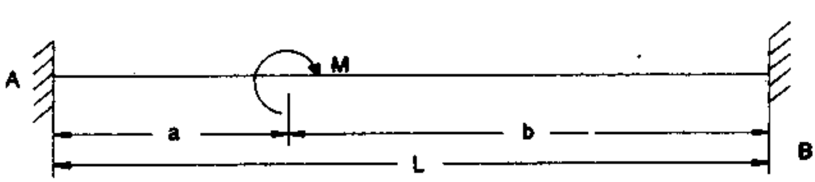


Figure 11.11 (a)

Solution

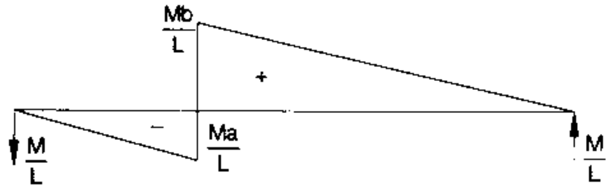


Figure 11.11 (b)

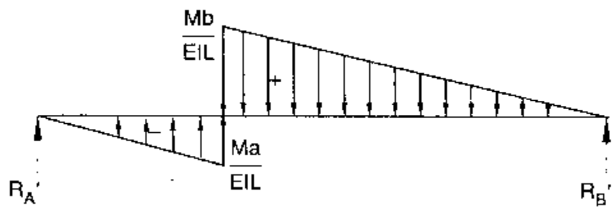


Figure 11.11 (c)

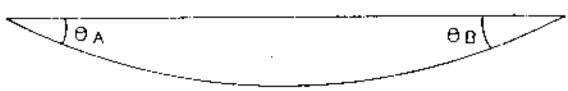


Figure 11.11 (d)

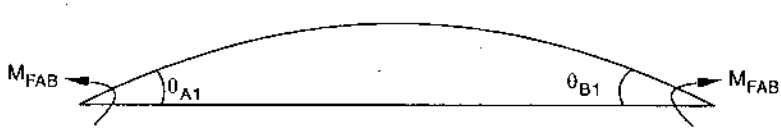


Figure 11.11 (e)

Taking a simply supported beam AB as the basic determinate structure, the bending moment diagram for the determinate structure is shown in Fig.11.11(b). The conjugate beam with $\frac{M}{EI}$ loading is as shown in Fig.11.11(c).

$$\theta_A = R_A' = \frac{1}{L} \text{ (Moment about B)}$$

$$= \frac{1}{L}\left[\frac{1}{2}\frac{Ma}{EIL}a\left(b+\frac{a}{3}\right)-\frac{Mb}{EIL}\frac{b}{2}\frac{2b}{3}\right]$$

$$= \frac{M}{6EIL^2}\left(a^2(3b+a)-2b^3\right) \qquad (1)$$

Similarly, $\quad _B = R_B' = \frac{1}{L}$ (Moment about A)

$$\theta_B = \frac{1}{L}\left[\frac{1}{2}\frac{Ma}{EIL}a\frac{2a}{3} - \frac{1}{2}b\frac{Mb}{EIL}(a+\frac{b}{3})\right]$$

$$= \frac{M}{6EIL^2}\ (2a^3 - b^2\ (3a + b) \qquad (2)$$

If the fixed end moments are M_{FAB} and M_{FBA} as shown in Fig.11.13(e), then from eqn.11.1 and 11.2,

$$\theta_{A1} = \frac{L}{6EI}(2M_{FAB} + M_{FBA}) \qquad (3)$$

$$\theta_{B1} = \frac{L}{6EI}\ (M_{FAB} + 2M_{FBA}) \qquad (4)$$

From the consistency condition

$$\theta_A = \theta_{A1} \qquad (5)$$

$$\theta_B = \theta_{B1} \qquad (6)$$

$$\frac{L}{6EI}\ (2M_{FAB} + M_{FBA}) = \frac{M}{6EIL^2}\ (\ a^2\ (3b + a) - 2b^3)$$

$$(2M_{FAB} + M_{FBA}) = \frac{M}{L^3}\ (a^2\ (3b + a) - 2b^3) \qquad (7)$$

From eqn.(6), $\quad M_{FAB} + 2M_{FBA} = \frac{M}{L^3}\ (2a^3 - b^2\ (3a + b) \qquad (8)$

Subtracting eqn.(8) from twice eqn.(7), we get

$$3M_{FAB} = \frac{M}{L^3}\ (2a^2\ (3b + a) - 4b^3 - 2a^3 + b^2\ (3a + b)$$

$$= \frac{M}{L^3}\ (6a^2b + 2a^3 - 4b^3 - 2a^3 + 3b^2a + b^3)$$

$$= \frac{M}{L^3}\ (\ 6a^2b + 3b^2a - 3b^3)$$

$$= \frac{3M}{L^3}\ (\ 2a^2b + b^2a - b^3)$$

$$M_{FAB} = \frac{M}{L^3}(\ 2a^2b + b^2a - b^3) \qquad (9)$$

$$= \frac{Mb}{L^3}(2a^2 + ab - b^2)$$

$$= \frac{Mb}{L^3}\ (a + b)\ (2a - b)$$

$$= \frac{Mb}{L^2}(2a - b), \text{ since } a + b = L \qquad (10)$$

To find M_{FBA}, eqn. (7) or (8) may be used.

$$M_{FBA} = \frac{M}{L^3}[a^2(3b + a) - 2b^3] - 2M_{FAB}$$

$$= \frac{M}{L^3}\ (3a^2b + a^3 - 2b^3 - 2\ (2a^2b + b^2a - b^3)$$

$$= \frac{Ma}{L^3}\ (a^2 - ab - 2b^2)$$

$$= \frac{Ma}{L^3}\ (a + b)\ (a - 2b)$$

$$= \frac{M}{L^3}(a+b)\ (a^2 - ab - ab)$$

$$= \frac{M}{L^2}\ a(a - 2b), \text{ since } a + b = L$$

Thus $$M_{FAB} = \frac{Mb}{L^2}\ (2a - b) \qquad (11.6)$$

and $$M_{FBA} = \frac{Ma}{L^2}\ (a - 2b) \qquad (11.7)$$

Example 11.10 Determine the fixed end moments developed in a fixed beam of span L and flexural rigidity EI when the right-hand side support settles down by Δ (refer Fig.11.12(a)).

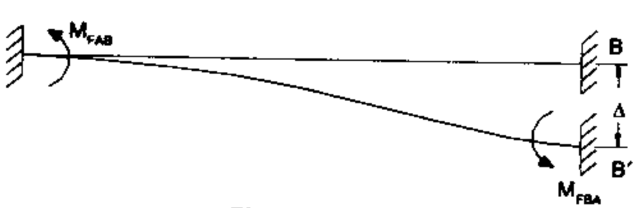


Figure 11.12 (a)

Solution

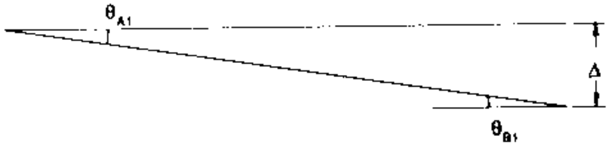


Figure 11.12 (b)

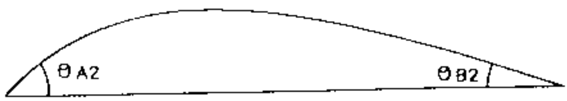


Figure 11.12 (c)

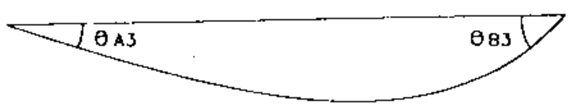


Figure 11.12 (d)

Let the beam settle from B to B′

To make zero slope at A and B′, fixed end moments M_{FAB} and M_{FBA} develop as shown in Fig.11.12(a).

Taking a simply supported beam as the basic determinate structure, let the slopes due to settlement, M_{FAB} and M_{FBA} be as shown in Fig.11.12b, c and d respectively.

$$\theta_{A1} = \frac{\Delta}{L} \qquad \theta_{B1} = \frac{\Delta}{L}$$

$$\theta_{A2} = \frac{M_{FAB}}{3EI} \qquad \theta_{B2} = M_{FAB}\frac{L}{6EI}$$

$$\theta_{A3} = M_{FBA}\frac{L}{6EI} \qquad \theta_{B3} = M_{FBA}\frac{L}{3EI}$$

$$\therefore \qquad \theta_A = M_{FAB}\frac{L}{3EI} - M_{FBA}\frac{L}{6EI} - \frac{\Delta}{L}$$

$$\theta_B = M_{FAB}\frac{L}{6EI} - M_{FBA}\frac{L}{3EI} + \frac{\Delta}{L}$$

From the consistency requirement, $\theta_A = \theta_B = 0$

(1)

$$\therefore \qquad M_{FAB}\frac{L}{3EI} - \frac{M_{FBA}L}{6EI} - \frac{\Delta}{L} = 0$$

and
$$\frac{M_{FAB}L}{6EI} - \frac{M_{FBA}L}{3EI} + \frac{\Delta}{L} = 0 \tag{2}$$

Adding eqns. (1) and (2) we get,

$$\frac{M_{FAB}L}{3EI}(1 + 0.5) - \frac{M_{FBA}L}{3EI}(1 + 0.5) = 0$$

or
$$M_{FAB} = M_{FBA} \tag{3}$$

Substituting eqn.(3) in eqn.(1), we get,

$$M_{FAB}\frac{L}{3EI} - M_{FAB}\frac{L}{6EI} = \frac{\Delta}{L}$$

$$M_{FAB}\frac{L}{6EI}(2 - 1) = \frac{\Delta}{L}$$

or
$$M_{FAB} = \frac{6EI\Delta}{L^2} = M_{FBA} \tag{3}$$

Note : Both M_{FAB} and M_{FBA} are in the anticlockwise direction. If the left-hand support settles down by Δ compared to the right-hand side support, both M_{FAB} and M_{FBA} will be $\frac{6EI\Delta}{L^2}$, but clockwise.

Example 11.11 Determine the fixed end moments developed in the beam shown in Fig.11.13.

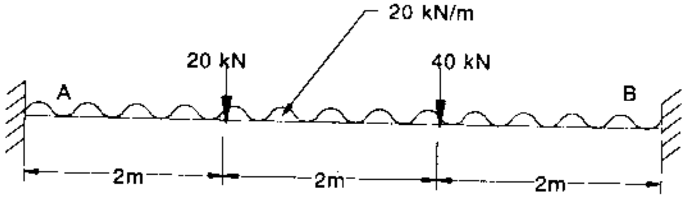


Figure 11.13

Solution

Taking clockwise moment as positive and anticlockwise as negative, the fixed end moments are written down below :

Table 11.1 Fixed end moments

Fixed end moments	*At end A*	*At end B*
(i) Due to 20 kN concentrated load	$\frac{-20\times2\times4^2}{6^2} = -17.78$	$\frac{20\times2^2\times4}{6^2} = 8.89$
(ii) Due to 40 kN concentated load	$\frac{-40\times4\times2^2}{6^2} = -17.8$	$\frac{40\times4^2\times2}{6^2} = 35.56$
(iii) Due to 20 kN/m udl	$\frac{-20\times6^2}{12} = -60$	$\frac{20\times6^2}{12} = 60$
Total	95.56 kN-m	104.45 kN-m

Note: Standard expressions $\frac{Wab^2}{L^2}$, $\frac{Wba^2}{L^2}$ and $\frac{WL^2}{12}$ derived earlier (equations 11.3, 11.4 and 11.5) are used here.

Example 11.12 In the problem 11.11, if end B settles by 1mm, determine the end moments developed, given that EI = 90000 kN-m^2.

Solution

Due to settlement, the end moments developed are

$$= -\frac{6EI\Delta}{L^2}, \text{ at both ends}$$

$$= -\frac{6\times90000\times0.001}{6^2} = -15 \text{ kN-m}$$

∴ Final end moments are

$$-95.56 - 15 = -110.56 \text{ kN-m at end A}$$

and

$$104.45 - 15 = 89.45 \text{ kN-m at end B}$$

Example 11.13 Determine the fixed end moments developed in the fixed beam of span L subjected to uniformly varying load as shown in Fig.11.14.

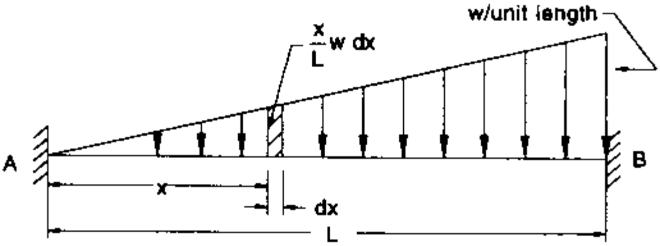


Figure 11.14

Solution

Consider an elemental load of width dx at distance 'x' from end A. Here, the intensity of the load is given by

$$= \frac{x}{L} \text{ w per unit length}$$

∴ Elemental load at distance x is $\frac{x}{L}$ w dx

$$\therefore \text{ FEM at A due to this load} = \frac{x}{L} \text{ wdx } \frac{(L-x)^2}{L^2}$$

$$= \frac{wx^2}{L^3}(L - x)^2 \, dx$$

$$\therefore \text{ FEM at A due to given load} = \int_0^L \frac{w}{L^3} x^2 (L-x)^2 dx$$

$$= \frac{w}{L^3}\int_0^L (x^2L^2 - 2Lx^3 + x^4)\, dx$$

$$= \frac{w}{L^3}\left[\frac{x^3L^2}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^L$$

$$= \frac{w}{L^3} L^5 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]$$

$$= wL^2 \frac{10-15+6}{30}$$

$$= \frac{wL^2}{30}, \text{ anticlockwise}$$

Similarly, at end B, FEM due to elemental load

$$= \frac{x}{L} w dx \frac{x^2(L-x)}{L^2}$$

$$= \frac{w}{L^3} x^3 (L-x)\, dx$$

$\therefore$ FEM at end B

$$= \int_0^L \frac{w}{L^3} x^3 (L-x)\, dx$$

$$= \frac{w}{L^3} \int_0^L (Lx^3 - x^4) dx$$

$$= \frac{w}{L^3} \left[\frac{Lx^4}{4} - \frac{x^5}{5} \right]_0^L$$

$$= wL^2 \left[\frac{1}{4} - \frac{1}{5} \right]$$

$$= \frac{wL^2}{20}, \text{ clockwise}$$

Example 11.14 The beam AB of span 6m is subjected to a uniformly distributed load of intensity 30 kN–m as shown in Fig.11.15. Determine the fixed end moments developed.

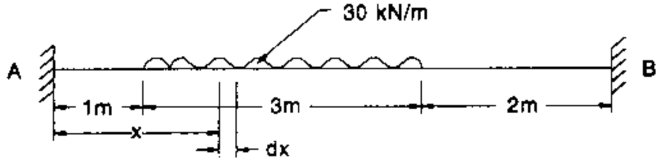


Figure 11.15

Solution

Consider an elemental length dx at x from end A.

Elemental load = 30 dx

It can be treated as a concentrated load at a distance x from end A. Hence, FEM due to this load at end A is

$$\frac{30dx\, x(6-x)^2}{6^2} = \frac{5}{6}\,(36x - 12x^2 + x^3)dx$$

$\therefore$ FEM at end A due to given loading

$$= \int_1^4 \frac{5}{6}\,(36x - 12x^2 + x^3)dx$$

$$= \frac{5}{6}\left[36\frac{x^2}{2} - 12\frac{x^3}{3} + \frac{x^4}{4}\right]_1^4$$

$$= \frac{5}{6}\left[18(4^2-1) - 4(4^3-1) + \frac{1}{4}(4^4-1)\right]$$

$= 68.125$ kN-m, anticlockwise

Similarly, FEM at end B due to elemental load

$$= \frac{30dx\,x^2(6-x)}{6^2} = \frac{5}{6}\,[6x^2 - x^3]\,dx$$

$\therefore$ FEM at end B due to given load

$$= \int_1^4 \frac{5}{6}\,(6x^2 - x^3)\,dx$$

$$= \frac{5}{6}\left[6\frac{x^3}{3} - \frac{x^4}{4}\right]_1^4$$

$$= \frac{5}{6}\left[2(4^3-1) - \frac{1}{4}(4^4-1)\right]$$

$= 51.875$ kN-m, clockwise

11.4 ANALYSIS OF CONTINUOUS BEAMS

To find deflections and rotations for a basic determinate structure, standard expressions may be used. If the case does not confirm to a standard one, conjugate beam method or unit load method can be used. The procedure is illustrated with the examples 11.15 to 11.18.

Example 11.15 Determine the reaction components in the beam shown in Fig.11.16(a). Assume flexural rigidity to be the same throughout.

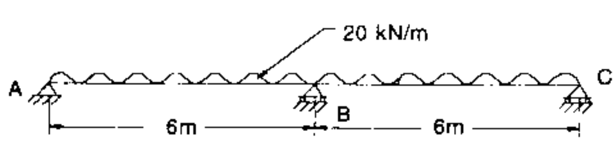


Figure 11.16 (a)

Solution

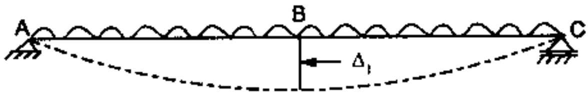


Figure 11.16 (b)

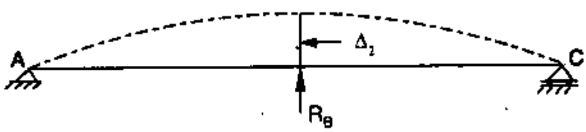


Figure 11.16 (c)

Reaction R_B is taken as the redundant force. Hence the basic determinate structure is a simply supported beam AC.

Downward deflection at B (Fig.11.16(b))

$$\Delta_1 = \frac{5}{384}\ \frac{WL^4}{EI} = \frac{5}{384} \times 20 \times \frac{12^4}{EI} \tag{a}$$

If R_B is applied as a load to the determinate structure AC, then the upward deflection is (Fig.11.16(c))

$$\Delta_1 = R_B\ \frac{L^3}{48EI} = R_B\ \frac{12^3}{48EI} \tag{b}$$

From the consistency condition, ,

$$\frac{5}{384} \times 20 \times \frac{12^4}{EI} = R_B\ \frac{12^3}{48EI}$$

$$R_B = 150 \text{ kN}$$

Now, considering the original beam ABC and taking moment about C, we get

$$R_A \times 12 - 20 \times 12 \times 6 + R_B \times 6 = 0$$

$$R_A = \frac{20 \times 12 \times 6 - 150 \times 6}{12}, \text{ since } R_B = 150 \text{ kN}$$

$$= 45 \text{ kN}$$

$$R_C = 20 \times 12 - R_B - R_A$$

$$= 240 - 150 - 45 = 45 \text{ kN}$$

Example 11.16 Determine the reaction components in the continuous beam shown in Fig.11.17(a). Flexural rigidity is constant throughout.

Solution

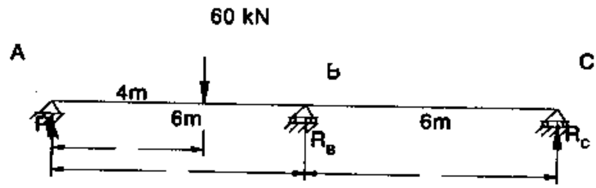


Figure 11.17 (a)

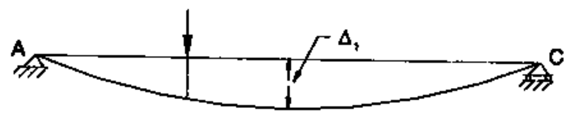


Figure 11.17 (b)

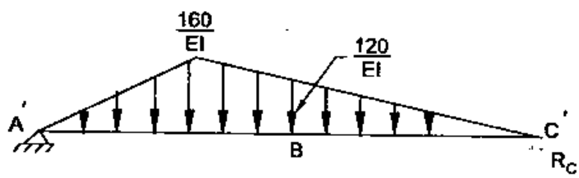


Figure 11.17 (c)

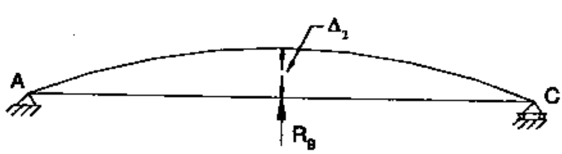


Figure 11.17 (d)

Reaction R_B is taken as the redundant force and hence a simply supported beam AC is the basic determinate structure.

To find downward deflection Δ_1 (Fig.11.17b) in the basic determinate structure AC, the conjugate beam method is used. The bending moment diagram in the basic structure is a triangle with maximum ordinate of

$$\frac{wab}{L} = \frac{60\times4\times8}{12} = 160 \text{ kN-m}$$

The conjugate beam is loaded as shown in Fig.11.17(c). In the conjugate beam, $\Sigma M_A = 0$ gives,

$$R_C' \times 12 = \frac{1}{2}\times 12\times \frac{160}{EI}\left(\frac{12+4}{3}\right)$$

$$R_C' = \frac{426.67}{EI}$$

$$\therefore \qquad M_B = R_C' \times 6 - \frac{1}{2} \times \frac{45}{EI} \times 6 \times 2,$$

since ordinate at B $= \frac{6}{8} \times \frac{60}{EI} = \frac{120}{EI}$

$$= \frac{426.67 \times 6}{EI} - \frac{120 \times 6}{EI}$$

$$= \frac{1840}{EI}$$

i.e., the downward deflection Δ_1 in the beam AC at B,

$$\Delta_1 = \frac{1840}{EI}$$

Due to R_B, upward deflection Δ_2 at B (refer Fig. 11.17(d)),

$$\Delta_2 = R_B \frac{12^3}{48EI}$$

From the consistency condition,

$$\Delta_1 = \Delta_2$$

i.e., $$\frac{1840}{EI} = R_B \frac{12^3}{48EI}$$

or $$R_B = 51.11 \text{ kN}$$

Taking the moment about A in the continuous beam ABC,

$$R_C \times 12 + R_B \times 6 - 60 \times 4 = 0$$

$$R_C = \frac{60 \times 4 - 51.11 \times 6}{12}, \text{ since } R_B = 63.61 \text{ kN}$$

$$= -5.56 \text{ kN}$$

$$= 5.56 \text{ kN, downward}$$

$$\therefore \qquad R_A = 60 - 51.11 + 5.56$$

$$= 14.45, \text{ upward}$$

Example 11.17 Determine the reaction components in the continuous beam shown in Fig. 11.18(a). EI is constant throughout.

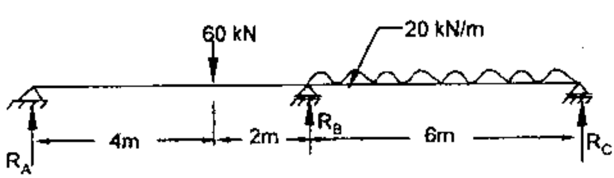


Figure 11.18 (a)

Solution

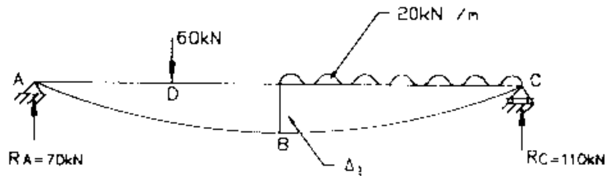


Figure 11.18 (b)

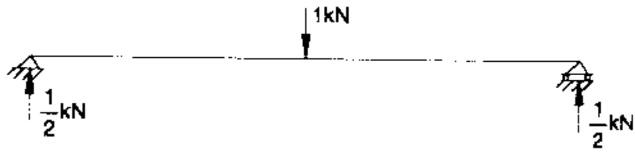


Figure 11.18 (c)

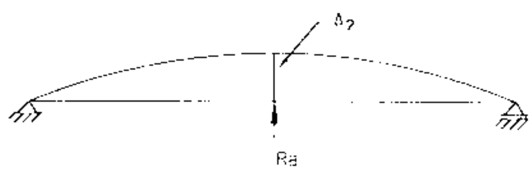


Figure 11.18 (d)

Taking reaction R_B as the redundant force, the resulting determinate structure is a simply supported beam ABC as shown in Fig.11.18(b). To find the deflection at B in this determinate beam, the unit load method is used.

In the determinate structure AC,

$$R_A = \frac{60\times8+20\times6\times3}{12} = 70 \text{ kN}$$

$$R_C = 60 + 20 \times 6 - 70 = 110 \text{ kN}$$

∴ Moments in the various portions are as shown in Table 11.1

Table 11.1 Moments in various portions

Portion	*AD*	*DB*	*BC*
Origin Limit	A 0– 4	A 4– 6	C 0– 6
M	70x	70x – 60(x–4)	$110x - \frac{20x^2}{2}$
m	$\frac{x}{2}$	$\frac{x}{2}$	$\frac{x}{2}$

Due to unit load at B (refer Fig.11.18(c)), $R_A = R_C = \frac{1}{2}$

Moments 'm' in various portions are as shown in Table 11.1.
Deflection at B,

$$\Delta_1 = \int_0^4 70x \, \frac{x}{2} \, \frac{dx}{EI} + \int_4^6 (10x + 240) \, \frac{x}{2} \, \frac{dx}{EI} + \int_0^6 (110x - 10x^2) \, \frac{x}{2} \, \frac{dx}{EI}$$

$$= \frac{35}{EI} \, \frac{4^3}{3} + \frac{1}{EI}\left[\frac{5}{3}(6^3 - 4^3) + 60(6^2 - 4^2)\right] + \frac{55}{EI} \, \frac{6^3}{3} - 5 \, \frac{6^4}{4}$$

$$= \frac{1}{EI} \, (746.67 - 1453.33 + 3960 - 1620)$$

$$= \frac{1633.34}{EI}$$

Upward deflection at B, due to R_B (Fig.11.18(d))

$$\Delta_2 = R_B \, \frac{12^3}{48EI}$$

From the consistency condition, $\Delta_1 = \Delta_2$

$$\frac{1633.34}{EI} = R_B \frac{12^3}{48EI}$$

$\therefore$ $$R_B = 45.37$$

$\therefore$ In the continuous beam ABC (Fig.11.18(a)), taking the moment about C,

$$R_A \times 12 - 60 \times 8 + R_B \times 6 - 20 \times 6 \times 3 = 0$$

$$R_A = \frac{60 \times 8 - 45.37 \times 6 + 20 \times 6 \times 3}{12}, \text{ since } R_B = 45.37 \text{ kN}$$

$$= 47.31 \text{ kN}$$

$$R_C = 60 + 20 \times 6 - (R_A + R_B)$$

$$= 60 + 120 - (47.31 + 45.37)$$

$$= 87.32 \text{ kN}$$

Example 11.18 Determine the reaction components in the continuous beam ABC shown in Fig.11.19(a). Flexural rigidity is constant throughout.

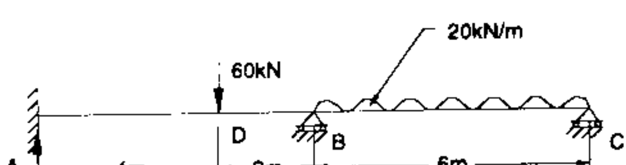


Figure 11.19 (a)

Solution

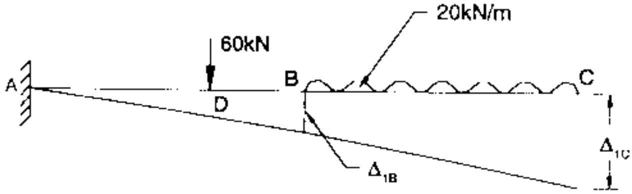


Figure 11.19 (b)

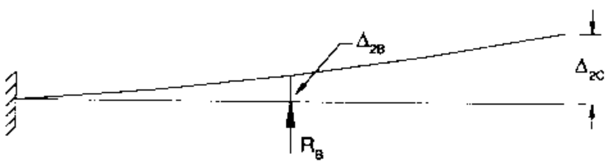


Figure 11.19 (c)

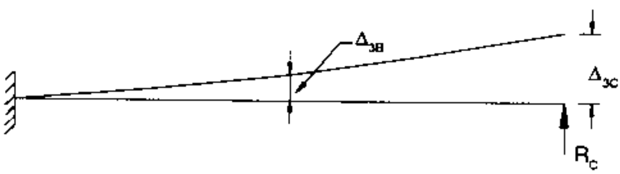


Figure 11.19 (d)

No. of reaction components = 3 + 1 + 1 = 5

No. of static equilibrium equations = 3

$\therefore$ Degree of indeterminacy = 5 – 3 = 2

Reactions R_B and R_C are taken as redundant forces. Hence the resulting basic determinate structure is a cantilever AC (Fig.11.19(b)). Let the deflections at B and C be Δ_{1B} and Δ_{1C} respectively. To find these deflections, the unit load method is used. Moments due to unit vertical downward load at B are noted down as m_1 and those due to unit load at C are noted down as m_2 and are shown in Table 11.2.

Table 11.2. *Moments due to unit vertical downward load at B*

Portion	*CB*	*BD*	*DA*
Origin	C	C	C
Limit	0–6	6–8	8–12
M	$-20\,\dfrac{x^2}{2}$	–(20)(6) (x–3)	–(20)(6)(x–3)–60(x–8)
	$= -10x^2$	= –120(x–3)	= – (180x – 840)
m_1	0	– 1(x – 6)	– 1 (x – 6)
m_2	– x	– x	– x

$$\Delta_{1B} = Mm_1 \frac{dx}{EI}$$

$$= \int_6^8 120(x-3)(x-6)\frac{dx}{EI} + \int_8^{12}(180x-840)(x-6)\frac{dx}{EI}$$

$$= \frac{120}{EI}\int_6^8 (x^2-9x+18)\,dx + \int_8^{12}(180x^2-1080x-840x-5040)\frac{dx}{EI}$$

$$= \frac{120}{EI}\left[\frac{x^3}{3}-\frac{9}{2}x^2+18x\right]_6^8 + \frac{1}{EI}\left[60x^3-540x^2-420x^2+5040x\right]_8^{12}$$

$$= \frac{1040}{EI} + \frac{16320}{EI} = \frac{17360}{EI}$$

$$\Delta_{1C} = Mm_2 \frac{dx}{EI}$$

$$= \int_0^6 10x^2 x\frac{dx}{EI} + \int_6^8 120(x-3)x\frac{dx}{EI} + \int_8^{12}(180x-840)x\frac{dx}{EI}$$

$$= \frac{1}{EI}\left[10\frac{x^4}{4}\right]_0^6 + \frac{120}{EI}\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_6^8 + \frac{1}{EI}\left[\frac{180x^3}{3}-\frac{840x^2}{2}\right]_8^{12}$$

$$= \frac{1}{EI}\,[3240 + 6800 + 39360]$$

$$= \frac{49400}{EI}$$

Now, using standard expressions, Δ_{2B} and Δ_{2C} (refer Fig.11.19(c)) due to R_B are calculated

$$\Delta_{2B} = R_B\frac{6^3}{3EI} = R_B\frac{72}{EI}$$

$$\Delta_{2C} = R_B\frac{6^3}{3EI} + R_B\frac{6^2}{2EI}\times 6$$

$$= R_B\frac{180}{EI}$$

Due to R_C, upward deflection at C (refer Fig.11.19(d)) is given by

$$\Delta_{3C} = R_C \times \frac{12^3}{3EI} = R_C\frac{576}{EI}$$

To find Δ_{3B}, Maxwell's reciprocal theorem can be used. From eqn. (iv), we find the deflection at C due to unit load at B is

$$= \frac{180}{EI}$$

∴ Deflection at B due to unit load at C

$$= \frac{180}{EI}$$

∴ Deflection at B due to redudant force R_C at C

$$\Delta_{3B} = R_C \frac{180}{EI}$$

Consistency of deflections at B needs,

$$\Delta_{1B} = \Delta_{2B} + \Delta_{3B}$$

$$\frac{17360}{EI} = \frac{72R_B}{EI} + \frac{180R_C}{EI}$$

i.e., $$17360 = 72R_B + 180R_C$$

Consistency of deflections at C needs,

$$\Delta_{1C} = \Delta_{2C} + \Delta_{3C}$$

$$\frac{49400}{EI} = \frac{180R_B}{EI} + \frac{576R_C}{EI}$$

i.e., $$49400 = 180R_B + 576R_C \qquad \text{...(b)}$$

Multiplying eqn. (a) by 2.5, we have

$$43400 = 180R_B + 450R_C \qquad \text{...(c)}$$

Subtracting eqn. (c) from eqn. (b), we get

$$6000 = 126R_C$$

or $$R_C = 47.62 \text{ kN}$$

∴ $$R_B = \frac{43400 - 450 \times 47.62}{180} = 122.06 \text{ kN}$$

$$\begin{aligned} R_A &= 60 + 20 \times 6 - R_B - R_C \\ &= 180 - 47.62 - 122.06 \\ &= 10.32 \text{ kN} \end{aligned}$$

$$\begin{aligned} M_A &= 47.62 \times 12 - 20 \times 6 \times 9 + 122.06 \times 6 - 60 \times 4 \\ &= -16.2 \text{ kN-m} \end{aligned}$$

Note: Consistent deformation method is not ideally suited for analysing structures with more than one degree indeterminacy.

11.5 ANALYSIS OF FRAMES

Consistent deformation method can be used for the analysis of indeterminate frames of degree one and two. Since this method is not ideally suited for higher degree indeterminate structures, they are not illustrated. To find deflections in basic determinate structures, the unit load method is best suited for frames. Application of consistent deformation method for frames is illustrated with the problems.

Example 11.19 In the frame ABCD shown in Fig. 11.20(a), end A is fixed and end D is roller. Analyse the frame for the loadings shown.

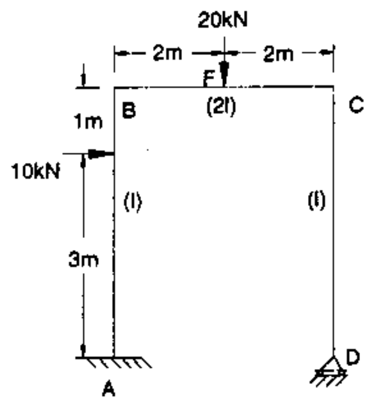


Figure 11.20 (a)

Solution

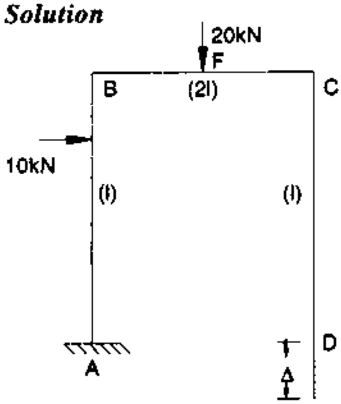


Figure 11.20 (b)

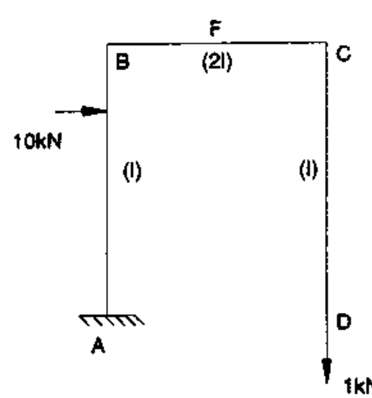


Figure 11.20 (c)

There are four reaction components and hence the degree of indeterminancy is one. Treating reaction V_D at D as the redundant force, we get the basic determinate structure as shown in Fig.11.20(b). The vertical downward deflection in this structure Δ is obtained by unit load method. Table 11.3 is prepared in which 'M' is the moment due to given loading, and 'm' is the moment due to unit downward load applied at D as shown in Fig.11.20(c).

Table 11.3

Portion	*DC*	*CF*	*FB*	*BE*	*EA*
Origin	D	C	F	B	E
Limit	0–4	0–2	0–2	0–1	0–3
EI	EI	2EI	2EI	EI	EI
M	0	0	– 20x	– 20(2)	– 20(2) – 10x
m	0	– x	– (x + 2)	– 4	– 4

$$\Delta = \int_0^4 0\ dx + \int_0^2 0\ dx + \int_0^2 20x\ (x+2)\ \frac{dx}{2EI} + \int_0^1 40\,(4)\ \frac{dx}{EI} + \int_0^3 4\ (40 + 10x)\ \frac{dx}{EI}$$

$$= \frac{10}{EI}\left[\frac{x^3}{3} + x^2\right]_0^2 + \left[\frac{160x}{EI}\right]_0^1 + \frac{1}{EI}\left[160x + 20x^2\right]_0^3$$

$$= \frac{1}{EI}\left[66.67 + 160 + 480 + 180\right]$$

$$= \frac{886.667}{EI}$$

Now, due to redundant force V_D, the moment M′ in various portions will be equal to V_Dm. Due to unit downward load at D, moments are m. Hence downward deflection, due to V_D is given by

$$\Delta' = M'm\ \frac{dx}{EI}$$

$$= -V_D m^2\ \frac{dx}{EI}$$

or $$\Delta = V_D\, m^2\ \frac{dx}{EI}, \text{ upward.}$$

$$\Delta\ = 0 + 0 + \int_0^2 V_D\,\frac{x^2}{2EI}\ dx + \int_0^2 V_D(x+2)^2\ \frac{dx}{2EI} + \int_0^1 V_D 16\frac{dx}{EI} + \int_0^3 V_D 16\frac{dx}{EI}$$

$$= \frac{V_D}{EI}\left[\frac{x^3}{6}\right]_0^2 + \frac{V_D}{2EI}\left[\frac{x^3}{3} + \frac{4x^2}{2} + 4x\right]_0^2 + \frac{16V_D}{EI}\,[x]_0^1 + \frac{V_D}{EI}\ 16[x]_0^3$$

$$= \frac{V_D}{EI}\left(\frac{8}{6} + \frac{8}{6} + 4 + 4 + 16 + 48\right)$$

$$= \frac{74.667}{EI}V_D$$

From the consistency condition, $\Delta' = \Delta$

i.e., $$\frac{74.667}{EI} V_D = \frac{886.667}{EI}$$

$$V_D = 11.875 \text{ kN}$$

Referring to the given frame (Fig.11.20(a)),

$$\Sigma V = 0$$

$$V_A + 11.875 - 20 = 0$$

$$V_A = 8.125 \text{ kN}$$

$$M_A = 11.875 \times 4 - 20 \times 2 - 10 \times 3$$

$$= -22.5 \text{ kN-m}$$

Example 11.20 Analyse the frame shown in Fig.11.21a by consistent deformation method.

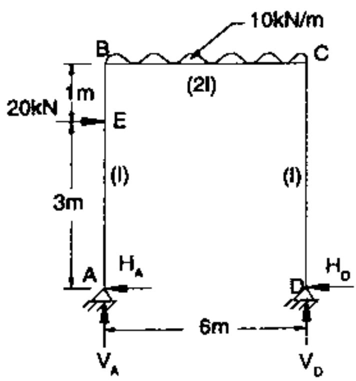


Figure 11.21 (a)

Solution

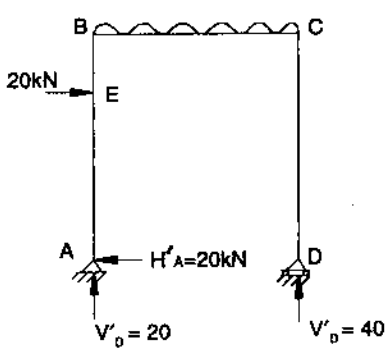


Figure 11.21 (b)

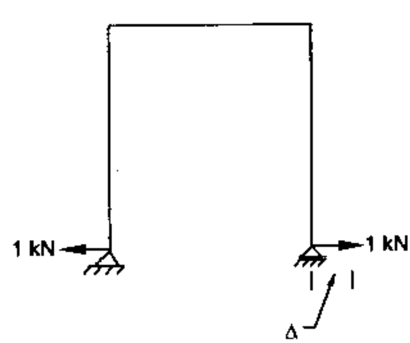


Figure 11.21 (c)

No. of unknown reaction components = 4
No. of static equations = 3
Hence, degree of indeterminacy = 1

Treating horizontal reaction at D as the redundant force, the basic determinate structure obtained is as shown in Fig.11.21(b). The horizontal displacement Δ, in this structure is obtained by unit load method. In Table 11.4, moments due to given loading are listed as M and those due to unit load at D (as shown in Fig.11.21(c)) are noted as m.

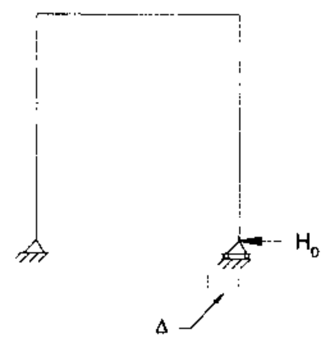


Figure 11.21 (d)

Taking the moment about A,

$$V_D \times 6 = 10 \times 6 \times 3 + 20 \times 3$$
$$V_D = 40 \text{ kN}$$
$$\therefore \quad V_A = 10 \times 6 - 40 = 20 \text{ kN}$$
$$\Sigma H = 0 \rightarrow H_A = 20 \text{ kN}$$

Table 11.4

Portion	*AE*	*EB*	*BC*	*CD*
Origin	A	E	C	D
Limit	0–3	0–1	0–6	0–4
EI	EI	EI	2EI	EI
M	20x	20(x+3)–20x = 60	$40x - \frac{10x^2}{2}$	0
m	x	1(3+x)	4	x

Horizontal displacement Δ in the basic determinate structure is given by

$$\Delta = Mm\,\frac{dx}{EI}$$

$$= \int_0^3 20x^2 \frac{dx}{EI} + \int_0^1 60(3+x)\frac{dx}{EI} + \int_0^6 4(40x-5x^2)\frac{dx}{2EI} + 0$$

$$= \frac{1}{EI}\left[20\frac{x^3}{3}\right]_0^3 + \frac{1}{EI}\left[180x+30x^2\right]_0^1 + \frac{1}{2EI}\left[80x^2 - 20\frac{x^3}{3}\right]_0^6$$

$$= \frac{1}{EI}\left[180+180+30+40\times 36 - \frac{10}{3}\times 6^3\right] = \frac{1110}{EI}$$

Referring to fig.11.21(d), the moment M' due to force H_D in various portions is given by

$$M' = -H_D m$$

$\therefore$ Displacement in the direction of unit load

$$\Delta' = \int M' m \frac{dx}{EI} = \int -H_D m^2 \frac{dx}{EI}$$

$$= H_D m^2 \frac{dx}{EI}, \text{ in the inward direction.}$$

$$\Delta' = \int_0^3 \frac{H_D}{EI} x^2\, dx + \int_0^1 H_D (3+x)^2 \frac{dx}{EI} + \int_0^6 H_D \frac{16dx}{2EI} + \int_0^4 H_D x^2 \frac{dx}{EI}$$

$$= \frac{H_D}{EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{H_D}{EI}\left[9x + \frac{6x^2}{2} + \frac{x^3}{3}\right]_0^1 + \frac{H_D}{2EI}\left[16x\right]_0^6 + \frac{H_D}{EI}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{H_D}{EI}\left[9+9+3+\frac{1}{3}+\frac{16\times 6}{2}+\frac{4^3}{3}\right]$$

$$= 90.667 \frac{H_D}{EI}$$

From the consistency requirement,

$$\Delta' = \Delta$$

$$90.667 \frac{H_D}{EI} = \frac{1110}{EI}$$

$$H_D = 12.24 \text{ kN}$$

Referring to the original structure (Fig.11.21(a)), we get,

$$H_A = 20 - 12.24 = 7.76 \text{ kN}$$

Example 11.21 Analyse the frame shown in Fig.11.22(a) using the consistent deformation method. Flexural rigidity is constant throughout.

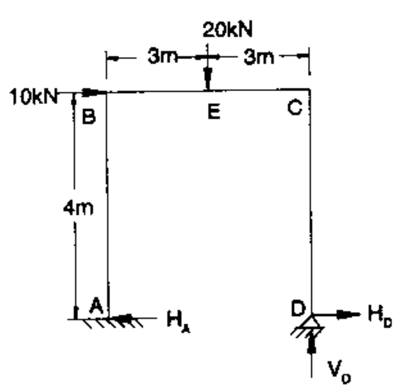


Figure 11.22 (a)

Solution

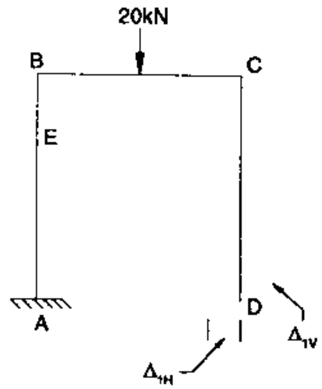


Figure 11.22 (b)

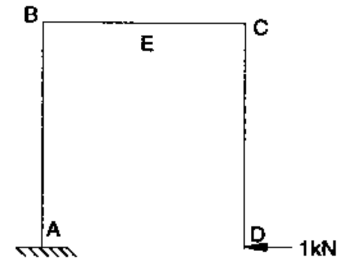


Figure 11.22 (c)

Number of unknown reaction components = 3 + 2 = 5

Number of static equations available = 3

∴ Degree of indeterminacy = 5 – 3 = 2

Treating horizontal and vertical reactions H_D and V_D as redundant forces, the basic determinate structure is as shown in Fig.11.21(b). In Table 11.5, the moments due to load in basic determinate structure are noted as M. The moments due to unit horizontal force 1 kN at D as shown in Fig.11.22(c) are noted down as 'm' and those due to unit vertical load at D as shown in Fig.11.22(d) are noted down as 'm''.

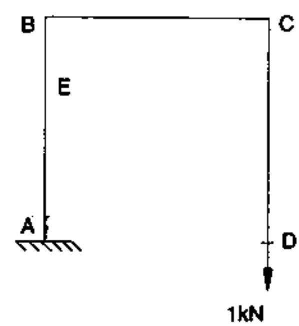


Figure 11.22 (d)

***Table 11.5** Moments due to load*

Portion	*DC*	*CE*	*EB*	*BA*
Origin	D	C	E	B
Limit	0–4	0–3	0–3	0–4
M	0	0	– 20x	–(60 + 10x)
m	– x	– 4	– 4	– (4–x)
m′	0	– x	– (x+3)	– 6

∴ Inward horizontal displacement of D,

$$\Delta_{1H} = Mm \frac{dx}{EI}$$

$$\Delta_{1H} = 0 + 0 + \int_0^3 20x4\frac{dx}{EI} + \int_0^4 (60+10x)(4-x)\frac{dx}{EI}$$

$$= \frac{1}{EI}\left[80\frac{x^2}{2}\right]_0^3 + \int_0^4 (240-20x-10x^2)\frac{dx}{EI}$$

$$= \frac{1}{EI}360 + \left[240-10x^2-10\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{EI}\left[360+960-160-\frac{640}{3}\right]$$

$$= \frac{946.67}{EI}$$

Vertical downward displacement of D is given by

$$\Delta_{1V} = Mm'\frac{dx}{EI}$$

$$\Delta_{1V} = 0+0+\int_0^3 20x(x+3)\frac{dx}{EI} + \int_0^4 (60+10x)6\frac{dx}{EI}$$

$$= \frac{1}{EI}\left[\frac{20x^3}{3}+30x^2\right]_0^3 + \frac{1}{EI}\left[360+30x^2\right]_0^4$$

$$= \frac{2730}{EI}$$

Due to horizontal force H_D acting in the outward direction, moments in the various portions of the basic determinate frame are $-H_D m$. Hence, due to H_D, the inward horizontal displacement (in the direction of unit force) is given by

$$= (-H_D m)\frac{dx}{EI}$$

i.e., horizontal outward displacement

$$\Delta_{2H} = H_D m^2 \frac{dx}{EI}$$

$$= \left[\int_0^4 x^2\frac{dx}{EI} + \int_0^3 4^2\frac{dx}{EI} + \int_0^3 4^2\frac{dx}{EI} + \int_0^4 (4-x)^2\frac{dx}{EI}\right]H_D$$

$$= \frac{H_D}{EI}\left[\frac{x3}{3}\right]_0^4 + \frac{H_D}{EI}[16x]_0^3 + \frac{H_D}{EI}[16x]_0^3$$

$$+ \frac{H_D}{EI}\left[16x-\frac{8x^2}{2}+\frac{x^3}{3}\right]_0^4$$

$$= \frac{H_D}{EI}\left[\frac{64}{3}+48+48+64-64+\frac{64}{3}\right]$$

$$= 138.67\ \frac{H_D}{EI}$$

Similarly, vertical upward displacement due to V_D is

$$\Delta_{2V} = V_D\, m'^2\, \frac{dx}{EI}$$

$$= 0 + \int_0^3 V_D x^2 \frac{dx}{EI} + \int_0^3 V_D (x+3)^2 \frac{dx}{EI} + \int_0^4 V_D (6)^2 \frac{dx}{EI}$$

$$= \frac{V_D}{EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{V_D}{EI}\left[\frac{x^3}{3}+3x^2+9x\right]_0^3 + \frac{V_D}{EI}\left[36x\right]_0^4$$

$$= \frac{V_D}{EI}[9 + 9 + 27 + 27 + 144]$$

$$= \frac{216}{EI}\ V_D$$

Vertical upward displacement of D due to horizontal force H_D is given by

$$\Delta_{3V} = H_D\, m\, m'\, \frac{dx}{EI}$$

$$= 0 + \int_0^3 \frac{H_D}{EI} 4x\,dx + \int_0^3 \frac{H_D}{EI} 4(x+3)dx + \int_0^4 \frac{H_D}{EI}(4-x)6dx$$

$$= \frac{H_D}{EI}\left[2x^2\right]_0^3 + \frac{H_D}{EI}\left[2x^2+12x\right]_0^3 + \frac{H_D}{EI}\left[24x-3x^2\right]_0^4$$

$$= \frac{H_D}{EI}\ [18 + 18 + 36 + 96 - 48]$$

$$= 120\ \frac{H_D}{EI}$$

Similarly, the horizontal outward displacement due to V_D

$$\Delta_{3H} = V_D\, m'\, m \frac{dx}{EI}$$

$$= 120 \frac{V_D}{EI}$$

From the consistency requirement,

$$\Delta_{2H} + \Delta_{3H} = \Delta_{1H}$$

$$138.67\ \frac{H_D}{EI} + 120 \frac{V_D}{EI} = \frac{946.67}{EI}$$

$$138.67\,H_D + 120\,V_D = 946.67 \quad \text{...(1)}$$

and

$$\Delta_{2V} + \Delta_{3V} = \Delta_{1V}$$

$$120\frac{H_D}{EI} + 216\frac{V_D}{EI} = \frac{2370}{EI}$$

$$120\,H_D + 216\,V_D = 2370 \quad \text{...(2)}$$

Multiplying eqn. (2) with $\frac{120}{216}$, we get

$$66.67\,H_D + 120\,V_D = 1316.67 \quad \text{...(3)}$$

Subtracting eqn.(3) from eqn.(1) we get,

$$72\,H_D = -370$$

$$H_D = -5.1389$$

From eqn.(1),

$$V_D = \frac{946.67 - 138.67(-5.1389)}{120}$$

$$= 13.827 \text{ kN}$$

Thus $H_D = \overleftarrow{5.1389} \text{ kN}$

and $V_D = 13.827 \text{ kN} \uparrow$

Considering the given frame, we get,

$$H_A = 10 - 5.1389 = \overleftarrow{4.8611} \text{ kN}$$

$$V_A = 20 - 13.827 = 6.173 \uparrow$$

$$M_A = 13.827 \times 6 - 20 \times 3 - 10 \times 4$$

$$= -17.038 \text{ kN-m}$$

$$= 17.038 \text{ kN-m, clockwise}$$

EXERCISES

11.1 A propped cantilever of span L is subjected to uniformly distributed load W per unit length over $\frac{3}{4}$ span from its fixed support. Determine the prop reaction.

Ans : 10.1714 WL

11.2 Determine the prop reaction in the beam shown in Fig.11.23.

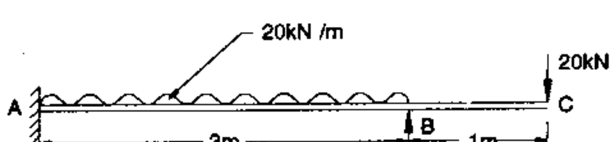


Figure 11.23

Ans : 152.5 kN

11.3 The support B in the overhanging beam shown in Fig. 11.24 sinks by 1mm. If EI = 300000 kN-m², determine the prop reaction.

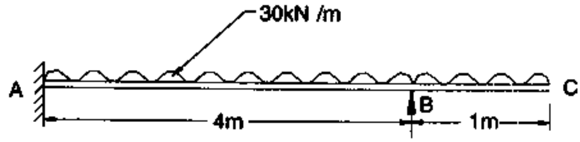
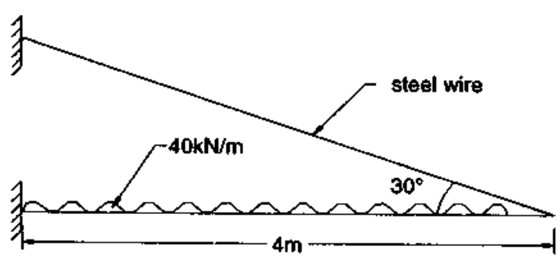


Figure 11.24

Ans : 119.94 kN

11.4 The free end of a cantilever is supported by a 10mm dia steel wire. Determine the force in wire and maximum bending moment in the beam when the cantilever is subjected to a uniformly distributed load of intensity 40 kN/m over its entire length (Fig. 11.25)
Given, flexural rigidity of beam
= 2×5^{13} N-mm²,
Young's Modulus of the steel wire = 2×10^5 N/mm².



Ans : P = 94.04 kN and M_{max} = 82.96 kN-m]

11.5 Determine the fixed end moments developed in the beam shown in Fig.11.26

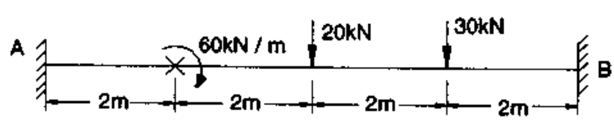


Figure 11.26

Ans : M_{FAB} = – 42.5 kN-m M_{FAB} = 35 kN-m

11.6 Determine the fixed end moments developed in the beam shown in Fig.11.27; if end B settles down by 1mm. Take EI = 60000 kN-m

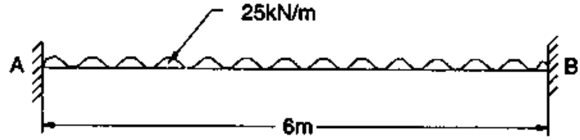


Figure 11.27

Ans : M_{FAB} = 85 kN-m; M_{FAB} = 65 kN-m

11.7 The beam AB shown in Fig.11.28 is subjected to uniformly varying symmetric load. Determine the fixed end moments developed.

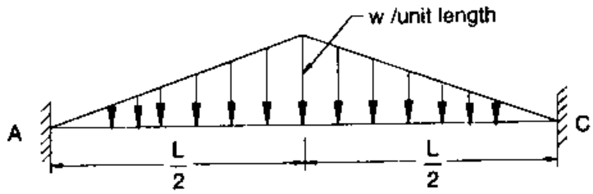


Figure 11.28

Ans : $M_{FAB} = -\frac{5}{96} wL^2$; $M_{FAB} = \frac{5}{96} wL^2$

11.8 Find the reactions in the beam shown in Fig.11.29

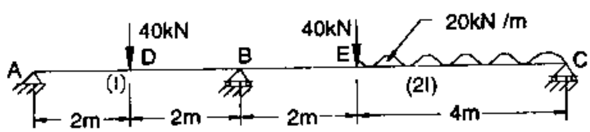


Figure 11.29

Ans : $R_A = 4.63$ kN; $R_B = 85.41$ kN; $R_C = 49.76$ kN

11.9 Determine the reactions at B and C in the continuous beam shown in Fig.11.30. Assume constant flexural rigidity throughout.

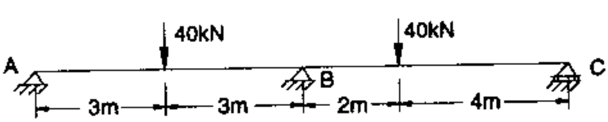


Figure 11.30

Ans : $R_B = 55.106$ kN; $R_C = 6.958$ kN

11.10 Analyse the frame shown in Fig.11.31.

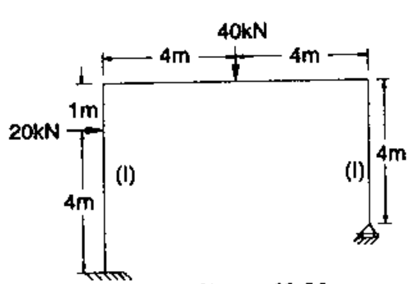


Figure 11.31

Ans : $V_C = 21.58$ kN

11.11 Determine the reaction components in the frame shown in Fig.11.32.

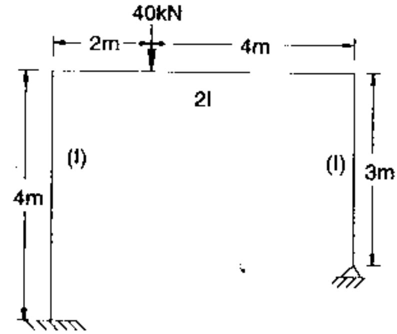


Figure 11.32

Ans : $V_A = 26.656$ kN; $H_A = \rightarrow 5.306$ kN
$V_D = 13.344$ kN; $H_D = \leftarrow 5.306$ kN

THREE MOMENT EQUATION 12

12.1 INTRODUCTION

The three moment equations express the relationship between the moments at three successive supports and the loading on the two spans between those three supports, with or without the unequal settlement of the supports. This relation is derived from the consistency condition that the slope at the middle support, calculated from the left span should be the same as the slope at the middle support calculated from the right span. Using the three moment equations, continuous beams can be analysed.

12.2 DERIVATION OF THREE MOMENT EQUATIONS

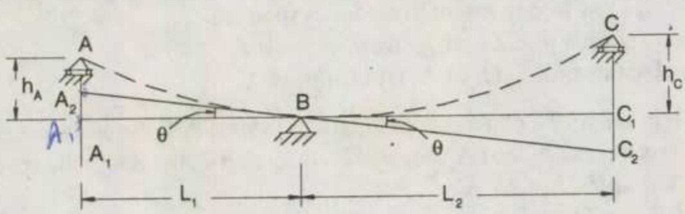


Figure 12.1 (a)

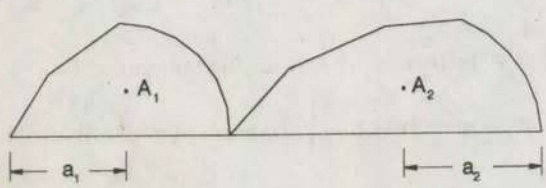


Figure 12.1 (b)

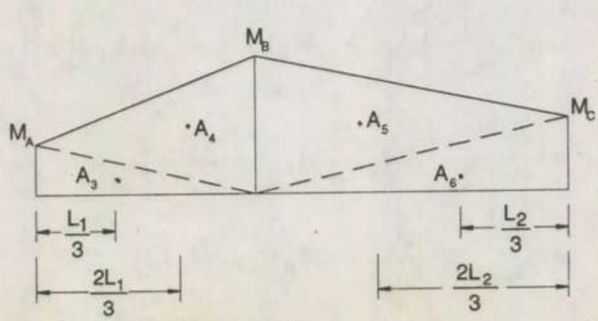


Figure 12.1 (c)

Fig. 12.1(a) shows the portion of a continuous beam between three successive supports A, B and C.

Let L_1 = Span of the beam AB
L_2 = Span of the beam BC
I_1 = Moment of inertia of the beam in span AB
I_2 = Moment of inertia of the beam in span BC
h_A = Relative position of support A with respect to support B after unequal settlement
h_C = Relative position of support C with respect to support B after unequal settlement

Let A_1 BC_1 be a horizontal line through B and A_2 BC_2 be the tangent to the elastic curve at B as shown in Fig.12.1(a).

Let M_A, M_B and M_C be the moments at supports A, B and C. They are taken positive, when they cause compression at the top (i.e., sagging moment is positive). Fig. 12.1(b) shows the free moment diagrams on spans AB and BC.

Let A_1 = Area of free moment diagram in span AB
A_2 = Area of free moment diagram in span BC
a_1 = Distance of C.G. of A_1 from support A
a_2 = Distance of C.G. of A_2 from support C

Fig.12.1(c) shows the end moment diagram. In span AB, the end moment diagram is split into two areas A_3 and A_4. Similarly, in span BC, the end moment diagram is split into two areas A_5 and A_6

Now, from Fig.12.1(a),

$$\frac{A_1A_2}{L_1} = \tan\theta = \frac{C_1C_2}{L_2} \qquad \text{(a)}$$

But, $A_1A_2 = h_A - AA_2$

$= h_A -$ deflection of A from the tangent of B

$= h_A -$ moment of $\frac{M}{EI}$ diagram between B and A about A

$$= h_A - \frac{1}{EI_1}\left[A_1a_1 + A_3\frac{L_1}{3} + A_4\frac{2L_1}{3}\right]$$

$$= h_A - \frac{1}{EI_1}\left[A_1a_1 + \frac{1}{2}M_AL_1\frac{L_1}{3} + \frac{1}{2}M_BL_1\frac{2L_1}{3}\right]$$

$$= h_A - \frac{1}{EI_1}\left[A_1a_1 + \frac{1}{6}M_AL_1^2 + \frac{M_BL_1^2}{3}\right]$$

$$= h_A - \frac{1}{6EI_1}\left[6A_1a_1 + M_AL_1^2 + 2M_BL_1^2\right] \qquad \text{(b)}$$

Similarly,

$$
\begin{aligned}
C_1C_2 &= CC_2 - h_C \\
&= \text{Deflection of C from the tangent of B} - h_C \\
&= \text{Moment of } \frac{M}{EI} \text{ diagram between B and C about C} - h_C \\
&= \frac{1}{EI_2}\left[A_2a_2 + A_5\frac{2L_2}{3} + A_6\frac{L_2}{3}\right] - h_C \\
&= \frac{1}{EI_2}\left[A_2a_2 + \frac{1}{2}L_2M_B\frac{2L_2}{3} + \frac{1}{2}M_CL_2\frac{L_2}{3}\right] - h_c \\
&= \frac{1}{6EI_2}\left[6A_2a_2 + 2M_BL_2^2 + M_CL_2^2\right] - h_c \qquad \text{(c)}
\end{aligned}
$$

Substituting eqns. (b) and (c) in eqn. (a), we get

$$
\frac{h_A}{L_1} - \frac{1}{6EI_1L_1}\left[6A_1a_1 + M_AL_1^2 + 2M_BL_1^2\right]
$$

$$
= \frac{1}{6EI_2L_2}\left[6A_2a_2 + 2M_BL_2^2 + M_CL_2^2\right] - \frac{h_c}{L_2}
$$

Multiplying it by 6E throughout and rearranging, we get

$$
M_A\left(\frac{L_1}{I_1}\right) + 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right)
$$

$$
= -\frac{6A_1a_1}{I_1L_1} - \frac{6A_2a_2}{I_2L_2} + \frac{6Eh_A}{L_1} + \frac{6Eh_c}{L_2}
$$

Equation 12.1 is known as *Three Moment Equation*. It is also referred as *Clapeyron's Theorem* of three moments equation, since the French investigator Clapeyron had a major contribution in deriving this equation.

12.3 APPLICATION OF THREE MOMENT EQUATIONS

The three moment equation may be used conveniently for the analysis of continuous beams. Consider the analysis of continous beam shown in Fig.12.2. The beam is having six unknown reactions.

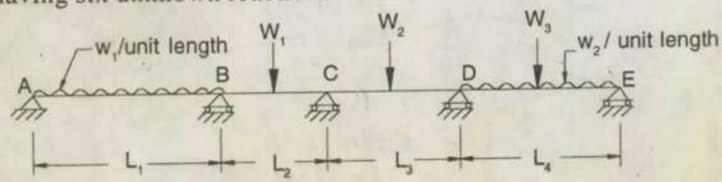


Figure 12.2

There are only three independent equations of equilibrium. Therefore, the degree of indeterminacy is three. Hence, we need three equations based on compatibility conditions. These three equations can be obtained by considering the slope continuity conditions at intermediate supports B, C and D i.e., by applying three moment equations for spans AB, BC; BC, CD and CD, DE. These equations helps in finding the moments at all supports. Once the bending moments at all supports are known, each span may be treated separately as being subjected to applied loadings on the span and the end moments to find the bending moment and shear force at any section desired.

If there is no settlement of supports, eqn.12.1 reduces to

$$M_A\left[\frac{L_1}{I_1}\right]+2M_B\left[\frac{L_1}{I_1}+\frac{L_2}{I_2}\right]+M_C\left[\frac{L_2}{I_2}\right]=-\frac{6A_1a_1}{I_1L_1}-\frac{6A_2a_2}{I_2L_2}$$

If the moment of inertia is same throughout, eqn.12.2 reduces to

$$M_AL_1+2M_B\left[L_1+L_2\right]+M_CL_2=-\frac{6A_1a_1}{L_1}-\frac{6A_2a_2}{L_2}$$

Example 12.1 Analyse the continuous beam ABC shown in Fig.12.3(a) using three moment equation and draw the bending moment diagram. Flexural rigidity EI is constant throughout.

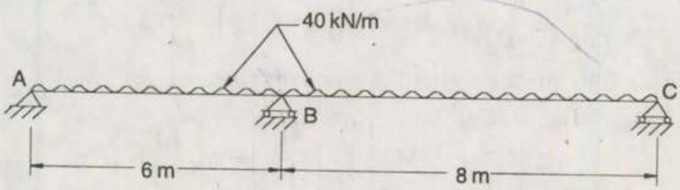


Figure 12.3 (a)

Solution

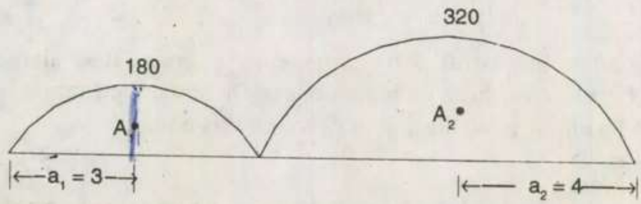


Figure 12.3 (b)

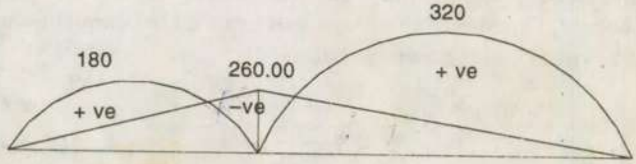


Figure 12.3 (c)

Free moment diagram for span AB is a parabola with maximum ordinate at mid-span

$$= \frac{40 \times 6^2}{8} = 180 \text{ kN-m}$$

Free moment diagram for span BC is also a parabola with maximum ordinate at mid-span

$$= \frac{40 \times 8^2}{8} = 320 \text{ kN-m}$$

These are shown in Fig. 12.3(b)

$$\therefore \quad A_1 = \frac{2}{3} \times 180 \times 6 = 720$$

$$a_1 = 3\text{m}$$

$$A_2 = \frac{2}{3} \times 320 \times 8 = 1706.67$$

$$a_2 = 4 \text{ m}$$

40kN/m × 6m² / 8

Three moment equation for spans AB and BC is

$$M_A L_1 + 2M_B [L_1 + L_2] + M_C L_2 = -\frac{6A_1 a_1}{L_1} - \frac{6A_2 a_2}{L_2}$$

Since $I_1 = I_2 = I$, $h_A = h_C = 0$

Now in beam ABC, since A and C are simple supports

$$M_A = M_C = 0$$

$\therefore$ Three moment equation reduces to

$$2M_B [L_1 + L_2] = -\frac{6A_1 a_1}{L_1} - \frac{6A_2 a_2}{L_2}$$

$$2M_B (6 + 8) = -\frac{6 \times 720 \times 3}{6} - \frac{6 \times 1706.67 \times 4}{8}$$

$$\therefore \quad M = -260.0 \text{ kN-m}$$

Hence the bending moment diagram for continuous beam is as shown in the Fig.12.3(c).

Example 12.2 Analyse the two span continuous beam shown in Fig.12.4(a) using three moment theorem.

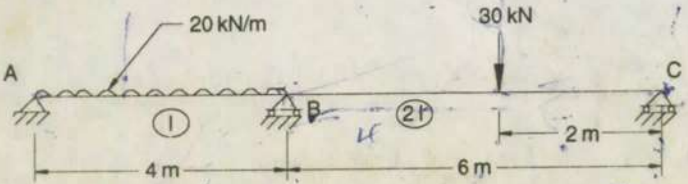


Figure 12.4 (a)

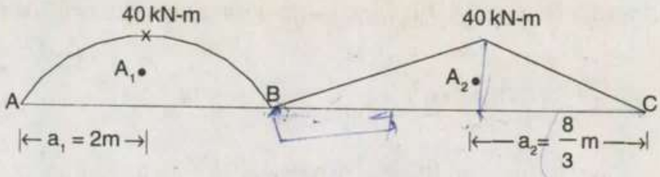


Figure 12.4 (b)

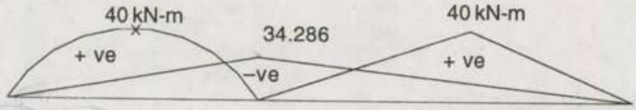


Figure 12.4 (c)

Solution

Free moment diagram for span AB is a parabola with maximum ordinate

$$= \frac{20\times 4^2}{8} = 40 \text{ kN-m at mid-psan}$$

Free moment diagram for span BC is a triangle with maximum ordinate

$$= \frac{30\times 4\times 2}{6} = 40 \text{ kN-m}$$

Area of free moment diagram in span AB,

$$A_1 = \frac{2}{3}\times 4\times 40 = 106.67$$

Distance of its C.G. from end A

$$a_1 = 2 \text{ m}$$

Area of free moment diagram in BC

$$A_2 = \frac{1}{2}\times 40\times 6 = 120$$

Distance of its C.G. from C

$$a_2 = \frac{6+2}{3} = \frac{8}{3} \text{ m}$$

Applying the three moment equation, we get

$$M_A \times \frac{L_1}{I_1} + 2M_B\left[\frac{L_1}{I_1}+\frac{L_2}{I_2}\right] + M_C\,\frac{L_2}{I_2} = -\frac{6A_1a_1}{I_1L_1} - \frac{6A_2a_2}{I_2L_2}$$

Since $M_A = M_C = 0$, we get

$$2M_B\left[\frac{4}{I}+\frac{6}{2I}\right] = -\frac{6\times 106.67}{I\times 4}\times 2 - \frac{6\times 120\times (8/3)}{2I\times 6}$$

$$M_B = -34.286 \text{ kN-m}$$

Hence, bending moment diagram is as shown in the Fig. 12.4(c).

Example 12.3 Analyse the continuous beam ABCD shown in Fig.12.5 if support C settles down by 5mm. Take E = 15 kN/mm². Moment of inertia is constant throughout and is equal to 5 x 10^9 mm⁴.

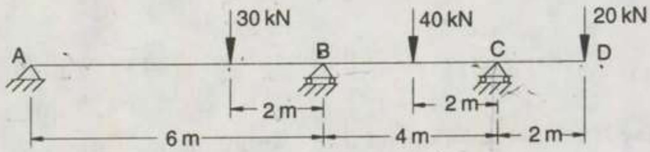


Figure 12.5 (a)

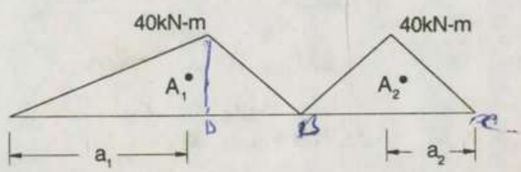


Figure 12.5 (b)

Solution

Free moment diagram for AB is a triangle with maximum ordinate under the load

$$= \frac{30\times4\times2}{6} = 40 \text{ kN-m}$$

Free moment diagram for BC is also a triangle with maximum ordinate under the load

$$= \frac{40\times2\times2}{4} = 40 \text{ kN-m}$$

Now, $M_A = 0$, since it is simply supported

Applying three moment theorem,

$$0 + 2M_B\frac{L_1+L_2}{1} + M_C\frac{L_2}{1} = -\frac{6A_1a_1}{IL_1} - \frac{6A_2a_2}{IL_2} + 0 + \frac{6Eh_C}{L_2}$$

Now, $L_1 = 6\text{m}, L_2 = 4\text{m}$

$$A_1 = \frac{1}{2}\times6\times40 = 120$$

$$a_1 = \frac{6+4}{3} = \frac{10}{3}\text{ m}$$

$$A_2 = \frac{1}{2}\times4\times40 = 80$$

$$a_2 = 2\text{m}$$
$$M_C = -20 \times 2 = -40 \text{ kN-m}$$
$$h_C = -5\text{mm},$$

(negative since C is below the mid-support B)

$$= -0.005 \text{ m}$$
$$E = 15 \text{ kN/mm}^2 = 15 \times 10^6 \text{ kN/m}^2$$
$$I = 5 \times 10^9 \text{ mm}^4 = 5 \times 10^{-3} \text{ m}^4$$

Multiplying by 'I' throughout and then substituting the various values, we get

$$2M_B\,(6 + 4) - 40 \times 4 = \frac{-6 \times 120(10/3)}{6} - \frac{6 \times 80 \times 2}{4}$$

$$+ \frac{6 \times 15 \times 10^6 (-0.005) \times 5 \times 10^{-3}}{4} = \frac{6EL}{L}$$

$$2M_B \times 10 = -10425$$

$$\therefore \quad M_B = -52.125 \text{ kN-m}$$

Thus, $M_A = 0$, $M_B = -58.125$ kN-m, $M_C = -40$ kN-m

Example 12.4 Determine the support moments in the continous beam shown in Fig. 12.6(a) by using three moment equations.

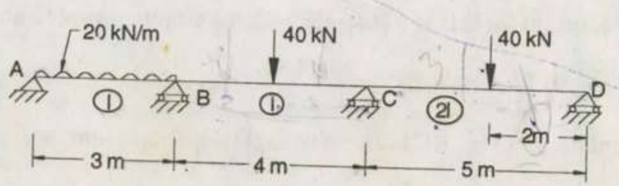


Figure 12.6 (a)

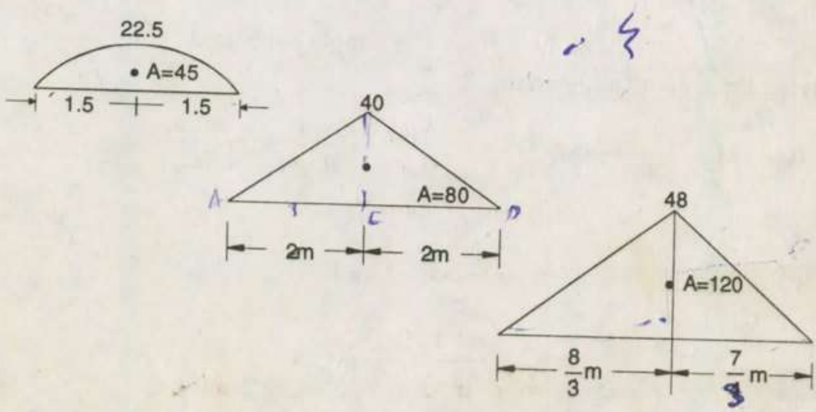


Figure 12.6 (b)

Solution

Free moment diagram :

In span AB, it is a parabola with its maximum ordinate at mid-span and it is given by

$$= \frac{20 \times 3^2}{8} = 22.5 \text{ kN-m}$$

$$\therefore \qquad \text{Area} = \frac{2}{3} \times 3 \times 22.5 = 45$$

In span BC, the free moment diagram is a triangle with its maximum ordinate under the load

$$= \frac{40 \times 4}{4} = 40 \text{ kN-m}$$

Hence its area $= \frac{1}{2} \times 40 \times 4 = 80$

In span CD, the free moment diagram is a triangle with its maximum ordinate under the load

$$= \frac{40 \times 3 \times 2}{5} = 48 \text{ kN-m}$$

Hence its area $= \frac{1}{2} \times 48 \times 5 = 120$

Distance of C.G. of this diagram, from C

$$= \frac{5+3}{3} = \frac{8}{3} \text{ m}$$

$\therefore$ Distance of C.G. of free moment diagram from D

$$= 5 - \frac{8}{3} = \frac{7}{3} \text{ m}$$

The free moment diagrams are shown in Fig. 12.6(b).

Now, applying three moment equation to span AB and BC, we get

$$M_A \frac{3}{I} + 2M_B\left(\frac{3}{I} + \frac{4}{I}\right) + M_C\left(\frac{4}{I}\right) = -\frac{6 \times 45 \times 1.5}{3 \times I} - \frac{6 \times 80 \times 2}{4 \times I}$$

But $M_A = 0$

Multiplying by 'I' throughout, we get

$$14M_B + 4M_C = -135 - 240$$

$$= -375 \qquad (1)$$

Applying three moment equation to span BC and CD, we get

$$M_B \frac{4}{I} + 2M_C\left(\frac{4}{I} + \frac{5}{2I}\right) + M_D\left(\frac{5}{2I}\right)$$

$$= -\frac{6 \times 80 \times 2}{4 \times I} - \frac{6 \times 120(8/3)}{2I \times 5}$$

Since D is simple support, $M_D = 0$. Multiplying by 'I' throughout, the above equation reduces to

$$\therefore \quad 4M_B + 13M_C = -240 - 192$$

$$= -432 \qquad (2)$$

Multiplying equation (2) by $\frac{4}{13}$ throughout, we get

$$1.231\,M_B + 4\,M_C = -132.921$$

Subtracting equation (3) from the equation (1) we get

$$12.769\,M_B = -242.079$$

$$\therefore \quad M_B = -18.958 \text{ kN-m}$$

Substituting it in eqn.(1), we get

$$-265.412 + 4\,M_C = -375$$

or $\quad M_C = -27.397$ kN-m

Thus $M_A = 0$, $M_B = -18.958$ kN-m, $M_C = -27.397$ and $M_D = 0$

Example 12.5 Analyse the continuous beam shown in Fig.12.7 and determine the moments at all supports.

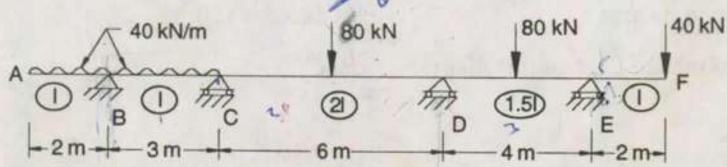


Figure 12.7 (a)

Solution

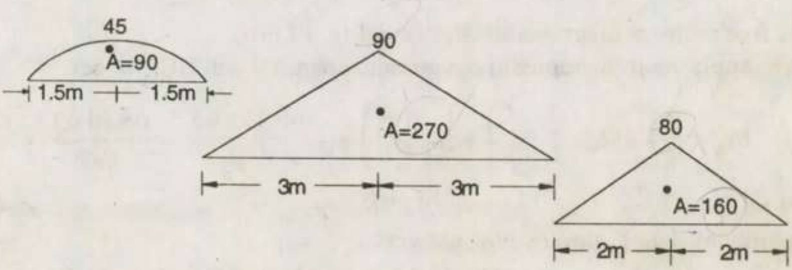


Figure 12.7 (b)

Free moment diagrams:

Span BC : It is a parabola with its C.G. at mid-span and maximum ordinate

$$= \frac{40 \times 3^2}{8} = 45 \text{ kN-m}$$

Area of moment diagram $= \dfrac{2}{3} \times 3 \times 45 = 90$

Span CD : It is a triangle with its C.G. at mid-span and maximum ordinate

$$\frac{60 \times 6}{4} = 90 \text{ kN-m} \qquad = \frac{80 \times 3 \times 3}{6} = 120 \text{ kNm}$$

Area of this diagram $= \dfrac{1}{2} \times 6 \times 90 = 270$ $\quad = \tfrac{1}{2} \times 6 \times 120 = 360$

Span DE : It is a triangle with its C.G. at mid-span and maximum ordinate

$$= 80 \times \frac{4}{4} = 80 \text{ kN-m}$$

$\therefore$ Area of moment diagram $= \dfrac{1}{2} \times 4 \times 80 = 160$

The details of free moment diagrams are shown in Fig.12.7(b). Applying the three moment equation to span BC and CD,

$$M_B\left(\frac{3}{I}\right) + 2M_C\left(\frac{3}{I} + \frac{6}{2I}\right) + M_D\left(\frac{6}{2I}\right)$$

$$= -\frac{6 \times 90 \times 1.5}{I \times 3} - \frac{6 \times 270 \times 3}{2I \times 6}$$

But $\qquad M_B = -\dfrac{40 \times 2^2}{2} = -80$ kN-m.

Substituting it in the above equation and multiplying by 'I' throughout, we get

$$-240 + 12\,M_C + 3M_D = -270 - 405$$

or $\qquad 4M_C + M_D = -145 \qquad \ldots(1)$

Applying the three moment equation to span CD and DE, we get

$$M_C\left(\frac{6}{2I}\right) + 2M_D\left(\frac{6}{2I} + \frac{4}{1.5I}\right) + M_E\frac{4}{1.5I} = -\frac{6 \times 270 \times 3}{2I \times 6} - \frac{6 \times 160 \times 2}{1.5I \times 4}$$

But $M_E = -40 \times 2 = -80$ kN-m. Substituting it in the above equation and multiplying 'I' throughout, we get

$$3M_C + 11.333M_D - 213.333 = -405 - 320$$

or $\qquad 3M_C + 11.333M_D = -511.667 \qquad \ldots(2)$

Multiplying eqn. (2) by $\dfrac{4}{3}$, we get

$$4M_C + 15.111M_D = -682.222 \qquad \ldots(3)$$

60×

Subtracting eqn.(1) from eqn.(3) we get

$$14.111M_D = -537.222$$

or $$M_D = -38.071 \text{ kN-m}$$

Substituting it in eqn.(1), we get

$$4M_C - 38.071 = -145$$

$$M_C = -26.732 \text{ kN-m}$$

Thus, $M_B = -80$ kN-m, $M_C = -26.732$ kN-m, $M_D = -38.071$ kN-m

and $$M_E = -80 \text{ kN-m}$$

Example 12.6 Analyse the continuous beam shown in Fig. 12.8, if support B sinks by 10mm, by using three moment equations. Draw the shear force diagram, bending diagram and elastic curve. Take E = 15 × 10[6] kN/m², I = 4 × 10[9] mm⁴.

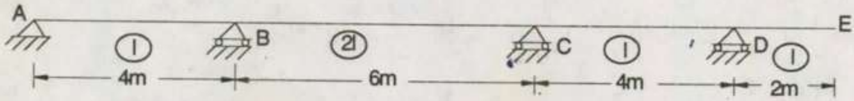


Figure 12.8 (a)

Solution

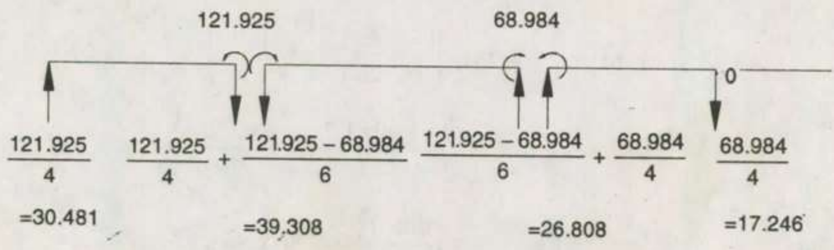


Figure 12.8 (b)

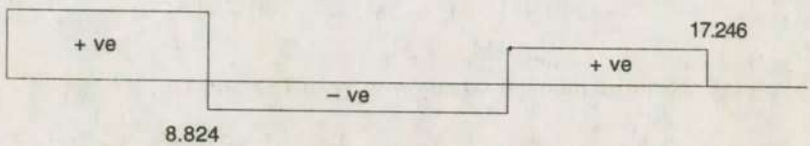


Figure 12.8 (c)

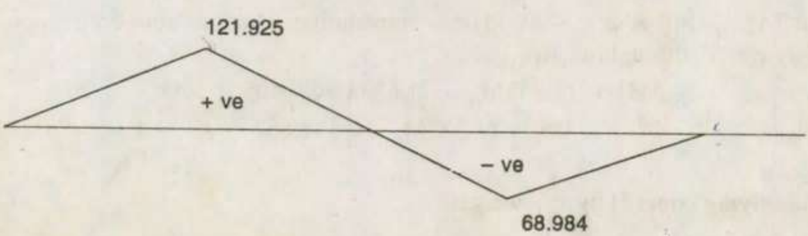


Figure 12.8 (d)

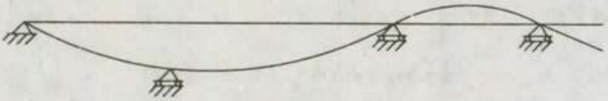


Figure 12.8 (e)

Consider the spans AB and BC.

Support B sinks. Hence $h_A = h_C = 10\text{mm} = 0.010$ m

$$E = 15 \times 10^6\ \text{kN/m}^2,$$

$$I = 4 \times 10^9\ \text{mm}^4 = 4 \times 10^{-3}\text{m}^4$$

There is no external loading. Hence the free moment diagrams are all equal to zero. Applying the three moment equation to spans AB and BC, we get

$$M_A\left(\frac{4}{I}\right) + 2M_B\left(\frac{4}{I}+\frac{6}{2I}\right) + M_C\left(\frac{6}{2I}\right)$$

$$= -0 - 0 + \frac{6\times15\times10^6\times0.010}{4} + \frac{6\times15\times10^6\times0.010}{6}$$

$$4M_A + 14M_B + 3M_C = 6\times15\times10^6\left(\frac{0.010}{4}+\frac{0.010}{6}\right)I$$

$$= 6\times15\times10^6\times0.010\left(\frac{1}{4}+\frac{1}{6}\right)I$$

$$= 6\times15\times10^6\times0.010\left(\frac{1}{4}+\frac{1}{6}\right)4\times10^{-3}$$

i.e., $14M_B + 3M_C = 1500,$ Since $M_A = 0$

Consider spans BC and CD. Support B is 10 mm below the middle support C.

Hence, $h = -10$ mm $= -0.010$ m

$$h_D = 0$$

Applying the three moment equation, we get

$$M_B\left(\frac{6}{2I}\right) + 2M_C\left(\frac{6}{2I}+\frac{4}{I}\right) + M_D\left(\frac{4}{I}\right)$$

$$= -0 - 0 + \frac{6\times15\times10^6\times(-0.010)}{6} + 0$$

$$3M_B + 14M_C + 4M_D = \frac{6\times15\times10^6\times(-0.010)}{6}\times4\times10^{-3}$$

Since $I = 4\times10^{-3}\,\text{m}^4$

But $M_D = 0$

$\therefore$ $3M_B + 14M_C = -600$ (2)

Multiplying eqn.(2) by $\frac{14}{3}$, we get

$$14\,M_B + 65.333\,M_C = -2800 \tag{3}$$

Subtracting eqn.(1) from eqn.(3), we get

$$62.333\,M_C = -4300$$

$\therefore$ $$M_C = -68.984 \text{ kN-m}$$

Substituting it in the eqn.(2), we get

$$3M_B - 965.776 = -600$$

$$M_B = 121.925 \text{ kN-m}$$

Fig. 12.8(b) shows the end moments in the various segments of continuous beam along with the reactions. Shear force diagram is shown in Fig.12.8(c) and the Bending moment and elastic curves are shown in Fig.12.8(d) and 12.8(e) respectively.

Example 12.7 Analyse the continuous beam ABCD shown in Fig. 12.9(a), if support C sinks by 5 mm.

Given E = 15 kN/mm², I = 5 × 10⁹ mm⁴.

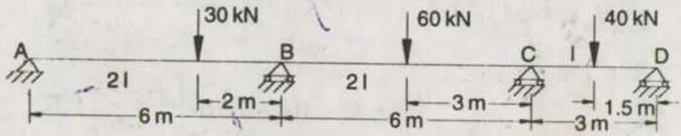


Figure 12.9 (a)

Solution

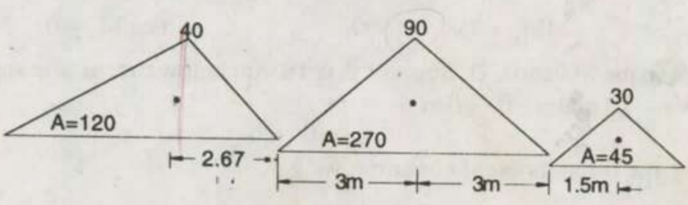


Figure 12.9 (b)

Free moment diagrams :

Span AB : It is a triangle with maximum ordinate under the load and its value

$$= \frac{30 \times 4 \times 2}{6} = 40 \text{ kN-m}$$

$\therefore$ $$A = \frac{1}{2} \times 40 \times 6 = 120$$

Distance of C.G. from B $= \frac{6+2}{3} = \frac{8}{3} = 2.667$ m

Span BC : It is a triangle with maximum ordinate under the load and its value,

$$\frac{60\times6}{4} = 90 \text{ kN-m}$$

$$\therefore \quad A = \frac{1}{2}\times6\times90 = 270$$

Span CD : Free moment diagram is a triangle, with its maximum ordinate at the centre of the span and its value

$$= \frac{40\times3}{4} = 30 \text{ kN-m}$$

$$\therefore \quad A = \frac{1}{2}\times3\times30 = 45$$

Condenser the spans AB and BC.

Support C is 5mm below the mid-support B.

Hence $h_C = -5\text{mm} = -0.005$ m

$E = 15$ kN/mm^2 $= 15 \times 10^6$ kN/m^2 and

$I = 5 \times 10^9$ mm^4 $= 5 \times 10^{-3}$ m^4

Applying three moment equation to spans AB and BC, we get

$$M_A\frac{6}{2I}+2M_B\left(\frac{6}{2I}+\frac{6}{2I}\right)+M_C\left(\frac{6}{2I}\right) = -6\times\frac{120\times 2.667}{2I\times6}$$ 5.33

$$-\frac{6\times270\times3}{2I\times6}+\frac{6\times15\times10^{6}(-0.005)}{6}$$

Noting $M_A = 0$ and multiplying throughout by 'I', we get

$$12M_B + 3M_C = -160 - 145 - \frac{6\times15\times10^{6}(0.005)\times5\times10^{-3}}{6}$$

Since $I = 5 \times 10^{-3}$ m^4

i.e. $12M_B + 3M_C = -160 - 405 - 375$

$= -940$

Consider the spans BC and CD.

Support B and D are 5mm above mid-support C

Hence $h_B = 5\text{mm} = 0.005 \text{ m} = h_D$

Applying three moment equation, we get

$$M_B\frac{6}{2I}+2M_C\left(\frac{6}{2I}+\frac{3}{I}\right)+M_D\left(\frac{3}{I}\right) = -\frac{6\times270\times3}{2I\times6}-\frac{6\times45\times1.5}{I\times3}+\frac{6\times15\times10^6\times0.005}{6}$$

$$+\frac{6\times15\times10^6\times0.005}{3}$$

Noting that M_D is zero and throughout multiplying by 'I' and substituting the value of 'I', we get

$$3M_B + 12M_C = -405 - 135 + \frac{6\times15\times10^6\times0.005\times5\times10^{-3}}{6}$$

$$+\frac{6\times15\times10^6\times0.005\times5\times10^{-3}}{3}$$

$$= -405 - 135 + 375 + 750$$

$$= 585 \tag{2}$$

Multiplying eqn.(2) by 4 we get

$$12M_B + 48M_C = 2340$$

Subtracting eqn.(1) from it, we get

$$45M_C = 2340 + 940$$

$$M_C = 72.889 \text{ kN-m}$$

Substituting the value of M_C in eqn.(2) we get

$$3M_B + 874.667 = 585$$

$$M_B = -96.556 \text{ kN-m}$$

Thus $M_A = 0, M_B = -96.556$ kN-m, $M_C = 72.889$ kN-m, $M_D = 0$

12.4 APPLICATION OF THREE MOMENT EQUATION TO PROBLEMS WITH FIXED END

Sometimes, the end support of continuous beams may be fixed support. At such support, the end moment exist but the slope is zero. Such problems also may be solved by using three moment equation treating such case as follows :

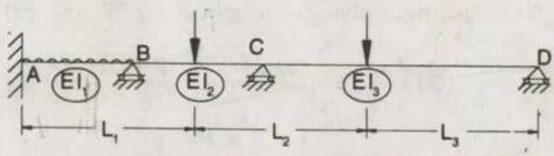


Figure 12.10 (a)

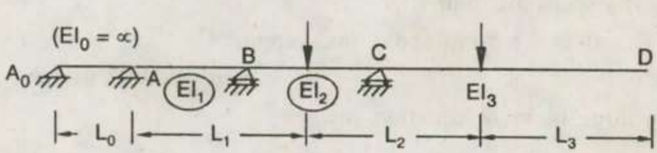


Figure 12.10 (b)

Support A is fixed as seen in Fig 12.10. Add an imaginary Span $A_o A$ of length L_o with infinite flexural rigidity (EI=∝). A_o being at the same level as A. Since EI is infinitely large, slope of beam A_oA at A is zero. If the three moment equation is applied to spans A_0A and AB, because of the continuity requirement at mid-span A, the slope of beam AB at A will also be zero. Thus, three moment equation for spans A_0A and AB is

$$M_o \frac{L_o}{\alpha} + 2M_A\left(\frac{L_o}{\alpha}+\frac{L_1}{I_1}\right) + \left(\frac{L_1}{I_1}\right)$$

$$= -\frac{6A_oa_o}{\alpha L_1} - \frac{6A_1a_1}{I_1L_1} + \frac{6Eh_o}{L_o} + \frac{6Eh_B}{L_1}$$

Noting that h_o is zero and any quantity when divided by ∝ is also zero, the three moment equation reduces to

$$2M_A\left(\frac{L_1}{I_1}\right) + MB\left(\frac{L_1}{I_1}\right) = \frac{6A_1a_1}{I_1L_1} + \frac{6Eh_B}{L_1} \qquad 12.4$$

Equation 12.4 satisfies the fixed end condition. Hence using it, fixed end cases may be handled. The procedure is illustrated with examples below

Example 12.8 Analyse the continuous beam shown in Fig.12.11(a), by using three moment equation and draw the bending moment diagram.

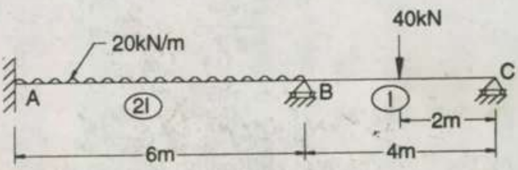


Figure 12.11 (a)

Solution

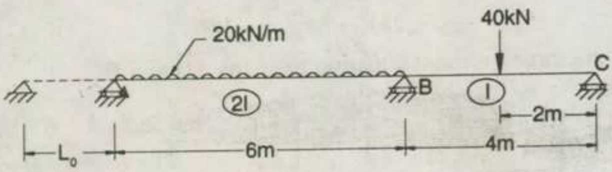


Figure 12.11 (b)

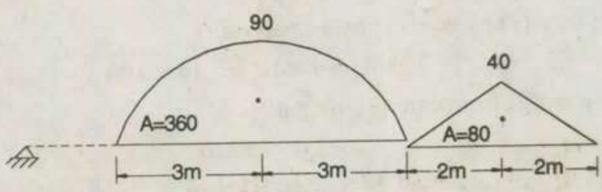


Figure 12.11 (c)

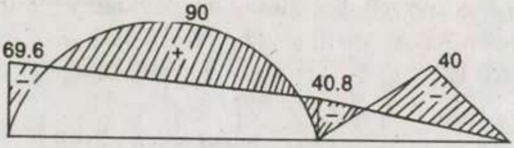


Figure 12.11 (d)

To take care of fixed end at A, an imaginary beam AA_o of span L_o and flexural rigidity $EI_o = \infty$ is considered as shown in Fig. 112.11(b).

Free moment diagrams :

On Span A_oA, A = 0

Span AB : It is a parabola, with maximum ordinate at mid-span

$$\frac{wL^2}{8} \qquad = \frac{20\times6^2}{8} = 90 \text{ kN-m}$$

$$\therefore \text{Area of moment diagram} \quad = \frac{2}{3}\times 6\times 90 = 360$$

Span BC : It is a triangle with maximum ordinate at mid-span and its value

$$= \frac{40\times4}{4} = 40 \text{ kN-m}$$

$$\text{Area of this moment diagram} \quad = \frac{1}{2}\times 4\times 40 = 80$$

Now, applying the three moment equation to span A_oA and AB, we get

$$M_o\left(\frac{L_o}{\alpha}\right) + 2M_A\left(\frac{L_o}{\alpha}+\frac{6}{2I}\right) + M_B\left(\frac{6}{2I}\right) = -\frac{6\times360\times3}{2I\times6}$$

$$6M_A + 3M_B = -540$$

$$2M_A + M_B = -180 \qquad ...(1)$$

Applying the three moment equation to spans AB and BC,

$$M_A\left(\frac{6}{2I}\right) + 2M_B\left(\frac{6}{2I}+\frac{4}{I}\right) + M_C\left(\frac{4}{I}\right) = -\frac{6\times360\times3}{2I\times6} - \frac{6\times80\times2}{I\times4}$$

Noting that $M_C = 0$, we get

$$3M_A + 14M_B = -780 \qquad ...(2)$$

Multiplying eqn. (1) by eqn. (1) by 14, we get

$$28M_A + 14MB = -14\times180 = -2520 \qquad ...(3)$$

Subtracting eqn. (2) from eqn. (3), we get

$$25M_A = -1740$$

$$M_A = -69.6 \text{ kN-m}$$

Substituting it in eqn.(1), we get

$$2 \times (-69.6) + M_B = -180$$

$$M_B = -40.8 \text{ kN-m}$$

The bending moment diagram is as shown in Fig. 12.11(d)

Example 12.9 Analyse the continuous beam ABC shown in Fig.12.12(a), if the support B sinks by 10 mm . Given E = 200 kN/mm² and I = 1 × 10⁸ mm⁴.

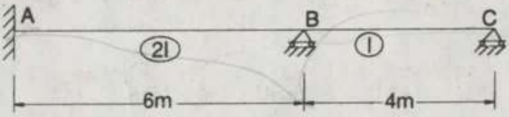


Figure 12.12 (a)

Solution

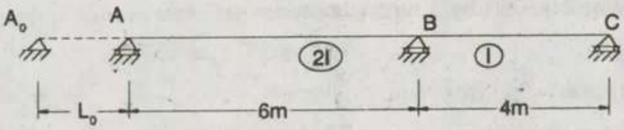


Figure 12.12 (b)

To handle fixed end case at end A, end A is made continous support, by adding imaginary beam A_0A of span L_0 and flexural rigidity $EI_0 = \infty$. Consider the spans A_0A and AB.

Support B is 10 mm below mid-support A.

Hence $h_B = -10 \text{ mm} = -0.010 \text{ m}$

$E = 200 \text{ kN/mm}^2 = 220 \times 10^6 \text{ kN/m}^2$

$I = 1 \times 10^8 \text{ mm} = 1 \times 10^{-4} \text{ m}^4$

The three moment equation for span A_0A and AB is

$$M_0 \frac{L_0}{\alpha} + 2M_A\left(\frac{L_0}{\alpha} + \frac{6}{2I}\right) + M_B\left(\frac{6}{2I}\right)$$

$$= -0 - 0 + 0 + \frac{6 \times 200 \times 10^6(-0.010)}{6}$$

$$6M_A + 3M_B = -200 \times 10^4 \times I$$

Substituting the values of I, we get ...(1)

$$6M_A + 3M_B = -200$$

Consider spans AB and BC. Since B sinks by 10mm, support A and C are 10mm above the position of mid-support B.

Hence $h_A = h_C = 10 \text{ mm} = 0.010 \text{ m}$

∴ The three moment equation for these two spans is

$$M_A \frac{6}{2I} + 2M_B\left(\frac{6}{2I}+\frac{4}{I}\right) + M_C\left(\frac{4}{I}\right)$$

$$= -0 - 0 + \frac{6\times 200\times 10^6 \times 0.010}{6} + \frac{6\times 200\times 10^6 \times 0.010}{4}$$

Noting that $M_C = 0$, and multiplying throughout by 'I', we get

$$3M_A + 14M_B = 6\times 200\times 10^6\times 0.001\left(\frac{1}{6}+\frac{1}{4}\right)\times I$$

$$= 6\times 200\times 10^6\times 0.001\left(\frac{1}{6}+\frac{1}{4}\right)\times 1\times 10^{-4}$$

i.e., $$3M_A + 14M_B = 50$$

Multiplying eqn. (2) by 2 throughout, we get

$$6M_A + 28M_B = 100$$

Subtracting eqn. (1) from eqn. (3), we get

$$25M_B = 300 \qquad ..(3)$$

$$M_B \doteq 12 \text{ kN-m}$$

Substituting it in eqn. (1), we get

$$6M_A + 3\times 12 = -200$$

∴ $$M_A = -27.333 \text{ kN-m}$$

Thus, $M_A = -27.333$ kN-m, $M_B = 12$ kN-m, $M_C = 0$.

EXERCISES

12.1 Analyse the continuous beam ABCDE shown in Fig.12.13 and draw the bending moment diagram.

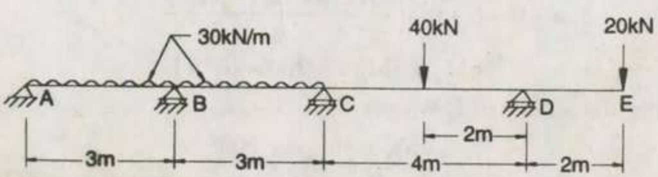


Figure 12.13

Ans : $M_B = -30.330$ kN-m, $M_C = -13.367$ kN-m

12.2 Analyse the continuous beam ABCDE shown in Fig.12.14 if support C sinks by 8mm. Given $E = 200\ kN/mm^2$ and $I = 0.8 \times 10^8\ mm^4$.

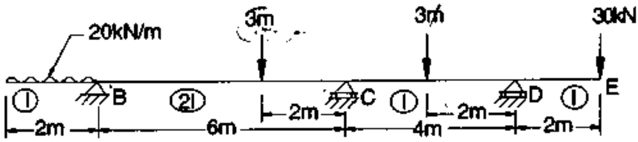


Figure 12.14

Ans : $M_C = -14.286$ kN-m

12.3 Analyse the beam ABC shown in Fig.12.15 and draw the bending moment diagram.

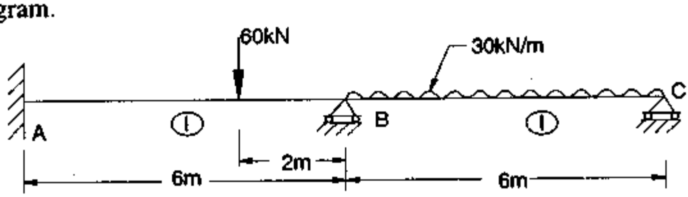


Figure 12.15

Ans : $M_A = -103.33$ kN-m; $M_B = -100$ kN-m

12.4 Analyse beam ABC shown in Fig.12.16 and draw the bending moment diagram if support B sinks by 10mm. Given $E = 15\ kN/mm^2$ and $I = 5 \times 10^9\ mm^4$

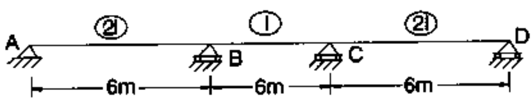


Figure 12.16

Ans : $M_B = 170.833$ kN-m, $M_C = -129.165$ kN-m

12.5 Analyse the beam shown in Fig.12.17 and draw the bending moment diagram, if support B yields by 10mm. Take $E = 15\ kN/mm^2$ and $I = 0.4 \times 10^9\ mm^4$.

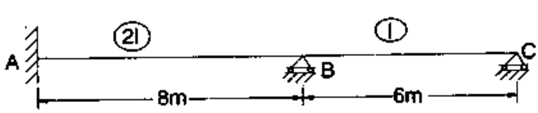


Figure 12.17

Ans : $M_A = -109.375$ kN-m, $M_B = 106.25$ kN-m

Appendix :

ANALYSIS OF PIN-JOINTED PLANE FRAMES

A1 INTRODUCTION

A **truss** is a structure made up of slender members pin-connected at ends. They are capable of taking the loads at joints. They are also known as **pin-connected frames**. Roof trusses are used to support sloping roofs. Bridge trusses support deck slabs. Many times trusses are used to support or lift machinery. Transmission line towers are also the examples of trusses. Since the members of trusses are slender and pin-connected they resist applied forces only by developing axial forces. No bending of members takes place.

The trusses in which all the members are in a single plane and the loads and reactions act in the plane of the truss are called *plane trusses*. Roof trusses and bridge trusses are the examples of *plane frames*. Tripod and transmission towers are the examples of *space trusses* since the members of these trusses do not fall in a single plane and they are subjected to forces in any plane. In this text only analysis of plane trusses is dealt.

A2 STABILITY OF PLANE FRAMES

When members are slender and are pin-connected at the ends, it may be easily observed that rectangle or quadrilateral (Fig. A1) is not a stable shape for the loads applied at the joints. A truss having such shape is called **deficient truss** and it cannot have any practical utility.

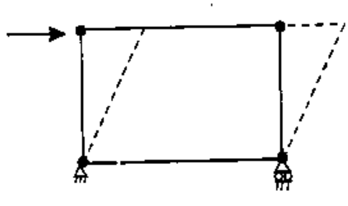


Figure A1 Deficient (Unstable) Truss

Triangle is a stable shape for pin-connected trusses. Figures A2 to A5 show some of such trusses. In these trusses load applied in the plane of frame at any joint in any direction is resisted without any visible changes in the shape. Such frames are called

perfect frames. In these trusses we observe that there are 3, 5 and 7 members when the number of joints are 3, 4 and 5 respectively. We need three members to create

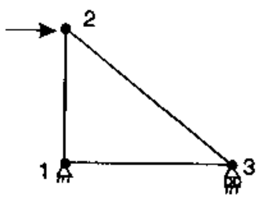


Figure A2 Stable Frame with 3 joints

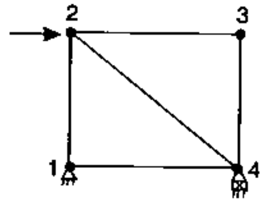


Figure A3 Stable Frame with 4 joints

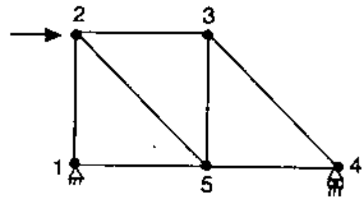


Figure A4 Stable Frame with 5 joints

three-joint frame. Then onward we need two members to add one joint. Hence, the relationship between the number of members and the number of joints in a pin-connected perfect plane truss is

$$m = 2j - 3 \qquad \text{...A1}$$

where m = number of members

j = number of joints.

However, the above relation is only a guideline for identifying a perfect frame, and it should not be taken as the criteria for identifying a perfect frame. For example, the trusses shown in Fig. A5(*a*) and A5(*b*) satisfy equation A1. But the truss in Fig. A5(b) is not a perfect frame. It is ultimately the stable triangular shapes in the truss which indicate the stability of the trusses.

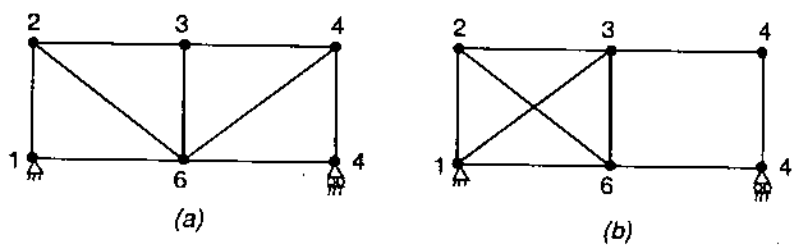
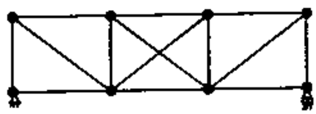


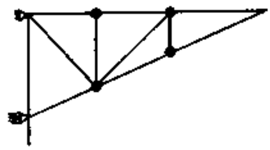
Figure A5

A3 EQUILIBRIUM OF PLANE FRAMES

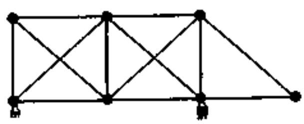
Treating entire frame as a rigid body, equilibrium equations can be applied to system of forces consisting of applied forces and reactions. If there are only three reaction components to be determined then it is possible to find them by using equations of equilibrium only. Such frames are called *externally determinate frames*. Figure A6 shows few cases of externally determinate frames. An externally determinate frame, if it belongs to the category of perfect frame, the forces in all the members of the truss can be determined using equations of equilibrium only. Hence, the perfect frame is said to possess statically zero internal indeterminancy. A truss is called a redundant truss if the number of members are more than the required number for a perfect frame. Each additional member contributes to one degree of statical internal indeterminancy. Thus, the truss shown in Fig. A6(*a*) is having an internal indeterminancy of one degree and the truss shown in Fig. A6(*c*) is having two degrees of statical internal indeterminancy.



(a) One End Hinged, another on Roller



(b) Cantilever Type Truss



(c) Overhanging Truss

***Figure A6** Externally Determinate Trusses*

Total Degree of indeterminancy of a truss

= Degree of external indeterminancy + Degree of internal indeterminancy.

∴ Degree of indeterminancy of truss shown in Fig. A6(*c*) = 0 + 2 = 2

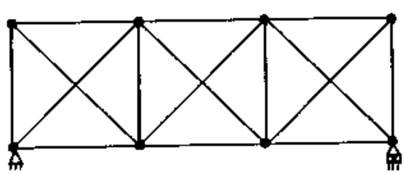


Figure A7

Degree of indeterminancy of frame shown in Fig. A7 = 3 + 2 = 5

A4 ANALYSIS OF FORCES IN TRUSS MEMBERS

In this chapter analysis of only determinate trusses is taken up. The following methods are available for the analysis:

(i) Method of Joints (ii) Method of Tension Coefficient
(iii) Method of Section (iv) Matrix method
(v) Graphical method.

The first three methods are explained here.

A5 METHOD OF JOINTS

In this method, first reaction components are found by applying equations of equilibrium to the entire truss. However, in cantilever frames one can straightaway go for joint analysis.

Imagine a cut in each member meeting at a joint. The forces in the members so cut along with the loads and reactions (if any) form a system of concurrent forces in equilibrium. Hence, at each joint two equations of equilibrium can be written. If there are only two unknown forces, they can be found using equations of equilibrium. Hence, start the analysis from the joint where there are only two unknown forces. Then take up another joint where now the number of unknowns have reduced to two. This way proceed from joint to joint and find the forces in all the members.

It may be noted that if there are 'j' number of joints, the number of equations of equilibrium available is $2j$. There are 'm' number of unknown member forces and 3 unknown reaction components. Thus, the total number of unknowns are $m + 3$. Hence, if $m + 3 = 2j$, it becomes a statically determinate problem. This condition is the same as eqn. A1 ($m = 2j - 3$). Hence, a perfect frame is a statically determinate frame.

Assumptions Made in the Analysis

The following assumptions are made in the analysis of plane trusses:

1. The ends of the members are having perfect pin connections.
2. The self weight of the truss is negligible.
3. Loads act at joints only.
4. At any joint, the axes of all members meeting pass through a single point.

With the above assumptions, member forces shall be only along the axis of the member. In other words, members are subjected to either axial tensile force or axial compressive forces. No bending of members takes place.

Identifying Force as Tensile or Compressive

Figure A8 shows a frame in which member BC is having compressive force and member AF is having tensile force. The joint forces are exactly equal but opposite to the member forces for the equilibrium state. During the analysis, the forces on the joints are indicated in the figures. Hence, if the joint forces are towards the joint [Ref. Fig. A8(b)], force in the member is compressive and if the member forces are away from the joint, the force in the member is tensile.

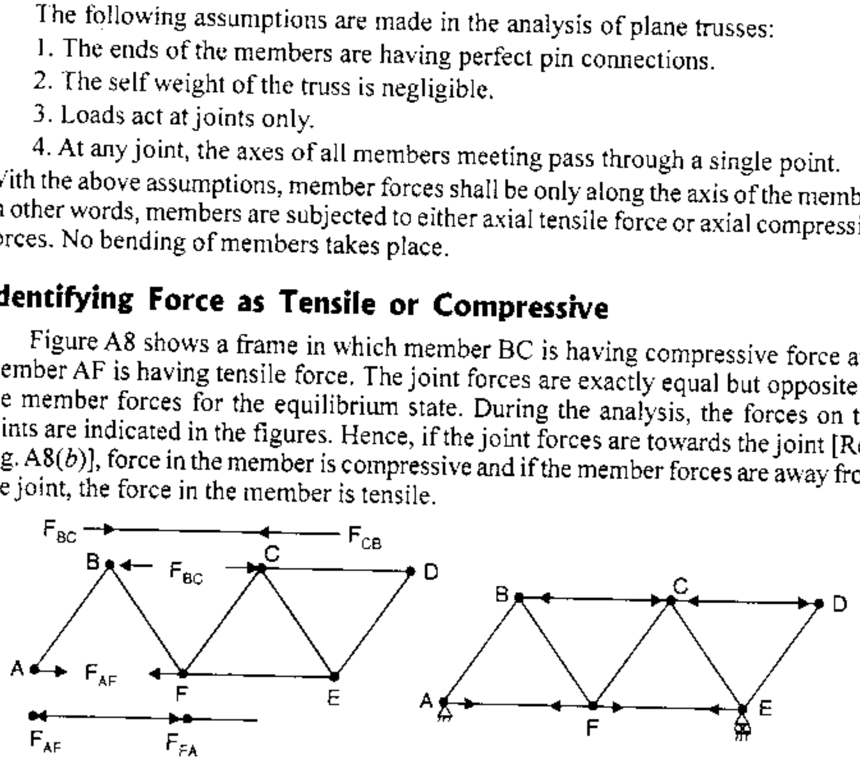


Figure A8

The analysis of pin-jointed trusses by method of joint is illustrated with a set of problems below:

***Example* A1** Determine the forces developed in the members of the truss shown in Fig. A9(a).

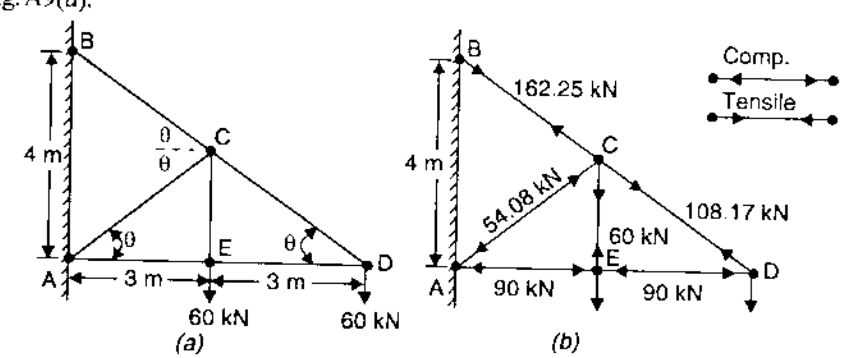


Figure A9(a&b)

Solution

In this case inclined members are at angle θ to the horizontal, where

$$\tan\theta = \frac{4}{6}$$

$$\therefore \qquad \theta = 33.69°$$

In this truss at joint D, there are only two unknown forces. Hence, the analysis can be started without finding the reactions. Free body diagram of joint D is as shown in Fig. A9(*c*). To balance the downward load of 60 kN the force on joint from the member CD should be away from the joint. Similarly, to balance the horizontal force from member CD, the force in member DE should be towards joint [*Note*: If the member forces are assumed the other way, the force comes out to be negative. Then, it is to be reversed and noted].

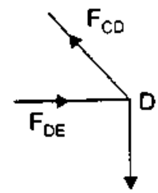


Figure A9(c)

At joint D:

Summation to forces in vertical directions = 0 (ΣV = 0) gives,

$$F_{CD} \sin\theta = 60$$

$$F_{CD} = \frac{60}{\sin 33.69} = 108.17 \qquad \text{(tensile, since it acts away from the joint.)}$$

Summation of forces in horizontal directions = 0 (ΣH = 0) gives

$$F_{DE} - F_{CD}\cos 33.69 = 0$$

i.e., $\quad F_{DE} = 108.17 \cos 33.69 = 90\text{ kN} \qquad$ (compressive)

These forces are marked in Fig. A9(*b*) near joint D. At the other end of the members forces are marked in opposite directions to give joint forces at C and D respectively.

Now, there are only two unknown forces at joint E as shown is Fig. A9(*d*). At this joint,

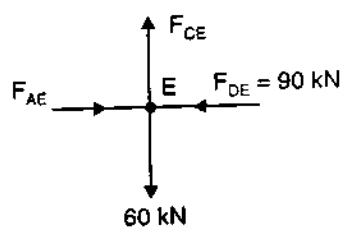


Figure A9(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{EC} = 60\text{ kN} \qquad \text{(tensile)}$$

Now, $\quad \Sigma H = 0 \rightarrow$

$$F_{AE} = F_{DE} = 90\text{ kN} \quad \text{(compressive)}$$

These forces are marked in Fig. A9(*b*). Now, there are only two unknown forces at joint C as shown is Fig. A9(*e*). Let their directions be assumed as shown in the figure.

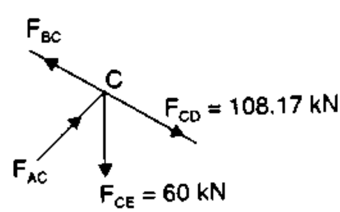


Figure A9(e)

Now, $\quad \Sigma V = 0 \rightarrow$

$$F_{BC} \sin\theta + F_{AC} \sin\theta - 60 - 108.17 \sin\theta = 0$$

or $\quad F_{BC} + F_{AC} = 216.33 \quad (\text{since } \theta = 33.69°) \quad \text{...(i)}$

$$\Sigma H = 0 \rightarrow$$

$$F_{BC} \cos\theta - F_{AC} \cos\theta - 108.17 \cos\theta = 0$$

or $\qquad F_{BC} - F_{AC} = 108.17 \qquad$ ...(ii)

Adding equations (i) and (ii) we get,

$$2F_{BC} = 324.5$$

$\therefore \qquad F_{BC} = 162.25$ kN $\qquad$ (compressive)

Substituting it in eqn (i), we get

$$F_{AC} = 216.33 - F_{AC}$$

$$= 216.33 - 162.25 = 54.08 \text{ kN} \qquad \text{(compressive)}$$

$\therefore$ Assumed directions of F_{BC} and F_{AC} are correct. These forces are marked in Fig. A9(b). Now all forces are known. They are listed below.

Member	*Magnitude*	*Nature*
BC	162.25 kN	Tensile
CD	108.17 kN	Tensile
DE	90 kN	Compressive
AE	90 kN	Compressive
CE	60 kN	Tensile
AC	54.08 kN	Compressive

Another method of showing member forces is to write the values near the members as shown in Fig. A9(b) and clearly indicate sign convention for tensile and compressive forces.

***Example* A2** Determine the forces in all members of the truss shown in Fig. A10(a). Indicate them on the sketch of the truss.

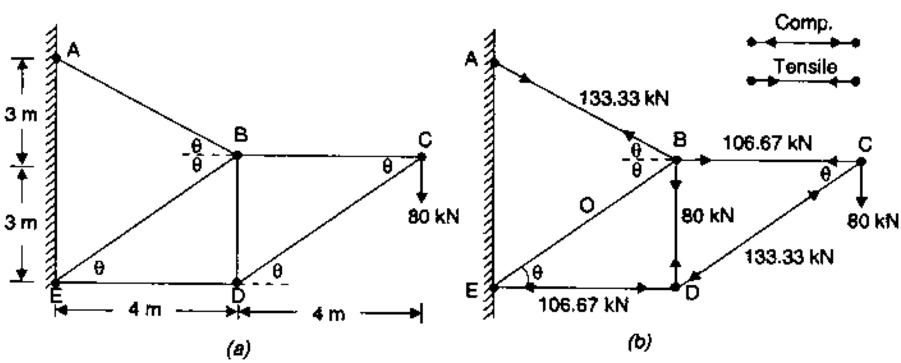


Figure A10(a&b)

Solution

Referring to Fig. A10(a),

$$\tan\theta = \frac{3}{4} \qquad \therefore \quad \theta = 36.87°$$

Consider the equilibrium of joint C (Ref. Fig. A 10(c)).

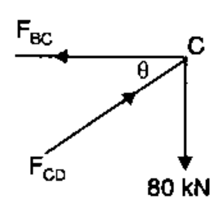


Figure A10(c)

$$\Sigma V = 0 \rightarrow$$

$$F_{CD} \sin 36.87 = 80$$

$\therefore$ $F_{CD} = 133.33$ kN (compressive)

$$\Sigma H = 0 \rightarrow$$

$$F_{BC} - 133.33 \cos\theta = 0$$

$\therefore$ $F_{BC} = 133.33 \cos 36.87 = 106.67$ kN (tensile)

Now consider the equilibrium of joint D (Ref. Fig. A10(*d*)).

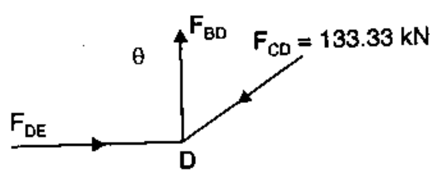


Figure A10(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{CD} \sin\theta - F_{BD} = 0$$

$$F_{BD} = F_{CD} \sin\theta = 133.33 \sin 36.87$$

$$= 80 \text{ kN} \quad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{DE} - F_{CD} \cos\theta = 0$$

$\therefore$ $F_{DE} = F_{CD} \cos\theta = 133.33 \cos 36.87 = 106.67$ kN (compressive)

Now consider the equilibrium of joint B (Fig. A10(*e*)):

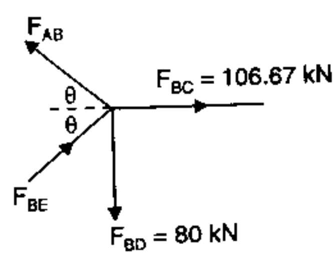


Figure A10(e)

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} \sin\theta + F_{BE} \sin\theta - 80 = 0$$

$\therefore$ $$F_{AB} + F_{BE} = \frac{80}{\sin\theta} = \frac{80}{\sin 36.85} = 133.33 \text{ kN} \qquad \text{...(i)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AB} \cos\theta - F_{BE} \cos\theta - 106.67 = 0$$

$\therefore$ $$F_{AB} - F_{BE} = \frac{106.67}{\cos 36.87} = 133.33 \qquad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$2F_{AB} = 2 \times 133.33$$

$$\therefore \quad F_{AB} = 133.33 \text{ kN} \qquad \text{(tensile)}$$

Substituting it in eqn (i) we get $F_{AB} = 0$.

All member forces are indicated in Fig. A10(*b*).

***Example* A3** Determine the forces in all the members of the truss shown in Fig. A11(*a*).

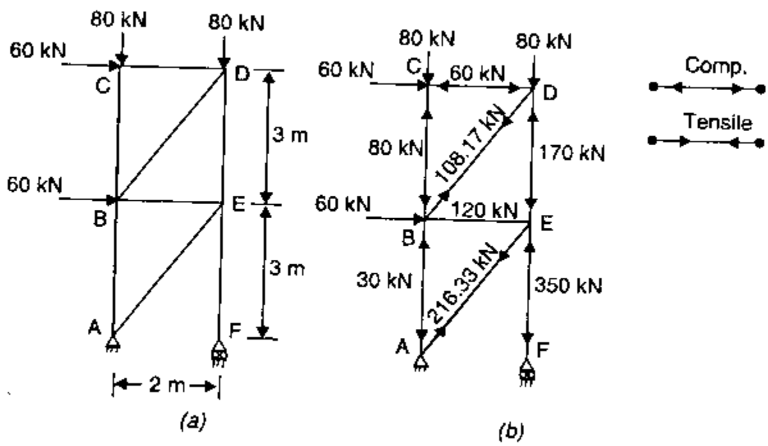


Figure A11(a&b)

Solution

Referring to Fig. A11(*a*),

$$\theta = \tan^{-1}\frac{3}{2} = 56.31°$$

The analysis may be started from joint C, where there are only two unknown forces. Two forces acting on joint C are as shown in Fig. A11(*c*).

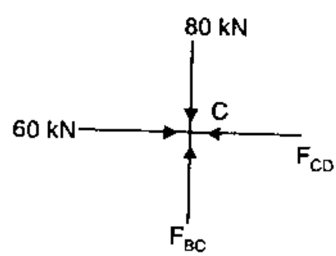


Figure A11(c)

$$\Sigma V = 0 \rightarrow$$

$$F_{BC} = 80 \text{ kN} \quad \text{(compressive)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{CD} = 60 \text{ kN} \quad \text{(compressive)}$$

Now, consider the equilibrium of joint D (Fig. A11(*d*))

$$\Sigma H = 0 \rightarrow$$

$$F_{BD} \cos\theta = 60$$

$$\therefore \quad F_{BD} = \frac{60}{\cos 56.31}$$

$$= 108.17 \text{ kN} \qquad \text{(tensile)}$$

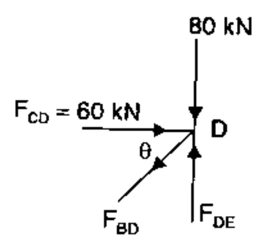


Figure A11(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{DE} - 80 - F_{BD} \sin\theta = 0$$

$$\therefore \quad F_{DE} = 80 + 108.17 \sin 56.31$$

$$= 170.0 \text{ kN} \qquad \text{(compressive)}$$

Now, consider the forces acting on joint B (Fig. A11(*e*)).

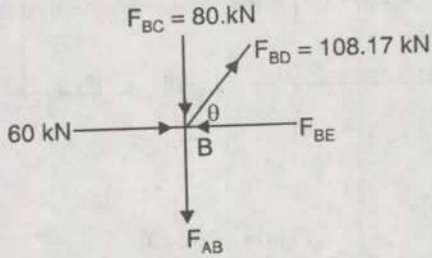


Figure A11(e)

$$\Sigma H = 0 \rightarrow$$

$$F_{BE} - 60 - 108.17 \cos\theta = 0$$

$$F_{BE} = 60 + 108.17 \cos 56.31$$

$$= 120 \text{ kN} \qquad \text{(compressive)}$$

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} + 80 - 108.17 \sin\theta = 0$$

$$F_{AB} = 108.17 \sin 56.31 - 80$$

$$= 30 \text{ kN} \qquad \text{(tensile)}$$

Consider the equilibrium of joint E (Fig. A11(*f*)).

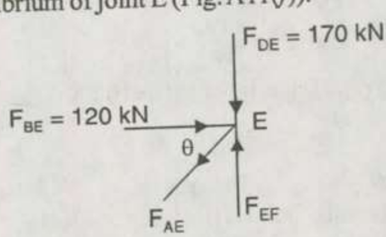


Figure A11(f)

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} \cos 56.31 = 120$$

$$\therefore \quad F_{AE} = 216.33 \text{ kN} \qquad \text{(tensile)}$$

$$\Sigma V = 0 \rightarrow$$

$$F_{EF} - 170 - F_{AE} \sin\theta = 0$$

$$\therefore \quad F_{EF} = 170 + 216.33 \sin 56.31$$

$$= 390 \text{ kN} \qquad \text{(compressive)}$$

Member forces are shown in Fig A11(b).

***Example* A4** Determine the forces in all the members of the truss shown in Fig. A12(a).

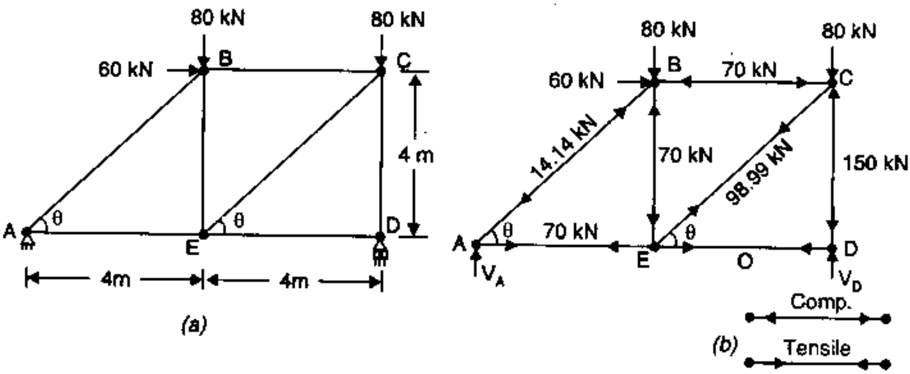


Figure A12(a&b)

Solution

Let AB and EC be at angle θ to the horizontal. Then

$$\tan\theta = \frac{4}{4} = 1$$

$$\therefore \qquad \theta = 45°$$

In this problem there is no joint where unknown forces are only two. Hence, reactions are to be found considering equilibrium of entire truss. Let the reactions be as shown in Fig. A12(b).

$$\Sigma \text{ Moments about A} = 0 \ (\Sigma M_A = 0), \text{ gives}$$

$$V_D \times 8 - 60 \times 4 - 80 \times 4 - 80 \times 8 = 0$$

$$\therefore \qquad V_D = 150 \text{ kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_D - 80 - 80 = 0$$

$$\therefore \qquad V_A = 160 - V_D = 160 - 150 = 10 \text{ kN}$$

$$\Sigma H = 0 \rightarrow$$

$$H_A = 60 \text{ kN}$$

Now, consider the equilibrium of joint A (Ref. Fig. A12(c)).

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} \sin 45 = 10$$

$$\therefore \qquad F_{AB} = 14.14 \text{ kN} \quad \text{(compressive)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} - 60 - F_{AB} \cos 45 = 0$$

$$\therefore \qquad F_{AE} = 61 + 14.14 \cos 45$$

$$= 70 \text{ kN} \qquad \text{(tensile)}$$

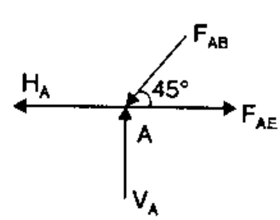


Figure A12(c)

Consider the equilibrium of joint B (Fig. A12(d)).

$$\Sigma V = 0 \rightarrow$$

$$F_{BE} - 80 + 14.14 \sin 45 = 0$$

$$F_{BE} = 70 \text{ kN} \quad \text{(compressive)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{BC} - 60 - 14.14 \cos 45 = 0$$

$\therefore \qquad F_{BF} = 70 \text{ kN} \qquad$ (tensile)

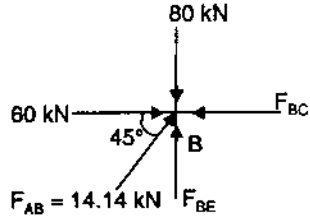


Figure A12(d)

Joint E: The forces acting on joint E are as shown in Fig. A12(e).

$$\Sigma V = 0 \rightarrow$$

$$F_{CE} \sin 45 = 70$$

$$\therefore \qquad F_{CE} = \frac{70}{\sin 45} = 98.99 \text{ kN} \quad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{DE} - 70 + F_{CE} \cos 45 = 0$$

$$F_{DE} = 70 - 98.94 \cos 45 = 0$$

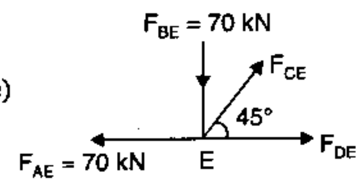


Figure A12(e)

Joint D: The forces acting at joint D are as shown is Fig A12 (f).

$$\Sigma V = 0 \rightarrow$$

$$F_{CD} = V_D = 190 \text{ kN} \quad \text{(compressive)}$$

[*Note:* Joint D could have been considered earlier also.]

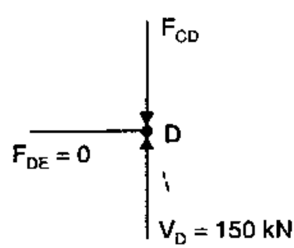


Figure A12(f)

The forces in all the members are shown in Fig. A12(b).

***Example* A5** Determine the forces developed in the members of the truss shown in Fig. A13(a).

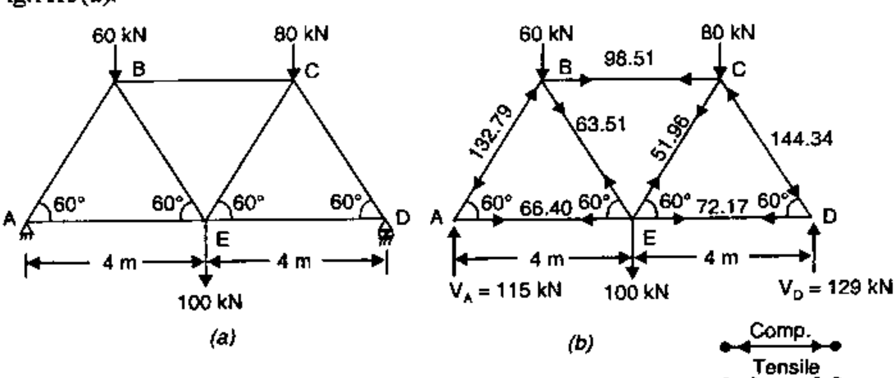


Figure A13(a&b)

Solution

Let the reactions developed at supports A be V_A and H_A and that developed at support D be V_D or shown is Fig. A13(b). Considering the equilibrium of entire truss,

$$\Sigma H = 0 \rightarrow$$

$$H_A = 0$$

$$\Sigma M_A = 0 \rightarrow$$

$$V_D \times 8 - 60 \times 2 - 100 \times 4 - 80 \times 6 = 0$$

$$\therefore \quad V_D = 125 \text{ kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_D - 60 - 100 - 80 = 0$$

$$V_A = 60 + 100 + 80 - 125 \qquad (\text{since } V_D = 125 \text{ kN})$$

$$= 115 \text{ kN}$$

Now consider the equilibrium of joint A (Fig. A13(c)).

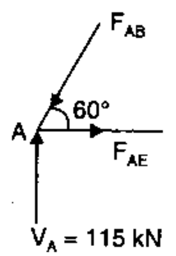


Figure A13(c)

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} \sin 60 = 115$$

$$\therefore \quad F_{AB} = 132.79 \text{ kN (compressive)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} - F_{AB} \cos 60 = 0$$

$$\therefore \quad F_{AE} = 132.79 \cos 60 = 66.4 \text{ kN (tensile)}$$

Joint B: The forces acting on this joint are as shown in Fig. A13(d).

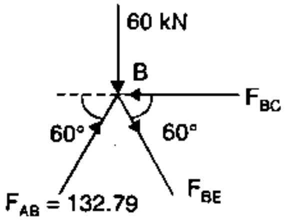


Figure A13(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{BE} \sin 60 + 60 - 132.79 \sin 60 = 0$$

$$F_{BE} = 63.51 \text{ kN} \qquad (\text{tensile})$$

$$\Sigma H = 0 \rightarrow$$

$$F_{BC} - 132.79 \cos 60 - F_{BE} \cos 60 = 0$$

$$\therefore \quad F_{BC} = 98.15 \text{ kN} \qquad (\text{compressive})$$

$$(\text{since, } F_{BE} = 63.5 \text{ kN})$$

Joint E: The forces acting on this joint are as shown in Fig. A13(e).

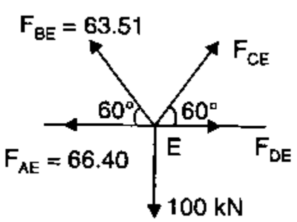


Figure A13(e)

$$\Sigma V = 0 \rightarrow$$

$$F_{CE} \sin 60 + 63.51 \sin 60 - 100 = 0$$

$$\therefore \quad F_{CE} = 51.96 \text{ kN} \qquad (\text{tensile})$$

$$\Sigma H = 0 \rightarrow$$

$$F_{DE} - 66.40 - 63.51 \cos 60 + F_{CE} \cos 60 = 0$$

$$F_{DE} = 66.40 + 63.51 \cos 60 - 51.96 \cos 60$$

$= 72.17 \text{ kN}$ (tensile)
(since, $F_{CE} = 91.9 \text{ kN}$)

Joint D: The forces acting on this joint are shown is Fig. A13(*f*).

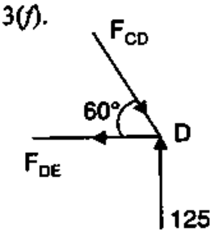


Figure A13(f)

$$\Sigma V = 0 \rightarrow$$

$$F_{CD} \sin 60 = 125$$

$$\therefore \quad F_{CD} = 144.34 \text{ kN.}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{DE} - F_{CD} \cos 60 = 0$$

$$\therefore \quad F_{DE} = 144.34 \cos 60 \quad (\text{since, } F_{CD} = 144.34 \text{ kN})$$

$= 72.17 \text{ kN}$ (tensile), same as calculated earlier.

It gives a check for the calculation. The member forces are shown in Fig. A13(*b*). The forces shown are in kN units.

***Example* A6** Determine the forces developed in the king post truss shown in Fig. A14(*a*).

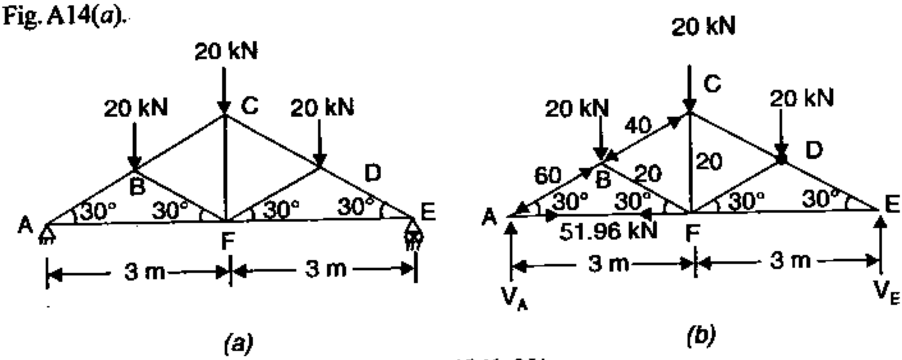


Figure A14(a&b)

Solution

Considering the equilibrium of entire truss, the reactions developed at supports can be found.

$$\Sigma H = 0 \rightarrow$$

$$H_A = 0$$

$$\Sigma M_A = 0 \rightarrow$$

$$V_E \times 6 - 20 \times 1.5 - 20 \times 3 - 20 \times 4.5 = 0$$

or $\quad V_E = 30 \text{ kN.}$

[This can be obtained making use of symmetry, i.e., due to symmetry $V_A = V_E = \frac{1}{2}$ total downward load]

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_E = 60$$

$$V_A = 60 - 30 \quad (\text{since } V_E = 30 \text{ kN})$$

$$= 30 \text{ kN}$$

Joint A: The forces acting on joint A are as shown in Fig. A14(*c*).

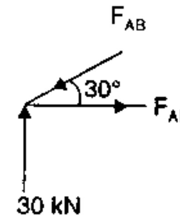


Figure A14(c)

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} \sin 30 = 30$$

$$\therefore \quad F_{AB} = 60 \text{ kN} \quad \text{(compressive)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AF} - F_{AB} \cos 30 = 0$$

$$F_{AF} = F_{AB} \cos 30 = 60 \cos 30$$

$$= 51.96 \text{ kN} \quad \text{(tensile)}$$

Joint B: The forces acting at joint B are as shown is Fig. A14(*d*).

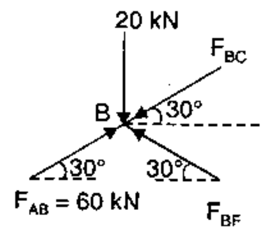


Figure A14(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{BF} \sin 30 - F_{BC} \sin 30 - 20 + F_{AB} \sin 30 = 0$$

$$\therefore \quad F_{BF} - F_{BC} = -60 + \frac{20}{5.430} = -20 \text{ kN} \quad \ldots\text{(i)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{BF} \cos 30 + F_{BC} \cos 30 - F_{AB} \cos 30 = 0$$

i.e. $\quad F_{BF} + F_{BC} = F_{AB} = 60 \quad \ldots\text{(ii)}$

Adding eqns. (i) and (ii), we get

$$2F_{BF} = 40$$

or $\quad F_{BF} = 20 \text{ kN} \quad$ (compressive)

Substituting it in eqn. (ii), we get

$$20 + F_{BC} = 60 \quad \text{or} \quad F_{BC} = 40 \text{kN}$$

(compressive)

Joint C: The forces acting at joint C are as shown in Fig. A14(*e*).

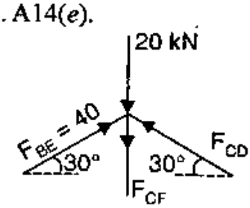


Figure A14(e)

$$\Sigma H = 0 \rightarrow$$

$$F_{CD} \cos 30 = F_{BF} \cos 30$$

$$\therefore \quad F_{CD} = F_{BE} = 40 \text{ kN} \quad \text{(compressive)}$$

$$\Sigma V = 0 \rightarrow$$

$$F_{BF} \sin 30 + F_{CD} \sin 30 - 20 - F_{CF} = 0$$

$$\therefore \quad F_{CF} = 40 \sin 30 + 40 \sin 30 - 20 = 20 \text{ kN}$$

(tensile)

Forces in half the truss are shown in Fig. A14(*b*). Using symmetry forces in the other half may be noted.

***Example* A7** Determine the forces in all the members of truss shown in Fig. A15(*a*).

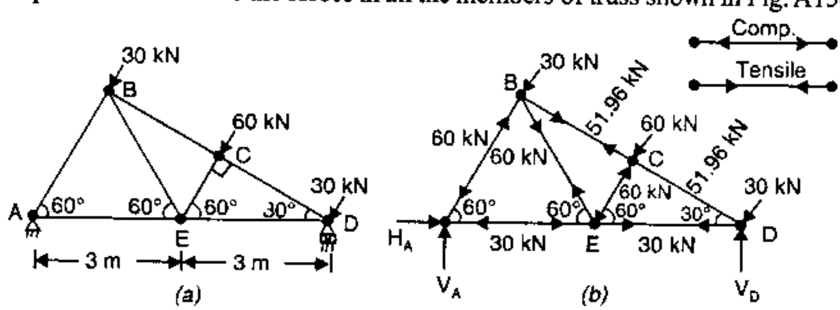


Figure A15(a&b)

Solution

Let the reactions developed at supports be V_A, H_A and V_D be as shown in Fig. A15(*b*).

$$\Sigma M_A = 0 \rightarrow$$

$$V_D \times 6 - 60 \sin 60 \times 3 - 30 \sin 60 \times 6 = 0$$

[60 kN load is resolved into its components at E—law of transmissibility is used]

$$\therefore \quad V_D = 51.96 \text{ kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_D - 30 \sin 60 - 60 \sin 60 - 30 \sin 60 = 0$$

$$V_A + 51.96 - 120 \sin 60 = 0$$

$$V_A = 51.96 \text{ kN}$$

$$\Sigma H = 0 \rightarrow$$

$$H_A - 30 \cos 60 - 60 \cos 60 - 30 \cos 60 = 0$$

$$\therefore \quad H_A = 60 \text{ kN}$$

Joint A: The forces acting on this joint are as shown in Fig. A15(*c*).

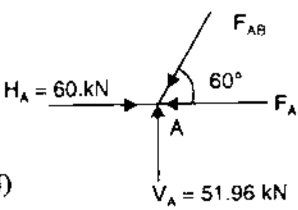


Figure A15(c)

$$\Sigma V = 0 \rightarrow$$

$$V_A - F_{AB} \sin 60 = 0$$

$$\therefore \quad F_{AB} = \frac{51.96}{\sin 60} \qquad (\text{since } V_A = 51.96 \text{ kN})$$

$$= 60 \text{ kN} \qquad (\text{compressive})$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} - 60 + F_{AB} \cos 60 = 0$$

$$\therefore \quad F_{AE} = 60 - 60 \cos 60 \qquad (\text{since } F_{AE} = 60 \text{ kN})$$

$$= 30 \text{ kN} \qquad (\text{tensile})$$

Joint D: The forces acting at joint D are shown in Fig. A15(*d*).

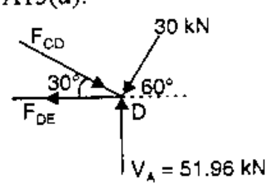


Figure A15(d)

$$\Sigma V = 0 \rightarrow$$

$$51.96 - F_{CD} \sin 30 - 30 \sin 60 = 0$$

$$F_{CD} = 51.96 \text{ kN} \qquad (\text{compressive})$$

$$\Sigma H = 0 \rightarrow$$

$$F_{DE} - F_{CD} \cos 30 + 30 \cos 60 = 0$$

$$\therefore \quad F_{DE} = F_{CD} \cos 30 - 30 \cos 60$$

$$= 30 \text{ kN}, \quad (\text{tensile})$$

$$(\text{since } F_{CD} = 51.96 \text{ kN})$$

Joint C: Referring to Fig. A15(*e*),

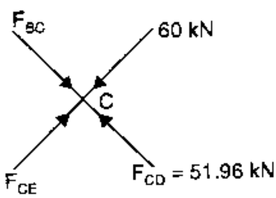


Figure A15(e)

$$\Sigma \text{Forces} \parallel \text{BCD} = 0 \rightarrow$$

$$F_{BC} = 51.96 \text{ kN} \quad (\text{compressive})$$

$$\Sigma \text{ Forces normal to BCD} = 0 \rightarrow$$

$$F_{CE} = 60 \text{ kN} \qquad (\text{compressive})$$

Joint E: The forces acting at joint E are as shown is Fig. A15(*f*).

$$\Sigma V = 0 \rightarrow$$

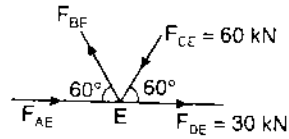


Figure A15(f)

$$F_{BE} \sin 60 - F_{CE} \sin 60 = 0$$

$$\therefore \quad F_{BE} = F_{CE} = 60 \text{ kN} \quad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} + 30 - F_{BE} \cos 60 - F_{CE} \cos 60 = 0$$

$$\therefore \quad F_{AE} = -30 + 60 \cos 60 + 60 \cos 60$$

$$= 30 \text{ kN} \qquad \text{(compressive)}$$

Forces in all members are shown in Fig. A15(*b*).

***Example* A8** Determine the forces in all the members of the truss shown in Fig. A16(*a*).

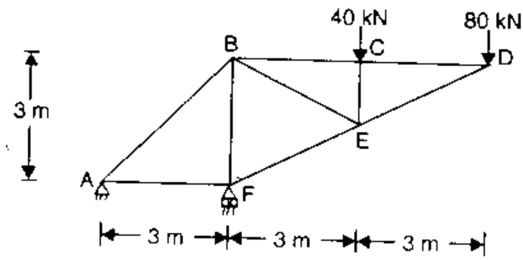
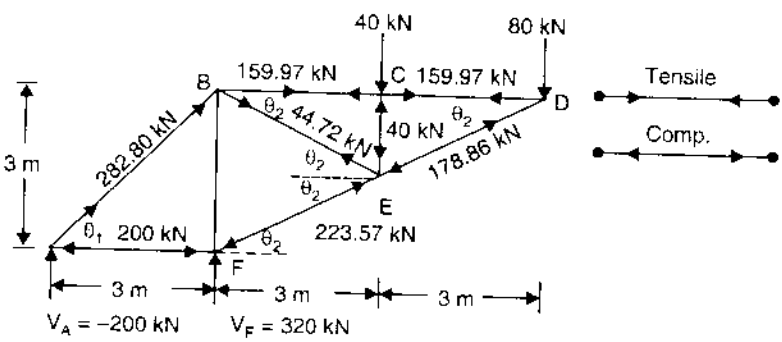


Figure A16(a)

Solution

Let the reactions developed at supports be V_A, H_A and V_F as shown in Fig. A16(*b*).



(*b*)

Figure A16(b)

$$\Sigma H = 0 \rightarrow \qquad H_A = 0$$

$$\Sigma M_A = 0 \rightarrow \qquad V_F \times 3 - 40 \times 6 - 80 \times 9 = 0$$

$$V_F = 320 \text{kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_F - 40 - 80 = 0$$

$\therefore \quad V_A = 40 + 80 - 320, \qquad$ (since $V_F = 320\,\text{kN}$)

$= -200\,\text{kN}$, i.e. upward direction assumed is not correct

$= 200\,\text{kN}$, downward.

Joint D: The forces acting at joint D are shown in Fig. A16(*c*).

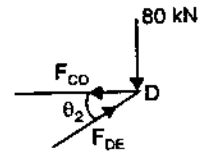


Figure A16(c)

Now, $\quad \tan\theta_2 = \dfrac{3}{6} = 0.5$

$\therefore \quad \theta_2 = 26.57°$

$\Sigma V = 0 \rightarrow$

$F_{DE} \sin\theta_2 = 80$

or $\quad F_{DE} = \dfrac{80}{\sin 26.57}$

$= 178.86\,\text{kN} \qquad$ (compressive)

$\Sigma H = 0 \rightarrow$

$F_{CD} - F_{DE}\ \cos\theta_2 = 0$

$\therefore \quad F_{CD} = 178.86 \cos\ 26.57$

$= 159.97\,\text{kN} \qquad$ (tensile)

Joint C: The forces acting on joint C are as shown in Fig. A16(*d*).

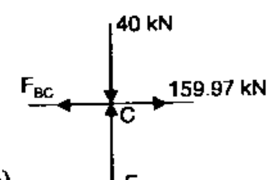


Figure A16(d)

$\Sigma V = 0 \rightarrow$

$F_{CE} = 40\,\text{kN} \qquad$ (compressive)

$\Sigma H = 0 \rightarrow$

$F_{BC} = 159.97\,\text{kN} \qquad$ (tensile)

Joint E: The forces acting at joint E are as shown in Fig. A16(*e*).

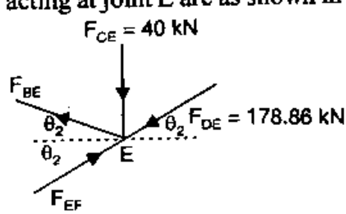


Figure A16(e)

$\Sigma V = 0 \rightarrow$

$F_{BE} \sin 26.57 + F_{EF} \sin\ 26.57 - 40 - 178.86 \sin 26.57 = 0$

$$F_{BC} + F_{EF} = 268.29 \qquad \text{...(i)}$$

$\Sigma H = 0 \rightarrow$

$F_{EF} \cos 26.57 - F_{BE} \cos 26.57 - 178.86 \cos 26.57 = 0$

i.e. $$F_{EF} - F_{BE} = 178.86 \qquad \text{...(ii)}$$

Adding eqns. (i) and (ii), we get

$$2F_{EF} = 268.29 + 178.86 = 447.15$$

$\therefore$ $F_{EF} = 223.57\,kN$ (compressive)

Substituting it in eqn. (*i*), we get

$$F_{BE} + 223.57 = 268.29$$

$\therefore$ $F_{BE} = 44.72\,kN$ (tensile)

Joint B: Consider the system of forces acting at joint B (Ref. Fig. A16(*f*)).

Now, $\theta_1 = \tan^{-1}\frac{3}{3} = 45°$

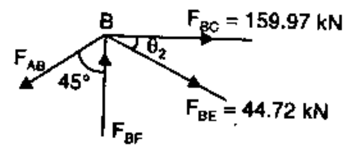


Figure A16(f)

$$\Sigma H = 0 \rightarrow$$

$$F_{AB}\sin 45 - 159.97 - 44.72\cos 26.57 = 0$$

$\therefore$ $F_{AB} = 282.79\,kN$ (tensile)

$$\Sigma V = 0 \rightarrow$$

$$F_{BF} - F_{AB}\cos 45 - 44.72\sin 26.57 = 0$$

$\therefore$ $F_{BE} = 219.97\,kN$ (compressive)

Joint A: The forces acting at joint A are as shown in Fig. A16(*g*).

$$\Sigma V = 0 \rightarrow$$

$$F_{AB}\sin 45 = 200$$

$$F_{AB} = 282.80\,kN \quad \text{(checked)}$$

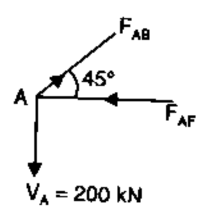


Figure A16(g)

$$\Sigma H = 0 \rightarrow$$

$$F_{AF} - F_{AB}\cos 45 = 0$$

$$F_{AF} = 282.8\cos 45$$

$$= 200\,kN \quad \text{(compresssive)}$$

All the forces in the members of the truss are shown in Fig. A16(*b*).

***Example* A9** Determine the forces developed in all members of the truss shown in Fig. A17(*a*).

Solution

Fig A17(*b*) shows FBD of the entire truss.

$$\Sigma M_A = 0 \rightarrow$$

$$H_C \times 6 - 20 \times 3 - 30 \times 3 - 30 \times 6 = 0$$

$$H_C = 55\,kN$$

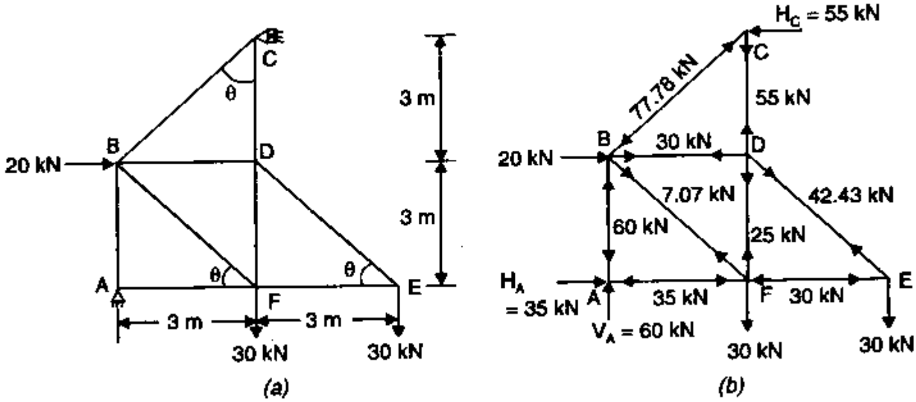


Figure A17(a&b)

$$\Sigma V = 0 \rightarrow$$

$$V_A - 30 - 30 = 0$$

$$\therefore \quad V_A = 60\,\text{kN}$$

$$\Sigma H = 0 \rightarrow$$

$$H_A + 20 - H_C = 0$$

$$\therefore \quad H_A = H_C - 20 = 55 - 20, \qquad \text{since } H_C = 55\,\text{kN}$$

$$= 35\,\text{kN}$$

Joint C: System of forces on joint C are as shown in Fig. A17(c).

Now $\theta = \tan^{-1}\dfrac{3}{3} = 45°$

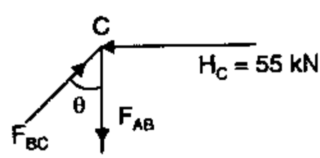


Figure A17(c)

$$\Sigma H = 0 \rightarrow$$

$$F_{BC}\sin 45 - 55 = 0$$

$$\therefore \quad F_{BC} = \frac{55}{\sin 45} = 77.78\,\text{kN}$$

(compressive)

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} - F_{BC}\cos 45 = 0$$

$$\therefore \quad F_{AB} = 77.78\cos 45, \text{ since } F_{BC} = 77.78\text{ kN}$$

(compressive)

Joint E: Consider the equilibrium of joint E (Ref. Fig. A17(*d*)).

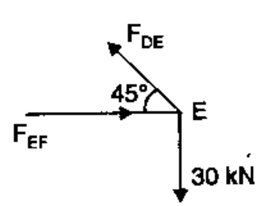


Figure A17(d)

$$\Sigma V = 0 \rightarrow$$

$$F_{DE}\sin 45 = 30$$

$$\therefore \quad F_{DE} = 42.43\,\text{kN} \quad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{EF} - F_{DE}\cos 45 = 0$$

$$\therefore \quad F_{EF} = 42.43\cos 45 = 30\,\text{kN}$$

(compressive)

Joint D: The forces acting on joint D are as shown in Fig. A17(*e*).

$$\Sigma H = 0 \rightarrow$$

$$F_{BD} = F_{DE}\sin 45 = 42.43\sin 45$$
$$=30 \text{ kN} \qquad \text{(tensile)}$$

$$\Sigma V = 0 \rightarrow$$

$$F_{DF} - 55 + 42.43\cos 45 = 0$$
$$F_{DF} = 25 \text{ kN} \qquad \text{(tensile)}$$

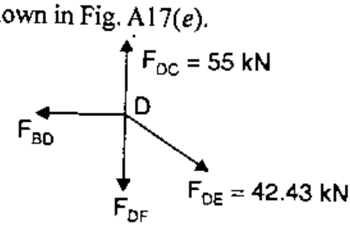


Figure A17(e)

Joint F: The forces acting on joint F are as shown in Fig. A17(*f*).

$$\Sigma V = 0 \rightarrow$$

$$F_{BF}\sin 45 + 25 - 30 = 0$$

$$\therefore \qquad F_{BF} = \frac{5}{\sin 45} = 7.07 \text{ kN} \quad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AF} - 30 - F_{BF}\cos 45 = 0$$

$$F_{AF} = 30 + 7.07\cos 45 = 35 \text{ kN} \text{ (compressive)}$$

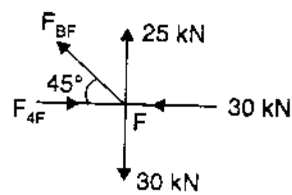


Figure A17(f)

Joint A: The forces acting at joint A are as shown in Fig. A17(*g*).

$$\Sigma H = 0 \rightarrow$$

$$F_{AF} = 35 \text{ kN} \qquad \text{(checked)}$$
$$\text{(compressive)}$$

$$\Sigma V = 0 \rightarrow$$

$$F_{AB} = 60 \text{ kN} \quad \text{(compressive)}$$

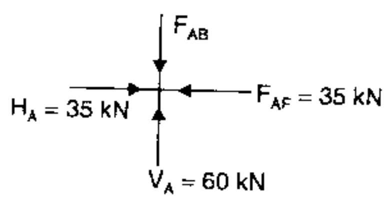


Figure A17(g)

Forces in all members are shown in Fig. A17(*b*).

A6 METHOD OF TENSION COEFFICIENT

A systematic approach to method of joints by Prof. Muller Breslan gave rise to method of tension coefficient in 1924. This method is very well suited to the analysis of three dimensional truss. In this method tensile forces are taken as positive. If a force works out to be negative it is compressive. Figure A18 shows a typical member L_{ij} oriented at $\theta_{x,ij}$ with x-axis and $\theta_{y,ij}$ with y-axis. Tensile force in the member be F_{ik}.

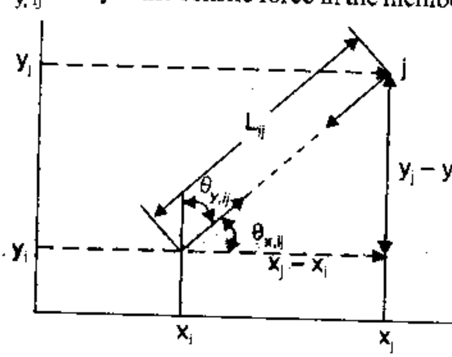


Figure A18

Now, length of the member is

$$L_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

and $$\cos\theta_{x,ij} = \frac{x_j - x_i}{L_{ij}} \qquad \text{...A2}$$

∴ The component force F_{ij} in x-direction acting on joint i is,

$$= F_{ij}\cos\theta_{x,ij} = F_{ij}\frac{x_j - x_i}{L_{ij}}$$

$$= t_{ij}(x_j - x_i) \qquad \text{...A3}$$

where $t_{ij} = \frac{F_{ij}}{L_{ij}}$ is tension coefficient of members. Thus, the tension coefficient of a member is defined as tensile force of the member divided by the length of the member.

Similarly, it can be shown that the component of force F_{ij} on joint i in *y*-direction

$$= F_{ij}\cos\theta_{y,ij} = F_{ij}\frac{y_j - y_i}{L_i}$$

$$= t_{ij}(y_j - y_i) \qquad \text{...A4}$$

It may be noted that the terms t_{ij}, and t_{ji} refer to the same quantity.

Now consider the equilibrium of joint A shown in Fig. A19, where a number of members are meeting. Let the external loads be X_A and Y_A.

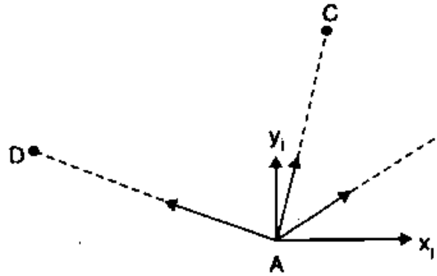


Figure A19

Then the equilibrium of forces in *x*-direction gives,

$$\Sigma X = 0 \rightarrow$$

$$X_A + t_{AB}(x_B - x_A) + t_{AC}(x_C - x_A) + t_{AD}(x_D - x_A) = 0$$

$$X_i + \Sigma t_{ij}(x_j - x_i) = 0 \qquad \text{...A5}$$

where *i* refers to the joint under consideration, *j* refers to the joint at the other end of member and summation is over the various members meeting at the joint.

Similarly, it can be shown that,

$$\Sigma Y = 0$$

$$Y_i + \Sigma t_{ij}(y_i - y_i) = 0 \qquad \text{...A6}$$

Equations A5 and A6 are applied joint by joint to get tension coefficients of all members. By multiplying these coefficients with the length of the member, the forces in the members are obtained. If the value works out to be positive, the force is tensile and it is compressive if it is negative.

This method is very useful in case of the analysis of space trusses in which case we get additional equation,

$$Z_i + \Sigma t_{ij}(z_j - z_i) = 0 \qquad \text{.... A7}$$

and the expression for length is

$$L_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \qquad \text{... A8}$$

However, in this chapter analysis is restricted to finding the member forces in plane trusses only.

***Example* A10** Analyse the truss shown in Fig A9 (example A1) by method of tension coefficient.

Solution

Taking A as origin, the coordinates of various joints in the truss are

A (0, 0), B (0, 4), C (3, 2), D (6, 0) and E (3, 0)

Hence, the lengths of various members are,

$$L_{BC} = \sqrt{(3-0)^2 + (2-4)^2} = 3.606\,\text{m}$$

$$L_{CD} = \sqrt{(6-3)^2 + (0-2)^2} = 3.606\,\text{m}$$

$$L_{DE} = 3\,\text{m}$$

$$L_{AE} = 3\,\text{m}$$

$$L_{CE} = 2\,\text{m}$$

and

$$L_{AC} = \sqrt{(3-0)^2 + (2-0)^2} = 3.606\,\text{m}$$

Now, the analysis is joint by joint as was done in the method of joint. Since we have only two equations of equilibrium in case of plane trusses, we select the joint where there are only two unknowns. In cantilever truss like this we can find such joints, even without going for finding reactions. Now we can start the analysis with joint D.

Joint D: From tension coefficient equations 3 and 4 we get

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{DC}(6-3) + t_{DE}(6-3) = 0$$

i.e.,

$$t_{DC} = -t_{DE} \qquad \text{... (i)}$$

$$\Sigma Y = 0 \rightarrow$$

and

$$-60 + t_{DC}(0-2) + t_{DE}(0-0) = 0$$

i.e.,

$$t_{DC} = 30 \qquad \text{... (ii)}$$

Hence from (*i*), $t_{DE} - 30$.

$\therefore$ $F_{DC} = 30\ L_{DC} = 30 \times 3.606 = 108.7\,\text{kN}$

$F_{DE} = -30\,L_{DE} = -30 \times 3 = -90\,\text{kN}$

Joint E :

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{ED}(6-3) + t_{EC}(3-3) + t_{EA}(0-3) = 0$$

i.e., $t_{ED} = t_{EA}$

i.e., $t_{EA} = -30$

$\therefore$ $F_{EA} = -30 \times 3 = -90\,\text{kN}$

$$\Sigma Y = 0 \rightarrow$$

$$-90 + t_{ED}(0-0) + t_{EC}(3-0) + t_{EA}(0-0) = 0$$

or $t_{EC} = 30$

$\therefore$ $F_{EC} = 30 \times 2 = 60\,\text{kN}$

Joint C:

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{CD}(6-3) + t_{CB}(0-3) + t_{CA}(0-3) + t_{CE}(3-3) = 0$$

i.e., $30 \times 3 - 3\ t_{CB} - 3t_{CA} = 0$

or $t_{CB} + t_{CA} = 30$ ...(iii)

$$\Sigma Y = 0 \rightarrow$$

$$0 + t_{CD}(0-2) + t_{CB}(4-2) + t_{CA}(0-2) + t_{CE}(0-2) = 0$$

$$-2t_{CD} + 2t_{CB} - 2t_{CA} - 2t_{CE} = 0$$

i.e., $t_{CB} - t_{CA} = t_{CD} + t_{CE} = 30 + 30 = 60$ ...(iv)

Adding equation (iii) and (iv) we get

$2t_{CB} = 90$ or $t_{CB} = 45$

$\therefore$ $F_{CB} = 162.25\,\text{kN}$

Substituting the value of t_{CB} is eqn. (iii), we get

$t_{CA} = 30 - 45 = -15$

$\therefore$ $F_{CA} = -15 \times 3.606 = -54.08\,\text{kN}$

Thus, the member forces are:

Member	*Force*
BC	162.25 kN
CD	108.17 kN
DE	–90.0 kN
EA	–90 kN
CE	60 kN
CA	–54.08 kN

Sign convention : + ve is tensile
– ve is compressive.

***Example* A11** Analyse the truss shown in Fig. A20(*a*) by the method of tension coefficient and determine the forces in all the members.

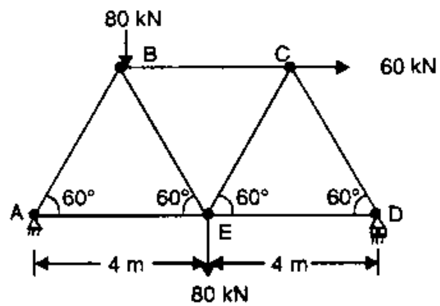


Figure A20(a)

Solution

Taking joint A as the origin the coordinates of various joints are

A (0, 0), B (2, 3.464), C (6, 3.464), D (8, 0), E (4, 0)

Since all triangles in the figure are equilateral, the length of all members is the same, that is, 4 m. Let the reactions at A and B be V_A, H_A and V_B as shown in Fig A20(*b*).

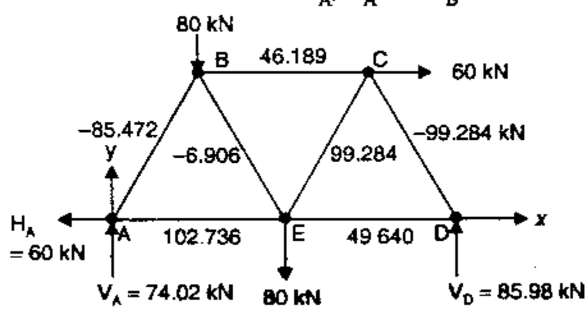


Figure A20(b)

$$\Sigma M_A = 0 \rightarrow$$

$$V_D \times 8 - 80 \times 2 - 80 \times 4 - 60 \times 3.464 = 0$$

$$\therefore \qquad V_D = 85.98 \text{ kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_D = 80 + 80$$

$$V_A = 160 - 85.98 = 74.02\,\text{kN}$$

$$\Sigma H = 0 \rightarrow$$

$$H_A = 60\,\text{kN}$$

Now, consider the analysis joint by joint.

Joint A: $\Sigma X = 0 \rightarrow$

$$-60 + t_{AB}(2-0) + t_{AE}(4-0) = 0$$

i.e., $t_{AB} + 2t_{AE} = 30$ ....(i)

$$\Sigma Y = 0 \rightarrow$$

$$74.02 + t_{AB}(3.464 - 0) + t_{AE}(0-0) = 0$$

$\therefore$ $t_{AB} = -\dfrac{74.02}{3.464} = -21.368, \ \therefore F_{AB} - 21.368 \times 4 = -85.472\,\text{kN}$

From eqn. (i),

$$-21.368 + 2t_{AE} = 30$$

or $t_{AE} = 25.684$ $\therefore$ $F_{AE} = 25.684 \times 4 = 102.736\,\text{kN}$

Joint D:

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{DC}(6-8) + t_{DE}(4-8) = 0$$

$$-2t_{DC} - 4t_{DE} = 0$$

or $t_{DC} + 2t_{DE} = 0$ ...(ii)

$$\Sigma Y = 0 \rightarrow$$

$$85.98 + t_{DC}(3.464 - 0) + t_{DE}(0-0) = 0$$

$\therefore$ $t_{DC} = -\dfrac{85.98}{3.646} = -24.821$

$$F_{DC} = 99.284\,\text{kN}.$$

Substituting the value of t_{DC} in eqn. (ii), we get

$$-24.84 + 2t_{DE} = 0$$

$\therefore$ $t_{DE} = 12.410$

$\therefore$ $F_{DE} = 49.640\,\text{kN}.$

Joint B:

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{BA}(2-0) + t_{BC}(6-2) + t_{BE}(4-2) = 0$$

or $t_{BA} + 2t_{BC} + t_{BE} = 0$ ...(iii)

i.e., $2t_{BA} + t_{BE} = 21.368,$ (since $t_{BA} = -21.368$)

$$\Sigma Y = 0 \rightarrow$$

$$-80 + t_{BA}(0 - 3.464) + t_{BC}(3.464 - 3.464) + t_{BE}(0 - 3.464) = 0$$

i.e., $$-80+(-21.368)(-3.464)+0-3.464\,t_{BE}=0$$

or $$t_{BE}=-\frac{5.981}{3.464}=-1.7266 \qquad \therefore\ F_{BE}=-6.906\,\text{kN}$$

Substituting it in (iii), we get

$$2t_{BC}-1.7266=-21.368$$

or $$t_{BC}=11.547 \qquad \therefore\ F_{BC}=11.547\times 4=46.189\,\text{kN.}$$

Joint C: $$\Sigma X=0\rightarrow$$

$$60+t_{CB}(2-6)+t_{CD}(8-6)+t_{CE}(4-6)=0$$

or $$-2t_{CB}+t_{CD}-t_{CE}=-60 \qquad \text{...(iv)}$$

$$\Sigma Y=0\rightarrow$$

$$0+t_{CB}(3.464-3.464)+t_{CD}(0-3.464)+t_{CE}(0-3.464)=0$$

$$t_{CD}=-t_{CE} \qquad \text{i.e., } t_{CE}=24.821 \quad (\text{since, } t_{CE}=-24.821)$$

$\therefore$ $$F_{CE}=-24.821\times 4=-99.284\,\text{kN.}$$

Substituting the value of t_{CE} in eqn (iv), we get,

$$-2\times 11.547-24.821-24.821=-60$$

This gives a check on the calculation. Member forces are shown in Fig. A20(*b*), with sign convention +ve meaning tensile and –ve meaning compressive.

***Example* A12** Analyse the truss shown in Fig A21(*a*) by the method of tension coefficient.

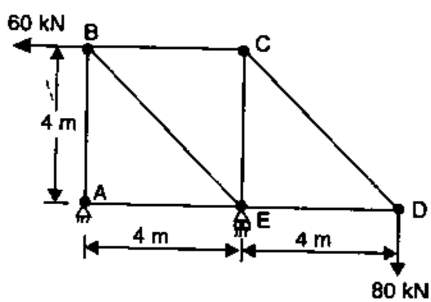


Figure A21(a)

Solution

Let the reactions developed at supports A and E be as shown in Fig. A21(*b*).

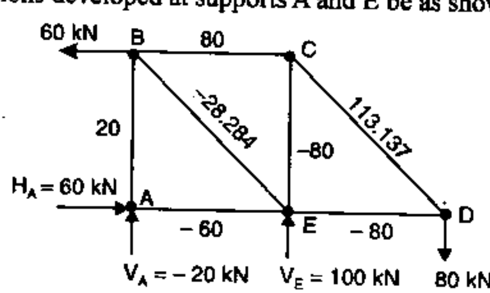


Figure A21(b)

$$\Sigma M_A = 0 \rightarrow$$

$$V_E \times 4 + 60 \times 4 - 80 \times 8 = 0$$

or $$V_E = 100\,\text{kN}$$

$$\Sigma V = 0 \rightarrow$$

$$V_A + V_E = 80$$

$\therefore$ $$V_A = 80 - V_A = 80 - 100$$
$$= -20\,\text{kN}$$

$$\Sigma H = 0 \rightarrow$$

$$H_A - 60 = 0 \qquad \text{or} \qquad H_A = 60\,\text{kN}.$$

Now, the coordinates of various joints are A(0, 0), B (0, 4), C(4, 4), D(8, 0), and E(4, 0).

Lengths of various members are,

$$L_{AB} = L_{BC} = L_{CE} = L_{ED} = 4\,\text{m}.$$

Inclined members' lengths are $4\sqrt{2} = 5.6569$ m.

Now consider the equilibrium of joints one by one till the forces in all members are determined.

Joint A:

$$\Sigma Y = 0 \rightarrow$$

$$-20 + t_{AB}(4-0) + t_{AE}(0-0) = 0$$

$\therefore$ $$t_{AB} = 5 \qquad \therefore\ F_{AB} = 5 \times L_{AB} = 5 \times 4 = 20\,\text{kN}.$$

$$\Sigma X = 0 \rightarrow$$

$$60 + t_{AB}(0-0) + t_{AE}(4-0) = 0$$

$\therefore$ $$t_{AE} = -15 \qquad \therefore\ F_{AE} = -15 \times 4 = -60\,\text{kN}.$$

Joint B:

$$\Sigma Y = 0 \rightarrow$$

$$0 + t_{BC}(4-4) + t_{BE}(0-4) + t_{BA}(0-4) = 0$$

$$t_{BE} = -t_{BA} = -5$$

$\therefore$ $$F_{BE} = 5 \times 4.52 = -28.284\,\text{kN}.$$

$$\Sigma X = 0 \rightarrow$$

$$-60 + t_{BC}(4-0) + t_{BE}(4-0) + t_{BA}(0-0) = 0$$

$$t_{BC} + t_{BE} = 15$$

$\therefore$ $$t_{BC} = 15 + 5 = 20 \qquad \text{(since } t_{BE} = -5\text{)}$$

$\therefore$ $$F_{BC} = 20 \times L_{BC} = 20 \times 4 = 80\,\text{kN}$$

Joint C:

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{CD}(8-4) + t_{CE}(4-4) + t_{CD}(0-4) = 0$$

$$\therefore \quad t_{CD} = t_{CB} = 20$$

$$\therefore \quad F_{CD} = 20 \times 4\sqrt{2} = 113.137\,\text{kN}$$

$$\Sigma Y = 0 \rightarrow$$

$$t_{CD}(0-4) + t_{CE}(0-4) + t_{CB}(4-4) = 0$$

$$t_{CD} + t_{CE} = 0$$

$$t_{CE} = -t_{CD} = -20$$

$$\therefore \quad F_{CE} = -20 \times 4 = -80\,\text{kN}$$

Joint D:

$$\Sigma Y = 0 \rightarrow$$

$$-80 + t_{DC}(4-0) + t_{DE}(0-0) = 0$$

$$t_{DC} = 20 \quad F_{CD} = 20 \times 4\sqrt{2} = 113.137\,\text{kN}, \text{ checked}$$

$$\Sigma X = 0 \rightarrow$$

$$0 + t_{DC}(8-4) + t_{DE}(4-8) = 0$$

$$\therefore \quad t_{DE} = t_{DC} = 20 \qquad \therefore\ F_{DE} = 20 \times 4 = 80\,\text{kN}$$

Forces in all members are shown in Fig. A21(*b*), positive sign indicating tensile force and negative sign indicating compressive force.

A7 METHOD OF SECTION

If we refer to truss shown in Fig. A22(*a*), we find even after finding reactions, we will not be able to find a joint where there are only two unknown forces. Hence we cannot analyse the truss by method of joint. Fig. A22(*b*) shows the truss in which, after finding the reactions we will be able to analyse the joints A, B, I, H, and J but will not be able to proceed further with analysis. In such situation, we use the method of section. In this method a cut is imagined through maximum of three members such that the truss is separated into two complete independent parts (section 1-1). Then, each part is in equilibrium with the reactions, applied load, and the forces in the members cut. The equilibrium of each part, thus obtained constitutes a system of non-concurrent forces. We have three equations of equilibrium for such system. Hence using them the forces in the members cut can be determined. It may be observed that the forces in the three members should be non-concurrent otherwise moment equilibrium conditions will not give useful equation. We may use this method advantageously, when there is need for finding force in only few members of a large truss. Instead of going joint by joint, we can use the section line through these members, and find the forces in these few members. Thus, the method of section is superior to method of joint in the following two situations.

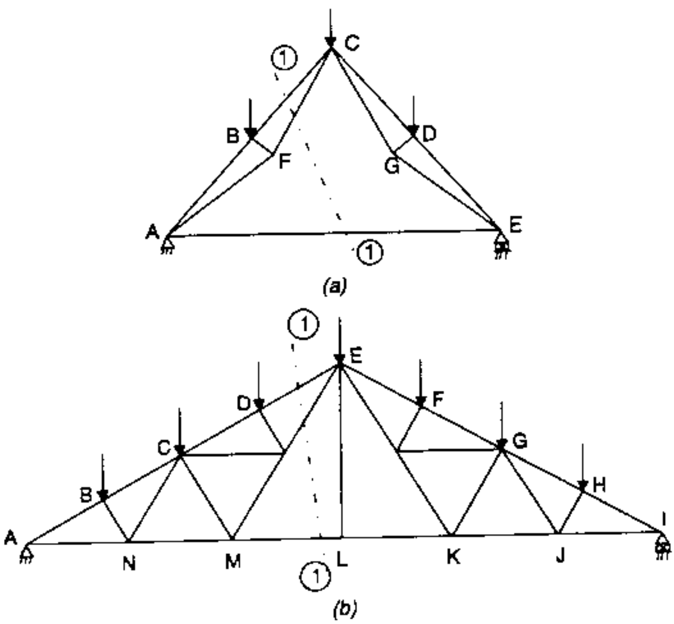


Figure A22(a&b)

1. When method of joint fails to start or proceed due to non-availability of a joint with only two unknown forces, even after finding reactions.
2. When forces in only few members of a large truss are required.

The method of section is illustrated below with examples.

***Example* A13** Determine the forces in all the members of the truss shown in Fig. A23(*a*).

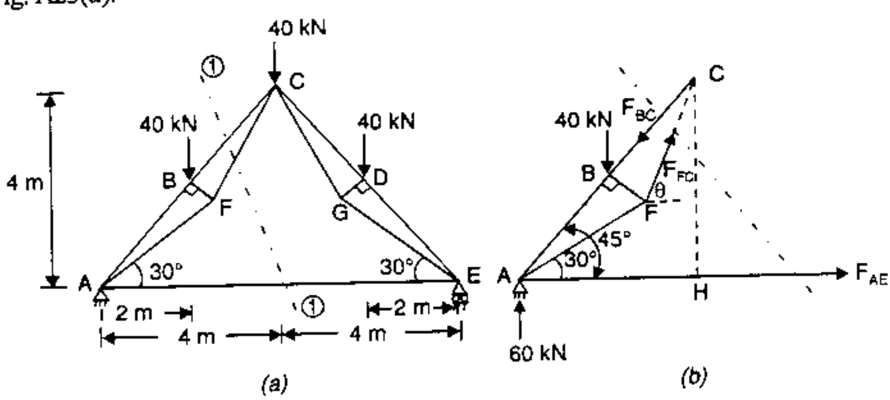


Figure A23(a&b)

Solution

$$\Sigma H = 0 \rightarrow$$

$$H_A = 0$$

Due to symmetry,

$$V_A = V_B = \frac{1}{2} \times \text{(total vertical load)}$$

$$= \frac{1}{2}(40+40+40) = 60\,\text{kN}.$$

Since there is no joint where there are only two unknown forces, method of joint fails to start. Consider the section line 1-1. It separates the truss into two parts, cutting only three members. Now consider the equilibrium of left side portion (Ref. Fig A23.(*b*))

$$\Sigma M_C = 0 \rightarrow$$

$$F_{AE} \times 4 = 40 \times 2 - 60 \times 4 = 0$$

$\therefore$ $\quad F_{AE} = 40\,\text{kN}$ (tensile)

Let FC be inclined at angle θ to horizontal.

Drop $\perp$ CH to AE.

In $\quad \Delta ACH,\ AH = CH = 4\text{ m}$

$\therefore \quad \angle CAH = \angle ACH = 45°$

$\therefore \quad \angle CAF = 45 - 30 = 15°$

As $\Delta ABF \equiv \Delta CBF$, we get

$$\angle BCF = \angle CAF = 15°$$

$\therefore \quad \angle FCH = \angle ACH - \angle BCF = 45 - 15 = 30°$

i.e., FC is at 30° to vertical.

Hence $\quad \theta = 60°$

Let the direction of forces F_{FC} and F_{BC} be as shown in Fig. A23(*b*)

$$\Sigma V = 0 \rightarrow$$

$$60 - 40 - F_{BC}\cos 45 + F_{FC}\sin 60 = 0$$

i.e., $$0.886 F_{FC} - \frac{F_{BC}}{\sqrt{2}} = -20 \qquad \text{...(i)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{AE} - F_{BC}\sin 45 + F_{FC}\cos 60 = 0$$

or $$-\frac{F_{BC}}{\sqrt{2}} + 0.5 F_{FC} = -F_{AE} = -40 \qquad \text{...(ii)}$$

Subtracting eqn. (ii) From eqn. (i), we get

$$(0.886 - 0.5)F_{FC} = -20 + 40$$

$\therefore \quad F_{FC} = 51.81\,\text{kN}$ (tensile)

Hence from eqn. (i),

$$0.886(51.81) - \frac{F_{BC}}{\sqrt{2}} = -20$$

or $\quad F_{BC} = \sqrt{2}[0.886 \times 51.81 + 20] = 65.91\,\text{kN}$ (compressive)

Now, the forces in other members can be found by the method of joint.

Joint B: The forces acting at joint B are as shown in Fig. A23(c).

Σ Forces normal to AC = 0 →

$F_{BF} - 40 \sin 45 = 0$

$F_{BF} = 28.28\,\text{kN}$ (compressive)

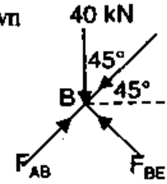


Figure A23(c)

Σ Forces parallel to AC = 0 →

$F_{AB} - 40 \cos 45 - F_{BC} = 0$

∴ $F_{AB} = 40 \cos 45 + 65.91$, (since $F_{BC} = 65.91\,\text{kN}$)

$= 94.19\,\text{kN}$ (compressive)

Joint A: Let the direction of force in AF be as shown in Fig A23(d);

$\Sigma V = 0$ →

$60 - F_{AB} \sin 45 + F_{AF} \sin 30 = 0$

$F_{AF} \sin 30 = -60 + 94.19 \sin 45$

$F_{AF} = 13.20\,\text{kN}$

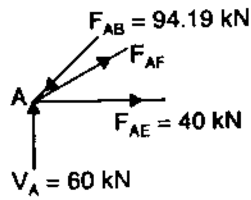


Figure A23(d)

Hence, assumed direction of AC is correct. F_{AC} is tensile.

Force in other members can be noted down, making use of symmetry. Final results are tabulated below.

Member	*Force in kN*	*Nature*
AB, DE	94.19	compressive
BC, CD	65.91	compressive
AF, EG	13.20	tensile
FC, CG	51.81	tensile
BF, DG	28.28	compressive
AE	40	tensile

***Example* A14** Determine the forces in the members CD, DE, EO and AL of truss shown in Fig. A24(a).

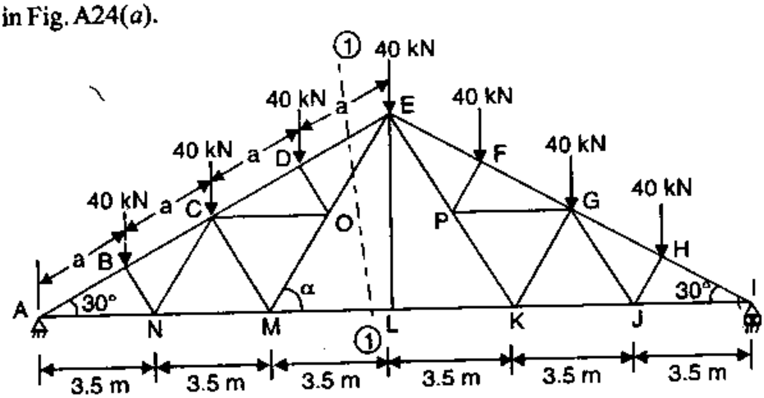


Figure A24(a)

Solution

Due to symmetry,

$$V_A = V_I = \frac{1}{2}\text{ total load} = \frac{1}{2}(40\times 7) = 140\text{ kN}$$

Fig. A24(*b*) shows the portion of the truss to the left of section 1-1. Let the forces in the members cut by section line be in the directions shown in Figure [Note the section line passes through only three members and separates the two parts completely]. Since AB = BC = CD = DE, the horizontal distances of B, C, D from E are,

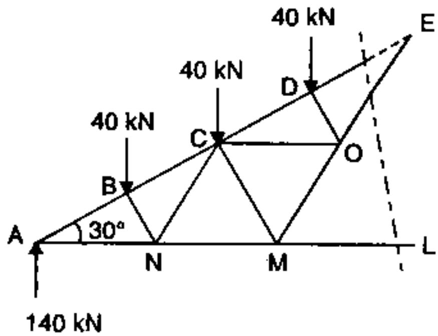


Figure A24(b)

$$\frac{3}{4}(10.5),\ \frac{1}{2}(10.5)\text{ and } 0.25\,(10.5)$$

$$\Sigma M_E = 0 \rightarrow$$

$$140\times(3.5\times 3) - 40\times\frac{3}{4}(10.5) - 40\times\frac{1}{2}(10.5) - 40\times 0.25(10.5)$$

$$-F_{ML}\times 10.5\ \tan 30 = 0$$

$\therefore$ $$F_{ML} = \frac{140-30-20-10}{\tan 30} = 138.56\text{ kN}$$

Now, $EL = 10.5\tan 30$

$ML = 3.5$

$\because$ $$\angle EML = \alpha = \tan^{-1}\left(\frac{EL}{ML}\right)$$

$$= \tan^{-1}\left(\frac{10.5\ \tan 30}{3.5}\right) = \tan^{-1}(1.732) = 60°$$

$$\Sigma V = 0 \rightarrow$$

$$140 - 40 - 40 - 40 + F_{DE}\sin 30 - F_{EO}\sin 60 = 0$$

$$0.5F_{DE} - 0.866\ F_{EO} = 20 \quad \text{...(i)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{ML} + F_{DE}\ \cos 30° - F_{EO}\ \cos 60 = 0$$

$$138.56 + 0.866\ F_{DE} - 0.5F_{EO} = 0$$

$$\therefore \quad F_{EO} = \frac{(138.56 + 0.866\ F_{DE})}{0.5} = 277.12 + 1.732\ F_{DE} \quad \text{...(ii)}$$

Substituting it in eqn. (i), we get,

$$0.5F_{DE} - 0.866\ (277.12 + 1.732\ F_{DE}) = 20$$

$$(0.5 - 1.5)F_{DE} - 239.98 = 20$$

or $\quad F_{DE} = -219.98,$ Direction is to be reversed.

$= 219.98$ kN (compressive)

$\therefore$ From eqn. (ii)

$$F_{EO} = 277.12 + 1.732(-219.98)$$

$= -103.88$ kN, Direction is to be reversed.

$= 103.88$ kN (tensile)

Now, consider the equilibrium for the joint D. The forces acting at the joints are as shown in Fig. A24(c).

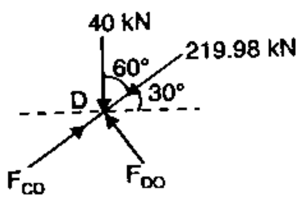


Figure A24(c)

Σ Forces Normal to AE = 0

$$F_{DO} - 40 \sin 60 = 0$$

$$F_{DO} = 34.64 \text{ kN}$$

Σ Forces Parallel to AE = 0

$$F_{CD} - 219.98 - 40 \cos 60 = 0$$

$F_{CD} = 239.98$ kN (compressive)

Thus,

$F_{DE} = 219.98$ (compressive)

$F_{EO} = 103.88$ kN (tensile)

$F_{CD} = 239.98$ kN (compressive)

$F_{ML} = 138.56$ kN (tensile)

Example **A15** Determine the forces in the members U_4U_5 U_4L_4, U_4L_5 and L_4L_5 of the truss shown in Fig. A25(a).

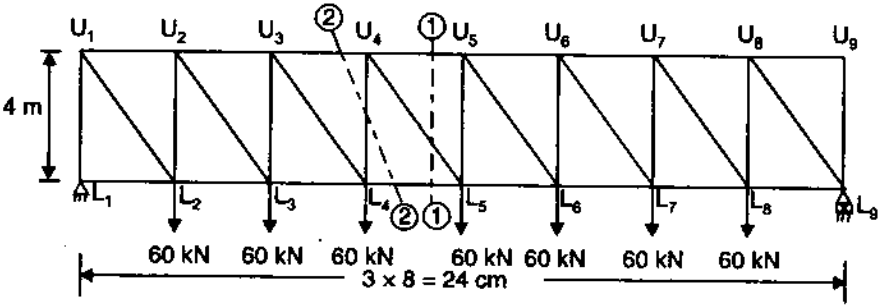


Figure A25(a)

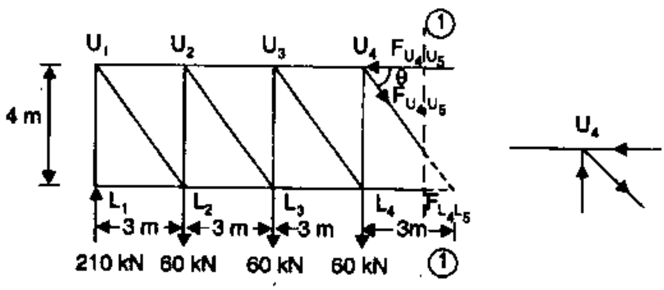
Solution

[**Note:** In this problem it is possible to get the solution by method of joint but it is time consuming.]

Due to symmetry

$$V_{L_1} = V_{L_9} = \frac{1}{2}\text{ (total load) } = \frac{1}{2}(60 \times 7) = 210\text{ kN.}$$

Section 1-1 as shown in Fig. A25(*a*) is taken so as to cut only three members and separate the two parts completely. Equilibrium of left side portion (Fig. A25(*b*)) is considered [*Note:* Consider the part which has less number of forces, for simplicity of calculation.)



(b)

Figure A25(b)

$$\Sigma M_{U4} = 0 \rightarrow$$

$$210 \times 9 - 60 \times 6 - 60 \times 3 - F_{L_4L_5} \times 4 = 0$$

$\therefore$ $$F_{L_4L_5} = 337.5\text{ kN} \qquad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{U_4U_5} = F_{L_4L_5}$$
$$= 337.5\text{ kN} \qquad \text{(compressive)}$$

Now, $$\theta = \tan^{-1}\frac{4}{3} = 53.13$$

$$\Sigma V = 0 \rightarrow$$

$$210 - 60 - 60 - 60 - F_{U_4L_5}\sin 53.13 = 0$$

$$F_{L_4L_5} = \frac{30}{\sin 53.13} = 37.5\text{ kN} \qquad \text{(tensile)}$$

Now, consider section 2-2 as shown in Fig. A25(c) and consider left side portion.

$$\Sigma V = 0 \rightarrow$$

$$210 - 60 - 60 - 60 + F_{U_4L_4} = 0$$

$$F_{U_4L_4} = 30\,\text{kN} \quad \text{(compressive)}$$

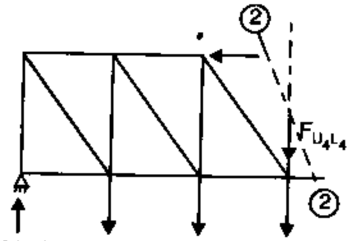


Figure A25(c)

Thus, the required member forces are

$$F_{U_4U_5} = 337.5\,\text{kN} \quad \text{(compressive)}$$
$$F_{U_4L_5} = 37.5\,\text{kN} \quad \text{(tensile)}$$
$$F_{U_4L_4} = 30\,\text{kN} \quad \text{(compressive)}$$
$$F_{L_4L_5} = 337.5\,\text{kN} \quad \text{(tensile)}$$

***Example* A16** Determine the forces in the members U_3U_4, L_3L_4 and U_4L_3 of the truss shown in Fig. A26(a).

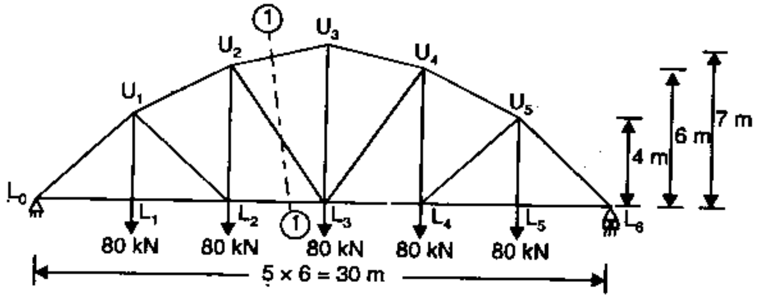


Figure A26(a)

Solution

Due to symmetry, vertical reactions at both supports are

$$\frac{1}{2}(80 \times 5) = 200\,\text{kN}$$

Consider the equilibrium of left side position of section 1-1. Let U_2U_3 and U_2L_3 make angles θ_1 and θ_2 respectively with the horizontal; then

$$\tan\theta_1 = \frac{7-6}{5} = 0.2 \qquad \theta_1 = 11.31°$$

$$\tan\theta_2 = \frac{6}{5} = 1.2 \qquad \theta_2 = 50.19°$$

Let the direction of forces of the joints in the cut members be as shown in Fig. A26(b).

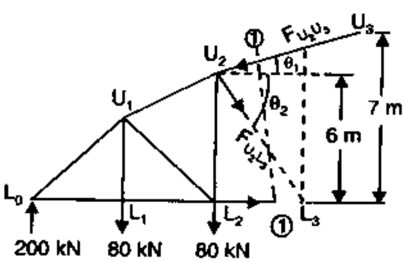


Figure A26(b)

$$\Sigma M_{U_2} = 0 \rightarrow$$

$$200 \times 10 - 80 \times 5 - F_{L_2L_3} \times 6 = 0$$

$$\therefore \quad F_{L_2L_3} = 266.67\,\text{kN} \qquad \text{(tensile)}$$

$$\Sigma M_{L_3} = 0 \rightarrow$$

$$200 \times 15 - 80 \times 10 - 80 \times 5 - F_{U_2U_3} \cos\theta_1 \times 7 = 0$$

[*Note:* $F_{U_2U_3}$ is transferred to U_3 and moments of its component forces about L_3 is considered.)

$$\therefore \quad F_{U_2U_3} = \frac{200 \times 15 - 800 - 400}{7\cos 11.31}, \text{ since } \theta_1 = 11.31°$$

$$= 262.24\,\text{kN} \qquad \text{(tensile)}$$

$$\Sigma H = 0 \rightarrow$$

$$F_{U_3L_3} \cos\theta_2 - F_{U_2U_3} \cos\theta_1 + F_{L_2L_3} = 0$$

$$F_{U_2L_3} \cos 50.14 - 262.24 \cos 11.31 + 266.67 = 0$$

$$F_{U_2L_3} = -14.88\,\text{kN}$$

The direction is to be reversed.

$$\therefore \quad F_{U_2L_3} = 14.88\,\text{kN} \qquad \text{(compression)}$$

IMPORTANT DEFINITIONS

A **perfect frame** is a pin-jointed truss which can resist the load applied at joint without undergoing visible changes in its shape and has just sufficient members to keep its shape. It consists of shape built up with triangles and has number of members three less than twice the total number of joints ($m = 2j - 3$).

A truss is said to be **deficient**, if it has number of members less than required for a perfect frame ($m < 2j - 3$). Its shape undergoes visible changes for the loads applied at joints. It is usually represented by shapes built of four or more members.

A redundant truss is a stable structure in which number of members are more than required for a perfect truss ($m > 2j - 3$). Such trusses cannot be analysed using the equations of equilibrium only.

Tension coefficeint of a member is the tensile force in the member divided by the length of the member. Hence, if the force is compressive the tension coefficient is negative.

IMPORTANT CONCEPTS AND FORMULAE

1. The number of members in a perfect frame are usually $m = 2j - 3$.
2. In the method of joints, at any time when the equilibrium equations are written for a joint, there should be only two unknown forces.
3. In the method of tension coefficient

$$L_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

and $$t_{ij} = \frac{F_{ij}}{L_{ij}}$$

where F_{ij} is positive if tensile, and negative if compressive.

4. In the method of section, the section line
   (i) should not pass through more than three members;
   (ii) the forces in the members cut by section should not be concurrent; and
   (iii) the truss should get completely separated into two parts.
5. The method of section is superior to the method of joints when the :
   (i) Method of join fails to start or proceed
   (ii) Forces only in few members are to be determined.

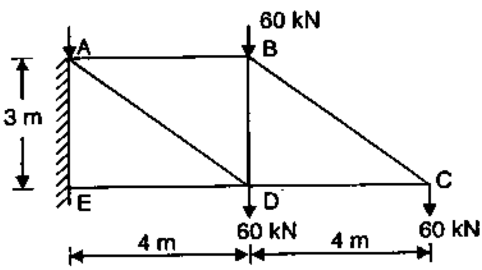
EXERCISES

I Determine the forces developed in all the members of the trusses shown in Fig. A27(*a*) to A27(*f*) by the :

(i) Method of joints

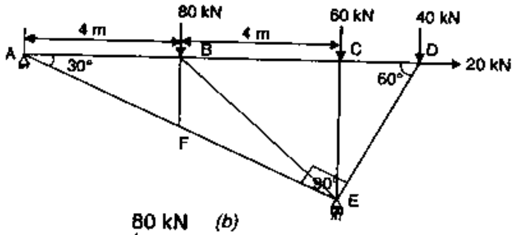
(ii) Method of tension coefficient.

[*Sign Convention (Answers):* The member forces are indicated with +ve sign to mean tension and –ve sign to mean compression.].



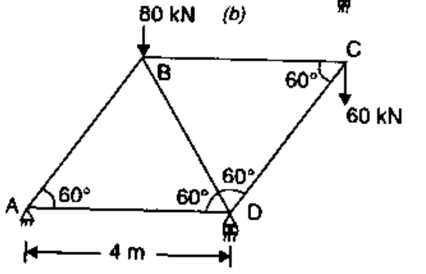
[***Ans***: F_{AB} = 80 kN, F_{BC} = 100 kN, F_{CD} = – 80 kN, F_{DE} = –320 kN, F_{BD} = –120 kN, F_{AD} = 300 kN]

Figure A27(a)



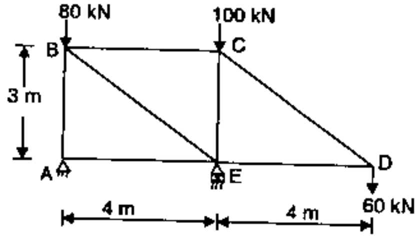
(b)

[**Ans.** $F_{AB} = 26.19$ kN,
$F_{BC} = 43.09$ kN,
$F_{CD} = 43.09$ kN, $F_{DE} = -46.19$ kN,
$F_{EF} = 53.34$ kN, $F_{AF} = 53.34$ kN,
$F_{CE} = -60$ kN, $F_{BF} = 0$,
$F_{BE} = -105.83$ kN]



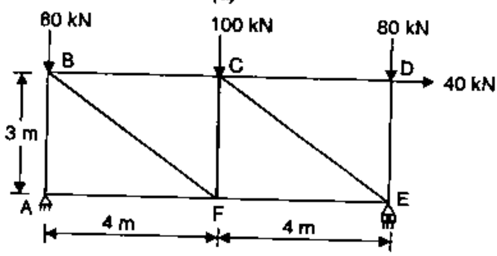
(c)

[**Ans.** $F_{AB} = -11.55$ kN,
$F_{BC} = 34.64$ kN, $F_{CD} = -69.28$ kN,
$F_{AD} = 5.77$ kN, $F_{BD} = 80.83$ kN]



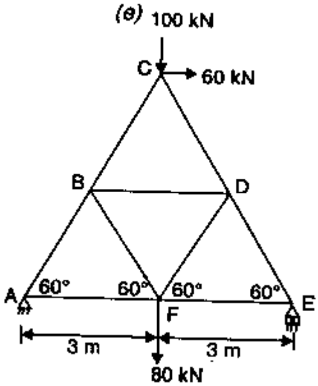
(d)

[**Ans:** $F_{AB} = -20$ kN, $F_{BC} = 80$ kN,
$F_{CD} = 100$ kN, $F_{DE} = -80$ kN,
$F_{AE} = 0$, $F_{CE} = -160$ kN,
$F_{BE} = -100$ kN]



(e)

[**Ans:** F_{AB} –115 kN, $F_{BC} = -46.67$ kN,
$F_{CD} = 40$ kN, $F_{DE} = -80$ kN,
$F_{EF} = 86.67$ kN, $F_{AF} = 40$ kN,
$F_{CF} = -35$ kN, $F_{BF} = 58.33$ kN,
$F_{CE} = -108.33$ kN]



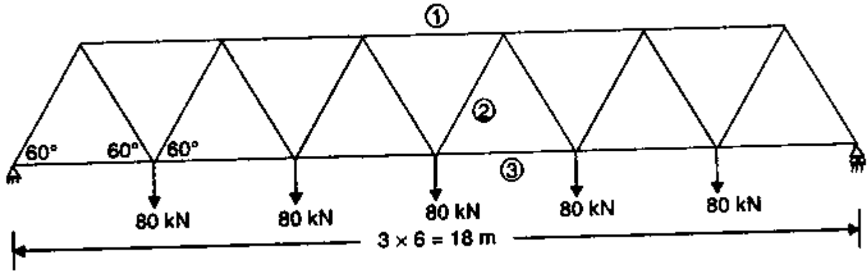
(f)

[**Ans:** $F_{AB} = -43.92$ kN, $F_{BC} = 2.27$ kN,
$F_{CD} = -117.73$ kN, $F_{DE} = -163.92$ kN,
$F_{EF} = 81.96$ kN, $F_{AF} = 81.96$ kN,
$F_{BF} = 46.19$ kN, $F_{DF} = 46.19$ kN,
$F_{BD} = 46.19$ kN]

Figure A27(b to f)

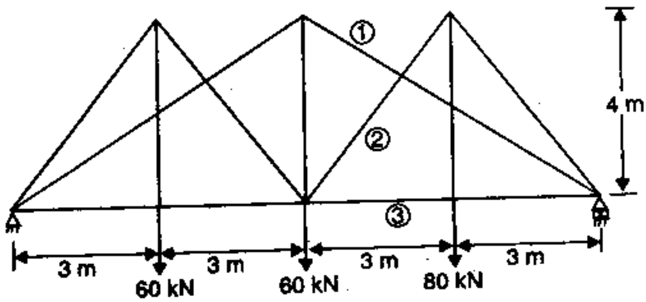
II Determine the forces in the members 1, 2, 3 of trusses shown in Figs. A28(*a*) to A28(*c*) by the method of section.

[*Sign Convention (Answers):* Forces are indicated with +ve sign for tension and –ve sign for compression.]



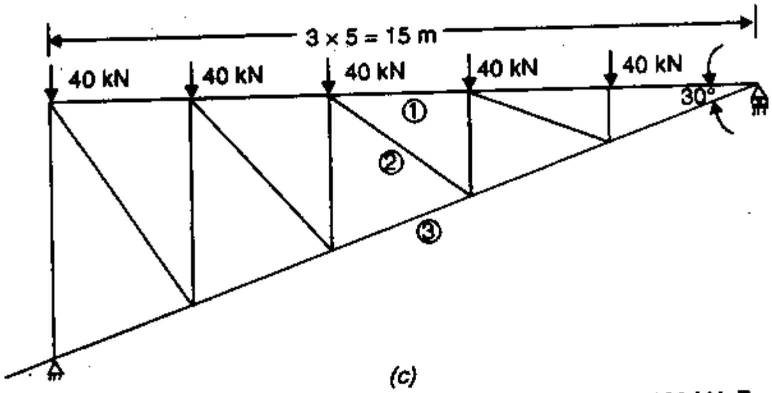
(a)

[**Ans:** $F_1 = 392.61$ kN, $F_2 = 46.19$ kN, $F_3 = -392.61$ kN]



(*b*)

[**Ans:** $F_1 = -117.18$ kN, $F_2 = 50$ kN, $F_3 = 67.5$ kN]



(*c*)

[**Ans:** $F_1 = -120$ kN, $F_2 = 56.57$ kN, $F_3 = 40$ kN]

Figure A28