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Submodular Functions and Optimization

Second Edition

SATORU FUJISHIGE

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Submodular Functions and Optimization

Second Edition

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Preface

Submodular functions frequently appear in the analysis of combinatorial systems such as graphs, networks, and algebraic systems, and reveal combinatorially nice and deep structures of the systems. The importance of submodular functions has widely been recognized in recent years in combinatorial optimization and other fields of combinatorial analysis. The present book provides the readers with an exposition of the theory of submodular functions from an elementary technical level to an advanced one.

The theory of submodular functions was developed in the earliest stage till 1950's by H. Whitney and W. T. Tutte for matroids, by G. Choquet for the capacity theory and by O. Ore for graphs. A flourishing stage of the theory came with J. Edmonds' work on matroids and polymatroids in 1960's. Related studies were also made in the theory of characteristic function games by L. S. Shapley and others. Since 1970, applications of (poly-)matroids to practical engineering problems have been extensively made by M. Iri, A. Recski and others, and also theoretical developments in submodular functions by W. H. Cunningham, J. Edmonds, A. Frank, M. Iri, E. L. Lawler, L. Lovász, A. Schrijver, É. Tardos, N. Tomizawa, D. J. A. Welsh, U. Zimmermann and others.

The theory of submodular functions is now becoming mature, but a lot of fundamental and useful results on submodular functions are scattered in the literature. The main purpose of the present book is to put these materials together and to show the author's unifying view of the theory of submodular functions by means of base polyhedra and duality for submodular and supermodular systems. Special emphasis is placed on the constructive aspects of the theory, which will lead us to practical efficient algorithms. No comprehensive survey of submodular functions is aimed at here. I had to omit important results on submodular functions such as a strongly polynomial time algorithm for minimizing submodular functions due to M. Grötschel, L. Lovász and A. Schrijver. This is mainly because the precise description and validation of the results would require further technical developments outside the mainstream of this book.

A sketch of the author's view of submodular functions was given in a survey paper [Fuji84c], which was written while I was visiting Professor Bernhard Korte's Institute in Bonn as an Alexander von Humboldt fellow in 1982-83, and laid a basis of the project of writing this book, which I gratefully acknowledge. I also acknowledge that part of my work, upon which the present book is primarily based, has been supported by grants-in-aid of the Ministry of Education, Science and Culture of Japan.

I would like to express my deep sincere thanks to Professor Masao Iri of the University of Tokyo who first drew my attention to the theory of matroids, a promising and enjoyable research field of combinatorics, in 1975 and has since then been keeping giving me invaluable advice and stimulating discussions on submodular functions and other related discrete systems. Without his advice most of my work would not have been accomplished. Thanks are also due to Professor Nobuaki Tomizawa, now at Niigata University, with whom I enjoyed inspiring discussions and joint work. I am also very much grateful to Professor Peter L. Hammer for his encouragement to write this book.

I used a preliminary version of this book for a lecture of the Doctoral Program in Socio-Economic Planning at the University of Tsukuba and I thank the students who attended the lecture for their useful comments.

I have also benefited from comments and communications received from Bill Cunningham, Tetsuo Ichimori, Naoki Katoh, Kazuo Murota, Masataka Nakamura, Hans Röck and Uwe Zimmermann, to name a few, in the course of my research on submodular functions and writing this book.

July 1990 S.F.

Preface to the Second Edition

When I finished my monograph (the first edition) in 1990, there was a polynomial-time algorithm for minimizing submodular functions by means of the ellipsoid method, due to Grötschel, Lovász, and Schrijver, and devising a combinatorial polynomial-time algorithm for minimizing submodular functions was still an open problem. Submodular function minimization is so fundamental in the theory of submodular functions and optimization that the monograph could not be completed without treating algorithms for submodular function minimization, but I hesitated to include the ellipsoid method for submodular function minimization due to Grötschel, Lovász, and Schrijver because of its non-combinatorial feature. Hence in the first edition submodular function minimization was treated algorithmically in a very unsatisfactory way. However, in 1999 the long-standing open problem of submodular function minimization was resolved independently by Satoru Iwata, Lisa Fleischer and myself, and by Lex Schrijver, in different ways though both algorithms were based on the framework of Bill Cunningham. I am very happy to add Chapter VI thereby including the combinatorial strongly polynomial algorithms for submodular function minimization.

Moreover, among other related developments after 1990 one of the most important results is the discrete convex analysis due to Kazuo Murota, which has been described in another new chapter, Chapter VII.

Chapters VI and VII form Part II of the second edition. Part I includes Chapters I$\sim$V, which formed the original edition of the monograph. In Chapters I$\sim$V of Part I typos and minor errors have been corrected. I appreciate comments received from readers and my friends and colleagues on the first edition of this monograph. I have also added remarks and references related to the developments after 1990, which are put between brackets [and].

I would like to mention here another important recent development related to submodular functions that I could not include in Part II. It is concerned with the connectivity augmentation problem for graphs and its generalization, which was pioneered by T. Watanabe and A. Nakamura [Watanabe+Nakamura87] and further developed by A. Frank, T. Jordán, H. Nagamochi, T. Ibaraki, and others. Readers should be referred to [Frank92, 94b, 05], [Frank+Jordán95], [Frank+Király02], [Jordán95], [Jackson+Jordán05], [Nagamochi+Ibaraki02], [Nagamochi00, 04] and the references therein for later and recent developments. I also would like to mention the source location problem, which is closely related to the connectivity augmentation problem. The source location problem was first considered in [Tamura+Sengoku+Shinoda+Abe92] and [Tamura+Sugawara+Sengoku+Shinoda98], and was further investigated in [Ito+Uehara+Yokoyama00], [Ito+IINUY02] and [Arata+Iwata+Makino+Fuji02] for undirected networks, and in [Nagamochi+Ishii+Ito01], [Ito+MAHIF03] and [Bárász+Becker+Frank05] for directed networks (also see the references therein).

I am very grateful to Hiroshi Hirai, Satoru Iwata, Tom McCormick, Kazuo Murota, Takeshi Naitoh, Akiyoshi Shioura, Akihisa Tamura, and Zaifu Yang for their useful comments on an earlier version of Chapters VI and VII in Part II, which helped rectifying errors and improved the presentation of Part II. Special thanks are also due to Kazuo Murota, who gave me valuable detailed comments on Chapter VII. I also thank András Frank and Tibor Jordán for useful information about recent developments in connectivity augmentation.

Kyoto, January 2005 S.F.

Contents

PART I

Part I consists of Chapters I$\sim$V that formed the main body of the first edition of this monograph. Comments and remarks added in the second edition are given within brackets [and].

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Chapter I. Introduction

In this chapter we give a short introduction of the structure of this book and briefly review elements of algebra, graphs, networks and linear inequalities for readers' convenience.

1. Introduction

1.1. Introduction

In 1935 H. Whitney [Whit35] introduced the concept of matroid as an abstraction of the linear dependence structure of a set of vectors. Several systems of axioms for defining a matroid are now known, each of which is simple but substantial enough to yield a deep theory in Combinatorial Optimization and to have a lot of applications in practical engineering problems (see [Iri83], [Iri+Fuji81], [Murota87, 00a], [Recski89]). Matroidal structures are closely related to a class of efficiently solvable combinatorial optimization problems; a careful examination of an efficiently solvable problem often reveals a matroidal structure which underlies the problem.

In 1970 J. Edmonds [Edm70] combined the matroid theory with polyhedral combinatorics and lead us to the concept of polymatroid. A polymatroid polyhedron, called an independence polyhedron, is expressed by a system of linear inequalities with $\{0,1\}$-coefficients and the right-hand sides given by a submodular function which is the rank function of the polymatroid. The relation between matroids and polymatroids is similar to that between matchings in bipartite graphs and flows in networks.

The rank function of any polymatroid is a monotone nondecreasing submodular function on a Boolean lattice 2^E for a finite set E. The monotonicity of the rank function does not play any essential rôle in characterizing the combinatorial structure of the polymatroid polyhedron since the monotonicity is not invariant with respect to translations of the polyhedron. The concepts of submodular and supermodular systems [Fuji78b,84c] naturally come up with this observation. The rank function of a submodular (or supermodular) system is a submodular (or supermodular) function on a distributive lattice (or a ring family), a sublattice of a Boolean lattice. The duality is defined between a submodular system and a supermodular system, which dissolves the clumsy definition of polymatroid duality [McDiarmid75]. Submodular systems are not only theoretical generalizations of matroids and polymatroids but also significantly extend the applicability in practical problems.

In Chapter II we first introduce the concepts of submodular and supermodular systems and their associated base polyhedra by following the historical generalization sequence of matroids, polymatroids and submodular systems. We then examine algorithmic aspects of submodular systems and basic structures of base polyhedra.

In Chapter III we consider a class of network flow problems with submodular boundary constraints, which we call the neoflow problem. It includes the (poly-)matroid intersection problem of J. Edmonds [Edm70], the submodular flow problem of J. Edmonds and R. Giles [Edm+Giles77], the independent flow problem of the author [Fuji78a] and the polymatroidal flow problem of R. Hassin [Hassin78,82] and E. L. Lawler and C. U. Martel [Lawler+Martel82b].

Submodular functions are discrete analogues of convex functions. In Chapter IV we develop a theory of submodular functions from the point of view of convex analysis [Rockafellar70], which we call the submodular analysis. We will make clear the close relationship between the submodular analysis and the results obtained in Chapter III.

Finally we consider nonlinear optimization problems with submodular constraints in Chapter V. A decomposition algorithm is shown for a separable convex optimization problem over a base polyhedron and it lays a basis for the algorithms of the other problems such as the lexicographically optimal base problem, the weighted max-min (min-max) problem and the fair resource allocation problem. We also consider a neoflow problem (the submodular flow problem) with a separable convex cost function.

1.2. Mathematical Preliminaries

(a) Sets

We denote the set of reals by $\mathbf{R}$, the set of rationals by $\mathbf{Q}$ and the set of integers by $\mathbf{Z}$. We also denote the set of nonnegative elements of $\mathbf{R}$ ($\mathbf{Q}$, $\mathbf{Z}$) by $\mathbf{R}_+$ ($\mathbf{Q}_+$, $\mathbf{Z}_+$).

For any finite set X we denote its cardinality by $|X|$. When X is a subset of a set Y, we write $X \subseteq Y$, and when X is a proper subset of Y (i.e., $X \subseteq Y$ and $X \neq Y$), we write $X \subset Y$. It should be noted that this is different from the conventional notation.

For subsets X and Y of a set E, when $X \cap Y = \emptyset$, we say X and Y are *disjoint*, and when $X \cup Y = E$ (or $(E - X) \cap (E - Y) = \emptyset$), we say X and Y are *codisjoint*.

A set of disjoint nonempty subsets B_i $(i \in I)$ of a set E is called a *partition* of E if $\bigcup_{i \in I} B_i = E$. If $\{B_i \mid i \in I\}$ is a partition of E, we call $\{E - B_i \mid i \in I\}$ a *copartition* of E. For sets A and B we call $\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$ the *direct sum* of A and B and denote it by $A \oplus B$. Also, we call $\{(a, b) \mid a \in A,\ b \in B\}$ the *direct product* of A and B and denote it by $A \times B$.

For a mapping $f\colon A \to B$ we often express f as $(f(a) \mid a \in A)$. For example, a family $\mathcal{F}$ of sets X_i $(i \in I)$ is written as $\mathcal{F} = (X_i \mid i \in I)$ and a matrix M with a row index set I, a column index set J and an (i, j)-element M_{ij} $(i \in I,\ j \in J)$ is expressed as $M = (M_{ij} \mid i \in I, j \in J)$. The set of all the mappings from A to B is denoted by B^A. The *characteristic vector* of a subset A of an underlying set E is the mapping $\chi_A : E \to \{0, 1\}$ such that $\chi_A(e) = 1$ for $e \in A$ and $\chi_A(e) = 0$ for $e \in E - A$.

(b) Algebraic structures For a set A we call a binary relation, denoted by $\preceq$, on A a *partial order* or simply an *order* on A if it satisfies

(i) (reflexivity) $\forall a \in A$: $a \preceq a$,

(ii) (antisymmetry) $a \preceq b,\ b \preceq a \implies a = b$,

(iii) (transitivity) $a \preceq b,\ b \preceq c \implies a \preceq c$.

We call the pair $(A, \preceq)$ a *partially ordered set* or a *poset* for short. Also, we often say that A is a poset, when the underlying partial order $\preceq$ is implicitly assumed. If $a \preceq b$ and $a \neq b$, we write $a \prec b$.

Moreover, if we have $a \preceq b$ or $b \preceq a$ for every $a, b \in A$, we call the partial order $\preceq$ a *total order* or a *linear order*. The ordinary orders on $\mathbf{R}$, $\mathbf{Q}$ and $\mathbf{Z}$ are total orders. A total order is usually denoted by $\leq$.

For a poset $\mathcal{P} = (A, \preceq)$ define a poset $\mathcal{P}^* = (A, \preceq^*)$ by

$$a \preceq b \iff b \preceq^* a \tag{1.1}$$

for any $a, b \in A$. We call $\mathcal{P}^* = (A, \preceq^*)$ the *dual poset* of $\mathcal{P}$. We often use $\succeq$ for $\preceq^*$.

For a poset $\mathcal{P} = (A, \preceq)$ a subset B of A is called an *ideal* (or a *lower ideal*) of $\mathcal{P}$ if $x \preceq y \in B$ implies $x \in B$. Also, $B \subseteq A$ is called a *dual ideal* (or an *upper ideal*) of $\mathcal{P}$ if $x \succeq y \in B$ implies $x \in B$.

For two elements x, y of a poset $\mathcal{P} = (A, \preceq)$, if $x \prec y$ and there exists no element z such that $x \prec z \prec y$, we say that y *covers* x.

Let $\mathcal{P} = (A, \preceq)$ be a poset. An element $a \in A$ is an *upper bound* of a subset $B \subseteq A$ if for each $b \in B$ we have $b \preceq a$. Similarly, an element $a \in A$ is a *lower bound* of $B \subseteq A$ if for each $b \in B$ $a \preceq b$. If a is an upper bound (lower bound) of B and a belongs to B, then a is called the *maximum element* (*minimum element*) of B. Note that the maximum (or minimum) element of B, if any exists, is unique. If the set of upper bounds (lower bounds) of B has the minimum element b_* (the maximum element b^*), we call b_* (b^*) the *supremum* (*infimum*) of B and denote it by $\sup B$ ($\inf B$). If for each $x, y \in A$ there exist $\sup\{x, y\}$ and $\inf\{x, y\}$, then we call poset $\mathcal{P} = (A, \preceq)$ a *lattice*, and we write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. These two binary operations, $\vee$ and $\wedge$, are called the *join* and the *meet*, respectively, and satisfy

(i) (idempotency) $\forall x \in A$: $x \vee x = x, \;\; x \wedge x = x$,

(ii) (commutativity) $\forall x, y \in A$: $x \vee y = y \vee x, \;\; x \wedge y = y \wedge x$,

(iii) (associativity) $\forall x, y, z \in A$: $x \vee (y \vee z) = (x \vee y) \vee z$,
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$$

(iv) (absorption) $\forall x, y \in A$: $x \wedge (x \vee y) = x, \;\; x \vee (x \wedge y) = x$.

Conversely, if a set A is given two binary operations $\vee$ and $\wedge$ which satisfy the above (i)$\sim$(iv), then, defining a binary relation $\preceq$ by "$x \preceq y \iff x \vee y = y$ (or $x \wedge y = x$)," we have a lattice $(A, \preceq)$ whose lattice operations are the given $\vee$ and $\wedge$. A lattice $(A, \preceq)$ with lattice operations $\vee$ and $\wedge$ is also expressed as $(A, \vee, \wedge)$. We often call A itself a lattice, assuming the underlying partial order or lattice operations.

A lattice $\mathcal{L} = (A, \vee, \wedge)$ is called a *distributive lattice* if it satisfies

(v) (distributivity) $\forall x, y, z \in A:\; x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let $\mathcal{D}$ be the set of (lower) ideals of a finite poset $\mathcal{P} = (E, \preceq)$. Then $\mathcal{D}$ is a distributive lattice with set union $\cup$ and intersection $\cap$ as the lattice operations $\vee$ and $\wedge$. Conversely, any finite distributive lattice is isomorphic with the one represented by a collection of sets as above (see [Birkhoff37,67]). Any sublattice of a distributive lattice is a distributive lattice. [A distributive lattice formed by ideals of a poset is often called a *ring family*.]

For posets $\mathcal{P}_i = (A_i, \preceq_i)$ $(i \in I)$, define a binary relation $\preceq$ on the direct product $\times_{i \in I} A_i$ of A_i $(i \in I)$ as follows: for $a = (a_i \mid i \in I)$ and $b = (b_i \mid i \in I)$ we have $a \preceq b$ if and only if $\forall i \in I$: $a_i \preceq_i b_i$. Then $\mathcal{P} = (\times_{i \in I} A_i, \preceq)$ is a poset, called the *direct product* of $\mathcal{P}_i = (A_i, \preceq_i)$ $(i \in I)$.

For a finite set E consider a vector space $\mathbf{R}^E$ with a partial order $\preceq$ such that for any $x, y \in \mathbf{R}^E$

$$x \preceq y \iff \forall e \in E : x(e) \leq y(e). \tag{1.2}$$

Since $(\mathbf{R}^E, \preceq)$ is the direct product of $|E|$ copies of $(\mathbf{R}, \leq)$ and a direct product of distributive lattices is a distributive lattice, $(\mathbf{R}^E, \preceq)$ is a distributive lattice. Also, note that for any $x, y, z \in \mathbf{R}^E$

(i) $x \preceq y \Longrightarrow x + z \preceq y + z$,

(ii) $x \preceq y, \lambda \in \mathbf{R}_+ \Longrightarrow \lambda x \preceq \lambda y$.

Therefore, $\mathbf{R}^E$ is a *vector lattice*. Following the convention, we also use $\leq$ for the partial order $\preceq$ on $\mathbf{R}^E$.

Consider a lattice $\mathcal{L} = (L, \vee, \wedge)$ with the minimum element O and the maximum element I. If $x, y \in L$ satisfy $x \vee y = I$ and $x \wedge y = O$, x (or y) is a *complement* of y (or x). Lattice $\mathcal{L}$ is said to be *complemented* if each $x \in L$ has its complement. A *Boolean* lattice is a lattice which is distributive and complemented.

A set G with an element $e \in G$, a binary operation $\cdot : G \times G \to G$ and a unary operation $(\ )^{-1}$: $G \to G$ is a *group* if it satisfies

(1) $\forall x, y, z \in G$: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(2) $\forall x \in G$: $e \cdot x = x \cdot e = x$,

(3) $\forall x \in G$: $x^{-1} \cdot x = x \cdot x^{-1} = e$.

Here, the element $e \in G$ satisfying (2) uniquely exists and is called the *unit element* (or the *neutral element*). Also, x^{-1} is called the *inverse element* of x. When G satisfies (1) but not necessarily (2) and (3), G is called a *semi-group*.

When a group G is *commutative*, i.e., $\forall x, y \in G$: $x \cdot y = y \cdot x$, G is called a *commutative group* or an *Abelian group*. For a commutative group G we often use $+$ instead of $\cdot$ for the binary operation, and denote the inverse of x by $-x$ and the unit (or neutral) element by 0. A commutative group G expressed in this way is called an *additive group*.

G is called a *totally ordered additive group* if G is an additive group with a total order $\leq$ defined on it and if the following (1) and (2) hold:

(1) $x \leq y \Longrightarrow -x \geq -y$,

(2) $x_1 \leq y_1,\ x_2 \leq y_2 \Longrightarrow x_1 + x_2 \leq y_1 + y_2$.

The set $\mathbf{R}$ of reals, $\mathbf{Q}$ of rationals and $\mathbf{Z}$ of integers are typical examples of a totally ordered additive group.

A *ring* A is an algebraic system with two operations $+$ (addition) and $\cdot$ (multiplication) such that

(i) A is an additive group with operation $+$,

(ii) A is a semi-group with operation $\cdot$,

(iii) $\forall x, y, z \in A$: $(x + y) \cdot z = x \cdot z + y \cdot z, \quad x \cdot (y + z) = x \cdot y + x \cdot z$.

A ring A is *commutative* if A is a commutative semi-group with respect to operation $\cdot$. The set $\mathbf{Z}$ of integers is an example of a commutative ring.

For a commutative ring A with a unit element $1 (\neq 0)$, an additive group G, and an operation $*$: $A \times G \to G$, we call G an *A-additive group* or an *A-module* if for each $\alpha, \beta \in A$ and $x, y \in G$ we have

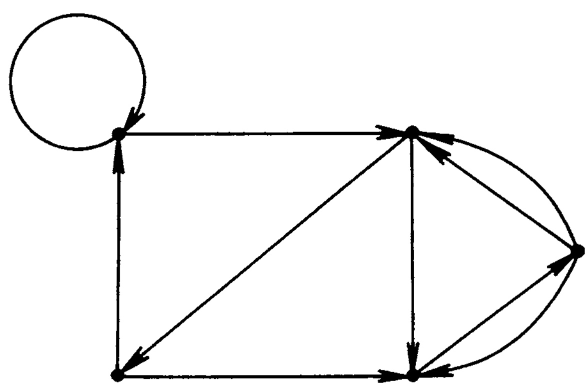
$$\alpha * (x + y) = \alpha * x + \alpha * y, \quad (\alpha + \beta) * x = \alpha * x + \beta * x, \tag{1.3a}$$
$$\alpha * (\beta * x) = (\alpha \cdot \beta) * x, \quad 1 * x = x. \tag{1.3b}$$

$\mathbf{Z}$ is a $\mathbf{Z}$-additive group.

A commutative ring K is called a *field* if $K - \{0\}$ is a commutative group with respect to operation $\cdot$. The set $\mathbf{Q}$ of rationals and $\mathbf{R}$ of reals are fields with ordinary operations of the addition and the multiplication. For two fields H and K with $H \subseteq K$ such that the operations in H are restrictions of those in K, we say H is a *subfield* of K and K is an *extension* of H. [A commutative ring with a unit element $1(\neq 0)$ (or a field) A is called a totally ordered ring (or field) if A is a totally ordered additive group and for any $x, y \in A$ with $x, y > 0$ we have $xy > 0$.]

(c) Graphs

Let V and A be finite sets, where V is called a *vertex set* and A an *arc set*. Each $v \in V$ is called a *vertex* and each $a \in A$ an *arc*. We are also given two functions $\partial^+, \partial^- \colon A \to V$. For each arc $a \in A$ $\partial^+ a$ is the *initial end-vertex* (or the *tail*) of a and $\partial^- a$ is the *terminal end-vertex* (or the *head*) of a. We call $G = (V, A; \partial^+, \partial^-)$ a *graph*. When there is no possibility of confusion, we also denote the graph by $G = (V, A)$. We often express an arc a by the ordered pair $(\partial^+ a, \partial^- a)$ of the end-vertices when such a pair uniquely determines the arc. A graph $G = (V, A)$ with arc set $A = \{(u, v) \mid u, v \in V, u \neq v\}$ is sometimes called a *complete directed graph*. See Fig. 1.1 for a geometric representation of a graph. For graphs see, e.g., [Berge73] and [Harary69].



$\partial^+ a$ a $\partial^- a$

Figure 1.1: A graph.

An arc a such that $\partial^+ a = \partial^- a$ is called a *selfloop* and arcs a_1, a_2 such that $\{\partial^+ a_1, \partial^- a_1\} = \{\partial^+ a_2, \partial^- a_2\}$ is called a *parallel arcs*.

We call a graph $H = (W, B; \partial_H^+, \partial_H^-)$ a *subgraph* of a graph $G = (V, A; \partial^+, \partial^-)$ if $W \subseteq V$, $B \subseteq A$ and ∂_H^+ and ∂_H^- are, respectively, the restrictions of ∂^+ and ∂^- to B. We often use the original ∂^+ and ∂^- for

∂_H^+ and ∂_H^-. Moreover, if B is the set of all the arcs $a \in A$ such that $\{\partial^+ a, \partial^- a\} \subseteq W$, then we call $H = (W, B)$ a subgraph *induced by* the vertex subset W; and if W is the set of all the end-vertices of arcs in B, then we call $H = (W, B)$ a subgraph *induced by* the arc subset B and denote it by $G \cdot B$. For any $B \subseteq A$ let $V(B)$ be the set of end-vertices of arcs of B. Consider a graph $H = (W, A - B; \partial_H^+, \partial_H^-)$ such that $W = (V - V(B)) \cup \{v_B\}$ (v_B: a new vertex), $\partial_H^\pm a = \partial^\pm a$ (if $\partial^\pm a \notin V(B)$) and $\partial_H^\pm a = v_B$ (if $\partial^\pm a \in V(B)$). We denote this graph H by G/B and call it a *contraction* of G by B. Informally, contraction G/B is obtained by shrinking B together with the end-vertices into a single vertex v_B.

When we do not distinguish $\partial^+ a$ from $\partial^- a$ for each a in A or we are not concerned with the orientations of the arcs, we call the graph $G = (V, A)$ an *undirected graph* and call each $a \in A$ an *edge* instead of an arc. When we emphasize the fact that a graph is not an undirected one, we call it a *directed* graph. A graph is called *simple* if it does not contain any selfloops or parallel arcs.

We define for each vertex $v \in V$

$$\delta^+ v = \{a \mid a \in A,\ \partial^+ a = v\}, \tag{1.4}$$

$$\delta^- v = \{a \mid a \in A,\ \partial^- a = v\}. \tag{1.5}$$

An arc a in $\delta^+ v \cup \delta^- v$ is said to be *incident to* v. For any $B \subseteq A$ and $U \subseteq V$ we define $\partial^\pm B = \{\partial^\pm a \mid a \in B\}$ and $\delta^\pm U = \bigcup_{v \in U} \delta^\pm v$. A vertex u in $(\partial^- \delta^+ v) \cup (\partial^+ \delta^- v)$ is said to be *adjacent to* v. Also define for each subset U of V

$$\Delta^+ U = \{a \mid a \in A,\ \partial^+ a \in U,\ \partial^- a \in V - U\}, \tag{1.6}$$

$$\Delta^- U = \{a \mid a \in A,\ \partial^- a \in U,\ \partial^+ a \in V - U\}. \tag{1.7}$$

If $\Delta^+ U \cup \Delta^- U \neq \emptyset$, we call the set $\Delta^+ U \cup \Delta^- U$ of arcs a *cutset*. A single-element cutset is called a *bridge*.

For a graph $G = (V, A)$ define a matrix $D = (D_v^a \mid v \in V, a \in A)$ with a row index set V and a column index set A by

$$D_v^a = \begin{cases} \ \ 1 & \text{if } v = \partial^+ a \text{ and } v \neq \partial^- a, \\ -1 & \text{if } v = \partial^- a \text{ and } v \neq \partial^+ a, \\ \ \ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

We call D the *incidence matrix* of G.

A *path* in G is an alternating sequence $(v_0, a_1, v_1, a_2, \cdots, v_{k-1}, a_k, v_k)$ of vertices v_i $(i = 0, 1, \cdots, k)$ and arcs a_i $(i = 1, 2, \cdots, k)$ such that $\{\partial^+ a_i, \partial^- a_i\}$ $= \{v_{i-1}, v_i\}$ $(i = 1, 2, \cdots, k)$. v_0 is the *initial vertex* of the path and v_k is the *terminal vertex*. We also say that the path is *from* v_0 *to* v_k. For an arc a_i such that $\partial^+ a_i = v_{i-1}$ and $\partial^- a_i = v_i$ we say a_i is *positively oriented* in the path. Also, if $\partial^+ a_i = v_i$, $\partial^- a_i = v_{i-1}$ and $v_i \neq v_{i-1}$, we say a_i is *negatively oriented* in the path. If all the arcs in the path are positively oriented, we call the path a *directed path*. A (directed) path is called a (*directed*) *cycle* if it contains at least one arc and its initial and terminal vertices coincide with each other. A path or cycle is called *elementary* if no vertices in it are repeated (except for the initial and terminal vertices when we consider a cycle).

A graph $G = (V, A)$ is called *connected* if for every two vertices $u, v \in V$ there exists a path from u to v. Connectedness is a property of G as an undirected graph. A maximal connected subgraph of G is called a *connected component* of G. G is decomposed into its connected components $H_i = (V_i, A_i)$ $(i \in I)$, where $\{V_i \mid i \in I\}$ is a partition of V and $\{A_i \mid i \in I\}$ is a partition of A. The *rank* of a graph $G = (V, A)$ is the number of its vertices minus the number of its connected components, which is equal to the matrix rank of the incidence matrix D of G. The *nullity* of G is equal to $|A|$ minus the rank of G.

A graph $G = (V, A)$ is called *strongly connected* if for every two vertices $u, v \in V$ there exists a directed path from u to v. A maximal strongly-connected subgraph of G is called a *strongly connected component* of G. G is decomposed into its strongly connected components $\hat{H}_i = (\hat{V}_i, \hat{A}_i)$ $(i \in \hat{I})$, where $\{\hat{V}_i \mid i \in \hat{I}\}$ is a partition of V but $\{\hat{A}_i \mid i \in \hat{I}\}$ is not a partition of A though $\hat{A}_i$ $(i \in \hat{I})$ are disjoint. For two strongly connected components $\hat{H}_{i_1}$ and $\hat{H}_{i_2}$ we write $\hat{H}_{i_1} \preceq \hat{H}_{i_2}$ if and only if there exists a directed path from a vertex of $\hat{H}_{i_2}$ to a vertex of $\hat{H}_{i_1}$. Note that if $\hat{H}_{i_1} \preceq \hat{H}_{i_2}$, there exists a directed path from any vertex of $\hat{H}_{i_2}$ to any vertex of $\hat{H}_{i_1}$. We can easily see that this binary relation $\preceq$ on the set of strongly connected components of G is a partial order. We say that the partial order is *naturally induced by the decomposition of* G *into strongly connected components*. (See Fig. 1.2.)

A graph $G = (V, A)$ is called a *tree* if it is connected and does not contain any cycle. For a graph $G = (V, A)$ and a subgraph $H = (W, B)$ of G, H is called a (*spanning*) *tree of* G if $W = V$ and H is a tree.

A tree $T = (V, A)$ is called a *directed tree* if $|\delta^- v| \leq 1$ for each $v \in V$.

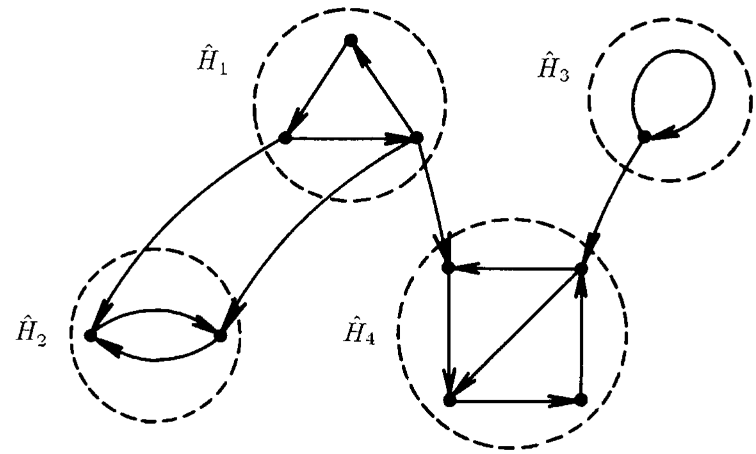


Figure 1.2: The decomposition into strongly connected components.

For a directed tree $T = (V, A)$ there uniquely exists a vertex $v_0 \in V$ such that $|\delta^- v_0| = 0$. We call v_0 the *root* of T. A vertex v of T with $|\delta^+ v| = 0$ is called a *leaf* of T. For vertices u, v of T such that for some arc a of T we have $\partial^+ a = u$ and $\partial^- a = v$, we call u the *parent* of v and v a *child* of u. Children of a common parent are called *siblings*.

A tree $\hat{T} = (V, A)$ is called a *directed tree toward the root* if $|\delta^+ v| \leq 1$ for each $v \in V$. The root of $\hat{T}$ is a vertex v_0 such that $|\delta^+ v_0| = 0$.

A *bipartite graph* $G = (V^+, V^-; A)$ is a graph with two disjoint vertex sets V^+ and V^- and with an arc set A consisting of arcs a such that $\partial^+ a \in V^+$ and $\partial^- a \in V^-$ alone. We often call V^+ the *left vertex set* and V^- the *right vertex set*. A *matching* M in the bipartite graph G is a subset of the arc set A such that for any distinct arcs a, a' in M we have $\partial^+ a \neq \partial^+ a'$ and $\partial^- a \neq \partial^- a'$. Also, a *cover* (U^+, U^-) of G is the ordered pair of $U^+ \subseteq V^+$ and $U^- \subseteq V^-$ such that for any arc $a \in A$ we have $\partial^+ a \in U^+$ or $\partial^- a \in U^-$.

Theorem 1.1 (König-Egerváry): *For a bipartite graph $G = (V^+, V^-; A)$ we have*

$$\begin{aligned} &\max\{|M| \mid M\text{: a matching of } G\} \\ &= \min\{|U^+| + |U^-| \mid (U^+, U^-)\text{: a cover of } G\}. \end{aligned} \tag{1.9}$$

For a simple graph $G = (V, A)$, not necessarily a bipartite graph, a subset

M of the arc set A is called a *matching* of G if for each distinct $a, a' \in M$ we have $\{\partial^+ a, \partial^- a\} \cap \{\partial^+ a', \partial^- a'\} = \emptyset$.

For a poset $\mathcal{P} = (V, \preceq)$ on a finite set V consider a graph $G(\mathcal{P}) = (V, A(\mathcal{P}))$ with the vertex set V and the arc set A defined by

$$A(\mathcal{P}) = \{(u, v) \mid u, v \in V,\ u \text{ covers } v \text{ in } \mathcal{P}\}. \tag{1.10}$$

The *Hasse diagram* of the poset $\mathcal{P} = (V, \preceq)$ is a planar drawing of the graph $G(\mathcal{P}) = (V, A(\mathcal{P}))$ in such a way that for each arc (u, v) of $G(\mathcal{P})$ u is located vertically higher than v. Since the orientation of each arc of the Hasse diagram is uniquely determined by the planar drawing, the Hasse diagrams are usually drawn as if they were undirected graphs (see Fig. 1.3).

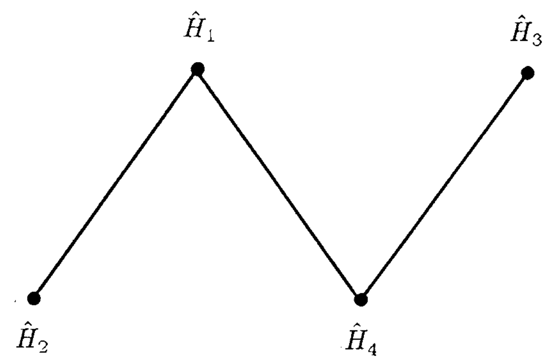


Figure 1.3: The Hasse diagram for the decomposition structure in Fig. 1.2.

For a finite set E and a family $\mathcal{F}$ of subsets of E we call $H = (E, \mathcal{F})$ a *hypergraph*, where each element of E is called a *vertex* of H and each element of $\mathcal{F}$ an *edge* of H. A hypergraph $H = (E, \mathcal{F})$ is said to be *connected* if for each distinct $e, e' \in E$ there exists a sequence $F_1, F_2, \cdots, F_k$ of edges of H such that $e \in F_1$, $e' \in F_k$ and $F_i \cap F_{i+1} \neq \emptyset$ $(i = 1, \cdots, k-1)$. A hypergraph $H' = (E', \mathcal{F}')$ is a *subhypergraph* of $H = (E, \mathcal{F})$ if $E' \subseteq E$ and $\mathcal{F}' \subseteq \mathcal{F}$. A *connected component* of $H = (E, \mathcal{F})$ is a maximal connected subhypergraph of H.

(d) Network flows

Consider a network with an underlying graph $G = (V, A)$ having two distinguished vertices $s^+, s^- \in V$ and with a nonnegative *capacity function*

$c: A \to \mathbf{R}_+$ on the arc set A. Vertices s^+ and s^- are, respectively, the *entrance* and the *exit* of the network. We call this network a *two-terminal network* and denote it by $\mathcal{N} = (G = (V, A), s^+, s^-, c)$. A function φ: $A \to \mathbf{R}$ is a *feasible flow* in the network $\mathcal{N} = (G = (V, A), s^+, s^-, c)$ if φ satisfies

$$\forall a \in A\colon\ 0 \le \varphi(a) \le c(a), \tag{1.11}$$

$$\forall v \in V - \{s^+, s^-\}\colon\ \partial\varphi(v) \equiv \sum_{a \in \delta^+ v} \varphi(a) - \sum_{a \in \delta^- v} \varphi(a) = 0. \tag{1.12}$$

The *value* of a feasible flow φ is given by

$$\begin{aligned} \partial\varphi(s^+) \quad &\equiv \sum_{a \in \delta^+ s^+} \varphi(a) - \sum_{a \in \delta^- s^+} \varphi(a) \\ &\qquad (= \sum_{a \in \delta^- s^-} \varphi(a) - \sum_{a \in \delta^+ s^-} \varphi(a) \equiv -\partial\varphi(s^-)). \end{aligned} \tag{1.13}$$

The function $\partial\varphi$: $V \to \mathbf{R}$ is called the *boundary* of φ. Note that regarding φ and $\partial\varphi$ as column vectors, we have $\partial\varphi = D\varphi$ for the incidence matrix D of G. A feasible flow of maximum value is called a *maximum flow* in $\mathcal{N}$.

We call a vertex subset U with $s^+ \in U$ and $s^- \notin U$ a *cut* of $\mathcal{N}$. The *capacity* of a cut U is defined by

$$\kappa_c(U) = \sum_{a \in \Delta^+ U} c(a). \tag{1.14}$$

Theorem 1.2 (Ford-Fulkerson): *For a two-terminal network* $\mathcal{N} = (G = (V, A), s^+, s^-, c)$

$$\begin{aligned} &\max\{\partial\varphi(s^+) \mid \varphi\colon \text{ a feasible flow in } \mathcal{N}\} \\ &= \min\{\kappa_c(U) \mid U\colon \text{ a cut of } \mathcal{N}\}. \end{aligned} \tag{1.15}$$

Moreover, if the capacity function c is integer-valued, there exists an integral maximum flow in $\mathcal{N}$.

Suppose that we are given a graph $G = (V, A)$ and lower and upper capacity functions $\underline{c}$, $\overline{c}$: $A \to \mathbf{R}$ with $\underline{c} \le \overline{c}$. Denote this network by $\mathcal{N}_0 = (G = (V, A), \underline{c}, \overline{c})$. A *feasible circulation* in network $\mathcal{N}_0$ is a function φ: $A \to \mathbf{R}$ such that

$$\forall a \in A\colon\ \underline{c}(a) \le \varphi(a) \le \overline{c}(a), \tag{1.16}$$

$$\forall v \in V\colon\ \partial\varphi(v) = 0. \tag{1.17}$$

Theorem 1.3 (Hoffman): *There exists a feasible circulation in network $\mathcal{N}_0$ if and only if for each $U \subseteq V$ we have*

$$\sum_{a\in\Delta^+U} \overline{c}(a) \geq \sum_{a\in\Delta^-U} \underline{c}(a). \tag{1.18}$$

Moreover, if the lower and upper capacity functions $\underline{c}$ and $\overline{c}$ are integer-valued and there exists a feasible circulation in $\mathcal{N}_0$, then there exists an integral feasible circulation in $\mathcal{N}_0$.

From Theorem 1.3 we can show

Theorem 1.4 (Gale): *Consider a bipartite graph $G = (V^+, V^-; A)$ and suppose we are given nonnegative (supply and demand) vectors $s \in \mathbf{R}_+^{V^+}$ and $d \in \mathbf{R}_+^{V^-}$. Then, there exists a nonnegative flow $\varphi\colon A \to \mathbf{R}_+$ in the bipartite network such that*

$$\forall v \in V^+\colon\ \partial\varphi(v) \leq s(v), \tag{1.19a}$$

$$\forall v \in V^-\colon\ -\partial\varphi(v) \geq d(v) \tag{1.19b}$$

if and only if for each $U^- \subseteq V^-$ we have

$$\sum_{v\in\partial^+\delta^-U^-} s(v) \geq \sum_{v\in U^-} d(v). \tag{1.20}$$

(e) Elements of convex analysis and linear inequalities

For more details of the subjects of this subsection see [Rockafellar70], [Stoer + Witzgall70], [Bachem + Grötschel82] and [Schrijver86].

Let E be a finite set. For any $x, y \in \mathbf{R}^E$ and nonnegative $\lambda, \mu \in \mathbf{R}$ such that $\lambda + \mu = 1$ we call $\lambda x + \mu y$ a *convex combination* of x and y. A set $A \subseteq \mathbf{R}^E$ is said to be *convex* if all the convex combinations of arbitrary two points in A belong to A. A typical example of a convex set is a solution set of a system of linear equations and linear inequalities:

$$\sum_{e\in E} a_i^1(e)x(e) = b_i^1 \quad (i \in I), \tag{1.21a}$$

$$\sum_{e\in E} a_j^2(e)x(e) \leq b_j^2 \quad (j \in J), \tag{1.21b}$$

where $a_i^1(e)$, $a_j^2(e)$, b_i^1, b_j^2 ($i \in I$, $j \in J$, $e \in E$) are real constants. Such a convex set is called a *polyhedral convex set* or a *convex polyhedron* or simply a *polyhedron*. We are mainly concerned with polyhedral convex sets. Note that different systems of linear equations and linear inequalities may give the same polyhedron.

A set $C \subseteq \mathbf{R}^E$ is called a *convex cone* if all the nonnegative combinations, $\lambda x + \mu y$ ($\lambda \geq 0$, $\mu \geq 0$; $x, y \in C$), of points in C belong to C. If C is the set of all the nonnegative combinations of points $a_i \in \mathbf{R}^E$ ($i \in I$), we say the set of points a_i ($i \in I$) *generates* the cone C.

An *affine set* or (*linear variety*) in $\mathbf{R}^E$ is a translation of a subspace of vector space $\mathbf{R}^E$. The *dimension* of an affine set A is the dimension of the subspace obtained by a translation of A. The dimension of a (polyhedral) convex set A in $\mathbf{R}^E$ is the dimension of the unique minimal affine set containing A. If the dimension of A is equal to $|E|$, A is said to be *full-dimensional*. A *line* is a one-dimensional affine set. A *halfline* (or *ray*) is a translation of a set given by $\{\lambda x \mid \lambda \geq 0\}$ for some nonzero vector $x \in \mathbf{R}^E$. We usually represent a halfline (or ray) from the origin by a nonzero vector contained in it.

A (polyhedral) convex set A is *bounded* if and only if it does not contain any halfline. A point of a (polyhedral) convex set A is called an *extreme point* of A if it can not be expressed as a convex combination of the other two points in A. A convex polyhedron (polyhedral convex set) is said to be *pointed* if it has at least one extreme point. For a convex polyhedron A described by (1.21) the *characteristic cone* (or *recession cone*) of A is the solution set of

$$\sum_{e \in E} a_i^1(e)x(e) = 0 \quad (i \in I), \tag{1.22a}$$

$$\sum_{e \in E} a_j^2(e)x(e) \leq 0 \quad (j \in J). \tag{1.22b}$$

Denote the characteristic cone of A by $\mathrm{C}(A)$. The characteristic cone $\mathrm{C}(A)$ does not depend on the choice of a representation (1.21) of A, as the following lemma shows.

Lemma 1.5: *For a convex polyhedron A, $x \in \mathrm{C}(A)$ if and only if for each $y \in A$ and nonnegative $\lambda \in \mathbf{R}$ we have $y + \lambda x \in A$. Moreover, A is bounded if and only if $\mathrm{C}(A) = \{\mathbf{0}\}$.*

Lemma 1.6: *A convex polyhedron A described by* (1.21) *is pointed if and* *only if A does not contain any line, or equivalently, the system of* (1.22a) *and*

$$\sum_{e \in E} a_j^2(e)x(e) = 0 \qquad (j \in J) \tag{1.23}$$

has the unique solution $x = \mathbf{0}$.

For any $J_0 \subseteq J$ denote by $A(J_0)$ the convex polyhedron described by (1.21a) and

$$\sum_{e \in E} a_j^2(e)x(e) \;=\; b_j^2 \qquad (j \in J_0), \tag{1.24}$$

$$\sum_{e \in E} a_j^2(e)x(e) \;\leq\; b_j^2 \qquad (j \in J - J_0). \tag{1.25}$$

Define

$$\mathcal{F}(A) = \{A(J_0) \mid J_0 \subseteq J\}. \tag{1.26}$$

Here, it should be noted that $A(J)$ may be empty. Each $F \in \mathcal{F}(A)$ is called a *face* of A. A face which is a proper subset of A is called a *proper face*. Faces of A are determined by A and are independent of the choice of a representation (1.21) of A.

For a nonzero vector $a \in \mathbf{R}^E$ and a scalar $b \in \mathbf{R}$, the set of points $x \in \mathbf{R}^E$ satisfying

$$(a, x) \equiv \sum_{e \in E} a(e)x(e) = b \tag{1.27}$$

is called a *hyperplane*, and the set of points $x \in \mathbf{R}^E$ satisfying

$$(a, x) \leq b \tag{1.28}$$

is a *halfspace*. A convex polyhedron is the intersection of a finite number of halfspaces. We denote by $\mathrm{H}(a, b)$ the hyperplane described by (1.27), and by $\mathrm{H}^+(a, b)$ the halfspace (1.28). Note that $\mathrm{H}^+(a, b) \cap \mathrm{H}^+(-a, -b) = \mathrm{H}(a, b)$. A hyperplane $\mathrm{H}(a, b)$ is called a *supporting hyperplane* of A if $\mathrm{H}(a, b) \cap A \neq \emptyset$ and if $A \subseteq \mathrm{H}^+(a, b)$ or $A \subseteq \mathrm{H}^+(-a, -b)$.

Lemma 1.7: *Suppose A is a full-dimensional convex polyhedron. A subset F of A is a nonempty proper face of A if and only if F is the intersection of A and a supporting hyperplane of A.*

A zero-dimensional face is called a *vertex* and a vector in a vertex is an *extreme point*. A one-dimensional face is called an *edge*. An edge of a convex cone is called an *extreme ray*. An extreme ray of a polyhedron is an extreme ray of the characteristic cone. An extreme ray is represented by a nonzero vector contained in it, which is called an *extreme vector*.

Lemma 1.8: *The set $\mathcal{F}(A)$ of faces of A is a lattice with respect to set inclusion as the partial order. For each $F_1, F_2 \in \mathcal{F}(A)$*

$$F_1 \vee F_2 = \bigcap\{F \mid F \in \mathcal{F}(A),\ F \supseteq F_1 \cup F_2\}, \tag{1.29}$$

$$F_1 \wedge F_2 = F_1 \cap F_2. \tag{1.30}$$

Lattice $\mathcal{F}(A)$ is called the *face lattice* of A. A maximal proper face of A is called a *facet* of A.

For a convex cone $C \subseteq \mathbf{R}^E$ define

$$C^* = \{x \mid x \in \mathbf{R}^E,\ \forall y \in C\colon\ (x, y) \leq 0\}, \tag{1.31}$$

which is called the *dual cone* of C. If C is represented by the system of inequalities

$$(c_i, x) \leq 0 \qquad (i \in I), \tag{1.32}$$

then the dual cone C^* is exactly the cone generated by the vectors $c_i \in \mathbf{R}^E$ $(i \in I)$.

A function $f\colon \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ is called a *convex function* if for each $x, y \in \mathbf{R}^E$ with $f(x), f(y) < +\infty$ and for each $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$ we have

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y). \tag{1.33}$$

[A function g such that $-g$ is a convex function is called a *concave function.* If f is given as a point-wise maximum of finitely many affine functions possibly restricted on a polyhedral convex set, f is called a *polyhedral convex function.*] A vector $a \in \mathbf{R}^E$ is called a *subgradient* of f at a point $x \in \mathbf{R}^E$ if for every $y \in \mathbf{R}^E$

$$f(y) - f(x) \geq (a, y - x). \tag{1.34}$$

The set of all the subgradients of f at x is the *subdifferential* of f at x and is denoted by $\partial f(x)$. Note that $x \in \mathbf{R}^E$ is a minimizer of f if and only if $\mathbf{0} \in \partial f(x)$. We use symbol ∂ for subdifferentials and for boundaries of flows as in (1.13), since there seems to be no possibility of confusion.

For a convex function f: $\mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ the *convex conjugate function* f^*: $\mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f^*(x) = \sup\{\langle x, y\rangle - f(y) \mid y \in \mathbf{R}^E\}. \tag{1.35}$$

[The transformation of f is often called the *Fenchel-Legendre transformation* or the *Legendre transformation.*]

Let us consider a *linear programming problem*:

$$(P) \quad \text{Maximize} \sum_{j=1}^{n} c_j x_j \tag{1.36a}$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, \cdots, m), \tag{1.36b}$$

where a_{ij}, b_i, $c_j \in \mathbf{R}$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ are given constants. The *dual* linear programming problem of (P) is given by

$$(D) \quad \text{Minimize} \sum_{i=1}^{m} b_i \lambda_i \tag{1.37a}$$

$$\text{subject to} \sum_{i=1}^{m} a_{ij} \lambda_i = c_j \quad (j = 1, 2, \cdots, n), \tag{1.37b}$$

$$\lambda_i \ge 0 \quad (i = 1, 2, \cdots, m). \tag{1.37c}$$

Problem (P) is called a *primal* problem and is the dual of (D). We say Problem (P) (or (D)) is *unbounded* if the problem has feasible solutions and the objective function can be made arbitrarily large (or small).

Lemma 1.9 (The weak duality for linear programming): *For any feasible solutions x of (P) and λ of (D),*

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i \lambda_i. \tag{1.38}$$

Theorem 1.10 (The duality for linear programming): *If one of the dual problems (P) and (D) has an optimal solution, then its dual problem also has an optimal solution and for any optimal solutions x^* of (P) and λ^* of (D) we have*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i \lambda_i^*. \tag{1.39}$$

Moreover, if one of the dual problems is unbounded, then its dual problem has no feasible solutions.

Concerning the feasibility of a system of linear inequalities we have the following.

Lemma 1.11 (Farkas): *There exists a feasible solution for the system of linear inequalities* (1.36b) *if and only if, for each nonnegative* $\lambda_i \in \mathbf{R}$ $(i = 1, 2, \cdots, m)$ *such that* $\sum_{i=1}^{m} \lambda_i a_{ij} = 0$ $(j = 1, 2, \cdots, n)$*, we have* $\sum_{i=1}^{m} \lambda_i b_i \geq 0$.

The "if" part is the essential part of the above lemma.

Suppose that a_{ij}, b_i, c_j $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ are given rationals. The system of linear inequalities (1.36b) is called *totally dual integral* if for each integral vector $c = (c_1, c_2, \cdots, c_n)$ such that (P) has an optimal solution there exists an integral optimal solution of the dual problem (D) (see [Hoffman74] and [Edm + Giles77]).

Theorem 1.12 (Hoffman and Edmonds-Giles): *If the system of linear inequalities* (1.36b) *is totally dual integral and* $b \equiv (b_1, b_2, \cdots, b_m)$ *is integral, then each nonempty face of the polyhedron described by* (1.36b) *contains at least one integral point and, in particular, each vertex of the polyhedron, if any, is integral.*

A version of the separation theorem for convex sets is given as follows.

Theorem 1.13: *For a convex cone* $C \subseteq \mathbf{R}^E$ *and a vector* $x \in \mathbf{R}^E$ *not belonging to* C *there exists a weight vector* $w \in \mathbf{R}^E$ *such that*

$$\forall y \in C\colon \sum_{e \in E} w(e)y(e) \geq 0, \tag{1.40}$$

$$\sum_{e \in E} w(e)x(e) < 0. \tag{1.41}$$

If C *is pointed, the inequalities in* (1.40) *can be made strict inequalities.*

[For more details about polyhedra and combinatorial optimization see, e.g., [Cook+Cunningham+Pulleyblank+Schrijver88], [Korte+Vygen00] and [Schrijver03].]

Chapter II.
Submodular Systems and Base Polyhedra

In this chapter we give basic concepts on matroids, polymatroids and submodular systems and show the natural generalization sequences of these concepts from matroids to submodular systems. We also examine the fundamental combinatorial structures of submodular systems and associated polyhedra.

For basic properties of matroids and polymatroids shown without proofs in this chapter, readers should be referred to [Tutte65,71], [Crapo + Rota70], [Lawler76], [Welsh76], [White86, 87] and [Oxley92]. The knowledge about matroids and polymatroids is not prerequisite to understanding submodular systems, though it certainly helps.

For general information on submodular and supermodular functions see, e.g., [Choquet55], [Edm70], [Faigle87], [Frank+Tardos88], [Fuji84c], [Lovász83], [Tomi + Fuji82] and [Welsh76] [(also [Narayanan97] and [Topkis98])].

2. From Matroids to Submodular Systems

2.1. Matroids

The concept of matroid was introduced in 1935 by H. Whitney [Whit35] and independently by B. L. van der Waerden [Waerden37]. The term “matroid” is due to Whitney. As the term indicates, a matroid is an abstraction of linear independence and dependence structure of the set of columns of a matrix.

Let E be a finite set. Suppose that a family $\mathcal{I}$ of subsets of E satisfies the following (I0)$\sim$(I2):

(I0) $\emptyset \in \mathcal{I}$.

(I1) $I_1 \subseteq I_2 \in \mathcal{I} \Longrightarrow I_1 \in \mathcal{I}$.

(I2) $I_1,\ I_2 \in \mathcal{I},\ |I_1| < |I_2| \Longrightarrow \exists e \in I_2 - I_1{:}\ I_1 \cup \{e\} \in \mathcal{I}$.

We call the pair $(E, \mathcal{I})$ a *matroid*. Each $I \in \mathcal{I}$ is called an *independent set* of matroid $(E, \mathcal{I})$ and $\mathcal{I}$ the *family of independent sets* of matroid $(E, \mathcal{I})$.

An independent set which is maximal in $\mathcal{I}$ with respect to set inclusion is called a *base*. The family $\mathcal{B}$ of bases satisfies the following (B0) and (B1):

(B0) $\mathcal{B} \neq \emptyset$.

(B1) $\forall B_1,\ B_2 \in \mathcal{B},\ \forall e_1 \in B_1 - B_2,\ \exists e_2 \in B_2 - B_1{:}\ (B_1 - \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

A subset of E which is not an independent set is called a *dependent set*. A dependent set which is minimal with respect to set inclusion is called a *circuit*. The family $\mathcal{C}$ of circuits satisfies the following (C0)$\sim$(C2):

(C0) $\mathcal{C} \neq \{\emptyset\}$.

(C1) $\forall C_1,\ C_2 \in \mathcal{C}{:}\ C_1 \subseteq C_2 \Longrightarrow C_1 = C_2$.

(C2) $\forall C_1,\ C_2 \in \mathcal{C}$ with $C_1 \neq C_2, \forall e \in C_1 \cap C_2,\ \exists C \in \mathcal{C}{:}\ C \subseteq (C_1 \cup C_2) - \{e\}$.

We define the *rank function* $\rho{:}\, 2^E \to \mathbf{Z}$ of matroid $(E, \mathcal{I})$ by

$$\rho(X) = \max\{|I| \mid I \subseteq X,\ I \in \mathcal{I}\} \tag{2.1}$$

for each $X \subseteq E$. The rank function ρ satisfies the following (ρ0)$\sim$(ρ2):

(ρ0) $\forall X \subseteq E{:}\ 0 \leq \rho(X) \leq |X|$.

(ρ1) $X \subseteq Y \subseteq E \Longrightarrow \rho(X) \leq \rho(Y)$.

(ρ2) $\forall X, Y \subseteq E{:}\ \rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$.

Any function ρ satisfying (ρ2) is called a *submodular function* on 2^E. From (ρ0)$\sim$(ρ2) we see that ρ has the *unit-increase property*, i.e., for any $X, Y \subseteq E$ with $X \subseteq Y$ and $|X|+1 = |Y|$ we have $\rho(Y) = \rho(X)$ or $\rho(Y) = \rho(X)+1$.

Also define the *closure function* $\mathrm{cl}{:}\, 2^E \to 2^E$ of matroid $(E, \mathcal{I})$ by

$$\mathrm{cl}(X) = \{e \mid e \in E,\ \rho(X \cup \{e\}) = \rho(X)\} \tag{2.2}$$

for each $X \subseteq E$. The closure function cl satisfies the following:

(cl0) $\forall X \subseteq E$: $X \subseteq \mathrm{cl}(X)$.

(cl1) $\forall X,\ Y \subseteq E$: $X \subseteq \mathrm{cl}(Y) \Longrightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.

(cl2) $\forall X \subseteq E$, $\forall e \in E$: $e' \in \mathrm{cl}(X \cup \{e\}) - \mathrm{cl}(X) \Longrightarrow e \in \mathrm{cl}(X \cup \{e'\}) - \mathrm{cl}(X)$.

For any independent set $I \in \mathcal{I}$ and any element $e \in \mathrm{cl}(I) - I$, there exists a unique circuit contained in $I \cup \{e\}$. Such a circuit is called *the fundamental circuit* with respect to I and e, and is denoted by $\mathrm{C}(I|e)$. For any $e' \in \mathrm{C}(I|e)$, $(I \cup \{e\}) - \{e'\}$ is an independent set.

It is well known (see [Welsh76] [and [Oxley92]]) that each of the family $\mathcal{I}$ of independent sets, the family $\mathcal{B}$ of bases, the family $\mathcal{C}$ of circuits, the rank function ρ and the closure function cl uniquely determines the matroid which defines it. Giving a system of axioms for each of the family $\mathcal{B}$ of bases, the family $\mathcal{C}$ of circuits, the rank function ρ and the closure function cl, we can define a matroid. We denote such a matroid by $(E, \mathcal{B})$, $(E, \mathcal{C})$, (E, ρ) and (E, cl), respectively. In fact, (B0) and (B1), (C0)$\sim$(C2), (ρ0)$\sim$(ρ2), and (cl0)$\sim$(cl2), respectively, give the systems of axioms for the family $\mathcal{B}$ of bases, the family $\mathcal{C}$ of circuits, the rank function ρ, and the closure function cl of a matroid on E. For example, any integer-valued function $\rho\colon 2^E \to \mathbf{Z}$ satisfying (ρ0)$\sim$(ρ2) defines a matroid (E, ρ) with the family $\mathcal{I}$ of independent sets given by

$$\mathcal{I} = \{I \mid I \subseteq E,\ \rho(I) = |I|\}. \tag{2.3}$$

For a matroid $\mathbf{M} = (E, \mathcal{B})$ with a family $\mathcal{B}$ of bases, the family $\mathcal{B}^*$ of the complements of bases of the matroid is also a family of bases of a matroid. The matroid $\mathbf{M}^* = (E, \mathcal{B}^*)$ is called the *dual matroid* of $\mathbf{M} = (E, \mathcal{B})$. The dual of $\mathbf{M}^*$ is equal to $\mathbf{M}$, i.e., $(\mathbf{M}^*)^* = \mathbf{M}$.

A matroidal term, say, "X" with respect to $\mathbf{M}^*$ is referred to as "co-X" with respect to $\mathbf{M}$. For example, bases and circuits of $\mathbf{M}^*$ are called *cobases* and *cocircuits* of $\mathbf{M}$.

A self-dual system of axioms for the family of bases is given by N. Tomizawa [Tomi77] as follows.

(B0$'$) $\mathcal{B} \neq \emptyset$.

(B1$'$) $\forall B_1,\ B_2 \in \mathcal{B}\ (B_1 \neq B_2)$, $\exists e_1 \in B_1 - B_2$, $\exists e_2 \in B_2 - B_1$:

$$(B_1 - \{e_1\}) \cup \{e_2\} \in \mathcal{B},\quad (B_2 - \{e_2\}) \cup \{e_1\} \in \mathcal{B}.$$

[The same system of axioms was found earlier by A. Kelmans [Kelmans73].]

Examples of a Matroid

(1) Graphs: For a graph $G = (V, E)$ with a vertex set V and an edge set E let $\mathcal{I}(G)$ be the set of those edge subsets each of which does not contain any cycle of G. Then $\mathbf{M}(G) = (E, \mathcal{I}(G))$ is a matroid with $\mathcal{I}(G)$ being the family of independent sets. A matroid which can be obtained in this way is called a *graphic matroid.* Given an independence oracle for the membership in the family of independent sets, we can efficiently discern whether a given matroid is graphic and, if graphic, construct a graph representing it, by combining the algorithms of P. D. Seymour [Seymour81] and the author [Fuji80a] (also see [Bixby+Wagner88]).

(2) Matrices: For a matrix $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ over some field with column vectors $\mathbf{a}_1$, $\cdots$, $\mathbf{a}_n$ let $E = \{1, \cdots, n\}$ be the column index set and $\mathcal{I}(A)$ be the set of those subsets of E each of which forms independent column vectors. Then $\mathbf{M}(A) = (E, \mathcal{I}(A))$ is a matroid with $\mathcal{I}(A)$ being the family of independent sets. We say the matrix A *represents* (or is a *representation* of) the matroid $\mathbf{M}(A)$. A matroid which can be obtained in this way is called a *matric matroid* or a *linear matroid.* A graphic matroid is a matric matroid represented by the incidence matrix of the associated graph. A *binary* matroid is a matroid which can be represented by a matrix over the field of $\{0, 1\}$. A *regular* matroid is a matroid which can be represented by a matrix over any field. A graphic matroid is a regular matroid. The dual of a binary (regular) matroid is also binary (regular). A characterization of regular matroids by decompositions is given in [Seymour80] which leads us to an efficient algorithm for discerning the regularity of a matroid. Extensive studies on matroid decompositions have been made by K. Truemper [Truemper92].

(3) Bipartite matchings: For a bipartite graph $G = (V^+, V^-; A)$ with left and right end-vertex sets V^+ and V^- and an arc set A, define

$$\mathcal{I}^+ = \{\partial^+ M \mid M\colon \text{ a matching of } G\}, \tag{2.4}$$

where $\partial^+ M$ is the set of left end-vertices of matching M. Then $\mathbf{M}^+ = (V^+, \mathcal{I}^+)$ is a matroid with $\mathcal{I}^+$ being the family of independent sets. $\mathbf{M}^+$ is called a *transversal matroid.* Transversal matroids are matric. Consider the matrix $P = (p_{ij} \mid i \in V^-,\ j \in V^+)$ with the row index set V^- and the column index set V^+ such that

(i) $p_{ij} \neq 0$ if $i \in V^-$ and $j \in V^+$ are adjacent in G and

(ii) $p_{ij} = 0$ otherwise.

Also, suppose that nonzero real elements p_{ij} are algebraically independent, i.e., they do not satisfy any nontrivial multi-variable polynomial equation with rational coefficients. Then matrix P represents the transversal matroid $\mathbf{M}^+$.

(4) General matchings: For a simple graph $G = (V, E)$ a subset W of the vertex set V is said to be *matchable* if W is the set of the end-vertices of a matching in G. Let $\mathcal{I}$ be the set of subsets W of the vertex set V such that W is a subset of some matchable vertex set. Then $(V, \mathcal{I})$ is a matroid, called a *matching matroid*.

(5) Algebraic matroids: Consider a field K and its extension H and let E be a set of a finite number of elements of H. Define $\mathcal{I}$ as the family of subsets X of E such that the elements of X do not satisfy any nontrivial multi-variable polynomial equation with coefficients chosen from K. Then $(E, \mathcal{I})$ is a matroid and is called an *algebraic* matroid.

(6) Uniform matroids: Let E be a finite set with cardinality $|E| = n > 0$. For any nonnegative integer $k \leq n$ define

$$\mathcal{I}_k = \{I \mid I \subseteq E,\ |I| \leq k\}. \tag{2.5}$$

Then $(E, \mathcal{I}_k)$ is a matroid. It is called a *uniform matroid* of rank k and is usually denoted by $\mathrm{U}_{k,n}$. In particular, $\mathrm{U}_{n,n} = (E, 2^E)$ is called a *free matroid* and $\mathrm{U}_{0,n} = (E, \{\emptyset\})$ a *trivial matroid*.

2.2. Polymatroids

Let E be a finite set and ρ be a function from 2^E to $\mathbf{R}$. Here, $\mathbf{R}$ is the set of reals but throughout this book $\mathbf{R}$ can be any totally ordered additive group such as the set $\mathbf{Z}$ of integers and the set $\mathbf{Q}$ of rationals unless otherwise stated. Suppose that the function ρ: $2^E \to \mathbf{R}$ satisfies

($\overline{\rho 0}$) $\rho(\emptyset) = 0$.

($\overline{\rho 1}$) $X \subseteq Y \subseteq E \Longrightarrow \rho(X) \le \rho(Y)$.

($\overline{\rho 2}$) $\forall X, Y \subseteq E$: $\rho(X) + \rho(Y) \ge \rho(X \cup Y) + \rho(X \cap Y)$.

The pair (E, ρ) is called a *polymatroid* and ρ the *rank function* of the polymatroid ([Edm70]). The rank function ρ is a monotone nondecreasing submodular function on 2^E with $\rho(\emptyset) = 0$ and does not necessarily have the unit-increase property as the rank function of a matroid. When ρ is the rank function of a matroid, polymatroid (E, ρ) is called *matroidal.*

Define

$$\mathrm{P}_{(+)}(\rho) = \{x \mid x \in \mathbf{R}^E,\ x \ge \mathbf{0},\ \forall X \subseteq E{:}\, x(X) \le \rho(X)\}, \tag{2.6}$$

where for each $X \subseteq E$ and $x \in \mathbf{R}^E$

$$x(X) = \sum_{e \in X} x(e). \tag{2.7}$$

We define $x(\emptyset) = 0$. $\mathrm{P}_{(+)}(\rho)$ is called the *independence polyhedron* (or *polymatroid polyhedron*) associated with polymatroid (E, ρ). Also define

$$\mathrm{B}(\rho) = \{x \mid x \in \mathrm{P}_{(+)}(\rho),\ x(E) = \rho(E)\}. \tag{2.8}$$

We call $\mathrm{B}(\rho)$ the *base polyhedron* associated with polymatroid (E, ρ). The base polyhedron $\mathrm{B}(\rho)$ is always nonempty, which will be shown in Theorem 2.3. (See Fig. 2.1.)

It can be shown (see [Edm70]) that the convex hull in R^E of the characteristic vectors of the independent sets (or bases) of a matroid on E with the rank function ρ, where $\mathbf{R}$ is the set of reals, is the independence polyhedron (or the base polyhedron) associated with the matroidal polymatroid (E, ρ) (see Corollary 3.25).

Each $x \in \mathrm{P}_{(+)}(\rho)$ is called an *independent vector* and each $x \in \mathrm{B}(\rho)$ a *base* of polymatroid (E, ρ).

For an independent vector $x \in \mathrm{P}_{(+)}(\rho)$ define

$$\mathcal{D}(x) = \{X \mid X \subseteq E,\ x(X) = \rho(X)\}. \tag{2.9}$$

$\mathcal{D}(x)$ is closed with respect to set union and intersection, i.e., $X,\ Y \in \mathcal{D}(x) \Longrightarrow X \cup Y,\ X \cap Y \in \mathcal{D}(x)$. For, if $X,\ Y \in \mathcal{D}(x)$, we have

$$\begin{aligned} 0 &= \rho(X) - x(X) + \rho(Y) - x(Y) \\ &\ge \rho(X \cup Y) - x(X \cup Y) + \rho(X \cap Y) - x(X \cap Y) \ \ge\ 0, \end{aligned} \tag{2.10}$$

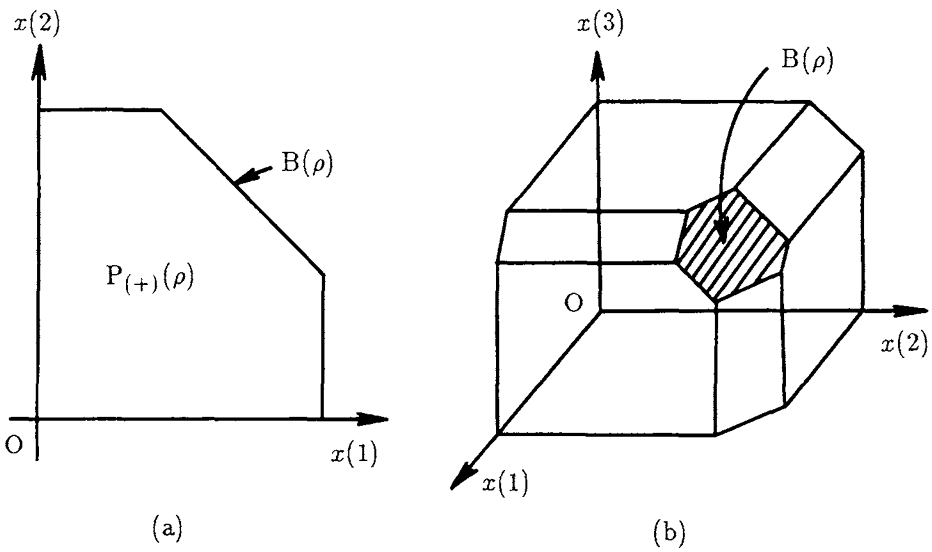


Figure 2.1: Polymatroids

where $\rho(X \cup Y) - x(X \cup Y) \geq 0$ and $\rho(X \cap Y) - x(X \cap Y) \geq 0$ since $x \in \mathrm{P}_{(+)}(\rho)$. It follows that $X \cup Y,\ X \cap Y \in \mathcal{D}(x)$. So, $\mathcal{D}(x)$ is a distributive lattice with set union and intersection as the lattice operations, join and meet. Denote the unique maximal element of $\mathcal{D}(x)$ by $\mathrm{sat}(x)$, i.e.,

$$\mathrm{sat}(x) = \bigcup\{X \mid X \subseteq E,\ x(X) = \rho(X)\}. \tag{2.11}$$

The function, sat: $\mathrm{P}_{(+)}(\rho) \to 2^E$, is called the *saturation function* ([Fuji78a]). Informally, $\mathrm{sat}(x)$ is the set of the saturated components of x. More precisely,

$$\mathrm{sat}(x) = \{e \mid e \in E,\ \forall \alpha > 0\colon x + \alpha\chi_e \notin \mathrm{P}_{(+)}(\rho)\}, \tag{2.12}$$

where χ_e is the unit vector with $\chi_e(e) = 1$ and $\chi_e(e') = 0$ $(e' \in E - \{e\})$. The saturation function is a generalization of the closure function of a matroid.

For an independent vector $x \in \mathrm{P}_{(+)}(\rho)$ and an element $e \in \mathrm{sat}(x)$ define

$$\mathcal{D}(x, e) = \{X \mid e \in X \subseteq E,\ x(X) = \rho(X)\}. \tag{2.13}$$

We have $\mathcal{D}(x, e) \subseteq \mathcal{D}(x)$ and $\mathcal{D}(x, e)$ is a sublattice of $\mathcal{D}(x)$. Denote the unique minimal element of $\mathcal{D}(x, e)$ by $\mathrm{dep}(x, e)$, i.e.,

$$\mathrm{dep}(x, e) = \bigcap\{X \mid e \in X \subseteq E,\ x(X) = \rho(X)\}. \tag{2.14}$$

For each $e \in E - \mathrm{sat}(x)$ we define $\mathrm{dep}(x, e) = \emptyset$. The function, dep: $\mathrm{P}_{(+)}(\rho) \times E \to 2^E$, is called the *dependence function* (see [Fuji78a]). For $x \in \mathrm{P}_{(+)}(\rho)$ and $e \in \mathrm{sat}(x)$,

$$\mathrm{dep}(x, e) = \{e' \mid e' \in E,\ \exists\alpha > 0\colon x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}_{(+)}(\rho)\}, \tag{2.15}$$

where note that for $e' \in \mathrm{dep}(x, e) - \{e\}$ we have $x(e') > 0$. We call an ordered pair (e, e') such that $e' \in \mathrm{dep}(x, e) - \{e\}$ an *exchangeable pair* associated with x. The dependence function is a generalization of a fundamental circuit of a matroid.

For $x \in \mathrm{P}_{(+)}(\rho)$ and $e \in E - \mathrm{sat}(x)$ define

$$\hat{\mathrm{c}}(x, e) = \max\{\alpha \mid \alpha \in \mathbf{R},\ x + \alpha\chi_e \in \mathrm{P}_{(+)}(\rho)\}\ (> 0), \tag{2.16}$$

which is called the *saturation capacity* associated with x and e. The saturation capacity is also expressed as

$$\hat{\mathrm{c}}(x, e) = \min\{\rho(X) - x(X) | \ e \in X \subseteq E\}. \tag{2.17}$$

For any α such that $0 \leq \alpha \leq \hat{\mathrm{c}}(x, e)$ we have $x + \alpha\chi_e \in \mathrm{P}_{(+)}(\rho)$.

For $x \in \mathrm{P}_{(+)}(\rho)$, $e \in \mathrm{sat}(x)$ and $e' \in \mathrm{dep}(x, e) - \{e\}$, define

$$\tilde{\mathrm{c}}(x, e, e') = \max\{\alpha \mid \alpha \in \mathbf{R},\ x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}_{(+)}(\rho)\}\ (> 0), \tag{2.18}$$

which is called the *exchange capacity* associated with x, e and e'. (See Fig. 2.2.) The exchange capacity is also expressed as

$$\tilde{\mathrm{c}}(x, e, e') = \min\{\rho(X) - x(X) \mid e \in X \subseteq E,\ e' \notin X\}, \tag{2.19}$$

where note that $x(e') \geq \tilde{\mathrm{c}}(x, e, e')$. For any α such that $0 \leq \alpha \leq \tilde{\mathrm{c}}(x, e, e')$ we have $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}_{(+)}(\rho)$.

A vector $v \in \mathbf{R}^E$ such that $x \leq v$ for any $x \in \mathrm{P}_{(+)}(\rho)$ is called a *dominating vector* of (E, ρ). Note that $v \in \mathbf{R}^E$ is a dominating vector of (E, ρ) if and only if $v(e) \geq \rho(\{e\})$ for all $e \in E$. For a dominating vector v of the polymatroid $\mathbf{P} = (E, \rho)$ define $\rho^*_{(v)}\colon 2^E \to \mathbf{R}$ by

$$\rho^*_{(v)}(X) = v(X) + \rho(E - X) - \rho(E) \quad (X \subseteq E). \tag{2.20}$$

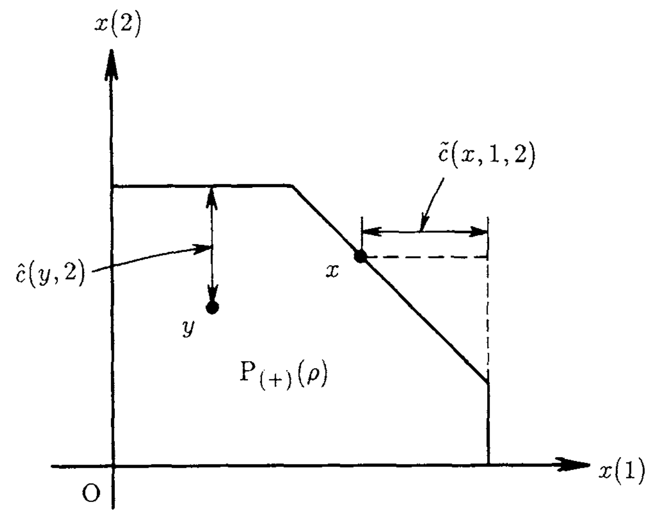


Figure 2.2: Exchange capacity $\tilde{c}(x,1,2)$ and saturation capacity $\hat{c}(y,2)$.

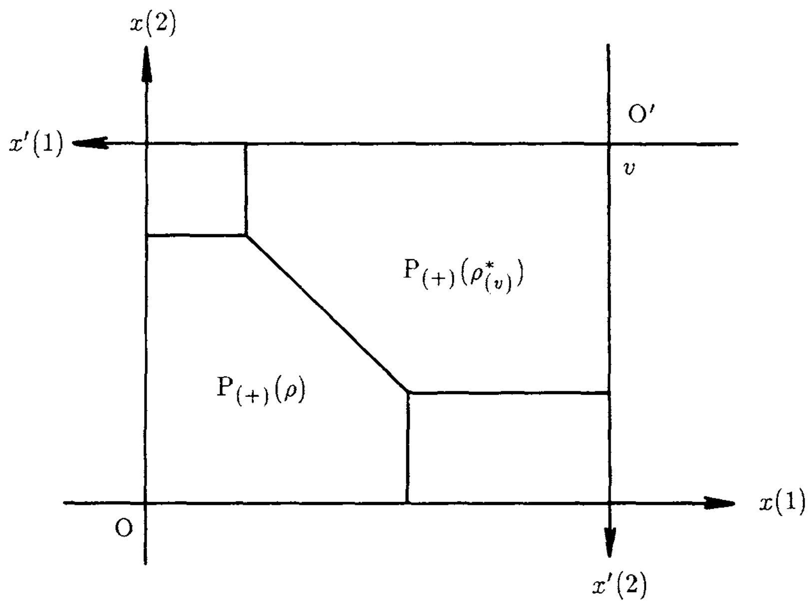


Figure 2.3: The duality in polymatroids.

Then $\mathbf{P}^*_{(v)} = (E, \rho^*_{(v)})$ is a polymatroid and is called the *dual polymatroid* of $\mathbf{P} = (E, \rho)$ with respect to v ([McDiarmid75]). (See Fig. 2.3.)

For most polymatroids there is no reasonable and physically meaningful way of choosing a dominating vector v and the arbitrariness of dual polymatroids remains, though the choice of $v = (\rho(\{e\}) \mid e \in E)$ may be reasonable. For matroidal polymatroids, the matroid duality corresponds to the polymatroid duality with respect to the vector $\mathbf{1}$ of all the components being equal to 1.

Examples of a Polymatroid

(1) Multiterminal flows: Consider a capacitated flow network $\mathcal{N} = (G = (V, A),\ c, S^+, S^-)$ with the underlying graph $G = (V, A)$, a nonnegative capacity function $c\colon A \to \mathbf{R}_+$, the set S^+ of entrances and the set S^- of exits, where S^+, $S^- \subseteq V$ and $S^+ \cap S^- = \emptyset$. We assume that there is no arc entering S^+ or leaving S^-. A function $\varphi\colon A \to \mathbf{R}$ is a *feasible flow* in $\mathcal{N}$ if

$$\forall a \in A\colon\ 0 \le \varphi(a) \le c(a), \tag{2.21}$$

$$\forall v \in V - (S^+ \cup S^-)\colon\ \partial\varphi(v) = 0, \tag{2.22}$$

where $\partial\varphi\colon V \to \mathbf{R}$ is the *boundary* of φ defined by

$$\partial\varphi(v) = \sum_{a \in \delta^+ v} \varphi(a) - \sum_{a \in \delta^- v} \varphi(a) \quad (v \in V). \tag{2.23}$$

It is shown (see [Megiddo74]) that the set $\{(\partial\varphi)^{S^+} \mid \varphi\colon$ a feasible flow in $\mathcal{N}\}$ is the independence polyhedron of a polymatroid on the set S^+ of entrances, where $(\partial\varphi)^{S^+}$ is the restriction of $\partial\varphi$ on S^+. Similarly, we have a polymatroid on the set S^- of exits.

(2) Linear polymatroids: Let A be a matrix with the column index set E. Suppose that E is partitioned into nonempty disjoint subsets F_1, F_2, $\cdots$, F_n, and define $\tilde{E} = \{1, 2, \cdots, n\}$. Also define for each $X \subseteq \tilde{E}$

$$\rho(X) = \mathrm{rank} A^X, \tag{2.24}$$

where A^X is the submatrix of A formed by the columns of A with the indices in $\bigcup\{F_i \mid i \in X\}$. We see that $(\tilde{E}, \rho)$ is a polymatroid, which is called a *linear polymatroid*.

(3) Entropy functions: Let x_1, x_2, $\cdots$, x_n be random variables taking on values in a finite set $\{1, 2, \cdots, N\}$. For the set $E = \{x_1, \cdots, x_n\}$ of the random variables, define for each subset X of E

$$h(X) = \begin{cases} \text{the entropy of } X \text{ in the Shannon sense} & \text{if } X \neq \emptyset, \\ 0 \quad \text{if } X = \emptyset. \end{cases} \tag{2.25}$$

For example, if $X = \{x_1, x_2, \cdots, x_k\}$ $(1 \leq k \leq n)$,

$$h(X) = -\sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} p(x_1 = i_1, \cdots, x_k = i_k) \log_2 p(x_1 = i_1, \cdots, x_k = i_k), \tag{2.26}$$

where $p(x_1 = i_1, \cdots, x_k = i_k)$ is the probability of the event that $x_1 = i_1$, $\cdots$, $x_k = i_k$. The function $h\colon 2^E \to \mathbf{R}_+$ is called an *entropy function.* Entropy function h is a monotone nondecreasing submodular function with $h(\emptyset) = 0$, i.e., (E, h) is a polymatroid. The submodularity of h is equivalent to the nonnegativity of conditional mutual information. See [Fuji78c] for polymatroidal problems in the Shannon information theory.

(4) Convex games: Consider a characteristic-function game (see, e.g., [Shubik82]). Let $E = \{1, 2, \cdots, n\}$ be a set of n persons, called *players.* A characteristic function v is a nonnegative function defined on the set of coalitions which are subsets of E, where we assume $v(\emptyset) = 0$.

A characteristic-function game (E, v) is called a *convex game* ([Shapley71]) if the characteristic function v is supermodular, i.e.,

$$\forall X, Y \subseteq E\colon\ v(X) + v(Y) \leq v(X \cup Y) + v(X \cap Y). \tag{2.27}$$

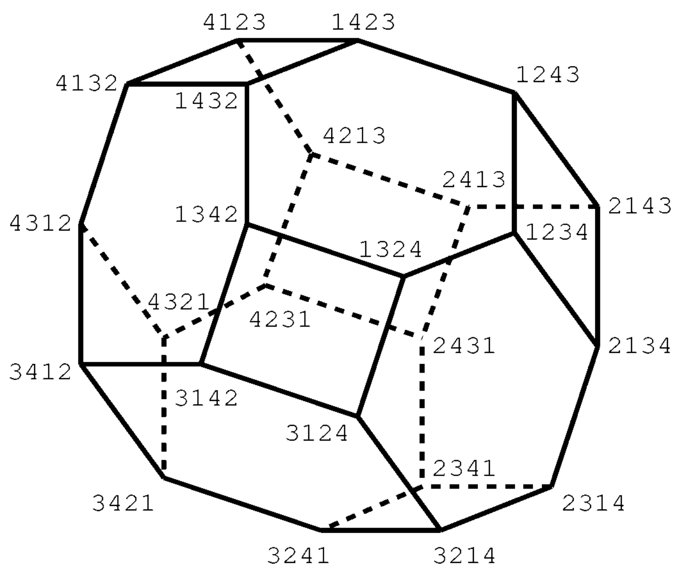
The *core* of the game (E, v) is the set of payoff vectors defined by

$$\{x \mid x \in \mathbf{R}^E,\ \forall X \subseteq E\colon x(X) \geq v(X), x(E) = v(E)\}. \tag{2.28}$$

Define the function $v^{\#}\colon 2^E \to \mathbf{R}$ by

$$v^{\#}(E - X) = v(E) - v(X) \quad (X \subseteq E). \tag{2.29}$$

Then we can show that $v^{\#}$ is a polymatroid rank function and that the core given by (2.28) coincides with the base polyhedron $\mathrm{B}(v^{\#})$ of the polymatroid (see Lemma 2.4). [Also see [Ichiishi81].]



A permutahedron

[(5) **Permutahedra**: Let $E = \{1, 2, \cdots, n\}$. Consider a nondecreasing concave function $g : \mathbf{R} \to \mathbf{R}$ with $g(0) = 0$ and define a function $\rho : 2^E \to \mathbf{R}$ by

$$\rho(X) = g(|X|) \qquad (X \subseteq E).$$

Then (E, ρ) is a polymatroid whose rank function value $\rho(X)$ depends only on the size of X. In particular, when ρ is given by

$$\rho(X) = \sum_{i=1}^{|X|}(n - i + 1) \qquad (X \subseteq E),$$

where $\rho(\emptyset) = 0$, the base polyhedron $\mathrm{B}(\rho)$ is called a *permutahedron* (or permutohedron). Every permutation or linear ordering $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ of integers $1, 2, \cdots, n$ can be identified with the vector $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ in $\mathbf{R}^E$. We can show that the set of such vectors for all permutations of $1, 2, \cdots, n$ is exactly the set of all extreme points of the permutahedron (we can see this fact through the discussions in Section 3.2). Note that for the permutahedron the slope $g(k) - g(k-1)$ decreases by one for $k = 1, 2, \cdots, n$. From a general nondecreasing concave function g with $g(0) = 0$ we have a nonincreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_n$ with $a_k = g(k) - g(k-1)$ for $k = 1, 2, \cdots, n$. Then the base polyhedron defined from g has extreme bases, each being a vector $(a_{\sigma(k)} \mid k = 1, 2, \cdots, n)$ $(\in \mathbf{R}^E)$ for a permutation σ of $1, 2, \cdots, n$, where an extreme base means an extreme point of the base polyhedron.

The concept of permutahedron can be traced back to [Schoute13] (also see [Berge68], [Yemelichev+Kovalev+Kravtsov81], [Ziegler95] and [Borovik +Gelfand+White03]). It may be worth pointing out that permutahedra are special cases of base polyhedra arising from multiterminal flows described above and are zonotopes, where a *zonotope* is an affine transformation of a unit hypercube.]

2.3. Submodular Systems

Let E be a nonempty finite set and $\mathcal{D}$ be a collection of subsets of E which forms a distributive lattice with set union and intersection as the lattice operations, join and meet, i.e., for each $X,\ Y \in \mathcal{D}$ we have $X \cup Y,\ X \cap Y \in \mathcal{D}$. Let $f\colon\ \mathcal{D} \to \mathbf{R}$ be a *submodular function* on the distributive lattice $\mathcal{D}$, i.e.,

$$\forall X,\ Y \in \mathcal{D}\colon\ f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{2.30}$$

We have the following fundamental lemma concerning submodular functions.

Lemma 2.1 ([Ore56]): *Let $f\colon\ \mathcal{D} \to \mathbf{R}$ be an arbitrary submodular function on a distributive lattice $\mathcal{D} \subseteq 2^E$. Then the set of all the minimizers of f given by*

$$\mathcal{D}_0 = \{X \mid X \in \mathcal{D},\ f(X) = \min\{f(Y) \mid Y \in \mathcal{D}\}\} \tag{2.31}$$

forms a sublattice of $\mathcal{D}$, i.e., for any $X,\ Y \in \mathcal{D}_0$ we have $X \cup Y,\ X \cap Y \in \mathcal{D}_0$.

(Proof) For any $X,\ Y \in \mathcal{D}_0$,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \tag{2.32}$$

where $\min\{f(X \cup Y), f(X \cap Y)\} \geq f(X) = f(Y)$. Hence we must have $f(X \cup Y) = f(X \cap Y) = f(X) (= f(Y))$, i.e., $X \cup Y,\ X \cap Y \in \mathcal{D}_0$. Q.E.D.

For a submodular function f on a distributive lattice $\mathcal{D} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{D}$ and $f(\emptyset) = 0$, we call the pair $(\mathcal{D}, f)$ a *submodular system* on E, and f the *rank function* of the submodular system (see [Fuji78b, 84c]). We call $f(E)$ the *rank* of $(\mathcal{D}, f)$.

Define a polyhedron in $\mathbf{R}^E$ by

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}\colon\ x(X) \leq f(X)\}. \tag{2.33}$$

We call $\mathrm{P}(f)$ the *submodular polyhedron* associated with submodular system $(\mathcal{D}, f)$. Also define

$$\mathrm{B}(f) = \{x \mid x \in \mathrm{P}(f),\ x(E) = f(E)\}. \tag{2.34}$$

We call $\mathrm{B}(f)$ the *base polyhedron* associated with submodular system $(\mathcal{D}, f)$. (See Fig. 2.4.)

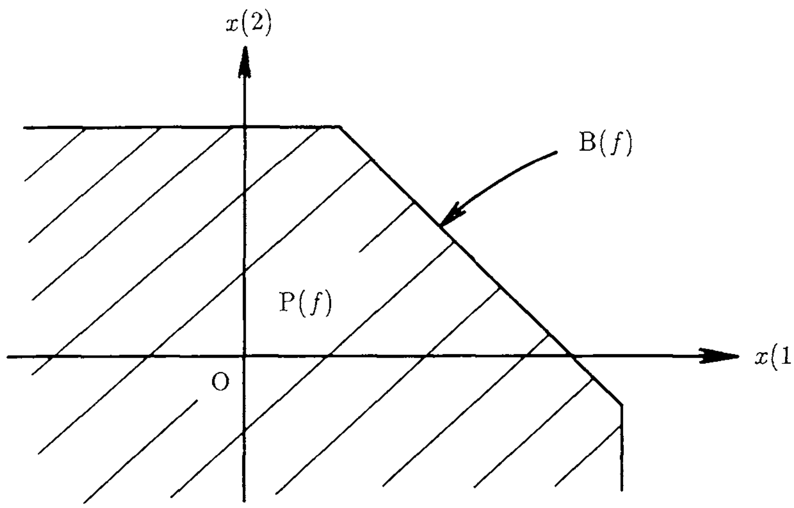


Figure 2.4: Submodular polyhedron $\mathrm{P}(f)$ and base polyhedron $\mathrm{B}(f)$.

A vector in the base polyhedron $\mathrm{B}(f)$ is called a *base* of $(\mathcal{D}, f)$ and a vector in the submodular polyhedron $\mathrm{P}(f)$ is called a *subbase* of $(\mathcal{D}, f)$.

The saturation function, the dependence function, the saturation capacity and the exchange capacity introduced for polymatroids can easily be extended for submodular systems.

Lemma 2.2: *For any subbase $x \in \mathrm{P}(f)$ define*

$$\mathcal{D}(x) = \{X \mid X \in \mathcal{D},\ x(X) = f(X)\}. \tag{2.35}$$

Then $\mathcal{D}(x)$ is a sublattice of $\mathcal{D}$.

(Proof) This follows from Lemma 2.1 since $\mathcal{D}(x)$ is the set of minimizers of the nonnegative submodular function $f - x$: $\mathcal{D} \to \mathbf{R}$. Q.E.D.

The unique maximal element of $\mathcal{D}(x)$ is denoted by $\mathrm{sat}(x)$. sat: $\mathrm{P}(f) \to 2^E$ is the *saturation function.*

For any subbase $x \in \mathrm{P}(f)$ and $e \in \mathrm{sat}(x)$, define

$$\mathcal{D}(x, e) = \{X \mid e \in X \in \mathcal{D},\ x(X) = f(X)\}. \tag{2.36}$$

Then $\mathcal{D}(x, e)$ is a sublattice of $\mathcal{D}$. (Note that $\mathcal{D}(x, e)$ is the set of minimizers of the nonnegative submodular function $f - x$ on the distributive lattice $\mathcal{D}(e) \equiv \{X \mid e \in X \in \mathcal{D}\}$.) The unique minimal element of $\mathcal{D}(x, e)$ is denoted by $\mathrm{dep}(x, e)$. For any $e \in E - \mathrm{sat}(x)$ we define $\mathrm{dep}(x, e) = \emptyset$. dep: $\mathrm{P}(f) \times E \to 2^E$ is the *dependence function.*

For any $x \in \mathrm{P}(f)$ and $e \in E - \mathrm{sat}(x)$ the *saturation capacity* $\hat{\mathrm{c}}(x, e)$ is defined by

$$\hat{\mathrm{c}}(x, e) = \min\{f(X) - x(X) \mid e \in X \in \mathcal{D}\}. \tag{2.37}$$

For a nonnegative α, we have $x + \alpha\chi_e \in \mathrm{P}(f)$ if and only if $0 \le \alpha \le \hat{\mathrm{c}}(x, e)$.

Moreover, for any $x \in \mathrm{P}(f)$, $e \in \mathrm{sat}(x)$ and $e' \in \mathrm{dep}(x, e) - \{e\}$ the *exchange capacity* $\tilde{\mathrm{c}}(x, e, e')$ is defined by

$$\tilde{\mathrm{c}}(x, e, e') = \min\{f(X) - x(X) \mid e \in X \in \mathcal{D},\ e' \notin X\}. \tag{2.38}$$

For a nonnegative α, we have $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}(f)$ if and only if $0 \le \alpha \le \tilde{\mathrm{c}}(x, e, e')$. (Note that if $x \in \mathrm{B}(f)$, then $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}(f)$ implies $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(f)$.)

Theorem 2.3: *The base polyhedron $\mathrm{B}(f)$ is the set of all the maximal subbases of $(\mathcal{D}, f)$. In particular, $\mathrm{B}(f) \neq \emptyset$. Here, the partial order $\le$ among vectors in $\mathbf{R}^E$ is the one defined for vector lattice $\mathbf{R}^E$ (i.e., for $x,\ y \in \mathbf{R}^E$ we have $x \le y$ if and only if $x(e) \le y(e)$ for all $e \in E$).*

(Proof) Denote by $\mathrm{B}'(f)$ the set of all the maximal subbases of $(\mathcal{D}, f)$. Any $x \in \mathrm{B}(f)$ is maximal in $\mathrm{P}(f)$ since $x(E) = f(E)$, so that we have $\mathrm{B}(f) \subseteq \mathrm{B}'(f)$. Conversely, for any $x \in \mathrm{B}'(f)$ we have $\mathrm{sat}(x) = E$ due to

the maximality of x. It follows that $x(E) = x(\mathrm{sat}(x)) = f(\mathrm{sat}(x)) = f(E)$, i.e., $x \in \mathrm{B}(f)$. Therefore, $\mathrm{B}'(f) \subseteq \mathrm{B}(f)$, and hence we have $\mathrm{B}'(f) = \mathrm{B}(f)$.
Q.E.D.

From Theorem 2.3 we see that for any subbase x of $(\mathcal{D}, f)$ there exists a base y of $(\mathcal{D}, f)$ such that $x \leq y$.

A function g: $\mathcal{D} \to \mathbf{R}$ on the distributive lattice $\mathcal{D}$ is called a *supermodular function* if $-g$ is a submodular function, i.e.,

$$\forall\, X,\ Y \in \mathcal{D}\colon\ g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y). \tag{2.39}$$

If $\emptyset, E \in \mathcal{D}$ and $g(\emptyset) = 0$, the pair $(\mathcal{D}, g)$ is called a *supermodular system* on E. Define

$$\mathrm{P}(g) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}\colon\ x(X) \geq g(X)\}, \tag{2.40}$$

$$\mathrm{B}(g) = \{x \mid x \in \mathrm{P}(g),\ x(E) = g(E)\}. \tag{2.41}$$

$\mathrm{P}(g)$ is called the *supermodular polyhedron* and $\mathrm{B}(g)$ the *base polyhedron* associated with the supermodular system $(\mathcal{D}, g)$. A vector in $\mathrm{P}(g)$ is called a *superbase* and a vector in $\mathrm{B}(g)$ a *base* of $(\mathcal{D}, g)$.

A function which is submodular and at the same time supermodular is called a *modular function*. For a modular function x: $\mathcal{D} \to \mathbf{R}$, $\mathrm{P}(x)$ should be considered as either a submodular polyhedron or a supermodular polyhedron according as we consider x as a submodular function or a supermodular function. There will be no confusion by the use of this notation.

If we consider the dual order $\leq^*$ of the ordinary order $\leq$ among $\mathbf{R}$ (where the dual order $\leq^*$ is the one such that $\alpha \leq^* \beta$ if and only if $\beta \leq \alpha$, for each $\alpha, \beta \in \mathbf{R}$), then a supermodular function g: $\mathcal{D} \to \mathbf{R}$ with respect to $(\mathbf{R}, \leq)$ is a submodular function with respect to $(\mathbf{R}, \leq^*)$. In the same way the supermodular polyhedron $\mathrm{P}(g)$ and a superbase $x \in \mathrm{P}(g)$ with respect to $(\mathbf{R}, \leq)$ is a submodular polyhedron and a subbase with respect to $(\mathbf{R}, \leq^*)$.

For a submodular system $(\mathcal{D}, f)$ on E define a function $f^{\#}$: $\overline{\mathcal{D}} \to \mathbf{R}$ as follows.

$$\overline{\mathcal{D}} = \{E - X \mid X \in \mathcal{D}\}, \tag{2.42}$$

$$f^{\#}(E - X) = f(E) - f(X) \quad (X \in \mathcal{D}). \tag{2.43}$$

$f^{\#}$ is a supermodular function on the dual lattice $\overline{\mathcal{D}}$ of $\mathcal{D}$. We call the pair $(\overline{\mathcal{D}}, f^{\#})$ the *dual supermodular system* of $(\mathcal{D}, f)$. (See Fig. 2.5.) Similarly, we define the *dual submodular system* $(\overline{\mathcal{D}}, g^{\#})$ of a supermodular system $(\mathcal{D}, g)$, where $\overline{\mathcal{D}}$ is defined by (2.42) and $g^{\#}$ by (2.43) with f replaced by g.

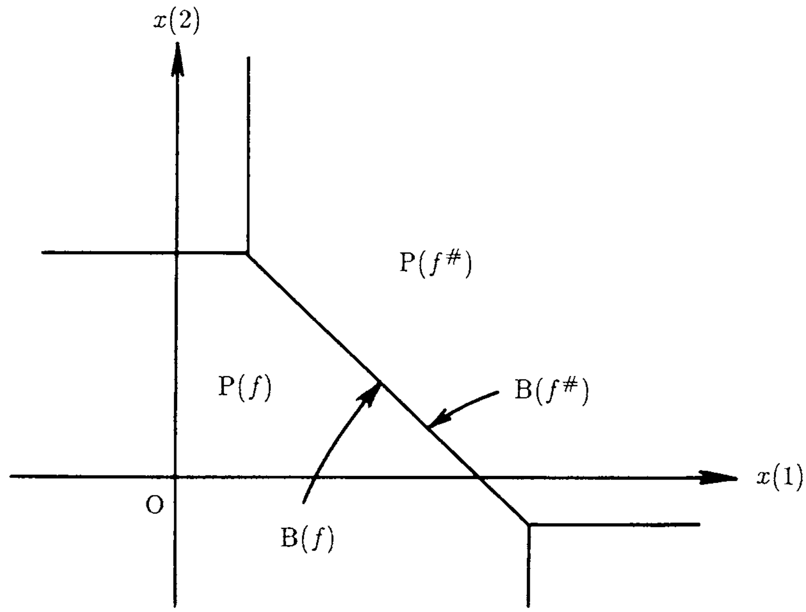


Figure 2.5: The duality in submodular and supermodular systems.

Lemma 2.4: *We have* $\mathrm{B}(f) = \mathrm{B}(f^{\#})$ *and* $(f^{\#})^{\#} = f$.

(Proof) Consider the system of inequalities and an equation for base polyhedron $\mathrm{B}(f)$:

$$x(X) \leq f(X) \quad (X \in \mathcal{D}), \tag{2.44a}$$

$$x(E) = f(E). \tag{2.44b}$$

This is equivalent to the following:

$$x(E - X) \geq f^{\#}(E - X)(= f(E) - f(X)) \quad (X \in \mathcal{D}), \tag{2.45a}$$

$$x(E) = f^{\#}(E)(= f(E)), \tag{2.45b}$$

which defines $\mathrm{B}(f^{\#})$. Furthermore, for $X \in \overline{\overline{\mathcal{D}}} = \mathcal{D}$

$$\begin{aligned} (f^{\#})^{\#}(X) &= f^{\#}(E) - f^{\#}(E - X) \\ &= f(E) - (f(E) - f(X)) \\ &= f(X). \end{aligned} \tag{2.46}$$

It follows that $(f^{\#})^{\#} = f$. Q.E.D.

It should be noted that in the above proof we do not use the submodularity of f and the lattice properties of $\mathcal{D}$. That is, Lemma 2.4 holds for *any* function f on *any* family $\mathcal{D}$ without the submodularity of f, where we define $\overline{\mathcal{D}}$ and $f^{\#}$ by (2.42) and (2.43).

The submodular/supermodular polyhedron and the base polyhedron of a submodular system also arise from more general functions. A family $\mathcal{F} \subseteq 2^E$ is called an *intersecting family* if for each intersecting $X,\ Y \in \mathcal{F}$ (i.e., $X,\ Y \in \mathcal{F}$ and $X \cap Y \neq \emptyset$) we have $X \cup Y,\ X \cap Y \in \mathcal{F}$. A function $f: \mathcal{F} \to \mathbf{R}$ on the intersecting family $\mathcal{F}$ is called *intersecting-submodular* if for each intersecting $X,\ Y \in \mathcal{F}$ we have the submodularity inequality

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{2.47}$$

Moreover, a family $\mathcal{F} \subseteq 2^E$ is called a *crossing family* if for each crossing $X,\ Y \in \mathcal{F}$ (i.e., $X,\ Y \in \mathcal{F}$, $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, $Y - X \neq \emptyset$, and $X \cup Y \neq E$) we have $X \cup Y,\ X \cap Y \in \mathcal{F}$. A function $f\colon \mathcal{F} \to \mathbf{R}$ on the crossing family $\mathcal{F}$ is called *crossing-submodular* if for each crossing $X,\ Y \in \mathcal{F}$ we have the submodularity inequality (2.47) (see [Edm+Giles77]). Note that the term, *submodular*, without any prefix is used for such a function $f\colon \mathcal{D} \to \mathbf{R}$ on a distributive lattice $\mathcal{D}$ that satisfies (2.47) for all $X,\ Y \in \mathcal{D}$. [See [Gabow95] for compact representations of intersecting and crossing families and their applications.]

The following theorems, Theorems 2.5 and 2.6, concerning intersecting- and crossing-submodular functions on intersecting and crossing families, play a very important rôle in the combinatorial optimization problems described by intersecting- or crossing-submodular functions on intersecting or crossing families, and reveal the essential combinatorial structures of the problems (which will also be seen in Chapter III). The proofs of Theorems 2.5 and 2.6 will be given in Section 3.4.

Theorem 2.5 [Fuji84b]:

(i) *Let f be an intersecting-submodular function on an intersecting family $\mathcal{F} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{F}$ and $f(\emptyset) = 0$. Define*

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \leq f(X)\}. \tag{2.48}$$

Then there exists a unique submodular system $(\mathcal{D}_1, f_1)$ on E such that

$$\mathrm{P}(f) = \mathrm{P}(f_1). \tag{2.49}$$

Moreover, if f is integer-valued, then f_1 is also integer-valued.

(ii) *Let f be a crossing-submodular function on a crossing family $\mathcal{F} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{F}$ and $f(\emptyset) = 0$. Define*

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \leq f(X),\ x(E) = f(E)\} \tag{2.50}$$

and suppose $\mathrm{B}(f) \neq \emptyset$. Then there exists a unique submodular system $(\mathcal{D}_2, f_2)$ on E such that

$$\mathrm{B}(f) = \mathrm{B}(f_2). \tag{2.51}$$

Moreover, if f is integer-valued, so is f_2.

Theorem 2.6 [Fuji84b]:

(i) *Let f be an intersecting-submodular function on an intersecting family $\mathcal{F} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{F}$ and $f(\emptyset) = 0$. The rank function f_1 of the submodular system $(\mathcal{D}_1, f_1)$ in* (i) *of Theorem* 2.5 *is given as follows. For each $Y \subseteq E$ define*

$$\begin{aligned} f_{\mathrm{P}}(Y) \;=\; \min\{\sum_{i \in I} f(X_i) \mid \{X_i \mid i \in I\}\colon &\ \text{a partition of } Y, \\ &\forall i \in I\colon\ X_i \in \mathcal{F}\}, \end{aligned} \tag{2.52}$$

where $f_{\mathrm{P}}(Y) = +\infty$ if the minimum in (2.52) *is taken over the empty set. Then, f_1 is the restriction of f_{P} on $\mathcal{D}_1 = \{Y \mid Y \subseteq E, f_{\mathrm{P}}(Y) < +\infty\}$.*

(ii) *Let f be a crossing-submodular function on a crossing family $\mathcal{F} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{F}$ and $f(\emptyset) = 0$. Define*

$$\begin{aligned} f_{\mathrm{P}}(E) \;=\; \min\{\sum_{i \in I} f(X_i) \mid \{X_i \mid i \in I\}\colon &\ \text{a partition of } E, \\ &\forall i \in I\colon\ X_i \in \mathcal{F}\}, \end{aligned} \tag{2.53}$$

$$\begin{aligned} (f^{\#})_{\mathrm{P}}(E) \;=\; \max\{\sum_{i \in I} f^{\#}(X_i) \mid \{X_i \mid i \in I\}\colon &\ \text{a partition of } E, \\ &\forall i \in I\colon\ E - X_i \in \mathcal{F}\}, \end{aligned} \tag{2.54}$$

where

$$f^{\#}(X) = f(E) - f(E - X) \quad (E - X \in \mathcal{F}). \tag{2.55}$$

Then $\mathrm{B}(f)$ *defined by* (2.50) *is nonempty if and only if*

$$f(E) = f_{\mathrm{P}}(E) = (f^{\#})_{\mathrm{P}}(E) \tag{2.56}$$

or equivalently

$$\sum_{j\in J} f^{\#}(Y_j) \le f(E) \le \sum_{i\in I} f(X_i) \tag{2.57}$$

for all partitions $\{X_i \mid i \in I\}$ *and* $\{Y_j \mid j \in J\}$ *of* E *with* $X_i \in \mathcal{F}$ $(i \in I)$ *and* $E - Y_j \in \mathcal{F}$ $(j \in J)$.

Moreover, if $\mathrm{B}(f) \ne \emptyset$, *the rank function* f_2 *of* $(\mathcal{D}_2, f_2)$ *in* (ii) *of Theorem* 2.5 *is given as follows. For each* $Y \subseteq E$ *define* $\hat{f}_2$ *in terms of its dual by*

$$\begin{aligned}\hat{f}_2{}^{\#}(Y) \ &(= f(E) - \hat{f}_2(E - Y)) \\ &= \max\{\sum_{i\in I}(f_{\mathrm{P}})^{\#}(X_i) \ \mid \ \{X_i \mid i \in I\}\colon \text{ a partition of } Y, \\ &\qquad\qquad \forall i \in I\colon\ E - X_i \in \mathcal{F}_{\mathrm{P}}\},\end{aligned} \tag{2.58}$$

where

$$\begin{aligned}f_{\mathrm{P}}(Y) &= \min\{\sum_{i\in I} f(X_i) \ \mid \ \{X_i \mid i \in I\}\colon \text{a partition of } Y, \\ &\qquad\qquad \forall i \in I\colon\ X_i \in \mathcal{F}\} \quad (Y \subseteq E),\end{aligned} \tag{2.59}$$

$$\mathcal{F}_{\mathrm{P}} = \{X \mid X \subseteq E,\ f_{\mathrm{P}}(X) < +\infty\}, \tag{2.60}$$

$$(f_{\mathrm{P}})^{\#}(X) = f(E) - f_{\mathrm{P}}(E - X) \quad (E - X \in \mathcal{F}_{\mathrm{P}}) \tag{2.61}$$

and the minimum (or maximum) taken over the empty set is equal to $+\infty$ *(or* $-\infty$*). The rank function* f_2 *is the restriction of* $\hat{f}_2$ *on* $\mathcal{D}_2 = \{X \mid X \subseteq E,\ \hat{f}_2(X) < +\infty\}$.

Function f_{P} defined by (2.52) is called the *Dilworth truncation* of the intersecting-submodular function $f\colon \mathcal{F} \to \mathbf{R}$ (see [Dilworth44], [Edm70], [Lovász77]). We also call f_{P} in (2.59) the *Dilworth truncation* of the crossing-submodular function $f\colon \mathcal{F} \to \mathbf{R}$. Note that $\hat{f}_2{}^{\#}$ in (2.58) is the Dilworth truncation of the intersecting-supermodular function $(f_{\mathrm{P}})^{\#}\colon \overline{\mathcal{F}_{\mathrm{P}}} \to \mathbf{R}$. f_2 is also called the *bi-truncation* of f in [Frank+Tardos88] (also see [Naitoh+Fuji92]).

Since $\mathrm{B}(f) = \mathrm{B}(f^{\#})$, we can also obtain a dual formula for f_2, beginning with $f^{\#}$ instead of f.

It should be noted that for a crossing-submodular function f on a crossing family $\mathcal{F}$ the polyhedron $\mathrm{P}(f)$ defined by (2.48) may not be a submodular polyhedron but that the intersection of $\mathrm{P}(f)$ with any hyperplane $x(E) = k$ (const.), if not empty, is a base polyhedron (see Fig. 2.6).

[Interesting applications of Theorem 2.6 (ii) can be found in connectivity augmentation problems for graphs and hypergraphs (see [Frank+Király03], [Frank+Király+ Király03] and [Berg+Jackson+Jordán03]).]

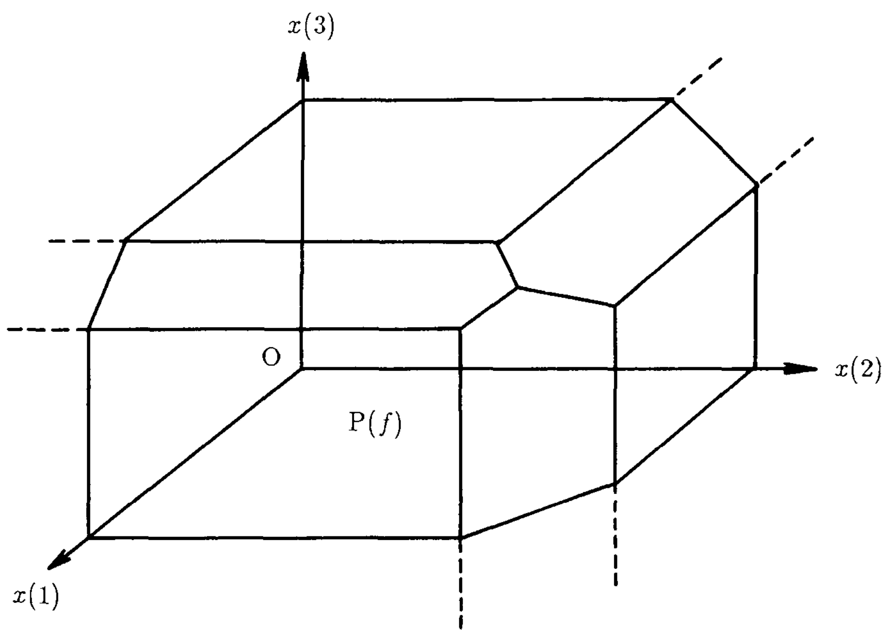


Figure 2.6: $\mathrm{P}(f)$ for a crossing-submodular function f.

Examples of a Submodular System

Matroids and polymatroids are examples of a submodular system. We show some non-polymatroidal submodular systems.

(1) Cut functions: A typical non-polymatroidal submodular system arises from network flows.

Let $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ be a capacitated network with the underlying graph $G = (V, A)$ and the lower and upper capacity functions $\underline{c}\colon A \to \mathbf{R} \cup \{-\infty\}$ and $\overline{c}\colon A \to \mathbf{R} \cup \{+\infty\}$ such that $\underline{c}(a) \leq \overline{c}(a)$ for each arc $a \in A$. Define a function $\kappa_{\underline{c},\overline{c}}\colon 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\kappa_{\underline{c},\overline{c}}(U) = \sum_{a \in \Delta^+ U} \overline{c}(a) - \sum_{a \in \Delta^- U} \underline{c}(a) \quad (U \subseteq V), \tag{2.62}$$

where $\Delta^+ U$ is the set of arcs leaving U in G and $\Delta^- U$ is the set of arcs entering U in G. For each U, $W \in 2^V$ such that $\kappa_{\underline{c},\overline{c}}(U) < +\infty$ and $\kappa_{\underline{c},\overline{c}}(W) < +\infty$, we have $\kappa_{\underline{c},\overline{c}}(U \cup W) < +\infty$, $\kappa_{\underline{c},\overline{c}}(U \cap W) < +\infty$ and

$$\begin{aligned}
&\kappa_{\underline{c},\overline{c}}(U) + \kappa_{\underline{c},\overline{c}}(W) - \kappa_{\underline{c},\overline{c}}(U \cup W) - \kappa_{\underline{c},\overline{c}}(U \cap W) \\
&= \sum \{\overline{c}(a) - \underline{c}(a) \mid a \in A,\ \partial^+ a \in U - W,\ \partial^- a \in W - U\} \\
&\quad + \sum \{\overline{c}(a) - \underline{c}(a) \mid a \in A,\ \partial^- a \in U - W,\ \partial^+ a \in W - U\} \\
&\geq 0.
\end{aligned} \tag{2.63}$$

Therefore, $\mathcal{D}(\underline{c}, \overline{c}) \subseteq 2^V$ defined by

$$\mathcal{D}(\underline{c}, \overline{c}) = \{U \mid U \subseteq V,\ \kappa_{\underline{c},\overline{c}}(U) < +\infty\} \tag{2.64}$$

is a distributive lattice with $\emptyset$, $V \in \mathcal{D}(\underline{c}, \overline{c})$. Denoting the restriction of $\kappa_{\underline{c},\overline{c}}$ to $\mathcal{D}(\underline{c}, \overline{c})$ by $\kappa_{\underline{c},\overline{c}}$ again, we have a submodular function $\kappa_{\underline{c},\overline{c}}$ on the distributive lattice $\mathcal{D}(\underline{c}, \overline{c})$, where $\kappa_{\underline{c},\overline{c}}(\emptyset) = \kappa_{\underline{c},\overline{c}}(V) = 0$. We call $\kappa_{\underline{c},\overline{c}}$ the *cut function* associated with network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$. The cut function $\kappa_{\underline{c},\overline{c}}$ is not monotone nondecreasing for nontrivial networks. A *feasible flow* φ in $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ is a function $\varphi\colon A \to \mathbf{R}$ such that $\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a)$ for any $a \in A$. The set of the boundaries $\partial\varphi$ of feasible flows φ in $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ is given by

$$\begin{aligned}
\partial\Phi &\equiv \{\partial\varphi \mid \varphi\colon A \to \mathbf{R},\ \forall a \in A\colon \underline{c}(a) \leq \varphi(a) \leq \overline{c}(a)\} \\
&= \mathrm{B}(\kappa_{\underline{c},\overline{c}}),
\end{aligned} \tag{2.65}$$

which is the base polyhedron associated with the submodular system $(\mathcal{D}(\underline{c}, \overline{c}), \kappa_{\underline{c},\overline{c}})$. (See (2.23) for the definition of the boundary $\partial\varphi$.) [Note that the permutahedron in $\mathbf{R}^V$ is a translation of a flow boundary base polyhedron in $\mathbf{R}^V$.]

The fact that each base in $\mathrm{B}(\kappa_{\underline{c},\overline{c}})$ is expressed as the boundary $\partial\varphi$ of a feasible flow φ in $\mathcal{N}$ can be shown by the use of the feasible circulation theorem (Theorem 1.3) of A. J. Hoffman [Hoffman60] as follows.

For any $x \in \mathrm{B}(\kappa_{\underline{c},\overline{c}})$, consider a new vertex $s \notin V$ and new arcs (s,v) $(v \in V)$, and define $\underline{c}(s,v) = \overline{c}(s,v) = x(v)$ $(v \in V)$. Denote the augmented network by $\mathcal{N}' = (G' = (V \cup \{s\}, A \cup \{(s,v) \mid v \in V\}), \underline{c}, \overline{c})$. There exists a feasible circulation (a feasible flow with the zero boundary) in $\mathcal{N}'$ if (and only if) for every $U \subseteq V \cup \{s\}$ we have

$$\sum_{a \in \Delta^+ U} \overline{c}(a) \geq \sum_{a \in \Delta^- U} \underline{c}(a), \tag{2.66}$$

where Δ^+ and Δ^- are with respect to G'. (2.66) is equivalent to $x \in \mathrm{B}(\kappa_{\underline{c},\overline{c}})$. Therefore, there exists a feasible circulation φ in $\mathcal{N}'$. Restricting φ to A, we obtain a required feasible flow in $\mathcal{N}$ whose boundary is equal to x.

The converse, $\partial\Phi \subseteq \mathrm{B}(\kappa_{\underline{c},\overline{c}})$, is immediate.

(2) Cross-free families: For a finite set E let $\mathcal{F} \subseteq 2^E$ be a cross-free family, i.e., for each X, $Y \in \mathcal{F}$ X and Y do not cross, where we assume $\emptyset$, $E \in \mathcal{F}$. Then for *any* function $f\colon \mathcal{F} \to \mathbf{R}$ with $f(\emptyset) = 0$, if $\mathrm{B}(f) \equiv \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon x(X) \leq f(X),\ x(E) = f(E)\} \neq \emptyset$, $\mathrm{B}(f)$ is a base polyhedron due to Theorem 2.5, since $\mathcal{F}$ is a crossing family and f is a crossing-submodular function on $\mathcal{F}$.

Moreover, if $\mathcal{F}$ is *laminar*, i.e., for each X, $Y \in \mathcal{F}$ $X \cap Y \neq \emptyset$ implies $X \subseteq Y$ or $X \supseteq Y$, then for any function f on $\mathcal{F}$ $\mathrm{P}(f) \equiv \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon x(X) \leq f(X)\}$ is a submodular polyhedron due to Theorem 2.5. Note that laminar $\mathcal{F}$ is an intersecting family and that f is an intersecting-submodular function on $\mathcal{F}$.

Also, if all the members of $\mathcal{F}$ form a chain $X_0 \subset X_1 \subset \cdots \subset X_n$, $\mathcal{F}$ is called *nested*. A nested family is laminar and any function f on a nested family $\mathcal{F}$ with $\emptyset, E \in \mathcal{F}$ and $f(\emptyset) = 0$ gives the submodular polyhedron $\mathrm{P}(f)$. It should also be noted that a nested family trivially forms a distributive lattice. See [Tamir80] for an application to a production-sales planning model.

[**(3) Submodular functions arising from concave functions**: Let $g : E \to \mathbf{R}$ be a concave function with $g(0) = 0$ and define a set function $f : 2^E \to \mathbf{R}$ by

$$f(X) = g(|X|) \quad (X \subseteq E).$$

Then we see that f is a submodular function and $(2^E, f)$ is a submodular system on E. Any submodular function $f : 2^E \to \mathbf{R}$ such that (1) $f(\emptyset) = 0$ and (2) for each $X \subseteq E$ $f(X)$ depends only on the cardinality $|X|$ is given in such a way. Recall that a permutahedron is a special case of a base polyhedron $\mathrm{B}(f)$ given as above. Such a submodular function also appears in connection with the concept of *majorization* as follows.

For any vector $x \in \mathbf{R}^E$ arrange all the components $x(e)$ $(e \in E)$ in descending order of magnitude and for $i = 1, 2, \cdots, |E|$ denote by $x_{[i]}$ the ith element in the linear ordering. For two vectors $x, y \in \mathbf{R}^E$, if we have

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad (k = 1, 2, \cdots, |E| - 1),$$
$$\sum_{i=1}^{|E|} x_{[i]} = \sum_{i=1}^{|E|} y_{[i]},$$

then x is said to be *majorized by* y ([Marshall+Olkin79]). Defining a submodular function

$$f_{(y)}(X) = \sum_{i=1}^{|X|} y_{[i]} \quad (X \subseteq E)$$

on 2^E, we see that x is majorized by y if and only if $x \in \mathrm{B}(f_{(y)})$. The concept of majorization plays a fundamental rôle in fair resource allocation and related problems.]

[**(4) Monge matrices**: An $m \times n$ matrix $A = [a_{ij}]$ is called a *Monge matrix* if for all p, q, r, s such that $1 \le p < q \le m$ and $1 \le r < s \le n$ we have $a_{ps} + a_{qr} \ge a_{qs} + a_{pr}$. Consider a poset $\mathcal{P} = (M, \preceq)$ such that M is the direct sum of the row set $R = \{1, 2, \cdots, m\}$ and the column set $C = \{1, 2, \cdots, n\}$, each forming a chain with respect to the ordinary order among integers. For each pair (i, j) of $i \in R$ and $j \in C$ define $R(i) = \{i' \mid 1 \le i' \le i\}$, $C(j) = \{j' \mid 1 \le j' \le j\}$, and $I(i, j) = R(i) \oplus C(j)$ (the direct sum), where each $I(i, j)$ is an ideal of poset $\mathcal{P}$ (see Section 3.2.a). Then $\mathcal{D} \equiv \{I(i, j) \mid 1 \le i \le m,\ 1 \le j \le n\} \cup \{\emptyset\}$ is a distributive lattice. Moreover, define $f(\emptyset) = 0$ and for each nonempty $X \in \mathcal{D}$

$$f(X) = a_{ij}$$

where $(i, j) \in R \times C$ is the pair of maximal elements of X in $\mathcal{P}$. Then $(\mathcal{D}, f)$ is a submodular system on $R \cup C$. (See [Hoffman63] and [Burkard+Klinz+Rudolf96] for more details about Monge properties, which have been investigated from the point of view of greedy algorithms. Also see [Queyranne+Spieksma+Tardella98], [Faigle+Kern96, 00a, 00b], [Krüger00], [Frank99], [Kashiwabara+Okamoto03] and [Fuji04] for related topics on dual greediness.)]

[Also see [Queyranne93] and [Queyranne+Schulz95] for supermodular structures in scheduling problems.]

3. Submodular Systems

In this section we show basic properties of submodular systems.

3.1. Fundamental Operations on Submodular Systems

We introduce several fundamental operations on a submodular system $(\mathcal{D}, f)$ on E.

(a) Reductions and contractions by sets

For any $A \in \mathcal{D}$ define

$$\mathcal{D}^A = \{X \mid A \supseteq X \in \mathcal{D}\}, \tag{3.1}$$

$$f^A(X) = f(X) \quad (X \in \mathcal{D}^A). \tag{3.2}$$

Then $(\mathcal{D}^A, f^A)$ is a submodular system on A and is called the *reduction* or the *restriction of* $(\mathcal{D}, f)$ *to* A. We denote $(\mathcal{D}^A, f^A)$ by $(\mathcal{D}, f) \cdot A$ or $(\mathcal{D}, f) - (E - A)$.

Also define for $A \in \mathcal{D}$

$$\mathcal{D}_A = \{X - A \mid A \subseteq X \in \mathcal{D}\}, \tag{3.3}$$

$$f_A(X) = f(X \cup A) - f(A) \quad (X \in \mathcal{D}_A). \tag{3.4}$$

Then we have a submodular system $(\mathcal{D}_A, f_A)$ on $E - A$, which is called the *contraction of* $(\mathcal{D}, f)$ *by* A and is denoted by $(\mathcal{D}, f)/A$ or $(\mathcal{D}, f) \times (E - A)$. [Note that f is submodular if and only if every contraction is subadditive.]

We call a submodular system obtained by repeated reductions and/or contractions of $(\mathcal{D}, f)$ by sets a *set minor* of $(\mathcal{D}, f)$.

Lemma 3.1: *For any $A \in \mathcal{D}$ let x^A be a base of the reduction $(\mathcal{D}, f) \cdot A$ of submodular system $(\mathcal{D}, f)$ to A and x_A be a base of the contraction $(\mathcal{D}, f)/A$ of $(\mathcal{D}, f)$ by A. Then the direct sum $\hat{x} = x^A \oplus x_A$ of x^A and x_A defined by*

$$(x^A \oplus x_A)(e) = \begin{cases} x^A(e) & (e \in A) \\ x_A(e) & (e \in E - A) \end{cases} \tag{3.5}$$

is a base of $(\mathcal{D}, f)$. Conversely, for any base $\hat{x}$ of $(\mathcal{D}, f)$ satisfying $\hat{x}(A) = f(A)$, restricting $\hat{x}$ on A (or on $E - A$) yields a base of $(\mathcal{D}, f) \cdot A$ (or $(\mathcal{D}, f)/A$).

(Proof) If x^A is a base of $(\mathcal{D}, f) \cdot A$ and x_A is a base of $(\mathcal{D}, f)/A$, we have for any $X \in \mathcal{D}$

$$\begin{aligned} \hat{x}(X) &= \hat{x}(X \cap A) + \hat{x}(X - A) \\ &= x^A(X \cap A) + x_A(X - A) \\ &\le f(X \cap A) + f(X \cup A) - f(A) \\ &\le f(X). \end{aligned} \tag{3.6}$$

Since $\hat{x}(E) = f(E)$, it follows that $\hat{x}$ is a base of $(\mathcal{D}, f)$. Conversely, if $\hat{x}$ is a base of $(\mathcal{D}, f)$ and satisfies $\hat{x}(A) = f(A)$, then for any $X \in \mathcal{D}^A$ and $Y \in \mathcal{D}_A$

$$x^A(X) = \hat{x}(X) \le f(X), \tag{3.7}$$

$$x_A(Y) = \hat{x}(Y \cup A) - \hat{x}(A) \le f(Y \cup A) - f(A). \tag{3.8}$$

Q.E.D.

Lemma 3.2: *For any $X \in \mathcal{D}$ there exists a subbase $x \in \mathrm{P}(f)$ such that $x(X) = f(X)$. Furthermore, for any $X \in 2^E - \mathcal{D}$ and any $K > 0$ there exists a subbase $x \in \mathrm{P}(f)$ such that $x(X) \ge K$.*

(Proof) The first half follows from Lemma 3.1. The second half is shown as follows. For any $X \in 2^E - \mathcal{D}$ there exist $e \in X$ and $e' \in E - X$ such that for each $Y \in \mathcal{D}$ with $e \in Y$ we have $e' \in Y$. (For, if for each $e \in X$ and $e' \in E - X$ there were $Y \in \mathcal{D}$ such that $e \in Y$ and $e' \notin Y$, denoting one such Y by $Y_{e,e'}$, we would have $X = \cup_{e \in X} \cap_{e' \in E-X} Y_{e,e'} \in \mathcal{D}$.) Hence for any $y \in \mathrm{P}(f)$, $y_d \equiv y + d(\chi_e - \chi_{e'})$ belongs to $\mathrm{P}(f)$ for any $d > 0$. Therefore, the value, $y_d(X) = y(X) + d$, can be made arbitrarily large. Q.E.D.

(b) Reductions and contractions by vectors

For any vector $x \in \mathbf{R}^E$ define a function $f^x: 2^E \to \mathbf{R}$ by

$$f^x(X) = \min\{f(Z) + x(X - Z) \mid X \supseteq Z \in \mathcal{D}\} \tag{3.9}$$

for each $X \subseteq E$. Then the function $f^x: 2^E \to \mathbf{R}$ is a submodular function on the Boolean lattice 2^E. For, denoting by Z_X a minimizer Z of the right-hand side of (3.9), we have for each $X,\ Y \subseteq E$

$$\begin{aligned} f^x(X) + f^x(Y) &= f(Z_X) + x(X - Z_X) + f(Z_Y) + x(Y - Z_Y) \\ &\geq f(Z_X \cup Z_Y) + x((X \cup Y) - (Z_X \cup Z_Y)) \\ &\quad + f(Z_X \cap Z_Y) + x((X \cap Y) - (Z_X \cap Z_Y)) \\ &\geq f^x(X \cup Y) + f^x(X \cap Y). \end{aligned} \tag{3.10}$$

We call the submodular system $(2^E, f^x)$ the *reduction of* $(\mathcal{D}, f)$ *by vector* x. A base of the reduction $(2^E, f^x)$ is called a *base of* x in $(\mathcal{D}, f)$.

Define

$$\mathrm{P}(f)^x = \{y \mid y \in \mathrm{P}(f),\ y \leq x\}, \tag{3.11}$$

which is the set of subbases of $(\mathcal{D}, f)$ smaller than or equal to x (see Fig. 3.1).

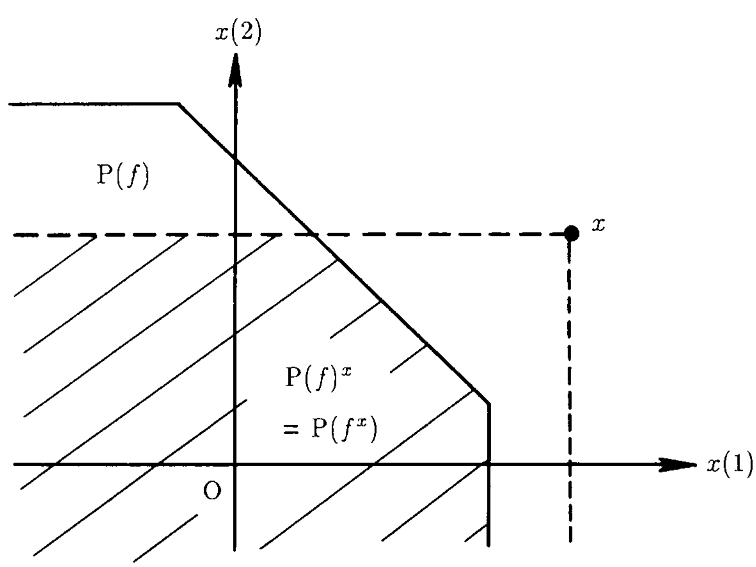


Figure 3.1: A reduction by a vector.

Theorem 3.3: *The submodular polyhedron associated with the reduction* $(2^E, f^x)$ *of* $(\mathcal{D}, f)$ *by vector* x *is given by*

$$\mathrm{P}(f^x) = \mathrm{P}(f)^x. \tag{3.12}$$

Also, if x *is a superbase of* $(\mathcal{D}, f)$, *i.e.* $x \in \mathrm{P}(f^{\#})$, *then*

$$\mathrm{B}(f^x) = \mathrm{B}(f)^x, \tag{3.13}$$

where $\mathrm{B}(f)^x = \{y \mid y \in \mathrm{B}(f),\ y \le x\}$.

(Proof) For any $y \in \mathrm{P}(f^x)$, we see from (3.9) with $X = Z \in \mathcal{D}$ or $X = \{e\} \supseteq Z = \emptyset$ that

$$\forall X \in \mathcal{D}\colon\ y(X) \le f(X), \tag{3.14}$$

$$\forall e \in E\colon\ y(e) \le x(e). \tag{3.15}$$

Hence $y \in \mathrm{P}(f)^x$. Conversely, for any $y \in \mathrm{P}(f)^x$ we have (3.14) and (3.15). For any $Z,\ X \subseteq E$ with $X \supseteq Z \in \mathcal{D}$,

$$y(X) = y(Z) + y(X - Z) \le f(Z) + x(X - Z). \tag{3.16}$$

Hence $y \in \mathrm{P}(f^x)$. Moreover, (3.13) follows from (3.12) and Theorem 2.3 since there exists a base $y \in \mathrm{B}(f)$ such that $y \le x$. Q.E.D.

For a supermodular system $(\mathcal{D}, g)$ on E we define the reduction of $(\mathcal{D}, g)$ by a vector $x \in \mathbf{R}^E$ in a dual manner. Define $g_x\colon 2^E \to \mathbf{R}$ by

$$g_x(X) = \max\{g(Z) + x(X - Z) \mid X \supseteq Z \in \mathcal{D}\}, \tag{3.17}$$

$$\mathrm{P}(g)_x = \{y \mid y \in \mathrm{P}(g),\ y \ge x\}. \tag{3.18}$$

[Similarly we define $\mathrm{B}(f)_x$ for $x \in \mathrm{P}(f)$.] Then $(2^E, g_x)$ is the *reduction* of $(\mathcal{D}, g)$ by x and its associated supermodular polyhedron is given by

$$\mathrm{P}(g_x) = \mathrm{P}(g)_x \tag{3.19}$$

due to Theorem 3.3.

From Lemma 3.2, Theorem 3.3 and (3.9) we have the following min-max relation.

Corollary 3.4:

$$\min\{f(Z) + x(E - Z) \mid Z \in \mathcal{D}\} = \max\{y(E) \mid y \in \mathrm{P}(f),\ y \le x\}. \tag{3.20}$$

(Proof) It is easy to see that

$$f(Z) + x(E - Z) \geq y(E) \tag{3.21}$$

for any $Z \in \mathcal{D}$ and any $y \in \mathrm{P}(f)$ with $y \leq x$. On the other hand, from (3.9) with $X = E$ and Lemma 3.2 there exists a subbase y of $(2^E, f^x)$ such that $y(E) = f^x(E)$. Since $\mathrm{P}(f^x) = \mathrm{P}(f)^x$ due to Theorem 3.3, this shows the existence of $Z \in \mathcal{D}$ and $y \in \mathrm{P}(f)$ with $y \leq x$ such that (3.21) holds with equality. Q.E.D.

[**Remark**: We should recall that Corollary 3.4 holds for any totally ordered additive group $\mathbf{R}$. Hence Corollary 3.4 implies a combinatorially deep statement that if the rank function f is integer-valued, x is integral, and we consider the set $\mathbf{R}$ of reals, then there exists an integral y that attains the maximum in the right-hand side of (3.20). The existence of such an integral y follows from the two propositions, one being Corollary 3.4 with the set $\mathbf{R}$ of reals and the other being Corollary 3.4 with the set $\mathbf{R}$ of integers. Readers should keep this fact in mind throughout this book although we will sometimes state such an integrality property explicitly.]

In particular, for $x = \mathbf{0}$ (3.20) becomes

Corollary 3.5:

$$\min\{f(Z) \mid Z \in \mathcal{D}\} = \max\{y(E) \mid y \in \mathrm{P}(f),\ y \leq \mathbf{0}\}. \tag{3.22}$$

Next, for any subbase $x \in \mathrm{P}(f)$ define a function $f_x: 2^E \to \mathbf{R}$ by

$$f_x(X) = \min\{f(Z) - x(Z - X) \mid X \subseteq Z \in \mathcal{D}\} \tag{3.23}$$

for each $X \subseteq E$. Similarly as in (3.10), we see that $f_x: 2^E \to \mathbf{R}$ is a submodular function on the Boolean lattice 2^E. Here, it should be noted that $f_x(\emptyset) = 0$ due to the fact that $x \in \mathrm{P}(f)$. We call the submodular system $(2^E, f_x)$ the *contraction of* $(\mathcal{D}, f)$ *by vector* $x \in \mathrm{P}(f)$.

Theorem 3.6: *For any subbase* $x \in \mathrm{P}(f)$ *we have*

$$\begin{aligned} (f_x)^{\#} &= (f^{\#})_x, && (3.24)\\ \mathrm{B}(f_x) &= \mathrm{B}(f)_x. && (3.25) \end{aligned}$$

(Proof) Since $x \in \mathrm{P}(f)$, we have $f_x(E) = f(E)$. (3.24) easily follows from (3.23) and the definition of the dual function. Furthermore, from (3.20), Theorem 3.3 and the duality shown in Lemma 2.4 together with the remarks given after it, we have

$$\mathrm{B}(f_x) = \mathrm{B}((f_x)^{\#}) = \mathrm{B}((f^{\#})_x) = \mathrm{B}(f^{\#})_x = \mathrm{B}(f)_x \tag{3.26}$$

Q.E.D.

The contraction of $(\mathcal{D}, f)$ by $x \in \mathrm{P}(f)$ corresponds to the reduction of its dual supermodular system $(\overline{\mathcal{D}}, f^{\#})$ by x (see Fig. 3.2).

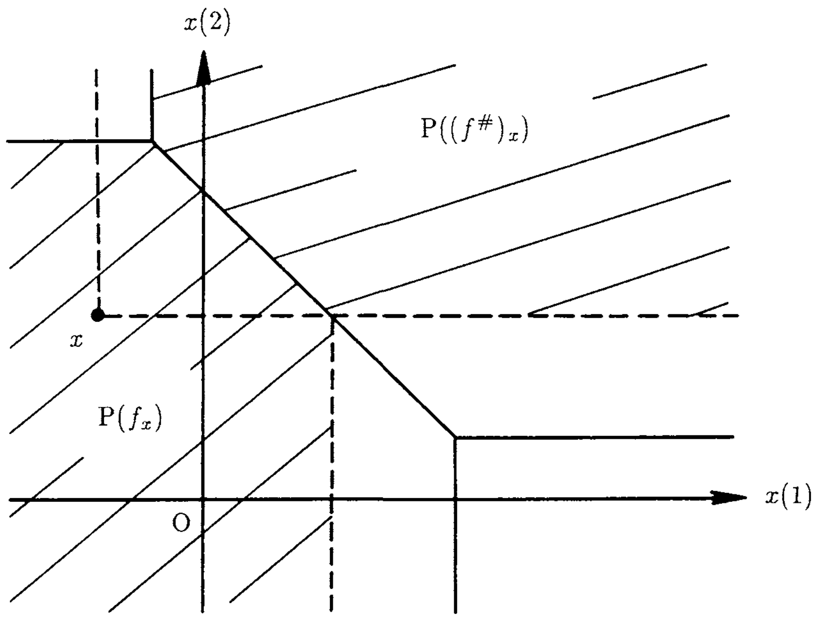


Figure 3.2: A contraction by a vector.

For any subbase $x \in \mathrm{P}(f)$ define

$$[\mathrm{P}(f)]_x = \{y \mid y \in \mathbf{R}^E,\ x \vee y \in \mathrm{P}(f)\}, \tag{3.27}$$

where $x \vee y$ is the vector in $\mathbf{R}^E$ defined by $(x \vee y)(e) = \max\{x(e), y(e)\}$ $(e \in E)$, i.e., the join of x and y in the vector lattice $\mathbf{R}^E$.

Lemma 3.7: *For any subbase* $x \in \mathrm{P}(f)$,

$$\mathrm{P}(f_x) = [\mathrm{P}(f)]_x. \tag{3.28}$$

(Proof) For any $y \in \mathrm{P}(f_x)$, we have from (3.23)

$$y(X) + x(Z - X) \le f(Z) \tag{3.29}$$

for any X and Z such that $X \subseteq Z \in \mathcal{D}$. This implies $x \vee y \in \mathrm{P}(f)$. Conversely, for any $y \in [\mathrm{P}(f)]_x$, we have (3.29) for any X and Z such that $X \subseteq Z \in \mathcal{D}$, from which follows the fact that $y \in \mathrm{P}(f_x)$. Q.E.D.

In a dual manner we define the contraction $(2^E, g^x)$ of a supermodular system $(\mathcal{D}, g)$ by a superbase $x \in \mathrm{P}(g)$, where $g^x = ((g^\#)^x)^\#$.

A submodular system obtained by repeated reductions and/or contractions of submodular system $(\mathcal{D}, f)$ by vectors in $\mathbf{R}^E$ is called a *vector minor* of $(\mathcal{D}, f)$.

Theorem 3.8: *If vectors* $x,\ y \in \mathbf{R}^E$ *satisfy* (i) $x \le y$, (ii) $\mathrm{B}(f)_x \ne \emptyset$ (*i.e.*, x *is a subbase*) *and* (iii) $\mathrm{B}(f)^y \ne \emptyset$ (*i.e.*, y *is a superbase*), *then we have* $\mathrm{B}(f)^y_x\ (= (\mathrm{B}(f)_x)^y = (\mathrm{B}(f)^y)_x) \ne \emptyset$.

(Proof) Since $x \in \mathrm{P}(f)$, $y \in \mathrm{P}(f^\#)$ and $x \le y$, we have $y \in \mathrm{P}(f^\#)_x = \mathrm{P}((f_x)^\#)$. Hence, $\mathrm{B}(f)^y_x = \mathrm{B}(f_x)^y = \mathrm{B}((f_x)^\#)^y \ne \emptyset$. Q.E.D.

For any $x \in \mathbf{R}^E$ the rank of the reduction of $(\mathcal{D}, f)$ by x is denoted by $\mathrm{r}_f(x) (= f^x(E))$. The function $\mathrm{r}_f\colon \mathbf{R}^E \to \mathbf{R}$ is called the *vector rank function* of $(\mathcal{D}, f)$. From the definition, r_f is a concave function (see (3.9)). We can show that r_f satisfies $\mathrm{r}_f(x) + \mathrm{r}_f(y) \ge \mathrm{r}_f(x \vee y) + \mathrm{r}_f(x \wedge y)$ for any $x, y \in \mathbf{R}^E$, where $\vee$ and $\wedge$ is the join and meet operations in vector lattice $\mathbf{R}^E$, i.e., r_f is a submodular function on $\mathbf{R}^E$ (see [Welsh76]). [The vector rank function r_f is an $\mathrm{M}^\natural$ concave function (see Section 15.2).]

(c) Translations and sums

For any vector $x \in \mathbf{R}^E$ the *translation* of a submodular system $(\mathcal{D}, f)$ by x is the submodular system whose rank function is given by $f + x\colon\ \mathcal{D} \to \mathbf{R}$, where x should be considered as a set function (a modular function) on 2^E by $x(X) = \sum_{e \in X} x(e)$ $(X \subseteq E)$. For the translation $(\mathcal{D}, f + x)$, we have

$$\mathrm{P}(f + x) = \mathrm{P}(f) + \{x\}, \tag{3.30}$$

$$\mathrm{B}(f + x) = \mathrm{B}(f) + \{x\}, \tag{3.31}$$

where x in the right-hand sides is in $\mathbf{R}^E$ and the sums in the right-hand sides denote the vector sum [(or the Minkowski sum)] (see Fig. 3.3).

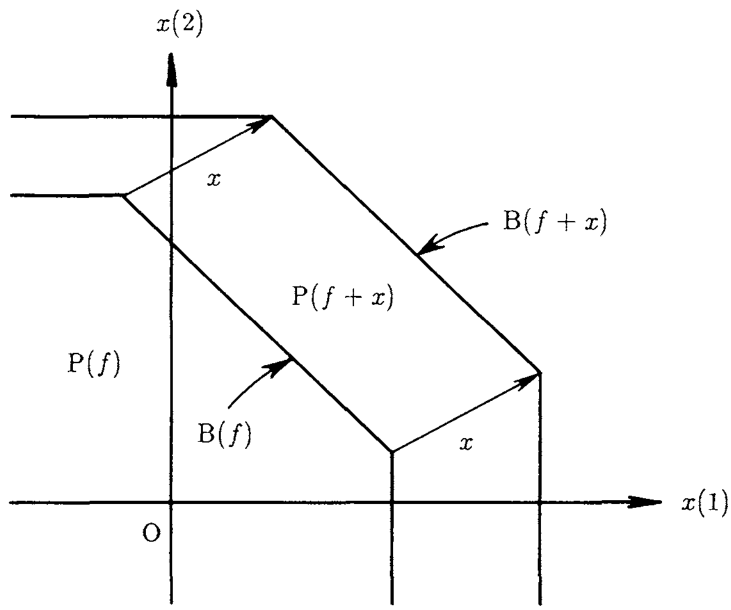


Figure 3.3: A translation.

It should be noted that a contraction of $(\mathcal{D}, f)$ by $x \in \mathrm{P}(f)$ followed by a translation by $-x$ corresponds to an ordinary contraction of a (poly-) matroid (see [Fuji80b]). The combinatorial structure of the submodular polyhedron and the base polyhedron is invariant with respect to translations and the rank function can be made monotone nondecreasing by an appropriate translation (see Lemma 3.23 in Section 3.3.a). Therefore, the monotonicity of the rank function plays no essential rôle in the theory of submodular systems but sometimes makes it easier for us to find an initial feasible solution in algorithms. Also, any results obtained in (poly-)matroid theory which are invariant with respect to translations can easily be extended to submodular systems. For example, we will generalize the polymatroid intersection theorem of Edmonds [Edm70] to submodular systems in Section 4.1.

For two submodular systems $(\mathcal{D}_1, f_1)$ and $(\mathcal{D}_2, f_2)$ on E the *sum* of the two submodular systems is defined as the submodular system $(\mathcal{D}_1 \cap \mathcal{D}_2, f_1 +$

f_2) on E. We have

$$\mathrm{P}(f_1+f_2) = \mathrm{P}(f_1)+\mathrm{P}(f_2), \tag{3.32}$$

$$\mathrm{B}(f_1+f_2) = \mathrm{B}(f_1)+\mathrm{B}(f_2), \tag{3.33}$$

where the sums in the right-hand sides denote the vector sum (or Minkowski sum). (Relations (3.32) and (3.33) will be shown in Section 4.2. [Note that we are considering any totally ordered additive group $\mathbf{R}$ such as the set $\mathbf{Z}$ of integers, which causes combinatorial difficulty in proving (3.32) and (3.33).]) Note that a translation is a special case of a sum.

(d) Other operations

For a submodular system $(\mathcal{D}, f)$ on E let r be an arbitrary nonnegative element in $\mathbf{R}$. Define

$$f_{-r}(X) = f(X) \quad (X \in \mathcal{D}-\{E\}), \tag{3.34}$$

$$f_{-r}(E) = f(E)-r. \tag{3.35}$$

Then $(\mathcal{D}, f_{-r})$ is also a submodular system on E and is called the *r-truncation* of $(\mathcal{D}, f)$. Similarly, we define the *r-truncation*, $(\mathcal{D}, g_{+r})$, of a supermodular system $(\mathcal{D}, g)$ by

$$g_{+r}(X) = g(X) \quad (X \in \mathcal{D}-\{E\}), \tag{3.36}$$

$$g_{+r}(E) = g(E)+r. \tag{3.37}$$

The dual of the r-truncation of the dual supermodular system of the submodular system $(\mathcal{D}, f)$ is called the *r-elongation* of $(\mathcal{D}, f)$ ([Tomi81a]) (see Fig. 3.4).

Denoting the r-elongation of $(\mathcal{D}, f)$ by $(\mathcal{D}, f_{+r})$, we have

$$f_{+r} = ((f^{\#})_{+r})^{\#}. \tag{3.38}$$

Similarly, in a dual manner we define the *r-elongation* of a supermodular system.

For a submodular system $(\mathcal{D}, f)$ on E and any $\alpha \in \mathbf{R}$ define

$$f^{(\alpha)}(X) = \begin{cases} f(X)+\alpha & (X \in \mathcal{D}-\{\emptyset, E\}) \\ f(X) & (X \in \{\emptyset, E\}). \end{cases} \tag{3.39}$$

Suppose $\alpha \le 0$. Then $f^{(\alpha)}\colon \mathcal{D} \to \mathbf{R}$ is a crossing-submodular function on $\mathcal{D}$. If $\mathrm{B}(f^{(\alpha)}) \ne \emptyset$, then by Theorem 2.5 there uniquely exists a submodular

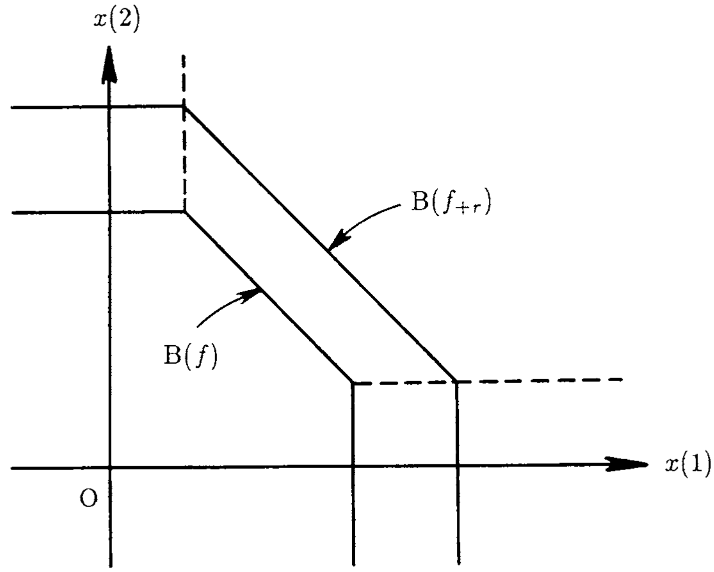


Figure 3.4: An r-elongation.

system $(\mathcal{D}, \tilde{f}^{(\alpha)})$ on E such that $\mathrm{B}(\tilde{f}^{(\alpha)}) = \mathrm{B}(f^{(\alpha)})$. We call $(\mathcal{D}, \tilde{f}^{(\alpha)})$ the $|\alpha|$-*abridgment* of $(\mathcal{D}, f)$. Also, suppose $\alpha \geq 0$. Then $(\mathcal{D}, f^{(\alpha)})$ is a submodular system and is called the α-*enlargement* of $(\mathcal{D}, f)$. (For the abridgment and enlargement also see [Tomi81b,81c].) The concepts of abridgment and enlargement also appear in the game theory as ϵ-*core* ($\epsilon \leq 0$ and $\epsilon \geq 0$); in a convex game an ϵ-core for $\epsilon \leq 0$ (or $\epsilon \geq 0$) is exactly the base polyhedron of the $|\epsilon|$-abridgment (or ϵ-enlargement) of the associated submodular system defining the convex game (see [Shubik82]).

For any partition $\Pi = \{A_1, A_2, \cdots, A_l\}$ of E a subset X of E is said to be *compatible with* Π if, for each $A_i \in \Pi$, $A_i \cap X \neq \emptyset$ implies $A_i \subseteq X$. Define

$$\mathcal{D}(\Pi) = \{X \mid X \in \mathcal{D},\ X \text{ is compatible with } \Pi\} \tag{3.40}$$

and denote the restriction of f to $\mathcal{D}(\Pi)$ by f_Π. The pair $(\mathcal{D}(\Pi), f_\Pi)$ is a submodular system on E [(or on Π)] and is called the *aggregation* of $(\mathcal{D}, f)$ by Π. Aggregations play a fundamental rôle in a decomposition theory for submodular systems ([Fuji83], [Cunningham83b]) which generalizes the decomposition theory of graphs by Tutte [Tutte66]. [Also note that any polymatroid is the aggregation of a matroid.]

3.2. Greedy Algorithm

In this section we consider a linear optimization problem over the base polyhedron and give an algorithm, called a *greedy algorithm*, for solving the problem. Before getting into the problem, we first examine the structure of the distributive lattice $\mathcal{D} \subseteq 2^E$ and show the one-to-one correspondence between the set of distributive lattices $\mathcal{D} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{D}$ and the set of partially ordered sets (posets) on partitions of E.

(a) Distributive lattices and posets

For a distributive lattice $\mathcal{D} \subseteq 2^E$ the cardinality $|\mathcal{D}|$ of $\mathcal{D}$ can be as large as $2^{|E|}$ and listing all the elements of $\mathcal{D}$ to represent it is not practical even for medium-sized E. We shall show how to efficiently express a distributive lattice as a structured system, a poset, on E.

Let $\mathcal{D} \subseteq 2^E$ be a distributive lattice with $\emptyset,\ E \in \mathcal{D}$. A sequence of monotone increasing elements of $\mathcal{D}$

$$\mathcal{C}\colon\ S_0 \subset S_1 \subset \cdots \subset S_k \tag{3.41}$$

is called a *chain* of $\mathcal{D}$ and k is the *length* of the chain $\mathcal{C}$. If there exists no chain which contains chain $\mathcal{C}$ as a proper subsequence, $\mathcal{C}$ is called a *maximal chain* of $\mathcal{D}$. If $\mathcal{C}$ given by (3.41) is a maximal chain, we have $S_0 = \emptyset$ and $S_k = E$.

For each $e \in E$ define

$$D(e) = \bigcap\{X \mid e \in X \in \mathcal{D}\}. \tag{3.42}$$

$D(e)$ is the unique minimal element of $\mathcal{D}$ containing e. Note that for any $e \in E$ and $e' \in D(e)$ we have

$$D(e') \subseteq D(e). \tag{3.43}$$

Also let $G(\mathcal{D}) = (E, A(\mathcal{D}))$ be a (directed) graph with a vertex set E and an arc set $A(\mathcal{D})$ given by

$$A(\mathcal{D}) = \{(e, e') \mid e \in E,\ e' \in D(e)\}. \tag{3.44}$$

Decompose the graph $G(\mathcal{D})$ into strongly connected components $G_i = (F_i, A_i)$ $(i \in I)$. Let $\preceq_{\mathcal{D}}$ be the partial order on the set of the strongly connected components $\{G_i \mid i \in I\}$ naturally induced by the decomposition (i.e., $G_{i_1} \preceq_{\mathcal{D}} G_{i_2}$ for $i_1,\ i_2 \in I$ if and only if there exists a directed path from a vertex of G_{i_2} to a vertex of G_{i_1}). Note that from (3.43) $G(\mathcal{D})$ is transitively closed (i.e., if there is a directed path from a vertex v_1 to a vertex v_2, then there is an arc (v_1, v_2) in $G(\mathcal{D})$). Therefore, if $G_{i_1} \preceq_{\mathcal{D}} G_{i_2}$, there exists an arc from any vertex of G_{i_2} to any vertex of G_{i_1}.

Denote the set of the vertex sets F_i $(i \in I)$ of the strongly connected components G_i $(i \in I)$ by

$$\Pi(\mathcal{D}) = \{F_i \mid i \in I\}. \tag{3.45}$$

$\Pi(\mathcal{D})$ is a partition of E. In the following we regard $\preceq_{\mathcal{D}}$ as a partial order on $\Pi(\mathcal{D})$ by identifying G_i with F_i for each $i \in I$.

Now, we have obtained a poset $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$, which is called the *poset derived from distributive lattice* $\mathcal{D}$. An example of a distributive lattice $\mathcal{D}$ and the poset $\mathcal{P}(\mathcal{D})$ derived from it is shown in Fig. 3.5.

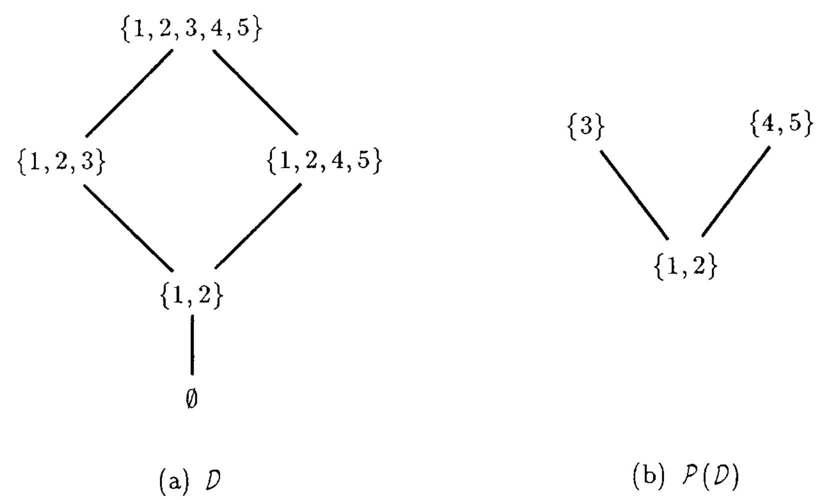


Figure 3.5: A distributive lattice and its representing poset.

For a (general) poset $\mathcal{P} = (P, \preceq)$, a set $J \subseteq P$ is called a (*lower*) *ideal* of $\mathcal{P}$ if for each $e, e' \in P$ we have

$$e \preceq e' \in J \Longrightarrow e \in J. \tag{3.46}$$

Theorem 3.9 [Birkhoff37]: *Let* $\mathcal{D} \subseteq 2^E$ *be a distributive lattice with* $\emptyset$, $E \in \mathcal{D}$. *Then, for the poset* $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$ *derived from* $\mathcal{D}$ *the following* (i) *and* (ii) *hold.*

(i) *For each ideal J of $\mathcal{P}(\mathcal{D})$,*

$$\bigcup\{F \mid F \in J\} \in \mathcal{D}. \tag{3.47}$$

(ii) *For each $X \in \mathcal{D}$, there exists an ideal J of $\mathcal{P}(\mathcal{D})$ such that*

$$X = \bigcup\{F \mid F \in J\}. \tag{3.48}$$

(Proof) (i): Put $X = \bigcup\{F \mid F \in J\}$. Since J is an ideal of $\mathcal{P}(\mathcal{D})$, it follows from the definition of $\mathcal{P}(\mathcal{D})$ that $D(e) \subseteq X$ for each $e \in X$. Then, $X = \bigcup\{D(e) \mid e \in X\}$ and we have $X \in \mathcal{D}$ since $D(e) \in \mathcal{D}$ from (3.42).

(ii): For a given $X \in \mathcal{D}$, X and any $F \in \Pi(\mathcal{D})$ do not cross, i.e., either $F \subseteq X$ or $F \subseteq E - X$, due to the definition of $\mathcal{P}(\mathcal{D})$. Therefore, $J = \{F \mid F \in \Pi(\mathcal{D}),\ F \subseteq X\}$ is a partition of X. Moreover, because of the definition of $\mathcal{P}(\mathcal{D})$ "$F_1 \preceq_{\mathcal{D}} F_2 \subseteq X$" implies "$F_1 \subseteq X$". Consequently, $J = \{F \mid F \in \Pi(\mathcal{D}),\ F \subseteq X\}$ is a desired ideal of $\mathcal{P}(\mathcal{D})$. Q.E.D.

From Theorem 3.9, (3.47) (or (3.48)) determines a one-to-one correspondence between $\mathcal{D}$ and the set of all the ideals of $\mathcal{P}(\mathcal{D})$. It should be noted that for any poset $\mathcal{P} = (P, \preceq)$ on a partition P of E the set $\mathcal{D}(\mathcal{P})$ defined by

$$\mathcal{D}(\mathcal{P}) = \{\tilde{J} \mid J\text{: an ideal of } \mathcal{P}\}, \tag{3.49}$$

$$\tilde{J} = \bigcup\{F \mid F \in J\} \tag{3.50}$$

forms a distributive lattice with $\emptyset,\ E \in \mathcal{D}(\mathcal{P})$. We can also see that the mapping which assigns each $\mathcal{D}$ to its derived poset $\mathcal{P}(\mathcal{D})$ is a one-to-one correspondence between the set of distributive lattices $\mathcal{D} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{D}$ and that of posets $\mathcal{P} = (P, \preceq)$ on partitions P of E.

From Theorem 3.9 we can easily show

Corollary 3.10: *Given a distributive lattice $\mathcal{D} \subseteq 2^E$ with $\emptyset,\ E \in \mathcal{D}$, let*

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{3.51}$$

be an arbitrary maximal chain of $\mathcal{D}$. Then we have

$$\Pi(\mathcal{D}) = \{S_i - S_{i-1} \mid i = 1, 2, \cdots, k\}. \tag{3.52}$$

In particular, the length of any maximal chain of $\mathcal{D}$ is independent of the choice of a maximal chain and is equal to $|\Pi(\mathcal{D})|$.

We call $\mathcal{D}$ *simple* if the partition $\Pi(\mathcal{D})$ is composed of singletons of E alone, i.e., $\Pi(\mathcal{D}) = \{\{e\} \mid e \in E\}$. For a simple distributive lattice $\mathcal{D}$ we regard $\mathcal{P}(\mathcal{D})$ as a poset on E and write $\mathcal{P}(\mathcal{D}) = (E, \preceq_{\mathcal{D}})$. Conversely, the set of all the (lower) ideals of a poset $\mathcal{P} = (E, \preceq)$ on E forms a simple distributive lattice $\mathcal{D} \subseteq 2^E$ and we denote such a simple $\mathcal{D}$ by $\mathbf{2}^{\mathcal{P}}$. A submodular system $(\mathcal{D}, f)$ with simple $\mathcal{D}$ is called *simple*.

For a non-simple submodular system $(\mathcal{D}, f)$ on E, define

$$\hat{X} = \{F \mid F \in \Pi(\mathcal{D}),\ F \subseteq X\} \quad (X \in \mathcal{D}), \tag{3.53}$$

$$\hat{\mathcal{D}} = \{\hat{X} \mid X \in \mathcal{D}\}, \tag{3.54}$$

$$\hat{f}(\hat{X}) = f(X) \quad (X \in \mathcal{D}). \tag{3.55}$$

Then we have a simple submodular system $(\hat{\mathcal{D}}, \hat{f})$ on $\Pi(\mathcal{D})$, which we call the *simplification* of $(\mathcal{D}, f)$.

[For a simple distributive lattice $\mathcal{D} \subseteq 2^E$ with $\emptyset, E \in \mathcal{D}$ a function $f : \mathcal{D} \to \mathbf{R}$ is submodular if and only if for each $X \in \mathcal{D}$ and distinct $e, e' \in E - X$ such that $X \cup \{e\}, X \cup \{e'\} \in \mathcal{D}$ we have

$$f(X \cup \{e\}) + f(X \cup \{e'\}) \geq f(X \cup \{e, e'\}) + f(X).$$

(The proof is left for readers.)]

(b) Greedy algorithm

For a submodular system $(\mathcal{D}, f)$ on E we consider a linear optimization problem described as follows.

$$P_w: \quad \text{Minimize} \sum_{e \in E} w(e)x(e) \tag{3.56a}$$

$$\text{subject to} \ \ x \in \mathrm{B}(f), \tag{3.56b}$$

where w: $E \to \mathbf{R}$ is a given weight function. [We assume that (3.56a) is well defined; we may assume that $w(e) \in \mathbf{Z}$ $(e \in E)$, or $\mathbf{R}$ is a totally ordered ring or field.] An optimal solution of P_w is called a *minimum-weight base* of $(\mathcal{D}, f)$ with respect to the weight function w. Similarly, a *maximum-weight base* of $(\mathcal{D}, f)$ with respect to the weight function w is an optimal solution of Problem P_{-w} with the weight function $-w$.

Fundamental structural properties of the base polyhedron $\mathrm{B}(f)$ are given by the following theorems.

Theorem 3.11: *The base polyhedron* $\mathrm{B}(f)$ *is pointed (or has at least one extreme point) if and only if* $\mathcal{D}$ *is simple, i.e.,* $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ *for some poset* $\mathcal{P} = (E, \preceq)$.

(Proof) A polyhedron is pointed if and only if its characteristic cone (or recession cone) does not contain any line ([Stoer+Witzgall70], [Rockafellar70]). The characteristic cone of $\mathrm{B}(f)$ is the solution set of the following system of inequalities and an equation:

$$
\begin{aligned}
x(X) &\le 0 \quad (X \in \mathcal{D}), && (3.57)\\
x(E) &= 0. && (3.58)
\end{aligned}
$$

Therefore, $\mathrm{B}(f)$ is pointed if and only if the system of equations

$$x(X) = 0 \quad (X \in \mathcal{D}) \tag{3.59}$$

has the unique solution $x = \mathbf{0}$, where note that $E \in \mathcal{D}$. We see from Corollary 3.10 that (3.59) is equivalent to

$$x(F) = 0 \quad (F \in \Pi(\mathcal{D})). \tag{3.60}$$

System (3.60) has the unique solution $x = \mathbf{0}$ if and only if $|F| = 1$ for each $F \in \Pi(\mathcal{D})$, i.e., $\mathcal{D}$ is simple. Q.E.D.

It should be noted that the rank of the coefficient matrix of the left-hand side of (3.59) is equal to the height of $\mathcal{D}$, the length of a maximal chain of $\mathcal{D}$.

Theorem 3.12: *The base polyhedron* $\mathrm{B}(f)$ *is bounded if and only if* $\mathcal{D}$ *is the Boolean lattice* 2^E*, i.e.,* $\mathcal{D}$ *is simple and complemented.*

(Proof) If $\mathcal{D} = 2^E$, then $\mathrm{B}(f)$ is included in the following bounded solution set of

$$x(e) \le f(\{e\}) \quad (e \in E), \qquad x(E) = f(E). \tag{3.61}$$

On the other hand, if $\mathcal{D} \neq 2^E$, there exist distinct elements $e,\ e' \in E$ such that for each $X \in \mathcal{D}$ $e \in X$ implies $e' \in X$. Then for any base $x \in \mathrm{B}(f)$ the ray or halfline

$$\{x + \alpha(\chi_e - \chi_{e'}) \mid \alpha \ge 0\} \tag{3.62}$$

is contained in $\mathrm{B}(f)$. Hence, $\mathrm{B}(f)$ is not bounded. Q.E.D.

When $(\mathcal{D}, f)$ is not simple, Problem P_w does not have a finite optimal solution if $w(e) \neq w(e')$ for any $e,\ e' \in F \in \Pi(\mathcal{D})$. Therefore, if P_w has an optimal solution, $w\colon E \to \mathbf{R}$ is constant on each $F \in \Pi(\mathcal{D})$, and hence it suffices to consider the simplification of $(\mathcal{D}, f)$.

We suppose without loss of generality that in the minimum-weight base problem P_w described by (3.56) $\mathrm{B}(f)$ is pointed, i.e., $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ with $\mathcal{P} = (E, \preceq)$.

Theorem 3.13 ([Fuji+Tomi83]): *Problem P_w in* (3.56) *has a finite optimal solution if and only if $w\colon E \to \mathbf{R}$ is a monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$, i.e., $\forall e,\ e' \in E$: $e \preceq e' \Longrightarrow w(e) \leq w(e')$.*

(Proof) *The "if" part*: Suppose that w is a monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$ and that the distinct values of weights $w(e)$ $(e \in E)$ are given by

$$w_1 < w_2 < \cdots < w_p. \tag{3.63}$$

Define

$$A_i = \{e \mid e \in E,\ w(e) \leq w_i\} \quad (i = 1, 2, \cdots, p), \tag{3.64}$$

where note that $A_p = E$. The sets A_i $(i = 1, 2, \cdots, p)$ form a chain $A_1 \subset A_2 \subset \cdots \subset A_p$ of $\mathcal{D}$. From Lemma 3.1 there exists a base $x \in \mathrm{B}(f)$ such that

$$x(A_i) = f(A_i) \quad (i = 1, 2, \cdots, p). \tag{3.65}$$

Then for any base $y \in \mathrm{B}(f)$ we have from (3.63)$\sim$(3.65)

$$\begin{aligned}
&\sum_{e \in E} w(e)y(e) - \sum_{e \in E} w(e)x(e) \\
&= \sum_{i=1}^{p} \sum_{e \in A_i - A_{i-1}} w_i(y(e) - x(e)) \\
&= \sum_{i=1}^{p} \{w_i(y(A_i) - x(A_i)) - w_i(y(A_{i-1}) - x(A_{i-1}))\} \\
&= \sum_{i=1}^{p-1} (w_i - w_{i+1})(y(A_i) - x(A_i)) + w_p(y(A_p) - x(A_p)) \\
&= \sum_{i=1}^{p-1} (w_{i+1} - w_i)(f(A_i) - y(A_i)) \\
&\geq 0,
\end{aligned} \tag{3.66}$$

where we define $A_0 = \emptyset$, and recall $A_p = E$. This shows the optimality of x.

The "only if" part: Suppose that w is not a monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$, i.e., for some k $(1 \leq k < p)$ A_k defined by (3.64) does not belong to $\mathcal{D}$. Then there exist elements $e \in A_k$ and $e' \in E - A_k$ such that for any $X \in \mathcal{D}$ with $e \in X$ we have $e' \in X$. Hence, for any base $x \in \mathrm{B}(f)$ we have $\tilde{c}(x, e, e') = +\infty$ and $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(f)$ for any $\alpha > 0$. Since $w(e) < w(e')$, Problem P_w does not have a finite optimal solution. Q.E.D.

Corollary 3.14: *Let $P_w{}'$ be the problem given by P_w where $\mathrm{B}(f)$ is replaced by $\mathrm{P}(f)$. Problem $P_w{}'$ has a finite optimal solution if and only if $w\colon E \to \mathbf{R}$ is a nonpositive monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$.*

(Proof) The nonpositiveness is imposed on w since for any $x \in \mathrm{P}(f)$ and $e \in E$ we have $x - \alpha\chi_e \in \mathrm{P}(f)$ for an arbitrary $\alpha \in \mathbf{R}_+$. For any nonpositive w, $P_w{}'$ has a finite optimal solution if and only if P_w has a finite optimal solution. Therefore, the present corollary follows from Theorem 3.13. Q.E.D.

Furthermore, we have the following

Theorem 3.15: *Suppose that w is a monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$, i.e., the sets A_i $(i = 1, 2, \cdots, p)$ defined by (3.64) form a chain of $\mathcal{D}$. For each $i = 1, 2, \cdots, p$ let f_i be the rank function of the set minor $(\mathcal{D}, f) \cdot A_i/A_{i-1}$, where $A_0 = \emptyset$. Then the set of all the optimal solutions of Problem P_w is given by*

$$\begin{aligned} &\mathrm{B}(f_1) \oplus \mathrm{B}(f_2) \oplus \cdots \oplus \mathrm{B}(f_p) \\ &= \{x_1 \oplus x_2 \oplus \cdots \oplus x_p \mid x_i \in \mathrm{B}(f_i) \ (i = 1, 2, \cdots, p)\}, \end{aligned} \tag{3.67}$$

where the direct sum $\oplus$ is defined by (3.5). That is, $x \in \mathrm{B}(f)$ is an optimal solution of P_w if and only if x restricted on $A_i - A_{i-1}$ is a base of $(\mathcal{D}, f) \cdot A_i/A_{i-1}$ for each $i = 1, 2, \cdots, p$.

(Proof) It follows from the proof of the "if" part of Theorem 3.13 that $x \in \mathrm{B}(f_1) \oplus \cdots \oplus \mathrm{B}(f_p)$ is an optimal solution of Problem P_w.

On the other hand, if x is an optimal solution, then we must have $\mathrm{dep}(x, e) \subseteq A_i$ for each $i = 1, 2, \cdots, p$ and $e \in A_i$. This implies

$x(A_i) = f(A_i)$ $(i = 1, 2, \cdots, p)$ since $A_i = \bigcup\{\mathrm{dep}(x, e) \mid e \in A_i\}$ (use Lemma 2.2). Therefore, we have $x \in \mathrm{B}(f_1) \oplus \cdots \oplus \mathrm{B}(f_p)$ due to Lemma 3.1. Q.E.D.

Theorem 3.16: *A base $x \in \mathrm{B}(f)$ is an optimal solution of Problem P_w if and only if for each $e,\ e' \in E$ such that $e' \in \mathrm{dep}(x, e)$ we have*

$$w(e) \geq w(e'). \tag{3.68}$$

(Proof) The "only if" part is trivial. The "if" part follows from Theorem 3.15. For, if (3.68) holds for each $e,\ e' \in E$ such that $e' \in \mathrm{dep}(x, e)$, then we have (3.65) for A_i $(i = 1, 2, \cdots, p)$ defined by (3.64). (Note that $A_i = \bigcup\{\mathrm{dep}(x, e) \mid e \in A_i\}$ for $i = 1, 2, \cdots, p$, and use Lemma 2.2.) Q.E.D.

Theorem 3.16 says that the local optimality with respect to (elementary) transformations from x to $x + \alpha(\chi_e - \chi_{e'})$ $(e \in E,\ e' \in \mathrm{dep}(x, e),\ \alpha \geq 0)$ implies the global optimality.

The proof of Theorem 3.13 provides us with an algorithm, called a *greedy algorithm*, for solving Problem P_w (see [Fuji+Tomi83]). The greedy algorithm was given by Edmonds [Edm70] for polymatroids and by Lovász [Lovász83] for submodular systems with $\mathcal{D} = 2^E$ [([Gale68] for matroids)].

A sequence $(e_1, e_2, \cdots, e_n)$ of all the elements of E (a linear or total ordering of E) is called a *linear extension* of $\mathcal{P} = (E, \preceq)$ if $e_i \preceq e_j$ implies $i \leq j$ $(i,\ j = 1,\ 2,\ \cdots,\ n)$. Furthermore, a linear extension $(e_1, e_2, \cdots, e_n)$ of $\mathcal{P} = (E, \preceq)$ is called *monotone nondecreasing* with respect to w: $E \to \mathbf{R}$ if $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_n)$. Such a monotone nondecreasing linear extension of $\mathcal{P} = (E, \preceq)$ exists if and only if w is a monotone nondecreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$.

Suppose we are given a monotone nondecreasing weight function w from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$.

Greedy algorithm I

Step 1: Find a monotone nondecreasing linear extension $(e_1, e_2, \cdots, e_n)$ of $\mathcal{P} = (E, \preceq)$ with respect to w.

Step 2: Compute a vector $x \in \mathbf{R}^E$ by

$$x(e_i) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \cdots, n), \tag{3.69}$$

where for each $i = 1, 2, \cdots, n$ S_i is the set of the first i elements of $(e_1, e_2, \cdots, e_n)$ and $S_0 = \emptyset$. Then x is a minimum-weight base of $(\mathcal{D}, f)$ with respect to weight w.

(End)

It should be noted that $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ is a maximal chain of $\mathcal{D}$ containing A_i $(i = 1, 2, \cdots, p)$ defined by (3.64) and that conversely such a maximal chain of $\mathcal{D}$ gives a monotone nondecreasing linear extension $(e_1, e_2, \cdots, e_n)$ of $\mathcal{P}$ by $\{e_i\} = S_i - S_{i-1}$ $(i = 1, 2, \cdots, n)$. Also note that due to Theorem 3.15 every extreme minimum-weight base can be obtained by the greedy algorithm by appropriately choosing a monotone nondecreasing linear extension $(e_1, e_2, \cdots, e_n)$ of $\mathcal{D}$ in Step 1.

Corollary 3.17: *Problem P_w has a unique optimal solution if w: $E \to \mathbf{R}$ is a one-to-one monotone increasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$.*

Theorem 3.18: *Let $f: \mathcal{D} \to \mathbf{R}$ be a function on a simple distributive lattice $\mathcal{D} \subseteq 2^E$ such that $\emptyset$, $E \in \mathcal{D}$ and $f(\emptyset) = 0$. Define*

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}\colon\ x(X) \leq f(X),\ x(E) = f(E)\}. \tag{3.70}$$

Then, the greedy algorithm described above works for $\mathrm{B}(f)$ *defined by* (3.70) *if and only if f is a submodular function on $\mathcal{D}$.*

(Proof) It suffices to show the "only if" part. Suppose that the greedy algorithm works for $\mathrm{B}(f)$. Then, for each maximal chain

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{3.71}$$

of $\mathcal{D}$ the vector $x \in \mathbf{R}^E$ defined by (3.69) belongs to $\mathrm{B}(f)$. For any incomparable X, $Y \in \mathcal{D}$ choose a maximal chain $\mathcal{C}$ of (3.71) containing $X \cap Y$ and $X \cup Y$ and define x by (3.69). Since $x \in \mathrm{B}(f)$, by the definition of x we have

$$x(X) \leq f(X),\ x(Y) \leq f(Y),\ x(X \cup Y) = f(X \cup Y),\ x(X \cap Y) = f(X \cap Y). \tag{3.72}$$

Hence we have

$$\begin{aligned} f(X) + f(Y) &\geq x(X) + x(Y) \\ &= x(X \cup Y) + x(X \cap Y) \\ &= f(X \cup Y) + f(X \cap Y). \end{aligned} \tag{3.73}$$

It follows that f is a submodular function on $\mathcal{D}$. Q.E.D.

The greediness of the algorithm may not be seen from the above description of the algorithm. We can also get the same optimal solution x as follows.

Let x_0 be a vector in $\mathbf{R}^E$ with sufficiently small (possibly negative) components $x_0(e)$ $(e \in E)$ such that $f - x_0 \colon \mathcal{D} \to \mathbf{R}$ is monotone nondecreasing. (This is satisfied if and only if $x_0 \leq \underline{\alpha}$, where $\underline{\alpha}$ will be defined by (3.95).)

Greedy algorithm II

Step 1: Find a monotone nondecreasing linear extension $(e_1, e_2, \cdots, e_n)$ of $\mathcal{P} = (E, \preceq)$ with respect to w. Put $x \leftarrow x_0$.

Step 2: For each $i = 1, 2, \cdots, n$ put

$$x \leftarrow x + \hat{c}(x, e_i)\chi_{e_i}. \tag{3.74}$$

Then x is a minimum-weight base of $(\mathcal{D}, f)$ with respect to weight w.
(End)

This algorithm is a coordinate-wise "steepest descent" method, which shows the greediness of the algorithm. The following theorem shows the validity of the algorithm.

Theorem 3.19: *Suppose that the same linear extension is chosen in Step* 1 *of greedy algorithms* I *and* II. *Then, starting from x_0 such that $f - x_0$: $\mathcal{D} \to$ $\mathbf{R}$ is monotone nondecreasing, greedy algorithm* II *finds the same minimum-weight base of $(\mathcal{D}, f)$ with respect to w that is obtained by greedy algorithm* I.

(Proof) It suffices to show that the finally obtained x satisfies (3.69). We show this by induction. Since $f - x_0$ is monotone nondecreasing, the minimum of

$$\min\{f(X) - x_0(X) \mid e_1 \in X \in \mathcal{D}\}(= \hat{c}(x_0, e_1)) \tag{3.75}$$

is attained by $\{e_1\}$ $(\in \mathcal{D})$. It follows that (3.69) is satisfied for $i = 1$. Suppose that we have just finished the i_0th cycle in Step 2 of greedy algorithm II for some i_0 with $1 \leq i_0 \leq n - 1$ and that (3.69) holds for $i = 1, 2, \cdots, i_0$. Let $\tilde{x}$ be the current x. Since $f(S_{i_0}) = \tilde{x}(S_{i_0})$, we have for any $X \in \mathcal{D}$

$$f(X) - \tilde{x}(X)$$

$$
\begin{aligned}
&= f(X) - \tilde{x}(X) + f(S_{i_0}) - \tilde{x}(S_{i_0}) \\
&\geq f(X \cup S_{i_0}) - \tilde{x}(X \cup S_{i_0}) + f(X \cap S_{i_0}) - \tilde{x}(X \cap S_{i_0}) \\
&\geq f(X \cup S_{i_0}) - \tilde{x}(X \cup S_{i_0}).
\end{aligned} \tag{3.76}
$$

Therefore, the minimum of

$$\min\{f(X) - \tilde{x}(X) \mid e_{i_0+1} \in X \in \mathcal{D}\}(= \hat{c}(\tilde{x}, e_{i_0+1})) \tag{3.77}$$

is attained by some $X \in \mathcal{D}$ such that $S_{i_0} \subseteq X$. For $X \in \mathcal{D}$ such that $S_{i_0} \subseteq X$,

$$f(X) - \tilde{x}(X) = f(X) - x_0(X) + x_0(S_{i_0}) - f(S_{i_0}). \tag{3.78}$$

Because of the monotonicity of $f - x_0$, the minimum of (3.78) for $X \in \mathcal{D}$ such that $S_{i_0} \subseteq X$ and $e_{i_0+1} \in X$ is attained by $X = S_{i_0} \cup \{e_{i_0+1}\} = S_{i_0+1}$. Consequently, (3.69) also holds for $i = i_0 + 1$. Q.E.D.

The linear programming dual of Problem P_w in (3.56) is given by

$$P_w{}^*: \quad \text{Maximize} \sum_{X \in \mathcal{D}} \lambda(X) f(X) \tag{3.79a}$$

$$\text{subject to } \forall e \in E\colon \sum\{\lambda(X) \mid X \in \mathcal{D},\ e \in X\} = w(e), \tag{3.79b}$$

$$\forall X \in \mathcal{D} - \{E\}\colon\ \lambda(X) \leq 0. \tag{3.79c}$$

For an optimal solution x obtained by the greedy algorithm we have

$$
\begin{aligned}
&\sum_{e \in E} w(e)x(e) \\
&= \sum_{i=1}^{p} w_i(f(A_i) - f(A_{i-1})) \\
&= \sum_{i=1}^{p-1} (w_i - w_{i+1}) f(A_i) + w_p f(E),
\end{aligned} \tag{3.80}
$$

where w_i, A_i $(i = 1, 2, \cdots, p)$ are defined by (3.63) and (3.64) and $A_0 = \emptyset$. Define

$$
\begin{aligned}
\lambda(A_i) &= w_i - w_{i+1} \quad (i = 1, 2, \cdots, p-1), && (3.81) \\
\lambda(E) &= w_p && (3.82)
\end{aligned}
$$

and $\lambda(X) = 0$ for other $X \in \mathcal{D}$. Then λ is a dual feasible solution and it follows from (3.80) that the values of the objective functions of the dual problems coincide with each other. Hence λ given above is an optimal solution of the dual problem $P_w{}^*$. From (3.81) and (3.82), for any *integral* w such that the primal problem P_w has an optimal solution there exists an *integral* optimal solution of the dual problem $P_w{}^*$. That is, we have proved the following.

Theorem 3.20: *The system of inequalities and an equation given by*

$$\begin{aligned} x(X) &\leq f(X) \quad (X \in \mathcal{D} - \{E\}), && (3.83) \\ x(E) &= f(E) && (3.84) \end{aligned}$$

is totally dual integral for any submodular system $(\mathcal{D}, f)$ on E.

Since the system of (3.83) and (3.84) is totally dual integral, if the rank function f is integer-valued, each face of $\mathrm{B}(f)$ contains at least one integral vector and in particular, each vertex is integral ([Hoffman74], [Edm+Giles77]), which can also be seen from the greedy algorithm (cf. Theorem 3.22 below).

Corollary 3.21: *The system of inequalities*

$$x(X) \leq f(X) \quad (X \in \mathcal{D}) \tag{3.85}$$

is totally dual integral for any submodular system $(\mathcal{D}, f)$.

(Proof) The proof is similar to that of Theorem 3.20. Use Corollary 3.14. Q.E.D.

3.3. Structures of Base Polyhedra

Suppose that $(\mathcal{D}, f)$ is a simple submodular system on E.

(a) Extreme points and rays

From the greedy algorithm and Corollary 3.17 we have

Theorem 3.22 (The extreme point theorem) [Fuji+Tomi83]: *A base $x \in \mathrm{B}(f)$ is an extreme point of the base polyhedron $\mathrm{B}(f)$ if and only if for a maximal chain*

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{3.86}$$

of $\mathcal{D}$ we have

$$x(e_i) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \cdots, n), \tag{3.87}$$

where $\{e_i\} = S_i - S_{i-1}$.

Theorem 3.22, when $\mathcal{D} = 2^E$, has been shown in [Edm70], [Shapley71] and [Lovász83].

Define a vector $\overline{\alpha} \in \mathbf{R}^E$ by

$$D(e) = \bigcap\{X \mid e \in X \in \mathcal{D}\}, \tag{3.88}$$

$$\overline{\alpha}(e) = f(D(e)) - f(D(e) - \{e\}) \quad (e \in E). \tag{3.89}$$

Then, by the submodularity of f we have

$$f(X) - f(X - \{e\}) \leq f(D(e)) - f(D(e) - \{e\}) \tag{3.90}$$

for any $e \in E$ and $X \in \mathcal{D}$ such that $e \in X$ and $X - \{e\} \in \mathcal{D}$. Hence, from Theorem 3.22, for any $e \in E$ $\overline{\alpha}(e)$ is the maximum value of the eth component of extreme bases of $(\mathcal{D}, f)$, i.e., $\overline{\alpha}$ is the least upper bound (or the join) of all the extreme points of $\mathrm{B}(f)$ in the vector lattice $\mathbf{R}^E$. In particular, for each $X \in \mathcal{D}$,

$$f(X) \leq \overline{\alpha}(X), \tag{3.91}$$

since there is an extreme base $x \in \mathrm{B}(f)$ such that $x(X) = f(X)$. Because of (3.91), $\overline{\alpha}$ can be used for estimating an upper bound of f, i.e.,

$$\begin{aligned} \max\{f(X) \mid X \in \mathcal{D}\} &\leq \max\{\overline{\alpha}(X) \mid X \in \mathcal{D}\} \\ &\leq \sum\{\overline{\alpha}(e) \mid e \in E,\ \overline{\alpha}(e) > 0\}. \end{aligned} \tag{3.92}$$

Also note that, given any base $x \in \mathrm{B}(f)$, a lower bound of f is computed as

$$\min\{f(X) \mid X \in \mathcal{D}\} \geq \min\{x(X) \mid X \in \mathcal{D}\} \geq \sum\{x(e) \mid e \in E,\ x(e) < 0\}. \tag{3.93}$$

Moreover, denote by $\underline{\alpha}$ the greatest lower bound (or the meet) of all the extreme points of $\mathrm{B}(f)$ in the vector lattice $\mathbf{R}^E$. $\underline{\alpha}$ is given similarly as (3.88) and (3.89) in a dual form:

$$D^*(e) = \bigcup\{X \mid e \notin X \in \mathcal{D}\} \quad (e \in E), \tag{3.94}$$

$$\underline{\alpha}(e) = f(D^*(e) \cup \{e\}) - f(D^*(e)) \quad (e \in E). \tag{3.95}$$

Lemma 3.23: *The rank function f of a simple submodular system $(\mathcal{D}, f)$ is monotone nondecreasing if and only if $\underline{\alpha} \geq \mathbf{0}$. In other words, f is monotone nondecreasing if and only if every extreme point of the base polyhedron $\mathrm{B}(f)$ belongs to the nonnegative orthant $\mathbf{R}^E_+$.*

(Proof) If f is monotone nondecreasing, then we have $\underline{\alpha} \geq \mathbf{0}$ by definition (3.95). Conversely, if $\underline{\alpha} \geq \mathbf{0}$, we have for any $e \in E$ and $X \in \mathcal{D}$ with $e \notin X$ and $X \cup \{e\} \in \mathcal{D}$,

$$f(X \cup \{e\}) - f(X) \geq f(D^*(e) \cup \{e\}) - f(D^*(e)) = \underline{\alpha}(e) \geq 0, \tag{3.96}$$

from which follows the monotonicity of f. We see from (3.96) that the minimum value of the eth component of bases of $(\mathcal{D}, f)$ is equal to $\underline{\alpha}(e)$. Q.E.D.

Lemma 3.24: *The rank function f of a simple submodular system $(\mathcal{D}, f)$ is a modular function if and only if for an extreme base x of $(\mathcal{D}, f)$ we have $x = \overline{\alpha}(= \underline{\alpha})$.*

(Proof) Since $(\mathcal{D}, f)$ is simple, the rank function f is modular if and only if $\mathrm{B}(f)$ has one and only one extreme base. The present lemma follows from this fact and the definitions of $\overline{\alpha}$ and $\underline{\alpha}$. Q.E.D.

It follows from Theorem 3.22 (also from Theorem 3.20) that if the rank function f of $(\mathcal{D}, f)$ is integer-valued, every extreme point of $\mathrm{B}(f)$ is integral. In particular, we have the following polyhedral characterization of matroids due to Edmonds [Edm70].

Corollary 3.25 [Edm70]: *For a matroidal submodular system $(2^E, \rho)$, where ρ is the rank function of a matroid $\mathbf{M}$ on E, the base polyhedron $\mathrm{B}(\rho)$ is the convex hull of the characteristic vectors of all the bases of the matroid $\mathbf{M}$.*

From Theorem 2.5 we can easily see that if f is a nonnegative integer-valued crossing-submodular function on a crossing family $\mathcal{F}$ and if the polyhedron given by

$$\mathrm{B}(f_{\mathbf{0}}^{\mathbf{1}}) = \{x \mid x \in \mathrm{B}(f),\ \forall e \in E{:}\, 0 \leq x(e) \leq 1\} \tag{3.97}$$

is nonempty, then $\mathrm{B}(f_{\mathbf{0}}^{\mathbf{1}})$ is the integral base polyhedron obtained by the reduction of $\mathrm{B}(f)$ by vector $\mathbf{1} = (\mathbf{1}(e) = 1 \mid e \in E)$ and the contraction by zero vector $\mathbf{0}$ and is a base polyhedron of a matroid. Hence, from Corollary 3.25 we see that

$$\{X \mid X \subseteq E,\ \forall Y \in \mathcal{F}{:}\ |X \cap Y| \leq f(Y),\ |X| = f(E)\} \tag{3.98}$$

is a family of bases of a matroid. This is a result by A. Frank and É. Tardos [Frank+Tardos81].

If the rank function f of a simple submodular system $(\mathcal{D}, f)$ is integer-valued and has the unit-increase property (i.e., for any $X,\ Y \in \mathcal{D}$ with $X \subseteq Y$ and $|X| + 1 = |Y|$ we have $f(Y) = f(X)$ or $f(Y) = f(X) + 1$), then all the extreme points of $\mathrm{B}(f)$ are $\{0,1\}$-vectors. Therefore, we can develop a matroid-like theory for such a submodular system $(\mathcal{D}, f)$, which corresponds to Faigle's geometry on the poset $\mathcal{P} = (E, \preceq)$ with $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ [Faigle79,80]. In fact, Theorem 3.22 gives a polyhedral characterization of the family of bases of Faigle's geometry.

Next, denote the characteristic cone of the base polyhedron $\mathrm{B}(f)$ by $\mathrm{C}(f)$, which is given by

$$\mathrm{C}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}{:}\ x(X) \leq 0,\ x(E) = 0\}. \tag{3.99}$$

Let $G = (E, A)$ be the graph with vertex set E and arc set A which represents the Hasse diagram of the poset $\mathcal{P} = (E, \preceq)$, i.e., $(e, e') \in A$ if and only if e covers e' (or $e' \prec e$ and there is no element $e'' \in E$ such that $e' \prec e'' \prec e$). Also define a capacity function c on A by

$$c(a) = +\infty \qquad (a \in A). \tag{3.100}$$

Then we easily see that $\mathrm{C}(f)$ in (3.99) coincides with the set of the boundaries $\partial\varphi$ of nonnegative flows φ in the network $\mathcal{N} = (G = (E, A), c)$. (Note that the cut function κ_c associated with network $\mathcal{N} = (G = (E, A), c)$ satisfies (i) $\kappa_c(X) = 0$ if X is an ideal of $\mathcal{P}$ and (ii) $\kappa_c(X) = +\infty$ otherwise. Hence, $\mathrm{C}(f)$ in (3.99) is the set of the boundaries $\partial\varphi$ of feasible flows (i.e., nonnegative flows) φ in $\mathcal{N}$.) Consequently, we have

Theorem 3.26 (The extreme ray theorem) [Tomi83]: *The extreme rays of the characteristic cone* $\mathrm{C}(f)$ *of the base polyhedron* $\mathrm{B}(f)$ *are exactly those represented by the vectors*

$$\chi_e - \chi_{e'} \tag{3.101}$$

for all $e,\ e' \in E$ *such that* e *covers* e' *in* $\mathcal{P} = (E, \preceq)$.

(Proof) Since the characteristic cone $\mathrm{C}(f)$ is the set of the boundaries of nonnegative flows in $\mathcal{N} = (G = (E, A), c)$ defined above, $\mathrm{C}(f)$ is generated by vectors $\chi_e - \chi_{e'}$ such that $(e, e') \in A$. Moreover, vector $\chi_e - \chi_{e'}$ for an arc $(e, e') \in A$ can not be expressed as a nonnegative linear combination of the other vectors $\chi_{e_1} - \chi_{e_1'}$ with $(e_1, e_1') \in A - \{(e, e')\}$. Q.E.D.

Theorem 3.13 characterizes the dual cone $\mathrm{C}^*(f)$ of $\mathrm{C}(f)$, where

$$\mathrm{C}^*(f) = \{y \mid y \in \mathbf{R}^E,\ \forall x \in \mathrm{C}(f)\colon \sum_{e \in E} x(e)y(e) \leq 0\}. \tag{3.102}$$

Theorem 3.13 says that $-\mathrm{C}^*(f)$ consists of monotone nondecreasing functions from $\mathcal{P} = (E, \prec)$ to $(\mathbf{R}, \leq)$ or that $\mathrm{C}^*(f)$ consists of monotone nonincreasing functions from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$. Therefore, $\mathrm{C}^*(f)$ is generated by the characteristic vectors of the (lower) ideals of $\mathcal{P}$ and $-\chi_E$.

(b) Elementary transformations of bases

For any base x of submodular system $(\mathcal{D}, f)$ on E and for $e,\ e' \in E$ such that $e' \in \mathrm{dep}(x, e) - \{e\}$ we have

$$x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(f) \qquad (0 \leq \alpha \leq \tilde{c}(x, e, e')). \tag{3.103}$$

The transformation of base $x \in \mathrm{B}(f)$ into such a base $x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(f)$ is called an *elementary transformation* of base $x \in \mathrm{B}(f)$.

The following theorem is important from an algorithmic point of view. For any real number α, $\lfloor \alpha \rfloor$ denotes the maximum integer less than or equal to α.

Theorem 3.27: *For any two bases* $x,\ y \in \mathrm{B}(f)$ *base* x *can be transformed into base* y *by at most* $\lfloor |E|^2/4 \rfloor$ *repeated elementary transformations such that each component* $x(e)$ *with* $x(e) < y(e)$ *monotonically increases and each component* $x(e)$ *with* $x(e) > y(e)$ *monotonically decreases.*

(Proof) Consider the following algorithm.

1° If $x = y$, then stop.

2° Choose any element $e \in E$ such that $x(e) < y(e)$.

3° Choose any element $e' \in \mathrm{dep}(x, e)$ such that $x(e') > y(e')$. Put $\alpha \leftarrow \min\{y(e) - x(e),\ x(e') - y(e'),\ \tilde{c}(x, e, e')\}$ and $x \leftarrow x + \alpha(\chi_e - \chi_{e'})$.

4° If $\alpha < y(e) - x(e)$, then go to Step 3°.

Otherwise ($\alpha = y(e) - x(e)$) go to Step 1°.

Note that if $x \neq y$, there is an element e such that $x(e) < y(e)$ and that if $x(e) < y(e)$, there is an element $e' \in \mathrm{dep}(x, e)$ such that $x(e') > y(e')$, since otherwise, putting $X = \mathrm{dep}(x, e)$, we would have $f(X) = x(X) < y(X)$, a contradiction. Also note that if $\alpha < y(e) - x(e)$ in Step 4°, we have $\alpha = \min\{x(e') - y(e'),\ \tilde{c}(x, e, e')\}$ and the number of elements in $\{e'' \mid e'' \in \mathrm{dep}(x, e),\ x(e'') > y(e'')\}$ decreases by at least one after the elementary transformation $x \leftarrow x + \alpha(\chi_e - \chi_{e'})$. Therefore, the case where $\alpha < y(e) - x(e)$ is repeated at most $|S^-|$ times, where $S^- = \{e \mid e \in E,\ x(e) > y(e)\}$ for the initial base x. Defining $S^+ = \{e \mid e \in E,\ x(e) < y(e)\}$ for the initial x, the total number of the elementary transformations is at most $|S^+| \times |S^-|$, which is bounded by $\lfloor |E|^2/4 \rfloor$. Q.E.D.

Consider a capacitated network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ with an underlying graph G and lower and upper capacity functions $\underline{c},\ \overline{c}\colon A \to \mathbf{R}$ with $\underline{c} \leq \overline{c}$. Let $\kappa_{\underline{c},\overline{c}}\colon 2^V \to \mathbf{R}$ be the cut function associated with the network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ (see (2.62) in Section 2.3). The set of the boundaries of feasible flows in $\mathcal{N}$ is the base polyhedron associated with the submodular system $(2^V, \kappa_{\underline{c},\overline{c}})$. Note that there exists a feasible circulation (a feasible flow φ such that $\partial\varphi = \mathbf{0}$) in $\mathcal{N}$ if and only if $\mathbf{0} \in \mathrm{B}(\kappa_{\underline{c},\overline{c}})$. Since $\partial\underline{c} \in \mathrm{B}(\kappa_{\underline{c},\overline{c}})$, there exists a feasible circulation in $\mathcal{N}$ if and only if the base $\partial\underline{c} \in \mathrm{B}(\kappa_{\underline{c},\overline{c}})$ is transformed into $\mathbf{0}$ by repeated elementary transformations as in Theorem 3.27. A standard algorithm for finding a feasible circulation by the use of a max-flow algorithm consists of such repeated elementary transformations (cf. [Hoffman60], [Ford+Fulkerson62]).

For a (directed) graph $G = (V, A)$ and a $\{0, 1\}$-valued function $\varphi\colon A \to \{0, 1\}$ define the graph G_φ as the one obtained from G by reorienting arcs $a \in A$ such that $\varphi(a) = 1$. We say φ *defines the reorientation* G_φ. A graph $G = (V, A)$ is *strongly k-connected* (k: a positive integer) if for each nonempty proper subset U of vertex set V there exist at least k arcs from

U to $V - U$. We call φ: $A \to \{0, 1\}$ a *strongly k-connected reorientation* of G if G_φ is strongly k-connected. Define a capacity function c: $A \to \mathbf{R}$ by

$$c(a) = 1 \qquad (a \in A), \tag{3.104}$$

and let κ: $2^V \to \mathbf{R}$ be the associated cut function, i.e., $\kappa(U) = c(\Delta^+U) = |\Delta^+U|$ for $U \subseteq V$. Also define

$$\kappa^{(k)}(U) = \begin{cases} \kappa(U) - k & (U \in 2^V - \{\emptyset, V\}) \\ 0 & (U \in \{\emptyset, V\}). \end{cases} \tag{3.105}$$

Then $\kappa^{(k)}: 2^V \to \mathbf{R}$ is a crossing-submodular function on 2^V and defines a base polyhedron $\mathrm{B}(\kappa^{(k)})$, if $\mathrm{B}(\kappa^{(k)}) \neq \emptyset$, due to Theorem 2.5. $\mathrm{B}(\kappa^{(k)})$ is called the *k-abridgment* of $\mathrm{B}(\kappa)$ in [Tomi81b, 81c]. It can easily be seen that

$$\mathrm{B}(\kappa^{(k)}) = \{\partial\varphi \mid \varphi \text{ defines a strongly } k\text{-connected reorientation of } G\}. \tag{3.106}$$

Here, we assume that the underlying totally ordered additive group $\mathbf{R}$ is the set $\mathbf{Z}$ of integers. Also note that the strong k-connectedness of a reorientation of G depends only on the boundary $\partial\varphi$ of φ which defines the reorientation. Because of this fact and Theorem 3.27 we obtain a theorem of Frank [Frank82a]:

> "Let G' and G'' be strongly k-connected reorientations of G. Then there exists a sequence of strongly k-connected reorientations $G' = G_0,\ G_1,\ \cdots,\ G_m = G''$ of G such that for each $i = 1, 2, \cdots, m$ G_i is obtained by reorienting arcs in a directed path or a directed cycle in G_{i-1}."

Note that an elementary transformation of a base (or a boundary $\partial\varphi$) in $\mathrm{B}(\kappa^{(k)})$ corresponds to a transformation of the reorientation of G (defined by φ) by reversing the arcs in a directed path and possibly directed cycles and that reversing arcs in a directed cycle does not change the boundary $\partial\varphi$.

(c) Tangent cones

For any base $x \in \mathrm{B}(f)$ associated with submodular system $(\mathcal{D}, f)$ on E the *tangent cone* of $\mathrm{B}(f)$ at x, denoted by $\mathrm{TC}(\mathrm{B}(f), x)$, is defined by

$$\mathrm{TC}(\mathrm{B}(f), x) = \{\lambda y \mid \lambda \geq 0,\ y \in \mathbf{R}^E,\ x + y \in \mathrm{B}(f)\}. \tag{3.107}$$

Here, the underlying totally ordered additive group $\mathbf{R}$ is assumed to be the set of reals (or rationals).

Given a base $x \in \mathrm{B}(f)$, we call an ordered pair (e, e') of elements of E an *exchangeable pair* associated with x if $e' \in \mathrm{dep}(x, e) - \{e\}$.

Theorem 3.28: *The tangent cone* $\mathrm{TC}(\mathrm{B}(f), x)$ *of* $\mathrm{B}(f)$ *at a base* x *is generated by the set of the following vectors:*

$$\chi_e - \chi_{e'} \qquad (e \in E,\ e' \in \mathrm{dep}(x, e) - \{e\}). \tag{3.108}$$

In other words, for any vector $y \in \mathrm{TC}(\mathrm{B}(f), x)$ *there exist some nonnegative coefficients* $\lambda(e, e')$ *for exchangeable pairs* (e, e') *such that*

$$y = \sum \{\lambda(e, e')(\chi_e - \chi_{e'}) \mid (e, e')\text{: an exchangeable pair associated with } x\}. \tag{3.109}$$

(Proof) Let C be the cone generated by the vectors in (3.108). It follows from the definition of dependence function that

$$C \ \subseteq\ \mathrm{TC}(\mathrm{B}(f), x). \tag{3.110}$$

Suppose that there exists a vector $y \in \mathrm{TC}(\mathrm{B}(f), x) - C$. Then, by the separation theorem (Theorem 1.13) there exists a vector $w \in \mathbf{R}^E$ such that

$$\forall z \in C\colon \sum_{e \in E} w(e) z(e) \geq 0, \tag{3.111}$$

$$\sum_{e \in E} w(e) y(e) < 0. \tag{3.112}$$

From Theorem 3.16 and (3.111) the base x is a minimum-weight base with respect to the weight function w but (3.112) implies that for a sufficiently small $\alpha > 0$ $x + \alpha y$ is a base and the weight of the base $x + \alpha y$ is smaller than that of x. This is a contradiction, so that we must have $C = \mathrm{TC}(\mathrm{B}(f), x)$. Q.E.D.

A constructive proof of this theorem for polymatroids was given in [Fuji78a, Lemma 9]. It should also be noted that Theorems 3.26 and 3.28 are closely related. We can show Theorem 3.28 by using Theorem 3.26 and vice versa.

Suppose that $(\mathcal{D}, f)$ is a simple submodular system on E and that x is an extreme point of $\mathrm{B}(f)$. Then,

$$\mathcal{D}(x) = \{X \mid X \in \mathcal{D},\ x(X) = f(X)\} \tag{3.113}$$

is also a simple distributive lattice since the length of a maximal chain of $\mathcal{D}(x)$ is equal to $|E|$ due to Theorem 3.22. This fact is crucial in the following argument. Let $\mathcal{P}(x) = (E, \preceq_x)$ be the poset representing $\mathcal{D}(x)$. An efficient method for finding $\mathrm{dep}(x, e)$ for all $e \in E$ is shown in [Bixby+Cunningham+Topkis85] (for polymatroids). Their algorithm also discerns whether a given vector is an extreme base. The algorithm adapted for submodular systems is given as follows.

Let x be any vector in $\mathbf{R}^E$.

Algorithm

Step 1: Put $S \leftarrow \emptyset$.

Step 2: For each $i = 1, 2, \cdots, |E|$ do the following (2-1) and (2-2).

(2-1) If there exists no element $e \in E - S$ such that $S \cup \{e\} \in \mathcal{D}$ and $x(S \cup \{e\}) = f(S \cup \{e\})$, then stop ($x$ is not an extreme base).
Otherwise find one such element $e \in E - S$ and put $S \leftarrow S \cup \{e\}$ and $e_i \leftarrow e$.

(2-2) Put $T \leftarrow S$.
For each $j = 1, 2, \cdots, i - 1$ do the following $(*)$.
$(*)$ If $x(T - \{e_{i-j}\}) = f(T - \{e_{i-j}\})$, then put $T \leftarrow T - \{e_{i-j}\}$.
Put $\mathrm{dep}(x, e_i) \leftarrow T$.

(End)

If x is an extreme base, then $\mathcal{D}(x) = \mathbf{2}^{\mathcal{P}(x)}$ with $\mathcal{P}(x) = (E, \preceq_x)$. Repeating Step (2-1), we have a maximal chain $S_0 \subset S_1 \subset \cdots \subset S_n$ $(n = |E|)$ of $\mathcal{D}(x)$, where $S_i = \{e_1, e_2, \cdots, e_i\}$. Note that a maximal chain of $\mathcal{D}(x)$ is obtained by augmenting elements of an ideal S of $\mathcal{P}(x)$ one by one in Step (2-1). Conversely, if we find a chain $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ such that $x(S_i) = f(S_i)$ $(i = 1, 2, \cdots, n)$, then x is an extreme base due to Theorem 3.22. This validates Step (2-1).

We now show the validity of Step (2-2). Because of the above argument we can assume that the given x is an extreme base. For the current i at the beginning of an execution of Step (2-2) we have a chain $S_0 \subset S_1 \subset \cdots \subset S_i$. Note that

$$\mathrm{dep}(x, e_i) \subseteq S_i, \tag{3.114}$$

$$\mathrm{dep}(x, e_i) \cup S_{i-j} \in \mathcal{D}(x) \qquad (j = 0, 1, \cdots, i). \tag{3.115}$$

Since distinct members of (3.115) form a maximal chain from $\mathrm{dep}(x, e_i)$ to S_i, the final T obtained in the current Step (2-2) is $\mathrm{dep}(x, e_i)$.

The Hasse diagram $G(x) = (E, A(x))$ of $\mathcal{P}(x) = (E, \preceq_x)$ can be constructed by the use of $\mathrm{dep}(x, e_i)$ $(i = 1, 2, \cdots, n)$. We can also modify the above algorithm so as to construct the Hasse diagram of $\mathcal{P}(x) = (E, \preceq_x)$ while executing the algorithm. In Step 1 we put $G(x) = (E, \emptyset)$. Modify Step (2-2) as follows.

(2-2′) Put $T \leftarrow S$ and $W \leftarrow \emptyset$.
For each $j = 1, 2, \cdots, i-1$ do the following $(*1)$ and $(*2)$.
$(*1)$ If $e_{i-j} \notin W$ and $x(T - \{e_{i-j}\}) = f(T - \{e_{i-j}\})$,
then put $T \leftarrow T - \{e_{i-j}\}$.
$(*2)$ If $e_{i-j} \notin W$ and $x(T - \{e_{i-j}\}) < f(T - \{e_{i-j}\})$,
then add to $G(x)$ an arc from e_i to e_{i-j} and
put $W \leftarrow W \cup \mathrm{dep}(x, e_{i-j})$.
Put $\mathrm{dep}(x, e_i) \leftarrow T$.

It should be noted that e_i is adjacent to e_{i-j} in the Hasse diagram if and only if $e_{i-j} \in \mathrm{dep}(x, e_i)$ and e_{i-j} is a maximal element of $\mathrm{dep}(x, e_i) - \{e_i\}$ in $\mathcal{P}(x)$. Set W in Step (2-2′) is introduced due to the fact that $e_{i-j} \in \mathrm{dep}(x, e_i)$ implies $\mathrm{dep}(x, e_{i-j}) \subseteq \mathrm{dep}(x, e_i)$. Set W can also be incorporated into Step (2-2), which may reduce the number of evaluations of f.

(d) Faces, dimensions and connected components [Fuji84d]

Consider a simple submodular system $(\mathcal{D}, f)$ on E. In the following, $\mathbf{R}$ is the set of reals (or rationals).

For any $\mathcal{F} \subseteq \mathcal{D}$ define

$$\begin{aligned} \mathrm{F}(\mathcal{F}) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) = f(X),& \\ \forall X \in \mathcal{D} - \mathcal{F}\colon\ x(X) \le f(X)\},& \end{aligned} \tag{3.116}$$

$$\begin{aligned} \mathrm{F}^\circ(\mathcal{F}) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) = f(X),& \\ \forall X \in \mathcal{D} - \mathcal{F}\colon\ x(X) < f(X)\}.& \end{aligned} \tag{3.117}$$

Also, define

$$\mathbf{D} = \{\mathcal{D}_0 \mid \mathcal{D}_0 \text{ is a sublattice of } \mathcal{D} \text{ with } \emptyset, E \in \mathcal{D}_0,\ \mathrm{F}^\circ(\mathcal{D}_0) \neq \emptyset\}. \tag{3.118}$$

Lemma 3.29: *The collection* $\mathbf{D}$ *of sublattices of* $\mathcal{D}$ *defined by* (3.118) *is given by*

$$\mathbf{D} = \{\mathcal{D}(x) \mid x \in \mathrm{B}(f)\}, \tag{3.119}$$

where for each $x \in \mathrm{B}(f)$ $\mathcal{D}(x)$ *is defined by* (3.113).

(Proof) If $\mathcal{D}_0 \in \mathbf{D}$, then for any $x \in \mathrm{F}^\circ(\mathcal{D}_0)$ we must have $\mathcal{D}_0 = \mathcal{D}(x)$ by definition (3.117). Conversely, for any $x \in \mathrm{B}(f)$ $\mathcal{D}(x)$ is a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}(x)$ and $x \in \mathrm{F}^\circ(\mathcal{D}(x))$. Hence we have $\mathcal{D}(x) \in \mathbf{D}$. Q.E.D.

It should be noted that for each $\mathcal{D}_0 \in \mathbf{D}$ $\mathrm{F}(\mathcal{D}_0)$ is a face of $\mathrm{B}(f)$ and $\mathrm{F}^\circ(\mathcal{D}_0)$ is the relative interior of the face $\mathrm{F}(\mathcal{D}_0)$.

We see from Lemma 3.29 that $\mathbf{D}$ is the collection of *equality sets*, each given by $\mathcal{D}(x)$, for $\mathrm{B}(f)$ expressed by

$$x(X) \le f(X) \quad (X \in \mathcal{D}), \tag{3.120}$$

$$x(E) = f(E). \tag{3.121}$$

From Lemma 1.8 we can easily show

Theorem 3.30: *The collection* $\mathbf{F}$ *of all the nonempty faces of* $\mathrm{B}(f)$ *is given by* $\{\mathrm{F}(\mathcal{D}_0) \mid \mathcal{D}_0 \in \mathbf{D}\}$. *Also,*

(i) *If* $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$ *and* $\mathcal{D}_1 \ne \mathcal{D}_2$, *then* $\mathrm{F}(\mathcal{D}_1) \ne \mathrm{F}(\mathcal{D}_2)$.

(ii) *For any* $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$, $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *if and only if* $\mathrm{F}(\mathcal{D}_2) \subseteq \mathrm{F}(\mathcal{D}_1)$.

In other words, F *in* (3.116) *determines an anti-isomorphism from* $\mathbf{D}$ *onto* $\mathbf{F}$, *where* $\mathbf{D}$ *and* $\mathbf{F}$ *are considered as posets relative to set inclusion.*

$\mathbf{F}$ (or $\mathbf{F} \cup \{\emptyset\}$ when $\mathbf{F}$ does not have a unique minimal element) is the *face lattice* of $\mathrm{B}(f)$.

For two posets $\mathcal{P}_i = (E_i, \preceq_i)$ $(i = 1, 2)$ we say $\mathcal{P}_2$ is a *homomorphic image* of $\mathcal{P}_1$ if there exists a mapping ψ from E_1 onto E_2 such that $e \preceq_1 e'$ implies $\psi(e) \preceq_2 \psi(e')$ for all e, $e' \in E_1$. Also, a partition $P_2 = \{Y_j \mid j \in J\}$ of E is a *refinement* of a partition $P_1 = \{X_i \mid i \in I\}$ of E if for each $Y_j \in P_2$ there exists an $X_i \in P_1$ such that $Y_j \subseteq X_i$.

Lemma 3.31: *For any distributive lattices* $\mathcal{D}_i$ $(i = 1, 2)$ *with* $\emptyset$, $E \in \mathcal{D}_i$, *we have* $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *if and only if* $\Pi(\mathcal{D}_2)$ *is a refinement of* $\Pi(\mathcal{D}_1)$ *and* $\mathcal{P}(\mathcal{D}_1) =$

$(\Pi(\mathcal{D}_1), \preceq_{\mathcal{D}_1})$ is a homomorphic image of $\mathcal{P}(\mathcal{D}_2) = (\Pi(\mathcal{D}_2), \preceq_{\mathcal{D}_2})$ under the natural mapping (i.e., $T_2 \in \Pi(\mathcal{D}_2)$ is made to correspond to T_1 if $T_2 \subseteq T_1$).

(Proof) Suppose $\mathcal{D}_1 \subseteq \mathcal{D}_2$. For each $i = 1, 2$, distinct elements e and e' of E belong to different components of $\Pi(\mathcal{D}_i)$ if and only if there exists an $X \in \mathcal{D}_i$ such that $|\{e, e'\} \cap X| = 1$. Therefore, since $\mathcal{D}_1 \subseteq \mathcal{D}_2$, $\Pi(\mathcal{D}_2)$ is a refinement of $\Pi(\mathcal{D}_1)$. Also, since we have $T_2 \preceq_{\mathcal{D}_\in} T_2'$ if and only if $T_2' \subseteq X \in \mathcal{D}_2$ implies $T_2 \subseteq X$ and since $\mathcal{D}_1 \subseteq \mathcal{D}_2$, $T_2' \subseteq X \in \mathcal{D}_1$ implies $T_2 \subseteq X$ if $T_2 \preceq_{\mathcal{D}_2} T_2'$. Consequently, for T_1, $T_1' \in \Pi(\mathcal{D}_1)$ such that $T_2 \subseteq T_1$ and $T_2' \subseteq T_1'$, we have $T_1 \preceq_{\mathcal{D}_1} T_1'$.

The converse is easy. Q.E.D.

Theorem 3.32: *For $\mathcal{D}_0 \in \mathbf{D}$ we have*

$$\mathrm{dimF}(\mathcal{D}_0) = |E| - |\Pi(\mathcal{D}_0)|, \tag{3.122}$$

where $\mathrm{dimF}(\mathcal{D}_0)$ is the dimension of the face $\mathrm{F}(\mathcal{D}_0)$.

(Proof) The dimension of the face $\mathrm{F}(\mathcal{D}_0)$ is equal to that of the affine set

$$\mathrm{M}(\mathcal{D}_0) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}_0\colon\ x(X) = f(X)\}. \tag{3.123}$$

Since the rank of the coefficient matrix in the right-hand side of (3.123) is equal to $|\Pi(\mathcal{D}_0)|$, we have (3.122). Q.E.D.

It may be noted that the extreme point theorem (Theorem 3.22) and the extreme ray theorem (Theorem 3.26) easily follow from Theorems 3.30 and 3.32 and Lemma 3.31.

Lemma 3.33: *Suppose $\mathcal{D}_0 \in \mathbf{D}$ and let*

$$\mathcal{C}_0\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{3.124}$$

be a maximal chain of $\mathcal{D}_0$. Then,

$$\mathrm{F}(\mathcal{C}_0) = \mathrm{F}(\mathcal{D}_0). \tag{3.125}$$

(Proof) Since $\Pi(\mathcal{C}_0) = \Pi(\mathcal{D}_0)$, we have $x \in \mathrm{F}(\mathcal{C}_0)$ if and only if $x \in \mathrm{F}(\mathcal{D}_0)$. Q.E.D.

Theorem 3.34: *Suppose $\mathcal{D}_0 \in \mathbf{D}$. Then a base $x \in \mathrm{B}(f)$ is an extreme point of the face $\mathrm{F}(\mathcal{D}_0)$ if and only if, for a maximal chain*

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{3.126}$$

of $\mathcal{D}$ which contains a maximal chain of $\mathcal{D}_0$ as a subchain, x is given by

$$x(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \cdots, n). \tag{3.127}$$

(Proof) *The "if" part*: Let $\mathcal{C}_0$ be a maximal chain of $\mathcal{D}_0$ contained in chain $\mathcal{C}$ of (3.126). Then, from (3.127) $x \in \mathrm{F}(\mathcal{C}_{\prime})$. Hence, from Lemma 3.33, $x \in \mathrm{F}(\mathcal{D}_0)$. Since x is an extreme point of $\mathrm{B}(f)$ due to Theorem 3.22, x must be an extreme point of $\mathrm{F}(\mathcal{D}_0)$.

The "only if" part: For any extreme point x of $\mathrm{F}(\mathcal{D}_0)$ we have $\mathcal{D}_0 \subseteq \mathcal{D}(x) \in \mathbf{D}$. Since x is also an extreme point of $\mathrm{B}(f)$, i.e., $\{x\}$ is a zero-dimensional face $\mathrm{F}(\mathcal{D}(x))$ of $\mathrm{B}(f)$, a maximal chain of $\mathcal{D}(x)$ is a maximal chain of $\mathcal{D}$ due to Theorem 3.32. It follows that a maximal chain of $\mathcal{D}(x)$ which contains a maximal chain of $\mathcal{D}_0$ as a subchain is a desired maximal chain of $\mathcal{D}$. Q.E.D.

Also, extreme rays of a face of $\mathrm{B}(f)$ are characterized by the following.

Theorem 3.35: *Suppose $\mathcal{D}_0 \in \mathbf{D}$ and $\Pi(\mathcal{D}_0) = \{T_i \mid i \in I\}$. Then the set of all the extreme rays of the characteristic cone of the face $\mathrm{F}(\mathcal{D}_0)$ is given by*

$$\{\chi_e - \chi_{e'} \mid e = \partial^+ a,\ e' = \partial^- a,\ a \in A^*(\mathcal{D}),\ \exists i \in I\colon\ \{e, e'\} \subseteq T_i\}, \tag{3.128}$$

where $A^*(\mathcal{D})$ is the arc set of the Hasse diagram $\mathrm{H}(\mathcal{P}(\mathcal{D})) = (E, A^*(\mathcal{D}))$ representing the poset $\mathcal{P}(\mathcal{D}) = (E, \preceq_{\mathcal{D}})$.

(Proof) The characteristic cone $\mathrm{C}(\mathcal{D}_0)$ of the face $\mathrm{F}(\mathcal{D}_0)$ is given by

$$\begin{aligned} \mathrm{C}(\mathcal{D}_0) = \{x \mid x \in \mathbf{R}^E,\ & \forall X \in \mathcal{D}_0\colon\ x(X) = 0, \\ & \forall X \in \mathcal{D} - \mathcal{D}_0\colon\ x(X) \le 0\}. \end{aligned} \tag{3.129}$$

It follows from (3.129) and the extreme ray theorem (Theorem 3.26) that the set of the extreme rays of $\mathrm{C}(f)(= \mathrm{C}(\{\emptyset, E\}))$ which belong to $\mathrm{C}(\mathcal{D}_0)$ is exactly given by (3.128). Q.E.D.

We say that a submodular system $(\mathcal{D}, f)$ on E is *connected* if there is no nonempty proper subset X of E such that $X, E - X \in \mathcal{D}$ and $f(X) + f(E - X) = f(E)$.

Theorem 3.36: *A submodular system $(\mathcal{D}, f)$ on E is connected if and only if $\{\emptyset, E\} \in \mathbf{D}$, i.e., there exists a base $x \in \mathrm{B}(f)$ such that*

$$x(X) < f(X) \tag{3.130}$$

for any $X \in \mathcal{D}$ *with* $\emptyset \neq X \neq E$.

(Proof) *The "if" part*: If there exists a base $x \in \mathrm{B}(f)$ satisfying (3.130) for each $X \in \mathcal{D}$ with $\emptyset \neq X \neq E$, then

$$f(E) = x(E) = x(X) + x(E - X) < f(X) + f(E - X) \tag{3.131}$$

for each $X \in \mathcal{D}$ with $E - X \in \mathcal{D}$ and $\emptyset \neq X \neq E$.

The "only if" part: Suppose that $(\mathcal{D}, f)$ is connected. We have $\mathrm{B}(f) = \mathrm{F}(\mathcal{D}_0)$ for some $\mathcal{D}_0 \in \mathbf{D}$. We show $\mathcal{D}_0 = \{\emptyset, E\}$. Note that we have $\{\emptyset, E\} \subseteq \mathcal{D}_0$. Suppose that there exists some $Y \in \mathcal{D}_0$ such that $\emptyset \neq Y \neq E$. Then, for any base x we must have

$$\forall e \in E - Y\colon\ \mathrm{dep}(x, e) \subseteq E - Y \tag{3.132}$$

since otherwise there would exist a base x' such that $x'(Y) < f(Y)$, which would contradict the fact that $Y \in \mathcal{D}_0$. It follows from (3.132) that

$$E - Y \in \mathcal{D}, \qquad x(E - Y) = f(E - Y). \tag{3.133}$$

Since $x(Y) = f(Y)$, we have from (3.133)

$$f(Y) + f(E - Y) = x(Y) + x(E - Y) = x(E) = f(E). \tag{3.134}$$

This implies that $(\mathcal{D}, f)$ is not connected, which contradicts the assumption. Therefore, we must have $\{\emptyset, E\} = \mathcal{D}_0$. Q.E.D.

The proof of the "only if" part shows the following.

Lemma 3.37: *Suppose that* $\mathrm{B}(f) = \mathrm{F}(\mathcal{D}_0)$ *for some* $\mathcal{D}_0 \in \mathbf{D}$. *Then* $\mathcal{D}_0$ *is a Boolean sublattice of* $\mathcal{D}$.

Theorem 3.38: *For a submodular system* $(\mathcal{D}, f)$ *on* E *there uniquely exists a partition* $P^* = \{T_1, T_2, \cdots, T_k\}$ $(k \geq 1)$ *of* E *such that*

(i) $T_i \in \mathcal{D} \quad (i = 1, 2, \cdots, k)$,

(ii) *each reduction* $(\mathcal{D}^{T_i}, f^{T_i}) \equiv (\mathcal{D}, f) \cdot T_i$ $(i = 1, 2, \cdots, k)$ *is connected, and*

(iii) $f(X) = f^{T_1}(T_1 \cap X) + \cdots + f^{T_k}(T_k \cap X) \quad (X \in \mathcal{D}).$ (3.135)

(Proof) Suppose that $\mathcal{D}_0 \in \mathbf{D}$ satisfies $\mathrm{F}(\mathcal{D}_0) = \mathrm{B}(f)$. From Lemma 3.37, $\mathcal{D}_0$ is a Boolean lattice. Let $T_1, \cdots, T_k$ be all the minimal nonempty elements (as subsets of E) of $\mathcal{D}_0$, i.e., $\Pi(\mathcal{D}_0) = \{T_1, \cdots, T_k\}$. Then,

$$T_i \in \mathcal{D} \quad (i = 1, 2, \cdots, k). \tag{3.136}$$

Since

$$\begin{aligned} kf(E) &= f(T_1) + f(E - T_1) + \cdots + f(T_k) + f(E - T_k) \\ &\geq f(T_1) + \cdots + f(T_k) + (k-1)f(E), \end{aligned} \tag{3.137}$$

we have

$$f(E) = f(T_1) + \cdots + f(T_k). \tag{3.138}$$

For any $X \in \mathcal{D}$ we have from (3.138)

$$\begin{aligned} f(X) + f(E) &= f(X) + f(T_1) + \cdots + f(T_k) \\ &\geq f(T_1 \cap X) + \cdots + f(T_k \cap X) + f(E) \\ &\geq f(X) + f(E). \end{aligned} \tag{3.139}$$

It follows from (3.139) that

$$f(X) = f(T_1 \cap X) + \cdots + f(T_k \cap X) \quad (X \in \mathcal{D}). \tag{3.140}$$

Also, for each T_i ($i \in \{1, 2, \cdots, k\}$) and $X \subseteq T_i$ such that $X \in \mathcal{D}$ and $T_i - X \in \mathcal{D}$, we have from (3.140) with X replaced by $E - X$

$$\begin{aligned} f(E) &\leq f(X) + f(E - X) \\ &= f(T_1) + \cdots + f(T_{i-1}) + f(X) + f(T_i - X) \\ &\qquad + f(T_{i+1}) + \cdots + f(T_k), \end{aligned} \tag{3.141}$$

where (3.141) holds with equality if and only if

$$f(T_i) = f(X) + f(T_i - X). \tag{3.142}$$

If (3.141) holds with equality or (3.142) holds, X must be a member of $\mathcal{D}_0$ and hence either $X = \emptyset$ or $X = T_i$ by the definition of T_i. Therefore, $(\mathcal{D}, f) \cdot T_i$ is connected. It follows that the partition $\{T_i \mid i = 1, 2, \cdots, k\}$ derived from $\mathcal{D}_0$ satisfies (i)$\sim$(iii) of the present theorem.

Moreover, if a partition $\{W_j \mid j \in J\}$ satisfies (i)$\sim$(iii) with T_i's replaced by W_j's, then from (iii) we have $W_j \in \mathcal{D}_0$ for each $j \in J$. If $\{W_j \mid j \in$

$J\} \neq \{T_i \mid i \in I\}$, there exists W_j such that $W_j = T_{i_1} \cup \cdots \cup T_{i_p}$ for some $i_1, \cdots, i_p \in I$ with $p \geq 2$. Then, we have

$$f(W_j) = f(T_{i_1}) + \cdots + f(T_{i_p}), \tag{3.143}$$

from which follows that $(\mathcal{D}, f) \cdot W_j$ is not connected. Consequently, we have $\{W_j \mid j \in J\} = \{T_i \mid i \in I\}$, which shows the uniqueness of the partition. Q.E.D.

Theorem 3.39: $\Pi(\mathcal{D}_0)$ *is the partition of* E *given by the decomposition of* $(\mathcal{D}, f)$ *into connected components. In particular, the number of connected components is equal to* $|\Pi(\mathcal{D}_0)|$.

(Proof) The present theorem follows from the proof of Theorem 3.38. Q.E.D.

From Theorems 3.32 and 3.39 we have

Corollary 3.40: *Let* n^* *be the number of the connected components of* $(\mathcal{D}, f)$. *Then,*

$$\mathrm{dim}\mathrm{B}(f) = |E| - n^*. \tag{3.144}$$

The dimension of $\mathrm{B}(f)$ was also given in [Shapley71], [Giles75], [Bixby+Cunningham+Topkis85] and [Tomi83] (when $\mathcal{D} = 2^E$).

The following lemma is useful for finding the partition $\Pi(\mathcal{D}_0)$ (see [Bixby+Cunningham+Topkis85]).

Lemma 3.41: *For any base* $x \in \mathrm{B}(f)$ *let* $G(x) = (E, A(x))$ *be the graph with vertex set* E *and arc set* $A(x) = \{(e, e') \mid e' \in \mathrm{dep}(x, e)\}$. *Then,* $\Pi(\mathcal{D}_0)$ *is the set of the vertex sets of the connected components of* $G(x)$.

(Proof) Let $\{W_j \mid j \in J\}$ be the set of the vertex sets of the connected components of $G(x)$. Since $\mathcal{D}_0 \subseteq \mathcal{D}(x) = \{X \mid X \in \mathcal{D},\ x(X) = f(X)\}$, partition $\{W_j \mid j \in J\}$ is a refinement of $\Pi(\mathcal{D}_0)$. However, for any $j \in J$ we have $W_j, E - W_j \in \mathcal{D}(x)$ by the definition of $G(x)$ and W_j, and

$$f(E) = x(E) = x(W_j) + x(E - W_j) = f(W_j) + f(E - W_j). \tag{3.145}$$

Hence, $W_j \in \mathcal{D}_0$ for any $j \in J$. Therefore, $\Pi(\mathcal{D}_0)$ is a refinement of $\{W_j \mid j \in J\}$. It follows that $\Pi(\mathcal{D}_0) = \{W_j \mid j \in J\}$. Q.E.D.

Since, in particular, Lemma 3.41 holds for an extreme base $x \in \mathrm{B}(f)$, the connected components of $(\mathcal{D}, f)$ can efficiently be found by the use of the algorithm shown in Section 3.3.c [Bixby + Cunningham + Topkis85].

Theorem 3.42: *For any $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$ we have $\mathcal{D}_1 \cap \mathcal{D}_2 \in \mathbf{D}$ and $\mathrm{F}(\mathcal{D}_1 \cap \mathcal{D}_2)$ is the unique minimal face which contains both $\mathrm{F}(\mathcal{D}_1)$ and $\mathrm{F}(\mathcal{D}_2)$. Moreover, if $\mathrm{F}(\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$, then $\mathrm{F}(\mathcal{D}_1 \cup \mathcal{D}_2)$ is the unique maximal face which is contained in both $\mathrm{F}(\mathcal{D}_1)$ and $\mathrm{F}(\mathcal{D}_2)$. $\mathcal{D}_3 \in \mathbf{D}$ satisfying $\mathrm{F}(\mathcal{D}_3) = \mathrm{F}(\mathcal{D}_1 \cup \mathcal{D}_2)$ is the unique minimal sublattice of $\mathcal{D}$ in $\mathbf{D}$ which contains both $\mathcal{D}_1$ and $\mathcal{D}_2$.*

(Proof) The present theorem follows from the general theory of convex polyhedra (see Lemma 1.8). Q.E.D.

It should be noted here that the unique minimal sublattice of $\mathcal{D}$ which contains both $\mathcal{D}_1$ and $\mathcal{D}_2$ does not necessarily belong to $\mathbf{D}$.

The following theorem gives a necessary and sufficient condition for a sublattice $\mathcal{D}_0$ of $\mathcal{D}$ to be a member of $\mathbf{D}$.

Theorem 3.43: *Let $\mathcal{D}_0$ be a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_0$ and f_0 be the restriction of f to $\mathcal{D}_0$. Then $\mathcal{D}_0 \in \mathbf{D}$ if and only if the following three statements hold:*

(i) *f_0 is a modular function.*

(ii) *For any maximal chain*

$$\mathcal{C}_0\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{3.146}$$

of $\mathcal{D}_0$ each minor $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ $(i = 1, 2, \cdots, k)$ is connected.

(iii) *Let $\mathcal{C}_0$ be any maximal chain of $\mathcal{D}_0$ as in (3.146) and $\hat{x}$ be any base of $(\mathcal{D}_0, f_0)$ such that*

$$\hat{x}(S_i) = f_0(S_i) \qquad (i = 1, 2, \cdots, k) \tag{3.147}$$

(i.e., $\mathcal{C}_0 \subseteq \mathcal{D}_0(\hat{x})$). Then for any $X \in \mathcal{D}$ such that (a) X is compatible with $\Pi(\mathcal{D}_0)$ and (b) $\hat{x}(X) = f(X)$, we have $X \in \mathcal{D}_0$.

(Proof) *The "if" part*: Suppose (i)$\sim$(iii) hold. For each $i = 1, 2, \cdots, k$ let x_i^* be a base of $(\mathcal{D}_i, f_i) \equiv (\mathcal{D}, f) \cdot S_i/S_{i-1}$ such that for any $X \in \mathcal{D}$ with $S_{i-1} \subset X \subset S_i$ we have

$$x_i^*(X - S_{i-1}) < f(X) - f(S_{i-1}) = f_i(X - S_{i-1}). \tag{3.148}$$

Such a base x_i^* exists due to Theorem 3.36. Let $x^* \in \mathbf{R}^E$ be the direct sum of x_i^* $(i = 1, 2, \cdots, k)$, i.e.,

$$x^*(e) = x_i^*(e) \tag{3.149}$$

for each $e \in E$ with $e \in S_i - S_{i-1}$ and $i \in \{1, 2, \cdots, k\}$. Then we have $x^* \in \mathrm{B}(f)$ due to Lemma 3.1. We show $x^* \in \mathrm{F}^\circ(\mathcal{D}_0)$, which will complete the proof of the "if" part.

Let X be any member of $\mathcal{D}_0$. Since f is modular on $\mathcal{D}_0$ and x^* satisfies

$$x^*(S_i) = f(S_i) \qquad (i = 1, 2, \cdots, k) \tag{3.150}$$

on the maximal chain of $\mathcal{D}_0$ given by (3.146), we have

$$x^*(X) = f(X) \qquad (X \in \mathcal{D}_0), \tag{3.151}$$

where recall that a simple submodular system with a modular rank function has a unique extreme base.

Now, suppose $X \in \mathcal{D} - \mathcal{D}_0$. If X is compatible with $\Pi(\mathcal{D}_0)$, then from (iii) we have

$$x^*(X) < f(X). \tag{3.152}$$

On the other hand, if X is not compatible with $\Pi(\mathcal{D}_0)$, i.e., for some $i_0 \in \{1, 2, \cdots, k\}$ $X - S_{i_0} \neq \emptyset$ and $S_{i_0+1} - X \neq \emptyset$, then defining $T_i = S_i - S_{i-1}$ $(i = 1, 2, \cdots, k)$, we have

$$\begin{aligned} & f(X) - x^*(X) \\ & = f(X) - x^*(X) + \sum_{i=1}^{k} \{f(S_i) - x^*(S_i)\} \\ & \geq \sum_{i=1}^{k} \{f((X \cap S_i) \cup S_{i-1}) - x^*((X \cap S_i) \cup S_{i-1})\} \\ & = \sum_{i=1}^{k} \{f((X \cap T_i) \cup S_{i-1}) - f(S_{i-1}) - x^*(X \cap T_i)\} \\ & > 0 \end{aligned} \tag{3.153}$$

due to (3.148) and (3.149). Therefore,

$$\forall X \in \mathcal{D} - \mathcal{D}_0\colon\ x^*(X) < f(X). \tag{3.154}$$

From (3.151) and (3.154) we have $x^* \in \mathrm{F}^\circ(\mathcal{D}_0)$.

The "only if" part: Suppose $\mathcal{D}_0 \in \mathbf{D}$. Then there exists a vector $x' \in \mathrm{F}^\circ(\mathcal{D}_0)$, from which follows (i). Also (ii) follows from Theorem 3.36. We show (iii). Let $\hat{x}$ be any base of $(\mathcal{D}_0, f_0)$ such that (3.147) holds. Suppose that $X_0 \in \mathcal{D}$ is compatible with $\Pi(\mathcal{D}_0)$ and that

$$\hat{x}(X_0) = f(X_0). \tag{3.155}$$

Since $x' \in \mathrm{F}^\circ(\mathcal{D}_0)$ also satisfies

$$x'(S_i) = f(S_i) \qquad (i = 1, 2, \cdots, k), \tag{3.156}$$

we have from (3.147) and (3.156)

$$\hat{x}(T) = x'(T) \qquad (T \in \Pi(\mathcal{D}_0)). \tag{3.157}$$

Since X_0 is compatible with $\Pi(\mathcal{D}_0)$, it follows from (3.155) and (3.157) that

$$x'(X_0) = \hat{x}(X_0) = f(X_0). \tag{3.158}$$

Since $x' \in \mathrm{F}^\circ(\mathcal{D}_0)$, we must have $X_0 \in \mathcal{D}_0$. Q.E.D.

From Theorems 3.32 and 3.43 we have the following.

Corollary 3.44: *Suppose that* $\mathrm{B}(f)$ *is full-dimensional, i.e.,* $\dim \mathrm{B}(f) = |E| - 1$, *and let* $\mathcal{D}_0$ *be a sublattice of* $\mathcal{D}$ *with* $\emptyset, E \in \mathcal{D}_0$. *Then,* $\mathrm{F}(\mathcal{D}_0)$ *is a facet of* $\mathrm{B}(f)$ *if and only if* $\mathcal{D}_0$ *forms a chain* $\emptyset = S_0 \subset S_1 \subset S_2 = E$ *of* $\mathcal{D}$ *(of length 2) and* $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ $(i = 1, 2)$ *are connected.*

We also have

Theorem 3.45: *Let* $\mathcal{D}_1$ *be a sublattice of* $\mathcal{D}$ *with* $\emptyset, E \in \mathcal{D}_1$. *Suppose that* f *is modular on* $\mathcal{D}_1$. *Also, let*

$$\mathcal{C}_1\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{3.159}$$

be any maximal chain of $\mathcal{D}_1$ *and for each* $i = 1, 2, \cdots, k$ *let* $P_i = \{T_i^1, T_i^2, \cdots, T_i^{r_i}\}$ *be the partition of* $S_i - S_{i-1}$ *given by the decomposition of* $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ *into connected components. Define* $P^* = \bigcup_{i=1}^k P_i$ *and let* $\hat{x}$ *be a base of* $(\mathcal{D}, f)$ *such that*

$$\hat{x}(S_i) = f(S_i) \qquad (i = 1, 2, \cdots, k). \tag{3.160}$$

Then $\mathcal{D}_2$ given by

$$\mathcal{D}_2 = \{X \mid X \in \mathcal{D},\ \hat{x}(X) = f(X),\ X \text{ is compatible with } P^*\} \tag{3.161}$$

belongs to **D** *and satisfies*

$$\mathrm{F}(\mathcal{D}_1) = \mathrm{F}(\mathcal{D}_2). \tag{3.162}$$

(Proof) We can easily see that $\mathcal{D}_2$ given by (3.161) satisfies (i)~(iii) of Theorem 3.43, where note that the set of minors $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ $(i = 1, 2, \cdots, k)$ does not depend on the choice of a maximal chain since f is modular on $\mathcal{D}_1$ (see Theorem 7.17). Hence we have $\mathcal{D}_2 \in \mathbf{D}$ due to Theorem 3.43. Moreover, for any $x' \in \mathrm{F}(\mathcal{D}_1)$ we have

$$\hat{x}(T_i^j) = x'(T_i^j) \qquad (j = 1, 2, \cdots, r_i;\ i = 1, 2, \cdots, k). \tag{3.163}$$

It follows that for any $X \in \mathcal{D}_2$

$$x'(X) = \hat{x}(X) = f(X), \tag{3.164}$$

i.e., $x' \in \mathrm{F}(\mathcal{D}_2)$. We thus have $\mathrm{F}(\mathcal{D}_1) \subseteq \mathrm{F}(\mathcal{D}_2)$. On the other hand, since f is modular on $\mathcal{D}_1$, from (3.160) and (3.161) we have $\mathcal{D}_1 \subseteq \mathcal{D}_2$ and $\mathrm{F}(\mathcal{D}_2) \subseteq \mathrm{F}(\mathcal{D}_1)$. Hence $\mathrm{F}(\mathcal{D}_2) = \mathrm{F}(\mathcal{D}_1)$. Q.E.D.

Note that the operation of getting $\mathcal{D}_2$ from $\mathcal{D}_1$ in Theorem 3.45 is a *closure* operation on sublattices of $\mathcal{D}$, containing $\emptyset$ and E, on which f is modular and that **D** is the collection of *closed* sublattices of $\mathcal{D}$.

Theorem 3.45 is closely related to the concept of *f-skeleton* introduced in [Nakamura + Iri81]. An f-skeleton is a sublattice of $\mathcal{D}$ on which f is modular. A sublattice $\mathcal{D}_1$ of $\mathcal{D}$ in Theorem 3.45 is an f-skeleton and there exists a unique maximal f-skeleton $\mathcal{D}_1^*$ (maximal with respect to set inclusion) such that $\Pi(\mathcal{D}_1^*) = \Pi(\mathcal{D}_1)$, which is called the *maximal* f-skeleton corresponding to $\mathcal{D}_1$ ([Nakamura + Iri81]). We can easily see that $\mathcal{D}_1^*$ may not be closed and that $\mathcal{D}_1^* \subseteq \mathcal{D}_2$, where $\mathcal{D}_2$ is the closure of $\mathcal{D}_1$ given by (3.161). In fact, $\mathcal{D}_1^*$ is the aggregation of $\mathcal{D}_2$ by $\Pi(\mathcal{D}_1)$. Therefore, $\mathcal{D}_2$ is a finer structure than the maximal f-skeleton $\mathcal{D}_1^*$. Theorem 3.45 gives a polyhedral interpretation of skeletons.

Theorem 3.46: *Let x be an extreme point of $\mathrm{B}(f)$ and K be an edge of $\mathrm{B}(f)$. Suppose $\{x\} = \mathrm{F}(\mathcal{D}_0)$ and $K = \mathrm{F}(\mathcal{D}_1)$ for $\mathcal{D}_0, \mathcal{D}_1 \in \mathbf{D}$. Then, we have $x \in K$ (i.e., x is an end point of K) if and only if for $\{e, e'\} \in \Pi(\mathcal{D}_1)$*

(i) *vertices e and e' in the Hasse diagram* $\mathrm{H}(\mathcal{P}(\mathcal{D}_0)) = (E, A^*(\mathcal{D}_0))$ *of* $\mathcal{P}(\mathcal{D}_0)$ *are adjacent and*

(ii) *the Hasse diagram* $\mathrm{H}(\mathcal{P}(\mathcal{D}_1))$ *of* $\mathcal{P}(\mathcal{D}_1)$ *is obtained by identifying e with e' in* $\mathrm{H}(\mathcal{P}(\mathcal{D}_0))$ *(i.e., contracting the arc which connects e and e').*

(For the definition of $\mathcal{P}(\mathcal{D}_i)$ $(i = 0, 1)$ see Section 3.2.a.)

(Proof) From Theorem 3.32, we have $|\Pi(\mathcal{D}_0)| = |E|$ and $|\Pi(\mathcal{D}_1)| = |E| - 1$, so that $\Pi(\mathcal{D}_1)$ consists of a two-element set and singletons. From Theorem 3.43 and Lemma 3.31, we have $x \in K$ if and only if (1) $\mathcal{D}_1 \subseteq \mathcal{D}_0$ and (2) for $T \in \Pi(\mathcal{D}_1)$ with $|T| = 2$ and for $S_1, S_2 \in \mathcal{D}_1$ with $S_1 \subset S_2$ and $T = S_2 - S_1$ $(\mathcal{D}, f) \cdot S_2/S_1$ is connected. The present theorem follows from this fact. Q.E.D.

Theorem 3.47: *Let x_i $(i = 1, 2)$ be distinct extreme points of* $\mathrm{B}(f)$ *and suppose* $\{x_i\} = \mathrm{F}(\mathcal{D}_i)$ *for* $\mathcal{D}_i \in \mathbf{D}$ $(i = 1, 2)$*. Then x_1 and x_2 are adjacent in* $\mathrm{B}(f)$ *if and only if there exist vertices e and e' in E connected by an arc both in* $\mathrm{H}(\mathcal{P}(\mathcal{D}_1))$ *and in* $\mathrm{H}(\mathcal{P}(\mathcal{D}_2))$ *such that the Hasse diagrams obtained by identifying e with e' in $H(\mathcal{P}(\mathcal{D}_1))$ and $H(\mathcal{P}(\mathcal{D}_2))$, respectively, coincide with each other.*

(Proof) From Theorems 3.32 and 3.43 and Lemma 3.31 we see that extreme bases x_1 and x_2 are adjacent in $\mathrm{B}(f)$ if and only if the following two statements hold:

(i) There exists a common sublattice $\mathcal{D}_3$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\emptyset, E \in \mathcal{D}_3$ and $|\Pi(\mathcal{D}_3)| = |E| - 1$.

(ii) For $T \in \Pi(\mathcal{D}_3)$ with $|T| = 2$ and for $S_1, S_2 \in \mathcal{D}_3$ with $S_1 \subset S_2$ and $T = S_2 - S_1$ $(\mathcal{D}, f) \cdot S_2/S_1$ is connected,

from which follows the present theorem. Q.E.D.

It should be noted that for extreme bases x_i $(i = 1, 2)$ we have $\mathcal{D}(x_i) \in \mathbf{D}$ (i.e., $\mathcal{D}_i = \mathcal{D}(x_i)$ in Theorem 3.47) and that if x_1 and x_2 are adjacent, we have $\mathcal{D}(x_1) \cap \mathcal{D}(x_2) \in \mathbf{D}$ and $\mathrm{F}(\mathcal{D}(x_1) \cap \mathcal{D}(x_2))$ is the edge of $\mathrm{B}(f)$ incident to x_1 and x_2.

3.4. Intersecting- and Crossing-Submodular Functions

In this section we prove Theorems 2.5 and 2.6 presented in Section 2.3 which relate intersecting- and crossing-submodular functions to submodular systems ([Fuji84b]).

(a) Tree representations of cross-free families

We first introduce the tree representation of cross-free families due to Edmonds and Giles [Edmonds+Giles77]. We call a set $\{X_i \mid i \in I\}$ of subsets X_i $(i \in I)$ of E a *copartition* of E if $\{E - X_i \mid i \in I\}$ is a partition of E.

Let $T = (V, A)$ be a tree with a vertex set V and an arc set $A = \{a_i \mid i \in I\}$. Also let $\mathcal{P} = (P_v \mid v \in V)$ be a family of subsets of E such that the nonempty P_v's form a partition of E. Possible repeated members in $\mathcal{P}$ are the empty set only. Note that each vertex $v \in V$ is given a subset P_v of E. Deleting an arc a_i $(i \in I)$ from tree T yields a graph with two connected components. Denote the vertex set of the connected component which contains the tail $\partial^+ a_i$ of a_i by $V(a_i)$, and define

$$X_i = \bigcup\{P_v \mid v \in V(a_i)\}. \tag{3.165}$$

We thus have a family

$$\mathcal{F} = (X_i \mid i \in I) \tag{3.166}$$

of subsets of E.

From this construction we can easily see that the family $\mathcal{F}$ is cross-free. We call the representation of $\mathcal{F}$ by the pair of the tree $T = (V, A)$ and the family $\mathcal{P} = (P_v \mid v \in V)$ a *tree representation* of $\mathcal{F}$. We show the converse that every cross-free family has one such tree representation.

First, consider a laminar family $\mathcal{F} = (X_i \mid i \in I)$ of subsets of E, i.e., for each X_i, X_j $(i, j \in I)$ we have at least one of the following three: (1) $X_i \cap X_j = \emptyset$, (2) $X_i \subseteq X_j$ and (3) $X_i \supseteq X_j$. For simplicity suppose that X_i $(i \in I)$ are distinct. Let i_0 be a new index not in I, and define a vertex set V by

$$V = \{v_i \mid i \in I \cup \{i_0\}\} \tag{3.167}$$

and an arc set

$$A = \{a_i \mid i \in I\}. \tag{3.168}$$

The incidence relation is defined as follows. For each $i \in I$ such that X_i is a nonempty and non-maximal member of $\mathcal{F}$, let X_k be the unique minimal member of $\mathcal{F}$ which properly includes X_i. (Note that the members of $\mathcal{F}$

which includes X_i form a chain due to the fact that $\mathcal{F}$ is a laminar family (without repeated members).) Then we define $\partial^+ a_i = v_i$ and $\partial^- a_i = v_k$. For each maximal member X_i of $\mathcal{F}$ we define $\partial^+ a_i = v_i$ and $\partial^- a_i = v_{i_0}$. Also, if $\mathcal{F}$ contains $X_i = \emptyset$, then for some nonempty minimal member X_l of $\mathcal{F}$ we define $\partial^+ a_i = v_i$ and $\partial^- a_i = v_l$. Consequently, the graph $T = (V, A)$ is a directed tree toward the root v_{i_0}. We define $X_{i_0} = E$ and for each $i \in I \cup \{i_0\}$

$$P_{v_i} = X_i - \bigcup \{X_k \mid k \in I,\ \partial^- a_k = v_i\}. \tag{3.169}$$

We can see that the pair of tree $T = (V, A = \{a_i \mid i \in I\})$ and $\mathcal{P} = (P_v \mid v \in V)$ gives a tree representation of the laminar family $\mathcal{F} = (X_i \mid i \in I)$.

If we replace an arc a_{i_1} by series arcs $a_{i_1{}'}$ and $a_{i_1{}''}$ with a vertex $v_{i_1{}'}$ between them such that $\partial^+ a_{i_1} = \partial^+ a_{i_1{}''}$, $\partial^- a_{i_1} = \partial^- a_{i_1{}'}$ and $\partial^+ a_{i_1{}'} = \partial^- a_{i_1{}''}$, and if we define $P_{v_{i_1{}'}} = \emptyset$, then the pair of the augmented tree $T' = (V \cup \{v_{i_1{}'}\}, (A - \{a_{i_1}\}) \cup \{a_{i_1{}'}, a_{i_1{}''}\})$ and $\mathcal{P}' = (P_{v_i} \mid i \in (I - \{i_1\}) \cup \{i_0, i_1{}', i_1{}''\})$ represents the laminar family $\mathcal{F}' = (X_i \mid i \in (I - \{i_1\}) \cup \{i_0, i_1{}', i_1{}''\})$ with repeated members $X_{i_1{}'} = X_{i_1{}''}$. Repeated members can be treated in this way.

We thus have

Lemma 3.48: *A family of subsets of a finite set E is laminar if and only if it has a tree representation such that the tree is a directed tree toward the root.*

Next, let $\mathcal{F} = (X_i \mid i \in I)$ be a cross-free family of subsets of E, i.e., for each X_i, X_j $(i, j \in I)$ we have at least one of the following four: (1) $X_i \cap X_j = \emptyset$, (2) $X_i \subseteq X_j$, (3) $X_i \supseteq X_j$ and (4) $X_i \cup X_j = E$.

Choose an arbitrary element $e_0 \in E$ and define

$$\hat{\mathcal{F}} = (\hat{X}_i \mid i \in I), \tag{3.170}$$

where, putting $I_0 = \{i \mid i \in I,\ e_0 \in X_i\}$,

$$\hat{X}_i = \begin{cases} E - X_i & \text{if } i \in I_0, \\ X_i & \text{if } i \in I - I_0 \end{cases} \tag{3.171}$$

for each $i \in I$. We can easily see that $\hat{\mathcal{F}}$ is a laminar family, since $\hat{\mathcal{F}}$ is cross-free and $\hat{X}_i \cup \hat{X}_j \subseteq E - \{e_0\}$ for any $i, j \in I$. Therefore, $\hat{\mathcal{F}}$ can be represented by a pair of a tree $\hat{T} = (V, \hat{A})$ and a family $\mathcal{P} = (P_v \mid v \in V)$, where $\hat{A} = \{\hat{a}_i \mid i \in I\}$. Construct a tree $T = (V, A)$ by reorienting the arcs

$\hat{a}_i$ ($i \in I_0$) of tree $\hat{T} = (V, \hat{A})$. Then, the pair of the tree T and the family $\mathcal{P}$ represents the original cross-free family $\mathcal{F}$.

Now, we have

Lemma 3.49 [Edm+Giles77]: *A family of subsets of E is cross-free if and only if it has a tree representation.*

The following lemma, concerning the decomposition of uniform cross-free families, is fundamental.

Lemma 3.50: *Let $\mathcal{G} = (X_i \mid i \in I)$ be a cross-free family of subsets of E which uniformly covers each element of E, i.e.,*

$$\forall e \in E\colon\ |\{i \mid i \in I,\ e \in X_i\}| = \text{const.} > 0. \tag{3.172}$$

Then, $\mathcal{G}$ is the direct sum of families each of which forms a partition or a copartition of E.

(Proof) Since $\{E\}$ is a partition and $\{\emptyset\}$ is a copartition of E, we suppose that $\emptyset, E \notin \mathcal{G}$ and that $\mathcal{G} \neq \emptyset$. From Lemma 3.49, the cross-free family $\mathcal{G} = (X_i \mid i \in I)$ can be represented by a pair of a tree $T = (V, A)$, with a vertex set V and an arc set $A = \{a_i \mid i \in I\}$, and a family

$$\mathcal{P} = (P_v \mid v \in V) \tag{3.173}$$

of subsets of E, where nonempty P_v's form a partition of E. Note that by the assumption we have

$$P_v \neq \emptyset \tag{3.174}$$

for any leaf v of T.

Now, from the assumption and (3.174) there exists at least one X_i such that $X_i \notin \{\emptyset, E\}$. Therefore, there exist distinct vertices v_1 and v_2 of T satisfying the conditions that

$$P_{v_1} \neq \emptyset, \qquad P_{v_2} \neq \emptyset \tag{3.175}$$

and that for any vertex $u \notin \{v_1, v_2\}$ lying on the unique path from v_1 to v_2, denoted by $Q(v_1, v_2)$, in T we have

$$P_u = \emptyset. \tag{3.176}$$

It follows from (3.172) that the number of positively oriented arcs in $Q(v_1, v_2)$ is equal to the number of negatively oriented arcs in $Q(v_1, v_2)$. (If this is not the case, the value of (3.172) for any $e_1 \in P_{v_1}$ can not be equal to that for any $e_2 \in P_{v_2}$.) In particular, there exists at least one positively oriented arc and at least one negatively oriented arc in $Q(v_1, v_2)$. Hence, there is at least one vertex $u \notin \{v_1, v_2\}$ on $Q(v_1, v_2)$. Let $\hat{u}$ be the vertex on $Q(v_1, v_2)$ adjacent to v_1 and let $\{v_2, v_3, \cdots, v_k\}$ $(k \geq 2)$ be the maximal set of vertices of T such that for each $l = 2, 3, \cdots, k$, (i) $P_{v_l} \neq \emptyset$, (ii) the vertex $\hat{u}$ lies on the unique path $Q(v_1, v_l)$ from v_1 to v_l in T and (iii) any vertex $u \notin \{v_1, v_l\}$ lying on $Q(v_1, v_l)$ satisfies (3.176). Moreover, let $a_{j(1)} \in A$ be the arc connecting v_1 with $\hat{u}$ and for each $l = 2, 3, \cdots, k$ let $a_{j(l)} \in A$ be the arc in $Q(v_1, v_l)$ such that the orientation of the arc $a_{j(l)}$ is opposite to the orientation of the arc $a_{j(1)}$ and any arc a_i $(\neq a_{j(l)})$ between $a_{j(1)}$ and $a_{j(l)}$ in $Q(v_1, v_l)$ has the same orientation as $a_{j(1)}$. (Note that arcs $a_{j(l)}$ $(l = 2, 3, \cdots, k)$ are not necessarily distinct.) (See Fig. 3.6.)

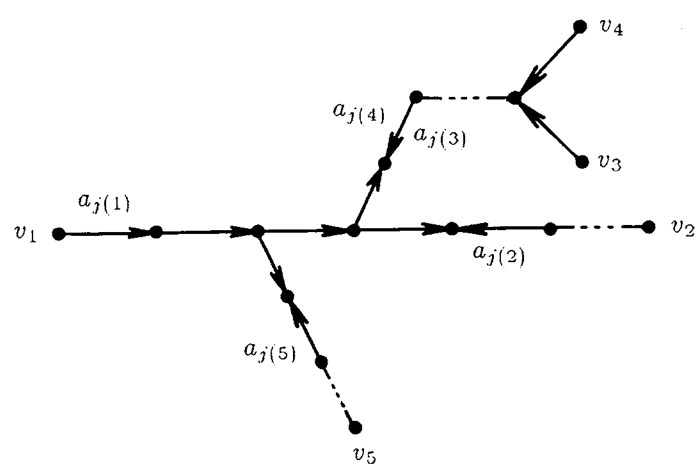


Figure 3.6: A tree representation.

Let us define

$$\Pi = \{X_{j(l)} \mid X_{j(l)} \in \mathcal{G},\ 1 \leq l \leq k\}. \tag{3.177}$$

Then,

(i) if $\partial^+ a_{j(1)} = v_1$, Π is a partition of E and

(ii) if $\partial^- a_{j(1)} = v_1$, Π is a copartition of E.

Therefore, the family $\mathcal{G}$ contains a subfamily which forms a partition or copartition of E. Define

$$\mathcal{G}' = (X_i \mid i \in I - \{j(l) \mid l = 1, 2, \cdots, k\}). \tag{3.178}$$

If $\mathcal{G}'$ is not empty, then $\mathcal{G}'$ also satisfies (3.172) and is cross-free. Repeating the above-mentioned argument, we see that $\mathcal{G}$ is the direct sum of families each of which forms a partition or a copartition of E. Q.E.D.

(b) Crossing-submodular functions

Let $\mathcal{F}$ be a crossing family of subsets of a finite set E with $\emptyset, E \in \mathcal{F}$ and f a crossing-submodular function on $\mathcal{F}$ with $f(\emptyset) = 0$ (see Section 2.3 for the terminology). We assume that $\mathbf{R}$ is the set of reals (or rationals).

Define

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \leq f(X)\}. \tag{3.179}$$

Note that $\mathrm{P}(f)$ is nonempty.

Now, we would like to find the tightest system of inequalities with $\{0, 1\}$-coefficients which represents the polyhedron $\mathrm{P}(f)$. Therefore, define

$$\hat{f}(Y) = \max\{x(Y) \mid x \in \mathrm{P}(f)\} \tag{3.180}$$

for any $Y \subseteq E$, where we put $\hat{f}(Y) = +\infty$ if the value of $x(Y)$ can be made arbitrarily large for $x \in \mathrm{P}(f)$. Note that by definition (3.180) we have $\hat{f}(\emptyset) = 0$.

By the linear programming duality theorem (Theorem 1.10) we have

$$\hat{f}(Y) = \min\{\sum_{X \in \mathcal{F}} f(X)c(X) \ \mid\ (3.182), (3.183)\}, \tag{3.181}$$

where

$$\sum_{e \in X \in \mathcal{F}} c(X) = \chi_Y(e) \equiv \begin{cases} 1 & (e \in Y) \\ 0 & (e \notin Y) \end{cases} \qquad (e \in E), \tag{3.182}$$

$$c(X) \geq 0 \quad (X \in \mathcal{F}). \tag{3.183}$$

We have $\hat{f}(Y) = +\infty$ if and only if there is no such $c(X)$ $(X \in \mathcal{F})$ that (3.182) and (3.183) are satisfied. We thus have a set function $\hat{f}\colon 2^E \to \mathbf{R} \cup \{+\infty\}$.

Since the minimum value of (3.181) can be attained by rational $c(X)$ $(X \in \mathcal{F})$, (3.181)~(3.183) can be rewritten as follows.

$$\hat{f}(Y) = \min\Big\{\frac{1}{\mu(\mathcal{G}, Y)} \sum_{i \in I} f(X_i) \ \Big|\ (3.185) - (3.187)\Big\}, \tag{3.184}$$

where

$$\mathcal{G} = (X_i \mid i \in I) \tag{3.185}$$

with

$$X_i \in \mathcal{F}, \quad X_i \subseteq Y \quad (i \in I) \tag{3.186}$$

and

$$|\{i \mid i \in I,\ e \in X_i\}| = \text{const.} \equiv \mu(\mathcal{G}, Y) > 0 \quad (e \in Y). \tag{3.187}$$

Note that $\mathcal{G} = (X_i \mid i \in I)$ is a family of subsets X_i of E, so that we may have $X_i = X_j$ for distinct $i, j \in I$. (3.187) means that the family $\mathcal{G}$ uniformly covers each element of Y.

By the definition of the set function $\hat{f}$: $2^E \to \mathbf{R} \cup \{+\infty\}$ we have

$$\hat{f}(X) \le f(X) \quad (X \in \mathcal{F}), \tag{3.188}$$

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \subseteq E\colon\ x(X) \le \hat{f}(X)\}. \tag{3.189}$$

If we decrease the value of any $\hat{f}(X)$ $(X \subseteq E)$, (3.189) does not hold. Since we have not used the crossing-submodularity of f in the above argument, (3.179)~(3.189) are valid for any set function on any family of subsets of E. In the following, we simplify (3.184)~(3.187) by use of the crossing-submodularity of f.

From the crossing-submodularity of f and (3.184)~(3.187), we can restrict admissible families $\mathcal{G}$ in (3.184)~(3.187) to those which satisfy

(i) $\mathcal{G}$ is a cross-free family, (3.190)

(ii) $\emptyset \notin \mathcal{G}$, (3.191)

to attain the minimum of (3.184). (For, let $\mathcal{G}$ be a family which satisfies (3.185)~(3.187) and attains the minimum of (3.184). If there exists a crossing pair of X_i and X_j in $\mathcal{G}$, then put

$$X_i \leftarrow X_i \cup X_j, \qquad X_j \leftarrow X_i \cap X_j. \tag{3.192}$$

After this replacement $\mathcal{G} = (X_i \mid i \in I)$ remains to satisfy (3.185)~(3.187), while the sequence of the values of $|X_i|$ $(i \in I)$ arranged in order of non-increasing magnitude becomes lexicographically greater than the previous one. Because of the finiteness character we can repeat the replacement (3.192) for crossing pairs in $\mathcal{G}$ only finitely many times and eventually we obtain a cross-free family $\mathcal{G}$ which satisfies (3.185)~(3.187) and attains the minimum of (3.184). Furthermore, if $\mathcal{G}$ contains the empty set, it is redundant and can be removed.)

Theorem 3.51: *Let $f_{\mathrm{P}}(E)$ and $f_{\mathrm{q}}(E)$ be defined by*

$$f_{\mathrm{P}}(E) = \min\Big\{\sum_{i \in I} f(X_i) \;\Big|\; \{X_i \mid i \in I\}\colon \text{ a partition of } E, \\ \forall i \in I\colon\ X_i \in \mathcal{F}\Big\}, \tag{3.193}$$

$$f_{\mathrm{q}}(E) = \min\Big\{\frac{1}{|J|-1}\sum_{j \in J} f(X_j) \;\Big|\; \{X_j \mid j \in J\}\colon \text{ a copartition of } E, \\ \forall j \in J\colon\ X_j \in \mathcal{F},\ |J| \geq 3\Big\}. \tag{3.194}$$

Then we have

$$\hat{f}(E) = \min\{f_{\mathrm{P}}(E), f_{\mathrm{q}}(E)\}. \tag{3.195}$$

(Proof) We can restrict admissible families $\mathcal{G} = (X_i \mid i \in I)$ in (3.185)~(3.187) with $Y = E$ to those further satisfying (3.190) and (3.191). For such a family $\mathcal{G}$, from Lemma 3.50, $\mathcal{G}$ is the direct sum of partitions and copartitions of E. Since every subfamily of $\mathcal{G}$ forming a partition or a copartition of E satisfies (3.185)~(3.187) with $Y = E$, (3.190) and (3.191), it follows that we can restrict admissible families $\mathcal{G} = (X_i \mid i \in I)$ to those which form partitions and copartitions of E with $X_i \in \mathcal{F}$ $(i \in I)$. Note that the copartition $\{\emptyset\}$ of cardinality 1 is excluded by (3.191) and that copartitions of cardinality 2 are also partitions. We thus have (3.193)~(3.195). Q.E.D.

From Theorem 3.51 we can show

Theorem 3.52: *The following four statements are equivalent*:

(i) *The polyhedron defined as the intersection of* $\mathrm{P}(f)$ *and the hyperplane* $x(E) = f(E)$, *i.e.,*

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \leq f(X),\ x(E) = f(E)\}, \tag{3.196}$$

is nonempty.

(ii) $\hat{f}(E) = f(E)$. $$\tag{3.197}$$

(iii) $f(E) = f_{\mathrm{P}}(E) = (f^{\#})_{\mathrm{P}}(E)$, $$\tag{3.198}$$

where

$$(f^{\#})_{\mathrm{P}}(E) = \max\Big\{\sum_{j \in J} f^{\#}(Y_j) \mid \{Y_j \mid j \in J\}\colon \text{ a partition of } E,\ \forall j \in J\colon\ E - Y_j \in \mathcal{F}\Big\}. \tag{3.199}$$

(iv) *For any partitions* $\{X_i \mid i \in I\}$ *and* $\{Y_j \mid j \in J\}$ *of* E *with* $X_i \in \mathcal{F}$ $(i \in I)$ *and* $E - Y_j \in \mathcal{F}$ $(j \in J)$ *we have*

$$\sum_{j \in J} f^{\#}(Y_j) \leq f(E) \leq \sum_{i \in I} f(X_i). \tag{3.200}$$

(Proof) The equivalence between (i) and (ii) follows from the definition of $\hat{f}(E)$. To show the equivalence between (ii) and (iii), recall (3.193)~(3.195). If $\hat{f}(E) = f(E)$, we have $f_{\mathrm{P}}(E) \geq f(E)$, while from (3.193) we have $f_{\mathrm{P}}(E) \leq f(E)$ since $E \in \mathcal{F}$. We thus have $f_{\mathrm{P}}(E) = f(E)$. Moreover, since $\emptyset \in \mathcal{F}$, we have from (3.199)

$$(f^{\#})_{\mathrm{P}}(E) \geq f^{\#}(E) = f(E). \tag{3.201}$$

If $\hat{f}(E) = f(E)$, we have from (3.194) and (3.195) $f(E) \leq f_{\mathrm{q}}(E)$, i.e.,

$$(|J| - 1)f(E) \leq \sum_{j \in J} f(X_j) \tag{3.202}$$

for any copartition $\{X_j \mid j \in J\}$ of E such that $X_j \in \mathcal{F}$ $(j \in J)$ and $|J| \geq 3$. Note that (3.202) holds for $|J| = 1$ and 2, due to the assumption that $f(E) \leq f_{\mathrm{P}}(E)$. (3.202) is rewritten as

$$\sum_{j \in J} f^{\#}(Y_j) \leq f(E) \tag{3.203}$$

for any partition $\{Y_j \mid j \in J\}$ of E such that $E - Y_j \in \mathcal{F}$ $(j \in J)$. Hence, $(f^{\#})_{\mathrm{P}}(E) \leq f(E)$. Consequently, from (3.201) we have $(f^{\#})_{\mathrm{P}}(E) = f(E)$. Conversely, suppose that $f(E) = f_{\mathrm{P}}(E) = (f^{\#})_{\mathrm{P}}(E)$. Then we have (3.203) and hence $f(E) \leq f_{\mathrm{Q}}(E)$. Therefore, $\hat{f}(E) = f(E)$.

Finally, we show the equivalence between (iii) and (iv). Since we always have

$$f_{\mathrm{P}}(E) \leq f(E) \leq (f^{\#})_{\mathrm{P}}(E), \tag{3.204}$$

(iv) implies (iii). The converse, (iii) $\Longrightarrow$ (iv), is immediate. Q.E.D.

Theorem 2.6 follows from Theorem 3.52.

For proper subsets of E the value of $\hat{f}$ is expressed as follows.

Theorem 3.53: *For each nonempty proper subset Y of E,*

$$\hat{f}(Y) = \min\Bigl\{\sum_{i \in I} f(X_i) \Bigm| \{X_i \mid i \in I\}\text{: a partition of } Y, \ \forall i \in I\text{: } X_i \in \mathcal{F}\Bigr\}. \tag{3.205}$$

(Proof) By the same argument as in the case of $Y = E$ in the proofs of Lemma 3.50 and Theorem 3.51, we see that we can restrict admissible families $\mathcal{G}$ in (3.185)$\sim$(3.187) to those which form partitions and copartitions of Y.

Let $\mathcal{P}^* = \{X_i \mid i \in I\}$ be a copartition of Y with $X_i \in \mathcal{F}$ $(i \in I)$ and $|I| \geq 3$. (Note that if $|I| = 2$, $\mathcal{P}^*$ is also a partition of Y.) Then, for any distinct $X_i, X_j \in \mathcal{P}^*$, X_i and X_j cross. It follows from (3.190) that we need not consider copartitions of Y. Q.E.D.

Note that $\hat{f}$ is the Dilworth truncation of f when $\mathrm{B}(f)$ is nonempty.

We also have

Theorem 3.54: *For any $X, Y \subseteq E$ with $X \cup Y \neq E$, if $\hat{f}(X), \hat{f}(Y) < +\infty$, then*

$$\hat{f}(X) + \hat{f}(Y) \geq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \tag{3.206}$$

(Proof) If $X \cap Y = \emptyset$, (3.206) is immediate. Therefore, suppose that $X, Y \subseteq E$ satisfy $X \cap Y \neq \emptyset$, $X \cup Y \neq E$ and $\hat{f}(X), \hat{f}(Y) < +\infty$. Then, for some partition $\{X_i \mid i \in I\}$ of X and some partition $\{Y_j \mid j \in J\}$ of Y such that $X_i \in \mathcal{F}$ $(i \in I)$ and $Y_j \in \mathcal{F}$ $(j \in J)$, we have

$$\hat{f}(X) + \hat{f}(Y) = \sum_{i \in I} f(X_i) + \sum_{j \in J} f(Y_j) \tag{3.207}$$

due to Theorem 3.53. Let $\mathcal{G} = (Z_k \mid k \in I \oplus J)$ be the direct sum of families $(X_i \mid i \in I)$ and $(Y_j \mid j \in J)$. Since $X \cup Y \neq E$, one of the following three statements holds for any $Z_i, Z_j \in \mathcal{G}$:

(i) Z_i and Z_j are disjoint, (3.208)

(ii) $Z_i \subseteq Z_j$ or $Z_j \subseteq Z_i$, (3.209)

(iii) Z_i and Z_j cross. (3.210)

If Z_i and Z_j cross, replace Z_i and Z_j as

$$Z_i \leftarrow Z_i \cup Z_j, \qquad Z_j \leftarrow Z_i \cap Z_j. \tag{3.211}$$

Repeat this replacement until the current family $\mathcal{G} = (Z_k \mid k \in I \oplus J)$ becomes a cross-free family and satisfies (3.208) and (3.209) for any $Z_i, Z_j \in \mathcal{G}$. By the same argument made after (3.192), this replacement process terminates in a finite number of steps. The finally obtained $\mathcal{G}$ is the direct sum of two families which form a partition $\{\hat{X}_i \mid i \in \hat{I}\}$ $(\hat{X}_i \in \mathcal{F}$ $(i \in \hat{I}))$ of $X \cup Y$ and a partition $\{\hat{Y}_j \mid j \in \hat{J}\}$ $(\hat{Y}_j \in \mathcal{F}$ $(j \in \hat{J}))$ of $X \cap Y$. Since the replacement (3.211) reduces the value of (3.207), we get

$$\begin{aligned} \hat{f}(X) + \hat{f}(Y) &\geq \sum_{i \in \hat{I}} f(\hat{X}_i) + \sum_{j \in \hat{J}} f(\hat{Y}_j) \\ &\geq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \end{aligned} \tag{3.212}$$

Q.E.D.

It follows from Theorem 3.54 that the family $\hat{\mathcal{F}}$ defined by

$$\hat{\mathcal{F}} = \{X \mid X \subseteq E,\ \hat{f}(X) < +\infty\} \tag{3.213}$$

is a *cointersecting family* (i.e., $X, Y \in \hat{\mathcal{F}},\ X \cup Y \neq E \implies X \cup Y, X \cap Y \in \hat{\mathcal{F}}$) and that $\hat{f}$ restricted to $\hat{\mathcal{F}}$ is a *cointersecting-submodular* function on $\hat{\mathcal{F}}$ (i.e., $\hat{f}(X) + \hat{f}(Y) \geq \hat{f}(X \cup Y) + \hat{f}(X \cap Y)$ for any $X, Y \in \hat{\mathcal{F}}$ with $X \cup Y \neq E$).

Now, let us consider the polyhedron

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \le f(X),\ x(E) = f(E)\} \tag{3.214}$$

which is the intersection of $\mathrm{P}(f)$ and the hyperplane $x(E) = f(E)$, and suppose that $\mathrm{B}(f)$ is not empty (see Theorem 3.52). We examine the structure of $\mathrm{B}(f)$.

For each proper subset $Y \subset E$, define

$$\hat{f}_2(Y) = \max\{x(Y) \mid x \in \mathrm{B}(f)\}. \tag{3.215}$$

Then, by the linear programming duality theorem,

$$\hat{f}_2(Y) = \min\{\sum_{X \in \mathcal{F}} f(X)c(X) \ \mid\ (3.217),\ (3.218)\}, \tag{3.216}$$

where

$$\sum_{e \in X \in \mathcal{F}} c(X) = \chi_Y(e) \qquad (e \in E), \tag{3.217}$$

$$c(X) \ge 0 \qquad (X \in \mathcal{F},\ X \ne E). \tag{3.218}$$

Similarly as (3.184), (3.216) can be rewritten as

$$\hat{f}_2(Y) = \min\Big\{\frac{1}{\mu(\mathcal{G}, Y) - \mu(\mathcal{G}, E - Y)}\Big[\sum_{i \in I} f(X_i) - \mu(\mathcal{G}, E - Y) f(E)\Big] \ \Big| \ (3.220) - (3.224)\Big\}, \tag{3.219}$$

where

$$\mathcal{G} = (X_i \mid i \in I), \tag{3.220}$$

$$X_i \in \mathcal{F}, \quad X_i \ne E \qquad (i \in I), \tag{3.221}$$

$$|\{i \mid e \in X_i,\ i \in I\}| = \text{const.} \equiv \mu(\mathcal{G}, Y) \qquad (e \in Y) \tag{3.222}$$

$$|\{i \mid e \in X_i,\ i \in I\}| = \text{const.} \equiv \mu(\mathcal{G}, E - Y) \qquad (e \in E - Y), \tag{3.223}$$

$$\mu(\mathcal{G}, Y) > \mu(\mathcal{G}, E - Y). \tag{3.224}$$

It should be noted that (3.219) is valid for any set function f.

By use of the set function $\hat{f}\colon 2^E \to \mathbf{R} \cup \{+\infty\}$ defined by (3.184) (or (3.195) and (3.205)), the polyhedron $\mathrm{B}(f)$ of (3.214) can also be expressed as

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \hat{\mathcal{F}}\colon\ x(X) \le \hat{f}(X),\ x(E) = \hat{f}(E)(= f(E))\}, \tag{3.225}$$

where $\hat{\mathcal{F}}$ is defined by (3.213). Therefore, we can replace f and $\mathcal{F}$ in (3.219) and (3.221) by $\hat{f}$ and $\hat{\mathcal{F}}$, respectively.

For $Y \subset E$, if $\{E - X_i \mid i \in I\}$ is a partition of $E - Y$, we call $\{X_i \mid i \in I\}$ a *copartition of $E - Y$ augmented by Y.*

Theorem 3.55: *For each nonempty $Y \subset E$,*

$$\hat{f}_2(Y) = \min\Big\{\sum_{i \in I} \hat{f}(X_i) - (|I| \quad -1)\hat{f}(E) \ \Big| \ \{E - X_i \mid i \in I\}\colon \text{ a partition of } E - Y, \ \forall i \in I\colon\ X_i \in \hat{\mathcal{F}}\Big\}. \tag{3.226}$$

(Proof) If f and $\mathcal{F}$ in (3.219) and (3.221) are replaced by $\hat{f}$ and $\hat{\mathcal{F}}$, we can restrict admissible families $\mathcal{G}$ in (3.220)$\sim$(3.224) to those which satisfy (3.220)$\sim$ (3.224) (with $\mathcal{F}$ replaced by $\hat{\mathcal{F}}$ in (3.221)) and the following (i)$\sim$(iv):

(i) $\mathcal{G}$ is a cross-free family, (3.227)

(ii) for any $X_i, X_j \in \mathcal{G}$ we have $X_i \cap X_j \neq \emptyset$, (3.228)

(iii) $\mathcal{G}$ does not contain a subfamily which forms a copartition of E, (3.229)

(iv) $E \notin \mathcal{G}$. (3.230)

(Here, (i)$\sim$(iii) follow from Theorems 3.51, 3.53 and 3.54. (iv) follows from the form of (3.219).) From (i), the family $\mathcal{G} = (X_i \mid i \in I)$ can be represented by a pair of a tree $T = (V, A)$ and a family

$$\mathcal{P} = (P_v \mid v \in V), \tag{3.231}$$

where $A = \{a_i \mid i \in I\}$ and nonempty P_v's form a partition of E as in Lemma 3.49. From (ii), T is a directed tree. (For, if there were distinct arcs a_i and a_j in T such that $\partial^- a_i = \partial^- a_j$, we would have $X_i \cap X_j = \emptyset$.)

Let v_0 be the root of T. If $P_{v_0} = \emptyset$, then $\mathcal{G}$ contains a subfamily which form a copartition of E. Therefore, $P_{v_0} \neq \emptyset$ due to (iii). Since for each $e \in E$ the number of i's for which $e \in X_i$ should be taken from the fixed set of two distinct values of (3.222) and (3.223), for any leaf u of T every vertex $w \notin \{u, v_0\}$ lying on the unique path $Q(v_0, u)$ connecting v_0 with u in T gives

$$P_w = \emptyset, \tag{3.232}$$

where note that $P_u \neq \emptyset$ due to (iv). It follows from (3.224) that

$$P_{v_0} = Y \tag{3.233}$$

and that

$$\{E - X_i \mid i \in I,\ \partial^+ a_i = v_0\} \tag{3.234}$$

is a partition of $E - Y$.

Let $\mathcal{G}'$ be the family obtained from $\mathcal{G}$ by deleting X_i's such that $\partial^+ a_i = v_0$. If $\mathcal{G}' \neq \emptyset$, then $\mathcal{G}'$ also satisfies (3.221)$\sim$(3.224) (with $\mathcal{F}$ replaced by $\hat{\mathcal{F}}$) and (3.227)$\sim$(3.230). (The tree representation of $\mathcal{G}'$ with root v_0 is obtained by contracting or shortcircuiting arcs a_i such that $\partial^+ a_i = v_0$ in T.) It follows that $\mathcal{G}$ is a direct sum of families which form copartitions of $E - Y$ augmented by Y. Since any family which forms a copartition of $E - Y$ augmented by Y also satisfies the conditions required for $\mathcal{G}$, the minimum of (3.219) is attained by a family $\mathcal{G}$ which forms a copartition of $E - Y$ augmented by Y. Q.E.D.

We define

$$\hat{f}_2(E) = \hat{f}(E)(= f(E)). \tag{3.235}$$

We now have a set function $\hat{f}_2$ from 2^E to $\mathbf{R} \cup \{+\infty\}$.

Theorem 3.56: *For any $X, Y \subseteq E$ such that $\hat{f}_2(X), \hat{f}_2(Y) < +\infty$, we have*

$$\hat{f}_2(X) + \hat{f}_2(Y) \geq \hat{f}_2(X \cup Y) + \hat{f}_2(X \cap Y). \tag{3.236}$$

(Proof) If $X \in \{\emptyset, E\}$ or $Y \in \{\emptyset, E\}$, then (3.236) is trivial. So, suppose $X \notin \{\emptyset, E\}$, $Y \notin \{\emptyset, E\}$ and $X, Y \in \hat{\mathcal{F}}$. Then, from Theorem 3.55, for some partition $\{E - X_i \mid i \in I\}$ of $E - X$ and some partition $\{E - Y_j \mid j \in J\}$ of $E - Y$ with $X_i \in \hat{\mathcal{F}}$ $(i \in I)$ and $Y_j \in \hat{\mathcal{F}}$ $(j \in J)$, we have

$$\begin{aligned} \hat{f}_2(X) + \hat{f}_2(Y) &= \sum_{i \in I} \hat{f}(X_i) - (|I| - 1)\hat{f}(E) \\ &\quad + \sum_{j \in J} \hat{f}(Y_j) - (|J| - 1)\hat{f}(E). \end{aligned} \tag{3.237}$$

Let $\mathcal{G} = (Z_k \mid k \in I \oplus J)$ be the direct sum of families $(X_i \mid i \in I)$ and $(Y_j \mid j \in J)$. If, for $Z_i, Z_j \in \mathcal{G}$, we have $(E - Z_i) \cap (E - Z_j) \neq \emptyset$, $Z_i \cap (E - Z_j) \neq \emptyset$ and $(E - Z_i) \cap Z_j \neq \emptyset$, then replace

$$Z_i \leftarrow Z_i \cup Z_j, \qquad Z_j \leftarrow Z_i \cap Z_j. \tag{3.238}$$

Repeat such a replacement until there is no such pair of Z_i and Z_j in $\mathcal{G}$. This process terminates in finitely many steps. We can easily see that the finally obtained $\mathcal{G}$ is the direct sum of families $\mathcal{G}_1 = (X_i^* \mid i \in I^*)$ and $\mathcal{G}_2 = (Y_j^* \mid j \in J^*)$ such that $\{E - X_i^* \mid i \in I^*\}$ is a partition of $E-(X \cup Y)$ and $\{E - Y_j^* \mid j \in J^*\}$ is a partition of $E-(X \cap Y)$, where, if $X \cap Y = \emptyset$, the family $\mathcal{G}_2$ may be composed of the empty set alone.

For any pair of Z_i and Z_j to be replaced by (3.238) we have $Z_i \cup Z_j \neq E$. It follows from Theorem 3.54 and (3.237) that

$$\begin{aligned}\hat{f}_2(X) + \hat{f}_2(Y) &\geq \sum_{i \in I^*} \hat{f}(X_i^*) - (|I^*|-1)\hat{f}(E) \\ &\quad + \sum_{j \in J^*} \hat{f}(Y_j^*) - (|J^*|-1)\hat{f}(E) \\ &\geq \hat{f}_2(X \cup Y) + \hat{f}_2(X \cap Y). \end{aligned} \tag{3.239}$$

Q.E.D.

From Theorem 3.56,

$$\mathcal{D}_2 = \{X \mid X \subseteq E,\ \hat{f}_2(X) < +\infty\} \tag{3.240}$$

is a distributive lattice with set union and intersection as the lattice operations, join and meet. Denote by f_2 the function obtained by restricting the domain 2^E of $\hat{f}_2$ to $\mathcal{D}_2$. Then, $(\mathcal{D}_2, f_2)$ is a submodular system on E and the polyhedron $\mathrm{B}(f)$ defined by (3.225) is also expressed as

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}_2\colon\ x(X) \leq f_2(X),\ x(E) = f_2(E)(= f(E))\} \tag{3.241}$$

which is the base polyhedron $\mathrm{B}(f_2)$ associated with the submodular system $(\mathcal{D}_2, f_2)$.

Since different submodular systems give different nonempty base polyhedra, nonempty $\mathrm{B}(f)$ uniquely determines a submodular system $(\mathcal{D}_2, f_2)$ such that $\mathrm{B}(f) = \mathrm{B}(f_2)$.

Moreover, we see from (3.235) and Theorems 3.53 and 3.55 that if $\mathrm{B}(f)$ is nonempty, each $f_2(X)$ $(X \in \mathcal{D}_2)$ is expressed as a linear combination of $f(Y)$'s $(Y \in \mathcal{F})$ with integer coefficients. Therefore, if f is integer-valued, so is f_2.

This completes the proof of (ii) of Theorem 2.5.

It should be noted that (3.226) can be rewritten in a dual form as

$$\hat{f}_2^{\#}(Z) = \max\{\sum_{j\in J} \hat{f}^{\#}(Y_j) \ \mid\ \{Y_j \mid j \in J\}\colon\ \text{a partition of } Z,$$
$$\forall j \in J\colon\ E - Y_j \in \hat{\mathcal{F}}\} \qquad (3.242)$$

for each nonempty proper subset $Z \subset E$. Here, $\hat{f}_2^{\#}$ is the Dilworth truncation of the intersecting-supermodular function $\hat{f}^{\#}$ on $\hat{\mathcal{F}}$.

(c) Intersecting-submodular functions

Let $f\colon \mathcal{F} \to \mathbf{R}$ be an intersecting-submodular function on an intersecting family $\mathcal{F}$. We assume that $f(\emptyset) = 0$ if $\emptyset \in \mathcal{F}$.

Consider the polyhedron

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{F}\colon\ x(X) \le f(X)\}. \qquad (3.243)$$

Since intersecting-submodular functions are crossing-submodular, $\mathrm{P}(f)$ is expressed as

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \subseteq E\colon\ x(X) \le \hat{f}(X)\}, \qquad (3.244)$$

where $\hat{f}$ is defined by (3.193)~(3.195) and (3.205).

Since $\mathcal{F}$ is an intersecting family and $f\colon \mathcal{F} \to \mathbf{R}$ is intersecting-submodular, we can restrict $\mathcal{G}$ in (3.184)~(3.187) to those which satisfy (3.186), (3.187), (3.190), (3.191) and the following:

(iii) for any $X_i, X_j \in \mathcal{G}$ such that $X_i \cap X_j \ne \emptyset$, we have $X_i \subseteq X_j$ or $X_j \subseteq X_i$,

(3.245)

i.e., $\mathcal{G}$ is a laminar family. In particular, $f_{\mathrm{P}}(E)$ and $f_{\mathrm{Q}}(E)$ defined by (3.193) and (3.194) satisfy

$$f_{\mathrm{P}}(E) \le f_{\mathrm{Q}}(E). \qquad (3.246)$$

It follows from (3.246) and Theorems 3.51 and 3.53 that for any nonempty $Y \subseteq E$

$$\hat{f}(Y) = \min\{\sum_{i\in I} f(X_i) \ \mid\ \{X_i \mid i \in I\}\colon\ \text{a partition of } Y,$$
$$\forall i \in I\colon\ X_i \in \mathcal{F}\}, \qquad (3.247)$$

and we have $\hat{f}(\emptyset) = 0$ by definition (3.180). Function $\hat{f}$ is the Dilworth truncation of f.

Theorem 3.57: *Let f: $\mathcal{F} \to \mathbf{R}$ be an intersecting-submodular function. For any $X, Y \subseteq E$ and $\hat{f}$: $2^E \to \mathbf{R} \cup \{+\infty\}$ defined by (3.247), if $\hat{f}(X), \hat{f}(Y) < +\infty$, then*

$$\hat{f}(X) + \hat{f}(Y) \geq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \tag{3.248}$$

(Proof) Since f is an intersecting-submodular function on $\mathcal{F}$, Theorem 3.54 holds for $X, Y \subseteq E$ with $X \cup Y = E$ as well. Q.E.D.

Define

$$\mathcal{D}_1 = \{X \mid X \subseteq E,\ \hat{f}(X) < +\infty\} \tag{3.249}$$

and let f_1 be the restriction of $\hat{f}$ to $\mathcal{D}_1$. Then, from Theorem 3.57, $\mathcal{D}_1$ is a distributive lattice and f_1 is a submodular function on $\mathcal{D}_1$. Since polyhedron $\mathrm{P}(f)$ in (3.243) is expressed in terms of f_1 as

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}_1\colon\ x(X) \leq f_1(X)\} \tag{3.250}$$

which is the submodular polyhedron $\mathrm{P}(f_1)$ associated with the submodular system $(\mathcal{D}_1, f_1)$ on E.

Since different submodular systems give different associated submodular polyhedra, the intersecting-submodular function f: $\mathcal{F} \to \mathbf{R}$ uniquely determines the submodular system $(\mathcal{D}_1, f_1)$ such that $\mathrm{P}(f) = \mathrm{P}(f_1)$. Moreover, from (3.247), f_1 is integer-valued if f is.

This completes the proof of (i) of Theorem 2.5.

3.5. Related Polyhedra

We show some polyhedra which are closely related to base polyhedra and submodular/supermodular polyhedra.

(a) Generalized polymatroids

A. Frank [Frank81b] introduced the concept of *generalized polymatroid.* Suppose that a submodular system $(\mathcal{D}_1, f')$ and a supermodular system $(\mathcal{D}_2, g')$ on E' satisfy

$$\begin{aligned}\forall X \in \mathcal{D}_1,\ \forall Y \in \mathcal{D}_2\colon\ & X - Y \in \mathcal{D}_1,\ Y - X \in \mathcal{D}_2, \\ & f'(X) - g'(Y) \geq f'(X - Y) - g'(Y - X).\end{aligned} \tag{3.251}$$

[Here, the unique maximal elements of $\mathcal{D}_1$ and $\mathcal{D}_2$ can be proper subsets of E', while we define submodular and supermodular polyhedra by systems of inequalities for $\mathcal{D}_1$ and $\mathcal{D}_2$ as in the original definitions.] Then the polyhedron $\mathrm{P}(f', g')$ defined by

$$\mathrm{P}(f', g') = \{x \mid x \in \mathbf{R}^{E'},\ \forall X \in \mathcal{D}_1 : x(X) \le f'(X),\\ \forall Y \in \mathcal{D}_2 : x(Y) \ge g'(Y)\} \quad (3.252)$$

is called a *generalized polymatroid* or *g-polymatroid*. (The same polyhedron is also considered by R. Hassin [Hassin78,82] for the case where $\mathcal{D}_1 = \mathcal{D}_2 = 2^{E'}$.)

Generalized polymatroids are characterized by the following theorem, which is implicit in [Frank81b] (also see [Schrijver84a]) (see Fig. 3.7).

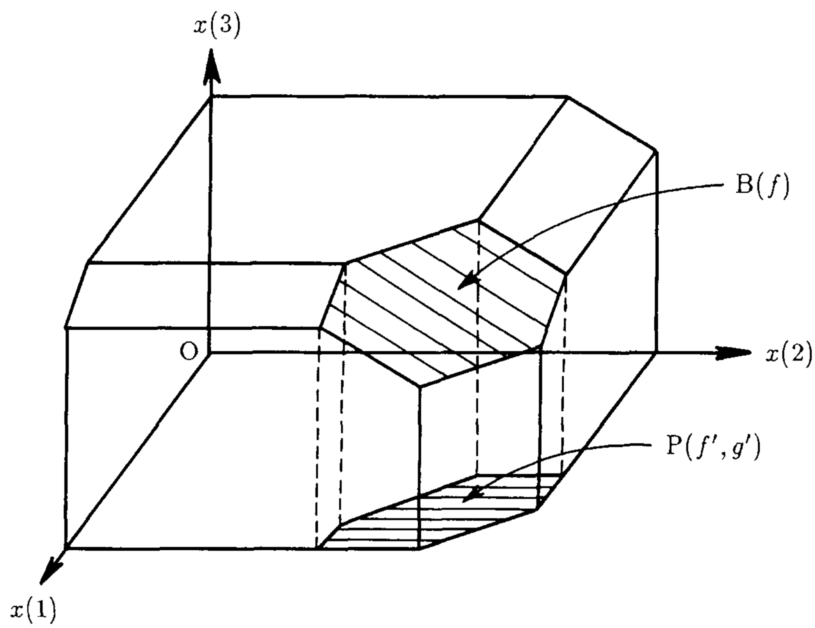


Figure 3.7: A generalized polymatroid as a projection of a base polyhedron.

Theorem 3.58 [Fuji84a]: *For the base polyhedron* $\mathrm{B}(f)$ *associated with a submodular system* $(\mathcal{D}, f)$ *on* E, *the projection of* $\mathrm{B}(f)$ *along an axis* $e \in E$ *on the hyperplane* $x(e) = 0$ *is a generalized polymatroid* $\mathrm{P}(f', g')$ *in* $\mathbf{R}^{E'}$

with $E' = E - \{e\}$, *where*

$$\mathcal{D}_1 = \{X \mid e \notin X \in \mathcal{D}\}, \tag{3.253}$$

$$\mathcal{D}_2 = \{E - X \mid e \in X \in \mathcal{D}\}, \tag{3.254}$$

f' *is the restriction of* f *to* $\mathcal{D}_1$, *and* g' *is the restriction of* $f^{\#}$ *to* $\mathcal{D}_2$.

Conversely, every generalized polymatroid in $\mathbf{R}^{E'}$ *is obtained in this way. For each generalized polymatroid* $\mathrm{P}(f', g')$ *with* $\mathcal{D}_i \subseteq 2^{E'}$ $(i = 1, 2)$ *such a base polyhedron* $\mathrm{B}(f)$ *in* $\mathbf{R}^E$ *with* $E = E' \cup \{e\}$ *is unique up to translation along the new axis* e, *and the two polyhedra* $\mathrm{P}(f', g')$ *and* $\mathrm{B}(f)$ *are isomorphic with each other under the projection of the hyperplane* $x(E) = f(E)$ *onto the hyperplane* $x(e) = 0$ *along the axis* e.

(Proof) The base polyhedron $\mathrm{B}(f)$ is the solution set of

$$x(X) \leq f(X) \quad (X \in \mathcal{D}), \tag{3.255}$$

$$x(E) = f(E). \tag{3.256}$$

Choose an element $e \in E$. From (3.256) we have

$$x(e) = f(E) - x(E - \{e\}). \tag{3.257}$$

Substituting (3.257) into (3.255), we have

$$\forall X \in \mathcal{D} \text{ with } e \notin X\colon\ x(X) \leq f(X), \tag{3.258}$$

$$\forall X \in \mathcal{D} \text{ with } e \in X\colon\ x(E - X) \geq f(E) - f(X). \tag{3.259}$$

(3.258) and (3.259) is rewritten as

$$\forall X \in \mathcal{D}_1\colon\ x(X) \leq f'(X), \tag{3.260}$$

$$\forall Y \in \mathcal{D}_2\colon\ x(Y) \geq g'(Y), \tag{3.261}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are, respectively, defined by (3.253) and (3.254), f' is the restriction of f to $\mathcal{D}_1$ and g' is the restriction of $f^{\#}$ to $\mathcal{D}_2$. It follows from (3.260) and (3.261) that the projection of $\mathrm{B}(f)$ along the axis e on the hyperplane $x(e) = 0$ is the generalized polymatroid $\mathrm{P}(f', g')$ in $\mathbf{R}^{E'}$ with $E' = E - \{e\}$. Note that (3.251) follows from the submodularity of f.

Now, we show the converse. For an arbitrary generalized polymatroid $\mathrm{P}(f', g')$ in $\mathbf{R}^{E'}$ with a submodular system $(\mathcal{D}_1, f')$ and a supermodular system $(\mathcal{D}_2, g')$ on E' let e be a new element not in E' and define

$$E = E' \cup \{e\}, \tag{3.262}$$

$$\overline{\mathcal{D}_2}^{\circ} = \{E - X \mid X \in \mathcal{D}_2\}, \tag{3.263}$$

$$\mathcal{D} = \mathcal{D}_1 \cup \overline{\mathcal{D}_2}^{\circ}. \tag{3.264}$$

Then, $\mathcal{D}$ is a distributive lattice with $\emptyset, E \in \mathcal{D}$ due to (3.251). Also, define a function $f\colon\ \mathcal{D} \to \mathbf{R}$ by

$$f(X) = f'(X) \qquad (X \in \mathcal{D}_1), \tag{3.265}$$

$$f(X) = f(E) - g'(E - X) \qquad (X \in \overline{\mathcal{D}_2}^{\circ}), \tag{3.266}$$

$$f(E) = c, \tag{3.267}$$

where $c \in \mathbf{R}$ is arbitrary but fixed. From (3.251), (3.265) and (3.266) we have for any $X \in \mathcal{D}_1$ and $Y \in \overline{\mathcal{D}_2}^{\circ}$

$$\begin{aligned} f(X) + f(Y) &= f'(X) + f(E) - g'(E - Y) \\ &\geq f'(X - (E - Y)) + f(E) - g'((E - Y) - X) \\ &= f'(X \cap Y) + f(E) - g'(E - (X \cup Y)) \\ &= f(X \cap Y) + f(X \cup Y). \end{aligned} \tag{3.268}$$

Therefore, $f\colon\ \mathcal{D} \to \mathbf{R}$ is a submodular function on the distributive lattice $\mathcal{D}$. Moreover, for some $\alpha \in \mathbf{R}$, $(x, \alpha) \in \mathrm{B}(f)$ if and only if

$$\forall X \in \mathcal{D}_1\colon\ x(X) \leq f(X) = f'(X), \tag{3.269}$$

$$\forall X \in \overline{\mathcal{D}_2}^{\circ}\colon\ x(X - \{e\}) + \alpha \leq f(X) = f(E) - g'(E - X), \tag{3.270}$$

$$x(E') + \alpha = f(E). \tag{3.271}$$

Eliminating α in (3.270) by using (3.271), we have

$$\forall X \in \overline{\mathcal{D}_2}^{\circ}\colon\ x(E - X) \geq g'(E - X) \tag{3.272}$$

or

$$\forall Y \in \mathcal{D}_2\colon\ x(Y) \geq g'(Y). \tag{3.273}$$

Therefore, for the submodular system $(\mathcal{D}, f)$ defined by (3.262)$\sim$(3.267)

$$\mathrm{P}(f', g') = \{x \mid x \in \mathbf{R}^E,\ \exists \alpha \in \mathbf{R}\colon\ (x, \alpha) \in \mathrm{B}(f)\}. \tag{3.274}$$

Moreover, it is clear that $\mathrm{P}(f', g')$ and $\mathrm{B}(f)$ are isomorphic with each other under the projection of the hyperplane $x(E) = f(E)$ onto the hyperplane $x(e) = 0$ along the axis e. Q.E.D.

It follows from Theorem 3.58 that extreme points, extreme rays, faces etc. of $\mathrm{P}(f', g')$ are characterized by the corresponding results for $\mathrm{B}(f)$ (cf. Section 3.3). Moreover, since the greedy algorithm works for $\mathrm{B}(f)$, it also works for $\mathrm{P}(f', g')$ *mutatis mutandis* (cf. [Hassin82]).

When $f'\colon 2^{E'} \to \mathbf{Z}_+$ is the rank function of a matroid $\mathbf{M}_1$ on E', $g'\colon 2^{E'} \to \mathbf{Z}_+$ is the dual supermodular function of the rank function of a matroid $\mathbf{M}_2$ on E' and the pair of f' and g' defines a generalized polymatroid, we call the ordered pair $(\mathbf{M}_1, \mathbf{M}_2)$ a *strong map*. This definition of strong map is equivalent to the ordinary one (see, e.g., [Welsh76]). If $(\mathbf{M}_1, \mathbf{M}_2)$ is a strong map and the rank of $\mathbf{M}_1$ is equal to the rank of $\mathbf{M}_2$ plus one, the submodular system $(\mathcal{D}, f)$ on $E = E' \cup \{e\}$ appearing in Theorem 3.58 is a matroid $\mathbf{M}_3$ such that $\mathbf{M}_3 - \{e\} = \mathbf{M}_1$ and $\mathbf{M}_3/\{e\} = \mathbf{M}_2$, if we define $f(E) = f'(E')$.

Frank [Frank81b] originally defined generalized polymatroids in terms of intersecting families. This corresponds to the fact that a crossing-submodular function on a crossing family determines the base polyhedron associated with a submodular system (see Theorem 2.5). Note that if $\mathcal{D} \subseteq 2^E$ is a crossing family, then $\mathcal{D}_i$ $(i = 1, 2)$ defined by (3.253) and (3.254) are intersecting families. (For more details on generalized polymatroids, see [Frank+Tardos88].)

(b) Polypseudomatroids

The concept of pseudomatroid was introduced by R. Chandrasekaran and S. N. Kabadi [Chandrasekaran+Kabadi88]. The same or similar concepts were independently considered by A. Bouchet [Bouchet87] as Δ-matroid, A. Dress and T. F. Havel [Dress+Havel86] as metroid, M. Nakamura [Nakamura88b] as universal polymatroid and L. Qi [Qi88] as ditroid. We shall consider (poly-)pseudomatroids of Chandrasekaran and Kabadi from the point of view of submodular systems.

Denote by 3^E the set of all the ordered pairs (X, Y) of disjoint subsets of E. [An element (X, Y) of 3^E is identified with a $\{0, \pm 1\}$-vector $\chi_{(X,Y)} \in \mathbf{R}^E$ defined by $\chi_{(X,Y)}(e) = 1$ for $e \in X$, $\chi_{(X,Y)}(e) = -1$ for $e \in Y$ and $\chi_{(X,Y)}(e) = 0$ for $e \in E - (X \cup Y)$. Such an ordered pair (X, Y) is called a *signed set*.] Let $f\colon 3^E \to \mathbf{R}$ be a function with $f(\emptyset, \emptyset) = 0$ such that for each $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$

$$
\begin{aligned}
f(X_1, Y_1) + f(X_2, Y_2) \geq\ & f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) \\
& + f(X_1 \cap X_2, Y_1 \cap Y_2). \qquad (3.275)
\end{aligned}
$$

Define a polyhedron

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^E,\ \forall (X,Y) \in 3^E\colon\ x(X) - x(Y) \le f(X,Y)\}. \quad (3.276)$$

The polyhedron $\mathrm{P}_*(f)$ is called a *polypseudomatroid* ([Chandrasekaran+Kabadi88], [Kabadi + Chandrasekaran90]) and f its *rank function* (see Fig. 3.8). Polyhedral studies are also made in [Nakamura88b] and [Qi88,89]. A *pseudomatroid* is a set-theoretical version of a polypseudomatroid. [Since $f(X,Y)$ is submodular both in X with any fixed Y and in Y with any fixed X, f is called a *bisubmodular function* (like 'bilinear' in a bilinear form) and a polypseudomatroid a *bisubmodular polyhedron* later in [Bouchet+Cunningham95], [Ando+Fuji96], [Fuji+Patkar95], etc. (Note that $f(X,Y)$ is not submodular in (X,Y) in general as a bilinear form is not linear.) Also see [Borovik+Gelfand+White03] for related topics in Coxeter matroids.]

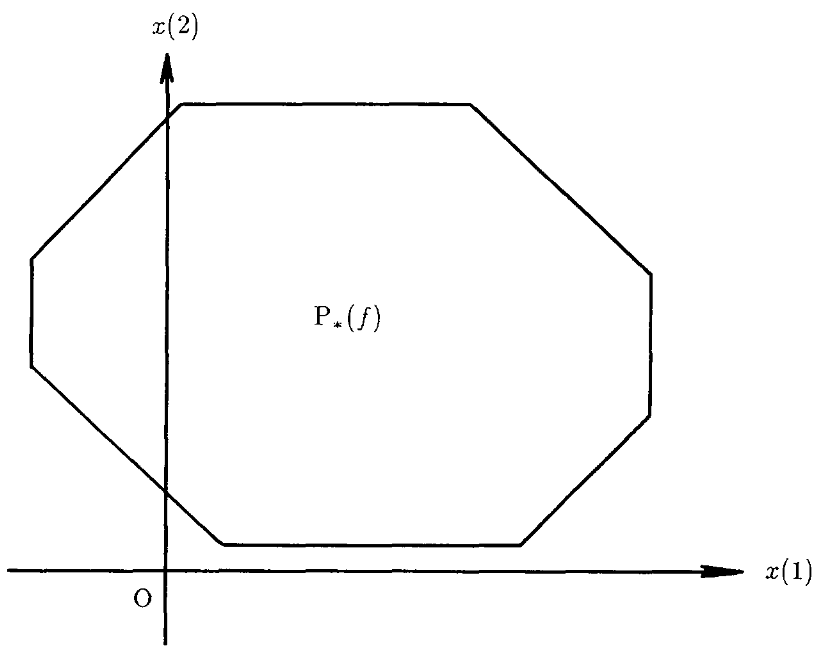


Figure 3.8: A polypseudomatroid.

We call a pair $(S,T) \in 3^E$ such that $S \cup T = E$ an *orthant* of $\mathbf{R}^E$. For each orthant (S,T) denote by $2^{(S,T)}$ the set of all the pairs (X,Y) such that $X \subseteq S$ and $Y \subseteq T$.

Now, choose an orthant (S,T). Then for each $(X_i, Y_i) \in 2^{(S,T)}$ $(i = 1,2)$ we have from (3.275)

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f(X_1 \cup X_2, Y_1 \cup Y_2) + f(X_1 \cap X_2, Y_1 \cap Y_2). \quad (3.277)$$

This means that for the orthant (S,T) the function f': $2^E \to \mathbf{R}$ defined by

$$f'(X) = f(S \cap X, T \cap X) \qquad (X \subseteq E) \quad (3.278)$$

is a submodular function on 2^E. Define

$$\mathrm{P}_{(S,T)}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall (X,Y) \in 2^{(S,T)}: x(X) - x(Y) \leq f(X,Y)\}. \quad (3.279)$$

The polyhedron $\mathrm{P}_{(S,T)}(f)$ is expressed by the submodular polyhedron $\mathrm{P}(f')$ associated with the submodular system $(2^E, f')$ as follows.

$$\mathrm{P}_{(S,T)}(f) = \{x \mid y \in \mathrm{P}(f'),\ \forall e \in S: x(e) = y(e), \forall e \in T: x(e) = -y(e)\}. \quad (3.280)$$

Therefore, the combinatorial properties of $\mathrm{P}_{(S,T)}(f)$ are the same as those of $\mathrm{P}(f')$ and the greedy algorithm described in Section 3.2.b works for $\mathrm{P}_{(S,T)}(f)$ *mutatis mutandis* as for $\mathrm{P}(f')$. Define

$$\mathrm{B}_{(S,T)}(f) = \{x \mid x \in \mathrm{P}_{(S,T)}(f),\ x(S) - x(T) = f(S,T)\}. \quad (3.281)$$

We call $\mathrm{B}_{(S,T)}(f)$ the *base polyhedron in the orthant* (S,T) of the polypseudomatroid $\mathrm{P}_*(f)$. When $f(S,T) = 0$ and $\mathrm{B}_{(S,T)}(f) \subseteq \mathbf{R}^E_+$, $\mathrm{B}_{(S,T)}(f)$ is the set of *linked vectors* of a *polylinking system* (or a *polybimatroid*); a set theoretical version of the system is called a *linking system* (or a *bimatroid*) (see [Schrijver78,79] and [Kung78]).

From Theorem 3.22 we have

Corollary 3.59: *A vector* $x \in \mathbf{R}^E$ *is an extreme point of* $\mathrm{B}_{(S,T)}(f)$ *in* (3.281) *if and only if for a maximal chain*

$$\mathcal{C}\colon\ \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = E \quad (3.282)$$

of 2^E *we have*

$$\begin{aligned} & f(S \cap A_i, T \cap A_i) - f(S \cap A_{i-1}, T \cap A_{i-1}) \\ &= \begin{cases} x(A_i - A_{i-1}) & \text{if } A_i - A_{i-1} \subseteq S \\ -x(A_i - A_{i-1}) & \text{if } A_i - A_{i-1} \subseteq T \end{cases} \end{aligned} \quad (3.283)$$

for each $i = 1, 2, \cdots, n$.

We also have

Lemma 3.60: *For each orthant* (S, T) *of* $\mathbf{R}^E$ *we have*

$$\mathrm{B}_{(S,T)}(f) \subseteq \mathrm{P}_*(f). \tag{3.284}$$

(Proof) Suppose $x \in \mathrm{B}_{(S,T)}(f)$. Then, for any $(X, Y) \in 3^E$ we have from (3.275)

$$\begin{aligned}
&x(X) - x(Y) - f(X, Y) \\
&\quad = \; x(X) - x(Y) - f(X, Y) + x(S) - x(T) - f(S, T) \\
&\quad \leq \; x(S - Y) - x(T - X) - f(S - Y, T - X) \\
&\qquad\qquad + x(S \cap X) - x(T \cap Y) - f(S \cap X, T \cap Y) \\
&\quad \leq \; 0.
\end{aligned} \tag{3.285}$$

Hence $x \in \mathrm{P}_*(f)$. Q.E.D.

We see from Lemma 3.60 that $\mathrm{B}_{(S,T)}(f)$ for each orthant (S, T) is a face of $\mathrm{P}_*(f)$. Since

$$\mathrm{P}_*(f) = \bigcap \{\mathrm{P}_{(S,T)}(f) \mid (S, T)\text{: an orthant of } \mathbf{R}^E\}, \tag{3.286}$$

for any w: $E \to \mathbf{R}$ we have from Lemma 3.60

$$\begin{aligned}
&\max\{\sum_{e \in E} w(e)x(e) \;\mid\; x \in \mathrm{P}_{(S,T)}(f)\} \\
&\quad \geq \; \max\{\sum_{e \in E} w(e)x(e) \;\mid\; x \in \mathrm{P}_*(f)\} \\
&\quad \geq \; \max\{\sum_{e \in E} w(e)x(e) \;\mid\; x \in \mathrm{B}_{(S,T)}(f)\}.
\end{aligned} \tag{3.287}$$

For an orthant (S, T) such that $w(e) \geq 0$ $(e \in S)$ and $w(e) \leq 0$ $(e \in T)$, (3.287) holds with equality. Therefore, the problem of maximizing the linear function $\sum_{e \in E} w(e)x(e)$ over the polypseudomatroid $\mathrm{P}_*(f)$ is solved by maximizing the same linear function over $\mathrm{P}_{(S,T)}(f)$ or $\mathrm{B}_{(S,T)}(f)$. Hence the greedy algorithm as in Corollary 3.59 works for the polypseudomatroid

$\mathrm{P}_*(f)$ and the union of all the extreme points of $\mathrm{B}_{(S,T)}(f)$ for all the orthants (S,T) is exactly the set of all the extreme points of $\mathrm{P}_*(f)$. Also, we see from Lemma 3.60 (and Lemma 3.2) that for each $(X,Y) \in 3^E$

$$f(X,Y) = \max\{x(X) - x(Y) \mid x \in \mathrm{P}_*(f)\}, \tag{3.288}$$

so that f is uniquely determined by $\mathrm{P}_*(f)$.

The greedy algorithm is given as follows. Here, note that we consider a maximization problem rather than a minimization problem (cf. Section 3.2.b).

Greedy algorithm for polypseudomatroids

Step 1: Find a sequence $(e_1, e_2, \cdots, e_n)$ of all the elements of E such that $|w(e_1)| \geq |w(e_2)| \geq \cdots \geq |w(e_n)|$. Choose an orthant (S,T) such that $w(e) \geq 0$ $(e \in S)$ and $w(e) \leq 0$ $(e \in T)$.

Step 2: Let A_i be the set of the first i elements of the sequence $(e_1, e_2, \cdots, e_n)$ for each $i = 1, 2, \cdots, n$ and compute a vector $x \in \mathbf{R}^E$ by (3.283). (x is a maximizer of the linear function $\sum_{e \in E} w(e)x(e)$ over the polypseudomatroid $P_*(f)$.)

(End)

Theorem 3.61 [Chandrasekaran+Kabadi88], [Nakamura88b]: *For any function f: $3^E \to \mathbf{R}$ define*

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^E,\ \forall (X,Y) \in 3^E : x(X) - x(Y) \leq f(X,Y)\}. \tag{3.289}$$

The greedy algorithm of the type described above works for $\mathrm{P}_*(f)$ if and only if f is the rank function of a polypseudomatroid.

(Proof) It suffices to show the "only if" part. Suppose that the greedy algorithm described above works for $\mathrm{P}_*(f)$. Then, for any $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$ choose a maximal chain $\mathcal{C}$ of (3.282) containing $(X_1 \cap X_2) \cup (Y_1 \cap Y_2)$ and $((X_1 \cup X_2) - (Y_1 \cup Y_2)) \cup ((Y_1 \cup Y_2) - (X_1 \cup X_2))$ in it and define the vector $x \in \mathbf{R}^E$ by (3.283), where (S,T) is an orthant such that $(X_1 \cup X_2) - (Y_1 \cup Y_2) \subseteq S$ and $(Y_1 \cup Y_2) - (X_1 \cup X_2) \subseteq T$. From the assumption we have $x \in \mathrm{P}_*(f)$. It follows from the definition of x that

$$x(X_i) - x(Y_i) \leq f(X_i, Y_i) \quad (i = 1, 2), \tag{3.290}$$

$$\begin{aligned} x((X_1 \cup X_2) - (Y_1 \cup Y_2)) - x((Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ = f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \end{aligned} \tag{3.291}$$

$$x(X_1 \cap X_2) - x(Y_1 \cap Y_2) = f(X_1 \cap X_2, Y_1 \cap Y_2). \tag{3.292}$$

From (3.290)~(3.292),

$$\begin{aligned} & f(X_1, Y_1) + f(X_2, Y_2) \\ \geq \;& x(X_1) - x(Y_1) + x(X_2) - x(Y_2) \\ = \;& x((X_1 \cup X_2) - (Y_1 \cup Y_2)) - x((Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ & \qquad + x(X_1 \cap X_2) - x(Y_1 \cap Y_2) \\ = \;& f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ & \qquad + f(X_1 \cap X_2, Y_1 \cap Y_2). \end{aligned} \tag{3.293}$$

It follows that f is the rank function of a polypseudomatroid. Q.E.D.

Theorem 3.62 [Chandrasekaran+Kabadi88]: *The system of inequalities*

$$x(X) - x(Y) \leq f(X, Y) \qquad ((X, Y) \in 3^E) \tag{3.294}$$

is totally dual integral.

(Proof) The present theorem follows from Lemma 3.60 and Corollary 3.21. Q.E.D.

The concept of polypseudomatroid explained above can be generalized as follows. Let $\mathcal{F}$ be a subset of 3^E such that for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ the two pairs $((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2))$ and $(X_1 \cap X_2, Y_1 \cap Y_2)$ belong to $\mathcal{F}$. Also let f: $\mathcal{F} \to \mathbf{R}$ be a function such that for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$ f satisfies (3.275). [We call $(\mathcal{F}, f)$ a *bisubmodular system* on E.] The class of polypseudomatroids [(bisubmodular polyhedra)] in this generalized sense includes as special cases submodular and supermodular polyhedra, base polyhedra and generalized polymatroids. Further generalization was considered by Qi [Qi88,89].

[For further developments and related results see [Cunningham + Green-Krótki91], [Bouchet + Cunningham95], [Ando + Fuji96], [Ando + Fuji + Nemoto96], [Fuji97], [Ando + Fuji + Naitoh95], [Fuji + Patkar95], [Lovász97] and [Geelen + Iwata + Murota03]. Similarly as submodular functions on distributive lattices (or ring families), bisubmodular functions on signed ring families are considered in [Ando + Fuji96], where a signed counterpart of Birkhoff's theorem on the poset representations of distributive lattices is also shown (also see [Reiner93]). Moreover, a class of related polyhedra, called polybasic polyhedra, that have edge vectors of support size at most two was investigated in [Fuji+ Makino+Takabatake+Kashiwabara04].]

(c) Ternary semimodular polyhedra

In the preceding subsection 3.5.b we considered the set 3^E of all the ordered pairs (X, Y) of disjoint subsets X and Y of E. 3^E is a distributive lattice with lattice operations $\vee$ and $\wedge$ defined as follows. For any $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$ define

$$(X_1, Y_1) \vee (X_2, Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2), \tag{3.295}$$

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cup Y_2). \tag{3.296}$$

Identifying each $(X, Y) \in 3^E$ with a $\{0, \pm 1\}$-vector $\chi(X, Y)$ defined by

$$\chi(X, Y)(e) = \begin{cases} 1 & (e \in X) \\ -1 & (e \in Y) \\ 0 & (e \in E - (X \cup Y)), \end{cases} \tag{3.297}$$

we see that operations $\vee$ and $\wedge$ in (3.295) and (3.296) are ordinary ones in vector lattice $\mathbf{R}^E$ and that 3^E is a finite sublattice of $\mathbf{R}^E$.

Let $\preceq$ be the partial order on 3^E associated with the distributive lattice 3^E. We call a pair of $(X_i, Y_i) \in 3^E$ $(i = 1, 2)$ *comparable* if we have $(X_1, Y_1) \preceq (X_2, Y_2)$ or $(X_2, Y_2) \preceq (X_1, Y_1)$. We also use the partial order $\preceq$ for $\{0, \pm 1\}$-vectors under the correspondence (3.297).

Let $\hat{\mathcal{D}}$ be a sublattice of the distributive lattice 3^E and $\hat{f}\colon \hat{\mathcal{D}} \to \mathbf{R}$ be a submodular function on $\hat{\mathcal{D}}$, i.e., for each $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i = 1, 2)$

$$\hat{f}(X_1, Y_1) + \hat{f}(X_2, Y_2) \geq \hat{f}(X_1 \cup X_2, Y_1 \cap Y_2) + \hat{f}(X_1 \cap X_2, Y_1 \cup Y_2), \tag{3.298}$$

where we write $\hat{f}(X, Y)$ instead of $\hat{f}((X, Y))$. Also, define a polyhedron $\hat{\mathrm{P}}(\hat{f})$ by

$$\hat{\mathrm{P}}(\hat{f}) = \{x \mid x \in \mathbf{R}^E,\ \forall (X, Y) \in \hat{\mathcal{D}}\colon\ x(X) - x(Y) \leq \hat{f}(X, Y)\}. \tag{3.299}$$

Without any additional condition the polyhedron $\hat{\mathrm{P}}(\hat{f})$ defined by (3.299) may be empty. The following theorem shows that a trivial necessary condition for $\hat{\mathrm{P}}(\hat{f})$ to be nonempty is also sufficient.

Theorem 3.63 [Fuji84e]: *$\hat{\mathrm{P}}(\hat{f})$ is nonempty if and only if*

$$\hat{f}(\emptyset, X) + \hat{f}(X, \emptyset) \geq 0 \tag{3.300}$$

for each $X \subseteq E$ such that $(\emptyset, X), (X, \emptyset) \in \hat{\mathcal{D}}$.

(Proof) The "only if" part is trivial. We show the "if" part. Suppose (3.300) holds for each $X \subseteq E$ such that $(\emptyset, X), (X, \emptyset) \in \hat{\mathcal{D}}$. It follows from the Farkas lemma or the linear programming duality theorem that $\hat{\mathrm{P}}(\hat{f}) \neq \emptyset$ if (and only if) for all positive $\alpha_i \in \mathbf{R}$ $(i \in I)$ and $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i \in I)$ with a finite index set I such that

$$\sum_{i \in I} \alpha_i \chi(X_i, Y_i) = \mathbf{0} \tag{3.301}$$

we have

$$\sum_{i \in I} \alpha_i \hat{f}(X_i, Y_i) \geq 0. \tag{3.302}$$

Here, note that $\chi(X_i, Y_i)$ is the coefficient vector of the inequality $x(X_i) - x(Y_i) \leq \hat{f}(X_i, Y_i)$. Since the vectors $\chi(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ are integral, we can restrict the positive real coefficients α_i $(i \in I)$ in (3.301) to positive integers. It follows that $\hat{\mathrm{P}}(f) \neq \emptyset$ if (and only if) for all $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i \in I)$ with a finite index set I such that

$$\sum_{i \in I} \chi(X_i, Y_i) = \mathbf{0} \tag{3.303}$$

we have

$$\sum_{i \in I} \hat{f}(X_i, Y_i) \geq 0, \tag{3.304}$$

where possibly $(X_i, Y_i) = (X_j, Y_j)$ for distinct $i, j \in I$. If the family of (X_i, Y_i) $(i \in I)$ in (3.303) contains an incomparable pair of (X_1, Y_1) and (X_2, Y_2), then put

$$(X_1, Y_1) \leftarrow (X_1 \cup X_2, Y_1 \cap Y_2), \tag{3.305a}$$

$$(X_2, Y_2) \leftarrow (X_1 \cap X_2, Y_1 \cup Y_2). \tag{3.305b}$$

After the replacement (3.305) relation (3.303) remains valid and the value of the left-hand side of (3.304) does not increase, due to the submodularity of f. Since we get a family of (X_i, Y_i) $(i \in I)$ composed of pairwise comparable elements of $\hat{\mathcal{D}}$ after a finite number of such replacements, it suffices to show that we have (3.304) for all $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i \in I)$ such that (X_i, Y_i) $(i \in I)$ are pairwise comparable and (3.303) holds. Therefore, suppose that (3.303) holds for (X_i, Y_i) $(i \in I = \{1, 2, \cdots, m\})$ and

$$(X_1, Y_1) \preceq (X_2, Y_2) \preceq \cdots \preceq (X_m, Y_m). \tag{3.306}$$

We see from (3.303) and (3.306) that

$$X_1 = \emptyset, \qquad Y_m = \emptyset. \tag{3.307}$$

Suppose $|I| \geq 2$. If $X_m - Y_1 \neq \emptyset$, then for any $e \in X_m - Y_1$ we have

$$\sum_{i \in I} \chi(X_i, Y_i)(e) > 0, \tag{3.308}$$

which contradicts (3.303). Similarly, $Y_1 - X_m \neq \emptyset$ leads to a contradiction. Hence, we must have

$$Y_1 = X_m. \tag{3.309}$$

Putting $Z_1 = Y_1 (= X_m)$, we have from (3.307) and (3.309)

$$\chi(X_1, Y_1) + \chi(X_m, Y_m) = \chi(\emptyset, Z_1) + \chi(Z_1, \emptyset) = \mathbf{0}. \tag{3.310}$$

From (3.310) and assumption (3.300),

$$\hat{f}(X_1, Y_1) + \hat{f}(X_m, Y_m) = \hat{f}(\emptyset, Z_1) + \hat{f}(Z_1, \emptyset) \geq 0. \tag{3.311}$$

Put $I \leftarrow I - \{1, m\}$. For the new index set I (3.303) still holds. Repeat the above argument until $|I| = 0$ or 1. When $|I| = 1$, suppose $I = \{i_0\}$. Then, from (3.303),

$$\chi(X_{i_0}, Y_{i_0}) = \mathbf{0}, \tag{3.312}$$

$$X_{i_0} = \emptyset, \qquad Y_{i_0} = \emptyset. \tag{3.313}$$

Hence, from (3.300),

$$\hat{f}(X_{i_0}, Y_{i_0}) = \hat{f}(\emptyset, \emptyset) \geq 0. \tag{3.314}$$

It follows from (3.311) and (3.314) that

$$\sum_{i \in I} \hat{f}(X_i, Y_i) \geq 0, \tag{3.315}$$

where $I = \{1, 2, \cdots, m\}$, the original index set. Q.E.D.

Corollary 3.64: *Define a sublattice $\mathcal{D}_0$ of $\hat{\mathcal{D}}$ by*

$$\mathcal{D}_0 = \{(X, \emptyset) \mid (X, \emptyset) \in \hat{\mathcal{D}}\} \cup \{(\emptyset, X) \mid (\emptyset, X) \in \hat{\mathcal{D}}\}. \tag{3.316}$$

Let $\hat{f}_0$ be the restriction of $\hat{f}$ to $\hat{\mathcal{D}}_0$, and $\hat{\mathrm{P}}(\hat{f}_0)$ be the polyhedron given by (3.299) with $\hat{\mathcal{D}}$ and $\hat{f}$ replaced by $\hat{\mathcal{D}}_0$ and $\hat{f}_0$, respectively. Then, $\hat{\mathrm{P}}(\hat{f}) \neq \emptyset$ if and only if $\hat{\mathrm{P}}(\hat{f}_0) \neq \emptyset$.

Corollary 3.64 means that when $\hat{\mathrm{P}}(\hat{f}_0) \neq \emptyset$, the inequalities in (3.299) with $X \neq \emptyset$ and $Y \neq \emptyset$ do not cut off $\hat{\mathrm{P}}(\hat{f}_0)$ too much to give empty $\hat{\mathrm{P}}(\hat{f})$.

Moreover, it should be noted that if $(\emptyset, \emptyset) \in \hat{\mathcal{D}}$ and $\hat{f}(\emptyset, \emptyset) = 0$, then for each $(X, Y) \in \hat{\mathcal{D}}$

$$\hat{f}(X, Y) = \hat{f}(X, Y) + \hat{f}(\emptyset, \emptyset) \geq \hat{f}(X, \emptyset) + \hat{f}(\emptyset, Y), \tag{3.317}$$

so that the inequalities appearing in the right-hand side of (3.299) for $(X, Y) \in \hat{\mathcal{D}}$ with $X \neq \emptyset$ and $Y \neq \emptyset$ are redundant, i.e., $\hat{\mathrm{P}}(\hat{f}) = \hat{\mathrm{P}}(\hat{f}_0)$, where $\hat{f}_0$ is defined in Corollary 3.64.

Now, let us consider the following linear programming problem (P) with $c \in \mathbf{R}^E$.

$$(P)\text{: Maximize } \sum_{e \in E} c(e)x(e) \tag{3.318a}$$

$$\text{subject to } \forall (X, Y) \in \hat{\mathcal{D}}\text{: } x(X) - x(Y) \leq \hat{f}(X, Y). \tag{3.318b}$$

In general, the system of linear inequalities (3.318b) is not totally dual integral, and even if $\hat{f}$ is integer-valued, $\hat{\mathrm{P}}(\hat{f})$ may have non-integral extreme points. For example, consider

$$E = \{1, 2\}, \tag{3.319a}$$

$$\hat{\mathcal{D}} = \{(\{1, 2\}, \emptyset), (\{1\}, \{2\})\}, \tag{3.319b}$$

$$\hat{f}(\{1, 2\}, \emptyset) = 1, \quad \hat{f}(\{1\}, \{2\}) = 0. \tag{3.319c}$$

Then $\hat{\mathrm{P}}(\hat{f})$ has the only one vertex given by $(\frac{1}{2}, \frac{1}{2})$.

The linear programming dual of Problem (P) is described as follows.

$$(P^*)\text{: Minimize } \sum_{(X,Y) \in \hat{\mathcal{D}}} \lambda(X, Y)\hat{f}(X, Y) \tag{3.320a}$$

$$\text{subject to } \sum_{\substack{(X,Y) \in \hat{\mathcal{D}} \\ e \in X}} \lambda(X, Y) - \sum_{\substack{(X,Y) \in \hat{\mathcal{D}} \\ e \in Y}} \lambda(X, Y) = c(e) \quad (e \in E), \tag{3.320b}$$

$$\lambda(X, Y) \geq 0 \quad ((X, Y) \in \mathcal{D}). \tag{3.320c}$$

Though the system of linear inequalities (3.318b) is not totally dual integral, we have the following.

Lemma 3.65: *Let $c \in \mathbf{R}^E$ be an integral vector and suppose that Problem (P^*) has an optimal solution. Then there exists a rational optimal solution $\lambda^*(X,Y)$ $((X,Y) \in \hat{\mathcal{D}})$ of (P^*) such that*

$$\mathcal{E} = \{(X,Y) \mid \lambda^*(X,Y) > 0\} \tag{3.321}$$

consists of pairwise comparable elements of $\hat{\mathcal{D}}$.

(Proof) By the assumption there exists a rational optimal solution $\lambda^*(X,Y)$ $((X,Y) \in \hat{\mathcal{D}})$ of (P^*). For some positive rational d_0 each $\lambda^*(X,Y)$ $((X,Y) \in \hat{\mathcal{D}})$ is a nonnegative integral multiple of d_0.

If $\mathcal{E}$ given by (3.321) contains an incomparable pair of elements (X_i, Y_i) $(i = 1, 2)$, then define

$$\alpha = \min\{\lambda^*(X_i, Y_i) \mid i = 1, 2\} > 0 \tag{3.322}$$

and change the values of $\lambda^*(X_i, Y_i)$ $(i = 1, 2)$, $\lambda^*(X_1 \cup X_2, Y_1 \cap Y_2)$ and $\lambda^*(X_1 \cap X_2, Y_1 \cup Y_2)$ as

$$\lambda^*(X_i, Y_i) \leftarrow \lambda^*(X_i, Y_i) - \alpha \quad (i = 1, 2), \tag{3.323}$$

$$\lambda^*(X_1 \cup X_2, Y_1 \cap Y_2) \leftarrow \lambda^*(X_1 \cup X_2, Y_1 \cap Y_2) + \alpha, \tag{3.324}$$

$$\lambda^*(X_1 \cap X_2, Y_1 \cup Y_2) \leftarrow \lambda^*(X_1 \cap X_2, Y_1 \cup Y_2) + \alpha. \tag{3.325}$$

Then, new λ^* satisfies (3.320b) and (3.320c) and is an optimal solution of (P^*) since the objective function is not made increase by the above replacement (3.323)$\sim$(3.325) due to the submodularity of $\hat{f}$. Repeat (3.322)$\sim$(3.325) so long as $\mathcal{E}$ given by (3.321) contains an incomparable pair. Only a finite number of repetitions of (3.322)$\sim$(3.325) are possible since (1) all the generated λ^*'s are distinct, (2) each $\lambda^*(X,Y)$ $((X,Y) \in \hat{\mathcal{D}})$ is nonnegative integral multiple of $d_0 > 0$ and (3) the sum of all $\lambda^*(X,Y)$ $((X,Y) \in \hat{\mathcal{D}})$ is constant. Q.E.D.

The above proof is a direct adaptation of a standard proof technique developed by N. Robertson, L. Lovász [Lovász76], J. Edmonds and R. Giles [Edm+Giles77] and A. J. Hoffman [Hoffman74].

Now, consider an additional structure of $\hat{\mathcal{D}}$ and $\hat{f}$ as follows.

(†) If for $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i = 1, 2)$

$$(X_1, Y_1) \succeq (X_2, Y_2), \tag{3.326}$$

$$X_1 \cap Y_2 \neq \emptyset, \tag{3.327}$$

and

$$X_2 \neq \emptyset \quad \text{or} \quad Y_1 \neq \emptyset, \tag{3.328}$$

then $(X_1 - Y_2, Y_1)$, $(X_2, Y_2 - X_1) \in \hat{\mathcal{D}}$ and

$$\hat{f}(X_1, Y_1) + \hat{f}(X_2, Y_2) \geq \hat{f}(X_1 - Y_2, Y_1) + \hat{f}(X_2, Y_2 - X_1). \tag{3.329}$$

Note that (3.326)∼(3.328) implies that for some $e, e' \in E$ we have $\chi(X_1, Y_1)(e) = \chi(X_2, Y_2)(e) \neq 0$ and $\chi(X_1, Y_1)(e') = -\chi(X_2, Y_2)(e') \neq 0$.

Theorem 3.66 [Fuji84e]: *Suppose $\hat{\mathcal{D}}$ and $\hat{f}$ satisfy property (†) described by (3.326)∼(3.329). Then system (3.318b) of linear inequalities is totally dual integral.*

To prove Theorem 3.66 we need some lemmas.

The proof of the following lemma is similar to the proof technique developed by Robertson et al.

Lemma 3.67: *Suppose that $c \in \mathbf{R}^E$ is an integral vector, that Problem (P^*) has an optimal solution, and that $\hat{\mathcal{D}}$ and $\hat{f}\colon \hat{\mathcal{D}} \to \mathbf{R}$ satisfy the above additional property (†). Then there exists a rational optimal solution $\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ of (P^*) such that $\mathcal{E}$ given by (3.321) consists of pairwise comparable elements of $\hat{\mathcal{D}}$ and does not contain any (X_i, Y_i) $(i = 1, 2)$ which satisfy (3.326)∼(3.328).*

(Proof) By Lemma 3.65, let $\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ be a rational optimal solution of (P^*) such that $\mathcal{E}$ given by (3.321) consists of pairwise comparable elements of $\hat{\mathcal{D}}$. Also let d_0 be a positive rational number such that each $\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ is a nonnegative integral multiple of d_0.

If $\mathcal{E}$ contains (X_i, Y_i) $(i = 1, 2)$ which satisfy (3.326)∼(3.328), then let (X_i, Y_i) $(i = 1, 2)$ be elements of $\mathcal{E}$ for which

(‡) (3.326)∼(3.328) hold and for each $(X_3, Y_3) \in \mathcal{E}$ such that $(X_1, Y_1) \succeq (X_3, Y_3) \succeq (X_2, Y_2)$ we have

$$X_1 \cap Y_2 \subseteq E - (X_3 \cup Y_3). \tag{3.330}$$

(Note that there always exists such a pair of (X_i, Y_i) $(i = 1, 2)$.) Define α by

$$\alpha = \min\{\lambda^*(X_i, Y_i) \mid i = 1, 2\} > 0. \tag{3.331}$$

Change the values of $\lambda^*(X_i, Y_i)$ $(i = 1, 2)$, $\lambda^*(X_1 - Y_2, Y_1)$ and $\lambda^*(X_2, Y_2 - X_1)$ as

$$\lambda^*(X_i, Y_i) \leftarrow \lambda^*(X_i, Y_i) - \alpha \quad (i = 1, 2), \tag{3.332}$$

$$\lambda^*(X_1 - Y_2, Y_1) \leftarrow \lambda^*(X_1 - Y_2, Y_1) + \alpha, \tag{3.333}$$

$$\lambda^*(X_2, Y_2 - X_1) \leftarrow \lambda^*(X_2, Y_2 - X_1) + \alpha. \tag{3.334}$$

The new λ^* satisfies (3.320b) and (3.320c) and is an optimal solution of (P^*) because of (3.329). It follows from (3.330) that the new λ^* gives $\mathcal{E}$ in (3.321) which consists of pairwise comparable elements of $\hat{\mathcal{D}}$.

Repeat (3.331)$\sim$(3.334) so long as $\mathcal{E}$ in (3.321) contains (X_i, Y_i) $(i = 1, 2)$ which satisfy the above (‡). Then, all the generated λ^*s are distinct, because each changing of λ^* by (3.332)$\sim$(3.334) increases the total sum of $|E - (X \cup Y)|\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ by $2|X_1 \cap Y_2|\alpha > 0$. Also each $\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ is a nonnegative integral multiple of $d_0 > 0$, and the sum of $\lambda^*(X, Y)$ $((X, Y) \in \hat{\mathcal{D}})$ is constant. Therefore, after a finite number of steps we get a desired optimal solution λ^* of (P^*). Q.E.D.

Let $A = (a(i, j) \mid i \in I,\ j \in J)$ be a $p \times q$ matrix with a row index set $I = \{1, 2, \cdots, p\}$ and a column index set $J = \{1, 2, \cdots, q\}$. For an integer t, we say A has the *upper consecutive t's property* (or the *lower consecutive t's property*) if $a(i_0, j_0) = t$ for a pair of $i_0 \in I$ and $j_0 \in J$ implies $a(i, j_0) = t$ for all $i \in I$ with $i \le i_0$ (or $i \ge i_0$). Also, we say A has the *consecutive t's property* if $a(i_1, j_0) = a(i_2, j_0) = t$ for some $i_1, i_2 \in I$ and $j_0 \in J$ with $i_1 < i_2$ implies $a(i, j_0) = t$ for all $i \in I$ with $i_1 \le i \le i_2$. Denote the ith row and the jth column of $A = (a(i, j) \mid i \in I,\ j \in J)$ by $a(i, \cdot)$ and $a(\cdot, j)$, respectively.

A rational matrix A is called *totally unimodular* if the determinant of every square submatrix of A is equal to 0, 1 or -1.

Lemma 3.68: *Let $I = \{1, 2, \cdots, p\}$ and $J = \{1, 2, \cdots, q\}$. Suppose $A = (a(i, j) \mid i \in I,\ j \in J)$ is a $\{0, \pm 1\}$-matrix such that*

(i) *any pair of rows $a(i_1, \cdot)$ and $a(i_2, \cdot)$ of A $(i_1, i_2 \in I)$ is comparable and*

(ii) *A does not contain, as a submatrix, any of the following* 2×2 *matrices*

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

and the ones obtained from these matrices by row and column permutations.

Then A is totally unimodular.

(Proof) Suppose that the rows of A are indexed so that

$$a(p,\cdot) \preceq a(p-1,\cdot) \preceq \cdots \preceq a(1,\cdot). \tag{3.335}$$

Also suppose $q = p$. We show that the possible values of the determinant of the (square) matrix A are 0, 1 and -1. Since the following argument is valid for any square submatrix of the original matrix A, this implies that A is totally unimodular.

Define

$$I(+) = \{i \mid i \in I,\ \mathbf{0} \prec a(i,\cdot)\}, \tag{3.336}$$

$$I(-) = \{i \mid i \in I,\ a(i,\cdot) \preceq \mathbf{0}\}, \tag{3.337}$$

$$I(+,-) = \{1,2,\cdots,p\} - (I(+) \cup I(-)), \tag{3.338}$$

where $\mathbf{0}$ is the zero row vector of dimension $|J|(=|I|)$. From (3.335), A has the upper consecutive 1's property and the lower consecutive -1's property.

If $I(+) \cup I(+,-) = \emptyset$, then A is totally unimodular and $\det A = 0$, 1 or -1, since A is a $\{0,-1\}$ matrix and has the consecutive -1's property[Hoffman+Kruskal56]. Therefore, suppose $I(+) \cup I(+,-) \neq \emptyset$. Let $a(\cdot, j_0)$ be a column of A such that $a(i,j_0) = 1$ for all $i \in I(+) \cup I(+,-)$. (Such a column exists because of (3.336), (3.338) and the upper consecutive 1's property of A.) If $a(i_0,j_0) = -1$ for some $i_0 \in I(-)$, then $I(+,-) = \emptyset$, since otherwise for an arbitrary $i_1 \in I(+,-)$ we have $a(i_1,j_0) = 1$ and there exists a column $a(\cdot,j_1)$ such that $a(i_1,j_1) = a(i_0,j_1) = -1$, which contradicts the assumption. When $I(+,-) = \emptyset$, A is totally unimodular and $\det A = 0$, 1 or -1, since by multiplying each row $a(i,\cdot)$ $(i \in I(-))$ by -1, A becomes a matrix which has the consecutive 1's property if the rows are appropriately reordered. Therefore, further suppose $a(i,j_0) = 0$ for all $i \in I(-)$.

Now, transform the matrix A by fundamental row operations with pivot $a(1,j_0) = 1$ in such a way that the j_0th column $a(\cdot,j_0)$ becomes a unit vector. Let A' be the matrix obtained from the resultant matrix by further removing the first row and the j_0th column, and let A'_1 and A'_2 be the submatrices of A' composed, respectively, of row vectors corresponding to $(I(+) \cup I(+,-)) - \{1\}$ and $I(-)$. Then, since A has the upper consecutive 1's property and the lower consecutive -1's property and since by the assumption for any $j \in J$ such that $a(1,j) = 1$ we have $a(i,j) = 0$ or 1 for all $i \in I(+) \cup I(+,-)$, both A'_1 and A'_2 have the lower consecutive -1's property. Hence, A' has the consecutive -1's property by appropriately reordering the rows (i.e., by reversing the row ordering of A'_2) and is totally unimodular. We thus have $\det A = \pm \det A' = 0,\ 1,$ or -1. Q.E.D.

We are now ready to prove Theorem 3.66.

(Proof of Theorem 3.66) Let $c \in \mathbf{R}^E$ be an integral vector, and suppose Problem (P^*) has an optimal solution. By Lemma 3.67 there exists an optimal solution λ^* of (P^*) such that $\mathcal{E}$ given by (3.321) consists of pairwise comparable elements of $\hat{\mathcal{D}}$ and does not contain any (X_i, Y_i) $(i = 1,2)$ which satisfy (3.326)$\sim$(3.328). Then it follows from Lemma 3.68 that the set of row vectors $\chi(X,Y)$ $((X,Y) \in \mathcal{E})$ forms a totally unimodular matrix, which implies the existence of an integral optimal solution of (P^*). Q.E.D.

We call the polyhedron $\hat{\mathrm{P}}(\hat{f})$ the *ternary semimodular polyhedron* associated with $(\hat{\mathcal{D}}, \hat{f})$ satisfying property (†).

A general framework for proving total dual integrality of systems of linear inequalities related to submodular functions is proposed by A. Schrijver [Schrijver84a,84b], which includes total dual integrality of the Edmonds-Giles submodular flow model [Edm + Giles77] and many other models. However, it is not clear whether the total dual integrality of (3.318b) with property (†) of (3.326)$\sim$(3.329) follows from Schrijver's framework.

For a submodular system $(\mathcal{D}', f')$ and a supermodular system $(\mathcal{D}'', g'')$ on E, define $\hat{\mathcal{D}} \subseteq 3^E$ by

$$\hat{\mathcal{D}} = \{(X, \emptyset) \mid X \in \mathcal{D}'\} \cup \{(\emptyset, Y) \mid y \in \mathcal{D}''\} \tag{3.339}$$

and for each $(X,Y) \in \hat{\mathcal{D}}$

$$\hat{f}(X,Y) = \begin{cases} f'(X) & \text{if } Y = \emptyset, \\ -g''(Y) & \text{if } X = \emptyset. \end{cases} \tag{3.340}$$

Then, $\hat{f}$: $\hat{\mathcal{D}} \to \mathbf{R}$ is a submodular function on $\hat{\mathcal{D}}$, and $\hat{\mathrm{P}}(\hat{f})$ defined by (3.299) is expressed as

$$\hat{\mathrm{P}}(\hat{f}) = \mathrm{P}(f') \cap \mathrm{P}(g''), \tag{3.341}$$

where $\mathrm{P}(f')$ is the submodular polyhedron associated with the submodular system $(\mathcal{D}', f')$ and $\mathrm{P}(g'')$ is the supermodular polyhedron associated with the supermodular system $(\mathcal{D}'', g'')$. Theorem 3.63 implies that $\mathrm{P}(f') \cap \mathrm{P}(g'')$ is nonempty if and only if $g''(X) \le f'(X)$ for each $X \in \mathcal{D}' \cap \mathcal{D}''$. Moreover, it should be noted that there exists no pair of (X_i, Y_i) in the present $\hat{\mathcal{D}}$ which satisfies (3.326)$\sim$(3.328). Therefore, if f' and g'' are integer-valued and $\mathrm{P}(f') \cap \mathrm{P}(g'') \neq \emptyset$, each face of $\mathrm{P}(f') \cap \mathrm{P}(g'')$ contains integral points due to Theorem 3.66. This leads to the discrete separation theorem (Theorem 4.12) due to [Frank82b].

It should be noted that a submodular polyhedron, a supermodular polyhedron, a base polyhedron, and the intersection of two base polyhedra are special cases of the intersection of submodular and supermodular polyhedra and that all of these polyhedra are ternary semimodular polyhedra. Hence, in general the greedy algorithm does not work for ternary semimodular polyhedra.

Let $(\mathcal{D}', f')$ be a submodular system and $(\mathcal{D}'', g'')$ a supermodular system, both on E, such that

$$\begin{aligned} \mathrm{P}(f', g'') = \{x \mid x \in \mathbf{R}^E, \ \forall X \in \mathcal{D}'\colon\ & x(X) \le f'(X), \\ \forall Y \in \mathcal{D}''\colon\ & x(Y) \ge g''(Y)\} \end{aligned} \tag{3.342}$$

is a generalized polymatroid, i.e., for each $X \in \mathcal{D}'$ and $Y \in \mathcal{D}''$ we have

$$X - Y \in \mathcal{D}', \qquad Y - X \in \mathcal{D}'', \tag{3.343}$$

$$f'(X) - g''(Y) \ge f'(X - Y) - g''(Y - X). \tag{3.344}$$

Define $\hat{\mathcal{D}}$ and $\hat{f}$: $\hat{\mathcal{D}} \to \mathbf{R}$ by (3.339) and (3.340), respectively. Since $\hat{f}(\emptyset, \emptyset) = f'(\emptyset) = g''(\emptyset) = 0$ by definition, we have from (3.344)

$$\hat{f}(\emptyset, X) + \hat{f}(X, \emptyset) \ge 2\hat{f}(\emptyset, \emptyset) = 0 \tag{3.345}$$

for each $X \in \mathcal{D}' \cap \mathcal{D}''$. It follows from Theorem 3.63 that generalized polymatroids are always nonempty (cf. Theorems 2.3 and 3.58). Also, the total dual integrality of (3.342) follows from Theorem 3.66.

Suppose that a distributive lattice $\hat{\mathcal{D}} \subseteq 3^E$ and a submodular function $\hat{f}$: $\hat{\mathcal{D}} \to \mathbf{R}$ satisfy the property that for any $(X_i, Y_i) \in \hat{\mathcal{D}}$ $(i = 1, 2)$ satisfying (3.326) and (3.327) we have $(X_1 - Y_2, Y_1)$, $(X_2, Y_2 - X_1) \in \hat{\mathcal{D}}$ and (3.329) holds. Then, $\hat{\mathrm{P}}(\hat{f})$ defined by (3.299) is called a *hybrid independence polyhedron* by Tomizawa [Tomi81a]. In this case, $\hat{\mathrm{P}}(\hat{f}) \neq \emptyset$ if and only if $\hat{f}(\emptyset, \emptyset) \geq 0$ (if $(\emptyset, \emptyset) \in \hat{\mathcal{D}}$), since (3.345) holds for each X such that $(\emptyset, X)$, $(X, \emptyset) \in \hat{\mathcal{D}}$.

It should be noted that the class of hybrid independence polyhedra includes the class of generalized polymatroids but not the class of intersections of submodular and supermodular polyhedra.

Readers should be referred to a nice review [Schrijver84a] for other related polyhedra and systems such as *lattice polyhedra* ([Hoffman+Schwartz 78], [Hoffman78], [Gröflin+Hoffman82]), Frank's *kernel system* ([Frank79]) and *Grishuhin's model* ([Grishuhin81]). [Also see a recent excellent book [Schrijver03].]

3.6. Submodular Systems of Network Type [Tomi+Fuji81]

Let $\mathcal{N} = (G = (V, A), c)$ be a capacitated network with an underlying graph $G = (V, A)$ and a nonnegative upper capacity function c: $A \to \mathbf{R}_+$, where the lower capacity function is regarded as the zero function. The cut function κ_c: $2^V \to \mathbf{R}$ associated with the network $\mathcal{N}$ is given by

$$\kappa_c(U) = \sum_{a \in \Delta^+ U} c(a) \qquad (U \subseteq V), \tag{3.346}$$

where $\Delta^+ U$ is the set of arcs leaving U (see Section 2.3). Without loss of generality we assume that G is a simple graph and that each arc $a \in A$ is identified with the ordered pair $(\partial^+ a, \partial^- a)$ of its end-vertices. (3.346) can be rewritten as

$$\kappa_c(U) = \sum_{u \in U} \sum_{v \in V - \{u\}} c(u, v) - \sum_{\{u,v\} \in \binom{U}{2}} (c(u, v) + c(v, u)) \qquad (U \subseteq V), \tag{3.347}$$

where for each arc $(u, v) \in A$ we write $c((u, v))$ as $c(u, v)$, $\binom{U}{2}$ is the set of all the two-element subsets of U, and we define $c(u, v) = 0$ for $(u, v) \notin A$. For any finite set X and any integer i with $0 \leq i \leq |X|$ we denote by $\binom{X}{i}$ the set of all the i-element subsets of X.

For any non-zero set function $f\colon 2^V \to \mathbf{R}$ there exist functions $f^{(i)}\colon \binom{V}{i} \to \mathbf{R}$ $(0 \le i \le |V|)$ such that

$$f(X) = \sum_{i=0}^{|X|} \sum_{Y \in \binom{X}{i}} f^{(i)}(Y) \qquad (X \subseteq V). \tag{3.348}$$

By the Möbius inversion formula $f^{(i)}$ $(0 \le i \le |V|)$ are uniquely determined from f as

$$f^{(i)}(X) = \sum_{Y \subseteq X} (-1)^{|X-Y|} f(Y). \tag{3.349}$$

Let k be an integer such that $0 \le k \le |V|$, $f^{(k)} \ne \mathbf{0}$ and $f^{(i)} = \mathbf{0}$ $(k+1 \le i \le |V|)$. The function f is called a set function *of order* k (see [Tomi80b]). We see from (3.347) that any cut function is of order 2 if $c \ne \mathbf{0}$ (also see (3.353) and (3.354) below).

A submodular system $(2^V, f)$ is said to be *of network type* if f is equal to the cut function $\kappa_c\colon 2^V \to \mathbf{R}$ associated with a network having a nonnegative capacity function c.

Theorem 3.69 [Tomi+Fuji81]: *Suppose that $(2^V, f)$ is a submodular system on V. $(2^V, f)$ is of network type if and only if the following* (i)$\sim$(iii) *hold:*

(i) *The order of f is less than or equal to* 2.

(ii) *For the functions $f^{(i)}$ $(0 \le i \le |V|)$ in* (3.348),

$$f^{(0)} = \mathbf{0}, \qquad f^{(1)} \ge \mathbf{0}, \tag{3.350}$$

$$\sum_{u \in V} f^{(1)}(\{u\}) = -\sum_{U \in \binom{V}{2}} f^{(2)}(U). \tag{3.351}$$

(iii) *For each $X \subseteq V$,*

$$\sum_{u \in X} f^{(1)}(\{u\}) \le -\sum_{\substack{U \in \binom{V}{2} \\ U \cap X \ne \emptyset}} f^{(2)}(U). \tag{3.352}$$

(Proof) *The "only if" part*: If $(2^V, f)$ is of network type and $f = \kappa_c$ for a network with a nonnegative capacity function c, then from (3.347) we have

$$f^{(0)} = \mathbf{0}, \qquad f^{(1)}(\{u\}) = \sum_{v \in V - \{u\}} c(u, v), \tag{3.353}$$

$$f^{(2)}(\{u,v\}) = -(c(u,v)+c(v,u)). \qquad (3.354)$$

Since the capacity function c is nonnegative, (i)$\sim$(iii) easily follow from (3.353) and (3.354).

The "if" part: Suppose (i)$\sim$(iii) hold. Consider the bipartite graph $\hat{G} = (W^+,\ W^-; \hat{A})$ with the left and right vertex sets W^+ and W^- and with the arc set $\hat{A}$ defined by

$$W^+ = V, \qquad W^- = \binom{V}{2}, \qquad (3.355)$$

$$\hat{A} = \left\{(u,U) \;\Big|\; u \in U \in \binom{V}{2}\right\}. \qquad (3.356)$$

Let $\hat{\mathcal{N}} = (\hat{G}, \hat{c})$ be a network with the underlying graph $\hat{G}$ and a nonnegative capacity function $\hat{c}$ such that for each $a \in \hat{A}$ $\hat{c}(a)$ is sufficiently large, where W^+ is the entrance vertex set and W^- the exit vertex set of $\hat{\mathcal{N}}$. Then, it follows from the assumption that there exists a flow φ: $\hat{A} \to \mathbf{R}$ in $\hat{\mathcal{N}}$ such that

$$\begin{aligned} \partial\varphi(u) &= f^{(1)}(\{u\}) \quad (u \in W^+(=V)), & (3.357)\\ \partial\varphi(U) &= f^{(2)}(U) \quad (U \in W^-(=\binom{V}{2})), & (3.358)\end{aligned}$$

due to the supply-demand theorem for bipartite networks (Theorem 1.4) ([Gale 57], [Ford + Fulkerson62]). Choose one such flow φ: $\hat{A} \to \mathbf{R}_+$. Define

$$A = \{(u,v) \mid u,v \in V,\ u \neq v\}, \qquad (3.359)$$

$$c(u,v) = \varphi((u,\{u,v\})) \quad ((u,v) \in A). \qquad (3.360)$$

Let $\mathcal{N}$ be the network with the underlying graph $G = (V, A)$ and the nonnegative capacity function c defined by (3.359) and (3.360). From (3.357)$\sim$(3.360) we have (3.353) and (3.354). Since the order of f is less than or equal to 2, f coincides with the cut function κ_c associated with the network $\mathcal{N}$. Q.E.D.

We see from this theorem that for a submodular system $(2^V, f)$ on V with the rank function of order at most 2 the problem of discerning whether the submodular system $(2^V, f)$ is of network type or not can be solved by a max-flow computation for the network $\hat{\mathcal{N}}$ defined in the above proof, since

(ii) in Theorem 3.69 can directly be checked and (iii) together with (3.351) is a necessary and sufficient condition for the existence of a feasible flow in the bipartite network $\hat{\mathcal{N}}$.

Moreover, when $(2^V, f)$ is of network type, consider the minimum-cost network-realization problem defined as follows:

> Find a network $\mathcal{N} = (G = (V, A), c)$ with $\kappa_c = f$ such that the cost
>
> $$\sum_{a \in A} \gamma(a)c(a) \tag{3.361}$$
>
> is as small as possible, where $\gamma(a)$ is the realization cost per unit capacity for each arc $a \in A$.

As is seen from the above proof, this problem is reduced to a Hitchcock transportation problem for the bipartite network $\hat{\mathcal{N}}$ defined in the above proof (see [Tomi+Fuji81]).

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Chapter III. Neoflows

In this chapter we consider generalizations of classical flow problems of Ford and Fulkerson to flow problems with boundary constraints described by submodular functions. The new flow problems to be treated are the submodular flow problem, the independent flow problem and the polymatroidal flow problem, and they are, in a sense, equivalent. Because of this we call the class of these flow problems and other possible equivalent ones *the neoflow problem* and each of them *a neoflow problem* (cf. [Fuji87a]). We give a theory and algorithms for the neoflow problem.

4. The Intersection Problem

In this section we consider the problem of finding a maximum common subbase of two submodular systems and some related problems.

4.1. The Intersection Theorem

Let $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ be two submodular systems on E and consider the following problem.

$$P_1\colon \ \text{Maximize } x(E) \tag{4.1a}$$
$$\text{subject to } x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2). \tag{4.1b}$$

Equivalently, we express this problem as follows. Let E' be a copy of E and we regard $(\mathcal{D}_2, f_2)$ as a submodular system on E'. Also let $G = (E, E'; A)$ be the bipartite graph with the left and right vertex sets E and E' and with the arc set $A = \{(e, e') \mid e \in E\}$, where $e' \in E'$ is a copy of $e \in E$, i.e., A gives a natural bijection between E and its copy E' (see Fig. 4.1). Furthermore, we consider a capacity function $c\colon A \rightarrow \mathbf{R} \cup \{+\infty\}$ such that $c(a) = +\infty$ $(a \in A)$. Then Problem P_1 in (4.1) is equivalent to the following problem

$$P_1{}'\colon \ \text{Maximize } \partial\varphi(E) \tag{4.2a}$$
$$\text{subject to } (\partial\varphi)^E \in \mathrm{P}(f_1), \tag{4.2b}$$
$$-(\partial\varphi)^{E'} \in \mathrm{P}(f_2), \tag{4.2c}$$

where $\varphi\colon A \rightarrow \mathbf{R}$ is a flow, $\partial\varphi$ is the boundary of φ in $\mathcal{N}$ and $(\partial\varphi)^E$ and $(\partial\varphi)^{E'}$ are, respectively, the restrictions of $\partial\varphi$ to E and E'. A flow

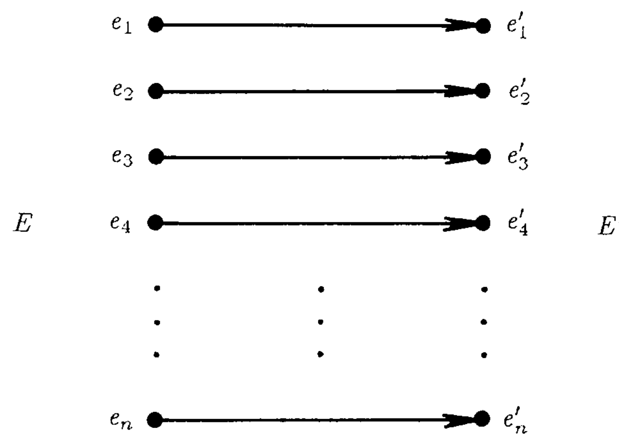

Figure 4.1: The intersection problem.

$\varphi: A \to \mathbf{R}$ satisfying (4.2b) and (4.2c) is called a *feasible flow* in $\mathcal{N} = (G = (E, E'; A), c, \mathbf{S}_1, \mathbf{S}_2)$, where $\mathbf{S}_i = (\mathcal{D}_i, f_i)$ $(i = 1, 2)$. As we will see later, Problem $P_1{}'$ is a special case of a neoflow problem.

(a) Preliminaries

We need some preliminaries to furnish an algorithm for solving the intersection problem described by (4.1) or (4.2).

Consider a submodular system $(\mathcal{D}, f)$ on E. In the following we give several lemmas which are obtained by a direct adaptation of the results shown in [Fuji78a] for polymatroids.

Lemma 4.1: *Suppose* $x \in \mathrm{P}(f)$, $u \in \mathrm{sat}(x)$ *and* $v \in \mathrm{dep}(x, u) - \{u\}$. *For any* $\alpha \in \mathbf{R}$ *such that* $0 < \alpha \leq \tilde{c}(x, u, v)$ *define* $y = x + \alpha(\chi_u - \chi_v)$. *Then,* $y \in \mathrm{P}(f)$ *and*

$$\mathrm{sat}(y) = \mathrm{sat}(x). \tag{4.3}$$

(Proof) From the definition of the exchange capacity we have $y \in \mathrm{P}(f)$. Also, since for any $X \in \mathcal{D}$ such that $X \supseteq \mathrm{sat}(x)$ we have $y(X) = x(X)$ and $\mathrm{sat}(x)$ is the unique maximal tight set X such that $x(X) = f(X)$, we have $\mathrm{sat}(y) = \mathrm{sat}(x)$. Q.E.D.

Lemma 4.2: *Under the same assumption as in Lemma* 4.1,

$$\hat{c}(y,w) = \hat{c}(x,w) \qquad (w \in E - \mathrm{sat}(x)). \tag{4.4}$$

(Proof) Put $X_0 = \mathrm{sat}(x)(= \mathrm{sat}(y))$. Since $x(X_0) = f(X_0)$ and $y(X_0) = f(X_0)$, we have for any $X \in \mathcal{D}$

$$\begin{aligned} f(X) - y(X) &= f(X) - y(X) + f(X_0) - y(X_0) \\ &\geq f(X \cup X_0) - y(X \cup X_0) + f(X \cap X_0) - y(X \cap X_0) \\ &\geq f(X \cup X_0) - y(X \cup X_0) \\ &= f(X \cup X_0) - x(X \cup X_0), \end{aligned} \tag{4.5}$$

and similarly,

$$f(X) - x(X) \geq f(X \cup X_0) - x(X \cup X_0). \tag{4.6}$$

Hence the lemma follows from (2.37). Q.E.D.

Lemma 4.3: *For any $x \in \mathrm{P}(f)$ let u_1, u_2 and v_2 be three distinct elements of E such that*

$$u_i \in \mathrm{sat}(x) \qquad (i = 1, 2), \tag{4.7}$$

$$v_2 \in \mathrm{dep}(x, u_2), \qquad v_2 \notin \mathrm{dep}(x, u_1). \tag{4.8}$$

For any $\alpha \in \mathbf{R}$ such that $0 < \alpha \leq \tilde{c}(x, u_2, v_2)$ define

$$y = x + \alpha(\chi_{u_2} - \chi_{v_2}). \tag{4.9}$$

Then we have $u_1 \in \mathrm{sat}(y)$ and

$$\mathrm{dep}(y, u_1) = \mathrm{dep}(x, u_1). \tag{4.10}$$

(Proof) From Lemma 4.1 we have $u_1 \in \mathrm{sat}(y)$. Also we have $u_2 \notin \mathrm{dep}(x, u_1)$, since otherwise we would have $\mathrm{dep}(x, u_2) \subseteq \mathrm{dep}(x, u_1)$ by the minimality of $\mathrm{dep}(x, u_2)$ and hence $v_2 \in \mathrm{dep}(x, u_1)$. Therefore, putting $X_0 = \mathrm{dep}(x, u_1)$, we have $y(X_0) = x(X_0) = f(X_0)$ and $y(X) = x(X)$ for any $X \in \mathcal{D}$ with $X \subseteq X_0$. (4.10) follows from the definition of dependence function. Q.E.D.

Lemma 4.4: *For any $x \in \mathrm{P}(f)$ let u_1, u_2, v_1 and v_2 be four distinct elements of E satisfying* (4.7), (4.8) *and*

$$v_1 \in \mathrm{dep}(x, u_1). \tag{4.11}$$

Then for the vector y defined by (4.9) for any $\alpha \in \mathbf{R}$ with $0 < \alpha \leq \tilde{c}(x, u_2, v_2)$, we have

$$\tilde{c}(y, u_1, v_1) = \tilde{c}(x, u_1, v_1). \tag{4.12}$$

(Proof) For any $z \in \mathrm{P}(f)$ and $X_0 \in \mathcal{D}$ such that $z(X_0) = f(X_0)$ we have

$$f(X) - z(X) \geq f(X \cap X_0) - z(X \cap X_0) \quad (X \in \mathcal{D}). \tag{4.13}$$

For $X_0 \equiv \mathrm{dep}(x, u_1)$ we have from Lemma 4.3

$$y(X_0) = x(X_0) = f(X_0) \tag{4.14}$$

and since $u_2,\ v_2 \notin \mathrm{dep}(x, u_1)$, we have

$$y(X) = x(X) \quad (X \subseteq X_0,\ X \in \mathcal{D}). \tag{4.15}$$

Since (4.13) holds for $z = x,\ y$, (4.12) follows from (4.14) and (4.15). Q.E.D.

Lemma 4.5: *For any $x \in \mathrm{P}(f)$ let $u_i,\ v_i$ $(i = 1, 2, \cdots, q)$ be $2q$ distinct elements of E such that*

$$u_i \in \mathrm{sat}(x), \qquad v_i \in \mathrm{dep}(x, u_i) \quad (i = 1, 2, \cdots, q), \tag{4.16}$$

$$v_j \notin \mathrm{dep}(x, u_i) \quad (1 \leq i < j \leq q). \tag{4.17}$$

For any α_i $(i = 1, 2, \cdots, q)$ satisfying $0 < \alpha_i \leq \tilde{c}(x, u_i, v_i)$ $(i = 1, 2, \cdots, q)$ define a vector $y \in \mathbf{R}^E$ by

$$y = x + \sum_{i=1}^{q} \alpha_i (\chi_{u_i} - \chi_{v_i}). \tag{4.18}$$

Then,

$$y \in \mathrm{P}(f), \tag{4.19}$$

$$\mathrm{sat}(y) = \mathrm{sat}(x), \tag{4.20}$$

$$\hat{c}(y, w) = \hat{c}(x, w) \quad (w \in E - \mathrm{sat}(x)). \tag{4.21}$$

(Proof) Considering the elementary transformations in the order of the pairs (u_q, v_q), (u_{q-1}, v_{q-1}), $\cdots$, (u_1, v_1), the present lemma can be shown by repeatedly applying Lemmas 4.1~4.4. Q.E.D.

It should be noted that we have (4.19)~(4.21) if (4.16) and (4.17) hold for an appropriate numbering of u_i's and v_i's.

Lemma 4.5 will play a very fundamental rôle in developing algorithms for solving the intersection problem and other related problems.

(b) An algorithm and the intersection theorem

We now consider Problem $P_1{}'$ described by (4.2). Given a feasible flow φ in network $\mathcal{N} = (G = (E, E'; A), c, \mathbf{S}_1, \mathbf{S}_2)$, define the *auxiliary network* $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$ associated with φ as follows. $G_\varphi = (V, A_\varphi)$ is a directed graph, called the *auxiliary graph* associated with φ, with vertex set V and arc set A_φ given by

$$\begin{aligned} V &= E \cup E' \cup \{s^+, s^-\}, &(4.22)\\ A_\varphi &= S_\varphi^+ \cup A_\varphi^+ \cup A^* \cup B^* \cup A_\varphi^- \cup S_\varphi^-, &(4.23)\end{aligned}$$

where

$$\begin{aligned} S_\varphi^+ &= \{(s^+, v) \mid v \in E - \mathrm{sat}^+(\partial^+\varphi)\}, &(4.24)\\ A_\varphi^+ &= \{(u, v) \mid v \in \mathrm{sat}^+(\partial^+\varphi),\ u \in \mathrm{dep}^+(\partial^+\varphi, v) - \{v\}\}, &(4.25)\\ A^* &= A, &(4.26)\\ B^* &= \{(e', e) \mid e \in E\}, &(4.27)\\ A_\varphi^- &= \{(u, v) \mid u \in \mathrm{sat}^-(\partial^-\varphi),\ v \in \mathrm{dep}^-(\partial^-\varphi, u) - \{u\}\}, &(4.28)\\ S_\varphi^- &= \{(v, s^-) \mid v \in E - \mathrm{sat}^-(\partial^-\varphi)\}. &(4.29)\end{aligned}$$

Here, $\partial^+\varphi = (\partial\varphi)^E$, $\partial^-\varphi = -(\partial\varphi)^{E'}$, and sat^+ and dep^+ (sat^- and dep^-) are, respectively, the saturation function and the dependence function defined with respect to submodular system $(\mathcal{D}_1, f_1)$ on E ($(\mathcal{D}_2, f_2)$ on E'). Note that B^* is the set of the reorientations of arcs of A. We also define the capacity function c_φ: $A_\varphi \to \mathbf{R} \cup \{+\infty\}$ by

$$c_\varphi(a) = \begin{cases} \hat{c}^+(\partial^+\varphi, v) & (a = (s^+, v) \in S_\varphi^+), \\ \tilde{c}^+(\partial^+\varphi, v, u) & (a = (u, v) \in A_\varphi^+), \\ +\infty & (a \in A^* \cup B^*), \\ \tilde{c}^-(\partial^-\varphi, u, v) & (a = (u, v) \in A_\varphi^-), \\ \hat{c}^-(\partial^-\varphi, v) & (a = (v, s^-) \in S_\varphi^-), \end{cases} \tag{4.30}$$

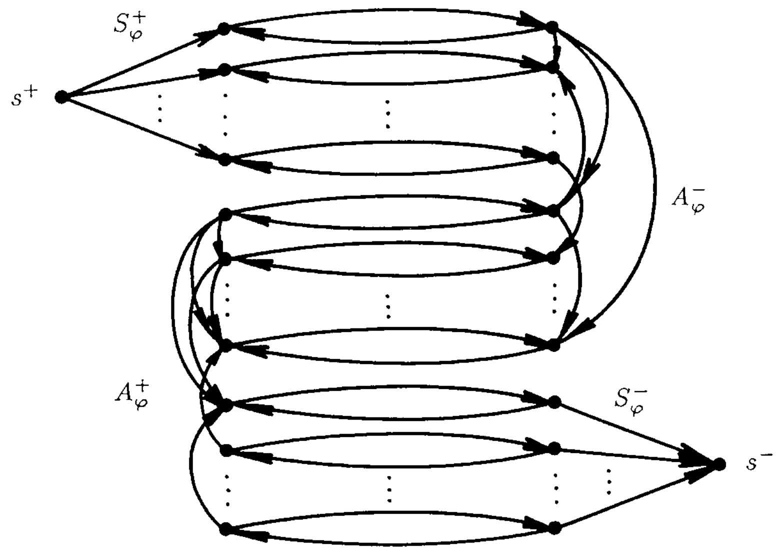

Figure 4.2: An auxiliary network.

where $\hat{c}^+$ and $\tilde{c}^+$ ($\hat{c}^-$ and $\tilde{c}^-$) are, respectively, the saturation capacity and the exchange capacity defined with respect to submodular system $(\mathcal{D}_1, f_1)$ on E ($(\mathcal{D}_2, f_2)$ on E'). Figure 4.2 shows the auxiliary network $\mathcal{N}_\varphi$.

Let us consider an algorithm described as follows. We call a directed path from s^+ to s^- of the minimum number of arcs a *shortest path* from s^+ to s^-. We assume an oracle for exchange capacities for $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$.

Algorithm for the intersection problem

Input: a feasible flow φ in $\mathcal{N} = (G = (E, E'; A), c, \mathbf{S}_1, \mathbf{S}_2)$.
Output: a maximum flow φ in $\mathcal{N}$.

Step 1: Construct the auxiliary network $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$.

Step 2: If there exists no directed path from s^+ to s^- in $\mathcal{N}_\varphi$, then the algorithm terminates and the current φ is a maximum flow in $\mathcal{N}$.
Otherwise, find a shortest path P from s^+ to s^- in $\mathcal{N}_\varphi$ and put

$$\alpha \leftarrow \min\{c_\varphi(a) \mid a \text{ is an arc in } P\}, \tag{4.31}$$

$$\varphi(a) \leftarrow \begin{cases} \varphi(a) + \alpha & (a = (e, e') \in A \text{ and } a \text{ is in } P), \\ \varphi(a) - \alpha & (a = (e, e') \in A \text{ and } (e', e) \text{ is in } P). \end{cases} \tag{4.32}$$

Go to Step 1.
(End)

Theorem 4.6: *The flows φ obtained in Step 2 of the algorithm are feasible flows in $\mathcal{N}$. Moreover, each time we carry out* (4.31) *and* (4.32), *the flow value of φ increases by $\alpha > 0$ given by* (4.31).

(Proof) In Step 2 we find a shortest path P from s^+ to s^- in the auxiliary network $\mathcal{N}_\varphi$. Let (u_1^+, v_1^+), $\cdots$, (u_p^+, v_p^+) be the arcs in A_φ^+ lying on P in this order and (u_1^-, v_1^-), $\cdots$, (u_q^-, v_q^-) be the arcs in A_φ^- lying on P in this order. By the way of choosing path P, for each i, j such that $1 \le i < j \le p$ (q) we have

$$u_i^+ \notin \mathrm{dep}^+(\partial^+\varphi, v_j^+) \qquad (v_j^- \notin \mathrm{dep}^-(\partial^-\varphi, u_i^-)). \tag{4.33}$$

Therefore, the present theorem follows from Lemma 4.5. Q.E.D.

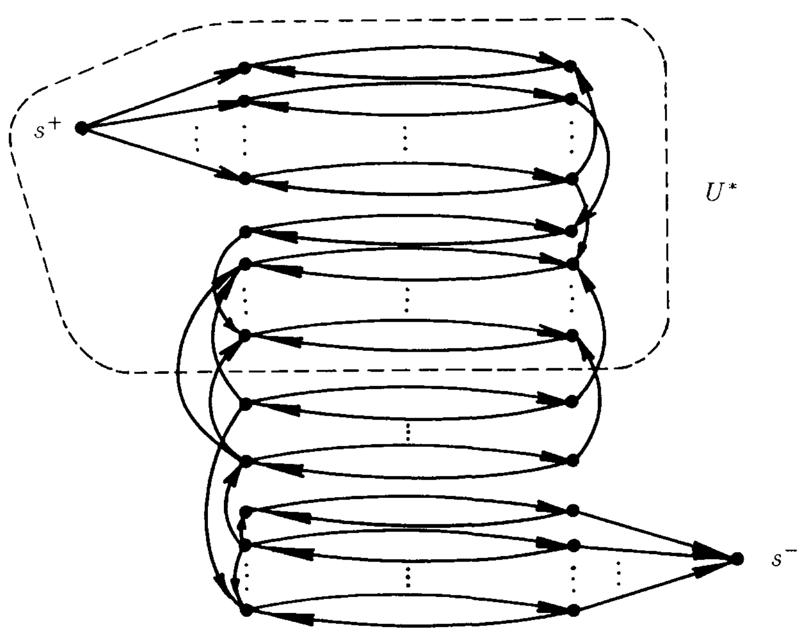

Figure 4.3: The reachable set $U*$.

Theorem 4.7: *If there exists no directed path from s^+ to s^- in the auxiliary network $\mathcal{N}_\varphi$ in Step 2 of the algorithm, the current flow φ is a maximum flow in $\mathcal{N}$.*

(Proof) Let U^* be the set of the vertices in V which are reachable from s^+ along directed paths in $\mathcal{N}_\varphi$ (see Fig. 4.3).

Then for each $v \in E - U^*$ and $u \in E' \cap U^*$ we have

$$\begin{aligned} \mathrm{dep}^+(\partial^+\varphi, v) &\subseteq E - U^*, && (4.34)\\ \mathrm{dep}^-(\partial^-\varphi, u) &\subseteq E' \cap U^*. && (4.35) \end{aligned}$$

Therefore,

$$\begin{aligned} \partial^+\varphi(E - U^*) &= f_1(E - U^*), && (4.36)\\ \partial^-\varphi(E' \cap U^*) &= f_2(E' \cap U^*). && (4.37) \end{aligned}$$

Moreover, from the definition of U^* and the auxiliary network $\mathcal{N}_\varphi$ we see

$$E \cap U^* = \{e \mid e' \in E' \cap U^*\} \ (\subseteq E). \tag{4.38}$$

From (4.36)$\sim$(4.38),

$$\begin{aligned} \partial^+\varphi(E) &= \partial^+\varphi(E - U^*) + \partial^-\varphi(E' \cap U^*)\\ &= f_1(E - U^*) + f_2(E' \cap U^*). && (4.39) \end{aligned}$$

On the other hand, for any feasible flow $\hat{\varphi}$ in $\mathcal{N}$ and for any $U \subseteq E \cup E'$ such that (1) $E - U \in \mathcal{D}_1$, (2) $E' \cap U \in \mathcal{D}_2$ and (3) $E \cap U = \{e \mid e' \in E' \cap U\}$ we have

$$\begin{aligned} \partial^+\hat{\varphi}(E) &= \partial^+\hat{\varphi}(E - U) + \partial^-\hat{\varphi}(E' \cap U)\\ &\leq f_1(E - U) + f_2(E' \cap U). && (4.40) \end{aligned}$$

It follows from (4.39) and (4.40) that the current flow φ is a maximum flow in $\mathcal{N}$. Q.E.D.

Since there exists a maximum flow in $\mathcal{N}$ (this fact will also algorithmically be proven later), Theorems 4.6 and 4.7 together with the proof of Theorem 4.7 show the following theorem. Note that when f_1 and f_2 are integer-valued, starting with an integral feasible flow in $\mathcal{N}$, the algorithm given above terminates after a finite number of steps and then gives an integral maximum flow.

Theorem 4.8: *For Problem $P_1{}'$ described by* (4.2) *we have*

$$\begin{aligned}&\max\{\partial^+\varphi(E) \mid \varphi\colon \text{ a feasible flow in } \mathcal{N}\}\\ &= \min\{f_1(X) + f_2(E' - X') \mid X \in \mathcal{D}_1,\ E' - X' \in \mathcal{D}_2\}, \quad (4.41)\end{aligned}$$

where $X' = \{x' \mid x \in X\} \subseteq E'$.

Moreover, if f_1 and f_2 are integer-valued, there exists an integral maximum flow in $\mathcal{N}$.

Rewriting Theorem 4.8 for Problem P_1, we also have

Theorem 4.9 (The Intersection Theorem): *For Problem P_1 described by* (4.1),

$$\begin{aligned}&\max\{x(E) \mid x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)\}\\ &= \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1,\ E - X \in \mathcal{D}_2\}. \quad (4.42)\end{aligned}$$

Moreover, if f_1 and f_2 are integer-valued, the maximum on the left-hand side of (4.42) *is attained by an integral vector $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$.*

Theorem 4.9 is a generalization of the intersection theorem for polymatroids due to Edmonds [Edm70]. The algorithm given in this section is a direct adaptation of the one given in [Fuji78a].

Denote by $\mathcal{L}$ the set of all the minimizers of the right-hand side of (4.42). $\mathcal{L}$ is a sublattice of $\mathcal{D}_1 \cap \overline{\mathcal{D}_2}$. Then, we have the following theorem. Recall that for a submodular system $(\mathcal{D}, f)$ and $X, Y \in \mathcal{D}$ with $X \subset Y$ f_X^Y denotes the rank function of the minor $(\mathcal{D}, f) \cdot Y/X$.

Theorem 4.10 (cf. [Iri79], [Nakamura+Iri81]): *Let*

$$\mathcal{C}\colon\ S^- = S_1 \subset S_2 \subset \cdots \subset S_k = S^+ \qquad (4.43)$$

be any (maximal) chain of $\mathcal{L}$ and define

$$S_0 = \emptyset, \qquad S_{k+1} = E. \qquad (4.44)$$

Then, the following two statements are equivalent:

(i) *x is a maximum common subbase of $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$.*

(ii) *For each* $i = 1, 2, \cdots, k+1$, $x^{S_i - S_{i-1}}$ *is a maximum common subbase of* $(\mathcal{D}_1, f_1) \cdot S_i/S_{i-1}$ *and* $(\mathcal{D}_2, f_2) \cdot (E - S_{i-1})/(E - S_i)$, *where for each* $i = 1, 2, \cdots, k$ $x^{S_i - S_{i-1}}$ *is a base of* $(\mathcal{D}_1, f_1) \cdot S_i/S_{i-1}$ *and for each* $i = 2, 3, \cdots, k+1$ $x^{S_i - S_{i-1}}$ *is a base of* $(\mathcal{D}_2, f_2) \cdot (E - S_{i-1})/(E - S_i)$. (We disregard the case of $i = 1$ (or $i = k+1$) if $S_1 = \emptyset$ (or $S_k = E$).)

(Proof) (i) $\Longrightarrow$ (ii): Suppose (i). From Theorem 4.9 we have

$$x(S_i) + x(E - S_i) = x(E) = f_1(S_i) + f_2(E - S_i) \quad (i = 1, 2, \cdots, k). \tag{4.45}$$

Since $x(S_i) \le f_1(S_i)$ and $x(E - S_i) \le f_2(E - S_i)$ for $i = 1, 2, \cdots, k$, we must have

$$x(S_i) = f_1(S_i), \quad x(E - S_i) = f_2(E - S_i) \quad (i = 1, 2, \cdots, k). \tag{4.46}$$

Since $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$, it follows from (4.43) and (4.46) that $x^{S_i - S_{i-1}}$ ($i = 1, 2, \cdots, k$) are bases of $(\mathcal{D}_1, f_1) \cdot S_i/S_{i-1}$ and that $x^{S_i - S_{i-1}}$ ($i = 2, 3, \cdots, k+1$) are bases of $(\mathcal{D}_2, f_2) \cdot (E - S_{i-1})/(E - S_i)$, where we disregard the case of $i = 1$ (or $i = k+1$) if $S_1 = \emptyset$ (or $S_k = E$). Hence, (ii) holds.

(ii) $\Longrightarrow$ (i): Suppose (ii). Then we have $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$. It follows from (ii) that

$$\begin{aligned} x(E) &= x(E - S_k) + \sum_{i=2}^{k} x(S_i - S_{i-1}) + x(S_1) \\ &= f_2(E - S_k) + \sum_{i=2}^{k} (f_1(S_i) - f_1(S_{i-1})) + f_1(S_1) \\ &= f_1(S_k) + f_2(E - S_k). \end{aligned} \tag{4.47}$$

From (4.47) and Theorem 4.9 x is a maximum common subbase of $(\mathcal{D}_i, f_i)$ ($i = 1, 2$). Q.E.D.

We can also show that the set of the minors of $(\mathcal{D}_1, f_1)$ and $(\mathcal{D}_2, f_2)$ appearing in (ii) of Theorem 4.10 does not depend on the choice of a maximal chain $\mathcal{C}$ of $\mathcal{L}$ ([Nakamura+Iri81], [Tomi+Fuji82]) (see Theorem 7.17).

Theorem 4.10 is a generalization of a structure theorem for bipartite graphs [Dulmage+Mendelsohn59].

(c) A refinement of the algorithm

When f_1 and f_2 are not integer-valued, the algorithm shown in Section 4.1.b may not terminate after finitely many steps. We shall show an algorithm due to Schönsleben [Schönsleben80] and Lawler and Martel [Lawler+Martel 82a] which modifies Step 2 of the algorithm and terminates after repeating Step 2 $\mathrm{O}(|E|^3)$ times.

Let π: $V(= E \cup E') \to \{1, 2, \cdots, 2n\}$ be a one-to-one mapping which denotes a numbering of the vertices in V, where $n = |E|$. We also define $\pi(s^+) = 0$ and $\pi(s^-) = 2n + 1$. We represent any directed path P from s^+ to s^- in $\mathcal{N}_\varphi$ by the sequence $(s^+, v_1, v_2, \cdots, v_p, s^-)$ of vertices in P and denote by $\pi(P)$ the sequence $(\pi(v_1), \pi(v_2), \cdots, \pi(v_p))$ of the numbering indices of the intermediate vertices. For any directed paths P_1 and P_2 from s^+ to s^- in $\mathcal{N}_\varphi$ we say P_1 is *lexicographically smaller than* P_2 if $\pi(P_1)$ is lexicographically smaller than $\pi(P_2)$. We call the lexicographically smallest one in the set of all the shortest paths from s^+ to s^- in $\mathcal{N}_\varphi$ the *lexicographically shortest path* from s^+ to s^- in $\mathcal{N}_\varphi$ (with respect to π).

Now, we modify the part of finding a shortest path P in Step 2 of the algorithm as follows.

(∗) If there exists a directed path from s^+ to s^- in $\mathcal{N}_\varphi$, find the lexicographically shortest path P from s^+ to s^- in $\mathcal{N}_\varphi$.

Theorem 4.11 ([Schönsleben80], [Lawler+Martel82a] for polymatroids): *If we modify Step* 2 *of the algorithm given in Section* 4.1.b *for Problem* $P_1{}'$ *as above, the algorithm finds a maximum flow in* $\mathcal{N}$ *after repeating Step* 2 $\mathrm{O}(|E|^3)$ *times.*

(Proof) Denote by $W_k \subseteq V \cup \{s^+, s^-\}$ the set of the vertices which are reachable by a directed path from s^+ having k arcs but not reachable by a directed path from s^+ having less than k arcs. Suppose $W_0 = \{s^+\}$ and $s^- \in W_p$. Let P be the lexicographically shortest path in $\mathcal{N}_\varphi$. Also suppose that the arcs of A_φ^+ lying on P appear in order as

$$(u_1^+, v_1^+),\ (u_2^+, v_2^+),\ \cdots,\ (u_l^+, v_l^+) \tag{4.48}$$

from s^+ to s^-. Put $x = \partial^+\varphi$ and define

$$y = x + \alpha(\chi_{v_1^+} - \chi_{u_1^+}), \tag{4.49}$$

where α is defined by (4.31) for the current φ. Consider vertices $w,\ z \in E$ such that

$$w \notin \mathrm{dep}^+(x, z), \qquad w \in \mathrm{dep}^+(y, z). \tag{4.50}$$

Then we must have

$$u_1^+ \in \mathrm{dep}^+(x, z), \qquad v_1^+ \notin \mathrm{dep}^+(x, z), \tag{4.51}$$

$$w \in \mathrm{dep}^+(x, v_1^+). \tag{4.52}$$

(See Fig. 4.4.) Here we may have $v_1^+ = w$ or $z = u_1^+$.

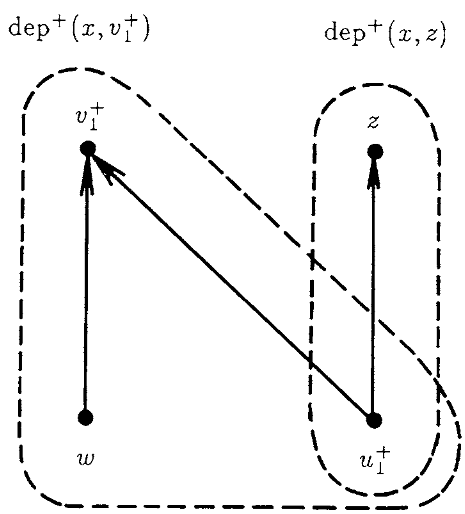

Figure 4.4: Exchangeable pairs.

Now, suppose

$$u_1^+ \in W_k, \qquad v_1^+ \in W_{k+1} \tag{4.53}$$

for some k $(1 \leq k \leq 2n)$. Then we have $z \in W_{k_1}$ for some positive integer k_1 with $k_1 \leq k+1$.

(1) If vertex w is not reachable from s^+ in $\mathcal{N}_\varphi$, then adding arc (w, z) to $\mathcal{N}_\varphi$ does not change the set of shortest paths from s^+ to s^- in $\mathcal{N}_\varphi$.

(2) Suppose $w \in W_{k_2}$ for some k_2 $(1 \leq k_2 \leq 2n)$. (Note that $k_2 \geq k$.)

(2-1) If $k_2 > k$ or $k_1 < k+1$, then adding arc (w, z) to $\mathcal{N}_\varphi$ does not change the set of shortest paths from s^+ to s^- in $\mathcal{N}_\varphi$.

(2-2) If $k_2 = k$, $k_1 = k+1$ and there exists no shortest paths from s^+ to s^- which include vertex z, then adding arc (w, z) to $\mathcal{N}_\varphi$ does not change the set of shortest paths from s^+ to s^- in $\mathcal{N}_\varphi$.

(2-3) If $k_2 = k$, $k_1 = k+1$ and there exists a shortest path from s^+ to s^- in $\mathcal{N}_\varphi$ which includes vertex z, then we have

$$\pi(v_1^+) < \pi(z) \tag{4.54}$$

since P is the lexicographically shortest path.

Because of Lemma 4.5 we can repeatedly apply the above argument to arcs (u_i^+, v_i^+) $(i = 2, \cdots, l)$ in (4.48).

We can also apply the argument for A_φ^+ to A_φ^- *mutatis mutandis*.

For each vertex $w \in V \cup \{s^+\}$ define

$$p_\varphi(w) \;=\; \min\{\pi(v) \mid v \in V,\ (w, v) \text{ lies on a shortest path from } s^+ \text{ to } s^- \text{ in } \mathcal{N}_\varphi\}. \tag{4.55}$$

Also denote by φ^* the feasible flow obtained from φ by transformation (4.32), and let P^* be the lexicographically shortest path in $\mathcal{N}_{\varphi^*}$. Since at least one arc lying on P is missing in $\mathcal{N}_{\varphi^*}$, it follows from the above argument that

(i) (the length of P^*) $\geq$ (the length of P)+1 or

(ii) (the length of P^*) $=$ (the length of P) and
$p_\varphi \leq p_{\varphi^*}$ with $p_\varphi(v) < p_{\varphi^*}(v)$ for some $v \in V \cup \{s^+\}$.

Case (ii) occurs consecutively $\mathrm{O}(|V|^2)$ times and hence Step 2 is executed $\mathrm{O}(|V|^3)$ $(=\mathrm{O}(|E|^3))$ times. Q.E.D.

The above proof technique is due to R. E. Bixby (cf. [Cunningham84]). It should be noted that Theorem 4.11 together with Theorem 4.7 shows the existence of a maximum flow in $\mathcal{N}$ for any totally ordered additive group $\mathbf{R}$ and that the modified algorithm finds a maximum flow in $\mathcal{N}$ by changing feasible flows $\mathrm{O}(|E|^3)$ times.

The above algorithm corresponds to the Edmonds-Karp algorithm for classical maximum flows [Edm + Karp72]. A complexity improvement over the above algorithm is shown in [Tardos + Tovey + Trick86] by generalizing the idea of layered networks due to E. A. Dinits [Dinits70]. [Currently the fastest algorithm for the intersection problem is given by Fujishige and Zhang [Fuji+Zhang92] by adapting the push-relabel algorithm of Goldberg and Tarjan [Goldberg+Tarjan88] for maximum flows.]

4.2. Discrete Separation Theorem

A. Frank showed the following theorem. We prove this theorem by the use of the intersection theorem (Theorem 4.9). (Note that we have already shown this theorem in Section 3.5.c.)

Theorem 4.12 (Discrete Separation Theorem) [Frank82b]: *Let $(\mathcal{D}_1, f)$ be a submodular system on E and $(\mathcal{D}_2, g)$ be a supermodular system on E. Then,*

$$g \le f \Longrightarrow \exists x \in \mathbf{R}^E\colon\ g \le x \le f. \tag{4.56}$$

Moreover, if f and g are integer-valued, there exists an integral $x \in \mathbf{R}^E$ such that $g \le x \le f$. Here, $g \le f$ means $\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2$: $g(X) \le f(X)$ and $g \le x$ $(x \le f)$ means $\forall X \in \mathcal{D}_2$: $g(X) \le x(X)$ $(\forall X \in \mathcal{D}_1$: $x(X) \le f(X))$.

(Proof) Suppose $g \le f$. Then from the intersection theorem,

$$\begin{aligned}&\max\{x(E) \mid x \in \mathrm{P}(f) \cap \mathrm{P}(g^{\#})\}\\ &= \min\{f(X) + g^{\#}(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\}\\ &= \min\{f(X) + g(E) - g(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\}\\ &= g(E).\end{aligned} \tag{4.57}$$

Therefore, there exists a vector x in $\mathrm{P}(f) \cap \mathrm{B}(g)$ $(= \mathrm{P}(f) \cap \mathrm{B}(g^{\#}))$. This vector x satisfies $g \le x \le f$. Moreover, if f and g are integer-valued, then from (4.57) and the intersection theorem there exists an integral $x \in \mathrm{P}(f) \cap \mathrm{B}(g)$, which satisfies $g \le x \le f$. Q.E.D.

We can also show the intersection theorem (Theorem 4.9) by using the discrete separation theorem (Theorem 4.12) as follows.

Let $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ be submodular systems on E. Define

$$k = \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1,\ E - X \in \mathcal{D}_2\}. \tag{4.58}$$

Also define $\hat{f}_2\colon \mathcal{D}_2 \to \mathbf{R}$ by

$$\hat{f}_2(X) = \begin{cases} f_2(X) & (X \in \mathcal{D}_2 - \{E\}),\\ k & (X = E).\end{cases} \tag{4.59}$$

Note that $\hat{f}_2\colon \mathcal{D}_2 \to \mathbf{R}$ is a submodular function since $k \le f_2(E)$; in fact, $(\mathcal{D}_2, \hat{f}_2)$ is the $(f_2(E) - k)$-truncation of $(\mathcal{D}_2, f_2)$. Then we have $\hat{f}_2{}^{\#}(\emptyset) =$

$0 = f_1(\emptyset)$ and for each $X \in \mathcal{D}_1 \cap \overline{\mathcal{D}_2} - \{\emptyset\}$

$$\begin{aligned} \hat{f}_2{}^{\#}(X) &= \hat{f}_2(E) - \hat{f}_2(E - X) \\ &= k - f_2(E - X) \\ &\leq f_1(X). \end{aligned} \tag{4.60}$$

It thus follows from the discrete separation theorem that there exists a vector $x \in \mathbf{R}^E$ such that $\hat{f}_2{}^{\#} \leq x \leq f_1$. Since $\mathrm{P}(f_1) \cap \mathrm{P}(\hat{f}_2{}^{\#}) \neq \emptyset$, we have $\mathrm{P}(f_1) \cap \mathrm{B}(\hat{f}_2)$ $(= \mathrm{P}(f_1) \cap \mathrm{B}(\hat{f}_2{}^{\#})) \neq \emptyset$. Consequently,

$$\begin{aligned} &\max\{x(E) \mid x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)\} \\ &\geq \max\{x(E) \mid x \in \mathrm{P}(f_1) \cap \mathrm{P}(\hat{f}_2)\} \\ &= k \\ &= \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1,\ E - X \in \mathcal{D}_2\} \\ &\geq \max\{x(E) \mid x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)\}, \end{aligned} \tag{4.61}$$

where the last inequality follows from the fact (the weak duality) that $x(E) \leq f_1(X) + f_2(E - X)$ for any $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$ and any $X \in \mathcal{D}_1$ with $E - X \in \mathcal{D}_2$. This proves (4.42).

Moreover, the integrality part of the intersection theorem follows from the integrality part of the discrete separation theorem and the above argument.

We have thus shown the equivalence between the intersection theorem and the discrete separation theorem.

Using the discrete separation theorem, we show (3.32) and (3.33) in Section 3.1.c of Chapter II. For two submodular systems $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ on E let f be the submodular function on $\mathcal{D}_1 \cap \mathcal{D}_2$ defined by

$$f(X) = f_1(X) + f_2(X) \qquad (X \in \mathcal{D}_1 \cap \mathcal{D}_2). \tag{4.62}$$

We write f as $f_1 + f_2$. We can easily see that the relation, $\mathrm{P}(f_1 + f_2) \supseteq \mathrm{P}(f_1) + \mathrm{P}(f_2)$, holds.

Conversely, for any $x \in \mathrm{P}(f_1 + f_2)$ we have

$$\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2\colon\ x(X) \leq f_1(X) + f_2(X), \tag{4.63}$$

which is rewritten as

$$\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2\colon\ x(X) - f_2(X) \leq f_1(X). \tag{4.64}$$

It follows from (4.64) and the discrete separation theorem that there exists a vector $y \in \mathbf{R}^E$ such that

$$\forall X \in \mathcal{D}_2\colon\ \ x(X) - f_2(X) \le y(X), \tag{4.65}$$

$$\forall X \in \mathcal{D}_1\colon\ \ y(X) \le f_1(X). \tag{4.66}$$

Defining $z = x - y$, we have from (4.65) $z \in \mathrm{P}(f_2)$ and from (4.66) $y \in \mathrm{P}(f_1)$. We thus have $x = y + z \in \mathrm{P}(f_1) + \mathrm{P}(f_2)$. Therefore, $\mathrm{P}(f_1 + f_2) \subseteq \mathrm{P}(f_1) + \mathrm{P}(f_2)$. This completes the proof of the fact that $\mathrm{P}(f_1 + f_2) = \mathrm{P}(f_1) + \mathrm{P}(f_2)$.

Since $(f_1 + f_2)(E) = f_1(E) + f_2(E)$, we also have $\mathrm{B}(f_1 + f_2) = \mathrm{B}(f_1) + \mathrm{B}(f_2)$.

4.3. The Common Base Problem

Let $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ be two submodular systems on E. The *common base problem* for submodular systems $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ is to discern whether there is a common base $x \in \mathrm{B}(f_1) \cap \mathrm{B}(f_2)$ and, if any, to find one such common base. Clearly, that $f_1(E) = f_2(E)$ is necessary for the existence of a common base.

Theorem 4.13: *Let $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ be submodular systems on E with $f_1(E) = f_2(E)$. Then there exists a common base $x \in \mathrm{B}(f_1) \cap \mathrm{B}(f_2)$ if and only if $f_2{}^{\#} \le f_1$, i.e.,*

$$\forall X \in \mathcal{D}_1 \cap \overline{\mathcal{D}_2}\colon\ \ f_2{}^{\#}(X) \le f_1(X). \tag{4.67}$$

Moreover, if f_i $(i = 1, 2)$ are integer-valued and a common base exists, then there exists an integral common base.

(Proof) If there exists a common base $x \in \mathrm{B}(f_1) \cap \mathrm{B}(f_2)$, then since $\mathrm{B}(f_1) \cap \mathrm{B}(f_2) = \mathrm{P}(f_1) \cap \mathrm{P}(f_2{}^{\#})$, we have $f_2{}^{\#} \le x \le f_1$. Conversely, if $f_2{}^{\#} \le f_1$, then there exists a vector $x \in \mathbf{R}^E$ such that $f_2{}^{\#} \le x \le f_1$, due to the discrete separation theorem (Theorem 4.12). Hence $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2{}^{\#}) = \mathrm{B}(f_1) \cap \mathrm{B}(f_2{}^{\#}) = \mathrm{B}(f_1) \cap \mathrm{B}(f_2)$.

Moreover, the integrality part also follows from the integrality part of the discrete separation theorem. Q.E.D.

Note that, in Theorem 4.13, $f_2{}^{\#} \le f_1$ if and only if $f_1{}^{\#} \le f_2$, since $f_1(E) = f_2(E)$. Also note that "$f_2{}^{\#} \le f_1$" is equivalent to:

$$\forall X \in \mathcal{D}_1 \cap \overline{\mathcal{D}}_2\colon\ f_1(X) + f_2(E - X) \geq f_2(E)\ (= f_1(E)). \tag{4.68}$$

From the intersection theorem, (4.68) means that there exists a vector $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$ such that $x(E) \geq f_2(E)$, and such a vector x is a common base since $f_1(E) = f_2(E)$. In this way Theorem 4.13 can also be shown by the intersection theorem.

The common base problem can be solved by finding a maximum common subbase $x \in \mathrm{P}(f_1) \cap \mathrm{P}(f_2)$ through the algorithm shown in Section 4.1. We shall also give an algorithm for the common base problem which deals only with bases in $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$.

Given bases $b_1 \in \mathrm{B}(f_1)$ and $b_2 \in \mathrm{B}(f_2)$, we denote $\beta = (b_1, b_2)$ and define the auxiliary network $\mathcal{N}_\beta = (G_\beta = (V, A_\beta), T_\beta^+, T_\beta^-, c_\beta)$ as follows. G_β is the underlying graph with the vertex set $V = E$ and the arc set A_β defined by

$$A_\beta = A_\beta^1 \cup A_\beta^2, \tag{4.69}$$
$$A_\beta^1 = \{(u, v) \mid u, v \in V,\ u \in \mathrm{dep}_1(b_1, v) - \{v\}\}, \tag{4.70}$$
$$A_\beta^2 = \{(u, v) \mid u, v \in V,\ v \in \mathrm{dep}_2(b_2, u) - \{u\}\}, \tag{4.71}$$

where dep_1 and dep_2 are the dependence functions associated with $(\mathcal{D}_1, f_1)$ and $(\mathcal{D}_2, f_2)$, respectively. $c_\beta\colon A_\beta \to \mathbf{R}$ is the capacity function defined by

$$c_\beta(a) = \begin{cases} \tilde{\mathrm{c}}_1(b_1, v, u) & (a = (u, v) \in A_\beta^1), \\ \tilde{\mathrm{c}}_2(b_2, u, v) & (a = (u, v) \in A_\beta^2). \end{cases} \tag{4.72}$$

Here, $\tilde{\mathrm{c}}_1$ and $\tilde{\mathrm{c}}_2$ are the exchange capacities associated with $(\mathcal{D}_1, f_1)$ and $(\mathcal{D}_2, f_2)$, respectively. Also, T_β^+ and T_β^- are subsets of V defined by

$$T_\beta^+ = \{v \mid v \in V,\ b_1(v) > b_2(v)\}, \tag{4.73}$$
$$T_\beta^- = \{v \mid v \in V,\ b_1(v) < b_2(v)\}. \tag{4.74}$$

T_β^+ is the set of entrances and T_β^- the set of exits in $\mathcal{N}_\beta$.

Now, an algorithm for finding a common base of $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$ is given as follows.

An algorithm for finding a common base

Input: Submodular systems $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$ on E with $f_1(E) = f_2(E)$ and initial bases b_i of $(\mathcal{D}_i, f_i)$ $(i = 1, 2)$.

Output: A common base b_1 $(= b_2)$ if any exists.

Step 1: While $b_1 \neq b_2$, do the following (a)$\sim$(c):

(a) Construct the auxiliary network $\mathcal{N}_\beta = (G_\beta = (V, A_\beta), T_\beta^+, T_\beta^-, c_\beta)$ associated with $\beta = (b_1, b_2)$. If there is no directed path from T_β^+ to T_β^-, then stop (there is no common base in $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$).

(b) Let P be a directed path from T_β^+ to T_β^- in G_β having the smallest number of arcs and put

$$\alpha \leftarrow \min\{\min\{c_\beta(a) \mid a \text{ is an arc on } P\}, \\ b_1(\partial^+ P) - b_2(\partial^+ P),\ b_2(\partial^- P) - b_1(\partial^- P)\}.$$

($\partial^+ P$ is the initial vertex of P and $\partial^- P$ the terminal vertex of P.)

(c) For each arc a on P,

if $a = (u, v) \in A_\beta^1$, then put

$$b_1(u) \leftarrow b_1(u) - \alpha, \qquad b_1(v) \leftarrow b_1(v) + \alpha,$$

if $a = (u, v) \in A_\beta^2$, then put

$$b_2(u) \leftarrow b_2(u) + \alpha, \qquad b_2(v) \leftarrow b_2(v) - \alpha.$$

Step 2: The current b_1 is a common base of $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$ and the algorithm terminates.

(End)

Because of the way of choosing a directed path P in (b) of Step 1, the vectors b_1 and b_2 remain bases in $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$, respectively, due to Lemma 4.5.

If the algorithm terminates at (a) of Step 1, then let U be the set of the vertices in G_β which are reachable by directed paths from T_β^+. It follows from the definition of the auxiliary network $\mathcal{N}_\beta$ that

$$b_1(U) = f_1(E) - f_1(E - U), \tag{4.75}$$

$$b_2(U) = f_2(U). \tag{4.76}$$

Since $b_1(U) > b_2(U)$ by the assumption and the definition of U, we have from (4.75) and (4.76)

$$f_1(E)(= f_2(E)) > f_1(E - U) + f_2(U). \tag{4.77}$$

From (4.77) (and Theorem 4.9) we see that there exists no common base in $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$.

When f_1 and f_2 are integer-valued and initial bases b_1 and b_2 are integral, bases b_1 and b_2 obtained during the execution of the above algorithm are integral and hence the algorithm terminates after repeating (a)~(c) of Step 1 at most $b_1(T_\beta^+) - b_2(T_\beta^+)$ times.

For general rank functions f_1 and f_2 we adopt the lexicographic ordering technique described in Section 4.1.c ([Schönsleben80], [Lawler + Martel82a]). When finding a shortest path from T_β^+ to T_β^- by the breadth-first search, for each $u \in V$ search arc (u, v) in A_β^1 (or A_β^2) earlier than arc (u, v') in A_β^1 (or A_β^2) if $\pi(v) < \pi(v')$, for a fixed numbering π: $V \to \{1, 2, \cdots, |V|\}$ of V. By this modification the algorithm terminates after repeating Cycle (a)~(c) of Step 1 $\mathrm{O}(|E|^3)$ times.

5. Neoflows

In this section we consider the submodular flow problem, the independent flow problem and the polymatroidal flow problem, which we call *neoflow problems*. We discuss the equivalence among these neoflow problems and give algorithms for solving them.

5.1. Neoflows

We first give the definitions of the submodular flow problem, the independent flow problem and the polymatroidal flow problem.

(a) Submodular flows

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. Also let $\overline{c}$: $A \to \mathbf{R} \cup \{+\infty\}$ be an *upper capacity function* and $\underline{c}$: $A \to \mathbf{R} \cup \{-\infty\}$ be a *lower capacity function*. A function γ: $A \to \mathbf{R}$ is a *cost function*. Let $\mathcal{F} \subseteq \in^V$ be a crossing family with $\emptyset$, $V \in \mathcal{F}$ and f: $\mathcal{F} \to \mathbf{R}$ be a crossing-submodular function on the crossing family $\mathcal{F}$ with $f(\emptyset) = f(V) = 0$. (See Section 2.3 for the definition of crossing-submodular function on a crossing family.) Denote this network by $\mathcal{N}_S = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{F}, f))$.

The *submodular flow problem* considered by Edmonds and Giles [Edm + Giles77] is described as follows.

$$P_S\colon\ \text{Minimize}\ \sum_{a\in A}\gamma(a)\varphi(a) \tag{5.1a}$$

$$\text{subject to}\ \ \underline{c}(a)\le\varphi(a)\le\overline{c}(a)\quad(a\in A), \tag{5.1b}$$

$$\partial\varphi\in \mathrm{B}(f). \tag{5.1c}$$

Here, $\partial\varphi$ is the boundary of φ with respect to G (see (2.23)) and $\mathrm{B}(f)$ is the base polyhedron associated with f (see Theorem 2.5), where we assume $\mathrm{B}(f)\neq\emptyset$.

A feasible $\varphi\colon A\to\mathbf{R}$ satisfying (5.1b) and (5.1c) is called a *submodular flow* in $\mathcal{N}_S$ and an optimal solution of the submodular flow problem P_S is called an *optimal submodular flow* in $\mathcal{N}_S$. (The term "submodular flow" was introduced by Zimmermann [Zimmermann82].)

(b) Independent flows

Let $G=(V,A;S^+,S^-)$ be a graph with a vertex set V, an arc set A, a set S^+ of *entrances* and a set S^- of *exits* such that $S^+,\ S^-\subseteq V$ and $S^+\cap S^-=\emptyset$. Also let $\overline{c}\colon A\to\mathbf{R}\cup\{+\infty\}$ be an *upper capacity function*, $\underline{c}\colon A\to\mathbf{R}\cup\{-\infty\}$ be a *lower capacity function*, and $\gamma\colon A\to\mathbf{R}$ be a *cost function*. Moreover, let $(\mathcal{D}^+,f^+)$ be a submodular system on S^+ and $(\mathcal{D}^-,f^-)$ be a submodular system on S^-. Denote this network by $\mathcal{N}_I=(G=(V,A;S^+,S^-),\underline{c},\overline{c},\gamma,(\mathcal{D}^+,f^+),(\mathcal{D}^-,f^-))$.

The *independent flow problem* [Fuji78a] is given as follows (also see [McDiarmid75]). For a given $v^*\in\mathbf{R}$,

$$P_I\colon\ \text{Minimize}\ \sum_{a\in A}\gamma(a)\varphi(a) \tag{5.2a}$$

$$\text{subject to}\ \ \underline{c}(a)\le\varphi(a)\le\overline{c}(a)\quad(a\in A), \tag{5.2b}$$

$$\partial\varphi(v)=0\quad(v\in V-(S^+\cup S^-)), \tag{5.2c}$$

$$(\partial\varphi)^{S^+}\in\mathrm{P}(f^+), \tag{5.2d}$$

$$-(\partial\varphi)^{S^-}\in\mathrm{P}(f^-), \tag{5.2e}$$

$$\partial\varphi(S^+)=v^*. \tag{5.2f}$$

Here, $(\partial\varphi)^{S^+}$ (or $(\partial\varphi)^{S^-}$) is the restriction of $\partial\varphi\colon V\to\mathbf{R}$ to S^+ (or S^-) and $\mathrm{P}(f^+)$ and $\mathrm{P}(f^-)$ are, respectively, the submodular polyhedra associated with $(\mathcal{D}^+, f^+)$ and $(\mathcal{D}^-, f^-)$. (The original independent flow problem considered in [Fuji78a] is described in terms of polymatroids and is slightly generalized here to submodular systems.)

It should also be noted that if the system of (5.2b)∼(5.2f) is feasible, we have $\min\{f^+(S^+), f^-(S^-)\} \geq v^*$ and that letting $(\mathcal{D}^+, \hat{f}^+)$ and $(\mathcal{D}^-, \hat{f}^-)$ be, respectively, the $(f^+(S^+) - v^*)$-truncation of $(\mathcal{D}^+, f^+)$ and the $(f^-(S^-) - v^*)$-truncation of $(\mathcal{D}^-, f^-)$, we can replace (5.2d)∼(5.2f) by

$$(\partial\varphi)^{S^+} \in \mathrm{B}(\hat{f}^+), \tag{5.2g}$$

$$-(\partial\varphi)^{S^-} \in \mathrm{B}(\hat{f}^-). \tag{5.2h}$$

Here, note that $\partial\varphi(S^+) = -\partial\varphi(S^-)$ due to (5.2c).

A feasible flow φ: $A \to \mathbf{R}$ satisfying (5.2b)∼(5.2f) is called an *independent flow* of value v^* in $\mathcal{N}_I$ and an optimal solution of the independent flow problem P_I is called an *optimal independent flow* of value v^* in $\mathcal{N}_I$.

(c) Polymatroidal flows

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. For each vertex $v \in V$ we consider distributive lattices $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$ and $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$ and submodular functions $f_v^+ \colon \mathcal{D}_v^+ \to \mathbf{R}$ and $f_v^- \colon \mathcal{D}_v^- \to \mathbf{R}$. Here, we assume $\emptyset \in \mathcal{D}_v^+$, $\emptyset \in \mathcal{D}_v^-$ but not necessarily $\delta^+ v \in \mathcal{D}_v^+$, $\delta^- v \in \mathcal{D}_v^-$. Also let $\underline{c}$: $A \to \mathbf{R} \cup \{-\infty\}$ be a lower capacity function and γ: $A \to \mathbf{R}$ be a cost function. Denote this network by $\mathcal{N}_P = (G = (V, A), \underline{c}, \gamma, (\mathcal{D}_v^+, f_v^+)(v \in V), (\mathcal{D}_v^-, f_v^-)(v \in V))$.

The *polymatroidal flow problem* [Hassin78,82], [Lawler + Martel82a,82b] is given as follows.

$$P_P\text{: Minimize } \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.3a}$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \quad (a \in A), \tag{5.3b}$$

$$\partial\varphi(v) = 0 \quad (v \in V), \tag{5.3c}$$

$$\varphi^{\delta^+ v} \in \mathrm{P}(f_v^+) \quad (v \in V), \tag{5.3d}$$

$$\varphi^{\delta^- v} \in \mathrm{P}(f_v^-) \quad (v \in V), \tag{5.3e}$$

where $\partial\varphi$ is the boundary of φ with respect to G, $\varphi^{\delta^+ v}$ (or $\varphi^{\delta^- v}$) is the restriction of φ: $A \to \mathbf{R}$ to $\delta^+ v$ (or $\delta^- v$), and

$$\mathrm{P}(f_v^+) = \{x \mid x \in \mathbf{R}^{\delta^+ v},\ \forall X \in \mathcal{D}_v^+ \colon x(X) \leq f_v^+(X)\}, \tag{5.4}$$

$$\mathrm{P}(f_v^-) = \{x \mid x \in \mathbf{R}^{\delta^- v},\ \forall X \in \mathcal{D}_v^- : x(X) \le f_v^-(X)\}. \tag{5.5}$$

(The problem is originally defined in terms of polymatroids and is slightly generalized here.)

A feasible flow φ: $A \to \mathbf{R}$ satisfying (5.3b)~(5.3e) is called a *polymatroidal flow* in $\mathcal{N}_P$ and an optimal solution of Problem P_P is called an *optimal polymatroidal flow* in $\mathcal{N}_P$.

5.2. The Equivalence of the Neoflow Problems

We show the equivalence of the submodular flow problem, the independent flow problem and the polymatroidal flow problem. Here, the equivalence is with respect to the capability of modeling flow problems. Different models may require different oracles for algorithms but we will not go into this matter here.

(a) From submodular flows to independent flows

Consider the submodular flow problem P_S defined by (5.1). From Theorem 2.5 and the definition of the submodular flow problem we can assume that f appearing in (5.1) is a submodular function on a distributive lattice $\mathcal{D} \subseteq 2^V$ such that $\emptyset$, $V \in \mathcal{D}$ and $f(\emptyset) = f(V) = 0$. Let s^- be a new vertex not in $G = (V, A)$ and define

$$S^+ = V, \qquad S^- = \{s^-\}. \tag{5.6}$$

Then the submodular flow problem P_S is rewritten as

$$\text{Minimize } \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.7a}$$

$$\text{subject to } \underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \tag{5.7b}$$

$$(\partial\varphi)^{S^+} \in \mathrm{B}(f), \tag{5.7c}$$

$$-(\partial\varphi)^{S^-} \in \{0\}. \tag{5.7d}$$

This is an independent flow problem with the underlying graph $G' = (V \cup \{s^-\}, A;\ S^+, S^-)$. (See (5.2b), (5.2c), (5.2g) and (5.2h), where constraint (5.2c) is void.)

It is interesting to see that the submodular flow problem looks like a very special case of the independent flow problem (see Fig. 5.1).

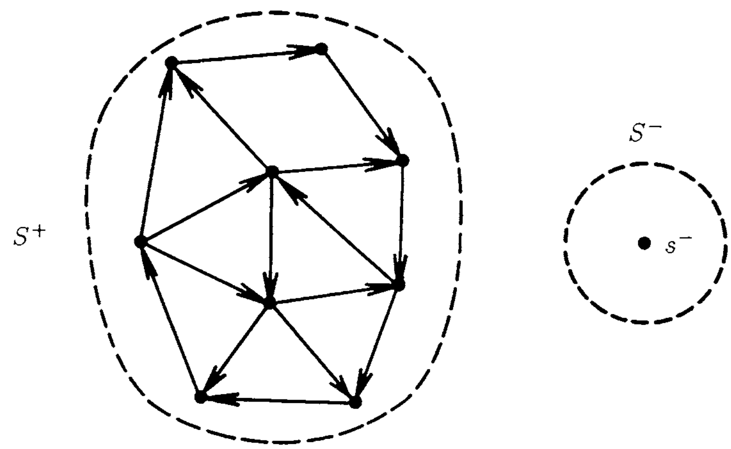

Figure 5.1: From submodular flows to independent flows.

(b) From independent flows to polymatroidal flows

Consider the independent flow problem P_I described by (5.2). Without loss of generality we assume that there is no arc entering S^+ or leaving S^- and that $f^+(S^+) = f^-(S^-) = v^*$. Let s^+ and s^- be new vertices not in $G = (V, A; S^+, S^-)$. Construct a graph $\hat{G} = (\hat{V}, \hat{A})$ with vertex set $\hat{V}$ and arc set $\hat{A}$ given by

$$\begin{aligned} \hat{V} &= V \cup \{s^+, s^-\}, && (5.8)\\ \hat{A} &= A \cup A^+ \cup A^- \cup A^0, && (5.9)\\ & \quad A^+ = \{(s^+, v) \mid v \in S^+\}, && (5.10)\\ & \quad A^- = \{(v, s^-) \mid v \in S^-\}, && (5.11)\\ & \quad A^0 = \{(s^-, s^+)\}. && (5.12) \end{aligned}$$

Also define a lower capacity function $\hat{\underline{c}}\colon \hat{A} \to \mathbf{R} \cup \{-\infty\}$ and a cost function $\hat{\gamma}\colon \hat{A} \to \mathbf{R}$ by

$$\hat{\underline{c}}(a) = \begin{cases} \underline{c}(a) & (a \in A), \\ v^* & (a = (s^-, s^+)), \\ -\infty & (a \in A^+ \cup A^-), \end{cases} \tag{5.13}$$

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & (a \in A), \\ 0 & (a \in A^+ \cup A^- \cup A^0). \end{cases} \tag{5.14}$$

Moreover, for each vertex $v \in V$ define polyhedra $P_v^+ \subseteq \mathbf{R}^{\delta^+ v}$ and $P_v^- \subseteq \mathbf{R}^{\delta^- v}$ by

$$P_v^+ = \begin{cases} \mathrm{P}(f^+) & (v = s^+), \\ (-\infty, +\infty) & (v \in \{s^-\} \cup S^-), \\ \{x \mid x \in \mathbf{R}^{\delta^+ v},\ \forall a \in \delta^+ v{:}\, x(a) \le \overline{c}(a)\} & (v \in V - S^-), \end{cases} \quad (5.15)$$

$$P_v^- = \begin{cases} \mathrm{P}(f^-) & (v = s^-), \\ (-\infty, +\infty) & (v \in \{s^+\} \cup S^+), \\ \{x \mid x \in \mathbf{R}^{\delta^- v},\ \forall a \in \delta^- v{:}\, x(a) \le \overline{c}(a)\} & (v \in V - S^+), \end{cases} \quad (5.16)$$

where $\mathrm{P}(f^+)$ and $\mathrm{P}(f^-)$ should, respectively, be regarded as polyhedra in $\mathbf{R}^{A^+}$ and $\mathbf{R}^{A^-}$ under the natural correspondence between S^+ and A^+ and between S^- and A^-.

Now, the independent flow problem P_I is rewritten as

$$\text{Minimize} \ \sum_{a \in \hat{A}} \hat{\gamma}(a)\varphi(a) \quad (5.17\text{a})$$

$$\text{subject to} \ \hat{\underline{c}}(a) \le \varphi(a) \quad (a \in \hat{A}), \quad (5.17\text{b})$$

$$\partial\varphi(v) = 0 \quad (v \in \hat{V}), \quad (5.17\text{c})$$

$$\varphi^{\delta^+ v} \in P_v^+ \quad (v \in \hat{V}), \quad (5.17\text{d})$$

$$\varphi^{\delta^- v} \in P_v^- \quad (v \in \hat{V}). \quad (5.17\text{e})$$

We can easily see that for each $v \in V$ there exist distributive lattices $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$ and $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$ and submodular functions $f_v^+{:}\, \mathcal{D}_v^+ \to \mathbf{R}$ and $f_v^-{:}\, \mathcal{D}_v^- \to \mathbf{R}$ such that $\emptyset \in \mathcal{D}_v^+ \cap \mathcal{D}_v^-$, $f_v^+(\emptyset) = f_v^-(\emptyset) = 0$ and

$$P_v^+ = \mathrm{P}(f_v^+), \qquad P_v^- = \mathrm{P}(f_v^-). \quad (5.18)$$

Therefore, the independent flow problem is reduced to a polymatroidal flow problem (see Fig. 5.2).

(c) From polymatroidal flows to submodular flows

Consider the polymatroidal flow problem P_P defined by (5.3). From the underlying graph $G = (V, A)$ we construct a graph $G' = (W, A)$ with the same arc set, where the vertex set W is given by

$$W = \{w_a^+ \mid a \in A\} \cup \{w_a^- \mid a \in A\} \quad (5.19)$$

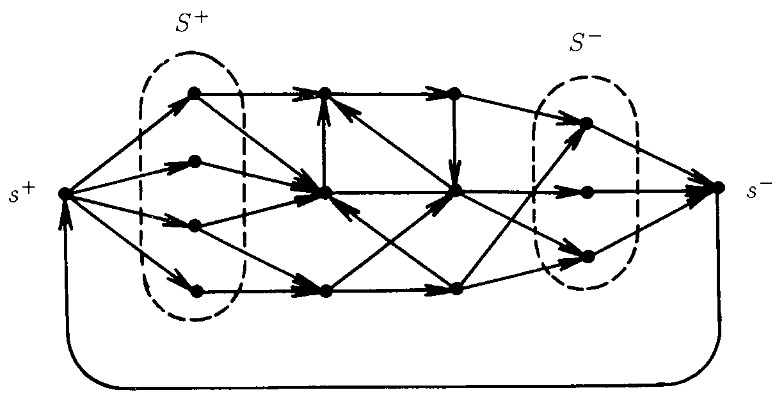

Figure 5.2: From independent flows to polymatroidal flows.

and we define $\partial^+ a = w_a^+$ and $\partial^- a = w_a^-$ for each $a \in A$ in G' (see Fig. 5.3). Each arc of G' forms a connected component of G'. Define an upper capacity function $\overline{c}'\colon A \to \mathbf{R} \cup \{+\infty\}$ and a lower capacity function $\underline{c}'\colon A \to \mathbf{R} \cup \{-\infty\}$ by

$$\overline{c}'(a) = +\infty \quad (a \in A), \qquad \underline{c}'(a) = \underline{c}(a) \quad (a \in A). \tag{5.20}$$

Also define

$$W_v^+ = \{w_a^+ \mid a \in \delta^+ v\}, \tag{5.21}$$

$$W_v^- = \{w_a^- \mid a \in \delta^- v\}, \tag{5.22}$$

where δ^+ and δ^- are with respect to G. Under the natural correspondences between W_v^+ and $\delta^+ v$ and between W_v^- and $\delta^- v$ for each $v \in V$ we regard $\mathrm{P}(f_v^+)$ and $\mathrm{P}(f_v^-)$ as polyhedra in $\mathbf{R}^{W_v^+}$ and $\mathbf{R}^{W_v^-}$, respectively.

Define for each $v \in V$

$$\begin{aligned} B_v = \{y \mid y \in \ & \mathbf{R}^{W_v^+ \cup W_v^-},\ y^{W_v^+} \in \mathrm{P}(f_v^+), \\ & -y^{W_v^-} \in \mathrm{P}(f_v^-),\ y(W_v^+) + y(W_v^-) = 0\} \end{aligned} \tag{5.23}$$

and let $B \subseteq \mathbf{R}^W$ be the direct sum of B_v $(v \in V)$:

$$B = \bigoplus_{v \in V} B_v. \tag{5.24}$$

Then, the polymatroidal flow problem is rewritten as

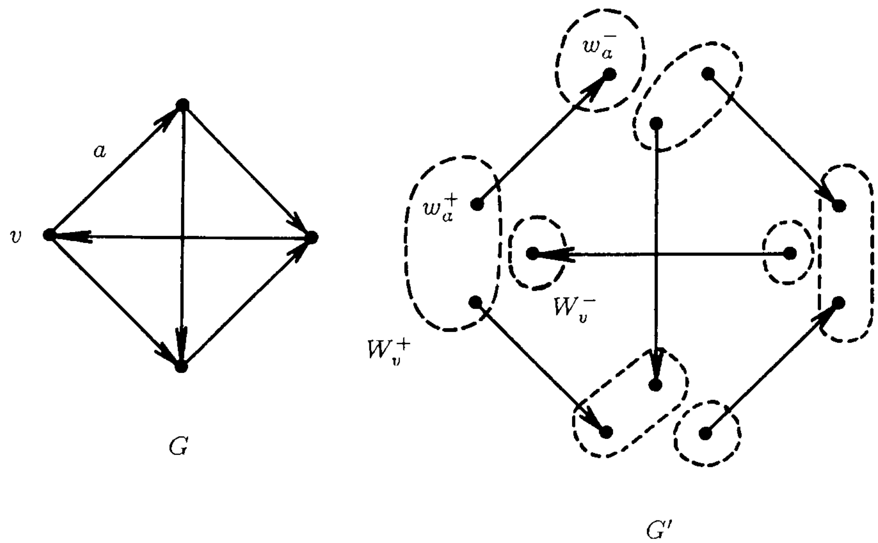

Figure 5.3: From polymatroidal flows to submodular flows.

$$\text{Minimize} \quad \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.25a}$$

$$\text{subject to} \quad \underline{c}'(a) \le \varphi(a) \le \overline{c}'(a) \quad (a \in A), \tag{5.25b}$$

$$\partial\varphi \in B. \tag{5.25c}$$

Note that $y \in B_v$ if and only if

$$\forall X \in \mathcal{D}_v^+: \quad y(X) \le f_v^+(X), \tag{5.26}$$

$$\forall X \in \mathcal{D}_v^-: \quad -y(X) \le f_v^-(X), \tag{5.27}$$

$$y(W_v^+) + y(W_v^-) = 0, \tag{5.28}$$

where we regard $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$ and $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$. The system of (5.26)~(5.28) is equivalent to that of (5.26), (5.28) and

$$\forall X \in \mathcal{D}_v^-: \ y(W_v^+) + y(W_v^- - X) \le f_v^-(X). \tag{5.29}$$

For each $v \in V$ let $\mathcal{F}_v$ be the crossing family defined by

$$\mathcal{F}_v = \mathcal{D}_v^+ \cup \overline{\mathcal{D}_v^-} \cup \{W_v^+ \cup W_v^-\}, \tag{5.30}$$

where $\overline{\mathcal{D}_v^-} = \{W_v^+ \cup (W_v^- - X) \mid X \in \mathcal{D}_v^-\}$, and let f_v: $\mathcal{F}_v \to \mathbf{R}$ be defined by

$$f_v(X) = \begin{cases} f_v^+(X) & (X \in \mathcal{D}_v^+), \\ f_v^-(W_v^- - X) & (X \in \overline{\mathcal{D}_v^-}), \\ 0 & (X = W_v^+ \cup W_v^-). \end{cases} \tag{5.31}$$

Then, f_v is a crossing-submodular function on the crossing family $\mathcal{F}_v$ and we have $B_v = \mathrm{B}(f_v)$, a base polyhedron. Therefore, it follows from (5.24) and (5.25) that the polymatroidal flow problem P_P is reduced to a submodular flow problem. It is interesting to see again that the polymatroidal flow problem looks like a very special case of the submodular flow problem.

5.3. Feasibility for Submodular Flows

In the previous section we have shown the equivalence of the submodular flow problem, the independent flow problem and the polymatroidal flow problem. Since the submodular flow problem is simple to describe, let us consider as a neoflow problem the submodular flow problem

$$P_S\text{: Minimize } \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.1a}$$

$$\text{subject to } \underline{c}(a) \le \varphi(a) \le \overline{c}(a) \ \ (a \in A), \tag{5.1b}$$

$$\partial\varphi \in \mathrm{B}(f). \tag{5.1c}$$

Due to Theorem 2.5 we assume for simplicity that f is a submodular function on a distributive lattice $\mathcal{D} \subseteq 2^V$ with $\emptyset,\ V \in \mathcal{D}$ and $f(\emptyset) = f(V) = 0$.

Recall (2.65), where we have shown

$$\begin{aligned} \partial\Phi &\equiv \{\partial\varphi \mid \varphi\colon A \to \mathbf{R},\ \forall a \in A\colon \underline{c}(a) \le \varphi(a) \le \overline{c}(a)\} \\ &= \mathrm{B}(\kappa_{\underline{c},\overline{c}}). \end{aligned} \tag{5.32}$$

Here, $\kappa_{\underline{c},\overline{c}}$: $2^V \to \mathbf{R} \cup \{+\infty\}$ is the cut function associated with the network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c})$ and $\partial\Phi$ is the base polyhedron associated with the cut function $\kappa_{\underline{c},\overline{c}}$. Hence, there exists a feasible flow φ for the submodular flow problem if and only if

$$\partial\Phi \cap \mathrm{B}(f) (= \mathrm{B}(\kappa_{\underline{c},\overline{c}}) \cap \mathrm{B}(f)) \ne \emptyset, \tag{5.33}$$

i.e., there exists a common base in $\mathrm{B}(\kappa_{\underline{c},\overline{c}})$ and $\mathrm{B}(f)$.

From Theorem 4.13 we have

Theorem 5.1 [Frank84]: *There exists a feasible flow for the submodular flow problem P_S satisfying* (5.1b) *and* (5.1c) *if and only if*

$$\forall X \in \mathcal{D}\colon\ (\kappa_{\underline{c},\overline{c}})^{\#}(X) \le f(X) \tag{5.34}$$

or

$$\forall X \in \mathcal{D}\colon\ \overline{c}(\Delta^{-}X) - \underline{c}(\Delta^{+}X) + f(X) \ge 0, \tag{5.35}$$

where for each $X \subseteq V$ $\Delta^{+}X = \{a \mid a \in A,\ \partial^{+}a \in X,\ \partial^{-}a \in V - X\}$ and $\Delta^{-}X = \{a \mid a \in A,\ \partial^{-}a \in X,\ \partial^{+}a \in V - X\}$.

Moreover, if $\overline{c}$, $\underline{c}$ and f are integer-valued and P_S is feasible, there exists an integral feasible flow.

(Proof) Immediate from Theorem 4.13. Q.E.D.

A feasible flow for the submodular flow problem can be obtained by the use of the algorithm shown in Section 4.3.

Frank [Frank84] showed feasibility theorems for the cases where f is an intersecting-submodular function and where f is a crossing-submodular function. We can give Frank's result by combining Theorems 5.1 and 2.6. That is,

Corollary 5.2 [Frank84]:

(i) *When f is an intersecting-submodular function on an intersecting family $\mathcal{F}$ such that $\emptyset, V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, the submodular flow problem P_S described by* (5.1) *has a feasible flow if and only if we have*

$$(\kappa_{\underline{c},\overline{c}})^{\#}(X) \le \sum_{i\in I} f(X_i) \tag{5.36}$$

for each $X \subseteq V$ and disjoint $X_i \in \mathcal{F}$ $(i \in I)$ such that $X = \bigcup_{i\in I} X_i$.

(ii) *When f is a crossing-submodular function on a crossing family $\mathcal{F}$ such that $\emptyset, V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, the submodular flow problem P_S has a feasible flow if and only if we have*

$$(\kappa_{\underline{c},\overline{c}})^{\#}(X) \le \sum_{i\in I}\sum_{j\in J_i} f(X_{ij}) \tag{5.37}$$

for each $X \subseteq V$, codisjoint $X_i \subseteq V$ $(i \in I)$ and disjoint X_{ij} $(j \in J_i)$ (for each $i \in I$) such that $X = \bigcap_{i\in I} X_i$ and $X_i = \bigcup_{j\in J_i} X_{ij}$ $(i \in I)$.

Note that (ii) of Corollary 5.2 is also expressed in a dual form as follows: Problem P_S has a feasible flow if and only if (5.37) holds for each $X \subseteq V$, disjoint $X_i \subseteq V$ $(i \in I)$ and codisjoint X_{ij} $(j \in J_i)$ (for each $i \in I$) such that $X = \bigcup_{i \in I} X_i$ and $X_i = \bigcap_{j \in J_i} X_{ij}$ $(i \in I)$ (see the remarks after Theorem 2.6).

Since the description of the algorithm for the common base problem given in Section 4.3 depends on the base polyhedron $\mathrm{B}(f)$ but not on the submodular function f or the system of linear inequalities expressing $\mathrm{B}(f)$, the algorithm also works even if f is a crossing-submodular function on a crossing family, provided that an initial base in $\mathrm{B}(f)$ and an oracle for exchange capacities are available. For such an f, however, finding a base in $\mathrm{B}(f)$ together with determining the nonemptiness of $\mathrm{B}(f)$ is itself a nontrivial problem.

For a fixed element $e \in E$, define families $\mathcal{F}_1 = \{X \mid e \notin X \in \mathcal{F}\} \cup \{V\}$, $\mathcal{F}_2 = \{X \mid e \in X \in \mathcal{F}\} \cup \{\emptyset\}$ and $\overline{\mathcal{F}_2} = \{E - X \mid X \in \mathcal{F}_2\}$, and let f_1 and f_2, respectively, be restrictions of f to $\mathcal{F}_1$ and $\mathcal{F}_2$. Then, $\mathcal{F}_1$ and $\overline{\mathcal{F}_2}$ are intersecting families and f_1 and $f_2^{\#}$ are, respectively, an intersecting-submodular function and an intersecting-supermodular function. Since $\mathrm{B}(f) = \mathrm{B}(f_1) \cap \mathrm{B}(f_2) = \mathrm{B}(f_1) \cap \mathrm{B}(f_2^{\#})$, a vector in $\mathrm{B}(f)$ is a common base in base polyhedra $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2^{\#})$, if both $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2^{\#})$ are nonempty. Since $\mathrm{P}(f_1)$ (or $\mathrm{P}(f_2^{\#})$) is a submodular (or supermodular) polyhedron, we can find a maximal (or minimal) vector in $\mathrm{P}(f_1)$ (or $\mathrm{P}(f_2^{\#})$) by adapting greedy algorithm II in Section 3.2.b. A vector in $\mathrm{B}(f)$, if any exists, is thus found by the algorithm for a common base problem (see [Frank84], [Fuji87b]). A more direct algorithm is given by [Frank + Tardos88] (also see [Naitoh + Fuji92]) as a bi-truncation algorithm.

5.4. Optimality for Submodular Flows

Consider the submodular flow problem P_S described by (5.1), where $(\mathcal{D}, f)$ is a submodular system on V with $f(V) = 0$.

We show the following optimality theorem.

Theorem 5.3: *A submodular flow φ: $A \rightarrow \mathbf{R}$ for Problem P_S is optimal if and only if there exists a function p: $V \rightarrow \mathbf{R}$ such that, defining γ_p: $A \rightarrow \mathbf{R}$ by*

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A), \tag{5.38}$$

we have for each $a \in A$

$$\gamma_p(a) > 0 \implies \varphi(a) = \underline{c}(a), \tag{5.39}$$

$$\gamma_p(a) < 0 \implies \varphi(a) = \overline{c}(a) \tag{5.40}$$

and such that the boundary $\partial\varphi\colon V \to \mathbf{R}$ is a maximum-weight base of $\mathrm{B}(f)$ *with respect to the weight function p.*

(Proof) *The "if" part*: By an elementary calculation we have

$$\sum_{a\in A}\gamma(a)\varphi(a) = \sum_{a\in A}\gamma_p(a)\varphi(a) - \sum_{v\in V}p(v)\partial\varphi(v). \tag{5.41}$$

The "if" part immediately follows from (5.41). We postpone the proof of the "only if" part. Q.E.D.

To prove the "only if" part of Theorem 5.3 we need some preliminaries.

Any function $p\colon V \to \mathbf{R}$ is called a *potential*. A potential p satisfying the conditions of Theorem 5.3 is called an *optimal potential*.

Given a submodular flow φ, we define the *auxiliary network* $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi, \gamma_\varphi)$, where G_φ is the graph with vertex set V and arc set A_φ given by

$$A_\varphi = A^*_\varphi \cup B^*_\varphi \cup C_\varphi, \tag{5.42}$$

$$A^*_\varphi = \{a \mid a \in A,\ \varphi(a) < \overline{c}(a)\}, \tag{5.43}$$

$$B^*_\varphi = \{\overline{a} \mid a \in A,\ \underline{c}(a) < \varphi(a)\} \quad (\overline{a}\text{: a reorientation of } a), \tag{5.44}$$

$$C_\varphi = \{(u,v) \mid u, v \in V,\ u \in \mathrm{dep}(\partial\varphi, v) - \{v\}\}, \tag{5.45}$$

$c_\varphi\colon A_\varphi \to \mathbf{R}$ is the capacity function given by

$$c_\varphi(a) = \begin{cases} \overline{c}(a) - \varphi(a) & (a \in A^*_\varphi) \\ \varphi(\overline{a}) - \underline{c}(\overline{a}) & (a \in B^*_\varphi,\ \overline{a}(\in A)\text{: a reorientation of } a) \\ \tilde{c}(\partial\varphi, v, u) & (a = (u,v) \in C_\varphi), \end{cases} \tag{5.46}$$

and $\gamma_\varphi\colon A_\varphi \to \mathbf{R}$ is the length function given by

$$\gamma_\varphi(a) = \begin{cases} \gamma(a) & (a \in A^*_\varphi) \\ -\gamma(\overline{a}) & (a \in B^*_\varphi,\ \overline{a}(\in A)\text{: a reorientation of } a) \\ 0 & (a = (u,v) \in C_\varphi). \end{cases} \tag{5.47}$$

We call the graph $G_{\partial\varphi} = (V, C_\varphi)$ the *exchangeability graph* associated with the base $\partial\varphi$ of $\mathrm{B}(f)$. Also, we call a directed cycle of negative length a *negative cycle*.

Lemma 5.4: *Let φ be an optimal submodular flow for Problem P_S. Then there exists no negative cycle, relative to the length function γ_φ, in the auxiliary network $\mathcal{N}_\varphi$.*

(Proof) Suppose, to the contrary, that there exists a negative cycle in $\mathcal{N}_\varphi$. Let Q be a negative cycle having the smallest number of arcs in $\mathcal{N}_\varphi$ and let the arcs in C_φ lying on Q be given by (u_i, v_i) $(i \in I)$.

We show that by an appropriate numbering of arcs (u_i, v_i) $(i \in I)$ the assumption of Lemma 4.5 is satisfied. Suppose, to the contrary, that the assumption of Lemma 4.5 cannot be satisfied by any numbering of arcs (u_i, v_i) $(i \in I)$, i.e., there are arcs (u_{i_k}, v_{i_k}) $(k = 1, 2, \cdots, p)$ such that $i_k \in I$ $(k = 1, 2, \cdots, p)$ and $(u_{i_k}, v_{i_{k+1}}) \in C_\varphi$ $(k = 1, 2, \cdots, p)$ with $i_{p+1} = i_1$. Then for each $k = 1, 2, \cdots, p$ let Q_k be the directed cycle formed by arc $(u_{i_k}, v_{i_{k+1}})$ and the path in Q from $v_{i_{k+1}}$ to u_{i_k} (see Fig. 5.4). Since $\gamma_\varphi((u_{i_k}, v_{i_k})) = \gamma_\varphi((u_{i_k}, v_{i_{k+1}})) = 0$ $(k = 1, 2, \cdots, p)$, we see that

$$\sum_{k=1}^{p} \gamma_\varphi(Q_k) = (p - q)\gamma_\varphi(Q) < 0 \tag{5.48}$$

for some integer q such that $1 \le q < p$, where $\gamma_\varphi(Q_k)$ and $\gamma_\varphi(Q)$ are the lengths of Q_k and Q relative to the length function γ_φ. It follows from (5.48) that there is at least one Q_k $(k = 1, 2, \cdots, p)$ having a negative length and such a directed cycle Q_k has a smaller number of arcs than Q. This contradicts the definition of Q, so that by an appropriate numbering of arcs (u_i, v_i) $(i \in I)$ the assumption of Lemma 4.5 is satisfied.

Define

$$\alpha = \min\{c_\varphi(a) \mid a \text{ lies on } Q\} \tag{5.49}$$

and modify the flow φ as

$$\varphi'(a) = \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(Q)) \\ \varphi(a) - \alpha & (\overline{a} \in B_\varphi^* \cap A_\varphi(Q),\ \overline{a}\text{: a reorientation of } a) \\ \varphi(a) & (\text{otherwise}) \end{cases} \tag{5.50}$$

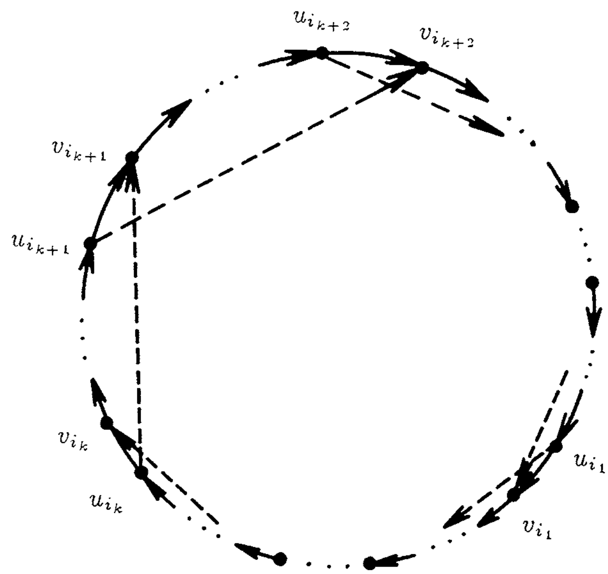

Figure 5.4: Q_k $(i = 1, 2, \cdots, p)$.

for each $a \in A$, where $A_\varphi(Q)$ denotes the set of arcs lying on Q. Then φ' is a submodular flow due to Lemma 4.5 and

$$\sum_{a \in A} \gamma(a)\varphi'(a) = \sum_{a \in A} \gamma(a)\varphi(a) + \alpha \cdot \gamma_\varphi(Q) < \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.51}$$

This contradicts the assumption that φ is an optimal submodular flow. Consequently, there is no negative cycle in $\mathcal{N}_\varphi$. Q.E.D.

The proof technique concerning (5.48) was originated by the author [Fuji77a, 77b] for matroids and may be interesting in its own right (also see [Zimmermann82]).

Now, we show the "only if" part of Theorem 5.3.

(Proof of the "only if" part of Theorem 5.3) Let φ be an optimal submodular flow for Problem P_S. Then, from Lemma 5.4 there exists no negative cycle in the auxiliary network $\mathcal{N}_\varphi$ relative to the length function γ_φ. This implies that there exists a potential p: $V \to \mathbf{R}$ such that

$$\gamma_{\varphi,p}(a) \equiv \gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \tag{5.52}$$

for each $a \in A_\varphi$. (Such a potential p can be found by a shortest path computation from a fixed origin s outside $\mathcal{N}_\varphi$, where the origin s is connected with each vertex of V by a new arc of any finite length, say, zero.) We can easily see that (5.52) for $a \in A^*_\varphi \cup B^*_\varphi$ implies (5.39) and (5.40) and that (5.52) for $a \in C_\varphi$ implies

$$p(u) \geq p(v) \qquad (u \in \mathrm{dep}(\partial\varphi, v) - \{v\}). \tag{5.53}$$

From (5.53) and Theorem 3.16 $\partial\varphi(\in \mathrm{B}(f))$ is a maximum-weight base of $\mathrm{B}(f)$ with respect to the weight function p. Q.E.D.

It should be emphasized here that Theorem 5.3 is independent of how the base polyhedron $\mathrm{B}(f)$ is represented by a system of linear inequalities. [Also note that the above proof shows that a submodular flow φ is optimal if and only if there is no negative cycle in the auxiliary network $\mathcal{N}_\varphi$.]

The proof of Lemma 5.4 suggests an algorithm for solving the submodular flow problem P_S as follows:

> Starting from an arbitrary submodular flow φ, (i) find a negative cycle Q having the smallest number of arcs in the auxiliary network $\mathcal{N}_\varphi$, (ii) modify the present flow φ along the negative cycle Q as in (5.49) and (5.50), where if $\alpha = +\infty$, the problem does not have a finite optimal solution, and (iii) repeat this process until there is no negative cycle in $\mathcal{N}_\varphi$.

This is the primal algorithm given in [Fuji78a] and [Zimmermann82] (also see [Cui + Fuji88] and [Zimmermann92]). When $\overline{c}$, $\underline{c}$, f and an initial φ are integer-valued, this algorithm terminates after a finite number of steps and finds an integer-valued optimal submodular flow if any optimal one exists. We shall discuss more efficient algorithms in the next section.

A necessary and sufficient condition for the existence of an optimal submodular flow is given as follows.

Theorem 5.5: *For the submodular flow problem P_S define a network $\hat{\mathcal{N}} = (\hat{G} = (V, \hat{A}), \hat{\gamma})$, where $\hat{G}$ is a graph with vertex set V and arc set $\hat{A}$ given by*

$$\hat{A} = A^* \cup B^* \cup C, \tag{5.54}$$

$$A^* = \{a \mid a \in A,\ \overline{c}(a) = +\infty\}, \tag{5.55}$$

$$B^* = \{\overline{a} \mid a \in A,\ \underline{c}(a) = -\infty\} \quad (\overline{a}\text{: a reorientation of } a), \tag{5.56}$$

$$C = \{(u, v) \mid u, v, \in V,\ \forall X \in \mathcal{D}\text{: } (v \in X \Rightarrow u \in X)\} \tag{5.57}$$

and $\hat{\gamma}\colon \hat{A} \to \mathbf{R}$ *is the length function defined by*

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & (a \in A^*) \\ -\gamma(\overline{a}) & (a \in B^*,\ \overline{a}(\in A)\text{: a reorientation of } a) \\ 0 & (a \in C). \end{cases} \tag{5.58}$$

Suppose that there exists a submodular flow for Problem P_S*. Then, there exists an optimal submodular flow for Problem* P_S *if and only if there is no negative cycle in* $\hat{\mathcal{N}}$ *relative to the length function* $\hat{\gamma}$*.*

(Proof) *The "if" part*: Suppose that there is no negative cycle in $\hat{\mathcal{N}}$ relative to $\hat{\gamma}$. Then there is a potential $p\colon V \to \mathbf{R}$ such that for each $\hat{a} \in \hat{A}$

$$\hat{\gamma}_p(\hat{a}) \equiv \hat{\gamma}(\hat{a}) + p(\partial^+\hat{a}) - p(\partial^-\hat{a}) \geq 0, \tag{5.59}$$

where ∂^+ and ∂^- are with respect to $\hat{G}$. (5.59) implies

$$\gamma(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \quad (a \in A,\ \overline{c}(a) = +\infty), \tag{5.60}$$

$$\gamma(a) + p(\partial^+ a) - p(\partial^- a) \leq 0 \quad (a \in A,\ \underline{c}(a) = -\infty), \tag{5.61}$$

$$p(u) \geq p(v) \qquad ((u,v) \in C). \tag{5.62}$$

Since (5.1c) is expressed as

$$\begin{aligned} \partial\varphi(X) &\leq f(X) \quad (X \in \mathcal{D}), &(5.63)\\ \partial\varphi(V) &= f(V)(=0), &(5.64)\end{aligned}$$

where (5.64) is void, and since

$$\partial\varphi(X) = \varphi(\Delta^+ X) - \varphi(\Delta^- X), \tag{5.65}$$

the linear-programming dual of Problem P_S with (5.1c) being replaced by (5.63) is described as

$$P_S{}^*\colon\ \text{Maximize} \sum_{a\in A} \underline{\xi}(a)\underline{c}(a) - \sum_{a\in A} \overline{\xi}(a)\overline{c}(a) - \sum_{X\in\mathcal{D}} \eta(X)f(X) \tag{5.66a}$$

$$\begin{aligned}\text{subject to }\ & \underline{\xi}(a) - \overline{\xi}(a) - \sum\{\eta(X) \mid X \in \mathcal{D},\ a \in \Delta^+ X\} \\ & + \sum\{\eta(X) \mid X \in \mathcal{D},\ a \in \Delta^- X\} = \gamma(a) \\ & \qquad (a \in A),\end{aligned} \tag{5.66b}$$

$$\underline{\xi}(a) = 0 \quad (a \in A,\ \underline{c}(a) = -\infty), \tag{5.66c}$$

$$\overline{\xi}(a) = 0 \quad (a \in A,\ \overline{c}(a) = +\infty), \tag{5.66d}$$

$$\underline{\xi},\ \overline{\xi},\ \eta \geq 0, \tag{5.66e}$$

where $\underline{\xi}$, $\overline{\xi}$: $A \to \mathbf{R}$, η: $\mathcal{D} \to \mathbf{R}$ and we should regard (5.66a) as the objective function with the terms $\underline{\xi}(a)\underline{c}(a)$ $(a \in A,\ \underline{c}(a) = -\infty)$ and $\overline{\xi}(a)\overline{c}(a)$ $(a \in A,\ \overline{c}(a) = +\infty)$ being suppressed.

Because of (5.62) there is a minimal chain

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = V \tag{5.67}$$

of $\mathcal{D}$ such that p is constant on each quotient $S_k - S_{k-1}$ $(k = 1, 2, \cdots, n)$. Using this chain $\mathcal{C}$ and the potential p, define

$$\eta(S_k) = p_k - p_{k+1} \quad (k = 1, 2, \cdots, n-1), \tag{5.68}$$

where p_k is the value of p taken in $S_k - S_{k-1}$ and note that $\eta(S_k) > 0$. Also define $\eta(X) = 0$ for other $X \in \mathcal{D}$. Moreover, define

$$\underline{\xi}(a) \;=\; \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A,\ \overline{c}(a) = +\infty), \tag{5.69}$$

$$\overline{\xi}(a) \;=\; -\gamma(a) - p(\partial^+ a) + p(\partial^- a) \quad (a \in A,\ \underline{c}(a) = -\infty), \tag{5.70}$$

and for each arc $a \in A$ with $\overline{c}(a) < +\infty$ and $\underline{c}(a) > -\infty$ define $\underline{\xi}(a)$ and $\overline{\xi}(a)$ such that $\underline{\xi}(a)$, $\overline{\xi}(a) \geq 0$ and (5.66b) holds. We can easily see that thus defined $\underline{\xi}$, $\overline{\xi}$, η satisfy (5.66b)$\sim$(5.66e), where note that for each arc $a \in A$ with $\overline{c}(a) = +\infty$ and $\underline{c}(a) = -\infty$ we have $\gamma(a) + p(\partial^+ a) - p(\partial^- a) = 0$ due to (5.60) and (5.61).

Since the dual of Problem P_S has a feasible solution and the feasibility of the primal problem P_S is assumed, there exists an optimal solution of Problem P_S.

The "only if" part: Suppose that there is a negative cycle in $\hat{\mathcal{N}}$ relative to the length function $\hat{\gamma}$, and let Q be such a negative cycle in $\hat{\mathcal{N}}$. Then for any positive α, if we define φ': $A \to \mathbf{R}$ by (5.50), φ' is feasible for Problem P_S because of the definition of $\hat{\mathcal{N}}$ and we have (5.51). Since α (> 0) is arbitrary and $\gamma_\varphi(Q) < 0$, Problem P_S does not have a finite optimal solution. Q.E.D.

We also have

Theorem 5.6 [Edm+Giles77]: *The system of linear inequalities*

$$\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \quad (a \in A), \tag{5.71}$$

$$\partial\varphi(X)(=\varphi(\Delta^+X)-\varphi(\Delta^-X))\le f(X)\quad (X\in\mathcal{D}) \tag{5.72}$$

for the submodular flow problem P_S is totally dual integral.

(Proof) Consider a submodular flow problem P_S such that the coefficients $\gamma(a)$ $(A\in A)$ of the objective function are integers and that there exists an optimal submodular flow φ. From the proof of the "only if" part of Theorem 5.3 there exists an integral potential p: $V\to\mathbf{Z}$ such that for each $a\in A$

$$\gamma_p(a)=\gamma(a)+p(\partial^+a)-p(\partial^-a)>0\Longrightarrow\varphi(a)=\underline{c}(a), \tag{5.73}$$

$$\gamma_p(a)=\gamma(a)+p(\partial^+a)-p(\partial^-a)<0\Longrightarrow\varphi(a)=\overline{c}(a) \tag{5.74}$$

and for each $u\in V$ and $v\in\mathrm{dep}(\partial\varphi,u)-\{u\}$

$$p(u)\le p(v). \tag{5.75}$$

Let $p_1>p_2>\cdots>p_l$ be the distinct values of $p(u)$ $(u\in V)$ and define

$$W_i=\{u\mid u\in V,\ p(u)\ge p_i\}\quad (i=1,2,\cdots,l). \tag{5.76}$$

From (5.75) we have

$$\partial\varphi(W_i)=f(W_i)\quad (i=1,2,\cdots,l). \tag{5.77}$$

For the dual problem $P_S{}^*$ in (5.66) of P_S define

$$\begin{aligned}
\underline{\xi}(a) &= \max\{0,\gamma_p(a)\}\quad (a\in A), && (5.78)\\
\overline{\xi}(a) &= \max\{0,-\gamma_p(a)\}\quad (a\in A), && (5.79)\\
\eta(W_i) &= p_i-p_{i+1}\quad (i=1,2,\cdots,l-1), && (5.80)\\
\eta(X) &= 0\quad \text{for other } X\in\mathcal{D}. && (5.81)
\end{aligned}$$

We can easily see that these $\underline{\xi}$, $\overline{\xi}$ and η satisfy (5.66b)∼(5.66e). Moreover, from (5.73), (5.74) and (5.76),

$$\begin{aligned}
&\sum_{a\in A}\underline{\xi}(a)\underline{c}(a)-\sum_{a\in A}\overline{\xi}(a)\overline{c}(a)-\sum_{X\in\mathcal{D}}\eta(X)f(X)\\
&\quad=\sum_{a\in A}\gamma_p(a)\varphi(a)-\sum_{i=1}^{l-1}(p_i-p_{i+1})f(W_i)\\
&\quad=\sum_{a\in A}\gamma_p(a)\varphi(a)-\sum_{i=1}^{l-1}(p_i-p_{i+1})\partial\varphi(W_i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a \in A} \gamma_p(a)\varphi(a) - \sum_{i=1}^{l} p_i \partial\varphi(W_i - W_{i-1}) \\
&= \sum_{a \in A} \gamma_p(a)\varphi(a) - \sum_{v \in V} p(v)\partial\varphi(v) \\
&= \sum_{a \in A} \gamma(a)\varphi(a), \qquad (5.82)
\end{aligned}
$$

where $W_0 = \emptyset$, we put $0 \times (\pm\infty) = 0$, and note that $\partial\varphi(W_l) = \partial\varphi(V) = 0$. Therefore, $\underline{\xi}$, $\overline{\xi}$ and η defined by (5.78)~(5.81) form an integral optimal solution of the dual problem ${P_S}^*$. Q.E.D.

It follows from Theorems 5.6 and 2.6 that if we replace f in (5.72) by a crossing-submodular function on a crossing family, the system of (5.71) and (5.72) is totally dual integral.

Corollary 5.7 [Edm+Giles77]: *For a crossing-submodular function f on a crossing family $\mathcal{F} \subseteq 2^V$, the system of inequalities*

$$\underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \qquad (5.83)$$

$$\partial\varphi(X) \le f(X) \quad (X \in \mathcal{F}) \qquad (5.84)$$

is totally dual integral.

The proof of Theorem 5.6 shows the way of constructing an optimal dual solution $\underline{\xi}$, $\overline{\xi}$ and η from an optimal potential p when f is a submodular function on the distributive lattice $\mathcal{D}$. When f is a crossing-submodular function on a crossing family $\mathcal{F} \subseteq 2^V$ with $\emptyset,\ V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, suppose that $\mathrm{B}(f) = \mathrm{B}(f_2)$ for a submodular system $(\mathcal{D}, f_2)$ (see Theorem 2.6) and that we are given an optimal submodular flow φ and an optimal potential p. Also suppose that for each W_i in (5.77) we have the expression of $f_2(W_i)$ in terms of f as follows (see Theorem 2.6).

$$f_2(W_i) = \sum_{j \in J_i} \sum_{k \in K_{ij}} f(X_{ijk}) \quad (i = 1, 2, \cdots, l-1), \qquad (5.85)$$

where $V - (\bigcup_{k \in K_{ij}} X_{ijk})$ $(j \in J_i)$ form a partition of $V - W_i$ for each $i = 1, 2, \cdots, l-1$ and X_{ijk} $(k \in K_{ij})$ are disjoint sets (as subsets of V) in

$\mathcal{F}$ for each $i = 1, 2, \cdots, l-1$ and $j \in J_i$. Here, note that $f_2(V) = 0$. Then define for each $X \in \mathcal{F}$

$$\eta(X) = \sum\{p_i - p_{i+1} \mid X = X_{ijk},\ i \in \{1, \cdots, l-1\},\ j \in J_i,\ k \in K_{ij}\}, \tag{5.86}$$

where the summation over the empty set is equal to zero. (Note that if all the X_{ijk} are distinct, we have $\eta(X_{ijk}) = p_i - p_{i+1}$ ($i \in \{1, \cdots, l-1\}$, $j \in J_i$, $k \in K_{ij}$) and $\eta(X) = 0$ for other $X \in \mathcal{F}$.) $\underline{\xi}$, $\overline{\xi}$ defined by (5.78) and (5.79) and this η form an optimal dual solution, since (5.82) with $\mathcal{D}$ replaced by $\mathcal{F}$ holds.

We show the way of finding an expression in (5.85) for each $i = 1,\ 2,\ \cdots,$ $l-1$. Put $x = \partial\varphi$ and define

$$\mathcal{F}(x) = \{X \mid X \in \mathcal{F},\ x(X) = f(X)\}. \tag{5.87}$$

We assume an oracle which, for each ordered pair (u, v) of distinct vertices $u,\ v \in V$, gives a u-v cut $X \in \mathcal{F}(x)$ or answers that there is no u-v cut $X \in \mathcal{F}(x)$, where a *u-v cut* is a set $X \subseteq V$ such that $u \in X$ and $v \notin X$. Let $G = (V, A(x))$ be the exchangeability graph associated with x, i.e.,

$$A(x) = \{(v, u) \mid u \in V,\ v \in \mathrm{dep}(x, u) - \{u\}\}, \tag{5.88}$$

where note that $(v, u) \in A(x)$ if and only if $u \neq v$ and there is no u-v cut $X \in \mathcal{F}(x)$. Choose any W_i ($i = 1, 2, \cdots, l-1$). Let G_i be the graph obtained from G by deleting all the vertices in W_i together with the arcs incident to W_i, and let U_{ij} ($j \in J_i$) be the vertex sets of the connected components of G_i. Note that

$$\Delta^+ U_{ij} = \emptyset \quad (j \in J_i) \tag{5.89}$$

in G, so that

$$f_2(V - U_{ij}) = x(V - U_{ij}) \quad (j \in J_i). \tag{5.90}$$

Therefore, for each $u \in V - U_{ij}$ and $v \in U_{ij}$ there exists a u-v cut $X \in \mathcal{F}(x)$. If $X \cap U_{ij} \neq \emptyset$, choose $v' \in X \cap U_{ij}$ such that for some $w \in U_{ij} - X$ we have $(v', w) \in A(x)$. Such a v' exists since $U_{ij} - X \neq \emptyset$ and U_{ij} is the vertex set of a connected component of G_i. There exists a u-v' cut $X' \in \mathcal{F}(x)$ and either X and X' cross or $X' \subseteq X$, due to the way of choosing v'. (Note that $u \in X \cap X'$, $v' \in X - X'$ and $w \notin X \cup X'$.) Since $\mathcal{F}(x)$ is a crossing family, we have $X \cap X' \in \mathcal{F}(x)$ and $u \in X \cap X' \subset X$. Put $X \leftarrow X \cap X'$.

If $X \cap U_{ij} \neq \emptyset$, then repeat this process until we have $X \cap U_{ij} = \emptyset$. After repeating this process at most $|U_{ij}|$ times we have $X \in \mathcal{F}(x)$ such that $u \in X \subseteq V - U_{ij}$. Denote this X by X_u.

In this way, for each $u \in V - U_{ij}$ we can find X_u such that $u \in X_u \subseteq V - U_{ij}$. Consider the hypergraph $H = (V - U_{ij}, \{X_u \mid u \in V - U_{ij}\})$ and let X_{ijk} $(k \in K_{ij})$ be the vertex sets of the connected components of H. Since $\mathcal{F}(x)$ is a crossing family of subsets of V, we have $X_{ijk} \in \mathcal{F}(x)$ $(k \in K_{ij})$. Consequently,

$$\begin{aligned} f_2(W_i) = x(W_i) &= \sum_{j \in J_i} \sum_{k \in K_{ij}} x(X_{ijk}) - (|J_i| - 1)x(V) \\ &= \sum_{j \in J_i} \sum_{k \in K_{ij}} f(X_{ijk}) \end{aligned} \tag{5.91}$$

since $x(V) = 0$.

When f is an intersecting-submodular function on an intersecting family $\mathcal{F} \subseteq 2^V$ with $\emptyset$, $V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, for each $u \in W_i$ we can find $X_u \in \mathcal{F}(x)$ such that $u \in X_u \subseteq W_i$. (Note that for any v, $v' \in V - W_i$ there exist a u-v cut $X \in \mathcal{F}(x)$ and a u-v' cut $X' \in \mathcal{F}(x)$ and that $u \in X \cap X' \in \mathcal{F}(x)$ since $\mathcal{F}(x)$ is an intersecting family.) Let X_{ij} $(j \in J_i)$ be the vertex sets of the connected components of the hypergraph $H = (W_i, \{X_u \mid u \in W_i\})$. Then $X_{ij} \in \mathcal{F}(x)$ and we have

$$f_1(W_i) = \sum_{j \in J_i} f(X_{ij}), \tag{5.92}$$

where f_1 is the submodular function appearing in (i) of Theorem 2.6.

Let $G = (V, A)$ be a connected (directed) graph. In Section 3.3.b we considered strongly k-connected reorientations of G for a positive integer k. For the crossing-submodular function $\kappa^{(k)}$: $2^V \to \mathbf{R}$ defined by (3.105), the problem of finding a minimum-cost strongly k-connected reorientation φ: $A \to \{0, 1\}$ is described in a relaxed form as

$$P^{(k)}: \quad \text{Minimize} \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.93a}$$

$$\text{subject to} \quad 0 \leq \varphi(a) \leq 1 \quad (a \in A), \tag{5.93b}$$

$$\partial\varphi(U) \leq \kappa^{(k)}(U) \quad (U \subseteq V), \tag{5.93c}$$

where γ: $A \to \mathbf{R}$ is a cost function and φ: $A \to \mathbf{R}$ is not restricted to integral vectors. Due to Corollary 5.7 the polyhedron of feasible flows φ is integral and there exists an integral optimal solution, if any optimal solution exists, which is a minimum-cost strongly k-connected reorientation of G. Problem (5.93) is a typical example of the submodular flow problem (see [Frank80,81c]). Note that Problem (5.93) with $k = 1$ has a feasible flow if and only if G is connected and has no bridge.

For a connected graph $G = (V, A)$ let $\mathcal{D}(G)$ be the set of all the ideals of G, i.e.,

$$\mathcal{D}(G) = \{W \mid W \subseteq V,\ \Delta^+ W = \emptyset\}. \tag{5.94}$$

For each $W \in \mathcal{D}(G) - \{\emptyset, V\}$ we call $\Delta^- W (\neq \emptyset)$ a *directed cut*. A subset B of arc set A is called a *directed-cut covering* of G if B has nonempty intersection with each directed cut of G. C. L. Lucchesi and D. H. Younger [Lucchesi + Younger78] consider the problem of finding a minimum directed-cut covering, which is formulated in a relaxed form as

$$DCC\colon\ \ \text{Minimize} \sum_{a \in A} \varphi(a) \tag{5.95a}$$

$$\text{subject to}\ \ 0 \leq \varphi(a) \leq 1 \quad (a \in A), \tag{5.95b}$$

$$\partial\varphi(W) \leq \kappa^{(1)}(W) \quad (W \in \mathcal{D}(G)), \tag{5.95c}$$

where $\kappa^{(1)}$ is defined by (3.105) with $k = 1$. It should be noted that $\kappa^{(1)}$ is a crossing-submodular function on $\mathcal{D}(G)$. The linear-programming dual of DCC is given by

$$DCC^*\colon\ \ \text{Maximize} - \sum_{a \in A} \xi(a) - \sum_{W \in \mathcal{D}(G)} \eta(W)\kappa^{(1)}(W) \tag{5.96a}$$

$$\text{subject to}\ \ -\xi(a) + \sum\{\eta(W) \mid W \in \mathcal{D}(G), a \in \Delta^- W\} \leq 1 \quad (a \in A), \tag{5.96b}$$

$$\xi,\ \eta \geq 0. \tag{5.96c}$$

Therefore, Problem DCC^* becomes

$$\text{Maximize} \sum_{a \in A} \min\{0, [1 - \sum\{\eta(W) \mid W \in \mathcal{D}(G), a \in \Delta^- W\}]\} + \sum_{W \in \mathcal{D}(G) - \{\emptyset, V\}} \eta(W) \tag{5.97a}$$

$$\text{subject to}\ \ \eta \geq 0. \tag{5.97b}$$

We see from (5.95)$\sim$(5.97) and Corollary 5.7 that there exists an integral optimal solution η of dual DCC^* such that (1) $\eta(W) \in \{0, 1\}$ ($W \in$

$\mathcal{D}(G) - \{\emptyset, V\}$) and (2) directed cuts $\Delta^- W$ with $\eta(W) = 1$ are disjoint. Consequently, we have the following theorem.

Theorem 5.8 [Lucchesi+Younger78]: *The minimum cardinality of a directed-cut covering of G is equal to the maximum cardinality of a set of disjoint directed cuts of G.*

5.5. Algorithms for Neoflows

We consider maximum neoflow and minimum-cost neoflow problems and show algorithms for these problems.

(a) Maximum independent flows

The maximum flow problem for neoflows seems to be easily formulated by the independent flow problem. The *maximum independent flow problem* is:

$$P_{MI}\text{: Maximize } \partial\varphi(S^+) \tag{5.98a}$$

$$\text{subject to } \underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \tag{5.98b}$$

$$\partial\varphi(v) = 0 \quad (v \in V - (S^+ \cup S^-)), \tag{5.98c}$$

$$(\partial\varphi)^{S^+} \in \mathrm{P}(f^+), \tag{5.98d}$$

$$-(\partial\varphi)^{S^-} \in \mathrm{P}(f^-), \tag{5.98e}$$

where we consider the same network described in Section 5.1.b. We denote $(\partial\varphi)^{S^+}$ and $-(\partial\varphi)^{S^-}$ by $\partial^+\varphi$ and $\partial^-\varphi$, respectively, in the following.

We suppose there exists a feasible flow. A feasible flow, if any exists, can be found by adapting the algorithm in Section 4.3 as discussed in Section 5.3 for submodular flows. Given a feasible flow φ, we define an auxiliary network $\mathcal{N}_\varphi = (G_\varphi = (V \cup \{s^+, s^-\}, A_\varphi), c_\varphi, s^+, s^-)$ with source s^+ and sink s^- as follows. G_φ is the underlying graph with vertex set $V \cup \{s^+, s^-\}$ and arc set A_φ defined by

$$A_\varphi = S_\varphi^+ \cup S_\varphi^- \cup A_\varphi^+ \cup A_\varphi^- \cup A_\varphi^* \cup B_\varphi^*, \tag{5.99}$$

$$S_\varphi^+ = \{(s^+, v) \mid v \in S^+ - \mathrm{sat}^+(\partial^+\varphi)\}, \tag{5.100}$$

$$S_\varphi^- = \{(v, s^-) \mid v \in S^- - \mathrm{sat}^-(\partial^-\varphi)\}, \tag{5.101}$$

$$A_\varphi^+ = \{(u, v) \mid v \in \mathrm{sat}^+(\partial^+\varphi),\ u \in \mathrm{dep}^+(\partial^+\varphi, v) - \{v\}\}, \tag{5.102}$$

$$A_\varphi^- = \{(v,u) \mid v \in \mathrm{sat}^-(\partial^-\varphi),\ u \in \mathrm{dep}^-(\partial^-\varphi, v) - \{v\}\}, \tag{5.103}$$
$$A_\varphi^* = \{a \mid a \in A,\ \varphi(a) < \overline{c}(a)\}, \tag{5.104}$$
$$B_\varphi^* = \{\overline{a} \mid a \in A,\ \varphi(a) > \underline{c}(a)\}, \tag{5.105}$$

where $\overline{a}$ denotes a reorientation of a, sat^+ (sat^-) is the saturation function with respect to $(\mathcal{D}^+, f^+)$ ($(\mathcal{D}^-, f^-)$) and dep^+ (dep^-) is the dependence function with respect to $(\mathcal{D}^+, f^+)$ ($(\mathcal{D}^-, f^-)$). Also, c_φ: $A_\varphi \to \mathbf{R}$ is defined by

$$c_\varphi(a) = \begin{cases} \hat{c}^+(\partial^+\varphi, v) & (a = (s^+, v) \in S_\varphi^+) \\ \hat{c}^-(\partial^-\varphi, v) & (a = (v, s^-) \in S_\varphi^-) \\ \tilde{c}^+(\partial^+\varphi, v, u) & (a = (u, v) \in A_\varphi^+) \\ \tilde{c}^-(\partial^-\varphi, v, u) & (a = (v, u) \in A_\varphi^-) \\ \overline{c}(a) - \varphi(a) & (a \in A_\varphi^*) \\ \varphi(\overline{a}) - \underline{c}(\overline{a}) & (a \in B_\varphi^*,\ \overline{a}(\in A):\ \text{a reorientation of } a), \end{cases} \tag{5.106}$$

where $\hat{c}^+$ (or $\hat{c}^-$) denotes the saturation capacity associated with $(\mathcal{D}^+, f^+)$ (or $(\mathcal{D}^-, f^-)$) and $\tilde{c}^+$ (or $\tilde{c}^-$) the exchange capacity associated with $(\mathcal{D}^+, f^+)$ (or $(\mathcal{D}^-, f^-)$).

We suppose that a feasible flow (an independent flow) φ is given. An algorithm for finding a maximum independent flow is furnished as follows (cf. [Fuji78a], [Lawler + Martel82a], [Schönsleben80]). We assume oracles for saturation capacities and exchange capacities.

An algorithm for finding a maximum independent flow

Input: an independent flow φ in network $\mathcal{N} = (G = (V, A; S^+, S^-), \underline{c}, \overline{c}, (\mathcal{D}^+, f^+), (\mathcal{D}^-, f^-))$ and a fixed numbering π: $V \to \{1, 2, \cdots, |V|\}$.

Output: a maximum independent flow φ in $\mathcal{N}$.

Step 1: While there exists a directed path from s^+ to s^- in the auxiliary network $\mathcal{N}_\varphi = (G_\varphi = (V \cup \{s^+, s^-\}, A_\varphi), c_\varphi, s^+, s^-)$, do the following.

(∗) Find a lexicographically shortest path P from s^+ to s^- in $\mathcal{N}_\varphi$ and put

$$\alpha \leftarrow \min\{c_\varphi(a) \mid a:\ \text{an arc lying on } P\}, \tag{5.107}$$

$$\varphi(a) \leftarrow \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(P)) \\ \varphi(a) - \alpha & (\overline{a} \in B_\varphi^* \cap A_\varphi(P), \end{cases} \tag{5.108}$$

$\overline{a}$: a reorientation of $a(\in A)$).

(End)

In this algorithm a lexicographically shortest path from s^+ to s^- in $\mathcal{N}_\varphi$ is a directed path from s^+ to s^- in $\mathcal{N}_\varphi$ which has the minimum number of arcs among directed paths from s^+ to s^- in $\mathcal{N}_\varphi$ and whose vertex sequence $(s^+, v_1, \cdots, v_p, s^-)$, say, gives the lexicographically minimum sequence $(\pi(v_1), \cdots, \pi(v_p))$ among directed paths from s^+ to s^- in $\mathcal{N}_\varphi$ having the minimum number of arcs. Also, $A_\varphi(P)$ is the set of arcs, in A_φ, lying on P.

The validity of the above algorithm can be shown almost in the same manner as in the proof of the validity of the algorithm for the intersection problem in Sections 4.1.b and 4.1.c. The algorithm described above finds a maximum independent flow after repeating $(*)$ at most $|V|^3$ times.

The maximum independent flow problem naturally includes the intersection problem, where the underlying graph is the bipartite graph representing the bijection between S^+ and S^-.

Theorem 5.9 (The maximum-independent-flow minimum-cut theorem) (cf. [McDiarmid75], [Fuji78a]): *Suppose that the maximum independent flow problem P_{MI} has a feasible flow. Then we have*

$$\begin{aligned}&\max\{\partial\varphi(S^+) \mid \varphi \text{ is an independent flow (satisfying (5.98b)}\sim\text{(5.98e))}\}\\&= \min\{f^+(S^+ - U) + \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U) \mid U \subseteq V\},\end{aligned} \tag{5.109}$$

where Δ^+ and Δ^- are with respect to G and we regard $f^+(X) = +\infty$ ($f^-(Y) = +\infty$) if $X \notin \mathcal{D}^+$ ($Y \notin \mathcal{D}^-$).

Moreover, if $\overline{c}$, $\underline{c}$, f^+ and f^- are integer-valued and Problem P_{MI} is feasible, then there exists an integral maximum independent flow for Problem P_{MI}.

(Proof) Let φ^* be a maximum independent flow for Problem P_{MI}, and consider the auxiliary network $\mathcal{N}_{\varphi^*}$ associated with φ^*. Let U^* be the set of vertices in V which are reachable by directed paths from s^+ in the auxiliary network $\mathcal{N}_{\varphi^*}$. Then we have

$$S^+ - U^* \subseteq \mathrm{sat}^+(\partial^+\varphi^*), \tag{5.110}$$

$$S^- \cap U^* \subseteq \mathrm{sat}^-(\partial^-\varphi^*) \tag{5.111}$$

since there is no directed path from s^+ to s^- in $\mathcal{N}_{\varphi^*}$. Hence, by the definition of U^*,

$$\mathrm{dep}^+(\partial^+\varphi^*, v) \subseteq S^+ - U^* \quad (v \in S^+ - U^*), \tag{5.112}$$

$$\mathrm{dep}^-(\partial^-\varphi^*, v) \subseteq S^- \cap U^* \quad (v \in S^- \cap U^*), \tag{5.113}$$

$$\varphi^*(a) = \overline{c}(a) \quad (a \in \Delta^+ U^*), \tag{5.114}$$

$$\varphi^*(a) = \underline{c}(a) \quad (a \in \Delta^- U^*). \tag{5.115}$$

From (5.112) and (5.113),

$$\partial^+\varphi^*(S^+ - U^*) = f^+(S^+ - U^*), \tag{5.116}$$

$$\partial^-\varphi^*(S^- \cap U^*) = f^-(S^- \cap U^*) \tag{5.117}$$

due to Lemma 2.1. From (5.114) and (5.115),

$$\varphi^*(\Delta^+ U^*) = \overline{c}(\Delta^+ U^*), \quad \varphi^*(\Delta^- U^*) = \underline{c}(\Delta^- U^*). \tag{5.118}$$

It follows from (5.116)$\sim$(5.118) that

$$\begin{aligned} \partial\varphi^*(S^+) &= \partial^+\varphi^*(S^+ - U^*) + \varphi^*(\Delta^+ U^*) - \varphi^*(\Delta^- U^*) + \partial^-\varphi^*(S^- \cap U^*) \\ &= f^+(S^+ - U^*) + \overline{c}(\Delta^+ U^*) - \underline{c}(\Delta^- U^*) + f^-(S^- \cap U^*). \end{aligned} \tag{5.119}$$

On the other hand, for any independent flow φ and any subset U of V we have

$$\begin{aligned} \partial\varphi(S^+) &= \partial^+\varphi(S^+ - U) + \varphi(\Delta^+ U) - \varphi(\Delta^- U) + \partial^-\varphi(S^- \cap U) \\ &\le f^+(S^+ - U) + \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U). \end{aligned} \tag{5.120}$$

(5.109) follows from (5.119) and (5.120).

Moreover, if $\overline{c}$, $\underline{c}$, f^+ and f^- are integer-valued and Problem P_{MI} is feasible, then there exists an integral feasible flow due to Theorem 5.1. Starting with an integral feasible flow, the algorithm finds an integral maximum independent flow in finitely many steps. Q.E.D.

We call any $U \subseteq V$ a *cut* and the value $f^+(S^+ - U) + \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U)$ the *capacity* of the cut U. Theorem 5.9 is a generalization of the classical max-flow min-cut theorem for ordinary capacitated networks [Ford + Fulkerson62].

Theorem 5.9 (for polymatroids) was shown non-algorithmically by McDiarmid [McDiarmid75] and algorithmically by the author [Fuji78a].

Now, consider a supermodular system $(\mathcal{D}^+, g^+)$ on S^+ instead of submodular system $(\mathcal{D}^+, f^+)$ and also consider the following system of inequalities.

$$\underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \tag{5.121}$$

$$\partial\varphi(v) = 0 \quad (v \in V - (S^+ \cap S^-)), \tag{5.122}$$

$$(\partial\varphi)^{S^+} \in \mathrm{P}(g^+), \tag{5.123}$$

$$-(\partial\varphi)^{S^-} \in \mathrm{P}(f^-), \tag{5.124}$$

where $\mathrm{P}(g^+)$ is the supermodular polyhedron associated with $(\mathcal{D}^+, g^+)$. Then we have the following theorem.

Theorem 5.10: *There exists a feasible flow φ satisfying* (5.121) – (5.124) *if and only if we have for each $U \subseteq V$ such that $S^+ \cap U \in \mathcal{D}^+$ and $S^- \cap U \in \mathcal{D}^-$*

$$g^+(S^+ \cap U) - f^-(S^- \cap U) \le \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) \tag{5.125}$$

and for each $U \subseteq V$ such that $S^+ \cup S^- \subseteq U$

$$0 \le \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U). \tag{5.126}$$

Moreover, if there exists a feasible flow and $\overline{c}$, $\underline{c}$, g^+ and f^- are integer-valued, then there exists an integral feasible flow.

(Proof) Define $\mathcal{D} \subseteq 2^V$ and g: $\mathcal{D} \to \mathbf{R}$ by

$$\mathcal{D} = \{U \mid U \subseteq V,\ S^+ \cap U \in \mathcal{D}^+,\ S^- \cap U \in \mathcal{D}^-\}, \tag{5.127}$$

$$g(U) = \begin{cases} g^+(S^+ \cap U) - f^-(S^- \cap U) & (U \in \mathcal{D},\ (S^+ \cup S^-) - U \ne \emptyset) \\ 0 & (U \in \mathcal{D},\ S^+ \cup S^- \subseteq U). \end{cases} \tag{5.128}$$

If there is a feasible flow, we must have

$$g^+(S^+) - f^-(S^-) \le 0. \tag{5.129}$$

Also, (5.125) with $U = V$ implies (5.129). Therefore, we assume (5.129). Due to (5.129), the function g: $\mathcal{D} \to \mathbf{R}$ defined by (5.128) is a supermodular function on the distributive lattice $\mathcal{D}$ with $\emptyset$, $V \in \mathcal{D}$ and $g(\emptyset) = g(V) = 0$. We have $x \in \mathrm{B}(g)$ if and only if

$$x^{S^+} \in \mathrm{P}(g^+), \ \ -x^{S^-} \in \mathrm{P}(f^-), \ \ x^{V-(S^+ \cup S^-)} = \mathbf{0}, \tag{5.130}$$

$$x(V) = 0. \tag{5.131}$$

It follows from (2.65) and Theorem 4.13 that there exists a feasible flow satisfying (5.121)∼(5.124) if and only if

$$g(U) \le \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) \quad (U \in \mathcal{D}). \tag{5.132}$$

We see that (5.132) is equivalent to (5.125) and (5.126), under condition (5.129).

The integrality part of the present theorem follows from the counterpart of Theorem 4.13. Q.E.D.

Theorem 5.10, where $\underline{c} = \mathbf{0}$, $\overline{c} \ge \mathbf{0}$, f^- is a polymatroid rank function and g^+ is the dual supermodular function of a polymatroid rank function, is shown in [Fuji78d].

The discrete separation theorem also follows from Theorem 5.10. The readers may deduce Theorem 5.10 from the feasibility theorem for submodular flows (Theorem 5.1).

(b) Maximum submodular flows

For the submodular flow problem with the network $\mathcal{N}_S = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{D}, f))$, disregarding the cost function γ, let us consider a "max-flow" problem P_{MS} given as follows [Cunningham + Frank85]. Let a_0 be a fixed reference arc in A.

$$P_{MS}\text{: Maximize } \ \varphi(a_0) \tag{5.133a}$$

$$\text{subject to } \ \underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \tag{5.133b}$$

$$\partial\varphi \in \mathrm{B}(f). \tag{5.133c}$$

We assume that there is a feasible flow in $\mathcal{N}_S$ and define the auxiliary network $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$ associated with φ as follows. G_φ is the graph with vertex set V and arc set A_φ defined by (5.42)∼(5.45) except that

$$B^*_\varphi = \{\overline{a} \mid a \in A - \{a_0\},\ \underline{c}(a) < \varphi(a)\} \quad (\overline{a}\text{: a reorientation of } a) \tag{5.134}$$

and c_φ: $A_\varphi \to \mathbf{R}$ is defined by (5.46).

An algorithm for finding a maximum submodular flow

Input: a feasible flow φ in $\mathcal{N}_S$ with reference arc a_0 and a vertex numbering π: $V \rightarrow \{1, 2, \cdots, |V|\}$ which defines the lexicographic ordering among directed paths from $\partial^- a_0$ to $\partial^+ a_0$ in $\mathcal{N}_\varphi$.

Output: a maximum submodular flow φ in $\mathcal{N}_S$ with reference arc a_0.

Step 1: While $\varphi(a_0) < \overline{c}(a_0)$ and there exists a directed path from $\partial^- a_0$ to $\partial^+ a_0$ in the auxiliary network $\mathcal{N}_\varphi$, do the following.

(∗) Find the lexicographically shortest path P from $\partial^- a_0$ to $\partial^+ a_0$ in $\mathcal{N}_\varphi$ and let Q be the directed cycle formed by P and reference arc a_0. Put

$$\alpha \leftarrow \min\{c_\varphi(a) \mid a \text{ lies on } Q\},$$

$$\varphi(a) \leftarrow \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(Q)) \\ \varphi(a) - \alpha & (\overline{a} \in B_\varphi^* \cap A_\varphi(Q), \end{cases}$$

$$\overline{a}\text{: a reorientation of } a \in A)$$

where if $\alpha = +\infty$, then stop (the flow value $\varphi(a_0)$ can be made arbitrarily large).

(End)

Here, $A_\varphi(Q)$ is the set of the arcs in A_φ lying on Q and the lexicographically shortest path P is the directed path from $\partial^- a_0$ to $\partial^+ a_0$ in $\mathcal{N}_\varphi$ which has the minimum number of arcs among directed paths from $\partial^- a_0$ to $\partial^+ a_0$ in $\mathcal{N}_\varphi$ and whose vertex sequence $(\partial^- a_0, v_1, \cdots, v_p, \partial^+ a_0)$, say, gives the lexicographically minimum sequence $(\pi(v_1), \cdots, \pi(v_p))$ among directed paths from $\partial^- a_0$ to $\partial^+ a_0$ in $\mathcal{N}_\varphi$ having the minimum number of arcs.

The algorithm terminates after repeating (∗) $\mathrm{O}(|V|^3)$ times. The analysis is by the same technique as shown in Section 4.1.c.

Theorem 5.11: *For the maximum submodular flow problem P_{MS} described by* (5.133),

$$\begin{aligned} &\max\{\varphi(a_0) \mid \varphi \text{ is a feasible flow in } \mathcal{N}_S\} \\ &= \min\{\overline{c}(a_0),\ \min\{\overline{c}(\Delta^- X) - \underline{c}(\Delta^+ X - \{a_0\}) \\ &\qquad + f(X) \mid X \in \mathcal{D},\ a_0 \in \Delta^+ X\}\}. \end{aligned} \tag{5.135}$$

Moreover, if $\underline{c}$, $\overline{c}$ and f are integer-valued and there exists a maximum submodular flow, then there exists an integral maximum submodular flow in $\mathcal{N}_S$.

(Proof) The maximum flow value is equal to the maximum value of $\underline{c}(a_0)$ under the constraint that the network $\mathcal{N}_S$ has a feasible flow, where $\underline{c}(a_0)$ is regarded as a variable. Therefore, relation (5.135) is deduced from Theorem 5.1. The integrality property also follows from Theorem 5.1. Q.E.D.

Theorem 5.11 can also be shown algorithmically. When the algorithm terminates with a maximum flow φ^* such that $\varphi^*(a_0) < \overline{c}(a_0)$, let U be the set of the vertices which are reachable by directed paths from $\partial^- a_0$ in the then obtained $\mathcal{N}_{\varphi^*}$. From the assumption, $\partial^- a_0 \in U$ and $\partial^+ a_0 \notin U$. Define $W = V - U$. It follows from the definition of $\mathcal{N}_{\varphi^*}$ that

$$\varphi^*(a) = \overline{c}(a) \quad (a \in \Delta^- W), \tag{5.136}$$

$$\varphi^*(a) = \underline{c}(a) \quad (a \in \Delta^+ W - \{a_0\}), \tag{5.137}$$

$$\mathrm{dep}(\partial\varphi^*, v) \subseteq W \quad (v \in W), \tag{5.138}$$

where Δ^+ and Δ^- are with respect to G. From (5.138) we have

$$\partial\varphi^*(W) = f(W). \tag{5.139}$$

and from (5.136) and (5.137)

$$\partial\varphi^*(W) = \varphi^*(a_0) + \underline{c}(\Delta^+ W - \{a_0\}) - \overline{c}(\Delta^- W). \tag{5.140}$$

Combining (5.139) with (5.140), we get

$$\varphi^*(a_0) = \overline{c}(\Delta^- W) - \underline{c}(\Delta^+ W - \{a_0\}) + f(W). \tag{5.141}$$

On the other hand, we can easily see that for any feasible flow φ in $\mathcal{N}_S$ and any $X \in \mathcal{D}$ with $a_0 \in \Delta^+ X$,

$$\varphi(a_0) \leq \overline{c}(\Delta^- X) - \underline{c}(\Delta^+ X - \{a_0\}) + f(X). \tag{5.142}$$

From (5.141) and (5.142) we have (5.135), where the case when $\varphi^*(a_0) = \overline{c}(a_0)$ is taken into account.

Moreover, the integrality part of Theorem 5.11 follows from the fact that if Problem P_{MS} is feasible and $\underline{c}$, $\overline{c}$ and f are integer-valued, there exists an integral feasible flow and, starting from such an integral feasible flow, we get an integral maximum submodular flow by the algorithm if a maximum submodular flow exists.

When $\varphi^*(a_0) < \overline{c}(a_0)$, the above defined $U(= V - W)$ is called a *minimum cut* in $\mathcal{N}_S$ with reference arc a_0.

We can also consider the *minimum submodular flow problem* which is to minimize $\varphi(a_0)$ subject to (5.133b) and (5.133c). Note that this problem is a maximum submodular flow problem when we consider the dual order $\leq^*$ among $\mathbf{R}$. Hence an algorithm for the minimum submodular flow problem is given *mutatis mutandis*.

(c) Minimum-cost submodular flows

As a minimum-cost flow problem for neoflows, consider the submodular flow problem P_S, described by (5.1), in network $\mathcal{N}_S = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{D}, f))$.

$$P_S\colon \quad \text{Minimize} \sum_{a \in A} \gamma(a)\varphi(a) \tag{5.1a}$$

$$\text{subject to} \quad \underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \quad (a \in A), \tag{5.1b}$$

$$\partial\varphi \in \mathrm{B}(f), \tag{5.1c}$$

where $(\mathcal{D}, f)$ is a submodular system on V, the vertex set of the underlying graph $G = (V, A)$. We suppose that Problem P_S has a feasible flow.

We shall show an algorithm which tries to find a feasible flow $\varphi\colon A \to \mathbf{R}$ and a potential $p\colon V \to \mathbf{R}$ satisfying the optimality condition of Theorem 5.3.

Suppose that we are given a feasible flow φ in $\mathcal{N}_S$. Choose a potential $p\colon V \to \mathbf{R}$ such that $\partial\varphi \in \mathrm{B}(f)$ is a maximum-weight base of $\mathrm{B}(f)$ with respect to the weight function p, i.e., $p(u) \geq p(v)$ for each $u,\ v \in V$ with $u \in \mathrm{dep}(\partial\varphi, v) - \{v\}$. For example, $p = \mathbf{0}$ (the zero function) satisfies this requirement. We define the auxiliary network $\mathcal{N}_{\varphi,p} = (G_\varphi = (V, A_\varphi), c_\varphi, \gamma_{\varphi,p})$ as follows. G_φ is the graph with vertex set V and arc set A_φ defined by (5.42)$\sim$(5.45) and $c_\varphi\colon A_\varphi \to \mathbf{R}$ is defined by (5.46). We define $\gamma_{\varphi,p}\colon A_\varphi \to \mathbf{R}$ by

$$\gamma_{\varphi,p}(a) = \gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A_\varphi), \tag{5.143}$$

where γ_φ is given by (5.47).

Now, an algorithm based on Cunningham and Frank's [Cunningham+Frank85] is given as follows. Also, compare it with an out-of-kilter method given in [Fuji87b].

An algorithm for finding an optimal submodular flow

Input: a feasible flow φ in $\mathcal{N}_S$ and a potential $p: V \to \mathbf{R}$ such that $p(u) \geq p(v)$ $(v \in V,\ u \in \mathrm{dep}(\partial\varphi, v) - \{v\})$.

Output: an optimal submodular flow φ and an optimal potential p.

Step 1: While $\gamma_{\varphi,p}(a) < 0$ for some $a \in A_\varphi$, choose an arc $a^0 \in A_\varphi$ such that $\gamma_{\varphi,p}(a^0) < 0$ and do (1-1) or (1-2) according as $a^0 \in A$ or $a^0 \notin A$:

(1-1) If $a^0 \in A$, then, while $\varphi(a^0) < \overline{c}(a^0)$ and $\gamma_p(a^0)(= \gamma(a^0) + p(\partial^+ a^0) - p(\partial^- a^0)) < 0$, do the following:

(∗) Starting with φ, find a maximum submodular flow φ^0 in the modified network $\mathcal{N}^0 = (G = (V, A), \underline{c}^0, \overline{c}^0, (\mathcal{D}^0, f^0))$ with the reference arc a^0, where the lower and upper capacity functions $\underline{c}^0$, $\overline{c}^0: A \to \mathbf{R}$ are defined by $\underline{c}^0(a^0) = \varphi(a^0)$, $\overline{c}^0(a^0) = \overline{c}(a^0)$ and for each $a \in A - \{a^0\}$

$$\underline{c}^0(a) = \begin{cases} \varphi(a) & (a \in A,\ \gamma_p(a) \neq 0) \\ \underline{c}(a) & (a \in A,\ \gamma_p(a) = 0), \end{cases} \tag{5.144}$$

$$\overline{c}^0(a) = \begin{cases} \varphi(a) & (a \in A,\ \gamma_p(a) \neq 0) \\ \overline{c}(a) & (a \in A,\ \gamma_p(a) = 0) \end{cases} \tag{5.145}$$

(where $\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a)$ with ∂^+ and ∂^- being defined with respect to G) and, letting $p_1 > p_2 > \cdots > p_t$ be the distinct values of $p(v)$ $(v \in V)$ and defining

$$S_i = \{v \mid v \in V,\ p(v) \geq p_i\} \quad (i = 1, 2, \cdots, t), \tag{5.146}$$

$$S_0 = \emptyset, \tag{5.147}$$

$(\mathcal{D}^0, f^0)$ is the submodular system given by the direct sum

$$\bigoplus_{i=1}^{t} (\mathcal{D}, f) \cdot S_i / S_{i-1} \tag{5.148}$$

of the set minors $(\mathcal{D}, f) \cdot S_i / S_{i-1}$ $(i = 1, 2, \cdots, t)$.

If the maximum flow value is equal to $+\infty$, then stop (Problem P_S does not have a finite optimal solution). Otherwise put $\varphi \leftarrow \varphi^0$.

If $\varphi^0(a^0) < \overline{c}(a^0)$, then for the auxiliary network $\mathcal{N}^0_{\varphi^0}$ associated with the current submodular flow φ^0 in $\mathcal{N}^0$ let U be the set of the vertices which are reachable by directed paths from $\partial^- a^0$ in $\mathcal{N}^0_{\varphi^0}$, define

$$\begin{aligned}
H_1 &= \{a \mid a \in \Delta^- U,\ \gamma_p(a) < 0\}, &(5.149)\\
H_2 &= \{a \mid a \in \Delta^+ U,\ \gamma_p(a) > 0\}, &(5.150)\\
&\quad (\Delta^+,\ \Delta^- \text{ in (5.149) and (5.150) are with respect to } G.)\\
H_3 &= \{(u,v) \mid u \in U,\ v \in V - U,\ u \in \mathrm{dep}(\partial\varphi, v) - \{v\}\}, &(5.151)\\
&\quad (\mathrm{dep} \text{ is with respect to } (\mathcal{D}, f).)\\
p^* &= \min\{\min\{|\gamma_p(a)| \mid a \in H_1 \cup H_2\},\\
&\qquad \min\{p(u) - p(v) \mid (u,v) \in H_3\}\}, &(5.152)
\end{aligned}$$

and put

$$p(u) \leftarrow p(u) - p^* \quad (u \in U). \tag{5.153}$$

(1-2) If $a^0 \notin A$, let $\overline{a}^0 \in A$ be the reorientation of a^0 and, starting with φ, find a minimum submodular flow φ^0, with reference arc $\overline{a}^0$, in the modified network $\mathcal{N}^0$ as defined in Step (1-1). Carry out Step (1-1) *mutatis mutandis.*

(End)

It should be noted that when we carry out (5.149)$\sim$(5.153), we have $H_1 \cup H_2 \cup H_3 \neq \emptyset$ since $a^0 \in H_1$, and that p^* in (5.152) is a finite positive number.

For a feasible flow φ and a potential p, if an arc $a \in A$ satisfies (i) $\gamma_p(a) > 0$ and $\underline{c}(a) < \varphi(a)$ or (ii) $\gamma_p(a) < 0$ and $\varphi(a) < \overline{c}(a)$, then arc a is said to be *out of kilter* and otherwise *in kilter* with respect to φ and p. Denote the set of all the out-of-kilter arcs by $A^O(\varphi, p)$ and that of all the in-kilter arcs by $A^I(\varphi, p)$.

During the execution of the algorithm,

(1) in-kilter arcs remain in kilter,

(2) the value of $|\gamma_p(a)|$ of each out-of-kilter arc a is monotone non-increasing,

(3) $|\gamma_p(a^0)|$ decreases every time the potential p is changed by (5.153) in Step (1-1) or (1-2), and

(4) φ is a submodular flow in $\mathcal{N}_S$ and $\partial\varphi$ is a maximum-weight base of $\mathrm{B}(f)$ with respect to weight function p.

Therefore, when the cost function γ is integer-valued, the algorithm terminates after at most $\sum_{a\in A}|\gamma(a)|$ maximum and minimum submodular-flow computations in Step (1-1) and Step (1-2) if we start with the initial potential $p = \mathbf{0}$. Moreover, for any real cost function γ the algorithm also terminates after finitely many steps, as shown below.

While the inner cycle $(*)$ of Step (1-1) is repeated, potential $p(\partial^+ a^0)$ remains fixed and $p(\partial^- a^0)$ is decreased by (5.153), where note that for each $(u,v) \in H_3$ we have $p(u) > p(v)$. Every time we change the potential p by (5.153), one of the following two cases occurs:

(i) at least one arc in $H_1 \cup H_2$ becomes in kilter,

(ii) the set of the vertices which are reachable from $\partial^- a^0$ in the new auxiliary network $\mathcal{N}^0_{\varphi^0}$ is augmented.

Note that the total number of the occurrences of Case (i) is at most $|A|$ and that Case (ii) occurs consecutively at most $|V|-1$ times without increasing $\varphi^0(a^0)$. While $\varphi(a^0) < \overline{c}(a^0)$ (when $\gamma_p(a^0) < 0$), the potential difference $p(\partial^- a^0) - p(\partial^+ a^0)$ is decreased by (5.153) between the successive maximum submodular flow computations. Since the potential difference $p(\partial^- a^0) - p(\partial^+ a^0)$ is equal to the length of a directed path from $\partial^- a^0$ to $\partial^+ a^0$ relative to the length function γ_φ (since the length of such a path relative to $\gamma_{\varphi,p}$ is equal to zero), and since the number of distinct lengths, relative to length function γ_φ, of elementary directed paths from $\partial^- a^0$ to $\partial^+ a^0$ in any possible auxiliary networks is finite, $\varphi(a^0)$ is increased only finitely many times. This argument can be applied to Step (1-2) *mutatis mutandis*. It follows that arc a^0 becomes in kilter or we discern that Problem P_S does not have a finite optimal solution, after finitely many steps.

If the algorithm terminates with φ and p such that $\gamma_{\varphi,p}(a) \geq 0$ for all $a \in A_\varphi$, then the obtained φ is an optimal submodular flow and p is an optimal potential due to Theorem 5.3.

If the maximum flow value for the modified network $\mathcal{N}^0$ in Step (1-1) is equal to $+\infty$, then there exists a directed cycle Q, containing a^0, in the auxiliary network $\mathcal{N}^0_\varphi = (G^0_\varphi = (V, A_\varphi), c^0_\varphi)$ associated with the current φ such that $c_\varphi(a) = +\infty$ for any arc a in Q. By the definition of $\mathcal{N}^0$, Q is also a directed cycle in the auxiliary network $\mathcal{N}_\varphi = (G_\varphi, c_\varphi, \gamma_\varphi)$ for the original submodular flow problem P_S and the length of Q relative to the length function γ_φ is negative. Moreover, φ remains to be a (feasible) submodular flow if we change φ along the directed cycle Q by an arbitrary positive amount of flow value. (Here, the condition in Lemma 4.5 for multiple exchanges is not required since $c_\varphi(a) = +\infty$ for any arc $a \in C_\varphi$ in Q.) Consequently, Problem P_S does not have a finite optimal solution. Also, the same argument is valid *mutatis mutandis* for Step (1-2).

Given an optimal submodular flow φ and an optimal potential p, we can find an optimal dual solution of P_S by the procedure given in Section 5.4 when the base polyhedron is expressed in terms of a submodular function on a distributive lattice, or an intersecting- or crossing-submodular function on an intersecting or crossing family.

When the cost function γ is integer-valued and an oracle for exchange capacities is assumed, the above algorithm requires time polynomial in $|V|$, $|A|$ and $\max\{|\gamma(a)| \mid a \in A\}$, i.e., it is a pseudopolynomial algorithm. Applying the cost-scaling technique for the ordinary min-cost flows of Röck [Röck80] to submodular flows, we obtain a polynomial algorithm for the submodular flow problem ([Cunningham + Frank85]), provided that an oracle for exchange capacities is available.

Without loss of generality we assume $\underline{c}(a) > -\infty$ for each $a \in A$.

A cost-scaling algorithm

Input: a feasible flow φ in $\mathcal{N}_S$, a potential $p = \mathbf{0}$, and $C \equiv \max\{|\gamma(a)| \mid a \in A\}$, where the cost function γ is integer-valued and $C > 0$.

Output: an optimal submodular flow φ and an optimal potential p.

Step 0: Discern whether Problem P_S does not have a finite optimal solution. If not, stop; otherwise put $k \leftarrow \lceil \log_2 C \rceil$.

Step 1: While $k \geq 0$, do the following:

(1-1) For each $a \in A$ put $\tilde{\gamma}(a) \leftarrow \lceil \gamma(a)/2^k \rceil$.

(1-2) By the pseudopolynomial algorithm given above, starting from the current flow φ and potential p, find an optimal submodular flow φ^* and an optimal potential p^* in the network $\tilde{\mathcal{N}}_S = (G = (V, A), \underline{c}, \overline{c}, \tilde{\gamma}, (\mathcal{D}, f))$.

(1-3) Put $\varphi \leftarrow \varphi^*$, $p \leftarrow 2p^*$, and $k \leftarrow k - 1$.

(End)

Step 0 can be carried out by a shortest path computation based on Theorem 5.5.

In Step (1-2) we have

$$\sum_{a \in A^O(\varphi,p)} |\tilde{\gamma}_p(a)| \leq 2|A| \tag{5.154}$$

for the current $\tilde{\gamma}$ and p. Hence each Step (1-2) requires at most $2|A|$ maximum and minimum submodular-flow computations. It follows that the cost-scaling algorithm requires $\mathrm{O}(|A| \log_2 C)$ maximum and minimum submodular flow computations and that the algorithm runs in polynomial time if an oracle for exchange capacities is available.

It should also be noted that, since in Step 1 we have

$$\tilde{\gamma}(a) \times 2^k \geq \gamma(a) \qquad (a \in A), \tag{5.155}$$

and since Problem P_S is bounded when we proceed to Step 1, the problem on network $\tilde{\mathcal{N}}_S = (G = (V, A), \underline{c}, \overline{c}, \tilde{\gamma}, (\mathcal{D}, f))$ with approximated cost function $\tilde{\gamma}$ is also bounded due to the assumption that $\underline{c}(a) > -\infty$ $(a \in A)$ (recall Theorem 5.5).

If we apply the cost-rounding and tree-projection technique of the author [Fuji86] (which is an improved version of Tardos's algorithm [Tardos 85]) for the ordinary minimum-cost flows, we can get a strongly polynomial algorithm for the submodular flow problem ([Fuji + Röck + Zimmermann89]). The first strongly polynomial algorithm for the submodular flow problem was given by Frank and Tardos [Frank + Tardos85] by the use of the simultaneous approximation algorithm of [Lenstra + Lenstra + Lovász82].

To give a strongly polynomial algorithm of [Fuji + Röck + Zimmermann89] we need a few preliminaries. For simplicity we assume that $\underline{c}$ and $\overline{c}$ take on only finite values, which guarantees the boundedness of Problem P_S.

For a cost function $\gamma\colon A \to \mathbf{R}$ and a positive real α, $\gamma'\colon A \to \mathbf{R}$ is called an *α-approximation* of γ if $|\gamma'(a) - \gamma(a)| \leq \alpha$ for all $a \in A$.

The following lemma is fundamental.

Lemma 5.12: *Let γ' be an α-approximation of γ and p' be an optimal potential in $\mathcal{N}' = (G = (V, A), \underline{c}, \overline{c}, \gamma', (\mathcal{D}, f))$. Then there exists an optimal potential p in $\mathcal{N} = (G, \underline{c}, \overline{c}, \gamma, (\mathcal{D}, f))$ such that we have for each $u, v \in V$*

$$|(p'(u) - p'(v)) - (p(u) - p(v))| \leq \alpha |V|. \tag{5.156}$$

(Proof) Without loss of generality we assume $\gamma'(a) \geq \gamma(a)$ for all $a \in A$ (by possible reorientations of arcs). If $\gamma' \neq \gamma$, let v^0 be a vertex in V such that $\gamma'(a) > \gamma(a)$ for an arc $a \in \delta^- v^0$. Define

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & (a \in \delta^- v^0) \\ \gamma'(a) & (a \in A - \delta^- v^0). \end{cases} \tag{5.157}$$

Starting from an optimal submodular flow φ' and an optimal potential p' in $\mathcal{N}' = (G, \underline{c}, \overline{c}, \gamma', (\mathcal{D}, f))$, we apply a slightly modified version of the (non-scaling) algorithm for submodular flows to $\hat{\mathcal{N}} = (G, \underline{c}, \overline{c}, \hat{\gamma}, (\mathcal{D}, f))$. The modification consists only in postponing potential changings as long as possible, which does not affect the finiteness of the algorithm.

Note that initially for any out-of-kilter arc $a \in A$ we have

$$a \in \delta^- v^0, \tag{5.158}$$

$$0 > \hat{\gamma}_{p'}(a) \geq -\alpha, \tag{5.159}$$

$$\varphi'(a) < \overline{c}(a). \tag{5.160}$$

In the original version of the algorithm an out-of-kilter arc $a^0 \in \delta^- v^0$ is chosen and $\varphi'(a^0)$ is increased by solving a maximum submodular flow problem with reference arc a^0, which is followed by a potential changing. In the modified version we repeatedly solve maximum submodular flow problems, each with an out-of-kilter arc $a \in \delta^- v^0$ as a reference arc, without any potential changings. If there exists at least one out-of-kilter arc after this process, then let U be the set of the vertices reachable from v^0 in the current auxiliary network and change the potential p by (5.149)$\sim$(5.153). Repeat this process until all the out-of-kilter arcs become in kilter. (This is repeated finitely many times.) Let p be the resultant potential and let a^0 be one of the arcs which became in kilter at the last potential-changing. We can see from (5.159) that

$$\begin{aligned} &\max\{|(p'(u) - p'(v)) - (p(u) - p(v))| \mid u, v \in V\} \\ &\leq \max\{|p'(u) - p(u)| \mid u \in V\} \\ &\leq |\hat{\gamma}_{p'}(a^0)| \\ &\leq \alpha, \end{aligned} \tag{5.161}$$

where note that $0 \leq p'(u) - p(u)$ for all $u \in V$ due to (5.153).

Putting $p' \leftarrow p$ and $\gamma' \leftarrow \hat{\gamma}$ and repeating the above argument for other vertices, we see that the total change of any potential difference is bounded by $\alpha|V|$. Q.E.D.

From this lemma we have

Theorem 5.13: *Let γ' be an α-approximation of γ and p' be an optimal potential in $\mathcal{N}' = (G = (V, A), \underline{c}, \overline{c}, \gamma', (\mathcal{D}, f))$. Then for any optimal submodular flow φ in $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{D}, f))$,*

(1) $\forall a \in A$: $\gamma_{p'}(a) > \alpha|V| \Longrightarrow \varphi(a) = \underline{c}(a)$,

(2) $\forall a \in A$: $\gamma_{p'}(a) < -\alpha|V| \Longrightarrow \varphi(a) = \overline{c}(a)$,

(3) $\forall u, v \in V$: $p'(u) - p'(v) > \alpha|V| \Longrightarrow v \notin \mathrm{dep}(\partial\varphi, u)$.

(Proof) The present theorem follows from Lemma 5.12 and Theorem 5.3. Q.E.D.

Theorem 5.13 gives a basis for a strongly polynomial algorithm for submodular flows.

We show a strongly polynomial algorithm which consists of the repeated applications of a procedure called *Fundamental Cycle*.

Fundamental Cycle

Input: Lower and upper capacity functions $\underline{c}$ and $\overline{c}$; a partition $\mathcal{W} = \{\mathcal{W}^{\rangle} \mid \rangle \in \mathcal{I}\}$ of V; a representation of $\mathbf{S} = \oplus_{i \in I} \mathbf{S}^i$ as a direct sum of submodular systems $\mathbf{S}^i = (\mathcal{D}^i, f^i)$ on W^i $(i \in I)$; a graph $H = (V, D)$ with connected components $H^i = (W^i, E^i)$ $(i \in I)$ which are strongly connected; and a set $A^0 = \{a \mid a \in A,\ \underline{c}(a) = \overline{c}(a)\}$ of all tight arcs. (*Comment*: At the initial application of this procedure we put $I = \{1\}$, $W^1 = V$ and $H = (V, D)$ with $D = \{(u, v) \mid u, v \in V,\ u \neq v\}$.)

Output: A nonnegative real M, and if $M \neq 0$, modified $\underline{c}$, $\overline{c}$, $\mathcal{W}$, $\mathbf{S}$, H and A^0. (*Comment*: When $M = 0$, the set of all the submodular flows in the current network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c}, \gamma, \mathbf{S})$ is exactly the set of all the optimal submodular flows in the original network. When $M \neq 0$, $\underline{c}$, $\overline{c}$, $\mathcal{W}$, H and A^0 have been modified in such a way that all the input characteristics are maintained and that at least one of the following two properties holds:

(a) A^0 is strictly larger than the input A^0,

(b) D is strictly smaller than the input D.)

Step 1 [Construction of a potential].

(1-1) Shrink the vertex subsets W^i $(i \in I)$ of $G = (V, A)$ into (pseudo-) vertices w^i $(i \in I)$ and denote the resultant graph by $G//W$. Denote by $\hat{G} = (\hat{V}, \hat{A})$ the graph obtained from $G//W$ by deleting all the arcs of A^0. Find a spanning forest T of $\hat{G}$ which is composed of rooted spanning trees of the connected components of $\hat{G}$.

(1-2) Put $\hat{p}(v) \leftarrow 0$ for all roots v of spanning trees in T and construct the potential $\hat{p}$: $\hat{V} \to \mathbf{R}$ in such a way that $\gamma_{\hat{p}}(a) \equiv \gamma(a) + \hat{p}(\partial^+ a) - \hat{p}(\partial^- a) = 0$ for all the arcs a of T.

(1-3) Define the potential p: $V \to \mathbf{R}$ by

$$p(v) = \hat{p}(w^i) \qquad (v \in W^i,\ i \in I). \tag{5.162}$$

Step 2 [Rounding the cost function].

(2-1) Put

$$M \leftarrow \max\{|\gamma_p(a)| \mid a \in A - A^0\}. \tag{5.163}$$

If $M = 0$, then stop.

(2-2) Put

$$\gamma_p^s(a) \leftarrow \gamma_p(a) \cdot |V|^2/M \qquad (a \in A),$$

$$\gamma'(a) \leftarrow \lceil \gamma_p^s(a) \rfloor \qquad (a \in A).$$

(*Comment*: $\lceil \cdot \rfloor$ means rounding to the nearest integer.)

Step 3 [Solving the approximate problem].
Find an optimal submodular flow φ' and an optimal potential p' in the network $\mathcal{N}' = (G = (V, A), \underline{c}, \overline{c}, \gamma', \mathbf{S})$ by the (cost-scaling) algorithm of Cunningham and Frank.

Step 4 [Extracting information on optimal submodular flows].

(4-1) For each arc $a \in A - A^0$,

(i) if $(\gamma_p^s)_{p'}(a) < -\frac{1}{2}|V|$, then put $\underline{c}(a) \leftarrow \overline{c}(a)$ and $A^0 \leftarrow A^0 \cup \{a\}$,

(ii) if $(\gamma_p^s)_{p'}(a) > \frac{1}{2}|V|$, then put $\overline{c}(a) \leftarrow \underline{c}(a)$ and $A^0 \leftarrow A^0 \cup \{a\}$,

where $(\gamma_p^s)_{p'}(a) = \gamma_p^s(a) + p'(\partial^+ a) - p'(\partial^- a)$.

(4-2) For each arc $(u, v) \in D$ of the graph H, if $p'(v) - p'(u) > \frac{1}{2}|V|$, then delete (u, v) from D and from E_i to which (u, v) belongs.

(4-3) For each $i \in I$ do the following (4-3-1)~(4-3-3):

(4-3-1) Find a maximal chain $\emptyset = U_0^i \subset U_1^i \subset \cdots \subset U_{k_i}^i = W^i$ of upper ideals of H^i. (*Comment*: An *upper ideal* of a graph is a vertex set which no arcs enter.)

(4-3-2) Define

$$\mathbf{S}_{i_s} (= (\mathcal{D}^{i_s}, f^{i_s})) = (\mathcal{D}^i, f^i) \cdot U_s^i / U_{s-1}^i \qquad (s = 1, 2, \cdots, k_i),$$

$$W^{i_s} = U_s^i - U_{s-1}^i \qquad (s = 1, 2, \cdots, k_i).$$

(*Comment*: W^{i_s} $(s = 1, 2, \cdots, k_i)$ are the vertex sets of the strongly connected components of H^i.)

(4-3-3) Delete from the graph H all the arcs connecting distinct subsets W^{i_s} $(s = 1, 2, \cdots, k_i)$.

(4-4) Put

$$\begin{aligned} I &\leftarrow \{i_s \mid s = 1, 2, \cdots, k_i,\ i \in I\}, \\ \mathbf{S} &\leftarrow \oplus_{i \in I} \mathbf{S}_i, \\ \mathcal{W} &\leftarrow \{\mathcal{W}^i \mid i \in I\}. \end{aligned}$$

(End)

To find an optimal submodular flow in the original network we repeatedly apply the procedure, Fundamental Cycle. We show the validity and the strong polynomiality of this algorithm.

At any stage of the algorithm the input to Fundamental Cycle is referred to as the current network with the current capacity functions, the current submodular systems, etc.

Theorem 5.14: *At any stage of the algorithm the following statements are valid.*

(1) *For any optimal submodular flow φ in the current network the exchangeability graph $G_{\partial\varphi}$ associated with base $\partial\varphi$ of the current submodular system is a subgraph of the current graph $H = (V, D)$.*

(2) *The set of all the optimal submodular flows in the original network coincides with the set of all the optimal submodular flows in the current network.*

(3) *While $M \neq 0$, each application of Fundamental Cycle strictly enlarges the set A^0 of the tight arcs or strictly reduces the set D.*

(Proof) We show the present theorem by induction on the number of applications of Fundamental Cycle.

Before the initial application of Fundamental Cycle the current network and the original network coincide and the current graph H contains all arcs (u, v) with $u, v \in V$ and $u \neq v$. Obviously, the statements of the theorem are valid.

Now, let us assume that the statements are valid before an application of Fundamental Cycle. We show their validity after this application.

Suppose $M \neq 0$; otherwise the algorithm terminates without modifying the input and we are done. Let a^* be an arc which maximizes (5.163) in Step (2-1). If we add the arc a^* to the forest T in $\hat{G} = (\hat{V}, \hat{A})$ (see Step (1-1)), then a unique cycle Q^* is created. Let Q^* be represented by a sequence of arcs as $Q^* = (a^0, a^1, \cdots, a^l)$ with $a^0 = a^*$. We assume that a^* is positively oriented in the cycle Q^*. Since the (pseudo-)vertices w^i correspond to the shrunken strongly connected graphs $H^i = (W^i, E^i)$ $(i \in I)$, we can extend Q^* to a cycle C in the graph $(V, A \cup D)$ such that all the arcs from $C \cap D$ are positively oriented. We define

$$\gamma^s(a) = \gamma(a) \cdot |V|^2/M \qquad (a \in A), \tag{5.164}$$

$$\gamma^s_p(a) = \gamma_p(a) \cdot |V|^2/M \qquad (a \in A), \tag{5.165}$$

$$p^s(v) = p(v) \cdot |V|^2/M \qquad (v \in V), \tag{5.166}$$

where we have

$$\gamma^s_p(a) = \gamma^s(a) + p^s(\partial^+ a) - p^s(\partial^- a) \qquad (a \in A). \tag{5.167}$$

We extend the domain of γ^s_p to $A \cup D$ by defining $\gamma^s_p(a) = 0$ $(a \in D)$. (Note that this is equivalent to defining $\gamma(a) = 0$ $(a \in D)$.)

In the following we assume that we have $\gamma_p(a^*) = -M$ in Step (2-1). (The case when $\gamma_p(a^*) = M$ can be treated similarly.) Then the length of the cycle C relative to the length function γ_p^s is given by

$$\gamma_p^s(C) = -|V|^2, \tag{5.168}$$

since $\gamma_p^s(a^*) = -|V|^2$ and $\gamma_p^s(a) = 0$ for any other arcs a in C.

In Step 3 of Fundamental Cycle we construct an optimal submodular flow φ' and an optimal potential p' in $\mathcal{N}' = (G = (V, A), \underline{c}, \overline{c}, \gamma', \mathbf{S})$. Since the length of C is invariant under the modification of the length function γ_p^s by any potential, there exists some arc $\hat{a}$ in C such that

$$\epsilon_C(\hat{a}) \cdot (\gamma_p^s(\hat{a}) + p'(\partial^+\hat{a}) - p'(\partial^-\hat{a})) \leq -|V|, \tag{5.169}$$

where $\epsilon_C(\hat{a})$ is equal to 1 or -1 according as $\hat{a}$ is positively oriented or negatively oriented in C. Since $\epsilon_C(\hat{a}) = 1$ if $\hat{a} \in D$, either (i) $|(\gamma_p^s)_{p'}(\hat{a})| > \frac{1}{2}|V|$ if $\hat{a} \in A - A^0$, or (ii) $p'(v) - p'(u) > \frac{1}{2}|V|$ if $\hat{a} = (u, v) \in D$.

Since γ' is a $\frac{1}{2}$-approximation of γ_p^s (see Step (2-2)), it follows from Theorem 5.13 that for any optimal submodular flow φ in $\mathcal{N}^s = (G, \underline{c}, \overline{c}, \gamma_p^s, \mathbf{S})$

(1) if $\hat{a} \in A - A^0$, then

$$\varphi(\hat{a}) = \begin{cases} \underline{c}(\hat{a}) & \text{if } (\gamma_p^s)_{p'}(\hat{a}) > \frac{1}{2}|V| \\ \overline{c}(\hat{a}) & \text{if } (\gamma_p^s)_{p'}(\hat{a}) < -\frac{1}{2}|V|, \end{cases} \tag{5.170}$$

(2) if $\hat{a} = (u, v) \in D$, then $p'(v) - p'(u) > \frac{1}{2}|V|$ and

$$u \notin \mathrm{dep}(\partial\varphi, v), \tag{5.171}$$

where the dependence function is defined with respect to the current base polyhedron.

Therefore, in Steps (4-1) and (4-2) A^0 is strictly enlarged or D is strictly reduced, which shows statement (3) in the present theorem.

Moreover, we have

$$\begin{aligned} (|V|^2/M) \cdot \sum_{a \in A} \gamma(a)\varphi(a) &= \sum_{a \in A} \gamma^s(a)\varphi(a) \\ &= \sum_{a \in A} \gamma_p^s(a)\varphi(a) - \sum_{v \in V} p(v)\partial\varphi(v) \\ &= \sum_{a \in A} \gamma_p^s(a)\varphi(a) - \sum_{i \in I} \hat{p}(w^i)\partial\varphi(W^i), \end{aligned} \tag{5.172}$$

where w^i is the (pseudo-)vertex representing the shrunken set W^i and $p(v) = p(w^i)$ ($v \in W^i$, $i \in I$) due to Step (1-2). Since $\partial\varphi(W^i)$ ($i \in I$) are independent of the choice of $\partial\varphi \in \mathbf{B} = \oplus_{i \in I}\mathbf{B}^i$, where $\mathbf{B}^i$ is the base polyhedron of submodular system $\mathbf{S}^i$ for each $i \in I$, it follows from (5.172) that the set of all the optimal submodular flows in $\mathcal{N}^s = (G, \underline{c}, \overline{c}, \gamma_p^s, \mathbf{S})$ coincides with that of all the optimal submodular flows in the current $\mathcal{N} = (G, \underline{c}, \overline{c}, \gamma, \mathbf{S})$. Consequently, (5.170) and (5.171) are valid for any optimal submodular flow φ in the current $\mathcal{N}$. Hence, the modifications of $\underline{c}$ and $\overline{c}$ in Step (4-1) do not change the set of all the optimal submodular flows in $\mathcal{N}$ and Step (4-2) maintains Property (1) given in the present theorem. Therefore, in Step (4-3) a maximal chain $\emptyset = U_0^i \subset U_1^i \subset \cdots \subset U_{k_i}^i = W^i$ of upper ideals of H^i is a chain of tight sets for $\partial\varphi$ restricted to W^i, i.e.,

$$\partial\varphi(U_s^i) = f^i(U_s^i) \qquad (s = 0, 1, \cdots, k_i). \tag{5.173}$$

Since Property (1) in the present theorem is valid before Step (4-3) is executed, (5.173) holds for any optimal submodular flow φ in the current $\mathcal{N}$. This validates the decomposition of $(\mathcal{D}^i, f^i)$ into minors $(\mathcal{D}^i, f^i) \cdot U_s^i / U_{s-1}^i$ ($s = 1, 2, \cdots, k_i$) for each $i \in I$ in Step (4-3). This shows the validity of (1) and (2) at the next stage. Q.E.D.

It follows from Theorem 5.14 that the algorithm terminates after at most $|A| + |V|(|V| - 1)$ applications of Fundamental Cycle. When $M = 0$, the set of all the submodular flows in the current network is exactly the set of all the optimal submodular flows in the original network. An optimal submodular flow is obtained by finding a (feasible) submodular flow in the current network by use of the common-base algorithm described in Section 4.3 (also see Section 5.3). All the steps of Fundamental Cycle can be carried out in strongly polynomial time, provided that an oracle for exchange capacities is available. [For recent developments in submodular flow algorithms see [Iwata97], [Wallacher + Zimmermann99], [Fleischer + Iwata + McCormick02] and [Iwata + McCormick + Shigeno03] (also see [Fuji + Iwata00]). Also, see [Frank93, 96, 97, 98, 05] for theory and applications of submodular flows and their generalizations.]

It should be noted that an exchange capacity can be computed in strongly polynomial time by applying the strongly polynomial algorithm [Grötschel + Lovász + Schrijver88] for submodular-function minimization, where only an oracle for function evaluations is assumed, but the algorithm heavily relies on the ellipsoid method [Khachiyan79, 80] and is not a combinatorial one. [See Section 14 in Chapter VI for recent developments in submodular function minimization.]

5.6. Matroid Optimization

We show some specializations of the results obtained in the previous sections to matroids, which is a retrospective view of matroid optimization.

(a) Maximum independent matchings

Let $G = (V^+, V^-; A)$ be a bipartite graph with the left and right end-vertex sets V^+ and V^- and the arc set A. Also let $\mathbf{M}^+ = (V^+, \mathcal{I}^+)$ and $\mathbf{M}^- = (V^-, \mathcal{I}^-)$, respectively, be matroids on V^+ and V^- with families $\mathcal{I}^+ \subseteq 2^{V^+}$ and $\mathcal{I}^- \subseteq 2^{V^-}$ of independent sets. Denote $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$.

An *independent matching* $M \subseteq A$ in $\mathcal{N}$ is a matching in G such that

$$\partial^+ M \in \mathcal{I}^+, \qquad \partial^- M \in \mathcal{I}^-, \tag{5.174}$$

where $\partial^+ M$ ($\partial^- M$) is the set of end-vertices in V^+ (V^-) of arcs in M. (We assume that for each arc $a \in A$ we have $\partial^+ a \in V^+$ and $\partial^- a \in V^-$.) The *maximum independent matching problem* is to find a maximum independent matching (i.e., an independent matching of maximum cardinality) in $\mathcal{N}$.

The maximum independent matching problem can naturally be reduced to a maximum independent flow problem as follows. Consider a network $\tilde{\mathcal{N}} = (G = (V^+, V^-; A), \underline{c}, \overline{c}, (2^{V^+}, \rho^+), (2^{V^+}, \rho^-))$, where V^+ (V^-) is the set of entrances (exits), $(2^{V^+}, \rho^+)$ ($(2^{V^-}, \rho^-)$) is the submodular system on V^+ (V^-) with ρ^+ (ρ^-) being the rank function of matroid $\mathbf{M}^+$ ($\mathbf{M}^-$), and

$$\begin{aligned} \underline{c}(a) &= 0 \qquad (a \in A), &(5.175)\\ \overline{c}(a) &= +\infty \qquad (a \in A). &(5.176) \end{aligned}$$

We see that any integral independent flow φ in $\tilde{\mathcal{N}}$ is $\{0,1\}$-valued and that $\{a \mid a \in A,\ \varphi(a) = 1\}$ is an independent matching in $\mathcal{N}$. From Theorem 5.9 an integral maximum independent flow in $\tilde{\mathcal{N}}$ gives a maximum independent matching in $\mathcal{N}$ and we have the following max-min theorem.

Theorem 5.15 (The maximum-independent-matching minimum-covering-rank theorem) [Edm70], [Rado42], [Welsh70]: *For network* $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$ *we have*

$$\max\{|M| \mid M \text{ is an independent matching in } \mathcal{N}\}$$
$$= \min\{\rho^+(U^+) + \rho^-(U^-) \mid (U^+, U^-) \text{ is a cover of } G\}. \quad (5.177)$$

(Proof) The present theorem immediately follows from Theorem 5.9, where a cut U of finite capacity of $\tilde{\mathcal{N}}$ corresponds to a cover (U^+, U^-) of G such that $U^+ = V^+ - U$ and $U^- = V^- \cap U$; this gives a one-to-one correspondence between the set of cuts of finite capacity of $\tilde{\mathcal{N}}$ and that of covers of G, and the capacity of a cut U of finite capacity of $\tilde{\mathcal{N}}$ is equal to $\rho^+(U^+) + \rho^-(U^-)$, the rank of the cover (U^+, U^-). Q.E.D.

The transformation of matroids by bipartite graphs

Let $G = (V^+, V^-; A)$ be a bipartite graph and $\mathbf{M}^+ = (V^+, \mathcal{I}^+)$ be a matroid with a family $\mathcal{I}^+$ of independent sets. Define

$$\mathcal{I}^- = \{\partial^- M \mid M\text{: a matching in } G,\ \partial^+ M \in \mathcal{I}^+\}, \quad (5.178)$$

where $\partial^- M = \{\partial^- a \mid a \in M\}$. We can easily see that $\mathcal{I}^-$ is a family of independent sets of a matroid. From Theorem 5.15 the rank function ρ^- of the matroid $\mathbf{M}^- = (V^-, \mathcal{I}^-)$ is given by

$$\rho^-(U^-) = \min\{\rho^+(X^+) + |Y^-| \mid (X^+, Y^- \cup (V^- - U^-)) \text{ is a cover of } G\} \quad (5.179)$$

for each $U^- \subseteq V^-$. We call $\mathbf{M}^-$ the matroid *induced from* $\mathbf{M}^+$ *by the bipartite graph* G.

The matroid intersection problem

When V^- is a copy of V^+ and a bipartite graph $G = (V^+, V^-; A)$ represents the natural bijection between V^+ and V^-, the maximum independent matching problem for the network $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$ becomes the problem of finding a maximum common independent set of the two matroid $\mathbf{M}^+$ and $\mathbf{M}^-$, where V^+ is identified with V^-. This problem is called the *matroid intersection problem.* From Theorem 5.15 we have

Theorem 5.16 (The matroid intersection theorem) [Edm70]: *For two matroids* $\mathbf{M}^+ = (E, \mathcal{I}^+)$ *and* $\mathbf{M}^- = (E, \mathcal{I}^-)$ *with* $\mathcal{I}^+$ *and* $\mathcal{I}^-$ *being families of independent sets,*

$$\max\{|I| \mid I \in \mathcal{I}^+ \cap \mathcal{I}^-\} \\ = \min\{\rho^+(X) + \rho^-(E - X) \mid X \subseteq E\}, \tag{5.180}$$

where ρ^+ (ρ^-) *is the rank function of* $\mathbf{M}^+$ ($\mathbf{M}^-$).

Theorem 5.16 also follows from Theorem 4.9.

The matroid intersection problem is a special case of the maximum independent matching problem in a natural way. Conversely, we can show that the maximum independent matching problem is reduced to a matroid intersection problem. Consider the maximum independent matching problem for a network $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$. From G define a bipartite graph $\hat{G} = (\hat{W}^+, \hat{W}^-; \hat{A})$, where

$$\hat{W}^+ = \{w_a^+ \mid a \in A\}, \qquad \hat{W}^- = \{w_a^- \mid a \in A\}, \tag{5.181}$$

$$\hat{A} = \{(w_a^+, w_a^-) \mid a \in A\}. \tag{5.182}$$

(See Fig. 5.5.)

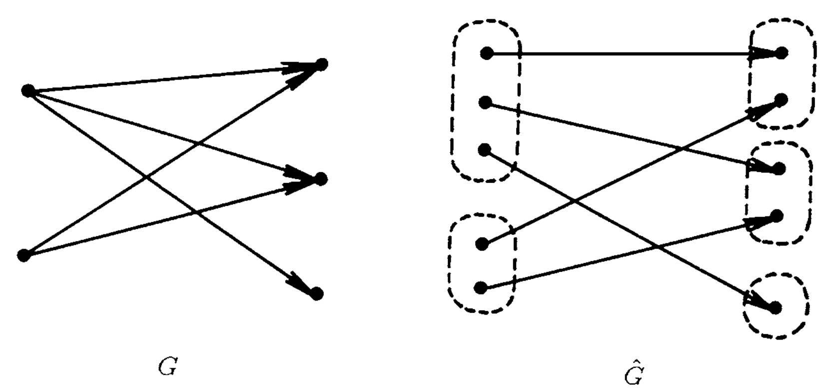

Figure 5.5: Graphs G and $\hat{G}$.

Moreover, define

$$\hat{\mathcal{I}}^+ = \{I^+ \mid I^+ \subseteq \hat{W}^+,\ \ \forall v \in V^+\colon |\{a \mid a \in \delta^+ v,\ w_a^+ \in I^+\}| \le 1,$$
$$\{\partial^+ a \mid a \in A,\ w_a^+ \in I^+\} \in \mathcal{I}^+\}, \quad (5.183)$$
$$\hat{\mathcal{I}}^- = \{I^- \mid I^- \subseteq \hat{W}^-,\ \ \forall v \in V^-\colon |\{a \mid a \in \delta^- v,\ w_a^- \in I^-\}| \le 1,$$
$$\{\partial^- a \mid a \in A,\ w_a^- \in I^-\} \in \mathcal{I}^-\}, \quad (5.184)$$

where ∂^+, ∂^-, δ^+ and δ^- are with respect to G. We can easily show that $\hat{\mathbf{M}}^+ = (\hat{W}^+, \hat{\mathcal{I}}^+)$ and $\hat{\mathbf{M}}^- = (\hat{W}^-, \hat{\mathcal{I}}^-)$ are matroids with families $\hat{\mathcal{I}}^+$ and $\hat{\mathcal{I}}^-$ of independent sets. The matroid $\hat{\mathbf{M}}^+$ ($\hat{\mathbf{M}}^-$) is regarded as the one induced from $\mathbf{M}^+$ ($\mathbf{M}^-$) by a bipartite graph, or as a composition of $\mathbf{M}^+$ ($\mathbf{M}^-$) and the direct sum of rank-one uniform matroids on $\delta^+ v$ ($v \in V^+$) ($\delta^- v$ ($v \in V^-$)). The maximum independent matching problem for network $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$ is thus reduced to a maximum independent matching problem for the new network $\hat{\mathcal{N}} = (\hat{G} = (\hat{W}^+, \hat{W}^-; \hat{A}), \hat{\mathbf{M}}^+, \hat{\mathbf{M}}^-)$, which is a matroid intersection problem.

The matroid union

Let $\mathbf{M}_i = (E_i, \mathcal{I}_i)$ $(i = 1, 2)$ be two matroids. Define

$$\mathcal{I}_{1\vee 2} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}. \quad (5.185)$$

Then $\mathbf{M}_{1\vee 2} = (E_1 \cup E_2, \mathcal{I}_{1\vee 2})$ is a matroid, which is the one induced from the direct sum $\mathbf{M}_1 \oplus \mathbf{M}_2$ of the two by the bipartite graph $G = (V^+, V^-; A)$, where $V^+ = E_1 \oplus E_2$, $V^- = E_1 \cup E_2$ and the arc set A consists of the natural bijections between $E_1 \subseteq V^+$ and $E_1 \subseteq V^-$ and between $E_2 \subseteq V^+$ and $E_2 \subseteq V^-$ (see Fig. 5.6). Therefore, from (5.179) we have

Theorem 5.17: *The rank function $\rho_{1\vee 2}$ of $\mathbf{M}_{1\vee 2}$ is given by*

$$\rho_{1\vee 2}(X) = \min\{\rho_1(Y \cap E_1) + \rho_2(Y \cap E_2) + |X - Y| \mid Y \subseteq X\} \quad (5.186)$$

for each $X \subseteq E_1 \cup E_2$.

The matroid $\mathbf{M}_{1\vee 2}$ is called the *union* (or *sum*) of $\mathbf{M}_i$ $(i = 1, 2)$.

Note that if $E_1 = E_2$, we have $\rho_{1\vee 2} = (\rho_1 + \rho_2)^{\mathbf{1}}$, which is the rank function of the reduction of the sum $(2^E, \rho_1 + \rho_2)$ of submodular systems $(2^E, \rho_i)$ $(i = 1, 2)$ by vector $\mathbf{1} = (\mathbf{1}(e) = 1 \mid e \in E)$ (see (3.9)).

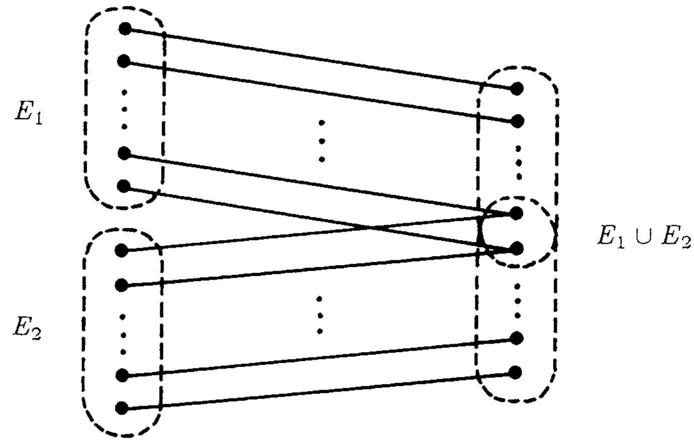

Figure 5.6: Matroid union.

A base of the union $\mathbf{M}_{1\vee 2}$ of $\mathbf{M}_1$ and $\mathbf{M}_2$ is given in the form of the union $B_1 \cup B_2$ of some bases B_i of matroids $\mathbf{M}_i$ $(i = 1, 2)$. When $E_1 = E_2$, for bases B_i $(i = 1, 2)$ of matroids $\mathbf{M}_i$ $B_1 \cup B_2$ is a base of $\mathbf{M}_{1\vee 2}$ if and only if $B_1 \cap (E_2 - B_2)$ is a maximum common independent set of $\mathbf{M}_1$ and the dual $\mathbf{M}_2{}^*$ of $\mathbf{M}_2$. In this sense the problem of finding a base of the union of two matroids is equivalent to the matroid intersection problem and hence to the maximum independent matching problem.

When we are given matroids $\mathbf{M}_i = (E_i, \mathcal{I}_i)$ $(i = 1, 2, \cdots, k)$ $(k \geq 2)$, we can define the union $\mathbf{M}_{1\vee 2\vee \cdots \vee k}$ of $\mathbf{M}_i$ $(i = 1, 2, \cdots, k)$ in the same way as in the case of $k = 2$. That is, an independent set of the union of $\mathbf{M}_i$ $(i = 1, 2, \cdots, k)$ is the union of independent sets $I_i \in \mathcal{I}_i$ $(i = 1, 2, \cdots, k)$. The rank function $\rho_{1\vee 2\vee \cdots \vee k}$ of $\mathbf{M}_{1\vee 2\vee \cdots \vee k}$ is given by

$$\rho_{1\vee 2\vee \cdots \vee k}(X) = \min\{\rho_1(Y \cap E_1) + \cdots + \rho_k(Y \cap E_k) + |X - Y| \mid Y \subseteq X\} \tag{5.187}$$

for each $X \subseteq E_1 \cup \cdots \cup E_k$.

Theorem 5.18: *There exist disjoint bases of* $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2, \cdots, k)$, *one from each* $\mathbf{M}_i$, *if and only if*

$$\rho_1(E) + \cdots + \rho_k(E) = \min\{\rho_1(X) + \cdots + \rho_k(X) + |E - X| \mid X \subseteq E\}. \tag{5.188}$$

(Proof) The present theorem follows from the fact that the rank of the union of $\mathbf{M}_i$ $(i = 1, 2, \cdots, k)$ is equal to the right-hand side of (5.188). Q.E.D.

The matroid partitioning

For matroids $\mathbf{M}_i = (E, \mathcal{I}_i)$ with rank functions ρ_i $(i = 1, 2, \cdots, k)$, the *matroid partitioning problem* is to find k disjoint subsets I_i of E such that $I_1 \cup I_2 \cup \cdots \cup I_k = E$ and $I_i \in \mathcal{I}_i$ $(i = 1, 2, \cdots, k)$. We can easily see that k such subsets I_i $(i = 1, 2, \cdots, k)$ exist if and only if the union of $\mathbf{M}_i$ $(i = 1, 2, \cdots, k)$ is the free matroid on E, i.e., from (5.187)

$$|E| = \min\{\rho_1(X) + \cdots + \rho_k(X) + |E - X| \mid X \subseteq E\}. \tag{5.189}$$

In other words,

Theorem 5.19 [Edm70]: *There exists a base B_i of $\mathbf{M}_i = (E, \mathcal{I}_i)$ for each $i = 1, 2, \cdots, k$ such that the union of B_i $(i = 1, 2, \cdots, k)$ is E if and only if*

$$\forall X \subseteq E\colon\ \rho_1(X) + \cdots + \rho_k(X) \geq |X|. \tag{5.190}$$

When matroids $\mathbf{M}_i = (E, \mathcal{I}_i)$ $(i = 1, 2, \cdots, k)$ are the same matroid $\mathbf{M} = (E, \mathcal{I})$ with the rank function ρ, E is partitioned into k independent sets $I_i \in \mathcal{I}$ $(i = 1, 2, \cdots, k)$ if and only if

$$\forall X \subseteq E\colon\ k\rho(X) \geq |X|. \tag{5.191}$$

From (5.191) we have

Corollary 5.20 [Edm65] (see [Tutte61], [Nash-Williams61] for graphs): *The minimum number k for which E is partitioned into k disjoint independent sets of $\mathbf{M} = (E, \mathcal{I})$ is equal to*

$$\max\{\lceil |X|/\rho(X) \rceil \mid \emptyset \neq X \subseteq E\}, \tag{5.192}$$

where we assume $\mathbf{M}$ does not have any selfloop, i.e., we assume $\rho(\{e\}) > 0$ $(e \in E)$.

Note that (5.191) is equivalent to that a uniform vector $(1/k)\mathbf{1}$ is an independent vector of the matroidal polymatroid $\mathbf{P} = (E, \rho)$ corresponding to $\mathbf{M}$.

If the matroidal polymatroid $\mathbf{P} = (E, \rho)$ has a uniform base $(l/k)\mathbf{1}$ for positive integers k, l such that $l|E| = k\rho(E)$, then there exist k bases B_i $(i = 1, 2, \cdots, k)$ of $\mathbf{M}$ such that each $e \in E$ is uniformly covered by B_i $(i = 1, 2, \cdots, k)$ l times, i.e., for each $e \in E$

$$|\{i \mid i \in \{1, 2, \cdots, k\},\ e \in B_i\}| = l, \tag{5.193}$$

due to the fact that $\mathrm{B}(\rho) + \cdots + \mathrm{B}(\rho)$ (the sum of k copies) $= \mathrm{B}(k\rho)$ with $\mathbf{Z}$ as the underlying totally ordered additive group (see Section 3.1.c). Such a family of bases B_i $(i = 1, 2, \cdots, k)$ is called a *complete family of bases* of $\mathbf{M}$ and a matroid having a complete family of bases is called *irreducible* [Tomi75,76]. The concept of irreducible matroid was introduced by N. Tomizawa for the analysis of *principal partition* of a matroid (see Sections 7.2.b.1 and 9.2). The same concept was also independently introduced by H. Narayanan [Narayanan74] and was called a *molecule* (also see [Narayanan + Vartak81] [and [Narayanan97]]).

An algorithm for the maximum independent matching problem by using auxiliary graphs is given by Tomizawa and Iri [Tomi+Iri74]. An algorithm for the matroid intersection problem is given by Edmonds [Edm79]. For other related algorithms see [Edm+Fulkerson65] for matroid partitionings and [Bruno+Weinberg71] for matroid unions.

(b) Optimal independent assignments

Consider a bipartite graph $G = (V^+, V^-; A)$, matroids $\mathbf{M}^+ = (V^+, \mathcal{I}^+)$ and $\mathbf{M}^- = (V^-, \mathcal{I}^-)$ and a weight function $w\colon A \to \mathbf{R}$. Denote such a network by $\mathcal{N} = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-, w)$. For a positive integer k, a *k-independent matching* M in $\mathcal{N}$ is an independent matching of cardinality k in $\mathcal{N}^0 = (G = (V^+, V^-; A), \mathbf{M}^+, \mathbf{M}^-)$. An *optimal k-independent assignment* in $\mathcal{N}$ is a k-independent matching M having the minimum weight $w(M) = \sum_{e \in E} w(e)$ among all the k-independent matchings in $\mathcal{N}$. The *independent assignment problem* is to find an optimal k-independent assignment in $\mathcal{N}$ for a given k. The independent assignment problem is a special case of a neoflow problem, especially of the independent flow problem in a natural way.

The independent assignment problem is equivalent to the weighted matroid intersection problem, which is to find a minimum-weight common independent set, of two matroids, having cardinality k for a given positive integer k.

For an independent matching M define the auxiliary network $\mathcal{N}_M = (G_M = (V^*, A_M), w_M)$ as follows. G_M is the graph with vertex set $V^* = V^+ \cup V^- \cup \{s^+, s^-\}$ and arc set $A_M = S_M^+ \cup A_M^+ \cup \hat{A}_M \cup \tilde{M} \cup A_M^- \cup S_M^-$, where

$$
\begin{aligned}
S_M^+ &= \{(s^+, v) \mid v \in V^+ - \mathrm{cl}^+(\partial^+ M)\} \cup \{(v, s^+) \mid v \in \partial^+ M\}, && (5.194)\\
A_M^+ &= \{(u, v) \mid v \in \mathrm{cl}^+(\partial^+ M) - \partial^+ M,\ u \in \mathrm{C}^+(\partial^+ M|v) - \{v\}\}, && (5.195)\\
\hat{A}_M &= A - M, && (5.196)\\
\tilde{M} &= \{\overline{a} \mid a \in M\} \quad (\overline{a}\text{: a reorientation of } a), && (5.197)\\
A_M^- &= \{(v, u) \mid v \in \mathrm{cl}^-(\partial^- M) - \partial^- M,\ u \in \mathrm{C}^-(\partial^- M|v) - \{v\}\}, && (5.198)\\
S_M^- &= \{(v, s^-) \mid v \in V^- - \mathrm{cl}^-(\partial^- M)\} \cup \{(s^-, v) \mid v \in \partial^- M\}. && (5.199)
\end{aligned}
$$

Here, cl^+ and cl^- are, respectively, the closure functions of $\mathbf{M}^+$ and $\mathbf{M}^-$, $\mathrm{C}^+(\partial^+ M|v)$ for $v \in \mathrm{cl}^+(\partial^+ M) - \partial^+ M$ is the fundamental circuit associated with $\partial^+ M \in \mathcal{I}^+$ and v which is the unique circuit of $\mathbf{M}^+$ contained in $\partial^+ M \cup \{v\}$, and $\mathrm{C}^-(\partial^- M|v)$ is similarly defined for $\mathbf{M}^-$. Also, $w_M\colon A_M \to \mathbf{R}$ is the length function defined by

$$
w_M(a) = \begin{cases} w(a) & (a \in \hat{A}_M) \\ -w(\overline{a}) & (a \in \tilde{M},\ \overline{a}(\in M)\text{: a reorientation of } a) \\ 0 & (a \in S_M^+ \cup A_M^+ \cup A_M^- \cup S_M^-). \end{cases} \qquad (5.200)
$$

A primal-dual algorithm for the independent assignment problem [Iri+Tomi76]

Step 1: Put $M \leftarrow \emptyset$.

Step 2: While $|M| < k$, do the following.

(2-1) Find a shortest path P, relative to the length function w_M, from s^+ to s^- in the auxiliary network $\mathcal{N}_M$ having the minimum number of arcs.

If there exists no directed path from s^+ to s^- in $\mathcal{N}_M$, then stop (there is no k-independent matching).

(2-2) Put $M \leftarrow M \triangle A(P)$ ($\triangle$: the symmetric difference), where

$$
\begin{aligned}
A(P) = \{a \mid a \in A,\ a \text{ lies on } P\} \\
\cup \{a \mid a \in M,\ \text{a reorientation of } a \text{ lies on } P\}. \quad (5.201)
\end{aligned}
$$

(End)

If we define in Step (2-1) a potential $p: V^* \to \mathbf{R}$ in such a way that for each $v \in V^*$ $p(v)$ is equal to the length of a shortest path from s^+ to v in $\mathcal{N}_M$, then using the potential p we can replace w_M in the next execution of Step (2-1) by $w_{M,p}$ defined by

$$w_{M,p}(a) = w_M(a) + p(\partial^+ a) - p(\partial^- a). \tag{5.202}$$

Here, we disregard those vertices which are not reachable from s^+ in $\mathcal{N}_M$, since vertices which are not reachable from s^+ will not become reachable from s^+ in $\mathcal{N}_M$ for new M. We can show that thus defined $w_{M,p}$ is nonnegative for those arcs reachable from s^+, so that we can reduce the complexity of finding a shortest path in $\mathcal{N}_M$ (see [Iri + Tomi76]). This is an adaptation of the technique for the classical min-cost flows developed by Tomizawa [Tomi71] and also independently by Edmonds and Karp [Edm + Karp72].

It should be noted that arcs entering s^+ or leaving s^- in $\mathcal{N}_M$ play no rôle in the primal-dual algorithm. These arcs are for the primal algorithm given as follows.

A primal algorithm for the independent assignment problem [Fuji 77a]

Step 1: Find a k-independent matching M in $\mathcal{N}$.

Step 2: While there exists a negative cycle, relative to the length function w_M, in the auxiliary network $\mathcal{N}_M$, do the following.

(2-1) Find a negative cycle Q in $\mathcal{N}_M$ having the minimum number of arcs.

(2-2) Put $M \leftarrow M \triangle A(Q)$, where $A(Q)$ is defined by (5.201) with P replaced by Q.

(End)

For other related algorithms, see [Edm79], [Lawler75], [Fuji77b], [Iri78], [Frank81a].

The problem of finding a minimum weight directed spanning tree in a graph is an example of the independent assignment problem or the weighted matroid intersection problem. Consider a graph $G = (V, A)$ and a weight function $w: A \to \mathbf{R}$. Let $\mathbf{M}_1$ be the graphic matroid $\mathbf{M}(G)$ represented by G

and $\mathbf{M}_2$ be the direct sum of the rank-one uniform matroids $\mathbf{M}^-(v)$ on $\delta^- v$ $(v \in V)$, i.e., $\mathbf{M}_2 = \oplus_{v \in V} \mathbf{M}^-(v)$. We see that $B \subseteq A$ with $|B| = |V| - 1$ is a common independent set of $\mathbf{M}_1$ and $\mathbf{M}_2$ if and only if B forms a directed spanning tree of G. Hence the problem of finding a minimum weight directed spanning tree is a weighted matroid intersection problem. If we want to find a minimum weight spanning tree with a fixed root $v_0 \in V$, then replace $\mathbf{M}^-(v_0)$ by the trivial matroid (rank-zero matroid) on $\delta^- v_0$. For other applications, see [Iri83], [Iri + Fuji81], [Murota87] and [Recski89] [(also see [Narayanan97] and [Murota00a])].

Finally, it should be noted that the problem of finding a common independent set of maximum cardinality for *three* matroids is NP-hard. For, consider a graph $G = (V, A)$, and let $\mathbf{M}_1$ be the 1-elongation of the graphic matroid $\mathbf{M}(G)$ and $\mathbf{M}_2$ be the direct sum of the rank-one uniform matroids $\mathbf{M}^-(v)$ on $\delta^- v$ $(v \in V)$, where the 1-elongation of $\mathbf{M}(G)$ is the union of $\mathbf{M}(G)$ and the rank-one uniform matroid on A (see Section 3.1.d). Also, let $\mathbf{M}_3$ be the direct sum of rank-one uniform matroids $\mathbf{M}^+(v)$ on $\delta^+ v$ $(v \in V)$. We can easily see that the maximum cardinality of common independent sets of $\mathbf{M}_i$ $(i = 1, 2, 3)$ is equal to $|V|$ if and only if there exists a directed Hamiltonian cycle in G, and that a common independent set of $\mathbf{M}_i$ $(i = 1, 2, 3)$ of cardinality $|V|$ forms a directed Hamiltonian cycle in G.

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Chapter IV. Submodular Analysis

Submodular (or supermodular) functions on distributive lattices share similar structures with convex (or concave) functions on convex sets. In this chapter we develop a theory of submodular and supermodular functions from the point of view of the duality in convex analysis ([Fuji84f, 84g, 84h]).

6. Submodular Functions and Convexity

We define the convex (concave) conjugate function of a submodular (supermodular) function and show a Fenchel-type duality theorem for submodular and supermodular functions. We also define the subgradients and subdifferentials of a submodular function and examine the relationship among these concepts and the polyhedra such as the submodular and supermodular polyhedra and the base polyhedron associated with the submodular function. The reason for the analogy between a submodular function and a convex function is nicely explained by the Lovász extension of a submodular function.

6.1. Conjugate Functions and a Fenchel-Type Min-Max Theorem for Submodular and Supermodular Functions

(a) Conjugate functions

Let $f: \mathcal{D} \to \mathbf{R}$ be a submodular function and $g: \mathcal{D} \to \mathbf{R}$ be a supermodular function on a distributive lattice $\mathcal{D} \subseteq 2^E$ with $\emptyset, E \in \mathcal{D}$. Here, we do not necessarily assume that $f(\emptyset) = g(\emptyset) = 0$.

Define the function $f^*: \mathbf{R}^E \to \mathbf{R}$ by

$$f^*(x) = \max\{x(X) - f(X) \mid X \in \mathcal{D}\} \quad (x \in \mathbf{R}^E) \tag{6.1}$$

and also, in a dual form, the function $g^*: \mathbf{R}^E \to \mathbf{R}$ by

$$g^*(x) = \min\{x(X) - g(X) \mid X \in \mathcal{D}\} \quad (x \in \mathbf{R}^E). \tag{6.2}$$

By the definition f^* is a convex function and g^* is a concave function. We call f^* the *convex conjugate function* of the submodular function f and g^* a *concave conjugate function* of the supermodular function g.

It may be noted that the terms $x(X)$ appearing in (6.1) and (6.2) can be regarded as the inner product of x and χ_X, the characteristic vector of X. Notice the analogy between the definition (6.1) and that of a convex conjugate function of an ordinary convex function (see (1.35)) ([Rockafellar70], [Stoer+Witzgall70]).

Theorem 6.1: *For a submodular function $f: \mathcal{D} \to \mathbf{R}$ and a supermodular function $g: \mathcal{D} \to \mathbf{R}$ we have for any $X \in \mathcal{D}$*

$$f(X) = \max\{x(X) - f^*(x) \mid x \in \mathbf{R}^E\}, \tag{6.3}$$

$$g(X) = \min\{x(X) - g^*(x) \mid x \in \mathbf{R}^E\}. \tag{6.4}$$

Moreover, for any $X \in 2^E - \mathcal{D}$ $x(X) - f^*(x)$ (or $x(X) - g^*(x)$) as a function of $x \in \mathbf{R}^E$ can be made arbitrarily large (or small).

(Proof) Without loss of generality we assume that $f(\emptyset) = g(\emptyset) = 0$. From (6.1),

$$f(X) \geq x(X) - f^*(x) \tag{6.5}$$

for any $X \in \mathcal{D}$ and $x \in \mathbf{R}^E$. We show that for any $X \in \mathcal{D}$ there exists a vector $x \in \mathbf{R}^E$ such that (6.5) holds with equality, from which (6.3) follows. From Lemma 3.2, for any $X \in \mathcal{D}$ there exists a subbase $\hat{x} \in \mathrm{P}(f)$ such that $\hat{x}(X) = f(X)$. Since $\hat{x} \in \mathrm{P}(f)$, we have from (6.1)

$$f^*(\hat{x}) = \hat{x}(X) - f(X)(= 0). \tag{6.6}$$

This implies (6.3).

Relation (6.4) is the same as (6.3) by considering the dual order on $\mathbf{R}$.

Moreover, it follows from Lemma 3.2 that for any $X \in 2^E - \mathcal{D}$ we can make $x(X)$ arbitrarily large (or small) subject to $x \in \mathrm{B}(f)$ (or $x \in \mathrm{B}(g)$). Since $f^*(x) = 0$ for $x \in \mathrm{B}(f)$ and $g^*(x) = 0$ for $x \in \mathrm{B}(g)$, this completes the proof. Q.E.D.

We see from Theorem 6.1 that the correspondence between a submodular (or supermodular) function f (or g) and its convex (or concave) conjugate function f^* (or g^*) is one to one.

The convex conjugate function f^* is closely related to the vector rank function r_f of the submodular system $(\mathcal{D}, f)$ when $f(\emptyset) = 0$. Recall that for each $x \in \mathbf{R}^E$

$$\mathrm{r}_f(x) = \min\{f(X) + x(E - X) \mid X \in \mathcal{D}\} \tag{6.7}$$

(see (3.9) or (3.20)).

Lemma 6.2: *For a submodular system $(\mathcal{D}, f)$ on E and a vector $x \in \mathbf{R}^E$ we have*

$$f^*(x) = x(E) - \mathrm{r}_f(x). \tag{6.8}$$

(Proof) Immediate from (6.1) and (6.7). Q.E.D.

Note that since r_f is submodular on the vector lattice $\mathbf{R}^E$ (see [Welsh76] when $\mathcal{D} = 2^E$), the convex function f^* is supermodular on $\mathbf{R}^E$, i.e., for any $x,\ y \in \mathbf{R}^E$

$$f^*(x) + f^*(y) \le f^*(x \vee y) + f^*(x \wedge y), \tag{6.9}$$

where $(x \vee y)(e) = \max(x(e), y(e))$, $(x \wedge y)(e) = \min(x(e), y(e))$ $(e \in E)$. (See [Topkis78] for submodular functions on general lattices.) Here, it may also be noted that vector lattice $\mathbf{R}^E$ is a distributive lattice.

(b) A Fenchel-type min-max theorem

We show a Fenchel-type min-max theorem for submodular and supermodular functions, which relates the difference between a submodular function f and a supermodular function g to that between the concave conjugate function g^* and the convex conjugate function f^*. (For Fenchel's duality theorem for ordinary convex and concave functions, see [Rockafellar70] and [Stoer + Witzgall70].)

Theorem 6.3 (A Fenchel-type min-max theorem) [Fuji84f]: *For a submodular function $f: \mathcal{D}_1 \to \mathbf{R}$ and a supermodular function $g: \mathcal{D}_2 \to \mathbf{R}$ on distributive lattices $\mathcal{D}_1,\ \mathcal{D}_2 \subseteq 2^E$ with $\emptyset, E \in \mathcal{D}_1 \cap \mathcal{D}_2$ we have*

$$\begin{aligned}&\min\{f(X) - g(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\}\\ &= \max\{g^*(x) - f^*(x) \mid x \in \mathbf{R}^E\}.\end{aligned} \tag{6.10}$$

Moreover, if f and g are integer-valued, the maximum on the right-hand side of (6.10) *can be attained by an integral vector x.*

(Proof) Without loss of generality we assume that $f(\emptyset) = g(\emptyset) = 0$. Hence $(\mathcal{D}_1, f)$ is a submodular system and $(\mathcal{D}_2, g)$ is a supermodular system, both on E. The min-max relation (6.10) is equivalent to

$$\begin{aligned} &\min\{f(X) + g^{\#}(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{g^*(x) + g(E) - f^*(x) \mid x \in \mathbf{R}^E\}, \end{aligned} \tag{6.11}$$

where recall that $g^{\#}$ is the dual submodular function of g. From (6.2), (6.7) and Lemma 6.2,

$$f^*(x) = x(E) - \mathrm{r}_f(x), \tag{6.12}$$

$$\begin{aligned} g^*(x) + g(E) &= \min\{x(X) + g^{\#}(E - X) \mid X \in \mathcal{D}_2\} \\ &= \mathrm{r}_{g^{\#}}(x). \end{aligned} \tag{6.13}$$

Substituting (6.12) and (6.13) into (6.11) yields

$$\begin{aligned} &\min\{f(X) + g^{\#}(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{\mathrm{r}_f(x) + \mathrm{r}_{g^{\#}}(x) - x(E) \mid x \in \mathbf{R}^E\}, \end{aligned} \tag{6.14}$$

which is equivalent to (6.10).

For any $x \in \mathbf{R}^E$ let y and z be bases of the reductions of $(\mathcal{D}_1, f)$ and $(\overline{\mathcal{D}}_2, g^{\#})$ by x, respectively, i.e.,

$$y \in \mathrm{P}(f), \quad y \leq x, \quad y(E) = \mathrm{r}_f(x), \tag{6.15}$$

$$z \in \mathrm{P}(g^{\#}), \quad z \leq x, \quad z(E) = \mathrm{r}_{g^{\#}}(x). \tag{6.16}$$

Also define

$$w = y \wedge z \ (\equiv (\min\{y(e), z(e)\} \mid e \in E)). \tag{6.17}$$

Then,

$$w \in \mathrm{P}(f) \cap \mathrm{P}(g^{\#}). \tag{6.18}$$

Because of (6.15)~(6.18),

$$\begin{aligned} &\mathrm{r}_f(x) + \mathrm{r}_{g^{\#}}(x) - x(E) \\ &= y(E) + z(E) - x(E) \\ &\leq w(E) + w(E) - w(E) \\ &= \mathrm{r}_f(w) + \mathrm{r}_{g^{\#}}(w) - w(E) \ (= w(E)). \end{aligned} \tag{6.19}$$

It follows from (6.18) and (6.19) that (6.14) (or (6.11)) is also equivalent to

$$\begin{aligned}&\min\{f(X) + g^{\#}(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\}\\&= \max\{x(E) \mid x \in \mathrm{P}(f) \cap \mathrm{P}(g^{\#})\}.\end{aligned} \tag{6.20}$$

The min-max relation (6.20) is exactly the intersection theorem (Theorem 4.9), so that (6.10) holds.

The integrality part of the present theorem follows from the counterpart of the intersection theorem. Q.E.D.

Note that if $x \in \mathbf{R}^E$ is a maximizer of the right-hand side of (6.10), then w given by (6.15)$\sim$(6.17) is a maximizer of the right-hand side of (6.20). If f and g are integer-valued functions and x is an integral vector, then such a vector w can also be integral. Conversely, any maximizer x of the right-hand side of (6.20) is a maximizer of the right-hand side of (6.10).

The above proof shows the equivalence between the Fenchel-type min-max theorem and the intersection theorem. Combining this with the results in Sections 4.1 and 4.2, we see that the three theorems, the intersection theorem (Theorem 4.9), the discrete separation theorem (Theorem 4.12) and the Fenchel-type min-max theorem (Theorem 6.3), are equivalent.

The Fenchel-type min-max theorem motivates further investigation of submodular and supermodular functions from the point of view of the duality theory in convex analysis ([Rockafellar70], [Stoer + Witzgall70]).

6.2. Subgradients of Submodular Functions

(a) Subgradients and subdifferentials

Consider a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ on a distributive lattice $\mathcal{D} \subseteq 2^E$ with $\emptyset, E \in \mathcal{D}$. For a vector $x \in \mathbf{R}^E$ and a set $X \in \mathcal{D}$, if

$$x(Y) - x(X) \le f(Y) - f(X) \tag{6.21}$$

holds for each $Y \in \mathcal{D}$, then we call x a *subgradient* of f at X. We denote by $\partial f(X)$ the set of all the subgradients of f at X and call $\partial f(X)$ the *subdifferential* of f at X. Previously we employed symbol ∂ as the boundary operator for flows in networks. Here we use the same symbol for subdifferentials, following the convention in convex analysis, because there seems to be no possibility of confusion.

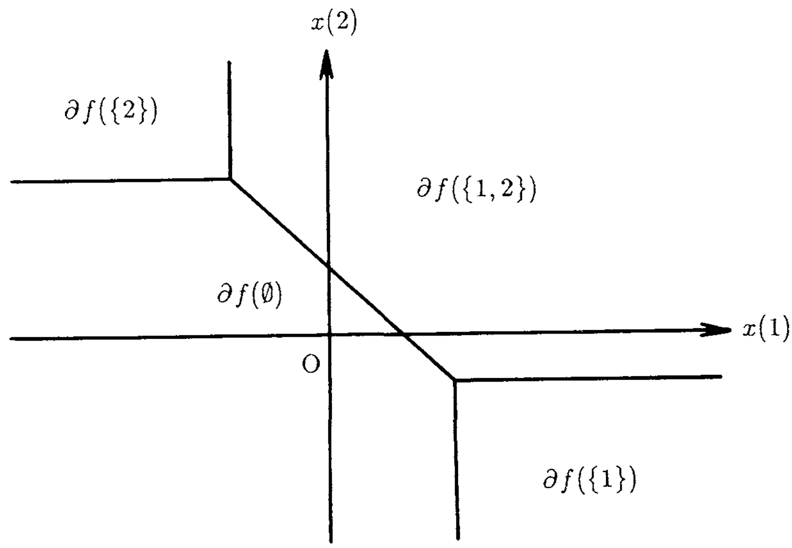

Figure 6.1: Subdifferentials.

Figure 6.1 shows a two-dimensional example of subdifferentials of $f: \mathcal{D} \to \mathbf{R}$ with $\mathcal{D} = 2^{\{1,2\}}$.

In general, $\mathbf{R}^E$ is divided into $|\mathcal{D}|$ unbounded polyhedra $\partial f(X)$ $(X \in \mathcal{D})$ and for distinct X, $Y \in \mathcal{D}$ the subdifferentials $\partial f(X)$ and $\partial f(Y)$ may have common faces but not common interior points. [Note that the convex conjugate function f^* of f is a convex function on $\mathbf{R}^E$ such that f^* is affine on $\partial f(X)$ for each $X \in \mathcal{D}$ and that $\partial f(X)$ $(X \in \mathcal{D})$ are g-polymatroids as will be seen below. That is, f^* is a special M$^\natural$-convex function, which will be considered in Section 17.]

It may be noted that a subgradient of any set function can be defined by (6.21). Some of the arguments in the following are valid for any set function.

Lemma 6.4: *For a submodular function $f: \mathcal{D} \to \mathbf{R}$ and a set $X \in \mathcal{D}$, we have $x \in \partial f(X)$ if and only if $x \in \mathbf{R}^E$ satisfies*

$$x(Y) - x(X) \leq f(Y) - f(X) \tag{6.22}$$

for each $Y \in [\emptyset, X]_{\mathcal{D}} \cup [X, E]_{\mathcal{D}}$, where

$$[\emptyset, X]_{\mathcal{D}} \;=\; \{Y \mid Y \in \mathcal{D},\ Y \subseteq X\}, \tag{6.23}$$

$$[X, E]_{\mathcal{D}} \quad = \quad \{Y \mid Y \in \mathcal{D},\ X \subseteq Y\}. \tag{6.24}$$

(Proof) It is sufficient to show the "if" part only. Suppose that (6.22) holds for each $Y \in [\emptyset, X]_{\mathcal{D}} \cup [X, E]_{\mathcal{D}}$. Then for any $Z \in \mathcal{D}$,

$$x(X \cup Z) - x(X) \le f(X \cup Z) - f(X), \tag{6.25}$$

$$x(X \cap Z) - x(X) \le f(X \cap Z) - f(X). \tag{6.26}$$

From (6.25), (6.26) and the submodularity of f,

$$\begin{aligned} x(Z) - x(X) &= x(X \cup Z) - x(X) + x(X \cap Z) - x(X) \\ &\le f(X \cup Z) + f(X \cap Z) - 2f(X) \\ &\le f(Z) - f(X). \end{aligned}$$

Q.E.D.

Lemma 6.5: *For a submodular function $f: \mathcal{D} \to \mathbf{R}$ and a set $X \in \mathcal{D}$ we have*

$$\partial f(X) = \partial f^X(X) \times \partial f_X(\emptyset), \tag{6.27}$$

where $\partial f^X(X) \subseteq \mathbf{R}^X$, $\partial f_X(\emptyset) \subseteq \mathbf{R}^{E-X}$, $\times$ denotes the direct product, and f^X and f_X are, respectively, the submodular functions on $[\emptyset, X]_{\mathcal{D}}$ and $[X, E]_{\mathcal{D}}/X = \{Y - X \mid Y \in [X, E]_{\mathcal{D}}\}$ defined by

$$f^X(Y) = f(Y) \quad (Y \in [\emptyset, X]_{\mathcal{D}}), \tag{6.28}$$

$$f_X(Y) = f(X \cup Y) - f(X) \quad (Y \in [X, E]_{\mathcal{D}}/X). \tag{6.29}$$

(Proof) From Lemma 6.4, we have $x \in \partial f(X)$ if and only if

$$\begin{aligned} x(Y) - x(X) &\le f(Y) - f(X) \\ &= f^X(Y) - f^X(X) \quad (Y \in [\emptyset, X]_{\mathcal{D}}), \end{aligned} \tag{6.30}$$

$$\begin{aligned} x(Z) - x(\emptyset) &= x(Z \cup X) - x(X) \\ &\le f(Z \cup X) - f(X) \\ &= f_X(Z) - f_X(\emptyset) \quad (Z \in [X, E]_{\mathcal{D}}/X). \end{aligned} \tag{6.31}$$

(6.30) means x^X $(= (x(e) \mid e \in X)) \in \partial f^X(X)$ and (6.31) means x^{E-X} $(= (x(e) \mid e \in E - X)) \in \partial f_X(\emptyset)$. We thus have (6.27). Q.E.D.

Lemma 6.6 (cf. [Rockafellar70, Theorem 23.5]): *For a (submodular) function $f: \mathcal{D} \to \mathbf{R}$, a vector $x \in \mathbf{R}^E$ and a set $X \in \mathcal{D}$, the following three are equivalent:*

$$\text{(i)} \qquad x \in \partial f(X), \tag{6.32}$$

$$\text{(ii)} \qquad \min\{f(Y) + x(E - Y) \mid Y \in \mathcal{D}\} = f(X) + x(E - X), \tag{6.33}$$

$$\text{(iii)} \qquad f(X) + f^*(x) = x(X). \tag{6.34}$$

(Proof) We can easily see that each of the above three is equivalent to $\min\{f(Y) - x(Y) \mid Y \in \mathcal{D}\} = f(X) - x(X)$. Q.E.D.

Lemma 6.6 holds for any set function.

Lemma 6.7: *For a submodular function $f: \mathcal{D} \to \mathbf{R}$ with $\emptyset,\ E \in \mathcal{D}$ and $f(\emptyset) = 0$,*

$$\text{(a)} \qquad \partial f(\emptyset) = \mathrm{P}(f), \tag{6.35}$$

$$\text{(b)} \qquad \partial f(E) = \mathrm{P}(f^{\#}), \tag{6.36}$$

$$\text{(c)} \qquad \partial f(X) \cap \mathrm{B}(f) \neq \emptyset \quad (X \in \mathcal{D}). \tag{6.37}$$

(Proof) (a) and (b) immediately follow from the definition of subdifferential. We prove (c). For any $X \in \mathcal{D}$ there is a base x of the submodular system $(\mathcal{D}, f)$ such that $x(X) = f(X)$ (see Lemma 3.2). Since $x \in \mathrm{B}(f)$ and $x(X) = f(X)$, it easily follows from (6.21) that $x \in \partial f(X)$. Hence, $\partial f(X) \cap \mathrm{B}(f) \neq \emptyset$. Q.E.D.

When $f(\emptyset) = 0$, (6.27) implies that the subdifferential $\partial f(X)$ is the direct product of the supermodular polyhedron $\partial f^X(X)$ and the submodular polyhedron $\partial f_X(\emptyset)$, due to Lemma 6.7.

The following theorem is a submodular analogue of [Rockafellar70, Theorem 23.8] in convex analysis.

Theorem 6.8: *Let $f_1: \mathcal{D}_1 \to \mathbf{R}$ and $f_2: \mathcal{D}_2 \to \mathbf{R}$ be submodular functions, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are distributive lattices with $\emptyset,\ E \in \mathcal{D}_1 \cap \mathcal{D}_2$. Then, for each $X \in \mathcal{D}_1 \cap \mathcal{D}_2$ we have*

$$\partial(f_1 + f_2)(X) = \partial f_1(X) + \partial f_2(X). \tag{6.38}$$

Also, for each $\lambda > 0$ and $X \in \mathcal{D}_1$,

$$\partial(\lambda f_1)(X) = \lambda \partial f_1(X). \tag{6.39}$$

Moreover, for $\lambda = 0$ and $X \in \mathcal{D}_1$,

$$\begin{aligned}\partial(0 \cdot f_1)(X) &= \{x \mid x \in \mathbf{R}^E,\ \forall Y \in \mathcal{D}_1\colon\ x(Y) - x(X) \le 0\} \\ &= 0^+\partial f_1(X), \qquad (6.40)\end{aligned}$$

where $0^+\partial f_1(X)$ is the characteristic cone (or recession cone) of $\partial f_1(X)$ (see (1.22)).

(Proof) From Lemma 6.5, for each $i = 1, 2$ and $X \in \mathcal{D}_i$ $\partial f_i(X)$ is the direct product of $\partial {f_i}^X(X)$ and $\partial f_{iX}(\emptyset)$. It follows from (3.32) (and its dual counterpart for supermodular functions) and from Lemma 6.7 that for each $X \in \mathcal{D}_1 \cap \mathcal{D}_2$ we have

$$\begin{aligned}\partial(f_1 + f_2)(X) &= \partial(f_1 + f_2)^X(X) \times \partial(f_1 + f_2)_X(\emptyset) \\ &= \partial({f_1}^X + {f_2}^X)(X) \times \partial(f_{1X} + f_{2X})(\emptyset) \\ &= (\partial {f_1}^X(X) + \partial {f_2}^X(X)) \times (\partial f_{1X}(\emptyset) + \partial f_{2X}(\emptyset)) \\ &= (\partial {f_1}^X(X) \times \partial f_{1X}(\emptyset)) + (\partial {f_2}^X(X) \times \partial f_{2X}(\emptyset)) \\ &= \partial f_1(X) + \partial f_2(X). \qquad (6.41)\end{aligned}$$

Relations (6.39) and (6.40) are immediate from the definition of subdifferential. Q.E.D.

Note that we have used the intersection theorem through (3.32) to prove Theorem 6.8.

For a convex conjugate function f^* of a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ and a vector $x \in \mathbf{R}^E$, define the set $\partial_2 f^*(x)$ of subsets of E as follows:

$$X \in \partial_2 f^*(x) \qquad (6.42)$$

if and only if $X \subseteq E$ and

$$y(X) - x(X) \le f^*(y) - f^*(x) \qquad (6.43)$$

for each $y \in \mathbf{R}^E$. We call $\partial_2 f^*(x)$ the *binary subdifferential* of f^* at x. Note that from (6.43) and Theorem 6.1 each $X \in \partial_2 f^*(x)$ belongs to $\mathcal{D}$.

Theorem 6.9 (cf. [Rockafellar70, Theorem 23.5]): *Consider a submodular function $f\colon \mathcal{D} \to \mathbf{R}$. For any vector $x \in \mathbf{R}^E$ and any set $X \in \mathcal{D}$ the following two are equivalent.*

(i) $x \in \partial f(X)$, (6.44)

(ii) $X \in \partial_2 f^*(x)$. (6.45)

Moreover, the binary subdifferential $\partial_2 f^*(x)$ is a sublattice of $\mathcal{D}$.

(Proof) From Theorem 6.1, Lemma 6.6 and the definition (6.43) of binary subdifferential of f^*, we see that both (i) and (ii) are equivalent to

$$f(X) + f^*(x) = x(X). \tag{6.46}$$

Moreover, (6.45) or equivalently (6.46) implies

$$x(X) - f(X) = \max\{x(Y) - f(Y) \mid Y \in \mathcal{D}\}. \tag{6.47}$$

Therefore, $\partial_2 f^*(x)$ is the set of the maximizers of the supermodular function $x - f$ on $\mathcal{D}$ and hence is a sublattice of $\mathcal{D}$. Q.E.D.

For two submodular functions $f_i: \mathcal{D}_i \to \mathbf{R}$ $(i = 1, 2)$ define the *convolution* $f_1{}^* \circ f_2{}^*: \mathbf{R}^E \to \mathbf{R}$ of the convex conjugate functions $f_i{}^*$ $(i = 1, 2)$ by

$$(f_1{}^* \circ f_2{}^*)(x) = \min\{f_1{}^*(x_1) + f_2{}^*(x_2) \mid x_1 + x_2 = x\} \tag{6.48}$$

for each $x \in \mathbf{R}^E$.

Theorem 6.10 (cf. [Rockafellar70, Theorem 16.4]): *For two submodular functions $f_i: \mathcal{D}_i \to \mathbf{R}$ $(i = 1, 2)$ with $\emptyset \in \mathcal{D}_1 \cap \mathcal{D}_2$ we have*

$$f_1{}^* \circ f_2{}^* = (f_1 + f_2)^*. \tag{6.49}$$

(Proof) For any $x \in \mathbf{R}^E$,

$$\begin{aligned}
&(f_1 + f_2)^*(x) \\
&= \max\{x(X) - f_1(X) - f_2(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\
&\le \min\{\max\{x_1(X) - f_1(X) + x_2(Y) - f_2(Y) \mid X \in \mathcal{D}_1,\ Y \in \mathcal{D}_2\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mid x_1 + x_2 = x\} \\
&= \min\{f_1{}^*(x_1) + f_2{}^*(x_2) \mid x_1 + x_2 = x\} \\
&= (f_1{}^* \circ f_2{}^*)(x).
\end{aligned} \tag{6.50}$$

On the other hand, let x be any vector in $\mathbf{R}^E$. There exists $X \in \mathcal{D}_1 \cap \mathcal{D}_2$ such that $x \in \partial(f_1 + f_2)(X)$. Furthermore, from Theorem 6.8 there exist

$x_1 \in \partial f_1(X)$ and $x_2 \in \partial f_2(X)$ such that $x_1 + x_2 = x$. Consequently, from Lemma 6.6

$$\begin{aligned}(f_1 + f_2)^*(x) &= x(X) - f_1(X) - f_2(X) \\ &= x_1(X) - f_1(X) + x_2(X) - f_2(X) \\ &= {f_1}^*(x_1) + {f_2}^*(x_2) \\ &\geq ({f_1}^* \circ {f_2}^*)(x).\end{aligned}$$

We thus have (6.49). Q.E.D.

(b) Structures of subdifferentials

Suppose that $\mathcal{D} \subseteq 2^E$ is a simple distributive lattice, i.e., $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ for a poset $\mathcal{P} = (E, \preceq)$. Consider a submodular function $f: \mathcal{D} \to \mathbf{R}$.

We first give a characterization of the extreme points of the subdifferential $\partial f(X)$ for $X \in \mathcal{D}$.

Theorem 6.11: *For each $X \in \mathcal{D}$, $x \in \mathbf{R}^E$ is an extreme point of $\partial f(X)$ if and only if there exists a maximal chain*

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{6.51}$$

of $\mathcal{D}$, including X in it, such that

$$x(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \cdots, n). \tag{6.52}$$

(Proof) We assume, without loss of generality, that $f(\emptyset) = 0$. From Lemma 6.7 we have $\partial f(\emptyset) = \mathrm{P}(f)$. Note that $\mathrm{P}(f)$ and $\mathrm{B}(f)$ have the same set of extreme points. Therefore, the present theorem for $X = \emptyset$ follows from Theorem 3.22 and, similarly, the present theorem for $X = E$ follows from Theorem 3.22, since $\partial f(E) = \mathrm{P}(f^\#)$ due to Lemma 6.7 and $\mathrm{P}(f^\#)$ and $\mathrm{B}(f^\#)$ $(= \mathrm{B}(f))$ have the same set of extreme points. Furthermore, for any $X \in \mathcal{D}$ we have $\partial f(X) = \partial f^X(X) \times \partial f_X(\emptyset)$ due to Lemma 6.5. Hence the extreme points of $\partial f(X)$ are given by the direct product of extreme points of $\partial f^X(X)$ and those of $\partial f_X(\emptyset)$. Since X is the unique maximal element of the domain $[\emptyset, X]_{\mathcal{D}}$ of f^X and $\emptyset$ is the unique minimal element of the domain $[X, E]_{\mathcal{D}}/X$ of f_X, the present theorem for $X \in \mathcal{D}$ with $\emptyset \subset X \subset E$ follows from that for $X = E$ and $X = \emptyset$ for f with domain $\mathcal{D}$. Q.E.D.

The characteristic cone (or the recession cone) of a subdifferential $\partial f(X)$ $(X \in \mathcal{D})$ is given by

$$\begin{aligned} \mathrm{C}_f(X) &= \{x \mid x \in \mathbf{R}^E,\ \forall Y \in \mathcal{D}\colon\ x(Y) - x(X) \le 0\} \\ &= \{x \mid x \in \mathbf{R}^E,\ \forall Y \in [\emptyset, X]_{\mathcal{D}} \cup [X, E]_{\mathcal{D}}\colon\ x(Y) - x(X) \le 0\}. \end{aligned} \tag{6.53}$$

Note that $\mathrm{C}_f(X)$ depends only on $\mathcal{D}$ and X. We next give a characterization of extreme rays of $\mathrm{C}_f(X)$.

Let $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$ be the directed graph with the vertex set E and the arc set $B^*(\mathcal{P})$ which represents the Hasse diagram of the poset $\mathcal{P} = (E, \preceq)$, i.e., $(e, e') \in B^*(\mathcal{P})$ if and only if e covers e' in $\mathcal{P}$. Denote by E^+ and E^-, respectively, the set of all the maximal elements of $\mathcal{P}$ and the set of all the minimal elements of $\mathcal{P}$. Note that $E^+ \cap E^-$ may be nonempty. For each $X \in \mathcal{D}$ denote by $\Delta^-(X)$ the set of all the arcs entering X in $G(\mathcal{P})$. Define vectors ξ_{p^+} $(p^+ \in E^+)$, η_{p^-} $(p^- \in E^-)$ and ζ_a $(a \in B^*(\mathcal{P}))$ in $\mathbf{R}^E$ by

$$\xi_{p^+}(e) = \begin{cases} -1 & (e = p^+) \\ 0 & (e \in E - \{p^+\}) \end{cases} \qquad (p^+ \in E^+), \tag{6.54}$$

$$\eta_{p^-}(e) = \begin{cases} 1 & (e = p^-) \\ 0 & (e \in E - \{p^-\}) \end{cases} \qquad (p^- \in E^-), \tag{6.55}$$

$$\zeta_a(e) = \begin{cases} 1 & (e = e') \\ -1 & (e = e'') \\ 0 & (e \in E - \{e', e''\}) \end{cases} \qquad (a = (e', e'') \in B^*(\mathcal{P})). \tag{6.56}$$

Also define for each $X \in \mathcal{D}$

$$\begin{aligned} \mathrm{ER}(X) = \{\ \xi_{p^+} \mid p^+ \in E^+ - X\} \cup \{\eta_{p^-} \mid p^- \in E^- \cap X\} \\ \cup \{\zeta_a \mid a \in B^*(\mathcal{D}) - \Delta^-(X)\}. \end{aligned} \tag{6.57}$$

Now, we are ready to show a theorem characterizing extreme rays of $\mathrm{C}_f(X)$ $(X \in \mathcal{D})$.

Theorem 6.12: *For each $X \in \mathcal{D}$, the set of all the extreme rays of the characteristic cone $\mathrm{C}_f(X)$ of the subdifferential $\partial f(X)$ is given by $\mathrm{ER}(X)$ in* (6.57).

(Proof) We can easily see from (6.53) that

$$\mathrm{ER}(X) \subseteq \mathrm{C}_f(X). \tag{6.58}$$

Since no vector in $\mathrm{ER}(X)$ can be expressed as a nonnegative linear combination of the other vectors in $\mathrm{ER}(X)$, it suffices to prove that every vector in $\mathrm{C}_f(X)$ can be expressed as a nonnegative linear combination of vectors in $\mathrm{ER}(X)$.

Let v be an arbitrary vector in $\mathrm{C}_f(X)$. From (6.53),

$$v(X - Y) \geq 0 \quad (X \supseteq Y \in \mathcal{D}), \tag{6.59}$$

$$v(Y - X) \leq 0 \quad (X \subseteq Y \in \mathcal{D}). \tag{6.60}$$

Suppose that each arc of $B^*(\mathcal{P}) - \Delta^-(X)$ has the infinite upper capacity and the zero lower capacity and that each arc of $\Delta^-(X)$ has the zero upper and lower capacities. Then it easily follows from (6.59), (6.60) and the feasibility theorem for network flows (Theorem 1.3) [Hoffman60] ([Ford + Fulkerson62]) that there exist a nonnegative flow $\varphi: B^*(\mathcal{P}) \to \mathbf{R}_+$ in $G(\mathcal{P})$ with $\varphi(a) = 0$ $(a \in \Delta^-(X))$, a nonpositive vector $x \in \mathbf{R}^E_-$ with $x(e) = 0$ $(e \notin E^+ - X)$ and a nonnegative vector $y \in \mathbf{R}^E_+$ with $y(e) = 0$ $(e \notin E^- \cap X)$ such that

$$v = \partial\varphi + x + y, \tag{6.61}$$

where $\partial\varphi$ is the boundary of φ in $G(\mathcal{P})$. (6.61) gives an expression of v as a nonnegative linear combination of vectors in $\mathrm{ER}(X)$. Q.E.D.

We see from Lemmas 6.5 and 6.7 that $\mathrm{C}_f(X)$ is the direct product of the characteristic cones of the supermodular polyhedron $\partial f^X(X)$ and the submodular polyhedron $\partial f_X(\emptyset)$, when $f(\emptyset) = 0$. Hence, Theorem 6.12 may also follow from Theorem 3.26.

It should be noted that if v in $\mathrm{C}_f(X)$ satisfies $v(X) = 0$, then $y = \mathbf{0}$ in (6.61) and that if v satisfies $v(E - X) = 0$, then $x = \mathbf{0}$ in (6.61). Theorem 3.26 (the extreme ray theorem for base polyhedra) also follows from this theorem.

6.3. The Lovász Extensions of Submodular Functions

Consider a submodular function $f: \mathcal{D} \to \mathbf{R}$ on a simple distributive lattice $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ with $\mathcal{P} = (E, \preceq)$. We assume $f(\emptyset) = 0$.

Define the convex function $\hat{f}: \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ by

$$\hat{f}(c) = \max\{(c, x) \mid x \in \mathrm{P}(f)\} \tag{6.62}$$

for each $c \in \mathbf{R}^E$, where

$$(c, x) = \sum_{e \in E} c(e)x(e). \tag{6.63}$$

Here, $\hat{f}$ is called the *support function* of $\mathrm{P}(f)$ and is a positively homogeneous function, i.e., $\hat{f}(\lambda c) = \lambda \hat{f}(c)$ for each $\lambda > 0$ and $c \in \mathbf{R}^E$ ([Rockafellar70], [Stoer + Witzgall70]). We see from Corollary 3.14 that $\hat{f}(c) < +\infty$ if and only if $c: E \to \mathbf{R}$ is a nonnegative monotone nonincreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$. Therefore, for any $c \in \mathbf{R}^E$ such that $\hat{f}(c) < +\infty$ there uniquely exist a chain

$$A_1 \subset A_2 \subset \cdots \subset A_k \tag{6.64}$$

of nonempty $A_i \in \mathcal{D}$ $(i = 1, 2, \cdots, k)$ and positive numbers $\lambda_i \in \mathbf{R}$ $(i = 1, 2, \cdots, k)$ such that

$$c = \sum_{i=1}^{k} \lambda_i \chi_{A_i} \tag{6.65}$$

where $k \geq 0$, $\chi_{A_i} \in \mathbf{R}^E$ is the characteristic vector of $A_i \subseteq E$ $(i = 1, 2, \cdots, k)$ and if $k = 0$ (i.e., $c = \mathbf{0}$), the empty sum is defined to be zero vector $\mathbf{0}$ in $\mathbf{R}^E$.

Moreover, we have

$$\hat{f}(c) = \sum_{i=1}^{k} \lambda_i f(A_i) \tag{6.66}$$

since the value of $\hat{f}(c)$ defined by (6.62) can be obtained by the greedy algorithm (see Section 3.2.b) and a maximizer x of (6.62) satisfies

$$x(A_i - A_{i-1}) = f(A_i) - f(A_{i-1}) \quad (i = 1, 2, \cdots, k) \tag{6.67}$$

with $A_0 = \emptyset$. If the right-hand side of (6.66) is the empty sum, it is defined to be zero.

Formula (6.66) was introduced by L. Lovász [Lovász83] for $\mathcal{D} = 2^E$. The construction of $\hat{f}$ through (6.64)$\sim$(6.66) can be applied to any function f on $\mathcal{D}$ with $f(\emptyset) = 0$ and $\hat{f}$ is an extension of f. We call such an extension $\hat{f}$ the *Lovász extension* of f [which is sometimes called the *Choquet integral* (see [Choquet55])].

Theorem 6.13 [Lovász83]: *A function $f: \mathcal{D} \to \mathbf{R}$ is submodular if and only if the Lovász extension $\hat{f}$ of f is convex.*

(Proof) If f is a submodular function, then its extension $\hat{f}$ is given by (6.62) and hence is a convex function. Conversely, suppose that the extension $\hat{f}$ of f is a convex function. By definition, for any $X,\ Y \in \mathcal{D}$

$$\begin{aligned} \hat{f}(\chi_X + \chi_Y) &= \hat{f}(\chi_{X\cup Y} + \chi_{X\cap Y}) \\ &= f(X \cup Y) + f(X \cap Y). \end{aligned} \tag{6.68}$$

Since $\hat{f}$ is a positively homogeneous convex function, we also have

$$\hat{f}(\chi_X + \chi_Y) \leq \hat{f}(\chi_X) + \hat{f}(\chi_Y) = f(X) + f(Y). \tag{6.69}$$

From (6.68) and (6.69) f is a submodular function on $\mathcal{D}$. Q.E.D.

Theorem 6.13 shows the close relationship between the submodularity and the convexity. The results in Sections 6.1 and 6.2 can be viewed from the theory of convex functions through this theorem. However, the integrality result in Theorem 6.3 does not follow directly from the ordinary convex analysis; it is truly a combinatorial deep result.

Define

$$\mathrm{P}(\mathcal{D}) = \text{ the convex hull of vectors } \chi_A \ (A \in \mathcal{D}). \tag{6.70}$$

Lemma 6.14 [Lovász83]: *For a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ we have*

$$\min\{f(X) \mid X \in \mathcal{D}\} = \min\{\hat{f}(c) \mid c \in \mathrm{P}(\mathcal{D})\}. \tag{6.71}$$

(Proof) Immediate from Theorem 6.13 and (6.66), the positive homogeneity of $\hat{f}$. Q.E.D.

Lemma 6.15: *For any $c \in \mathrm{P}(\mathcal{D})$ there uniquely exists a nonempty chain*

$$B_1 \subset B_2 \subset \cdots \subset B_p \tag{6.72}$$

of $\mathcal{D}$ such that c is expressed as a convex combination

$$c = \sum_{i=1}^{p} \mu_i \chi_{B_i} \tag{6.73}$$

with $\mu_i > 0$ $(i = 1, 2, \cdots, p)$ and $\sum_{i=1}^{p} \mu_i = 1$.

(Proof) Any $c \in \mathrm{P}(\mathcal{D})$ is a nonnegative monotone nonincreasing function from $\mathcal{P} = (E, \preceq)$ to $(\mathbf{R}, \leq)$. Therefore, there uniquely exists a chain (6.64) of nonempty $A_i \in \mathcal{D}$ $(i = 1, 2, \cdots, k)$ and $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$ such that (6.65) holds. Since $c \in \mathrm{P}(\mathcal{D})$, we have

$$\sum_{i=1}^{k} \lambda_i \leq 1. \tag{6.74}$$

If $\sum_{i=1}^{k} \lambda_i = 1$, then (6.65) is the desired unique expression. Otherwise put $\lambda_0 = 1 - \sum_{i=1}^{k} \lambda_i$ and $A_0 = \emptyset$. This yields a desired unique expression

$$c = \sum_{i=0}^{k} \lambda_i \chi_{A_i} \tag{6.75}$$

with $\lambda_i > 0$ $(i = 0, 1, \cdots, k)$ and $\sum_{i=0}^{k} \lambda_i = 1$. Q.E.D.

Now, let $f : \mathcal{D} \to \mathbf{R}$ be a submodular function and define $\tilde{f} : \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(c) = \begin{cases} \hat{f}(c) & (c \in \mathrm{P}(\mathcal{D})) \\ +\infty & (c \in \mathbf{R}^E - \mathrm{P}(\mathcal{D})), \end{cases} \tag{6.76}$$

where $\hat{f}$ is the Lovász extension of f. Call $\tilde{f}$ the *truncated Lovász extension* of f.

Theorem 6.16: *For each $A \in \mathcal{D}$ we have*

$$\partial f(A) = \partial \tilde{f}(\chi_A), \tag{6.77}$$

where $\partial \tilde{f}(\chi_A)$ denotes the subdifferential of the convex function $\tilde{f}$ at χ_A in an ordinary sense of convex analysis (see (1.34)).

(Proof) By definition, we have $x \in \partial \tilde{f}(\chi_A)$ if and only if

$$\forall c \in \mathbf{R}^E : (c - \chi_A, x) \leq \tilde{f}(c) - \tilde{f}(\chi_A). \tag{6.78}$$

Since $\tilde{f}(\chi_A) = f(A)$, (6.78) is rewritten as

$$\begin{aligned} f(A) - x(A) &\leq \min\{\tilde{f}(c) - (c, x) \mid c \in \mathbf{R}^E\} \\ &= \min\{\hat{f}(c) - (c, x) \mid c \in \mathrm{P}(\mathcal{D})\} \\ &= \min\{f(X) - x(X) \mid X \in \mathcal{D}\}, \end{aligned} \tag{6.79}$$

where the last equality follows from Lemma 6.14 with f replaced by $f - x$. (6.79) is equivalent to $x \in \partial f(A)$. Q.E.D.

Theorem 6.17: *Let c be an arbitrary vector in* $\mathrm{P}(\mathcal{D})$ *and suppose that c is expressed as* (6.73) *with* (6.72)*. Then, we have*

$$\partial \tilde{f}(c) = \bigcap \{\partial f(B_i) \mid i = 1, 2, \cdots, p\}. \tag{6.80}$$

(Proof) We have $x \in \partial \tilde{f}(c)$ if and only if

$$\forall\, b \in \mathrm{P}(\mathcal{D})\colon\ (b - c, x) \leq \hat{f}(b) - \hat{f}(c). \tag{6.81}$$

From (6.72) and (6.73), (6.81) is rewritten as

$$\begin{aligned} \sum_{i=1}^{p} \mu_i (f(B_i) - x(B_i)) &\leq \min\{\hat{f}(b) - (b, x) \mid b \in \mathrm{P}(\mathcal{D})\} \\ &= \min\{f(X) - x(X) \mid X \in \mathcal{D}\} \end{aligned} \tag{6.82}$$

due to Lemma 6.14. Furthermore, since $\sum_{i=1}^{p} \mu_i = 1$ and $\mu_i > 0$ $(i = 1, 2, \cdots, p)$, (6.82) is equivalent to

$$f(B_i) - x(B_i) = \min\{f(X) - x(X) \mid X \in \mathcal{D}\} \quad (i = 1, 2, \cdots, p) \tag{6.83}$$

or

$$x \in \bigcap \{\partial f(B_i) \mid i = 1, 2, \cdots, p\}. \tag{6.84}$$

Q.E.D.

For any maximal chain $\mathcal{C}\colon \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ of $\mathcal{D}$, denote by $\mathrm{P}(\mathcal{C})$ the n-simplex with vertices χ_{S_i} $(i = 0, 1, \cdots, n)$.

Lemma 6.18: *The collection of* $\mathrm{P}(\mathcal{C})$*'s for all the maximal chains $\mathcal{C}$ of $\mathcal{D}$ forms a simplicial subdivision of* $\mathrm{P}(\mathcal{D})$*. Moreover, for two maximal chains $\mathcal{C}^i$: $\emptyset = S_0^i \subset S_1^i \subset \cdots \subset S_n^i = E$ $(i = 1, 2)$ the n-simplices* $\mathrm{P}(\mathcal{C}^i)$ $(i = 1, 2)$ *have a common facet if and only if for some k with $1 \leq k \leq n - 1$ we have*

$$S_j^1 = S_j^2 \quad (0 \leq j \leq n,\ j \neq k). \tag{6.85}$$

(Proof) The first half of this lemma follows from the uniqueness property of Lemma 6.15. Moreover, any facet of the n-simplex $\mathrm{P}(\mathcal{C}^i)$ corresponds to a subchain of $\mathcal{C}^i$ with length $n-1$. From this follows the second half of the lemma. Q.E.D.

Lemma 6.19: *For any maximal chain $\mathcal{C}$: $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ of $\mathcal{D}$ and any interior point c of $\mathrm{P}(\mathcal{C})$, $\tilde{f}$ has a unique subgradient x at c given by*

$$x(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \cdots, n). \tag{6.86}$$

(Proof) From Theorem 6.17 vector x given by (6.86) is the unique subgradient of $\tilde{f}$ at c. Q.E.D.

Note that the subgradient x of $\tilde{f}$ given by (6.86) is an extreme point of the base polyhedron $\mathrm{B}(f)$.

We can see that for a submodular function $f: \mathcal{D} \to \mathbf{R}$, the convex conjugate function f^* and the truncated Lovász extension $\tilde{f}$ of f are the convex conjugate functions of each other in an ordinary sense of convex analysis (see (1.35)), since

$$\begin{aligned} f^*(x) &= \max\{x(X) - f(X) \mid X \in \mathcal{D}\} \\ &= \max\{(c, x) - \tilde{f}(x) \mid x \in \mathbf{R}^E\}. \end{aligned} \tag{6.87}$$

Consequently, the Fenchel-type min-max theorem for submodular and supermodular functions (Theorem 6.3), except for the integrality property, follows from Fenchel's duality theorem for ordinary convex and concave functions.

Finally, it should be noted that the extension of the rank function of a polypseudomatroid can be defined by generalizing the Lovász extension of a submodular function, due to the greediness property of polypseudomatroids (see [Qi88]).

7. Submodular Programs

In this section we consider optimization problems with objective functions and constraints described by submodular functions, which we call *submodular programs* ([Fuji84f]).

7.1. Submodular Programs – Unconstrained Optimization

Let $f: \mathcal{D} \to \mathbf{R}$ be a submodular function on a simple distributive lattice $\mathcal{D} \subseteq 2^E$ with $f(\emptyset) = 0$. We consider the problem of minimizing the submodular function $f: \mathcal{D} \to \mathbf{R}$ without any constraints. It should, however, be noted that the underlying distributive lattice $\mathcal{D}$ itself may be regarded as a constrained feasible region. [We treat submodular function minimization in more detail in Chapter VI.]

(a) Minimizing submodular functions

From the definition of subdifferential of a submodular function we have the following trivial but fundamental lemma.

Lemma 7.1: *A set $A \in \mathcal{D}$ is a minimizer of $f: \mathcal{D} \to \mathbf{R}$ if and only if*

$$\mathbf{0} \in \partial f(A), \tag{7.1}$$

where $\mathbf{0}$ is the zero vector in $\mathbf{R}^E$.

From Lemmas 6.4 and 7.1 we get the following theorem which means that some "local" or "partial" minimality implies "global" minimality.

Theorem 7.2: *A set $A \in \mathcal{D}$ is a minimizer of $f: \mathcal{D} \to \mathbf{R}$ if and only if $A \in \mathcal{D}$ minimizes f restricted to the sublattice $[\emptyset, A]_{\mathcal{D}} \cup [A, E]_{\mathcal{D}}$.*

Grötschel, Lovász and Schrijver [Grötschel + Lovász + Schrijver88] have devised a strongly polynomial algorithm for minimizing a submodular function $f: \mathcal{D} \to \mathbf{R}$ which requires time polynomial in $|E|$. Their algorithm heavily depends on the so-called ellipsoid method [Khachiyan79, 80] for linear programs and is not a combinatorial one. A combinatorial but pseudopolynomial algorithm is proposed by Cunningham [Cunningham85a], where the submodular function f is integer-valued and the required running time is polynomial in $|E|$ and $\max |f(X)|$ but not in $\log(\max |f(X)|)$. [Combinatorial strongly polynomial algorithms for minimizing submodular functions were devised by Iwata, Fleischer and Fujishige [IFF01] and Schrijver [Schrijver00] in 1999 independently and in different ways based on Cunningham's framework [Cunningham84], [Cunningham85a]. See Section 14 for more details.]

For special classes of submodular functions the problem of minimizing a submodular function can be reduced to network optimization problems such as the minimum cut problem [Picard76] and the maximum-weight stable-set problem in a bipartite graph [Billionnet + Minoux85].

We show a "practical" algorithm for minimizing submodular functions based on the following lemma (see (3.22)).

Lemma 7.3:

$$\min\{f(X) \mid X \in \mathcal{D}\} = \max\{y(E) \mid y \in \mathrm{P}(f),\ y \leq \mathbf{0}\}. \tag{7.2}$$

Consider a special quadratic programming problem over the base polyhedron $\mathrm{B}(f)$ described as

$$\text{Minimize} \ \ \|x\|^2 \ (= \sum_{e \in E} x(e)^2) \tag{7.3a}$$

$$\text{subject to} \ \ x \in \mathrm{B}(f) \tag{7.3b}$$

(see [Fuji80b]).

In the present subsection we assume that the underlying totally ordered additive group $\mathbf{R}$ is the set of reals or rationals. (In Chapter V we will consider in detail a class of nonlinear optimization problems over the base polyhedron which includes Problem (7.3).)

Let x^* be the optimal solution of Problem (7.3) and define

$$y^* = x^* \wedge \mathbf{0} \ (= (\min\{x^*(e), 0\} \mid e \in E)), \tag{7.4}$$

$$A_- = \{e \mid e \in E,\ x^*(e) < 0\}, \tag{7.5}$$

$$A_0 = \{e \mid e \in E,\ x^*(e) \leq 0\}. \tag{7.6}$$

Lemma 7.4: *y^* in (7.4) is a maximizer of the right-hand side of (7.2). Also, A_- is the unique minimal minimizer of f and A_0 is the unique maximal minimizer of f.*

(Proof) Because of the optimality of x^* we have

$$\forall\, e \in A_-\colon \ \ \mathrm{dep}(x^*, e) \subseteq A_-, \tag{7.7}$$

$$\forall\, e \in A_0\colon \ \ \mathrm{dep}(x^*, e) \subseteq A_0. \tag{7.8}$$

From (7.4)~(7.8), we have $A_-,\ A_0 \in \mathcal{D}$ and

$$f(A_-) = f(A_0) = y^*(E) \ (= x^*(A_-) = x^*(A_0)). \tag{7.9}$$

Since for any $X \in \mathcal{D}$ and any $y \in \mathrm{P}(f)$ with $y \leq \mathbf{0}$ we have $f(X) \geq y(E)$, it follows from (7.9) that y^* is a maximizer of the right-hand side of (7.2) and A_- and A_0 are minimizers of f. Moreover, for any $X \in \mathcal{D}$ such that $X \subset A_-$,

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-) \tag{7.10}$$

and for any $X \in \mathcal{D}$ such that $A_0 \subset X$,

$$f(X) \geq x^*(X) > x^*(A_0) = f(A_0). \tag{7.11}$$

From (7.9)$\sim$(7.11), A_- is the unique minimal minimizer of f and A_0 is the unique maximal minimizer of f. Q.E.D.

Let $c_1 < c_2 < \cdots < c_p$ be the distinct values of $x^*(e)$ $(e \in E)$ and define

$$B_i = \{e \mid e \in E,\ x^*(e) \leq c_i\} \quad (i = 1, 2, \cdots, p), \tag{7.12}$$

$$B_0 = \emptyset. \tag{7.13}$$

By a similar argument as (7.7)$\sim$(7.9) we see that

$$\emptyset = B_0 \subset B_1 \subset \cdots \subset B_p = E \tag{7.14}$$

is a chain of $\mathcal{D}$ and that

$$f(B_i) = x^*(B_i) \quad (i = 0, 1, \cdots, p). \tag{7.15}$$

Consequently, we have

$$c_i = \frac{f(B_i) - f(B_{i-1})}{|B_i - B_{i-1}|} \quad (i = 1, 2, \cdots, p). \tag{7.16}$$

Problem (7.3) is to find the minimum-norm point in $\mathrm{B}(f)$. For simplicity, let us suppose $\mathrm{B}(f)$ is bounded, i.e., $\mathcal{D} = 2^E$. A solution algorithm for the minimum norm-point problem is proposed by P. Wolfe [Wolfe76] (also see [von Hohenbalken75]). We can adopt Wolfe's algorithm for Problem (7.3). We express a simplex S in $\mathbf{R}^E$ by the set of its extreme points.

An algorithm for finding the minimum-norm point in B(f)

Step 1: Let x^* be any extreme point of $\mathrm{B}(f)$. Put $S \leftarrow \{x^*\}$.

Step 2: Using the greedy algorithm, find a minimum-weight base $\hat{x}$ of $\mathrm{B}(f)$ with respect to the weight function x^*. If $(x^*, \hat{x} - x^*) = 0$, then stop (x^* is the minimum-norm point). Otherwise put $S \leftarrow S \cup \{\hat{x}\}$.

Step 3: Find the minimum-norm point x_0 in the affine space generated by S. If x_0 is in the relative interior of the simplex S, then put $x^* \leftarrow x_0$ and go to Step 2. Otherwise let $S' \subset S$ be the unique minimal face of simplex S which has the nonempty intersection with the line segment $[x^*, x_0]$. Put $x^* \leftarrow$ the intersection point of the face S' and the line segment $[x^*, x_0]$, and also put $S \leftarrow S'$. Go to the beginning of Step 3.
(End)

Step 3 is consecutively repeated at most $|E| - 1$ times, since the cardinality of S monotonically decreases. Each simplex S available when executing Step 2 uniquely determines x^* lying in the relative interior of simplex S and the norm $\|x^*\|$ monotonically decreases every time x^* is renewed. Therefore, all the simplices S which we encounter during the execution of the algorithm are different, so that the algorithm terminates after a finite number of steps.

Example: As an illustrative example, consider the minimum-cut problem for the two-terminal network $\mathcal{N} = (G = (V, A), s^+, s^-, c)$ shown in Fig. 7.1, where $V = \{s^+, s^-, 1, 2, 3\}$. The numbers attached to the arcs denote the capacities $c(a)$ ($a \in A$). The problem is to find a cut $U \subseteq V$ with $s^+ \in U$ and $s^- \notin U$ which minimizes its capacity $\sum_{a \in \Delta^+ U} c(a)$, where $\Delta^+ U$ is the set of arcs leaving U. Putting $V^* = V - \{s^+, s^-\}$, we define a submodular function $f\colon 2^{V^*} \to \mathbf{R}$ as follows. For each $W \subseteq V^*$,

$$f(W) = \sum_{a \in \Delta^+(W \cup \{s^+\})} c(a) - \sum_{a \in \Delta^+\{s^+\}} c(a). \tag{7.17}$$

The second constant term is for the normalization, $f(\emptyset) = 0$.

Now, the minimum-cut problem is reduced to the problem of minimizing the submodular function $f\colon 2^{V^*} \to \mathbf{R}$.

Let us start with the extreme point x^* of $\mathrm{B}(f)$ corresponding to the sequence $(1, 2, 3)$ of the vertices in V^*, i.e.,

$$x^* \ (= (x^*(1), x^*(2), x^*(3))) = (0, 10, -20). \tag{7.18}$$

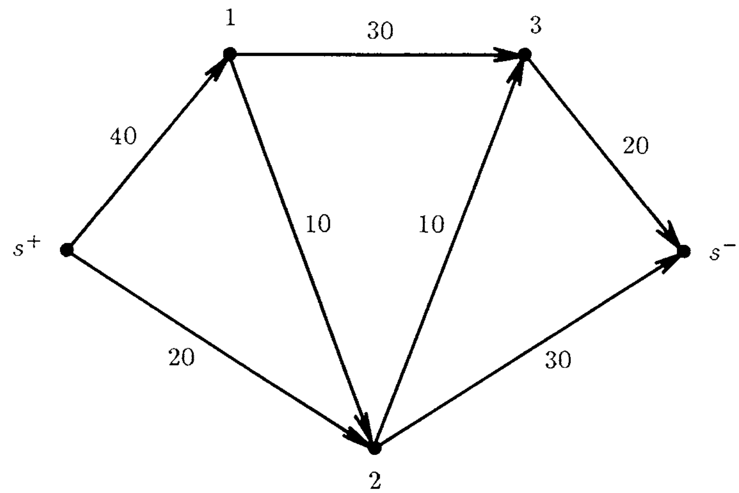

Figure 7.1: An example of a flow network.

Next, in Step 2 we find by the greedy algorithm the minimum-weight base $\hat{x}$ of $\mathrm{B}(f)$ with respect to weight x^*, which is the extreme point of $\mathrm{B}(f)$ corresponding to the vertex sequence $(3, 1, 2)$, i.e.,

$$\hat{x} = (-30, 0, 20). \tag{7.19}$$

The minimum-norm point x_0 in the line through the two points of (7.18) and (7.19) is given by

$$\begin{aligned} x_0 &= \frac{17}{26}(0, 10, -20) + \frac{9}{26}(-30, 0, 20) \\ &= \frac{1}{26}(-270, 170, -160). \end{aligned} \tag{7.20}$$

Since $0 < \frac{17}{26}, \frac{9}{26}$, we put $x^* \leftarrow x_0$ and find a new extreme point, denoted by $\hat{x}$, of $\mathrm{B}(f)$ corresponding to the vertex sequence $(1, 3, 2)$ determined by x^* ($= x_0$ in (7.20)). $\hat{x}$ is given by

$$\hat{x} = (0, 0, -10). \tag{7.21}$$

The minimum-norm point x_0 in the two-dimensional affine space generated by the three points of (7.18), (7.19) and (7.21) is given by

$$x_0 = -\frac{1}{3}(0, 10, -20) + \frac{1}{9}(-30, 0, 20) + \frac{11}{9}(0, 0, -10). \tag{7.22}$$

Since $-\frac{1}{3} < 0 < \frac{1}{9}, \frac{11}{9}$, the minimum-norm intersection point of the present simplex formed by points of (7.18), (7.19) and (7.21) and the line segment between points of (7.20) and (7.22) lies in the face formed by points of (7.19) and (7.21). Hence, we discard point (7.18) from the present simplex S and find the minimum-norm point, denoted by x_0 again, in the line through points of (7.19) and (7.21).

$$\begin{aligned} x_0 &= \frac{1}{6}(-30, 0, 20) + \frac{5}{6}(0, 0, -10) \\ &= (-5, 0, -5). \end{aligned} \tag{7.23}$$

Since the present x_0 is in the relative interior of the simplex formed by points of (7.19) and (7.21), we put $x^* \leftarrow x_0$ and go back to Step 2. Now, we find a minimum-weight base $\hat{x}$ with respect to the weight given by (7.23). Such a base $\hat{x}$ is given by (7.19) (or (7.21)) and the algorithm terminates.

From the minimum-norm base (7.23) we see that $U_0 = \{s^+, 1, 2, 3\}$ is the unique maximal minimum-cut and $U_- = \{s^+, 1, 3\}$ is the unique minimal minimum-cut of the network, due to Lemma 7.4.

When the algorithm for finding the minimum-norm point in $\mathrm{B}(f)$ terminates, we are given the minimum-norm point x^* and a set of extreme points x_i $(i \in I)$ of $\mathrm{B}(f)$ such that x^* is expressed as a convex combination

$$x^* = \sum_{i \in I} \lambda_i x_i \tag{7.24}$$

of x_i $(i \in I)$. For each $i \in I$ let $\mathcal{P}_i = (E, \preceq_i)$ be the poset which corresponds to the distributive lattice

$$\mathcal{D}(x_i) = \{X \mid X \in \mathcal{D},\ x_i(X) = f(X)\} \tag{7.25}$$

with $\mathcal{D}(x_i) = \mathbf{2}^{\mathcal{P}_i}$, and define a distributive lattice $\hat{\mathcal{D}}$ by

$$\hat{\mathcal{D}} = \bigcap_{i \in I} \mathbf{2}^{\mathcal{P}_i}. \tag{7.26}$$

Then,

$$\hat{\mathcal{D}} = \mathcal{D}(x^*)\ (= \{X \mid X \in \mathcal{D},\ x^*(X) = f(X)\}). \tag{7.27}$$

An algorithm for finding the poset $\mathcal{P}_i = (E, \preceq_i)$ for each extreme point $x_i \in \mathrm{B}(f)$ is proposed by Bixby, Cunningham and Topkis [Bixby+Cunningham+Topkis85] (see Section 3.3.c). The poset $\hat{\mathcal{P}} = (\hat{E}, \preceq)$ on a partition of

E which corresponds to the distributive lattice $\hat{\mathcal{D}}$ is easily obtained by superimposing the posets $\mathcal{P}_i$ $(i \in I)$.

For any maximal chain

$$\emptyset = \hat{A}_0 \subset \hat{A}_1 \subset \cdots \subset \hat{A}_q = E \tag{7.28}$$

of $\hat{\mathcal{D}}$, the minimum-norm point x^* satisfies

$$x^*(e) = \frac{f(\hat{A}_j) - f(\hat{A}_{j-1})}{|\hat{A}_j - \hat{A}_{j-1}|} \tag{7.29}$$

for each $e \in \hat{A}_j - \hat{A}_{j-1}$ $(j = 1, 2, \cdots, q)$ (cf. (7.16)). Furthermore, all the minimizers of f are given by the interval $[A_-, A_0]_{\hat{\mathcal{D}}}$ of $\hat{\mathcal{D}}$, where A_- and A_0 are defined by (7.5) and (7.6).

(b) Minimizing modular functions

Let $\mu\colon \mathcal{D} \to \mathbf{R}$ be a modular function on a simple distributive lattice $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ with $\mathcal{P} = (E, \preceq)$ and $\mu(\emptyset) = 0$.

Lemma 7.5: *There exists a unique vector $\nu \in \mathbf{R}^E$ such that for each $X \in \mathcal{D}$*

$$\mu(X) = \sum_{e \in X} \nu(e). \tag{7.30}$$

(Proof) Choose any maximal chain $\mathcal{C}\colon \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ of $\mathcal{D}$ and define a vector $\nu \in \mathbf{R}^E$ by

$$\nu(e_i) = \mu(S_i) - \mu(S_{i-1}) \quad (i = 1, 2, \cdots, n), \tag{7.31}$$

where $\{e_i\} = S_i - S_{i-1}$ $(i = 1, 2, \cdots, n)$. For any $X \in \mathcal{D}$ there exist integers $1 \le j_1 < \cdots < j_p \le n$ with $|X| = p$ such that

$$X = \{e_{j_k} \mid \{e_{j_k}\} = S_{j_k} - S_{j_k - 1},\ k = 1, 2, \cdots, p\}. \tag{7.32}$$

Since μ is a modular function and $S_{j_1} \subset S_{j_2} \subset \cdots \subset S_{j_p}$, we have

$$\begin{aligned} &\mu(X) + \sum_{k=1}^{p} \mu(S_{j_k - 1}) \\ &= \mu(X \cup S_{j_p - 1}) + \sum_{k=2}^{p} \mu((X \cap S_{j_k - 1}) \cup S_{j_{k-1} - 1}) + \mu(X \cap S_{j_1 - 1}) \\ &= \sum_{k=1}^{p} \mu(S_{j_k}), \end{aligned} \tag{7.33}$$

where use is made of the fact that $X \cup S_{j_p-1} = S_{j_p}$, $X \cap S_{j_k-1} = X \cap S_{j_{k-1}}$ $(k = 2, \cdots, p)$ and $X \cap S_{j_1-1} = \emptyset$. (7.30) follows from (7.31) and (7.33). The uniqueness of ν is clear from (7.30). Q.E.D.

We see from this lemma that the problem of minimizing the modular function $\mu: \mathcal{D} \to \mathbf{R}$ is equivalent to the problem of finding a minimum-weight (lower) ideal of the poset $\mathcal{P} = (E, \preceq)$ with respect to the weight function $\nu: E \to \mathbf{R}$. The latter problem was solved by J. C. Picard [Picard76] by reducing it to a minimum-cut problem. Conversely, the reducibility of the minimum-cut problem to a problem of minimizing a modular function was shown by W. H. Cunningham [Cunningham85b].

We first show Picard's approach. Let $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$ be the graph representing the Hasse diagram of $\mathcal{P} = (E, \preceq)$, i.e., $(e_1, e_2) \in B^*(\mathcal{P})$ if and only if e_1 covers e_2 in $\mathcal{P}$. (The following procedure works if we take, as $G(\mathcal{P})$, any graph G' whose transitive closure coincides with that of $G(\mathcal{P})$, and a (transitively) closed set of G' is an ideal of $\mathcal{P}$.) Consider new vertices s^+ and s^- and sets of new arcs

$$S^+ = \{(s^+, e) \mid e \in E,\ \nu(e) < 0\}, \tag{7.34}$$

$$S^- = \{(e, s^-) \mid e \in E,\ \nu(e) > 0\}. \tag{7.35}$$

Also define the capacities $c(a)$ of arcs a in $B^*(\mathcal{P}) \cup S^+ \cup S^-$ by

$$c(a) = \begin{cases} +\infty & (a \in B^*(\mathcal{P})) \\ -\nu(e) & ((s^+, e) \in S^+) \\ \nu(e) & ((e, s^-) \in S^-). \end{cases} \tag{7.36}$$

Denote this network by $\mathcal{N} = (\hat{G} = (\hat{E}, \hat{A}), s^+, s^-, c)$, where $\hat{E} = E \cup \{s^+, s^-\}$ and $\hat{A} = B^*(\mathcal{P}) \cup S^+ \cup S^-$ (see Fig. 7.2).

For any cut $U \subseteq \hat{E}$ of the network $\mathcal{N}$ (i.e., $s^+ \in U$ and $s^- \notin U$), the capacity of the cut U is given by

$$\begin{aligned} \kappa_c(U) = \sum & \{-\nu(e) \mid e \in E - U,\ \nu(e) < 0\} \\ & + \sum \{\nu(e) \mid e \in E \cap U,\ \nu(e) > 0\} \\ & + \sum \{c(a) \mid a \in B^*(\mathcal{P}),\ \partial^+ a \in U,\ \partial^- a \in E - U\}. \end{aligned} \tag{7.37}$$

Note that for any cut U of finite capacity $\kappa_c(U)$, the third term in (7.37) vanishes and $U - \{s^+\}$ is an ideal of $\mathcal{P} = (E, \preceq)$ and, conversely, that for any ideal $I \subseteq E$ of $\mathcal{P}$ $I \cup \{s^+\}$ is a cut of finite capacity in $\mathcal{N}$.

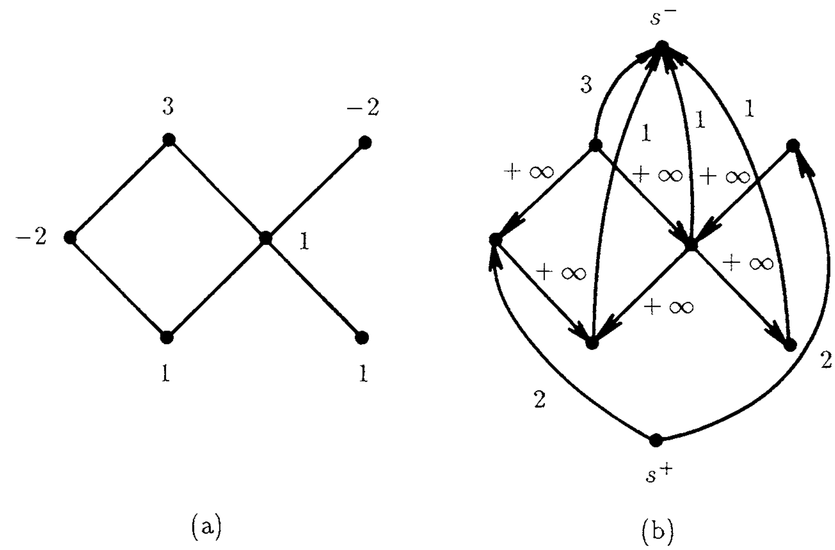

Figure 7.2: (a) A poset $\mathcal{P}$ with weights. (b) Network $\mathcal{N}$.

Furthermore, for any ideal I of $\mathcal{P}$ we have

$$\begin{aligned}\kappa_c(I \cup \{s^+\}) &= \sum\{-\nu(e) \mid e \in E - I,\ \nu(e) < 0\} \\ &\quad + \sum\{\nu(e) \mid e \in I,\ \nu(e) > 0\} \\ &= \nu(I) + \sum\{-\nu(e) \mid e \in E,\ \nu(e) < 0\}. \qquad (7.38)\end{aligned}$$

Therefore, minimizing $\kappa_c(I \cup \{s^+\})$ for cuts $I \cup \{s^+\}$ of $\mathcal{N}$ is equivalent to minimizing $\nu(I)$ for ideals I of $\mathcal{P}$.

See [Picard + Queyranne82] for practical applications of the problem of finding a minimum- or maximum-weight ideal of a poset or a minimum- or maximum-weight closed set of a graph.

Next, we consider the problem of minimizing a modular function $\mu: \mathcal{D} \to \mathbf{R}$ from the point of view of submodular analysis.

Given a submodular function $f: \mathcal{D} \to \mathbf{R}$ and a vector $x \in \mathbf{R}^E$, consider the following problem. It includes the problem of minimizing the submodular function f.

(∗) Find $A \in \mathcal{D}$ such that $x \in \partial f(A)$ and then find an expression

$$x = x_1 + x_2 \qquad (7.39)$$

such that x_1 is a convex combination of extreme points of $\partial f(A)$ and x_2 is a nonnegative linear combination of extreme vectors of $\mathrm{C}_f(A)$, the characteristic cone of $\partial f(A)$.

Note that the problem of minimizing f is equivalent to that of finding a set $A \in \mathcal{D}$ such that $\mathbf{0} \in \partial f(A)$. We consider the above problem $(*)$ in the special case when f is a modular function $\mu: \mathcal{D} \to \mathbf{R}$ with $\mu(\emptyset) = 0$.

For modular function μ, the subdifferential $\partial\mu(A)$ for each $A \in \mathcal{D}$ has a unique extreme point ν due to Theorem 6.11 and Lemma 7.5, where ν is the vector appearing in Lemma 7.5. Hence Problem $(*)$ is reduced to the problem of finding $A \in \mathcal{D}$ such that $x - \nu \in \mathrm{C}_f(A)$ and of finding a nonnegative linear combination, of $\mathrm{ER}(A)$ given by (6.57), which expresses $x - \nu$.

The proof of Theorem 6.12 suggests an algorithm as follows. The graph $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$ represents the Hasse diagram of the poset $\mathcal{P} = (E, \preceq)$.

An algorithm for Problem $(*)$

Step 1: Find a nonnegative flow $\varphi: B^*(\mathcal{P}) \to \mathbf{R}_+$ in $G(\mathcal{P})$ and nonnegative coefficients $\alpha(p^+)$ $(p^+ \in E^+)$ and $\beta(p^-)$ $(p^- \in E^-)$ such that

$$\sum_{p^+ \in E^+} \alpha(p^+)\xi_{p^+} + \sum_{p^- \in E^-} \beta(p^-)\eta_{p^-} + \partial\varphi = x - \nu. \tag{7.40}$$

(Here, ξ_{p^+} $(p^+ \in E^+)$ and η_{p^-} $(p^- \in E^-)$ are vectors in $\mathbf{R}^E$ defined by (6.54) and (6.55).) For each p^+ and p^- such that $p^+ = p^- \in E^+ \cap E^-$ we choose the values of $\alpha(p^+)$ and $\beta(p^-)$ so that $\alpha(p^+)\beta(p^-) = 0$.

Step 2: Construct a network $\hat{\mathcal{N}} = (\hat{G}(\mathcal{P}), \hat{c})$ with an underlying graph $\hat{G}(\mathcal{P}) = (E, \hat{B}(\mathcal{P}))$ and a capacity function $\hat{c}: \hat{B}(\mathcal{P}) \to \mathbf{R}_+ \cup \{+\infty\}$ defined as follows. The arc set $\hat{B}(\mathcal{P})$ of $\hat{G}(\mathcal{P})$ is given by

$$\hat{B}(\mathcal{P}) = B^*(\mathcal{P}) \cup \{(e, e') \mid (e', e) \in B^*(\mathcal{P})\} \tag{7.41}$$

and the capacity function $\hat{c}$ by

$$\hat{c}(a) = \begin{cases} \varphi(a) & (a \in B^*(\mathcal{P})) \\ +\infty & (a = (e, e'),\ (e', e) \in B^*(\mathcal{P})). \end{cases} \tag{7.42}$$

Then find a maximum flow $\psi: \hat{B}(\mathcal{P}) \to \mathbf{R}_+$ in $\hat{\mathcal{N}}$ from the entrance vertex set $E^+ - E^-$ to the exit vertex set $E^- - E^+$ such that

$$0 \le \psi(a) \le \hat{c}(a) \qquad (a \in \hat{B}(\mathcal{P})), \tag{7.43}$$

$$\partial\psi(e) = 0 \qquad (e \in E - (E^+ \cup E^-)), \tag{7.44}$$

$$\partial\psi(p^+) \leq \alpha(p^+) \qquad (p^+ \in E^+ - E^-), \tag{7.45}$$

$$-\partial\psi(p^-) \leq \beta(p^-) \qquad (p^- \in E^- - E^+). \tag{7.46}$$

(Here, the boundary operator ∂ is defined with respect to $\hat{G}(\mathcal{P})$.)

Step 3: Put

$$\begin{aligned} \varphi((e,e')) &\leftarrow \varphi((e,e')) - \psi((e,e')) + \psi((e',e)) \quad ((e,e') \in B^*(\mathcal{P})), & (7.47)\\ \alpha(p^+) &\leftarrow \alpha(p^+) - \partial\psi(p^+) \quad (p^+ \in E^+ - E^-), & (7.48)\\ \beta(p^-) &\leftarrow \beta(p^-) + \partial\psi(p^-) \quad (p^- \in E^- - E^+). & (7.49) \end{aligned}$$

Then find $A \in \mathcal{D}$ such that

$$\text{(i)}\quad \varphi(a) = 0 \quad (a \in \Delta^-(A)), \tag{7.50}$$

$$\text{(ii)}\quad \alpha(p^+) = 0 \quad (p^+ \in E^+ \cap A), \tag{7.51}$$

$$\text{(iii)}\quad \beta(p^-) = 0 \quad (p^- \in E^- - A). \tag{7.52}$$

(End)

For any $A \in \mathcal{D}$ satisfying (7.50)~(7.52) we have $x \in \partial\mu(A)$ and x is expressed as

$$x = \nu + \sum_{p^+ \in E^+} \alpha(p^+)\xi_{p^+} + \sum_{p^- \in E^-} \beta(p^-)\eta_{p^-} + \sum_{a \in B^*(\mathcal{P})} \varphi(a)\zeta_a, \tag{7.53}$$

using the obtained α, β and φ.

Step 1 can be carried out by the breadth-first search and requires linear time. Step 2 is performed by any maximum flow algorithm. Any minimum cut $U \subseteq E$ obtained by the maximum flow algorithm in Step 2 gives a desired $A \in \mathcal{D}$ to be found in Step 3 as $A = E - U$. From (7.50)~(7.52), $x - \nu$ in (7.53) is a nonnegative linear combination of $\mathrm{ER}(A)$.

When $x = \mathbf{0}$, the above algorithm solves the problem of minimizing the modular function μ. In this case $A \in \mathcal{D}$ found in Step 3 is a minimizer of μ since $\mathbf{0} \in \partial\mu(A)$.

Let us consider the problem of minimizing a modular function $\mu\colon \mathcal{D} \to \mathbf{R}$ from a polyhedral point of view. Denote by $\mathrm{P}(\mathcal{D})$ the convex hull of all the characteristic vectors χ_X $(X \in \mathcal{D})$.

Lemma 7.6: *For any $A \in \mathcal{D}$, the inequalities of all the facets of the polyhedron* $\mathrm{P}(\mathcal{D})$ *which include the vertex* χ_A *are given by*

$$x(e) \geq 0 \quad (e \in E^+ - A), \tag{7.54}$$

$$x(e) - x(e') \leq 0$$
$$(e \text{ covers } e' \text{ in } \mathcal{P} \text{ and either } e,\ e' \in A \text{ or } e,\ e' \notin A), \tag{7.55}$$

$$x(e) \leq 1 \quad (e \in E^- \cap A). \tag{7.56}$$

(Proof) The lemma follows from the fact that the dual cone (of the tangent cone) of $\mathrm{P}(\mathcal{D})$ at χ_A (regarded as the origin) is the characteristic cone $\mathrm{C}_\mu(A)$ of $\partial\mu(A)$. Notice the one-to-one correspondence between the set of facets (7.54)$\sim$(7.56) and the set $\mathrm{ER}(A)$ of the extreme rays of $\mathrm{C}_\mu(A)$ given by (6.57) with (6.54)$\sim$(6.56). Q.E.D.

Corollary 7.7: *All the facet inequalities of* $P(\mathcal{D})$ *are given by*

$$x(e) \geq 0 \quad (e \in E^+), \tag{7.57}$$

$$x(e) - x(e') \leq 0 \quad (e \text{ covers } e' \text{ in } \mathcal{P}), \tag{7.58}$$

$$x(e) \leq 1 \quad (e \in E^-). \tag{7.59}$$

The problem of minimizing the modular function $\mu\colon \mathcal{D} \to \mathbf{R}$ is reduced to the problem of minimizing the linear function

$$(\nu, x) = \sum_{e \in E} \nu(e)x(e) \tag{7.60}$$

over $\mathrm{P}(\mathcal{D})$, where ν is the vector appearing in Lemma 7.5. Therefore, it follows from Lemma 7.6 that $A \in \mathcal{D}$ is a minimizer of μ (or χ_A is a minimizer of (ν, x) over $\mathrm{P}(\mathcal{D})$) if and only if the vector $-\nu$ is expressed as a nonnegative linear combination of vectors ξ_{p^+} $(p^+ \in E^+ - A)$, ζ_a $(a \in B^*(\mathcal{P}) - \Delta^-(A))$ and η_{p^-} $(p^- \in E^- \cap A)$ which are coefficient vectors of (7.54)$\sim$(7.56), where inequalities (7.54) should be considered in the form of $-x(e) \leq 0$ $(e \in E^+ - A)$.

7.2. Submodular Programs – Constrained Optimization

We consider the problem of minimizing a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ with constraints on the domain $\mathcal{D}$ of f and discuss some other related problems. [Here we assume that $\mathbf{R}$ is the set of rationals or reals.]

(a) Lagrangian functions and optimality conditions

Suppose that a sublattice $\mathcal{D}_0$ of $\mathcal{D}$ is given. We say that a vector $a \in \mathbf{R}^E$ is *normal to* $\mathcal{D}_0$ at $A \in \mathcal{D}_0$ if for each $X \in \mathcal{D}_0$

$$a(X) - a(A) \le 0. \tag{7.61}$$

The following theorem characterizes the minimizers of f when the domain of f is restricted to a sublattice of $\mathcal{D}$.

Theorem 7.8 (cf. [Rockafellar70, Theorem 27.4]): *Let $f\colon \mathcal{D} \to \mathbf{R}$ be a submodular function and $\mathcal{D}_0$ be a sublattice of $\mathcal{D}$. Then, for $A \in \mathcal{D}_0$ we have*

$$f(A) = \min\{f(X) \mid X \in \mathcal{D}_0\} \tag{7.62}$$

if and only if there exists a subgradient $a \in \partial f(A)$ such that $-a$ is normal to $\mathcal{D}_0$ at A.

(Proof) *The "if" part*: From the assumption, we have for any $X \in \mathcal{D}_0$

$$0 \le a(X) - a(A) \le f(X) - f(A), \tag{7.63}$$

from which (7.62) follows.

The "only if" part: Define a modular function $\mu_0\colon \mathcal{D}_0 \to \mathbf{R}$ by

$$\mu_0(X) = 0 \quad (X \in \mathcal{D}_0). \tag{7.64}$$

Then, by the assumption A is a minimizer of $f_0 \equiv f + \mu_0\colon \mathcal{D}_0 \to \mathbf{R}$. It follows from Theorem 6.8 and Lemma 7.1 that

$$\mathbf{0} \in \partial f_0(A) = \partial f(A) + \partial \mu_0(A). \tag{7.65}$$

Hence, there exists a vector $a \in \partial f(A)$ such that

$$-a \in \partial \mu_0(A). \tag{7.66}$$

From (7.64), (7.66) implies that $-a$ is normal to $\mathcal{D}_0$ at A. Q.E.D.

Next, consider a constrained minimization problem for which the "feasible region" $\mathcal{D}_0$ in (7.62) is defined by a set of equations. Suppose that we are given submodular functions $f_i: \mathcal{D} \to \mathbf{R}$ $(i = 0, 1, \cdots, m)$ and that the minimum value of f_i for each $i = 1, 2, \cdots, m$ is known and equal to α_i. The feasible region $\mathcal{D}_0$ is given by

$$\mathcal{D}_0 = \{X \mid X \in \mathcal{D},\ \forall i \in \{1, 2, \cdots, m\}\colon\ f_i(X) = \alpha_i\}. \tag{7.67}$$

Here, we assume $\mathcal{D}_0 \neq \emptyset$. From Lemma 2.1, $\mathcal{D}_0$ is a sublattice of $\mathcal{D}$.

Let us consider the following constrained minimization problem:

$$\text{Minimize} \quad f_0(X) \tag{7.68a}$$

$$\text{subject to} \quad f_i(X) = \alpha_i \quad (i = 1, 2, \cdots, m). \tag{7.68b}$$

Define a function $L: \mathbf{R}^m_+ \times \mathcal{D} \to \mathbf{R}$ by

$$L(\lambda, X) = f_0(X) + \sum_{i=1}^{m} \lambda_i (f_i(X) - \alpha_i) \tag{7.69}$$

for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbf{R}^m_+$ and $X \in \mathcal{D}$. We call L the *Lagrangian function* associated with Problem (7.68). Also, we call $\lambda \in \mathbf{R}^m_+$ an *optimal Lagrange multiplier* if

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} \tag{7.70}$$

is equal to the optimal value of the objective function of Problem (7.68).

Theorem 7.9 (cf. [Rockafellar70, Theorem 28.3]): *For Problem* (7.68),

(1) $\hat{\lambda} = (\hat{\lambda}_1, \cdots, \hat{\lambda}_m)$ *is an optimal Lagrange multiplier and*

(2) $\hat{X}$ *is an optimal solution of Problem* (7.68)

if and only if

(i) $\hat{\lambda} \in (\hat{\lambda}_1, \cdots, \hat{\lambda}_m) \in \mathbf{R}^m_+$,

(ii) $\hat{X}$ *is a feasible solution of Problem* (7.68) *and*

(iii) $\mathbf{0} \in \partial f_0(\hat{X}) + \hat{\lambda}_1 \partial f_1(\hat{X}) + \cdots + \hat{\lambda}_m \partial f_m(\hat{X})$.

(Proof) First, note that for each $i = 1, \cdots, m$, $\partial f_i(X)$ and $\partial f_0(X)$ has the same characteristic cone, since the characteristic cone is determined by the underlying distributive lattice $\mathcal{D}$ alone. Therefore,

$$\partial f_0(X) = \partial f_0(X) + 0^+ \partial f_i(X) \quad (i = 1, 2, \cdots, m). \tag{7.71}$$

Hence, from Theorem 6.8 and Lemma 7.1, (iii) is equivalent to

$$\mathbf{0} \in \partial_X L(\hat{\lambda}, \hat{X}), \tag{7.72}$$

where $\partial_X L(\hat{\lambda}, \hat{X})$ denotes the subdifferential of the submodular function $L(\hat{\lambda}, \cdot)\colon \mathcal{D} \to \mathbf{R}$ at $\hat{X}$.

The "if" part: From (i)$\sim$(iii) and (7.72) we have

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} = L(\hat{\lambda}, \hat{X}) = f_0(\hat{X}), \tag{7.73}$$

while we have for any feasible solution Y

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} \le L(\hat{\lambda}, Y) = f_0(Y). \tag{7.74}$$

Hence (1) and (2) follow.

The "only if" part: From (1) and (2) we have

$$\hat{X} \in \mathcal{D}_0, \tag{7.75}$$

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} = f_0(\hat{X}) = L(\hat{\lambda}, \hat{X}). \tag{7.76}$$

Therefore, we have (7.72), or (iii). (i) and (ii) trivially follow from (1) and (2). Q.E.D.

The above proof almost parallels that of the corresponding theorem for ordinary convex functions ([Rockafellar70, Theorem 28.3]). It should, however, be noted that the above proof heavily depends on the results in Section 6.2, especially Theorem 6.8.

Define a function $p\colon \mathbf{R}^m_+ \to \mathbf{R} \cup \{+\infty\}$ by

$$p(u) = \min\{f_0(X) \mid X \in \mathcal{D},\ \forall i \in \{1, \cdots, m\}\colon\ f_i(X) - \alpha_i \le u_i\}, \tag{7.77}$$

where $u = (u_1, \cdots, u_m) \in \mathbf{R}^m_+$. We define $p(u) = +\infty$ for u for which there exists no $X \in \mathcal{D}$ such that $f_i(X) - \alpha_i \le u_i$ for all $i = 1, \cdots, m$. We call p the *perturbation function* associated with Problem (7.68).

Theorem 7.10: *For the perturbation function p associated with Problem (7.68) we have for each $\lambda \in \mathbf{R}_+^m$*

$$\begin{aligned}&\min\{p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m \mid u \in \mathbf{R}_+^m\}\\&= \min\{L(\lambda, X) \mid X \in \mathcal{D}\}.\end{aligned} \tag{7.78}$$

(Proof) Suppose that $\hat{X} \in \mathcal{D}$ is a minimizer of $L(\lambda, X)$ in $X \in \mathcal{D}$. Then, from the definition of $p(u)$,

$$L(\lambda, \hat{X}) = p(\overline{u}) + \lambda_1 \overline{u}_1 + \cdots + \lambda_m \overline{u}_m, \tag{7.79}$$

where $\overline{u}_i = f_i(\hat{X}) - \alpha_i$ $(i = 1, \cdots, m)$. Hence we have

$$\begin{aligned}&\min\{p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m \mid u \in \mathbf{R}_+^m\}\\&\leq \min\{L(\lambda, X) \mid X \in \mathcal{D}\}.\end{aligned} \tag{7.80}$$

On the other hand, suppose that $\hat{u} \in \mathbf{R}_+^m$ is a minimizer of $p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m$ in $u \in \mathbf{R}_+^m$. Then, there exists an $X_0 \in \mathcal{D}$ such that

$$p(\hat{u}) = f_0(X_0), \tag{7.81}$$

$$f_i(X_0) - \alpha_i \leq \hat{u}_i \quad (i = 1, \cdots, m). \tag{7.82}$$

Since $\lambda \in \mathbf{R}_+^m$, we have from (7.81) and (7.82)

$$p(\hat{u}) + \lambda_1 \hat{u}_1 + \cdots + \lambda_m \hat{u}_m \geq L(\lambda, X_0). \tag{7.83}$$

Therefore,

$$\begin{aligned}&\min\{p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m \mid u \in \mathbf{R}_+^m\}\\&\geq \min\{L(\lambda, X) \mid X \in \mathcal{D}\}.\end{aligned} \tag{7.84}$$

The present theorem follows from (7.80) and (7.84). Q.E.D.

In the proof of Theorem 7.10 we have already shown the following.

Theorem 7.11: *For each $\lambda \in \mathbf{R}_+^m$,*

(a) *if $\hat{X} \in \mathcal{D}$ is a minimizer of $L(\lambda, X)$ in $X \in \mathcal{D}$, then $\hat{u} = (\hat{u}_1, \cdots, \hat{u}_m)$ given by*

$$\hat{u}_i = f_i(\hat{X}) - \alpha_i \quad (i = 1, \cdots, m) \tag{7.85}$$

is a minimizer of $p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m$ in $u \in \mathbf{R}_+^m$;

(b) *if* $\hat{u} = (\hat{u}_1, \cdots, \hat{u}_m)$ *is a minimizer of* $p(u) + \lambda_1 u_1 + \cdots + \lambda_m u_m$ *in* $u \in \mathbf{R}^m_+$*, then there exists an* $\hat{X} \in \mathcal{D}$ *such that*

$$p(\hat{u}) = f_0(\hat{X}), \tag{7.86}$$

$$f_i(\hat{X}) - \alpha_i \le \hat{u}_i \quad (i = 1, \cdots, m), \tag{7.87}$$

and $\hat{X}$ *is a minimizer of* $L(\lambda, X)$ *in* $X \in \mathcal{D}$*. Here, for each* $i = 1, \cdots, m$*,* (7.87) *holds with equality if* $\lambda_i > 0$*, while* $\lambda_i = 0$ *if* (7.87) *for* i *holds with strict inequality.*

Furthermore, we have

Theorem 7.12 (cf. [Rockafellar70, Theorem 29.1]): *A vector* $\hat{\lambda} \in \mathbf{R}^m_+$ *is an optimal Lagrange multiplier of Problem* (7.68) *if and only if*

$$\min\{p(u) + \hat{\lambda}_1 u_1 + \cdots + \hat{\lambda}_m u_m \mid u \in \mathbf{R}^m_+\} = p(\mathbf{0}). \tag{7.88}$$

(Proof) (7.88) means that the zero vector $\mathbf{0} \in \mathbf{R}^m_+$ is a minimizer of $p(u) + \hat{\lambda}_1 u_1 + \cdots + \hat{\lambda}_m u_m$ in $u \in \mathbf{R}^m_+$. Hence, the present theorem follows from Theorems 7.9$\sim$7.11. Q.E.D.

We see from Theorems 7.11 and 7.12 that if $\hat{\lambda} \in \mathbf{R}^m_+$ is an optimal Lagrange multiplier, then any $\lambda \in \mathbf{R}^m_+$ such that $\hat{\lambda} \le \lambda$ is also an optimal one. Note that (7.87) for i such that $\hat{u}_i = 0$, holds with equality since $f_i(X) - \alpha_i \ge 0$ for any $X \in \mathcal{D}$.

An algorithm for finding an optimal solution and an optimal Lagrange multiplier for Problem (7.68) is given as follows. For simplicity, we consider the case when $m = 1$.

An algorithm for solving Problem (7.68) (with $m = 1$)

Step 1: Let a_0 be an upper bound of f_0 such that $a_0 > f_0(X)$ for all $X \in \mathcal{D}$ (with strict inequality) and let α_0 be a lower bound of f_0. Also let a_1 be an upper bound of f_1 such that $a_1 > \alpha_1$. Put $\lambda \leftarrow (a_0 - \alpha_0)/(a_1 - \alpha_1)$.

Step 2: Put $\hat{X} \leftarrow$ a minimizer of $L(\lambda, X) = f_0(X) + \lambda(f_1(X) - \alpha_1)$ in $X \in \mathcal{D}$.

Step 3: If $f_1(\hat{X}) - \alpha_1 = 0$, then stop ($\hat{X}$ is an optimal solution and λ is an optimal Lagrange multiplier). Otherwise, put $\lambda \leftarrow (a_0 - f_0(\hat{X}))/(f_1(\hat{X}) - \alpha_1)$ and go back to Step 2.

(End)

The validity of the above algorithm follows from Theorems 7.8, 7.11 and 7.12. The case when $m > 1$ can also be treated by the algorithm by setting $f_1 \leftarrow f_1 + f_2 + \cdots + f_m$. If $\min\{f_1(X) + f_2(X) + \cdots + f_m(X) \mid X \in \mathcal{D}\} \neq \alpha_1 + \alpha_2 + \cdots + \alpha_m$, then there is no feasible solution.

The upper bounds of the submodular functions required in Step 1 are obtained by adapting (3.89). When f_i $(i = 0, 1)$ are integer-valued, the above algorithm terminates after repeating the cycle of Steps 2 and 3 at most $2\min\{a_0 - \alpha_0, a_1 - \alpha_1\}$ times, where upper bounds a_i $(i = 0, 1)$ and a lower bound α_0 are chosen to be integral.

For each $\lambda \in \mathbf{R}_+^m$ denote by $\mathcal{L}(\lambda)$ the set of minimizers of the Lagrangian function

$$L(\lambda, X) = f_0(X) + \lambda_1 f_1(X) + \cdots + \lambda_m f_m(X) \tag{7.89}$$

in $X \in \mathcal{D}$. $\mathcal{L}(\lambda)$ is a sublattice of $\mathcal{D}$. Because of the finiteness character of the problem there are a finite number of distinct $\mathcal{L}(\lambda)$'s $(\lambda \in \mathbf{R}_+^m)$. The structure of these $\mathcal{L}(\lambda)$'s is closely related to the concept of *principal partition* ([Kishi + Kajitani68], [Iri71], [Tomi76], [Narayanan74], [Fuji78c], [Iri79], [Nakamura + Iri81], [Iri84], [Nakamura88a], [Tomi + Fuji82]), which will be discussed in the subsequent subsection.

(b) Related problems

We discuss other problems related to submodular programs.

(b.1) The principal partition [Iri79], [Nakamura+Iri81], [Tomi+Fuji82]

Let us consider the Lagrangian function $L(\lambda, X)$ of (7.69) for the case when $m = 1$ and we suppress the term α_1 in $L(\lambda, X)$. We suppose that $(\mathcal{D}_i, f_i)$ $(i = 0, 1)$ are submodular systems on E, that $\mathcal{D}_1$ is a Boolean lattice (i.e., $\mathcal{D}_1 = \overline{\mathcal{D}_1}$ $(= \{E - X \mid X \in \mathcal{D}_1\})$), and that f_1 is monotone nonincreasing. The monotonicity of f_1 plays an essential rôle in the following argument. Define $\mathcal{D} = \mathcal{D}_0 \cap \mathcal{D}_1$.

Consider the Lagrangian function

$$L(\lambda, X) = f_0(X) + \lambda f_1(X) \tag{7.90}$$

for $\lambda \geq 0$ and $X \in \mathcal{D}$. We also define $L(\lambda, X)$ for $\lambda < 0$ as

$$L(\lambda, X) = f_0(X) + \lambda {f_1}^{\#}(X), \tag{7.91}$$

where ${f_1}^{\#}$ is the dual supermodular function of f_1, i.e., ${f_1}^{\#}(X) = f_1(E) - f_1(E - X)$ $(X \in \mathcal{D}_1)$. Note that for any $\lambda \in \mathbf{R}$ $L(\lambda, X)$ is a submodular function in $X \in \mathcal{D}$. For each $\lambda \in \mathbf{R}$ let $\mathcal{L}(\lambda)$ be the set of minimizers of $L(\lambda, X)$ in $X \in \mathcal{D}$. $\mathcal{L}(\lambda)$ is a sublattice of $\mathcal{D}$.

Theorem 7.13 ([Tomi + Fuji82]): *Define*

$$\mathcal{L}^* = \bigcup_{\lambda \in \mathbf{R}} \mathcal{L}(\lambda). \tag{7.92}$$

$\mathcal{L}^*$ is a sublattice of $\mathcal{D}$. More precisely, for any λ and λ' with $\lambda \leq \lambda'$ and for any $X \in \mathcal{L}(\lambda)$ and $X' \in \mathcal{L}(\lambda')$, we have

$$X \cap X' \in \mathcal{L}(\lambda), \qquad X \cup X' \in \mathcal{L}(\lambda'). \tag{7.93}$$

(Proof) If $\lambda = \lambda'$, (7.93) holds.

Case 1: $0 \leq \lambda < \lambda'$

For any $X \in \mathcal{L}(\lambda)$ and $X' \in \mathcal{L}(\lambda')$ we have

$$\begin{aligned}
&L(\lambda, X) + L(\lambda', X') \\
&= f_0(X) + \lambda f_1(X) + f_0(X') + \lambda' f_1(X') \\
&\geq f_0(X \cup X') + \lambda' f_1(X \cup X') + f_0(X \cap X') + \lambda f_1(X \cap X') \\
&\qquad + (\lambda' - \lambda)(f_1(X \cap X') - f_1(X)) \\
&\geq L(\lambda', X \cup X') + L(\lambda, X \cap X')
\end{aligned} \tag{7.94}$$

due to the submodularity of f_0 and f_1 and the monotonicity of f_1. It follows from (7.94) that

$$L(\lambda', X \cup X') = L(\lambda', X'), \tag{7.95}$$

$$L(\lambda, X \cap X') = L(\lambda, X) \tag{7.96}$$

(and $f_1(X) = f_1(X \cap X')$). We thus have (7.93).

Case 2: $\lambda < 0 \leq \lambda'$

Similarly as (7.94), we have

$$
\begin{aligned}
&L(\lambda, X) + L(\lambda', X') \\
&= f_0(X) + \lambda f_1{}^{\#}(X) + f_0(X') + \lambda' f_1(X') \\
&\geq f_0(X \cup X') + \lambda' f_1(X \cup X') + f_0(X \cap X') + \lambda f_1{}^{\#}(X \cap X') \\
&\qquad + \lambda'(f_1(X \cap X') - f_1(X)) + \lambda(f_1{}^{\#}(X) - f_1{}^{\#}(X \cap X')) \\
&\geq L(\lambda', X \cup X') + L(\lambda, X \cap X'). \qquad (7.97)
\end{aligned}
$$

From this we have (7.93).

Case 3: $\lambda < \lambda' < 0$

Similarly, we have

$$
\begin{aligned}
&L(\lambda, X) + L(\lambda', X') \\
&= f_0(X) + \lambda f_1{}^{\#}(X) + f_0(X') + \lambda' f_1{}^{\#}(X') \\
&\geq f_0(X \cup X') + \lambda' f_1{}^{\#}(X \cup X') + f_0(X \cap X') + \lambda f_1{}^{\#}(X \cap X') \\
&\qquad + (\lambda' - \lambda)(f_1{}^{\#}(X') - f_1{}^{\#}(X \cup X')) \\
&\geq L(\lambda', X \cup X') + L(\lambda, X \cap X'). \qquad (7.98)
\end{aligned}
$$

Hence we have (7.93). Q.E.D.

Denote by $S^+(\lambda)$ the unique maximal element of $\mathcal{L}(\lambda)$ and by $S^-(\lambda)$ the unique minimal element of $\mathcal{L}(\lambda)$ for each $\lambda \in \mathbf{R}$.

Theorem 7.14: *For any λ and λ' with $\lambda < \lambda'$ we have*

$$
S^+(\lambda) \subseteq S^+(\lambda'), \qquad S^-(\lambda) \subseteq S^-(\lambda'). \qquad (7.99)
$$

(Proof) For any $X \in \mathcal{L}(\lambda)$ and $X' \in \mathcal{L}(\lambda')$ we have from Theorem 7.13

$$
X \cup S^+(\lambda') \in \mathcal{L}(\lambda'), \qquad S^-(\lambda) \cap X' \in \mathcal{L}(\lambda). \qquad (7.100)
$$

This implies $X \subseteq S^+(\lambda')$ $(X \in \mathcal{L}(\lambda))$ and $S^-(\lambda) \subseteq X'$ $(X' \in \mathcal{L}(\lambda'))$. Hence we have (7.99). Q.E.D.

We further suppose that f_1 is monotone decreasing. Then, for a sufficiently large $\lambda > 0$ we have $\mathcal{L}(\lambda) = \{E\}$ and for a sufficiently small $\lambda < 0$, $\mathcal{L}(\lambda) = \{\emptyset\}$.

Theorem 7.15: *Suppose that f_1 is monotone decreasing. Then, there exists a sequence of reals*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p \tag{7.101}$$

with $p \leq |E|$ such that the distinct $\mathcal{L}(\lambda)$ $(\lambda \in \mathbf{R})$ are given by

$$\mathcal{L}(\lambda_i) \quad (i = 1, 2, \cdots, p), \tag{7.102}$$

$$\{S^+(\lambda_i)\}\ (= \{S^-(\lambda_{i+1})\}) \quad (i = 1, 2, \cdots, p-1), \tag{7.103}$$

$$\{S^-(\lambda_1)\}\ (= \{\emptyset\}), \qquad \{S^+(\lambda_p)\}\ (= \{E\}). \tag{7.104}$$

For any $i \in \{1, 2, \cdots, p-1\}$ and any λ such that $\lambda_i < \lambda < \lambda_{i+1}$ we have

$$\mathcal{L}(\lambda) = \{S^+(\lambda_i)\} = \{S^-(\lambda_{i+1})\}. \tag{7.105}$$

Also,

$$\mathcal{L}(\lambda) = \begin{cases} \{\emptyset\} & (\lambda < \lambda_1) \\ \{E\} & (\lambda_p < \lambda). \end{cases} \tag{7.106}$$

(Proof) Because of the finiteness character there exist finitely many distinct $\mathcal{L}(\lambda)$ $(\lambda \in \mathbf{R})$. Choose any $\hat{\lambda} \in \mathbf{R}$. If $\mathcal{L}(\hat{\lambda})$ contains more than one element, there exist some distinct $X,\ X' \in \mathcal{L}(\hat{\lambda})$ with $X \subset X'$ and

$$f_0(X) + \hat{\lambda} f_1(X) = f_0(X') + \hat{\lambda} f_1(X'). \tag{7.107}$$

Since $f_1(X) > f_1(X')$ by the monotonicity assumption, the value of $\hat{\lambda}$ is uniquely determined from (7.107). If $\mathcal{L}(\hat{\lambda})$ contains only one element, then from the finiteness character there exists an open interval (λ', λ'') such that $\hat{\lambda} \in (\lambda', \lambda'')$ and

$$\mathcal{L}(\lambda) = \mathcal{L}(\hat{\lambda}) \quad (\lambda \in (\lambda', \lambda'')). \tag{7.108}$$

It follows that there exists a finite sequence of reals

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p \tag{7.109}$$

such that distinct $\mathcal{L}(\lambda)$ $(\lambda \in \mathbf{R})$ are given by

$$\mathcal{L}(\lambda_i) \quad (i = 1, 2, \cdots, p), \tag{7.110}$$

$$\mathcal{L}(\lambda) \quad (\lambda \in (\lambda_i, \lambda_{i+1}),\ i = 0, 1, \cdots, p), \tag{7.111}$$

where $\lambda_0 \equiv -\infty$, $\lambda_{p+1} \equiv +\infty$, $\mathcal{L}(\lambda)$'s are the same in each interval $(\lambda_i, \lambda_{i+1})$ $(i = 0, 1, \cdots, p)$, $|\mathcal{L}(\lambda_i)| \geq 2$ $(i = 1, 2, \cdots, p)$ and $|\mathcal{L}(\lambda)| = 1$ $(\lambda \in (\lambda_i, \lambda_{i+1})$, $i = 0, 1, \cdots, p)$. Moreover, for each $i = 1, 2, \cdots, p$, because of the finiteness character there exists a (sufficiently small) positive number ϵ such that

$$\mathcal{L}(\lambda_i - \epsilon) \subseteq \mathcal{L}(\lambda_i), \tag{7.112}$$

$$\mathcal{L}(\lambda_i + \epsilon) \subseteq \mathcal{L}(\lambda_i). \tag{7.113}$$

Since f_1 is monotone decreasing, we have from (7.112) and (7.113)

$$S^-(\lambda_i) \in \mathcal{L}(\lambda_i - \epsilon), \tag{7.114}$$

$$S^+(\lambda_i) \in \mathcal{L}(\lambda_i + \epsilon). \tag{7.115}$$

From (7.114) and (7.115) we have (7.103)$\sim$(7.106). Also, since the set of the quotients $S^+(\lambda_i) - S^+(\lambda_{i-1})$ $(i = 1, 2, \cdots, p)$ with $S^+(\lambda_0) \equiv \emptyset$ is a partition of E into nonempty subsets of E due to Theorem 7.14 and (7.103)$\sim$(7.106), we have $p \leq |E|$. Q.E.D.

The λ_i $(i = 1, 2, \cdots, p)$ in (7.101) are called *critical values* for the pair of submodular systems $(\mathcal{D}_0, f_0)$ and $(\mathcal{D}_1, f_1)$. Denote $\mathbf{S}_0 = (\mathcal{D}_0, f_0)$ and $\mathbf{S}_1 = (\mathcal{D}_1, f_1)$. Submodular systems $\mathbf{S}_i$ $(i = 0, 1)$ are decomposed according to the distributive lattice $\mathcal{L}^* = \bigcup_{\lambda \in \mathbf{R}} \mathcal{L}(\lambda)$ as follows. Choose any maximal chain

$$\mathcal{C}\colon\ \emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = E \tag{7.116}$$

of $\mathcal{L}^*$ and then decompose $\mathbf{S}_i$ $(i = 0, 1)$ into their minors

$$\mathbf{S}_i \cdot A_j / A_{j-1} \quad (j = 1, 2, \cdots, k), \tag{7.117}$$

where $\mathbf{S}_i \cdot A_j / A_{j-1}$ is the set minor of $\mathbf{S}_i$ obtained by restricting $\mathbf{S}_i$ to A_j and contracting A_{j-1}. Such a set of decompositions of $\mathbf{S}_i$ $(i = 0, 1)$ is called the *principal partition* of the pair of $\mathbf{S}_i$ $(i = 0, 1)$. By the poset on the partition $\{A_j - A_{j-1} \mid j = 1, 2, \cdots, k\}$ of E which is uniquely determined by $\mathcal{L}^*$ (see Section 3.2.a), the corresponding poset structure is defined on the set of minors (7.117) for each $i = 0, 1$. We can show that the decompositions (7.117) do not depend on the choice of a maximal chain in $\mathcal{L}^*$ ([Nakamura + Iri81], [Tomi + Fuji82]), due to Theorem 7.17 shown below.

Lemma 7.16: *Let* $\mu\colon \mathcal{D}_0 \to \mathbf{R}$ *be a modular function on a distributive lattice* $\mathcal{D}_0 \subseteq 2^E$ *with* $\emptyset, E \in \mathcal{D}_0$. *We have* $\mu(X) = 0$ *for all* $X \in \mathcal{D}$ *if* *and only if* $\mu(S_i) = 0$ $(i = 0, 1, \cdots, k)$ *for an arbitrary maximal chain* $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E$ *of* $\mathcal{D}_0$.

(Proof) This lemma immediately follows from Lemma 7.5 and Corollary 3.10. Q.E.D.

Theorem 7.17 [Nakamura+Iri81] (also see [Tomi+Fuji82]): *For a submodular system* $(\mathcal{D}, f)$ *on* E *suppose that* f *is modular on a sublattice* $\mathcal{D}_0$ *of* $\mathcal{D}$ *with* $\emptyset, E \in \mathcal{D}_0$. *For a maximal chain* $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E$ *of* $\mathcal{D}_0$ *consider minors* $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ $(i = 1, 2, \cdots, k)$. *Then, the set of these minors is independent of the choice of a maximal chain of* $\mathcal{D}_0$.

(Proof) For a vector $x \in \mathbf{R}^E$ the following two statements are equivalent (see Lemma 3.1):

(i) $x^{S_i - S_{i-1}}$ is a base of $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ for each $i = 1, 2, \cdots, k$.

(ii) x is a base of $(\mathcal{D}, f)$ and $f(S_i) - x(S_i) = 0$ for each $i = 0, 1, \cdots, k$.

Since $f - x$: $\mathcal{D}_0 \to \mathbf{R}$ is modular on $\mathcal{D}_0$, we see from Lemma 7.16 that (ii) is also equivalent to

(iii) x is a base of $(\mathcal{D}, f)$ and $f(X) - x(X) = 0$ for each $X \in \mathcal{D}_0$.

It follows from the equivalence between (i) and (iii) that the set of minors $(\mathcal{D}, f) \cdot S_i/S_{i-1}$ $(i = 1, 2, \cdots, k)$ does not depend on the choice of a maximal chain of $\mathcal{D}_0$. Q.E.D.

The concept of principal partition was originated by G. Kishi and Y. Kajitani [Kishi + Kajitani68] for graphs, where $\mathbf{S}_0$ is a graphic matroid and $\mathbf{S}_1 = (2^E, -\mathbf{1})$ with uniform modular function $-\mathbf{1}(X) = -|X|$ $(X \subseteq E)$. It was generalized to a pair of graphs [Ozawa74]; to a pair of a matrix and a uniform modular function [Iri69b,71]; to a pair of a matroid and a uniform modular function [Narayanan74], [Tomi76] (also [Bruno + Weinberg71]); to a pair of a polymatroid and a positive modular function [Fuji78c], [Fuji80b]; to a pair of polymatroids [Iri79]; to a pair of a submodular system and a modular function [Fuji80c]; and to a pair of submodular systems [Nakamura + Iri81], [Tomi80d], [Tomi + Fuji82]. The concept has effectively been applied to electrical network analysis [Iri + Tomi75], [Narayanan74], network flows [Fuji80b], structure and scene analyses [Sugihara82,86], the determination of the concept structure [Sugihara + Iri80], etc. (Also see [Iri79], [Iri83], [Iri + Fuji81], [Tomi + Fuji82]).

Example b.1: The principal partition of a graph

Consider a graph $G = (V, E)$ shown in Fig. 7.3. Let f_0: $2^E \to \mathbf{Z}_+$ be the rank function r of the graph and let $f_1: 2^E \to \mathbf{Z}_+$ be the modular function which corresponds to the negative of the uniform vector $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbf{R}^E$. We have $L(\lambda, X) = r(X) - \lambda|X|$. The set (7.117) of minors $\mathbf{S}_0 \cdot A_j/A_{j-1}$ $(j = 1, 2, \cdots, 10)$ together with the poset structure is given in Fig. 7.4.

The principal partition shown in Fig. 7.4 is the decomposition in the sense of Tomizawa and Narayanan. Note that each minor $\mathbf{S}_0 \cdot A_j/A_{j-1}$, considered as a polymatroid with a graphic rank function, has the uniform base $u_{\lambda_j} \equiv (\lambda_j, \lambda_j, \cdots, \lambda_j) \in \mathbf{R}^{A_j - A_{j-1}}$ with the critical value λ_j corresponding to the minor. The direct sum of these bases u_{λ_j} of minors $\mathbf{S}_0 \cdot A_j/A_{j-1}$ $(j = 1, 2, \cdots, 10)$ is a base of the original graphic polymatroid and, decomposing the exchangeability graph associated with this base into strongly connected components, we have the principal partition shown in Fig. 7.4 (see [Fuji80c]).

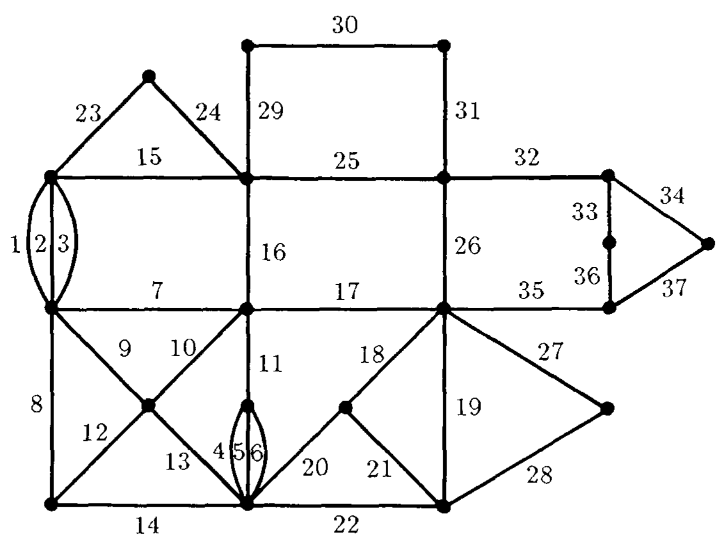

Figure 7.3: A graph.

For each graph $G_j = (V_j, E_j)$ representing the minor $\mathbf{S}_0 \cdot A_j/A_{j-1}$ $(j = 1, 2, \cdots, 10)$, the critical value λ_j is given by

$$\lambda_j = (|V_j| - 1)/|E_j|, \tag{7.118}$$

so that $1/\lambda_j$ can be regarded as the "density" of $G_j = (V_j, E_j)$. A minor with the smallest critical value gives a subgraph of G with the maximum density (cf. [Goldberg83]).

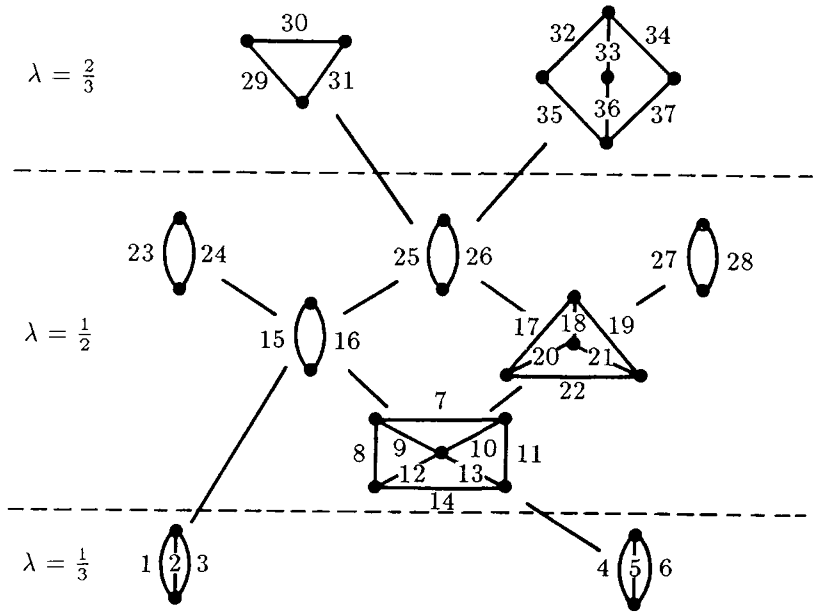

Figure 7.4: The principal partition of the graph in Fig. 7.3.

The principal partition of a graph $G = (V, E)$ in the sense of Kishi and Kajitani is the partition of E into three parts E_+, E_0, E_- such that $G_+ = G \times E_+$, $G_0 = (G - E_+)/E_-$, and $G_- = G \cdot E_-$ have the critical values greater than $\frac{1}{2}$, equal to $\frac{1}{2}$, and less than $\frac{1}{2}$, respectively. Based on this tri-partition, we can determine the *topological degree of freedom* of the (electrical) network represented by G. The topological degree of freedom is the minimum number of current variables and voltage variables whose values uniquely determine the value of the current variable or the voltage variable of each arc through Kirchhoff's current and voltage laws. The topological degree of freedom of G is given by $\min\{r(X) + |E - X| - r^{\#}(E - X) \mid X \subseteq E\}$ in terms of the rank function r of G. Note that $|E - X| - r^{\#}(E - X)$ is the nullity of the contraction G/X and that $r(X) + |E - X| - r^{\#}(E - X) = 2r(X) - |X| + |E| - r(E)$. Therefore, the problem is reduced to minimize $2r(X) - |X|$ or $r(X) - \frac{1}{2}|X|$ and the topological degree of freedom of G is equal to the sum of the rank of G_-, the nullity of G_+, and the rank (or nullity) of G_0 (see [Iri68,69], [Kishi + Kajitani68] and [Ohtsuki + Ishizaki + Watanabe68]).

Kishi and Kajitani's tri-partition also furnishes the solution of *Shannon's switching game* (see [Edm65b], [Bruno + Weinberg71]). Shannon's switching game is described as follows. Two players play on a graph G with a special reference edge e_0. Alternately choosing one edge of G, one player, called a short player, tries to construct a cycle consisting of the reference edge e_0 and some of the arcs already chosen by the player, and the other player, called a cut player, tries to construct a cutset (cocircuit) containing e_0. The short (or cut) player wins if he succeeds in constructing such a cycle (or cutset). If e_0 belongs to G_- (or G_+), then the short (or cut) player can win; and if e_0 belongs to G_0, the first-move player can win (see [Bruno + Weinberg71]).

Example b.2: The principal partition of a polymatroid induced by a multi-terminal network

Consider a capacitated single-source multiple-sink network $\mathcal{N} = (G = (V, A), c, s^+, S^-)$ shown in Fig. 7.5, where the label attached to an arc $a \in A$ denotes its capacity $c(a)$, s^+ is the source and $S^- = \{s_i^- \mid i = 1, 2, \cdots, 6\}$ is the set of the sinks.

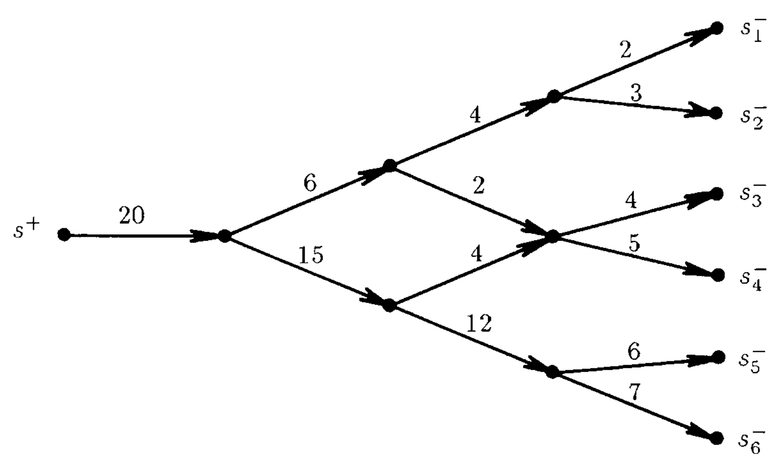

Figure 7.5: A capacitated single-source multiple-sink network.

For each feasible flow φ from s^+ to S^- in $\mathcal{N}$ define $\partial^-\varphi \in \mathbf{R}^{S^-}$ by

$$\partial^-\varphi(s_i^-) = -\partial\varphi(s_i^-) \quad (i = 1, 2, \cdots, 6). \tag{7.119}$$

Then the set of vectors $\partial^-\varphi$ for all feasible flows φ forms the independence polyhedron of a polymatroid (see Section 2.2). Denote this polymatroid by $\mathbf{P}(\mathcal{N})$. Let f_0: $2^{S^-} \to \mathbf{R}_+$ be the rank function of $\mathbf{P}(\mathcal{N})$ and f_1: $2^{S^-} \to \mathbf{R}_+$ be the modular function corresponding to the negative of the uniform vector $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbf{R}^{S^-}$. Hence, we have

$$L(\lambda, X) = f_0(X) - \lambda|X|. \tag{7.120}$$

Then, associated with this pair of f_0 and f_1, the principal partition of $\mathbf{P}(\mathcal{N})$ is given as in Fig. 7.6. A base $\partial^-\varphi$ of $\mathbf{P}(\mathcal{N})$ is shown in Fig. 7.7. Note that the principal partition of $\mathbf{P}(\mathcal{N})$ is obtained by decomposing the exchangeability graph associated with the base $\partial^-\varphi$. A characterization of such a base will be given in Section 9.2. Informally, such a base has as "uniform" components as possible.

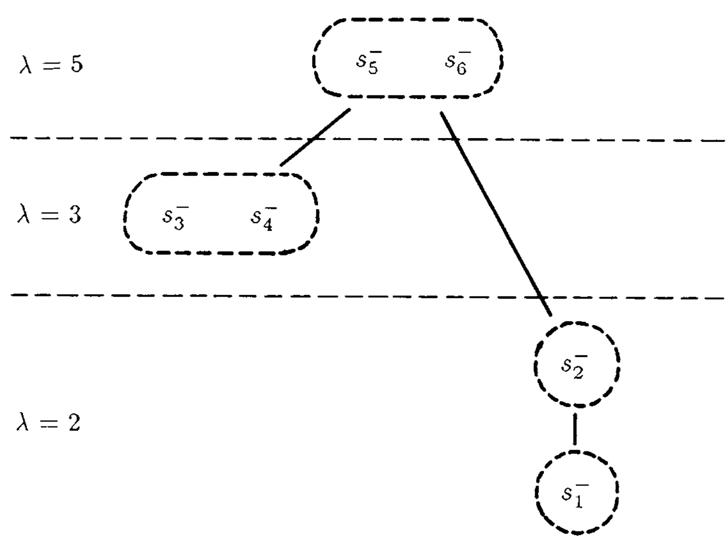

Figure 7.6: The principal partition of polymatroid $\mathbf{P}(\mathcal{N})$.

Example b.3: The principal partition of a pair of polymatroids ([Iri79, 84], [Nakamura + Iri81], [Nakamura86b])

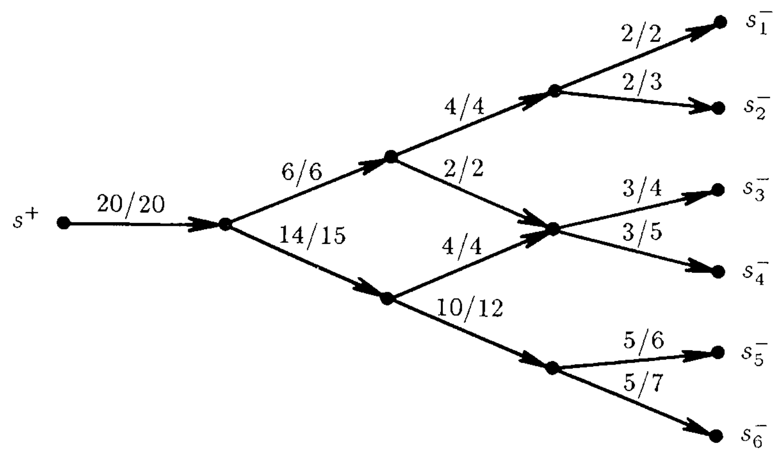

Figure 7.7: A maximal flow φ in $\mathcal{N}$ (each arc label denotes $\varphi(a)/c(a)$ $(a \in A)$).

Let (E, ρ_i) $(i = 0, 1)$ be polymatroids and suppose that $\rho_1\colon 2^E \to \mathbf{R}$ is monotone increasing. Define

$$L(\lambda, X) = \rho_0(X) + \lambda\rho_1(E - X) \tag{7.121}$$

for $\lambda \geq 0$ and $X \subseteq E$, where we may consider (7.121) as the Lagrangian function $L(\lambda, X) = \rho_0(X) + \lambda(\rho_1(E - X) - \rho_1(E))$ with the term $-\lambda\rho(E)$ being omitted. Let the critical values be given by

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p. \tag{7.122}$$

Choose a maximal chain

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{7.123}$$

of $\mathcal{L}^*$ in (7.92) for (7.121). Note that $\mathcal{C}$ is a concatenation of maximal chains of $\mathcal{L}(\lambda_i)$ $(i = 1, 2, \cdots, p)$, due to Theorem 7.15. It follows from Theorem 4.10 that there exists a base b_i of polymatroid (E, ρ_i) for each $i = 0, 1$ such that for each $j = 1, 2, \cdots, k$ we have $b_0^{S_j - S_{j-1}} = \lambda_{j^*} b_1^{S_j - S_{j-1}}$ and this vector $b_0^{S_j - S_{j-1}}$ is a common base of the minors $(E, \rho_0) \cdot S_j / S_{j-1}$ and $(E, \lambda_{j^*}\rho_1) \cdot (E - S_{j-1})/(E - S_j)$, where j^* denotes the integer in $\{1, 2, \cdots, p\}$ such that $S_{j-1}, S_j \in \mathcal{L}(\lambda_{j^*})$. We can easily see from Theorem 4.10 that for any $\lambda \geq 0$ $b_0 \wedge \lambda b_1 = (\min\{b_0(e), \lambda b_1(e)\} \mid e \in E)$ is a maximum common subbase (independent vector) of polymatroids (E, ρ_0) and (E, ρ_1). The pair of the bases b_0 and b_1 is called a *universal pair of bases* ([Nakamura81] and [Tomi80d]). Such a universal pair is not necessarily unique.

Note that

$$b_0^{S^-(\lambda)} < \lambda b_1^{S^-(\lambda)}, \quad b_0^{S^+(\lambda)-S^-(\lambda)} = \lambda b_1^{S^+(\lambda)-S^-(\lambda)}, \quad b_0^{E-S^+(\lambda)} > \lambda b_1^{E-S^+(\lambda)} \tag{7.124}$$

and

(i) $b_0^{S^-(\lambda)}$ is a base of $(E, \rho_1) \cdot S^-(\lambda)$ and a subbase of $(E, \lambda\rho_2)/(E - S^-(\lambda))$,

(ii) $b_0^{S^+(\lambda)-S^-(\lambda)} (= \lambda b_2^{S^+(\lambda)-S^-(\lambda)})$ is a common base of $(E, \rho_0) \cdot S^+(\lambda)/S^-(\lambda)$ and $(E, \lambda\rho_1) \cdot (E - S^-(\lambda))/(E - S^+(\lambda))$, and

(iii) $\lambda b_1^{E-S^+(\lambda)}$ is a subbase of $(E, \rho_0)/S^+(\lambda)$ and a base of $(E, \lambda\rho_1) \cdot (E - S^+(\lambda))$.

The collection of minors $(E, \rho_0) \cdot S_j/S_{j-1}$ and $(E, \rho_1) \cdot (E - S_{j-1})/(E - S_j)$ $(j = 1, 2, \cdots, p)$ does not depend on the choice of a maximal chain $\mathcal{C}$ of $\mathcal{L}^*$ ([Nakamura+ Iri81], [Tomi + Fuji82]), due to Theorem 7.17.

An efficient algorithm for the principal partition of a graph is given by H. Imai [Imai83] and it requires $\mathrm{O}(|E|^3 \log |V|)$ time. Also, the principal partition of a polymatroid induced by a multi-terminal network can be found in $\mathrm{O}(|V|^3)$ time (see [Gallo + Grigoriadis + Tarjan89] and also see [Fuji80b]).

The concept of principal partition is also closely related to convex optimization problems over base polyhedra, which will be treated in Chapter V.

Finally, it should also be noted that the above argument is also valid with an appropriate modification if f_1 is a monotone nondecreasing submodular function.

(b.2) The principal structures of submodular systems

Related to the principal partition, we introduce the concept of *principal structure of* a submodular system ([Fuji80c]).

Consider a submodular system $(\mathcal{D}, f)$ on E and define for any $e \in E$

$$\mathcal{D}_f(e) = \{X \mid e \in X \in \mathcal{D},\ f(X) = \min\{f(Y) \mid e \in Y \in \mathcal{D}\}\}. \tag{7.125}$$

Then, $\mathcal{D}_f(e)$ is a distributive lattice. We denote by $D_f(e)$ the unique minimal element of $\mathcal{D}_f(e)$.

Lemma 7.18: *For any* $e_1, e_2 \in E$, *if* $e_2 \in D_f(e_1)$, *then*

$$D_f(e_2) \subseteq D_f(e_1). \tag{7.126}$$

(Proof) Put $A_i = D_f(e_i)$ $(i = 1, 2)$. Because of the definition of $A_1 = D_f(e_1)$,

$$f(A_1) \le f(A_1 \cup A_2). \tag{7.127}$$

From (7.127) and the submodularity of f we have

$$\begin{aligned} f(A_2) &\ge f(A_1 \cap A_2) + (f(A_1 \cup A_2) - f(A_1)) \\ &\ge f(A_1 \cap A_2). \end{aligned} \tag{7.128}$$

Since $e_2 \in A_1 \cap A_2$, it follows from (7.128) and the minimality of $A_2 = D_f(e_2)$ that we have (7.126). Q.E.D.

Now, let us define a directed graph $G_f = (E, A_f)$ with vertex set E and arc set A_f by

$$A_f = \{(e_1, e_2) \mid e_1, e_2 \in E,\ e_2 \in D_f(e_1)\}. \tag{7.129}$$

It follows from Lemma 7.18 that graph $G_f = (E, A_f)$ is transitive and that every strongly connected component of G_f is a complete directed graph with a selfloop at each vertex.

Decompose $G_f = (E, A_f)$ into strongly connected components $G_f^{(i)} = (E^{(i)}, A^{(i)})$ $(i = 1, 2, \cdots, p)$. This induces the partition $\mathcal{E} = \{E^{(i)} \mid i = 1, 2, \cdots, p\}$ of set E and the partial order $\preceq$ on $\mathcal{E}$ such that $E^{(i)} \preceq E^{(j)}$ if and only if there exists at least one directed path in G_f from a vertex in $E^{(j)}$ to a vertex in $E^{(i)}$. The *principal structure* of the submodular system $(\mathcal{D}, f)$ is the partition $\mathcal{E}$ together with the partial order $\preceq$ on $\mathcal{E}$. We can easily see from the definitions of $D_f(e)$ and $E^{(i)}$'s that for any $e \in E^{(j)} \in \mathcal{E}$

$$D_f(e) = \bigcup\{E^{(i)} \mid E^{(i)} \in \mathcal{E},\ E^{(i)} \preceq E^{(j)}\}. \tag{7.130}$$

Example b.4: The Dulmage-Mendelsohn decomposition of bipartite graphs

Let $G = (V^+, V^-; A)$ be a bipartite graph with left and right vertex sets V^+ and V^- and arc set $A \subseteq V^+ \times V^-$. Define a set function $f^+\colon 2^{V^+} \to \mathbf{R}$ by

$$f^+(U) = |\Gamma^+ U| - |U| \quad (U \subseteq V^+) \tag{7.131}$$

where $\Gamma^+ U = \{v^- \mid (v^+, v^-) \in A,\ v^+ \in U\}$ for $U \subseteq V^+$. Then f^+ is a submodular function, the negative of the so-called *deficiency function*. The principal structure of $(2^{V^+}, f^+)$ expresses a finer structure than the Dulmage-Mendelsohn decomposition of G ([Dulmage + Mendelsohn59]). See Fig. 7.8 for an example of (a) the decomposition of a bipartite graph

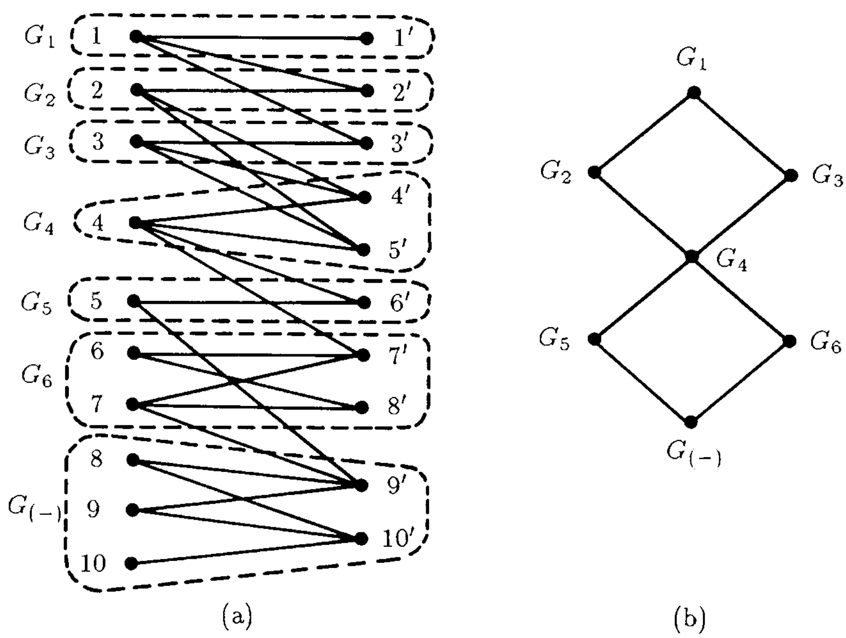

Figure 7.8: The principal structure of a bipartite graph.

based on the principal structure of $(2^{V^+}, f^+)$ and (b) the Hasse diagram representing the poset structure. Components G_1, G_2, G_3 and G_4 constitute a single component in the Dulmage-Mendelsohn decomposition.

It may also be noted that for a (directed) graph $G = (V, A)$, if we define $f(U) = |(\partial^- \delta^+ U) \cup U| - |U|$ $(U \subseteq V)$, then the principal structure of $(2^V, f)$ is the decomposition of G into strongly connected components with the associated poset structure. Also see [Murota90] for an application to certain matrices.

Example b.5: The principal partition of a polymatroid

Let $b \in \mathbf{R}^E$ be a base of a polymatroid (E, ρ) with rank function ρ and define a submodular function $f\colon 2^E \to \mathbf{R}$ by

$$f(X) = \rho(X) - b(X) \qquad (X \subseteq E). \tag{7.132}$$

We call the principal structure of submodular system $(2^E, f)$ on E the *principal structure of polymatroid (E, ρ) with respect to base b*. The principal partition of a matroid due to [Tomi76] and [Narayanan74] is a special case of the principal structure of the submodular system $(2^E, f)$ having the rank function f defined by (7.132) with an appropriately chosen base b.

(b.3) The minimum-ratio problem

Suppose that $f\colon \mathcal{D} \to \mathbf{R}$ is a nonnegative submodular function with $f(\emptyset) = 0$ and that $g\colon \mathcal{D} \to \mathbf{R}$ is a nonnegative supermodular function with $g(\emptyset) = 0$ and $g(X) > 0$ for some $X \in \mathcal{D}$.

Consider the following problem:

$$\text{Minimize} \quad \frac{f(X)}{g(X)} \tag{7.133a}$$

$$\text{subject to} \quad X \in \mathcal{D}, \quad g(X) > 0. \tag{7.133b}$$

This is called a *minimum-ratio problem*. Special cases of the problem have been treated in [Brown79] and [Ichimori + Ishii + Nishida82] as a sharing problem in a network (also see [Megiddo74], [Fuji80b]), and in [Cunningham85c] as a minimum-cost problem of disconnecting a network. The minimum-ratio problem given by (7.133) is closely related to the principal partition discussed in the preceding subsection 7.2.b.1.

Define a Lagrangian function for f and $-g$ associated with Problem (7.133) by

$$L(\lambda, X) = f(X) - \lambda g(X) \tag{7.134}$$

for $\lambda \geq 0$ and $X \in \mathcal{D}$.

Theorem 7.19: *A nonnegative $\hat{\lambda}$ is the minimum value of the ratio of Problem* (7.133) *if and only if*

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} = 0 \quad (0 \leq \lambda \leq \hat{\lambda}), \tag{7.135}$$

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} < 0 \quad (\hat{\lambda} < \lambda). \tag{7.136}$$

(Proof) Suppose that $\hat{\lambda}$ is the minimum value of the ratio of Problem (7.133). Then we have

$$L(\hat{\lambda}, X) = f(X) - \hat{\lambda} g(X) \geq 0 \quad (X \in \mathcal{D}), \tag{7.137}$$

where (7.137) holds with equality for some $\hat{X} \in \mathcal{D}$ such that $g(\hat{X}) > 0$. It follows that

$$L(\lambda, X) \geq L(\hat{\lambda}, X) \geq 0 \quad (0 \leq \lambda \leq \hat{\lambda},\ X \in \mathcal{D}), \tag{7.138}$$

$$L(\lambda, \hat{X}) < L(\hat{\lambda}, \hat{X}) = 0 \quad (\hat{\lambda} < \lambda). \tag{7.139}$$

Since $L(\lambda, \emptyset) = 0$ for any $\lambda \geq 0$, from (7.138) and (7.139) we have (7.135) and (7.136).

Conversely, suppose (7.135) and (7.136) hold for a nonnegative $\hat{\lambda}$. Then, there exists an $\hat{X} \in \mathcal{D}$ with $g(\hat{X}) > 0$ which attains the minimum of $L(\hat{\lambda}, X)$ in X, since otherwise we would have $\min\{L(\hat{\lambda} + \epsilon, X) \mid X \in \mathcal{D},\ g(X) > 0\} > 0$ for a sufficiently small $\epsilon > 0$, which would contradict (7.136). It follows from (7.135) that

$$L(\hat{\lambda}, X) \geq L(\hat{\lambda}, \hat{X}) = 0 \quad (X \in \mathcal{D}), \tag{7.140}$$

i.e., for any $X \in \mathcal{D}$ with $g(X) > 0$ we have

$$f(X)/g(X) \geq \hat{\lambda} \tag{7.141}$$

which holds with equality for $X = \hat{X}$. Q.E.D.

It should be noted that Theorem 7.19 holds for f and g without submodularity or supermodularity, though the problem would become hard without submodularity or supermodularity. A minimum-ratio problem for general set functions has also been discussed by Cunningham [Cunningham83a]. For minimum-ratio problems also see [Gondran + Minoux84].

From Theorem 7.19, the minimum-ratio problem (7.133) is reduced to the problem of finding the minimum critical value λ_1 $(= \hat{\lambda})$ for $L(\lambda, X)$. The minimum critical value λ_1 can be obtained by a binary search in $\mathbf{R}_+$ based on Theorem 7.19 (cf. [Cunningham85c] and [Imai83] for special cases). Also a dichotomy works for finding an element of $\mathcal{L}(\lambda_1)$ from among $\mathcal{L}^* = \bigcup\{\mathcal{L}(\lambda) \mid \lambda \geq 0\}$ by searching a chain of $\mathcal{L}^*$ (cf. [Fuji80b], [Nakamura + Iri81], [Tomi80d], [Tomi + Fuji82]).

Example b.6: **The strength of a weighted graph** [Cunningham85c]

Let $G = (V, E)$ be a connected undirected graph with a positive weight function w: $E \to \mathbf{R}$ on the edge set E. For each edge $e \in E$ $w(e)$ represents the *strength* of e. A measure of network invulnerability, called the *strength* of the weighted graph G, is defined by

$$\sigma(G, w) = \min\left\{\frac{w(X)}{k(X)} \;\middle|\; X \subseteq E,\ k(X) > 0\right\}, \tag{7.142}$$

where for $X \subseteq E$ $k(X)$ is the number of the connected components of $G - X$ minus one ([Cunningham85c] and also [Gusfield83] when $w = \mathbf{1}$). Let r be the rank function of G. Since G is connected, we have

$$k(X) = r(E) - r(E - X) = r^{\#}(X) \quad (X \subseteq E). \tag{7.143}$$

Hence, k: $2^E \to \mathbf{R}$ is a nonnegative supermodular function with $k(\emptyset) = 0$. Also, w: $2^E \to \mathbf{R}$ is a modular function. Therefore, the problem of determining the strength $\sigma(G, w)$ in (7.142) is a minimum-ratio problem. From Theorem 7.19,

$$\begin{aligned} \sigma(G, w) &= \max\{\lambda \mid \lambda \geq 0,\ \forall X \subseteq E\colon\ w(X) - \lambda r^{\#}(X) \geq 0\} \\ &= \max\{\lambda \mid \lambda > 0,\ \frac{1}{\lambda} w \in \mathrm{P}(r^{\#})\}, \end{aligned} \tag{7.144}$$

where $\mathrm{P}(r^{\#})$ is the supermodular polyhedron associated with the supermodular function $r^{\#}$: $2^E \to \mathbf{R}$.

Let f_0 be the rank function r of graph G, f_1 be the modular function $-w$, and λ_i $(i = 1, 2, \cdots, p)$ be the critical values, in Theorem 7.15, for the pair of submodular systems $(2^E, f_i)$ $(i = 0, 1)$. We can show from (7.144) that

$$\sigma(G, w) = \frac{1}{\lambda_p} \tag{7.145}$$

(see Corollary 9.6 in Section 9.2).

Example b.7: **Finding a maximum density subgraph**

For a graph $G = (V, E)$ without selfloops define the *density* of G by

$$d(G) = \frac{|E|}{(|V| - 1)}. \tag{7.146}$$

Consider the problem of finding a maximum density subgraph of G. Since a maximum density subgraph of G is connected, it suffices to find a maximum-density connected subgraph of G. For a connected subgraph $H = (W, F)$ of G the density is given by

$$d(H) = \frac{|F|}{|W| - 1} = \frac{|F|}{r(F)}, \tag{7.147}$$

where r is the rank function of G. Conversely, if an edge subset F_0 attains the maximum of (7.147), then the edge set of each connected component of the subgraph formed by F_0 also attains the maximum of (7.147). Therefore, the problem is reduced to solving the following problem:

$$\text{Maximize } \frac{|F|}{r(F)} \quad (F \subseteq E, F \neq \emptyset) \tag{7.148}$$

or a minimum-ratio problem:

$$\text{Minimize } \frac{r(F)}{|F|} \quad (F \subseteq E, F \neq \emptyset). \tag{7.149}$$

The minimum value of (7.149) is equal to the minimum critical value λ_1 of the principal partition of G in the sense of Tomizawa and Narayanan (see Corollary 9.6).

The maximum density subgraph problem is also considered by [Goldberg83], where the density of a graph $G = (V, E)$ is defined by $|E|/|V|$.

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Chapter V.

Nonlinear Optimization with Submodular Constraints

We consider a class of nonlinear optimization problems with constraints described by submodular functions which includes problems of minimizing separable convex functions over base polyhedra with and without integer constraints (see [Fuji89]). We also give efficient algorithms for solving these problems.

8. Separable Convex Optimization

Let $(\mathcal{D}, f)$ be a submodular system on E with a real-valued (or rational-valued) rank function f. The underlying totally ordered additive group is assumed to be the set $\mathbf{R}$ of reals (or the set $\mathbf{Q}$ of rationals) unless otherwise stated.

For each $e \in E$ let $w_e: \mathbf{R} \to \mathbf{R}$ be a real-valued convex function on $\mathbf{R}$, and consider the following problem

$$P_1\colon \text{ Minimize } \sum_{e \in E} w_e(x(e)) \tag{8.1a}$$

$$\text{subject to } \ x \in \mathrm{B}(f). \tag{8.1b}$$

Problem P_1 was first considered by the author [Fuji80b] for the case where for each $e \in E$ $w_e(x(e))$ is a quadratic function given by $x(e)^2/w(e)$ with a positive real weight $w(e)$ and f is a polymatroid rank function. H. Groenevelt [Groenevelt85] also considered Problem P_1 where each w_e is a convex function and f is a polymatroid rank function.

8.1. Optimality Conditions

It is almost straightforward to generalize the result of [Fuji80b] and [Groenevelt85] to Problem P_1 for a general submodular system.

Optimal solutions of Problem P_1 are characterized as follows.

Theorem 8.1 ([Groenevelt85]; also see [Fuji80b]): *A base $x \in \mathrm{B}(f)$ is an optimal solution of Problem P_1 if and only if for each exchangeable pair (e, e') associated with base x (i.e., $e \in E$ and $e' \in \mathrm{dep}(x, e) - \{e\}$), we have*

$$w_e{}^+(x(e)) \geq w_{e'}{}^-(x(e')), \tag{8.2}$$

where $w_e{}^+$ denotes the right derivative of w_e and $w_{e'}{}^-$ the left derivative of $w_{e'}$.

(Proof) *The "if" part*: Denote by A_x the set of all the exchangeable pairs (e, e') associated with x. Suppose that (8.2) holds for each $(e, e') \in A_x$. From Theorem 3.28, for any base $z \in \mathrm{B}(f)$ there exist some nonnegative coefficients $\lambda(e, e')$ $((e, e') \in A_x)$ such that

$$z = x + \sum \{\lambda(e, e')(\chi_e - \chi_{e'}) \mid (e, e') \in A_x\}. \tag{8.3}$$

For each $e \in E$ define

$$\overline{w}_e(x(e)) = \max\{w_{e'}{}^-(x(e')) \mid e' \in \mathrm{dep}(x, e)\}. \tag{8.4}$$

We see from (8.2) and (8.4) that

$$w_e{}^-(x(e)) \leq \overline{w}_e(x(e)) \leq w_e{}^+(x(e)) \quad (e \in E), \tag{8.5}$$

$$\overline{w}_e(x(e)) \geq \overline{w}_{e'}(x(e')) \quad ((e, e') \in A_x), \tag{8.6}$$

where recall that $e' \in \mathrm{dep}(x, e)$ implies $\mathrm{dep}(x, e') \subseteq \mathrm{dep}(x, e)$. From (8.3)$\sim$(8.6) and the convexity of w_e $(e \in E)$ we have

$$\begin{aligned}
\sum_{e \in E} w_e(z(e)) &= \sum_{e \in E} w_e(x(e) + \partial\lambda(e)) \\
&\geq \sum_{e \in E} \{w_e(x(e)) + \partial\lambda(e) \cdot \overline{w}_e(x(e))\} \\
&= \sum_{e \in E} w_e(x(e)) + \sum_{(e, e') \in A_x} \lambda(e, e')(\overline{w}_e(x(e)) - \overline{w}_{e'}(x(e'))) \\
&\geq \sum_{e \in E} w_e(x(e)),
\end{aligned} \tag{8.7}$$

where $\partial\lambda\colon E \to \mathbf{R}$ is the boundary of λ defined by

$$\partial\lambda(e) = \sum_{(e, e') \in A_x} \lambda(e, e') - \sum_{(e', e) \in A_x} \lambda(e', e) \tag{8.8}$$

for each $e \in E$. (8.7) shows the optimality of x.

The "only if" part: Suppose that for a base $x \in \mathrm{B}(f)$ there exists an exchangeable pair (e, e') associated with x such that

$$w_e{}^+(x(e)) < w_{e'}{}^-(x(e')). \tag{8.9}$$

Then for a sufficiently small $\alpha > 0$ we have

$$w_e(x(e)) + w_{e'}(x(e')) > w_e(x(e) + \alpha) + w_{e'}(x(e') - \alpha), \tag{8.10}$$

$$x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{B}(f). \tag{8.11}$$

Therefore, x is not an optimal solution. Q.E.D.

Theorem 8.1 generalizes Theorem 3.16 for the minimum-weight base problem to that with a convex weight function.

For each $e \in E$ and $\xi \in \mathbf{R}$ define the interval

$$J_e(\xi) = [w_e{}^-(\xi), w_e{}^+(\xi)]. \tag{8.12}$$

$J_e(\xi)$ is the subdifferential of w_e at ξ. Conversely, for each $e \in E$ and $\eta \in \mathbf{R}$ define

$$I_e(\eta) = \{\xi \mid \xi \in \mathbf{R},\ \eta \in J_e(\xi)\}. \tag{8.13}$$

Because of the convexity of w_e, $I_e(\eta)$, if nonempty, is a closed interval in $\mathbf{R}$ and we express it as

$$I_e(\eta) = [i_e{}^-(\eta), i_e{}^+(\eta)]. \tag{8.14}$$

Note that $\eta \in J_e(\xi)$ if and only if $\xi \in I_e(\eta)$.

Theorem 8.2: *A base $x \in \mathrm{B}(f)$ is an optimal solution of Problem P_1 if and only if there exists a chain*

$$\mathcal{C}\colon\ \emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = E \tag{8.15}$$

of $\mathcal{D}$ such that

(i) $x(A_i) = f(A_i) \quad (i = 0, 1, \cdots, k),$ (8.16)

(ii) *for each $i = 1, \cdots, k$,*

$$\bigcap\{J_e(x(e)) \mid e \in A_i - A_{i-1}\} \neq \emptyset, \tag{8.17}$$

(iii) *for each* $i, j = 1, \cdots, k$ *such that* $i < j$, *we have*

$$w_{e'}{}^{-}(x(e')) \le w_e{}^{+}(x(e)) \tag{8.18}$$

for any $e' \in A_i - A_{i-1}$ *and* $e \in A_j - A_{j-1}$.

(Proof) *The "if" part*: Suppose there exists a chain $\mathcal{C}$ of (8.15) satisfying (i)∼(iii). For any exchangeable pair (e', e'') for base x it follows from (i) that for some $i, j \in \{1, 2, \cdots, k\}$ with $i \le j$ we have $e' \in A_j - A_{j-1}$ and $e'' \in A_i - A_{i-1}$. If $i = j$, (ii) implies

$$w_{e'}{}^{+}(x(e')) \ge \eta \ge w_{e''}{}^{-}(x(e'')) \tag{8.19}$$

for any $\eta \in \bigcap\{J_e(x(e)) \mid e \in A_i - A_{i-1}\}$. From (8.19), (iii) and Theorem 8.1, x is an optimal solution of Problem P_1.

The "only if" part: Let $x \in \mathrm{B}(f)$ be an optimal solution of Problem P_1. The sublattice

$$\mathcal{D}(x) = \{X \mid X \in \mathcal{D},\ x(X) = f(X)\} \tag{8.20}$$

defines a poset $\mathcal{P}(\mathcal{D}(x)) = (\Pi(\mathcal{D}(x)), \preceq_{\mathcal{D}(x)})$ (see Section 3.2.a). Suppose that $\Pi(\mathcal{D}(x)) = \{E_1, E_2, \cdots, E_k\}$. For each $i = 1, 2, \cdots, k$ define

$$J_i = \bigcap\{J_e(x(e)) \mid e \in E_i\}. \tag{8.21}$$

For each distinct $e,\ e' \in E_i$ (e, e') is an exchangeable pair associated with x due to the definition of E_i. Hence, because of the optimality of x we have

$$w_e{}^{+}(x(e)) \ge w_{e'}{}^{-}(x(e')) \tag{8.22}$$

for each $e,\ e' \in E_i$ due to Theorem 8.1. Therefore,

$$J_i \ne \emptyset \quad (i = 1, 2, \cdots, k). \tag{8.23}$$

For each $i = 1, 2, \cdots, k$ we have

$$J_i = [\eta_i^-, \eta_i^+], \tag{8.24}$$

where $\eta_i^- = \max\{w_e{}^{-}(x(e)) \mid e \in E_i\}$ and $\eta_i^+ = \min\{w_e{}^{+}(x(e)) \mid e \in E_i\}$. Also define for each $i = 1, 2, \cdots, k$

$$\overline{\eta}_i = \max\{\eta_j^- \mid j\colon\ E_j \preceq_{\mathcal{D}(x)} E_i\}. \tag{8.25}$$

For any i, j such that $E_j \preceq_{\mathcal{D}(x)} E_i$ we have

$$\eta_j^- \leq \eta_i^+ \tag{8.26}$$

due to Theorem 8.1. From (8.25) and (8.26),

$$\overline{\eta}_i \in [\eta_i^-, \eta_i^+] \quad (i = 1, 2, \cdots, k) \tag{8.27}$$

and we have the following monotonicity:

$$E_j \preceq_{\mathcal{D}(x)} E_i \implies \overline{\eta}_j \leq \overline{\eta}_i. \tag{8.28}$$

We assume without loss of generality that

$$\overline{\eta}_1 \leq \overline{\eta}_2 \leq \cdots \leq \overline{\eta}_k, \tag{8.29}$$

and that, defining

$$A_i = E_1 \cup E_2 \cup \cdots \cup E_i \quad (i = 1, 2, \cdots, k), \tag{8.30}$$
$$A_0 = \emptyset, \tag{8.31}$$

we have a (maximal) chain

$$\mathcal{C}\colon\ \emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = E \tag{8.32}$$

of $\mathcal{D}(x)$. In particular, (i) holds. Also, (ii) is exactly (8.23). Moreover, from (8.27) and (8.29) we have (iii). Q.E.D.

Theorem 8.2 generalizes the greedy algorithm given in Section 3.2.b.

8.2. A Decomposition Algorithm

In the following we assume that $I_e(\eta) \neq \emptyset$ for every $\eta \in \mathbf{R}$, to simplify the following argument. It should also be noted that this assumption guarantees the existence of an optimal solution even if $\mathrm{B}(f)$ is unbounded. When $\mathrm{B}(f)$ is bounded, there is no loss of generality with this assumption.

We first describe an algorithm for Problem P_1 in (8.1). Here, x^* is the output vector giving an optimal solution.

A decomposition algorithm

Step 1: Choose $\eta \in \mathbf{R}$ such that

$$\sum_{e \in E} i_e{}^-(\eta) \leq f(E) \leq \sum_{e \in E} i_e{}^+(\eta) \tag{8.33}$$

(see (8.14)).

Step 2: Find a base $x \in \mathrm{B}(f)$ such that for each $e,\ e' \in E$

(1) if $w_e{}^+(x(e)) < \eta$ and $w_{e'}{}^-(x(e')) > \eta$, then we have $e' \notin \mathrm{dep}(x, e)$,

(2) if $w_e{}^+(x(e)) < \eta$, $w_{e'}{}^-(x(e')) = \eta$ and $e' \in \mathrm{dep}(x, e)$, then for any $\alpha > 0$ we have $w_{e'}{}^-(x(e') - \alpha) < \eta$, i.e., $x(e') = i_{e'}{}^-(\eta)$,

(3) if $w_e{}^+(x(e)) = \eta$, $w_{e'}{}^-(x(e')) > \eta$ and $e' \in \mathrm{dep}(x, e)$, then for any $\alpha > 0$ we have $w_e{}^+(x(e) + \alpha) > \eta$, i.e., $x(e) = i_e{}^+(\eta)$.

Put

$$E_- = \bigcup \{\mathrm{dep}(x, e) \mid e \in E,\ w_e{}^+(x(e)) < \eta\}, \tag{8.34}$$
$$E_+ = \bigcup \{\mathrm{dep}^\#(x, e) \mid e \in E,\ w_e{}^-(x(e)) > \eta\}, \tag{8.35}$$
$$E_0 = E - (E_+ \cup E_-), \tag{8.36}$$

where $\mathrm{dep}^\#$ is the *dual dependence function* defined by

$$\mathrm{dep}^\#(x, e) = \bigcap \{X \mid e \in X \in \overline{\mathcal{D}},\ x(X) = f^\#(X)\}. \tag{8.37}$$

Put $x^*(e) = x(e)$ for each $e \in E_0$.

Step 3: If $E_- \neq \emptyset$, then apply the present algorithm recursively to the problem with E and f, respectively, replaced by E_- and f^{E_-} and with the base polyhedron associated with the reduction $(\mathcal{D}, f) \cdot E_-$. Also, if $E_+ \neq \emptyset$, then apply the present algorithm recursively to the problem with E and f, respectively, replaced by E_+ and f_{E_+} and with the base polyhedron associated with the contraction $(\mathcal{D}, f) \times E_+$.

(End)

The present algorithm is adapted from the algorithms in [Fuji80b] and [Groenevelt85]. Here, it is described in a self-dual form.

It should be noted that for the dual dependence function $\mathrm{dep}^\#$ we have $e' \in \mathrm{dep}^\#(x, e)$ if and only if $e \in \mathrm{dep}(x, e')$, for $x \in \mathrm{B}(f)$ $(= \mathrm{B}(f^\#))$.

Now, we show the validity of the decomposition algorithm described above. First, note that $E_- \cap E_+ = \emptyset$ in Step 2 since otherwise there exist $e, e' \in E$ such that $w_e{}^+(x(e)) < \eta$, $w_{e'}{}^-(x(e')) > \eta$ and $e' \in \mathrm{dep}(x, e)$, which contradicts (1) of Step 2.

Suppose we have $E_- = E$ in Step 2. Then, from (8.34)

$$x(e) \leq i_e{}^-(\eta) \qquad (e \in E) \tag{8.38}$$

and $x(e) < i_e^-(\eta)$ for at least one $e \in E$. Therefore,

$$f(E) = x(E) < \sum_{e \in E} i_e{}^-(\eta), \tag{8.39}$$

which contradicts (8.33). Hence $E_- \neq E$. Similarly, we have $E_+ \neq E$. Consequently, the total number of executions of Step 2 is at most $|E| - 1$. When the algorithm terminates, then obtained vector x^* is an optimal solution of Problem P_1 due to Theorem 8.2. Note that in Step 2 $x(E_-) = f(E_-)$ and $x(E - E_+) = f(E - E_+)$ from (8.34) and (8.35) and that E_- and $E - E_+$, if nonempty proper subsets of E, will be members of the chain $\mathcal{C}$ in (8.15).

In Step 1 a desired η may be found by a binary search but Step 1 heavily depends on the structure of the given functions w_e $(e \in E)$. A base $x \in \mathrm{B}(f)$ satisfying (1)$\sim$(3) of Step 2 is obtained by $\mathrm{O}(|E|^2)$ elementary transformations if an oracle for exchange capacities for $\mathrm{B}(f)$ is available. If an oracle for saturation capacities for $\mathrm{P}(f)$ is available, Step 2 can be executed by calling the oracle $\mathrm{O}(|E|)$ times.

Remark 8.1: In Step 3 the original problem on E is decomposed into two problems, one being on E_- and the other on E_+, and the values of $x^*(e)$ $(e \in E_0)$ are fixed. The decomposition relation obtained through the algorithm recursively defines a binary tree in such a way that E_- is the left child and E_+ the right child of E. The decomposition algorithm described above traverses the binary tree by the depth-first search where the search of the left child is prior to that of its sibling, the right child. The above algorithm is also valid if we modify Step 3 according to any efficient way of traversing the binary tree from the root.

Remark 8.2: If w_e is strictly convex for each $e \in E$, then conditions (2) and (3) in Step 2 are always satisfied, so that we have only to consider condition (1). Moreover, if w_e is strictly convex and differentiable for each

$e \in E$, then the above algorithm will further be simplified. This is the case to be treated in Section 9.

The decomposition algorithm for the separable convex optimization problem P_1 lays a basis for the algorithms for the other problems to be considered in Sections 9~11.

8.3. Discrete Optimization

Suppose that the rank function f of the submodular system $(\mathcal{D}, f)$ is integer-valued. We consider a discrete optimization problem which is Problem P_1 with variables being restricted to integers.

For each $e \in E$ let $\hat{w}_e$ be a real-valued function on $\mathbf{Z}$ such that the piecewise linear extension, denoted by w_e, of $\hat{w}_e$ on $\mathbf{R}$ is a convex function, where $w_e(\xi) = \hat{w}_e(\xi)$ for $\xi \in \mathbf{Z}$ and w_e restricted on each unit interval $[\xi, \xi+1]$ ($\xi \in \mathbf{Z}$) is a linear function. Consider a discrete optimization problem described as

$$IP_1\colon \ \text{Minimize} \ \sum_{e \in E} \hat{w}_e(x(e)) \tag{8.40a}$$

$$\text{subject to} \ \ x \in \mathrm{B}_{\mathbf{Z}}(f), \tag{8.40b}$$

where

$$\mathrm{B}_{\mathbf{Z}}(f) = \{x \mid x \in \mathbf{Z}^E, \ \forall\, X \in \mathcal{D}\colon \ x(X) \le f(X), \ x(E) = f(E)\}. \tag{8.41}$$

$\mathrm{B}_{\mathbf{Z}}(f)$ is the base polyhedron associated with $(\mathcal{D}, f)$ where the underlying totally ordered additive group is the set $\mathbf{Z}$ of integers. For the same f we also denote by $\mathrm{B}_{\mathbf{R}}(f)$ the base polyhedron $\mathrm{B}(f)$ associated with $(\mathcal{D}, f)$ where the underlying totally ordered additive group is the set $\mathbf{R}$ of reals.

Also consider the continuous version of IP_1:

$$P_1\colon \ \text{Minimize} \ \sum_{e \in E} w_e(x(e)) \tag{8.42a}$$

$$\text{subject to} \ \ x \in \mathrm{B}_{\mathbf{R}}(f). \tag{8.42b}$$

Recall that w_e is the piecewise linear extension of $\hat{w}_e$ for each $e \in E$.

Theorem 8.3 (cf. [Groenevelt85]): *If there exists an optimal solution for Problem P_1 of (8.42), there exists an integral optimal solution for Problem P_1.*

(Proof) Suppose that x^* is an optimal solution of Problem P_1. Define vectors $l,\ u \in \mathbf{R}^E$ by

$$l(e) = \lfloor x^*(e) \rfloor \quad (e \in E), \tag{8.43}$$

$$u(e) = \lceil x^*(e) \rceil \quad (e \in E). \tag{8.44}$$

(Here, for each $\xi \in \mathbf{R}$, $\lfloor \xi \rfloor$ is the maximum integer less than or equal to ξ and $\lceil \xi \rceil$ is the minimum integer greater than or equal to ξ.) Consider the following problem

$$P_1{}'\text{: Minimize } \sum_{e \in E} w_e(x(e)) \tag{8.45a}$$

$$\text{subject to } \ x \in \mathrm{B}_{\mathbf{R}}(f)_l^u, \tag{8.45b}$$

where $\mathrm{B}_{\mathbf{R}}(f)_l^u$ is the base polyhedron of the submodular system $(\mathcal{D}, f)_l^u$ which is the vector minor of $(\mathcal{D}, f)$ obtained by the restriction by u and the contraction by l. Note that an optimal solution of $P_1{}'$ is an optimal solution of P_1. Since f is integer-valued and l and u are integral, $\mathrm{B}_{\mathbf{R}}(f)_l^u$ is an integral base polyhedron. Also, the objective function in (8.45a) is linear on $\mathrm{B}_{\mathbf{R}}(f)_l^u$. Therefore, by the greedy algorithm given in Section 3.2.b we can find an integral optimal solution for Problem $P_1{}'$, which is also optimal for Problem P_1. Q.E.D.

We can also prove Theorem 8.3 by using the decomposition algorithm given in Section 8.2. From the assumption, for each $e \in E$ and $\eta \in \mathbf{R}$ $i_e{}^-(\eta)$ and $i_e{}^+(\eta)$ are integers. We can choose an integral base $x \in \mathrm{B}(f)$ as the base x required in Step 2 of the decomposition algorithm. Note that $w_e{}^+(x(e)) < \eta$ ($w_e{}^-(x(e)) > \eta$) is equivalent to $x(e) < i_e{}^-(\eta)$ ($x(e) > i_e{}^+(\eta)$).

An integral optimal solution of Problem P_1 of (8.42) is an optimal solution of Problem IP_1 of (8.40), and vice versa.

An incremental algorithm is also given in [Federgruen + Groenevelt86].

9. The Lexicographically Optimal Base Problem

We consider a submodular system $(\mathcal{D}, f)$ on E with the set $\mathbf{R}$ of reals (or the set $\mathbf{Q}$ of rationals) as the underlying totally ordered additive group. Throughout this section $\mathbf{R}$ may be replaced by $\mathbf{Q}$ but not by the set $\mathbf{Z}$ of integers.

9.1. Nonlinear Weight Functions

For each $e \in E$ let h_e be a continuous and monotone increasing function from $\mathbf{R}$ onto $\mathbf{R}$. For any vector $x \in \mathbf{R}^E$ we denote by $\mathrm{T}(x)$ the sequence of the components $x(e)$ $(e \in E)$ of x arranged in order of increasing magnitude, i.e., $\mathrm{T}(x) = (x(e_1), x(e_2), \cdots, x(e_n))$ with $x(e_1) \le x(e_2) \le \cdots \le x(e_n)$, where $|E| = n$ and $E = \{e_1, e_2, \cdots, e_n\}$.

For real n-sequences $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ α is *lexicographically greater than or equal to* β if and only if (1) $\alpha = \beta$ or (2) $\alpha \ne \beta$ and for the minimum index i such that $\alpha_i \ne \beta_i$ we have $\alpha_i > \beta_i$.

Consider the following problem.

$$P_2\text{: Lexicographically maximize } \ \mathrm{T}((h_e(x(e)) \mid e \in E)) \tag{9.1a}$$

$$\text{subject to } \ x \in \mathrm{B}(f). \tag{9.1b}$$

We call an optimal solution of Problem P_2 a *lexicographically optimal base of* $(\mathcal{D}, f)$ *with respect to nonlinear weight functions* h_e $(e \in E)$. Informally, Problem P_2 is to find a base x which is as close as possible to a vector which equalizes the values of $h_e(x(e))$ $(e \in E)$ on the basis of the lexicographic ordering in (9.1a).

Theorem 9.1 (cf. [Fuji80b]): *Let x be a base in $\mathrm{B}(f)$. Define a vector $\eta \in \mathbf{R}^E$ by*

$$\eta(e) = h_e(x(e)) \quad (e \in E) \tag{9.2}$$

and let the distinct values of $\eta(e)$ $(e \in E)$ be given by

$$\eta_1 < \eta_2 < \cdots < \eta_p. \tag{9.3}$$

Also, define

$$A_i = \{e \mid e \in E,\ \eta(e) \le \eta_i\} \quad (i = 1, 2, \cdots, p). \tag{9.4}$$

Then the following four statements are equivalent:

(i) *x is a lexicographically optimal base of $(\mathcal{D}, f)$ with respect to h_e $(e \in E)$.*

(ii) *$A_i \in \mathcal{D}$ and $x(A_i) = f(A_i)$ $(i = 1, 2, \cdots, p)$.*

(iii) *$\mathrm{dep}(x, e) \subseteq A_i$ $(e \in A_i,\ i = 1, 2, \cdots, p)$.*

(iv) *x is an optimal solution of Problem P_1 in* (8.1) *where for each $e \in E$ the derivative of w_e coincides with h_e.*

(Proof) The equivalence, (ii) $\Longleftrightarrow$ (iii), immediately follows from the definition of dependence function. Also, the equivalence, (ii) (or (iii)) $\Longleftrightarrow$ (iv), follows from Theorem 8.2. Therefore, (ii)$\sim$(iv) are equivalent. We show the equivalence, (i) $\Longleftrightarrow$ (ii)$\sim$(iv).

(i) $\Longrightarrow$ (iii): Suppose (i). If there exist $i \in \{1, 2, \cdots, p\}$ and $e \in A_i$ such that $e' \in \mathrm{dep}(x, e) - A_i$, then for a sufficiently small $\alpha > 0$ the vector given by

$$y = x + \alpha(\chi_e - \chi_{e'}) \tag{9.5}$$

is a base in $\mathrm{B}(f)$ and $\mathrm{T}((h_e(y(e)) \mid e \in E))$ is lexicographically greater than $\mathrm{T}((h_e(x(e)) \mid e \in E))$. This contradicts (i). So, (iii) holds.

(ii), (iii) $\Longrightarrow$ (i): Suppose (ii) (and (iii)). Let $\overline{x}$ be an arbitrary base such that $\mathrm{T}((h_e(\overline{x}(e)) \mid e \in E))$ is lexicographically greater than or equal to $\mathrm{T}((h_e(x(e)) \mid e \in E))$. Define a vector $\overline{\eta} \in \mathbf{R}^E$ by

$$\overline{\eta}(e) = h_e(\overline{x}(e)) \quad (e \in E). \tag{9.6}$$

Also define $A_0 = \emptyset$. We show by induction on i that

$$x(e) = \overline{x}(e) \quad (e \in A_i) \tag{9.7}$$

for $i = 0, 1, \cdots, p$, from which the optimality of x follows. For $i = 0$ (9.7) trivially holds. So, suppose that (9.7) holds for some $i = i_0 < p$. Since $\mathrm{T}(\overline{\eta})$ is lexicographically greater than or equal to $\mathrm{T}(\eta)$, we have from (9.3) and (9.4)

$$\overline{\eta}(e) \geq \eta(e) = \eta_{i_0+1} \quad (e \in A_{i_0+1} - A_{i_0}). \tag{9.8}$$

From (9.8) and the monotonicity of h_e $(e \in E)$,

$$\overline{x}(e) \geq x(e) \quad (e \in A_{i_0+1} - A_{i_0}). \tag{9.9}$$

Since $\overline{x} \in \mathrm{B}(f)$, it follows from (9.7) with $i = i_0$, (9.9) and assumption (ii) that

$$f(A_{i_0+1}) \geq \overline{x}(A_{i_0+1}) \geq x(A_{i_0+1}) = f(A_{i_o+1}). \tag{9.10}$$

From (9.9) and (9.10) we have $\overline{x}(e) = x(e)$ $(e \in A_{i_0+1})$. Q.E.D.

We see from Theorem 9.1 that the lexicographically optimal base is unique and that the problem can be solved by the decomposition algorithm given in Section 8.2.

For a vector $x \in \mathbf{R}^E$ denote by $\mathrm{T}^*(x)$ the sequence of the components $x(e)$ $(e \in E)$ of x arranged in order of decreasing magnitude. We call a base $x \in \mathrm{B}(f)$ which lexicographically minimizes $\mathrm{T}^*((h_e(x(e)) \mid e \in E))$ a *co-lexicographically optimal base* of $(\mathcal{D}, f)$ with respect to h_e $(e \in E)$.

Theorem 9.2: *x is a lexicographically optimal base of $(\mathcal{D}, f)$ with respect to h_e $(e \in E)$ if and only if it is a co-lexicographically optimal base of $(\mathcal{D}, f)$ with respect to h_e $(e \in E)$.*

(Proof) Using $\eta(e)$ $(e \in E)$ and η_i $(i = 1, 2, \cdots, p)$ appearing in Theorem 9.1, define

$$A_i{}^* = \{e \mid e \in E,\ \eta(e) \geq \eta_{p-i+1}\} \quad (i = 1, 2, \cdots, p). \tag{9.11}$$

Also define $A_0 = \emptyset = A_0{}^*$. Since $A_i{}^* = E - A_{p-i}$ $(i = 0, 1, \cdots, p)$ and $x(E) = f(E)$, we can easily see that for a base $x \in \mathrm{B}(f)$ x satisfies (ii) of Theorem 9.1 if and only if x satisfies

(ii*) $A_i{}^* \in \overline{\mathcal{D}}$ and $x(A_i{}^*) = f^{\#}(A_i{}^*)$ $(i = 1, 2, \cdots, p)$.

Consequently, the present theorem follows from Theorem 9.1 and the duality shown in Lemma 2.4. Q.E.D.

Recall the self-dual structure of Problem P_1 shown in the decomposition algorithm in Section 8.2.

9.2. Linear Weight Functions

Let us consider Problem P_2 in (9.1) for the case when $h_e(x(e))$ is a linear function expressed as $x(e)/w(e)$ with $w(e) > 0$ for each $e \in E$. Such a lexicographically optimal base is called a *lexicographically optimal base with respect to the weight vector* $w = (w(e) \mid e \in E)$ (see [Fuji80b]). This is a generalization of the concept of (lexicographically) optimal flow introduced by N. Megiddo [Megiddo74] concerning multiple-source multiple-sink networks. A polymatroid induced on the set of sources (or sinks) is considered in [Megiddo74] (see Section 2.2). [The concept of lexicographically optimal base was re-discovered in economic theory by B. Dutta and D. Ray ([Dutta+Ray89] and [Dutta90]) and is often called the Dutta-Ray solution (also see [Hokari02] and [Hokari+van Gellekom02]).]

From Theorem 9.1 the lexicographically optimal base problem with respect to the weight vector w is equivalent to the following separable quadratic optimization problem (see [Fuji80b]).

$$P_1{}^w\text{: Minimize } \sum_{e \in E} x(e)^2/w(e) \tag{9.12a}$$

$$\text{subject to } \ x \in \mathrm{B}(f). \tag{9.12b}$$

This is a minimum-norm point problem for $\mathrm{B}(f)$ (see Section 7.1.a, where $w(e) = 1$ $(e \in E)$). Problem $P_1{}^w$ can be solved by the decomposition algorithm shown in Section 8.2.

The following procedure was also given in [Fuji80b] for polymatroids and in [Megiddo74] for multiterminal networks.

A Monotone Algorithm

Step 1: Put $i \leftarrow 1$ and $F \leftarrow E$.

Step 2: Compute

$$\lambda^* = \max\{\lambda \mid \lambda w^F \in \mathrm{P}(f)\}. \tag{9.13}$$

Put $E_i \leftarrow \mathrm{sat}(\lambda^* w^F)$ and $x^*(e) \leftarrow \lambda^* w(e)$ for each $e \in E_i$.

Step 3: If $\lambda^* w^F \in \mathrm{B}(f)$ (or $E_i = F$), then stop. Otherwise put $(\mathcal{D}, f) \leftarrow (\mathcal{D}, f)/E_i$ and $F \leftarrow F - E_i$. Put $i \leftarrow i + 1$ and go to Step 2.
(End)

Note that w^F appearing in Step 2 is the restriction of w to $F \subseteq E$.

The above monotone algorithm can also be viewed as follows. Start from a subbase $x = \lambda w \in \mathrm{P}(f)$ for a sufficiently small λ, where one such λ can be given in terms of the greatest lower bound $\underline{\alpha}$ defined by (3.94) and (3.95) when $(\mathcal{D}, f)$ is simple, since $\lambda w \leq \underline{\alpha}$ implies $\lambda w \in \mathrm{P}(f)$; or take any subbase $y \in \mathrm{P}(f)$ and choose λ such that $\lambda w \leq y$. Increase all the components of x $(= \lambda w)$ proportionally to w as far as x belongs to $\mathrm{P}(f)$. Then fix the saturated components of x and increase all the other components of x proportionally to w as far as x belongs to $\mathrm{P}(f)$. Repeat this process until x becomes a base in $\mathrm{B}(f)$.

The above monotone algorithm can also be adapted to Problem P_2 with nonlinear weight functions h_e $(e \in E)$.

Theorem 9.3 ([Fuji80b]): *Let x^* be the base in $\mathrm{B}(f)$ obtained by the monotone algorithm described above. Then x^* is the lexicographically optimal base with respect to the weight vector w.*

(Proof) We see from the monotone algorithm that for any $e,\ e' \in E$ such that

$$x^*(e)/w(e) < x^*(e')/w(e') \tag{9.14}$$

(e, e') is not an exchangeable pair associated with x^*. The optimality of x^* follows from Theorems 8.1 and 9.1. Q.E.D.

It should be noted that the above-described monotone algorithm traverses the leaves, from the leftmost to the rightmost one, of the binary tree mentioned in Remark 8.1 in Section 8.2. For a general submodular system $(\mathcal{D}, f)$, computing λ^* in (9.13) is not easy even if we are given oracles for saturation capacities and exchange capacities for $\mathrm{P}(f)$. One exception is the case where $\mathcal{D}$ is induced by a laminar family of subsets of E expressed as a tree structure. Also, in case of multiple-source multiple-sink networks considered in [Megiddo74] and [Fuji80b], the problem can be solved in time proportional to that required for finding a maximum flow in the same network (see [Gallo + Grigoriadis + Tarjan89], where use is made of the distance labeling technique introduced by A. V. Goldberg and R. E. Tarjan ([Goldberg + Tarjan88])).

For efficient algorithms when f is a graphic rank function, see [Imai83] and [Cunningham85b]. [Also see [Fleischer+Iwata03].]

Lemma 9.4 (see [Fuji79]): *Let x^* be the lexicographically optimal base with respect to the weight vector w. Then, for any $\lambda \in \mathbf{R}$ $x^* \wedge (\lambda w) = (\min\{x^*(e),\ \lambda w(e)\} \mid e \in E)$ is a base of λw (i.e., a maximal vector in $\mathrm{P}(f)^{\lambda w}$).*

(Proof) Since the lexicographically optimal base x^* is unique and the above monotone algorithm finds it, the present lemma follows from the algorithm. Q.E.D.

In the sense of Lemma 9.4 we also call x^* the *universal base* with respect to w (cf. [Nakamura + Iri81]).

Now, let us examine the relationship between the lexicographically optimal base and the subdifferentials of f.

Theorem 9.5: *Let x^* be the lexicographically optimal base of $(\mathcal{D}, f)$ with respect to w. For $\lambda \in \mathbf{R}$ and $A \in \mathcal{D}$ we have $\lambda w \in \partial f(A)$ if and only if, defining*

$$B_\lambda{}^+ = \{e \mid e \in E,\ x^*(e) \le \lambda w(e)\}, \tag{9.15}$$

$$B_\lambda{}^- = \{e \mid e \in E,\ x^*(e) < \lambda w(e)\}, \tag{9.16}$$

we have $B_\lambda{}^- \subseteq A \subseteq B_\lambda{}^+$ *and* $A \in \mathcal{D}(x^*)$ *(i.e.,* $x^*(A) = f(A)$*).*

(Proof) *The "if" part*: Suppose that $B_\lambda{}^- \subseteq A \subseteq B_\lambda{}^+$ and $A \in \mathcal{D}(x^*)$. Then for any $X \in \mathcal{D}$ such that $X \subseteq A$ or $A \subseteq X$, we have

$$\begin{aligned} \lambda w(X) - \lambda w(A) &\leq x^*(X) - x^*(A) \\ &\leq f(X) - f(A). \end{aligned} \tag{9.17}$$

From Lemma 6.4 we have $\lambda w \in \partial f(A)$.

The "only if" part: Suppose $\lambda w \in \partial f(A)$. From the monotone algorithm x^* satisfies

$$x^*(B_\lambda{}^-) = f(B_\lambda{}^-). \tag{9.18}$$

From (9.18) and the assumption we have

$$\begin{aligned} \lambda w(B_\lambda{}^-) - \lambda w(A) &\leq f(B_\lambda{}^-) - f(A) \\ &\leq x^*(B_\lambda{}^-) - x^*(A) \end{aligned} \tag{9.19}$$

or

$$\lambda w(B_\lambda{}^-) - x^*(B_\lambda{}^-) \leq \lambda w(A) - x^*(A). \tag{9.20}$$

Since from (9.16) $\lambda w(X) - x^*(X)$ is maximized at $X = B_\lambda{}^-$, it follows from (9.20) that $X = A$ also maximizes $\lambda w(X) - x^*(X)$. This implies $B_\lambda{}^- \subseteq A \subseteq B_\lambda{}^+$ because of (9.15) and (9.16). Since we have

$$\lambda w(B_\lambda{}^-) - x^*(B_\lambda{}^-) = \lambda w(A) - x^*(A), \tag{9.21}$$

from (9.18), (9.21) and the fact that $x^*(A) \leq f(A)$ we have

$$\lambda w(B_\lambda{}^-) - \lambda w(A) \geq f(B_\lambda{}^-) - f(A). \tag{9.22}$$

Since $\lambda w \in \partial f(A)$, we must have from (9.22)

$$x^*(A) = f(A). \tag{9.23}$$

Q.E.D.

Corollary 9.6: *Under the same assumption as in Theorem 9.5, let* $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ *be the distinct values of* $x^*(e)/w(e)$ $(e \in E)$*. Then we have*

$$\begin{aligned} \lambda_1 &= \min\{f(X)/w(X) \mid X \in \mathcal{D},\ X \neq \emptyset\} \\ &= \max\{\lambda \mid \lambda w \in \mathrm{P}(f)\}, \end{aligned} \tag{9.24}$$

$$\begin{aligned} \lambda_p &= \max\{f^{\#}(X)/w(X) \mid X \in \overline{\mathcal{D}},\ X \neq \emptyset\} \\ &= \min\{\lambda \mid \lambda w \in \mathrm{P}(f^{\#})\}. \end{aligned} \tag{9.25}$$

(Proof) Note that $\partial f(\emptyset) = \mathrm{P}(f)$ and $\partial f(E) = \mathrm{P}(f^{\#})$. It follows from Theorem 9.5 that $\lambda w \in \mathrm{P}(f)$ if and only if $B_\lambda^- = \emptyset$ or $x^*(e) \geq \lambda w(e)$ for all $e \in E$. Therefore,

$$\max\{\lambda \mid \lambda w \in \mathrm{P}(f)\} = \min\{x^*(e)/w(e) \mid e \in E\} = \lambda_1. \tag{9.26}$$

Also, since $\lambda w \in \mathrm{P}(f)$ if and only if $\lambda w(X) \leq f(X)$ for all $X \in \mathcal{D}$, (9.24) follows from (9.26). Similarly, from Theorem 9.5, we have $\lambda w \in \mathrm{P}(f^{\#})$ if and only if $B_\lambda^+ = E$ or $x^*(e) \leq \lambda w(e)$ for all $e \in E$. This implies (9.25). Q.E.D.

Corollary 9.7: *Under the same assumption as in Theorem 9.5, λw belongs to the interior of $\partial f(A)$ if and only if $A = {B_\lambda}^- = {B_\lambda}^+$.*

(Proof) Since B_λ^-, $B_\lambda^+ \in \mathcal{D}$, the present corollary immediately follows from Theorem 9.5, but we will give an alternative proof.

Suppose that λw is in the interior of $\partial f(A)$. If $A \neq {B_\lambda}^-$, then

$$\begin{aligned} \lambda w({B_\lambda}^-) - \lambda w(A) &< f({B_\lambda}^-) - f(A) \\ &\leq x^*({B_\lambda}^-) - x^*(A). \end{aligned} \tag{9.27}$$

This contradicts the fact that $X = {B_\lambda}^-$ maximizes $\lambda w(X) - x^*(X)$. Therefore, we have $A = {B_\lambda}^-$. Similarly, we have $A = {B_\lambda}^+$ and hence ${B_\lambda}^- = A = {B_\lambda}^+$.

Conversely, suppose $A = {B_\lambda}^- = {B_\lambda}^+$. Then $X = A$ $(= {B_\lambda}^- = {B_\lambda}^+)$ is the unique maximizer of $\lambda w(X) - x^*(X)$, so that

$$\lambda w(X) - x^*(X) < \lambda w(A) - x^*(A) \quad (X \in \mathcal{D},\ X \neq A). \tag{9.28}$$

From (9.28),

$$\lambda w(X) - \lambda w(A) < x^*(X) - x^*(A) \leq f(X) - f(A) \quad (X \in \mathcal{D},\ X \neq A), \tag{9.29}$$

where note that $x^*(A) = f(A)$. Hence λw belongs to the interior of $\partial f(A)$. Q.E.D.

Let

$$\partial f(A_0) = \partial f(\emptyset),\ \partial f(A_1),\ \cdots,\ \partial f(A_p) = \partial f(E) \tag{9.30}$$

be the subdifferentials of f whose interiors have nonempty intersections with the line $L = \{\lambda w \mid \lambda \in \mathbf{R}\}$, where the subdifferentials in (9.30) are arranged in order of increasing $\lambda \in \mathbf{R}$. Suppose that for each $i = 0, 1, \cdots, p$ $(\lambda_i, \lambda_{i+1})$ is the open interval consisting of those λ for which λw belongs to the interior of $\partial f(A_i)$, where $\lambda_0 \equiv -\infty$ and $\lambda_{p+1} = +\infty$. We see from Theorem 9.5 and Corollary 9.6 that for each $i = 1, 2, \cdots, p$

$$x^*(e) = \lambda_i w(e) \quad (e \in A_i - A_{i-1}). \tag{9.31}$$

Compare the present results with those in Sections 7.1.a and 7.2.b.1 (also see [Fuji84c]).

Recall that $\lambda w \in \partial f(A)$ if and only if $X = A$ is a minimizer of $f(X) - \lambda w(X)$ $(X \in \mathcal{D})$. Therefore, λw belongs to the interior of $\partial f(A)$ if and only if $X = A$ is a unique minimizer of $f(X) - \lambda w(X)$ $(X \in \mathcal{D})$ (see Corollary 9.7). Hence, λ_1, λ_2, $\cdots$, λ_p are exactly the critical values associated with the principal partition of the pair of submodular systems $(\mathcal{D}_i, f_i)$ $(i = 0, 1)$ with $\mathcal{D}_0 = \mathcal{D}$, $\mathcal{D}_1 = 2^E$, $f_0 = f$ and $f_1 = -w$.

The concept of lexicographically optimal base is generalized by M. Nakamura [Nakamura81] and N. Tomizawa [Tomi80d]. Suppose that we are given two (polymatroid) base polyhedra $\mathrm{B}(f_i)$ $(i = 1, 2)$ such that every base of $\mathrm{B}(f_i)$ $(i = 1, 2)$ consists of positive components alone. If b_1 is the lexicographically optimal base of $\mathrm{B}(f_1)$ with respect to a weight vector $b_2 \in \mathrm{B}(f_2)$ and b_2 is the lexicographically optimal base of $\mathrm{B}(f_2)$ with respect to b_1, then the pair (b_1, b_2) is called a *universal pair of bases* (the original definition in [Nakamura81] and [Tomi80d] is different from but equivalent to the present one (see Section 7.2.b.1)). Some characterizations of universal pairs are given by K. Murota [Murota88].

10. The Weighted Max-Min and Min-Max Problems

We consider the problem of maximizing the minimum (or minimizing the maximum) of a nonlinear objective function over the base polyhedron $\mathrm{B}(f)$.

10.1. Continuous Variables

For each $e \in E$ let $h_e\colon \mathbf{R} \to \mathbf{R}$ be a right-continuous and monotone nondecreasing function such that $\lim_{\xi \to +\infty} h_e(\xi) = +\infty$ and $\lim_{\xi \to -\infty} h_e(\xi) = -\infty$. Consider the following max-min problem with the nonlinear weight function h_e $(e \in E)$.

$$P_*\colon \text{ Maximize } \quad \min_{e\in E} h_e(x(e)) \tag{10.1a}$$

$$\text{subject to } \quad x \in \mathrm{B}_{\mathbf{R}}(f), \tag{10.1b}$$

where $\mathrm{B}_{\mathbf{R}}(f)$ is the base polyhedron associated with a submodular system $(\mathcal{D}, f)$ on E and the underlying totally ordered additive group is assumed to be the set $\mathbf{R}$ of reals.

For each $e \in E$ let $w_e\colon \mathbf{R} \to \mathbf{R}$ be a convex function whose right derivative ${w_e}^+$ is given by h_e.

Theorem 10.1: *Consider Problem P_1 in (8.1) with w_e ($e \in E$) such that ${w_e}^+ = h_e$. Let x be an optimal solution of Problem P_1. Then x is an optimal solution of Problem P_* in (10.1).*

(Proof) Define

$$\eta_1 = \min\{h_e(x(e)) \mid e \in E\}, \tag{10.2}$$

$$S_1 = \{e \mid e \in E,\ h_e(x(e)) = \eta_1\}, \tag{10.3}$$

$${S_1}^* = \bigcup\{\mathrm{dep}(x, e) \mid e \in S_1\}. \tag{10.4}$$

We have from (10.4)

$$x({S_1}^*) = f({S_1}^*). \tag{10.5}$$

It follows from Theorem 8.1 that

$${w_e}^-(x(e)) \le \eta_1 \quad (e \in {S_1}^*). \tag{10.6}$$

If there were a base $y \in \mathrm{B}_{\mathbf{R}}(f)$ such that

$$\eta_1 < \min\{h_e(y(e)) \mid e \in E\}, \tag{10.7}$$

then from (10.2)$\sim$(10.7) we would have

$$x(e) < y(e) \quad (e \in S_1), \tag{10.8}$$

$$x(e) \le y(e) \quad (e \in {S_1}^* - S_1), \tag{10.9}$$

since $h_e = {w_e}^+$. Hence, from (10.5), (10.8) and (10.9),

$$f({S_1}^*) = x({S_1}^*) < y({S_1}^*), \tag{10.10}$$

which contradicts the fact that $y \in \mathrm{B}_{\mathbf{R}}(f)$. Q.E.D.

We see from the above proof that the decomposition algorithm given in Section 8.2 can be simplified for solving Problem P_* as follows. We may put

$x^*(e) = x(e)$ for each $e \in E_0 \cup E_+$ in Step 2 and apply the decomposition algorithm recursively to the problem on E_- but not to the one on E_+ in Step 3 (cf. [Ichimori + Ishii + Nishida82]). In other words, we only go down the leftmost path in the binary decomposition tree mentioned in Remark 8.1 in Section 8.2.

Next, consider the following min-max problem

$$P^*\colon \text{ Minimize } \max_{e\in E} h_e(x(e)) \tag{10.11a}$$

$$\text{subject to } x \in \mathrm{B}_{\mathbf{R}}(f). \tag{10.11b}$$

Here, we assume that h_e is left-continuous rather than right-continuous for each $e \in E$.

For each $e \in E$ let $w_e\colon \mathbf{R} \to \mathbf{R}$ be a convex function whose left derivative ${w_e}^-$ is given by h_e. Then we have

Corollary 10.2: *Consider Problem P_1 in (8.1) with w_e $(e \in E)$ such that ${w_e}^- = h_e$. Let x be an optimal solution of Problem P_1. Then x is an optimal solution of Problem P^* in (10.11).*

The proof of Corollary 10.2 is similar to that of Theorem 10.1 by duality. An optimal solution of Problem P^* can be obtained by the decomposition algorithm given in Section 8.2, where we only go down the rightmost path of the binary decomposition tree mentioned in Remark 8.1.

Moreover, suppose that for each $e \in E$ h_e is continuous monotone nondecreasing function such that $\lim_{\xi\to+\infty} h_e(\xi) = +\infty$ and $\lim_{\xi\to-\infty} h_e(\xi) = -\infty$. From Theorem 10.1 and Corollary 10.2 we have the following.

Corollary 10.3: *Consider Problem P_1 in (8.1) with w_e $(e \in E)$ such that the derivative of w_e is equal to h_e for each $e \in E$. Let x be an optimal solution of Problem P_1. Then x is an optimal solution of Problem P^* of (10.1) and, at the same time, an optimal solution of Problem P_* in (10.11).*

A common optimal solution of Problems P_* and P^* can be obtained by the decomposition algorithm given in Section 8.2, where we only go down the leftmost and rightmost paths of the binary decomposition tree mentioned in Remark 8.1.

Problems P_* and P^* are sometimes called *sharing problems* in the literature ([Brown79], [Ichimori + Ishii + Nishida82]). The sharing problems with more general objective functions and feasible regions are considered by U. Zimmermann [Zimmermann86a,86b].

10.2. Discrete Variables

For each $e \in E$ let $\hat{h}_e: \mathbf{Z} \to \mathbf{R}$ be a monotone nondecreasing function on $\mathbf{Z}$ such that $\lim_{\xi \to +\infty} \hat{h}_e(\xi) = +\infty$ and $\lim_{\xi \to -\infty} \hat{h}_e(\xi) = -\infty$ for each $e \in E$. Consider

$$IP_*\text{: Maximize} \quad \min_{e \in E} \hat{h}_e(x(e)) \tag{10.12a}$$

$$\text{subject to} \quad x \in \mathrm{B}_{\mathbf{Z}}(f), \tag{10.12b}$$

where $\mathrm{B}_{\mathbf{Z}}(f)$ is the base polyhedron associated with an integral submodular system $(\mathcal{D}, f)$ on E and the underlying totally ordered additive group is the set $\mathbf{Z}$ of integers.

For each $e \in E$ let $w_e: \mathbf{R} \to \mathbf{R}$ be a piecewise-linear convex function such that the following two hold:

(i) its right derivative ${w_e}^+$ satisfies ${w_e}^+(\xi) = \hat{h}_e(\xi) \quad (\xi \in \mathbf{Z})$, (10.13a)

(ii) w_e is linear on each unit interval $[\xi, \xi + 1]$ $(\xi \in \mathbf{Z})$. (10.13b)

Theorem 10.4: *Let x_* be an integral optimal solution of Problem P_1 in* (8.1) *with w_e $(e \in E)$ defined as above. Then x_* is an optimal solution of Problem IP_* in* (10.12).

(Proof) For each $e \in E$ let $h_e: \mathbf{R} \to \mathbf{R}$ be a right-continuous piecewise-constant nondecreasing function such that $h_e(\eta) = \hat{h}_e(\xi)$ $(\eta \in [\xi, \xi+1),\ \xi \in \mathbf{Z})$. It follows from Theorem 10.1 that an integral optimal solution of Problem P_1 with w_e $(e \in E)$ defined by (10.13) is an integral optimal solution of Problem P_* in (10.1) with h_e $(e \in E)$ defined as above. Therefore, x_* is an optimal solution of Problem IP_*. Q.E.D.

It should be noted that there exists an integral optimal solution x_* of Problem P_1 in Theorem 10.4 due to Theorem 8.3. The reduction of Problem IP_* to Problem P_1 was also communicated by N. Katoh [Katoh85]. A direct algorithm for Problem IP_* is given in [Fuji + Katoh + Ichimori88].

Moreover, consider the weighted min-max problem

$$IP^*\text{: Minimize} \quad \max_{e \in E} \hat{h}_e(x(e)) \tag{10.14a}$$

$$\text{subject to} \quad x \in \mathrm{B}_{\mathbf{Z}}(f). \tag{10.14b}$$

For each $e \in E$ let $w_e: \mathbf{R} \to \mathbf{R}$ be a piecewise-linear convex function such that

(i) its left derivative ${w_e}^-$ satisfies ${w_e}^-(\xi) = \hat{h}_e(\xi) \quad (\xi \in \mathbf{Z})$, (10.15a)

(ii) w_e is linear on each unit interval $[\xi, \xi+1]$ $(\xi \in \mathbf{Z})$. (10.15b)

Similarly as Theorem 10.4 we have

Corollary 10.5: *Let x^* be an integral optimal solution of Problem P_1 in* (8.1) *with w_e $(e \in E)$ defined by* (10.15)*. Then x^* is an optimal solution of Problem IP^* in* (10.14)*.*

The discrete max-min and min-max problems are thus reduced to continuous ones.

11. The Fair Resource Allocation Problem

In this section we consider the problem of allocating resources in a fair manner which generalizes the max-min and min-max problems treated in the preceding section. The readers should also be referred to the book [Ibaraki + Katoh88] by T. Ibaraki and N. Katoh for resource allocation problems and related topics.

11.1. Continuous Variables

Let $g\colon \mathbf{R}^2 \to \mathbf{R}$ be a function such that $g(u, v)$ is monotone nondecreasing in u and monotone nonincreasing in v. Typical examples of such a function g are the following.

$$g(u,v) = u - v, \tag{11.1}$$
$$g(u,v) = u/v \quad (u,\ v > 0). \tag{11.2}$$

Also, for each $e \in E$ let h_e be a continuous monotone nondecreasing function from $\mathbf{R}$ onto $\mathbf{R}$.

Consider

$$P_3\colon \ \text{Minimize} \ \ g(\max_{e\in E} h_e(x(e)), \min_{e \in E} h_e(x(e))) \tag{11.3a}$$
$$\text{subject to} \ \ x \in \mathrm{B}_{\mathbf{R}}(f). \tag{11.3b}$$

We call Problem P_3 the *continuous fair resource allocation problem with* *submodular constraints.* This type of objective function was first considered by N. Katoh, T. Ibaraki and H. Mine [Katoh + Ibaraki + Mine85].

Using the same functions h_e $(e \in E)$ appearing in (11.3), let us consider Problems P_* and P^* described by (10.1) and (10.11), respectively. Denote the optimal values of the objective functions of Problems P_* and P^* by v_* and v^*, respectively, and define vectors $l,\ u \in \mathbf{R}^E$ by

$$l(e) = \min\{\alpha \mid \alpha \in \mathbf{R},\ h_e(\alpha) \geq v_*\} \quad (e \in E), \tag{11.4}$$

$$u(e) = \max\{\alpha \mid \alpha \in \mathbf{R},\ h_e(\alpha) \leq v^*\} \quad (e \in E). \tag{11.5}$$

Theorem 11.1: *Suppose that values v_* and v^* and vectors l and u are defined as above. Then we have $v_* \leq v^*$ and $l \leq u$. Moreover, $\mathrm{B}(f)_l^u$ is nonempty and any $x \in \mathrm{B}(f)_l^u$ is an optimal solution of Problem P_3 in* (11.3), *where* $\mathrm{B}(f)_l^u = \{x \mid x \in \mathrm{B}(f),\ l \leq x \leq u\}$ (see Section 3.1.b).

(Proof) Let x_* and x^*, respectively, be optimal solutions of Problems P_* and P^*. If $v_* > v^*$, then we have

$$x^*(e) \leq u(e) < l(e) \leq x_*(e) \quad (e \in E), \tag{11.6}$$

which contradicts the fact that $x^*(E) = f(E) = x_*(E)$. Therefore, we have $v_* \leq v^*$. This implies $l \leq u$. Moreover, since $x_* \in \mathrm{B}(f)_l$, $x^* \in \mathrm{B}(f)^u$ and $l \leq u$, from Theorem 3.8 we have $\mathrm{B}(f)_l^u \neq \emptyset$. For any $x \in \mathrm{B}(f)_l^u$ and $y \in \mathrm{B}(f)$ we have

$$\begin{aligned} & g(\max_{e\in E} h_e(y(e)), \min_{e\in E} h_e(y(e))) \\ & \geq g(v^*, v_*) \\ & = g(\max_{e\in E} h_e(x(e)), \min_{e\in E} h_e(x(e))), \end{aligned} \tag{11.7}$$

due to the monotonicity of g. This shows that any $x \in \mathrm{B}(f)_l^u$ is an optimal solution of Problem P_3. Q.E.D.

The continuous fair resource allocation problem P_3 is thus reduced to Problems P_* and P^* and can be solved by the decomposition algorithm given in Section 8.2.

11.2. Discrete Variables

We consider the *discrete fair resource allocation problem*, which is a discrete version of the continuous fair resource allocation problem P_3 treated in Section 11.1 (see [Fuji + Katoh + Ichimori88]).

Let $g: \mathbf{R}^2 \to \mathbf{R}$ be a function such that $g(u, v)$ is monotone nondecreasing in u and monotone nonincreasing in v. Also, for each $e \in E$ let $\hat{h}_e: \mathbf{Z} \to \mathbf{R}$ be a monotone nondecreasing function. We assume for simplicity that $\lim_{\xi \to +\infty} \hat{h}_e(\xi) = +\infty$ and $\lim_{\xi \to -\infty} \hat{h}_e(\xi) = -\infty$.

For a submodular system $(\mathcal{D}, f)$ on E with an integer-valued rank function f, consider the problem

$$IP_3\text{: Minimize } \; g(\max_{e \in E} \hat{h}_e(x(e)), \min_{e \in E} \hat{h}_e(x(e))) \tag{11.8a}$$

$$\text{subject to } \; x \in \mathrm{B}_{\mathbf{Z}}(f). \tag{11.8b}$$

Problem IP_3 is not so easy as its continuous version P_3 because of the integer constraints.

Using the same functions $\hat{h}_e$ $(e \in E)$, consider the weighted integral max-min problem IP_* in (10.12) and the weighted integral min-max problem IP^* in (10.14). Let $\hat{v}_*$ and $\hat{v}^*$, respectively, be the optimal values of the objective functions of IP_* and IP^*. Define vectors $\hat{l}, \; \hat{u} \in \mathbf{Z}^E$ by

$$\hat{l}(e) = \min\{\alpha \mid \alpha \in \mathbf{Z}, \; \hat{h}_e(\alpha) \geq \hat{v}_*\}, \tag{11.9}$$

$$\hat{u}(e) = \max\{\alpha \mid \alpha \in \mathbf{Z}, \; \hat{h}_e(\alpha) \leq \hat{v}^*\} \tag{11.10}$$

for each $e \in E$. We have $\hat{v}_* \leq \hat{v}^*$ but, unlike the continuous version of the problem, we may not have $\hat{l} \leq \hat{u}$ in general. However, we have

$$\hat{l}(e) \leq \hat{u}(e) + 1 \quad (e \in E). \tag{11.11}$$

Lemma 11.2: *If we have $\hat{l} \leq \hat{u}$, then $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}^{\hat{u}}$ is nonempty and any $x \in \mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}^{\hat{u}}$ is an optimal solution of Problem IP_3 in* (11.8).

(Proof) Since the vector minors $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}$, $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}$ and $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}^{\hat{u}}$ are integral, the present lemma can be shown similarly as Theorem 11.1. Q.E.D.

Now, let us suppose that we do not have $\hat{l} \leq \hat{u}$. Define

$$D = \{e \mid e \in E, \; \hat{l}(e) > \hat{u}(e)\}. \tag{11.12}$$

It follows from (11.9)$\sim$(11.12) that

$$\hat{l}(e) = \hat{u}(e) + 1 \quad (e \in D), \tag{11.13}$$

$$\hat{l}(e) \le \hat{u}(e) \quad (e \in E - D), \tag{11.14}$$

$$\hat{h}_e(\hat{u}(e)) < \hat{v}_* \le \hat{v}^* < \hat{h}_e(\hat{l}(e)) \quad (e \in D), \tag{11.15}$$

$$\hat{v}_* \le \hat{h}_e(\hat{l}(e)) \le \hat{h}_e(\hat{u}(e)) \le \hat{v}^* \quad (e \in E - D). \tag{11.16}$$

Moreover, define $\hat{l}\wedge\hat{u} = (\min\{\hat{l}(e), \hat{u}(e)\} \mid e \in E)$ and $\hat{l}\vee\hat{u} = (\max\{\hat{l}(e), \hat{u}(e)\} \mid e \in E)$. Then, all the four sets $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}$, $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}$, $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}\wedge\hat{u}}$ and $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{l}\vee\hat{u}}$ are nonempty since $\mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}\wedge\hat{u}} \supseteq \mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}} \neq \emptyset$ and $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{l}\vee\hat{u}} \supseteq \mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}} \neq \emptyset$. Therefore, from Theorem 3.8 $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l}\wedge\hat{u}}$ and $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{l}\vee\hat{u}}_{\hat{l}}$ are nonempty. Choose any bases $\hat{x} \in \mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l}\wedge\hat{u}}$ and $\hat{y} \in \mathrm{B}_{\mathbf{Z}}(f)^{\hat{l}\vee\hat{u}}_{\hat{l}}$. From (11.12) we have

$$\hat{x}(e) = \hat{u}(e), \quad \hat{y}(e) = \hat{l}(e) \quad (e \in D). \tag{11.17}$$

Hence, from (11.13)$\sim$(11.16),

$$\hat{y}(e) = \hat{x}(e) + 1 \quad (e \in D), \tag{11.18}$$

$$\hat{h}_e(\hat{x}(e)) < \hat{v}_* \le \hat{v}^* < \hat{h}_e(\hat{y}(e)) \quad (e \in D), \tag{11.19}$$

$$\hat{v}_* \le \min\{\hat{h}_e(\hat{x}(e)), \hat{h}_e(\hat{y}(e))\} \le \max\{\hat{h}_e(\hat{x}(e)), \hat{h}_e(\hat{y}(e))\} \le \hat{v}^* \quad (e \in E-D). \tag{11.20}$$

Let the distinct values of $\hat{h}_e(\hat{x}(e))$ $(e \in D)$ be given by

$$d_1 < d_2 < \cdots < d_k, \tag{11.21}$$

where note that $d_k < \hat{v}_*$. Also define

$$A_i = \{e \mid e \in D,\ \hat{h}_e(\hat{x}(e)) \le d_i\} \quad (i = 1, 2, \cdots, k). \tag{11.22}$$

We consider a parametric problem $IP_3{}^{\lambda}$ with a real parameter λ associated with the original problem IP_3.

$$IP_3{}^{\lambda}\text{: Minimize } \ g(\max_{e\in E} \hat{h}_e(x(e)), \lambda) \tag{11.23a}$$

$$\text{subject to } \ x \in \mathrm{B}_{\mathbf{Z}}(f), \tag{11.23b}$$

$$\hat{h}_e(x(e)) \ge \lambda \quad (e \in E), \tag{11.23c}$$

where $\lambda \le \hat{v}_*$. Denote the minimum of the objective function (11.23a) of

$IP_3{}^\lambda$ by $\gamma(\lambda)$. Then, we have

Lemma 11.3: *Suppose that $\lambda = \lambda^*$ attains the minimum of $\gamma(\lambda)$ for $\lambda \le \hat{v}_*$. Then any optimal solution of $IP_3{}^{\lambda^*}$ is an optimal solution of the original problem IP_3 and the minimum of the objective function* (11.8a) *of IP_3 is equal to $\gamma(\lambda^*)$.*

(Proof) Let x^* and x^0, respectively, be any optimal solutions of Problems $IP_3{}^{\lambda^*}$ and IP_3. Denote by v^0 be the minimum of the objective function (11.8a) of Problem IP_3, and define $\lambda^0 = \min_{e \in E} \hat{h}_e(x^0(e))$. Then

$$v^0 = g(\max_{e \in E} \hat{h}_e(x^0(e)), \lambda^0) \ge \gamma(\lambda^0) \ge \gamma(\lambda^*). \tag{11.24}$$

On the other hand, we have

$$\gamma(\lambda^*) \ge g(\max_{e \in E} \hat{h}_e(x^*(e)), \min_{e \in E} \hat{h}_e(x^*(e))) \ge v^0. \tag{11.25}$$

From (11.24) and (11.25) we have $v^0 = \gamma(\lambda^*)$ and x^* is also an optimal solution of IP_3. Q.E.D.

We determine the function $\gamma(\lambda)$ to solve the original problem IP_3 with the help of Lemma 11.3. In the following arguments, $\hat{x}$, $\hat{y}$, d_i, A_i $(i = 1, 2, \cdots, k)$ are those appearing in (11.17)$\sim$(11.22). We consider the two cases when $\lambda \le d_1$ and when $d_i < \lambda \le d_{i+1}$ for some $i \in \{1, 2, \cdots, k\}$, where $d_{k+1} \equiv \hat{v}_*$.

Case I: $\lambda \le d_1$.

Because of the definition (11.21) of d_1, $\hat{x}$ is a feasible solution of $IP_3{}^\lambda$ for $\lambda \le d_1$. Since $\hat{x} \in \mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l} \wedge \hat{u}}$, $\hat{x}$ is also an optimal solution of IP^*. Therefore, it follows from the monotonicity of g that

$$\gamma(\lambda) = g(\hat{v}^*, \lambda). \tag{11.26}$$

Case II: $d_i < \lambda \le d_{i+1}$ for some $i \in \{1, 2, \cdots, k\}$ $(d_{k+1} \equiv \hat{v}_*)$.

From (11.18)$\sim$(11.22) and the monotonicity of g we have

$$\gamma(\lambda) \ge g(\max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1), \lambda). \tag{11.27}$$

It follows, from (11.18)$\sim$(11.20) for two bases $\hat{x}$ and $\hat{y}$, that by repeated elementary transformations of $\hat{x}$ we can have a base $\hat{z} \in \mathrm{B}_{\mathbf{Z}}(f)$ such that

$$\hat{z}(e) = \hat{x}(e) \quad (e \in D - A_i), \tag{11.28}$$

$$\hat{z}(e) = \hat{y}(e) \ (= \hat{x}(e) + 1) \quad (e \in A_i), \tag{11.29}$$

$$\min\{\hat{x}(e), \hat{y}(e)\} \le \hat{z}(e) \le \max\{\hat{x}(e), \hat{y}(e)\} \quad (e \in E - D). \tag{11.30}$$

We see from (11.18)∼(11.20) and (11.28)∼(11.30) that $\hat{z}$ is a feasible solution of $IP_3{}^\lambda$ for the present λ and that

$$\max_{e \in E} \hat{h}_e(\hat{z}(e)) = \max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1). \tag{11.31}$$

From (11.27) and (11.31) $\hat{z}$ is an optimal solution of $IP_3{}^\lambda$ and we have

$$\gamma(\lambda) = g(\max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1), \lambda). \tag{11.32}$$

The function $\gamma(\lambda)$ is thus given by (11.26) for $\lambda \le d_1$ and by (11.32) for $d_i < \lambda \le d_{i+1}$ $(i = 1, 2, \cdots, k)$.

Theorem 11.4: *Suppose that we do not have $\hat{l} \le \hat{u}$. Then, the minimum of the objective function* (11.8a) *of Problem IP_3 is equal to the minimum of the following $k+1$ values*

$$g(\hat{v}^*, d_1), \quad g(\max_{e \in A_i} \hat{h}_e(\hat{u}(e) + 1), d_{i+1}) \quad (i = 1, 2, \cdots, k). \tag{11.33}$$

(Proof) The present theorem follows from Lemma 11.3, (11.17), (11.26), (11.32) and the monotonicity of g. Q.E.D.

It should be noted that because of (11.17) we can employ $\hat{u}$ instead of $\hat{x}$ in the definitions of d_i and A_i $(i = 1, 2, \cdots, k)$ in (11.21) and (11.22).

An algorithm for solving Problem IP_3 in (11.8) is given as follows.

An algorithm for the discrete fair resource allocation problem

Step 1: Solve Problems IP_* and IP^* given by (10.12) and (10.14), respectively. Let $\hat{v}_*$ and $\hat{v}^*$, respectively, be the optimal values of the objective functions of IP_* and IP^* and determine vectors $\hat{l}$ and $\hat{u}$ by (11.9) and (11.10). If $\hat{l} \le \hat{u}$, then any base $x \in \mathrm{B}_{\mathbf{Z}}(f)_{\hat{l}}^{\hat{u}}$ is an optimal solution of IP_3 and the algorithm terminates.

Step 2: Let $D \subseteq E$ be defined by (11.12) and $d_1 < d_2 < \cdots < d_k$ be the distinct values of $\hat{h}_e(\hat{u}(e))$ $(e \in D)$. Also determine sets A_i $(i = 1, 2, \cdots, k)$ by (11.22) with $\hat{x}$ replaced by $\hat{u}$.

Step 3: Find the minimum of the following $k+1$ values.

$$g(\hat{v}^*, d_1), \quad g(\max_{e \in A_i} \hat{h}_e(\hat{u}(e) + 1), d_{i+1}) \quad (i = 1, 2, \cdots, k). \tag{11.34}$$

(3-1) If $g(\hat{v}^*, d_1)$ is minimum, then find any base $\hat{x} \in \mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l} \wedge \hat{u}}$ and $\hat{x}$ is an optimal solution of IP_3.

(3-2) If $g(\max_{e \in A_{i^*}} \hat{h}_e(\hat{u}(e)+1), d_{i^*+1})$ for $i^* \in \{1, 2, \cdots, k\}$ is minimum, then putting $w^* = \max_{e \in A_{i^*}} \hat{h}_e(\hat{u}(e)+1)$, define $l^0,\ u^0 \in \mathbf{Z}^E$ by

$$\begin{aligned} l^0(e) &= \min\{\alpha \mid \alpha \in \mathbf{Z},\ \hat{h}_e(\alpha) \geq d_{i^*+1}\}, && (11.35)\\ u^0(e) &= \max\{\alpha \mid \alpha \in \mathbf{Z},\ \hat{h}_e(\alpha) \leq w^*\} && (11.36)\end{aligned}$$

for each $e \in E$, and any base $\hat{z} \in \mathrm{B}_{\mathbf{Z}}(f)^{u^0}_{l^0}$ is an optimal solution of IP_3.

(End)

It should be noted that from the arguments preceding Theorem 11.4 we have $\mathrm{B}_{\mathbf{Z}}(f)^{u^0}_{l^0} \neq \emptyset$ in Step 3-2.

Let us consider the computational complexity of the algorithm when $\mathrm{B}_{\mathbf{Z}}(f)$ is bounded, i.e., $\mathcal{D} = 2^E$. Denote by $T(IP_*, IP^*)$ the time required for solving Problems IP_* and IP^*. Let M be an integral upper bound of $|f(X)|$ $(X \subseteq E)$. Then we have

$$-2M \leq f(E) - f(E - \{e\}) \leq x(e) \leq f(\{e\}) \leq M \tag{11.37}$$

for each $x \in \mathrm{B}_{\mathbf{Z}}(f)$ and $e \in E$. Therefore, given $\hat{v}_*$ and $\hat{v}^*$, we can determine the values of $\hat{l}(e)$ and $\hat{u}(e)$ in (11.9) and (11.10) by the binary search, which requires $\mathrm{O}(\log M)$ time for each $e \in E$, where we assume that the function evaluation of $\hat{h}_e$ requires unit time for each $e \in E$. Also, finding a base in $\mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l}}$ requires not more than $\mathrm{O}(T(IP_*, IP^*))$ time. Hence, Step 1 runs in $\mathrm{O}(T(IP_*, IP^*) + |E| \log M)$ time.

Determining set D in Step 2 requires $\mathrm{O}(|E|)$ time. The values of d_1, d_2, $\cdots$, d_k are found and sorted in $\mathrm{O}(|E| \log |E|)$ time. Instead of having sets A_i $(i = 1, 2, \cdots, k)$ we compute differences $\tilde{A}_i = A_i - A_{i-1}$ $(i = 1, 2, \cdots, k)$ with $A_0 = \emptyset$, which requires $\mathrm{O}(|E|)$ time. Note that having the differences $\tilde{A}_i = A_i - A_{i-1}$ $(i = 1, 2, \cdots, k)$ is enough to carry out Step 3. Then, Step 2 requires $\mathrm{O}(|E| \log |E|)$ time.

In Step 3 we can compute all the values of

$$\max_{e \in A_i} \hat{h}_e(\hat{u}(e)+1) \quad (i = 1, 2, \cdots, k) \tag{11.38}$$

in $\mathrm{O}(|E|)$ time, using the differences $\tilde{A}_i = A_i - A_{i-1}$ $(i = 1, 2, \cdots, k)$. Moreover, for each $e \in E$, $l^0(e)$ and $u^0(e)$ can be computed in $\mathrm{O}(\log M)$ time, and a base $\hat{x} \in \mathrm{B}_{\mathbf{Z}}(f)^{\hat{u}}_{\hat{l} \wedge \hat{u}}$ or $\hat{z} \in \mathrm{B}_{\mathbf{Z}}(f)^{u^0}_{l^0}$ can be found in $\mathrm{O}(T(IP_*, IP^*))$ time. Consequently, Step 3 runs in $\mathrm{O}(T(IP_*, IP^*) + |E| \log M)$ time.

The overall running time of the algorithm is thus $\mathrm{O}(T(IP_*, IP^*) + |E| \cdot (\log M + \log |E|))$ for general (bounded) submodular constraints. When specialized to the problem of Katoh, Ibaraki and Mine [Katoh + Ibaraki + Mine85], where

$$\mathrm{B}_{\mathbf{Z}}(f) = \{x \mid x \in \mathbf{Z}^E_+, \ l \le x \le u, \ x(E) = c\}, \tag{11.39}$$

$f(E)$ $(= c)$ can be chosen as M, $T(IP_*, IP^*)$ is $\mathrm{O}(|E| \log(f(E)/|E|)$ by the algorithm of G. N. Frederickson and D. B. Johnson [Frederickson + Johnson82], and hence the complexity becomes the same as the algorithm of [Katoh + Ibaraki + Mine85].

Very recently, K. Namikawa and T. Ibaraki [Namikawa + Ibaraki91] have improved the algorithm of [Fuji + Katoh + Ichimori88] by the use of an optimal solution of the continuous version of the original problem.

12. A Neoflow Problem with a Separable Convex Cost Function

In this section we consider the submodular flow problem (see Section 5.1) where the cost function is given by a separable convex function.

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A, $\overline{c}: A \to \mathbf{R} \cup \{+\infty\}$ be an upper capacity function and $\underline{c}: A \to \mathbf{R} \cup \{-\infty\}$ be a lower capacity function. Also, for each arc $a \in A$ let $w_a: \mathbf{R} \to \mathbf{R}$ be a convex function. Suppose that $(\mathcal{D}, f)$ is a submodular system on V such that $f(V) = 0$. Denote this network by $\mathcal{N}_{SS} = (G = (V, A), \underline{c}, \overline{c}, w_a(a \in A), (\mathcal{D}, f))$.

Consider the following flow problem in $\mathcal{N}_{SS}$.

$$P_{SS}\text{: Minimize } \sum_{a \in A} w_a(\varphi(a)) \tag{12.1a}$$

$$\text{subject to } \ \underline{c}(a) \le \varphi(a) \le \overline{c}(a) \quad (a \in A), \tag{12.1b}$$

$$\partial\varphi \in \mathrm{B}_{\mathbf{R}}(f). \tag{12.1c}$$

We call a function $\varphi: A \to \mathbf{R}$ satisfying (12.1b) and (12.1c) a *submodular flow* in $\mathcal{N}_{SS}$.

Optimal solutions of Problem P_{SS} in (12.1) are characterized by the following.

Theorem 12.1: *A submodular flow φ in $\mathcal{N}_{SS}$ is an optimal solution of Problem P_{SS} in* (12.1) *if and only if there exists a potential $p\colon V \to \mathbf{R}$ such that the following* (i) *and* (ii) *hold. Here, ${w_a}^+$ and ${w_a}^-$ denote, respectively, the right derivative and the left derivative of w_a for each $a \in A$.*

(i) *For each $a \in A$,*

$$ {w_a}^+(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) < 0 \Longrightarrow \varphi(a) = \overline{c}(a), \tag{12.2} $$

$$ {w_a}^-(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) > 0 \Longrightarrow \varphi(a) = \underline{c}(a). \tag{12.3} $$

(ii) *$\partial\varphi$ is a maximum-weight base of $\mathrm{B}_{\mathbf{R}}(f)$ with respect to the weight vector p.*

(Proof) *The "if" part*: Suppose that a potential p satisfies (i) and (ii) for a submodular flow φ in $\mathcal{N}_{SS}$. Note that (i) implies the following:

(a) If $\varphi(a) < \overline{c}(a)$, we have

$$ p(\partial^- a) - p(\partial^+ a) \le {w_a}^+(\varphi(a)). \tag{12.4} $$

(b) If $\underline{c}(a) < \varphi(a)$, we have

$$ {w_a}^-(\varphi(a)) \le p(\partial^- a) - p(\partial^+ a). \tag{12.5} $$

Hence, for any submodular flow $\hat{\varphi}$ in $\mathcal{N}_{SS}$ we have

$$ \begin{aligned} \sum_{a\in A} w_a(\hat{\varphi}(a)) &\ge \sum_{a\in A} w_a(\varphi(a)) + \sum_{a\in A} (p(\partial^- a) - p(\partial^+ a))(\hat{\varphi}(a) - \varphi(a)) \\ &= \sum_{a\in A} w_a(\varphi(a)) + \sum_{v\in V} p(v)(\partial\varphi(v) - \partial\hat{\varphi}(v)) \\ &\ge \sum_{a\in A} w_a(\varphi(a)). \end{aligned} \tag{12.6} $$

Therefore, φ is an optimal solution of Problem P_{SS} in (12.1).

The "only if" part: Let φ be an optimal solution of Problem P_{SS} in (12.1). Construct an auxiliary network $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), \gamma_\varphi)$ associated with φ as follows. The arc set A_φ is defined by (5.42)~(5.45) and $\gamma_\varphi\colon A_\varphi \to \mathbf{R}$ is the length function defined by

$$\gamma_\varphi(a) = \begin{cases} w_a{}^+(\varphi(a)) & (a \in A_\varphi{}^*) \\ -w_{\overline{a}}{}^-(\varphi(\overline{a})) & (a \in B_\varphi{}^*,\ \overline{a}\ (\in A)\text{: a reorientation of } a) \\ 0 & (a \in C_\varphi), \end{cases} \tag{12.7}$$

where $A_\varphi{}^*$, $B_\varphi{}^*$ and C_φ are defined by (5.43)~(5.45). Since there is no directed cycle of negative length relative to the length function γ_φ due to the optimality of φ, there exists a potential $p\colon V \to \mathbf{R}$ such that

$$\gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \quad (a \in A_\varphi), \tag{12.8}$$

where ∂^+, ∂^- are with respect to G_φ. (12.8) is equivalent to (i) and (ii). (ii) is due to Theorem 3.16. Q.E.D.

It should be noted that Theorem 12.1 generalizes Theorem 5.3. If w_a is differentiable for each $a \in A$, then (i) in Theorem 12.1 becomes as follows. For each $a \in A$,

$$w_a{}'(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) < 0 \implies \varphi(a) = \overline{c}(a) \tag{12.9}$$

$$w_a{}'(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) > 0 \implies \varphi(a) = \underline{c}(a). \tag{12.10}$$

Here, $w_a{}'$ denotes the derivative of w_a for each $a \in A$.

Given a submodular flow φ in $\mathcal{N}_{SS}$ and a potential $p\colon V \to \mathbf{R}$, we say that an arc $a \in A$ is *in kilter* if (12.2) or (12.3) in Theorem 12.1 is satisfied and that an ordered pair (u, v) of vertices $u,\ v \in V$ $(u \neq v)$ is *in kilter* if (1) $p(u) \geq p(v)$ or (2) $u \notin \mathrm{dep}(\partial\varphi, v)$. Also, to be *out of kilter* is to be not in kilter. We see from Theorem 12.1 that if all the arcs and all the pairs of distinct vertices are in kilter, then the given submodular flow φ is optimal. For each arc $a \in A$ we call the set of points $(\varphi(a), p(\partial^- a) - p(\partial^+ a))$ in $\mathbf{R}^2$ such that arc a is in kilter a *characteristic curve* (see Fig. 12.1).

Theorem 12.1 suggests an out-of-kilter method which keeps in-kilter arcs in kilter and monotonically decreases the *kilter number* of each out-of-kilter arc, which is a "distance" to the characteristic curve for $a \in A$ or $\max\{0, p(v) - p(u)\}$ for (u, v) such that $u \in \mathrm{dep}(\partial\varphi, v)$. When w_a is piecewise linear for each $a \in A$, the characteristic curve for each $a \in A$ consists of vertical and horizontal line segments and the out-of-kilter method described in Section 5.5.c can easily be adapted so as to be a finite algorithm.

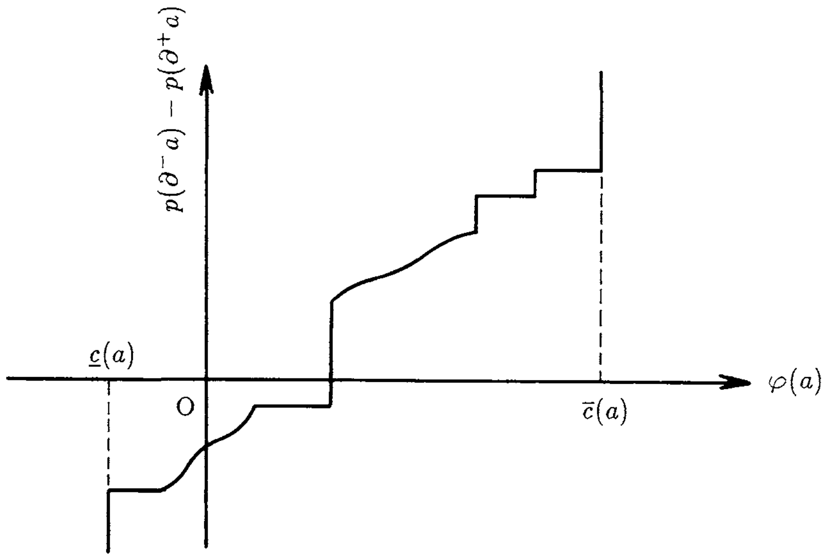

Figure 12.1: A characteristic curve.

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PART II

Part II deals with development of the theory of submodular functions after the publication of the first edition of this monograph in 1991. Since it is hard to give a comprehensive survey of the recent developments, we concentrate on the topics on submodular function minimization and discrete convex analysis, which we consider the most essential in the future developments in the theory and algorithms of submodular functions and their applications.

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Chapter VI.
Submodular Function Minimization

The present chapter deals with developments in algorithms for submodular function minimization. Readers should also be referred to a nice survey [McCormick03] for submodular function minimization.

13. Symmetric Submodular Function Minimization: Queyranne's Algorithm

H. Nagamochi and T. Ibaraki [Nagamochi+Ibaraki92] devised an efficient algorithm for finding a minimum cut for a capacitated undirected network without using any flow algorithms, where a cut means a nonempty proper subset U of the vertex set V, the capacity of a cut U is the sum of the capacities of edges between U and $V - U$, and a minimum cut is a cut of minimum capacity. Then, A. Frank [Frank94a] and M. Stoer and F. Wagner [Stoer+Wagner95] independently gave simple proofs of the validity of the Nagamochi-Ibaraki min-cut algorithm. Based on the results of [Frank94a] and [Stoer+Wagner95], M. Queyranne [Queyranne95] extended the Nagamochi-Ibaraki algorithm to a combinatorial polynomial algorithm for minimizing symmetric submodular functions.

Although the problem of minimizing symmetric submodular functions is quite different from that of minimizing submodular functions, Queyranne's result recaptured researchers' attention to submodular function minimization.

Let V be a nonempty finite set of cardinality $|V| = n \geq 2$ and $f : 2^V \to \mathbf{R}$ be a submodular function. A submodular function f is called *symmetric* if f satisfies

$$f(X) = f(V - X) \qquad (X \subseteq V). \tag{13.1}$$

Throughout this section we assume that $f : 2^V \to \mathbf{R}$ is a symmetric submodular function with $f(\emptyset) = f(V) = 0$. Note that under this assumption $f(X) \geq 0$ $(X \subseteq V)$ since $2f(X) = f(X) + f(V - X) \geq f(V) + f(\emptyset) = 0$. Hence $\emptyset$ and V are minimizers of f on 2^V. The problem of minimizing a symmetric submodular function f is to minimize f over $2^V - \{\emptyset, V\}$. We call such a minimizer a *min-cut* of f.

Remark: The term "symmetric submodular function" was introduced in [Fuji83]. But the author thought that the term "symmetric" was confusing, and hence tacitly avoided the description of symmetric submodular functions in the first edition of this monograph. (In the ordinary terminology a set function $h : 2^V \to \mathbf{R}$ is called symmetric if for each $X \subseteq V$ the value of $h(X)$ depends only on the cardinality of X.) However, since Queyranne's paper [Queyranne95] appeared, the term "symmetric submodular function" has widely been accepted, so the author also adopts it here.

For any distinct $s, t \in V$ a set $U \subset V$ with $|\{s,t\} \cap U| = 1$ is called an *s-t cut* and a *minimum s-t cut* U is an s-t cut of minimum value $f(U)$. In the same way as Nagamochi and Ibaraki's min-cut algorithm, Queyranne's algorithm picks up elements of V one by one, which determines a linear ordering $(v_1, v_2, \cdots, v_n)$ of V, in such a way that $\{v_n\}$ is a minimum v_{n-1}-v_n cut. The ordering $(v_1, v_2, \cdots, v_n)$ is called a *maximum-adjacency ordering* (or an *MA-ordering* for short) in Nagamochi and Ibaraki's algorithm, so we also call this ordering an *MA-ordering* (also see [Nagamochi00, 04] and [Nagamochi+Ibaraki02] for related topics). The following argument is based on [Fuji98].

While determining an MA-ordering $(v_1, v_2, \cdots, v_n)$, we define $U_k = \{v_1, v_2, \cdots, v_k\}$ $(k = 0, 1, \cdots, n)$ and also for each $k = 1, 2, \cdots, n-1$ define $w_k : 2^V \to \mathbf{R}$ by

$$w_k(C) = f(C) - (1/2)\{f(C \cap \overline{U_{k-1}}) + f(\overline{C} \cap \overline{U_{k-1}}) - f(\overline{U_{k-1}})\} \tag{13.2}$$

for all $C \subseteq V$, where $\overline{X}$ for $X \subseteq V$ denotes the complement of X in V. It should be noted that w_k is symmetric (i.e., $w_k(C) = w_k(V - C)$) but not necessarily submodular.

Queyranne's Algorithm for Finding an MA-ordering

Step 0: Choose an element $v_1 \in V$. Put $U_1 \leftarrow \{v_1\}$ and $k \leftarrow 2$.

Step 1: Let v_k be an element of $V - U_{k-1}$ that attains the maximum value of $w_k(\{u\})$ over $u \in V - U_{k-1}$, where w_k is defined by (13.2).

Step 2: If $k = n$, then return $(v_1, v_2, \cdots, v_n)$. Otherwise put $U_k \leftarrow U_{k-1} \cup \{v_k\}$ and $k \leftarrow k + 1$, and go to Step 1.
(End)

Then we have the following.

Lemma 13.1: *For any $k = 1, 2, \cdots, n-1$ and any $u \in V - U_k$, $\{u\}$ minimizes $w_k(C)$ over u-v_k cuts C.*

(Proof) We show this lemma by induction. For $k = 1$, we have $U_0 = \emptyset$, so that $w_1(C) = 0$ for all $C \subseteq V$. Hence the statement for $k = 1$ holds. Now, suppose that it holds for $k = l$ with $1 \leq l < n-1$. Consider any $u \in V - U_{l+1}$ and any u-v_{l+1} cut C. If C is a u-v_l cut, we assume without loss of generality that $\{u, v_l, v_{l+1}\} \cap C = \{v_l, v_{l+1}\}$. Then, by the induction hypothesis ($w_l(C) \geq w_l(\{u\})$) and by the definitions of w_l and w_{l+1} we have

$$w_{l+1}(C) - w_{l+1}(\{u\}) \geq w_l(C) - w_l(\{u\}) \geq 0, \tag{13.3}$$

where the first inequality follows from the symmetry and submodularity of f as

$$\begin{aligned}
&2(w_{l+1}(C) - w_{l+1}(\{u\}) - w_l(C) + w_l(\{u\})) \\
&= f(C \cap \overline{U_l} \cup \{v_l\}) - f(C \cap \overline{U_l}) + f(\overline{\{u\}} \cap \overline{U_l}) - f(\overline{\{u\}} \cap \overline{U_{l-1}}) \\
&= f(C \cap \overline{U_l} \cup \{v_l\}) + f(\overline{U_l} - \{u\}) - f(C \cap \overline{U_l}) - f(\overline{U_{l-1}} - \{u\}) \\
&\geq f(\overline{U_{l-1}} - \{u\}) + f(C \cap \overline{U_l}) - f(C \cap \overline{U_l}) - f(\overline{U_{l-1}} - \{u\}) \\
&= 0.
\end{aligned} \tag{13.4}$$

Next, if C is a v_l-v_{l+1} cut, we assume that $\{u, v_l, v_{l+1}\} \cap C = \{v_{l+1}\}$. Then, by the symmetry and submodularity of f we have

$$w_{l+1}(C) - w_l(C) \geq w_{l+1}(\{v_{l+1}\}) - w_l(\{v_{l+1}\}), \tag{13.5}$$

since

$$\begin{aligned}
&2(w_{l+1}(C) - w_l(C) - w_{l+1}(\{v_{l+1}\}) + w_l(\{v_{l+1}\})) \\
&= f(\overline{C} \cap \overline{U_{l-1}}) + f(\overline{U_l} - \{v_{l+1}\}) - f(\overline{C} \cap \overline{U_l}) - f(\overline{U_{l-1}} - \{v_{l+1}\}) \\
&\geq 0.
\end{aligned} \tag{13.6}$$

It follows from the induction hypothesis ($w_l(C) \geq w_l(\{v_{l+1}\})$) and (13.5) that

$$w_{l+1}(C) \geq w_{l+1}(\{v_{l+1}\}). \tag{13.7}$$

Moreover, by the definition of v_{l+1} we have $w_{l+1}(\{v_{l+1}\}) \geq w_{l+1}(\{u\})$. Hence, from (13.7) we get

$$w_{l+1}(C) \geq w_{l+1}(\{u\}). \tag{13.8}$$

Consequently, the statement for $k = l+1$ holds. Q.E.D.

For $k = n-1$ and any v_{n-1}-v_n cut C we have

$$w_{n-1}(C) = f(C) - (1/2)\{f(\{v_{n-1}\}) + f(\{v_n\}) - f(\{v_{n-1}, v_n\})\}. \quad (13.9)$$

Hence from Lemma 13.1 we have

Theorem 13.2: *For the MA-ordering $(v_1, v_2, \cdots, v_n)$ obtained by Queyranne's algorithm, $\{v_n\}$ is a minimum v_{n-1}-v_n cut for f.*

After finding a minimum v_{n-1}-v_n cut $C_0 \equiv \{v_n\}$ for f, we restrict the domain 2^V of f to $\mathcal{D}_1 \equiv \{X \mid X \subseteq V,\ |\{v_{n-1}, v_n\} \cap X| \neq 1\}$. Denote the restriction of f to $\mathcal{D}_1$ by f_1. The new symmetric submodular system $(\mathcal{D}_1, f_1)$ is the aggregation of $(2^V, f)$ by the partition $\{\{v_1\}, \cdots, \{v_{n-2}\}, \{v_{n-1}, v_n\}\}$ (see Section 3.1.d). Considering the simplification $(\hat{\mathcal{D}}_1, \hat{f}_1)$ of $(\mathcal{D}_1, f_1)$, or regarding $\{v_{n-1}, v_n\}$ as a single element, we apply Queyranne's algorithm described above to the simplification $(\hat{\mathcal{D}}_1, \hat{f}_1)$ to find a minimum cut $\hat{C}_1$ of $\hat{f}_1$. The minimum cut $\hat{C}_1$ gives a cut C_1 of f that attains the minimum of $f(C)$ over $C \in \mathcal{D}_1$. We repeat this process $n-1$ times to get a set of cuts $C_0, C_1, \cdots, C_{n-1}$ of f, where each C_{i-1} $(i = 1, 2, \cdots, n)$ is a cut obtained by the ith application of Queyranne's algorithm. A cut C_{i^*} that attains the minimum of $f(C_0), f(C_1), \cdots, f(C_{n-1})$ is a minimum cut of the original f.

Theorem 13.3: *Queyranne's algorithm finds a minimum cut of f in* $\mathrm{O}(n^3)$ *time, where we assume a function evaluation oracle for f.*

Remark: Queyranne [Queyranne95] has also shown that for a symmetric submodular function $f : 2^V \to \mathbf{R}$ and a pair of distinct $s, t \in V$, finding a minimum s-t cut is as hard as minimizing a (general nonsymmetric) submodular function. Nagamochi and Ibaraki [Nagamochi+Ibaraki98] have shown that Queyranne's algorithm also works for submodular functions satisfying $f(X) + f(Y) \geq f(X - Y) + f(Y - X)$ $(X, Y \subseteq V)$, which are slightly more general than symmetric submodular functions (also see [Rizzi00]). Also see [Baïou+Barahona+Mahjoub00] for an application of Queyranne's algorithm.

It may be worth pointing out that an MA-ordering algorithm provides us with a new maximum-flow algorithm for directed capacitated networks ([Fuji03b], [Fuji+Isotani03]).

14. Submodular Function Minimization

Grötschel, Lovász and Schrijver devised the first weakly polynomial algorithm for submodular function minimization [Grötschel + Lovász + Schrijver81] and also the first strongly polynomial algorithm [Grötschel + Lovász + Schrijver88], both based on the ellipsoid method [Khachiyan79, 80] (see Section 7.1.a). It had been an open problem since 1981 to find a 'combinatorial' polynomial algorithm for minimizing submodular functions (see earlier work of [Cunningham84, 85a], [Narayanan95] and [Sohoni92]). This long-standing open problem was resolved in 1999 independently by S. Iwata, L. Fleischer and S. Fujishige [IFF01] and A. Schrijver [Schrijver00] in different ways but based on the same framework due to W. H. Cunningham [Cunningham84, 85a]. We describe their polynomial algorithms for minimizing submodular functions.

It should be noted that there are problems that heavily rely on algorithms for general submodular function minimization. Such problems are found in [Hoppe+Tardos00] for dynamic flows, in [Tamir93] for facility location, in [Han79] for multiterminal source coding (also see [Fuji78c]), etc.

Let V be a nonempty finite set and consider a submodular function $f : 2^V \to \mathbf{R}$ with $f(\emptyset) = 0$. The following min-max relation due to Edmonds [Edm70] is essential in submodular function minimization.

Lemma 14.1:

$$\min\{f(X) \mid X \subseteq V\} = \max\{x(V) \mid x \leq \mathbf{0},\ x \in \mathrm{P}(f)\}, \tag{14.1}$$

where $\mathrm{P}(f)$ *is the submodular polyhedron associated with* f. (*Moreover, if* f *is integer-valued, there exists an integral maximizer* x *of the right-hand side of* (14.1).)

The min-max relation (14.1) was explicitly stated in [Fuji84c] for the first time in the literature, to the author's knowledge, though it easily follows from [Edm70]. Equivalently it can be rewritten in terms of the associated base polyhedron $\mathrm{B}(f)$ as follows.

Lemma 14.2:

$$\min\{f(X) \mid X \subseteq V\} = \max\{x^-(V) \mid x \in \mathrm{B}(f)\}, \tag{14.2}$$

where x^- *for* $x \in \mathbf{R}^V$ *is a vector defined by*

$$x^-(v) = \min\{0, x(v)\} \qquad (v \in V). \tag{14.3}$$

(*Moreover, if* f *is integer-valued, there exists an integral maximizer* x *of the right-hand side of* (14.2).)

Note that the min-max relation (14.1) (or (14.2)) is a *strong duality*. We have the following easy *weak duality*:

Lemma 14.3:

$$\forall X \subseteq V,\ \forall x \in \mathrm{B}(f):\ f(X) \geq x^-(V). \tag{14.4}$$

Because of this weak duality, if f is integer-valued and the *duality gap* $f(X) - x^-(V)$ is less than one, then we see that X is a minimizer of f. This lays a basis for obtaining a weakly polynomial algorithm for submodular function minimization of Iwata, Fleischer and Fujishige [IFF01].

Given a base x, consider a set $\hat{X} \subseteq V$ of nonpositive components of x such that

$$\{v \in V \mid x(v) < 0\} \subseteq \hat{X} \subseteq \{v \in V \mid x(v) \leq 0\}. \tag{14.5}$$

Then we see from (14.2) or (14.4) that if $\hat{X}$ is *x-tight*, i.e., $x(\hat{X}) = f(\hat{X})$, then

$$f(\hat{X}) = \min\{f(X) \mid X \subseteq V\}, \tag{14.6}$$

i.e., $\hat{X}$ is a minimizer of f. If there is no x-tight set $\hat{X}$ satisfying (14.5), we can increase some negative component of base x and simultaneously decrease some positive component of x by the same amount to get a new base. We may repeat this process until we eventually get an x-tight set $\hat{X}$ satisfying (14.5). This is, however, a generic algorithm that is not easy to implement in an efficient way.

Hence, instead of directly treating a base, we express a base as a convex combination of extreme bases since extreme bases are easy to transform. This is the framework of Cunningham ([Cunningham84, 85a]).

Let $L = (v_1, v_2, \cdots, v_n)$ be a linear ordering of V. For each $i = 1, 2, \cdots, n$ denote by $L(v_i)$ the set of the first i elements of L, i.e., $L(v_i) = \{v_1, v_2, \cdots, v_i\}$. Then, linear ordering L determines an extreme base $y \in \mathrm{B}(f)$ by the greedy algorithm of Edmonds [Edm70] and Shapley [Shapley71] as

$$y(v_i) = f(L(v_i)) - f(L(v_{i-1})) \qquad (i = 1, 2, \cdots, n), \tag{14.7}$$

where $L(v_0) = \emptyset$ (see Section 3.2.b). We see from the greedy algorithm that each initial segment $L(v_i)$ of L is y-tight, i.e., $y(L(v_i)) = f(L(v_i))$, for $i = 1, 2, \cdots, n$.

When a base x is expressed as a convex combination of extreme bases y_i $(i \in I)$ as

$$x = \sum_{i \in I} \lambda_i y_i \tag{14.8}$$

with $\lambda_i > 0$ ($i \in I$) and $\sum_{i \in I} \lambda_i = 1$, we can easily see that for a set $W \subseteq V$

$$W \text{ is } x\text{-tight} \iff \forall i \in I: \ W \text{ is } y_i\text{-tight}. \tag{14.9}$$

14.1. The Iwata-Fleischer-Fujishige Algorithm

In this section we give the algorithm devised by Iwata, Fleischer and Fujishige [IFF01] (also see [Fuji03a]). We assume that we are given an oracle for the function evaluation of a submodular function f, i.e., to compute $f(X)$ for any $X \subseteq V$ requires O(1) time.

(a) A weakly polynomial algorithm

We describe a weakly polynomial algorithm for submodular function minimization of [IFF01]. The key techniques are augmenting-path and scaling techniques [Iwata97] and an exchange operation technique to search for augmenting paths [Fleischer+Iwata+McCormick02] both developed for submodular flows. The former technique of [Iwata97] overcomes the difficulty arising in rounding base polyhedra and the latter technique of [Fleischer+Iwata+McCormick02] avoids exchange operations on an augmenting path. It should be mentioned that a technique related to the former was also proposed in [Narayanan95] and one related to the latter in [Goldfarb+Jin99].

We first consider an integer-valued submodular function f defined on 2^V. Let $\mathcal{N}_V$ be the complete directed network with vertex set V and arc set $V \times V$. For a given parameter $\delta > 0$ a flow $\varphi : V \times V \to \mathbf{R}$ in $\mathcal{N}_V$ is called *δ-feasible* if it satisfies

$$0 \le \varphi(u, v) \le \delta \qquad (u, v \in V). \tag{14.10}$$

For any δ-feasible flow φ in $\mathcal{N}_V$ we assume that $\varphi(v, v) = 0$ for $v \in V$ and

$$\forall u, v \in V: \ (\varphi(u, v) > 0 \implies \varphi(v, u) = 0). \tag{14.11}$$

Furthermore, define

$$\partial\Phi_\delta = \{\partial\varphi \mid \varphi: \ \text{a } \delta\text{-feasible flow in } \mathcal{N}_V\}, \tag{14.12}$$

where $\partial\varphi$ is the boundary of flow φ in $\mathcal{N}_V$. Recall that the flow boundary polyhedron $\partial\Phi_\delta$ is the base polyhedron associated with the cut function

κ_δ of network $\mathcal{N}_V$ with a uniform capacity δ, i.e., $\kappa_\delta(X) = \delta|X||V - X|$ $(X \subseteq V)$ (see (2.65)).

Now, instead of directly treating the dual pair of the min-max problems in (14.2) we consider perturbed dual pair of problems associated with the Minkowski sum (vector sum) of the original base polyhedron $\mathrm{B}(f)$ and the flow boundary polyhedron $\partial\Phi_\delta (= \mathrm{B}(\kappa_\delta))$. Note that the Minkowski sum $\mathrm{B}(f) + \partial\Phi_\delta$ is equal to $\mathrm{B}(f + \kappa_\delta)$.

We try to solve (approximately) the following dual min-max problems:

$$\begin{array}{ll} \text{Minimize} & f(X) + \kappa_\delta(X) \\ \text{subject to} & X \subseteq V, \end{array} \tag{14.13}$$

$$\begin{array}{ll} \text{Maximize} & (x + \partial\varphi)^-(V) \\ \text{subject to} & x \in \mathrm{B}(f), \\ & \varphi\text{: a } \delta\text{-feasible flow} \end{array} \tag{14.14}$$

by repeating a δ-*augmentation* (precisely defined below). Each δ-augmentation increases a negative component of $x + \partial\varphi$ by δ and simultaneously decreases a positive component of it by δ. The detail of δ-augmentation is described below.

Suppose that we are given a base $x \in \mathrm{B}(f)$ and a δ-feasible flow φ in $\mathcal{N}_V$. Define the *residual graph* $G(\varphi)$ to be a graph $(V, E(\varphi))$ with an arc set

$$E(\varphi) = \{(u, v) \mid u, v \in V,\ u \neq v,\ \varphi(u, v) = 0\}. \tag{14.15}$$

The arcs of the residual graph $G(\varphi)$ are exactly those (non-selfloop) arcs in which flow φ can be increased by δ without destroying the δ-feasibility of φ. Also define

$$S = \{v \in V \mid x(v) + \partial\varphi(v) \leq -\delta\}, \tag{14.16}$$

$$T = \{v \in V \mid x(v) + \partial\varphi(v) \geq \delta\}. \tag{14.17}$$

A directed path from S to T in residual graph $G(\varphi)$ is called a δ-*augmenting path*. If there exists a δ-augmenting path P, augment the current flow φ by δ along path P as

$$\varphi(u, v) \leftarrow \delta - \varphi(v, u), \tag{14.18}$$

$$\varphi(v, u) \leftarrow 0 \tag{14.19}$$

for each arc (u, v) in P. This results in increasing $(x + \partial\varphi)^-(V)$ in (14.14) by δ and the updated flow φ is δ-feasible. This operation is called a δ-*augmentation*.

If there does not exist such a δ-augmenting path, then let W be the set of vertices in residual graph $G(\varphi)$ that are reachable along directed paths from S.

If W is x-tight (i.e., $x(W) = f(W)$), then we finish what we call the *δ-scaling phase* for a current $\delta > 0$ (and, if necessary, we put $\delta \leftarrow \frac{1}{2}\delta$ and $\varphi \leftarrow \frac{1}{2}\varphi$ and start the next δ-scaling phase).

If W is not x-tight, we transform extreme bases y_i that express the current base x as a convex combination, as follows. Recall that W is x-tight if and only if W is y_i-tight for all $i \in I$ and that if W is an initial segment of linear ordering L_i that determines extreme base y_i, then W is y_i-tight. Hence, when W is not x-tight, there exists an index $i \in I$ such that W is not an initial segment of L_i. Find, in such an L_i, elements $u \in W$ and $v \in V - W$ such that v is immediately before u (see Fig. 14.1). We call such a triple (i, u, v) an *active triple.*

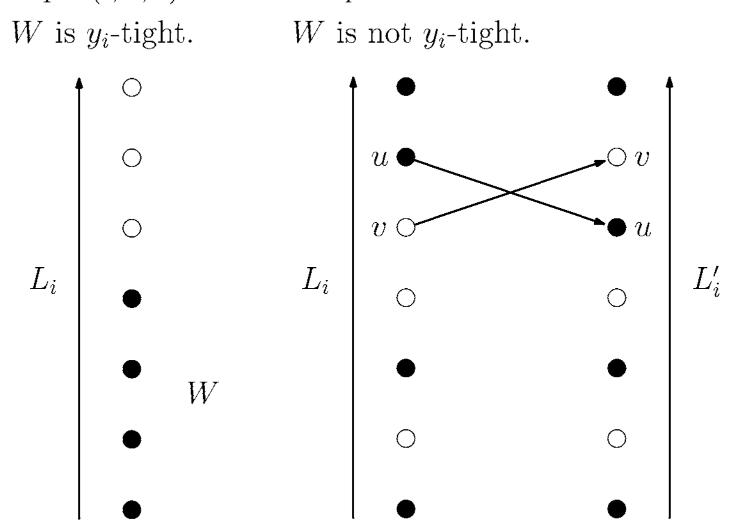

Figure 14.1: An example of exchanging (the dark circles are elements of W and the light circles are of $V - W$).

We shift forward a W-element u by interchanging a non$W$-element v before u in L_i. By this transformation we get a new linear ordering L'_i and a new extreme base y'_i. Its u component is increased and v component is decreased by the same amount, say $\alpha(\geq 0)$, as

$$y'_i \leftarrow y_i + \alpha(\chi_u - \chi_v), \tag{14.20}$$

which can easily be seen from the greedy algorithm (see Section 3.2). Here, α can be computed as

$$\alpha = f(L_i(u) - \{v\}) - f(L_i(u)) + y_i(v) \tag{14.21}$$

(recall that this is equal to the exchange capacity $\tilde{c}(y_i, u, v)$ (see (2.38)). Also note that if $y_i \neq y_i'$, then y_i and y_i' are adjacent vertices (extreme bases) of base polyhedron $\mathrm{B}(f)$.

We keep $x + \partial\varphi$ invariant and also keep the δ-feasibility of φ. To achieve this we compute new x and φ as follows (we denote the procedure by Double-Exchange). Putting $\beta = \min\{\delta, \lambda_i\alpha\}$,

$$x \leftarrow x + \beta(\chi_u - \chi_v) \tag{14.22}$$

and

$$\begin{aligned} \varphi(v,u) &\leftarrow \max\{0, \beta - \varphi(u,v)\}, && (14.23)\\ \varphi(u,v) &\leftarrow \max\{0, \varphi(u,v) - \beta\}. && (14.24) \end{aligned}$$

If $\lambda_i\alpha \leq \delta$, the new y_i replaces the old y_i. We put

$$y_i \leftarrow y_i + \alpha(\chi_u - \chi_v) \tag{14.25}$$

and update L_i by interchanging u and v. In this case the present operation of updating x, φ, y_i and L_i is called a *saturating push.*

If $\lambda_i\alpha > \delta$, then we need both new and old extreme bases to express the new x in (14.22) as a convex combination of currently available extreme bases. We put

$$\begin{aligned} &k \leftarrow \text{ a new index}, && (14.26)\\ &I \leftarrow I \cup \{k\}, && (14.27)\\ &\lambda_k \leftarrow \lambda_i - \beta/\alpha, && (14.28)\\ &\lambda_i \leftarrow \beta/\alpha, && (14.29)\\ &y_k \leftarrow y_i, && (14.30)\\ &L_k \leftarrow L_i && (14.31) \end{aligned}$$

and update y_i by (14.25) and L_i by interchanging u and v, where note that we have $\alpha > 0$. This operation is called a *nonsaturating push.*

(1) After a nonsaturating push, the value of $\varphi(u,v)$ becomes zero by (14.24). Hence arc (u,v) appears in the updated residual graph $G(\varphi)$ and W gets enlarged. Consequently, there are at most n nonsaturating pushes before the next δ-augmentation or the end of the current scaling phase.

(2) $|I|$ is increased by one after a nonsaturating push and hence $|I| \leq 2n$, where initially and every time we finish a δ-augmentation or a current scaling phase, we get $|I| \leq n$ by expressing a current base x as a convex combination of affinely independent extreme bases. (We denote the procedure to reduce $|I|$ by $\mathsf{Reduce}(x, I)$.)

(3) There are $\mathrm{O}(n^2)$ saturating/nonsaturating pushes for each y_i $(i \in I)$ since for each i every W-element is shifted forward in the list (linear ordering) L_i.

(4) Each saturating push requires $\mathrm{O}(1)$ time and each nonsaturating push $\mathrm{O}(n)$ time.

It follows that we find a δ-augmenting path (or finish the δ-scaling phase) in $\mathrm{O}(n^3)$ time. As mentioned in (2) above, after a δ-augmentation (or at the end of δ-scaling phase) we perform $\mathsf{Reduce}(x, I)$, i.e., we express a current base $x = \sum_{i \in I} \lambda_i y_i$ as a convex combination of an affinely independent subset of $\{y_i \mid i \in I\}$, which requires $\mathrm{O}(n^3)$ time by Gaussian elimination.

Let us now examine how many δ-augmentations and how many scaling phases we need to get a minimizer of f.

We have the following *relaxed weak duality.*

Theorem 14.4: *For any base $x \in \mathrm{B}(f)$ and for any δ-feasible flow φ we have*

$$(x + \partial\varphi)^-(V) \leq f(X) + n^2\delta/4 \qquad (X \subseteq V). \tag{14.32}$$

(Proof) For any $X \subseteq V$ we have

$$\begin{aligned}(x + \partial\varphi)^-(V) &\leq (x + \partial\varphi)^-(X) \leq (x + \partial\varphi)(X) = x(X) + \partial\varphi(X) \\ &\leq f(X) + \delta|X||V - X| \leq f(X) + n^2\delta/4. \end{aligned} \tag{14.33}$$

Q.E.D.

We also have a *relaxed strong duality.*

Theorem 14.5: *At the end of each δ-scaling phase we get a set*

$$X = \begin{cases} \emptyset & \textit{if } S = \emptyset \\ V & \textit{if } T = \emptyset \\ W & \textit{otherwise} \end{cases} \tag{14.34}$$

such that

$$(x + \partial\varphi)^-(V) \geq f(X) - n\delta, \tag{14.35}$$

which also implies

$$x^-(V) \geq f(X) - n^2\delta. \tag{14.36}$$

(Proof) Let X be the set defined by (14.34). If $S = \emptyset$, it follows from the definition of S in (14.16) that $(x + \partial\varphi)(v) \geq -\delta$ $(v \in V)$. Hence we have (14.35) for $X = \emptyset$. If $T = \emptyset$, then we see from (14.17) that $(x+\partial\varphi)^-(V) \geq (x+\partial\varphi)(V) - n\delta = f(V) - n\delta$. If $S \neq \emptyset$ and $T \neq \emptyset$, then $S \subseteq W \subseteq V - T$. Hence, putting $z = x + \partial\varphi$, we have $z^-(V) = z^-(W) + z^-(V - W) \geq z(W) - \delta|W| - \delta|V - W| = x(W) + \partial\varphi(W) - n\delta \geq f(W) - n\delta$, where the last inequality follows from the fact that W is x-tight and $\partial\varphi(W) \geq 0$. This shows (14.35).

Moreover, since $\partial\varphi(v) \leq (n-1)\delta$ for each $v \in V$, we have from (14.35) $x^-(V) \geq z^-(V) - n(n-1)\delta \geq f(X) - n^2\delta$. Q.E.D.

It follows from (14.36) that if $\delta < 1/n^2$, then X obtained after the current δ-scaling phase through (14.34) is a minimizer of f. If $\delta \geq 1/n^2$, then we proceed to the next scaling phase by putting $\delta \leftarrow \frac{1}{2}\delta$ and $\varphi \leftarrow \frac{1}{2}\varphi$.

In the beginning of the next δ-scaling phase we have from Theorems 14.4 and 14.5

$$f(X) - 2n\delta - n^2\delta/4 \leq (x + \partial\varphi)^-(V) \leq f(X) + n^2\delta/4. \tag{14.37}$$

This implies that the current duality gap is at most $(2n + n^2/2)\delta$, so that there are $\mathrm{O}(n^2)$ δ-augmentations in the δ-scaling phase. Moreover, we initialize the input so that the number of δ-augmentations is $\mathrm{O}(n^2)$ in the initial scaling phase, as follows. We put

$$L \leftarrow \text{ a linear ordering of } V, \tag{14.38}$$

$$x \leftarrow \text{ an extreme base determined by } L, \tag{14.39}$$

$$\delta \leftarrow \min\{|x^-(V)|, x^+(V)\}/n^2, \tag{14.40}$$

$$I \leftarrow \{k\},\ y_k \leftarrow x,\ \lambda_k \leftarrow 1,\ L_k \leftarrow L, \tag{14.41}$$

$$\varphi \leftarrow \mathbf{0}, \tag{14.42}$$

where x^+ is a vector defined as $x^+(v) = \max\{0, x(v)\}$ $(v \in V)$. It follows from (14.40) that there are $\mathrm{O}(n^2)$ δ-augmentations in the initial scaling phase.

Summing up the above arguments, we describe the IFF algorithm as follows.

The Weakly Polynomial IFF Algorithm $\mathsf{SFM}(f)$

Step 0: Initialize L, x, δ, I, φ as (14.38)$\sim$(14.42).
Step 1: **While** $\delta \geq 1/n^2$, **do** the following (1)$\sim$(6):
(1) $S \leftarrow \{v \mid x(v) + \partial\varphi(v) \leq -\delta\}$,
(2) $T \leftarrow \{v \mid x(v) + \partial\varphi(v) \geq \delta\}$,
(3) $W \leftarrow$ the set of vertices reachable from S in $G(\varphi)$,
(4) **While** $W \cap T \neq \emptyset$ or there is an active triple **do**
    While $W \cap T = \emptyset$ and there is an active triple **do**
        Apply $\mathsf{Double\text{-}Exchange}$ to an active triple (i, u, v).
        Update W.
    If $W \cap T \neq \emptyset$ **then**
        Augment flow φ along a δ-augmenting path P.
        Update $G(\varphi), S, T, W$.
    Apply $\mathsf{Reduce}(x, I)$.
(5) $\delta \leftarrow \delta/2$
(6) $\varphi \leftarrow \varphi/2$
Step2: **Return** W.

Now the complexity of the IFF algorithm is given as follows.

Theorem 14.6: *Define* $M = \max\{f(X) \mid X \subseteq V\}$ *and suppose* $M > 1$. *There are* $\mathrm{O}(\log M)$ *scaling phases till* $\delta < 1/n^2$. *In each* δ*-scaling phase (with* $\delta \geq 1/n^2$*) there are* $\mathrm{O}(n^2)$ δ*-augmentations. Each* δ*-augmentation requires* $\mathrm{O}(n^3)$ *time. Hence we can find a minimizer of* f *in* $\mathrm{O}(n^5 \log M)$ *time.*

(Proof) It suffices to show the first statement. The others follow from the above argument. From the definition of the initial δ in (14.40) we see that the initial δ satisfies $n^2\delta = \min\{|x^-(V)|, x^+(V)\} \leq x^+(V) \leq M$. Hence after $\mathrm{O}(\log M)$ scaling phases δ becomes less than $1/n^2$ and we find a minimizer of f. Q.E.D.

Remark: Without Gaussian eliminations the Iwata-Fleischer-Fujishige (IFF) algorithm is still a polynomial algorithm and requires $\mathrm{O}(n^7 \log M)$

time. On the other hand, Schrijver's algorithm described later does not enjoy this property. This is a crucial point in Iwata's fully combinatorial polynomial algorithm [Iwata02] for submodular function minimization derived from the IFF algorithm.

(b) A strongly polynomial algorithm

In the IFF paper [IFF01] a technique is given to obtain a strongly polynomial algorithm for submodular function minimization by using the weakly polynomial algorithm as a subroutine: the weakly polynomial algorithm is modified so that we perform $\mathrm{O}(\log n)$ scaling phases, and we compute a minimizer of f after invoking the weakly polynomial IFF algorithm $\mathrm{O}(n^2)$ times. Hence the strongly polynomial IFF algorithm runs in $\mathrm{O}(n^7 \log n)$ time.

Let $f : 2^V \to \mathbf{R}$ be a real-valued submodular function. Note that we can perform the weakly polynomial IFF algorithm for f, discarding the stopping criterion $\delta < 1/n^2$. We denote this procedure by $\mathsf{SFM}(f)$.

The following lemma is crucial to get a strongly polynomial algorithm for submodular function minimization.

Lemma 14.7: *At the end of a δ-scaling phase in $\mathsf{SFM}(f)$ the following two statements hold:*

(a) *If $x(w) < -n^2\delta$, then w is contained in every minimizer of f.*

(b) *If $x(w) > n^2\delta$, then w is not contained in any minimizer of f.*

(Proof) Let X be the set and x the base appearing in Theorem 14.5. Then, for any minimizer Y of f we have

$$f(X) \geq f(Y) \geq x(Y) \geq x^-(Y). \tag{14.43}$$

It follows from (14.43) and Theorem 14.5 that at the end of the δ-scaling phase

$$x^-(V) \geq f(X) - n^2\delta \geq x^-(Y) - n^2\delta. \tag{14.44}$$

Hence, if $x(w) < -n^2\delta$, then $w \in Y$. On the other hand we have

$$x^-(Y) \geq x^-(V) \geq f(X) - n^2\delta \geq x(Y) - n^2\delta. \tag{14.45}$$

Theorefore, if $x(w) > n^2\delta$, then $w \notin Y$. Q.E.D.

Lemma 14.7 will be employed to get information about elements that are contained in every minimizer of f and about a binary relation $R \subseteq V \times V$ such that $(v, w) \in R$ implies that any minimizer of f containing v also contains w.

We keep

(1) a set $X \subseteq V$ that is included in every minimizer of f,

(2) a partition $\Pi = \{V_1, V_2, \cdots, V_l\}$ of $V - X$ and a set $U = \{u_1, u_2, \cdots, u_l\}$ such that each $u_i \in U$ represents component $V_i \in \Pi$,

(3) a submodular function $\hat{f}$ on 2^U,

(4) a directed acyclic graph $D = (U, F)$, where the existence of an arc $(v, w) \in F$ means that any minimizer of $\hat{f}$ containing v also contains w.

Initially, we have

$$X = \emptyset, \quad \Pi = \{\{v\} \mid v \in V\}, \quad U = V, \quad F = \emptyset, \quad \hat{f} = f. \tag{14.46}$$

For each $u \in U$ let $R(u)$ be the set of the vertices of D that are reachable from vertex u by directed paths in D. Also let $\hat{f}_u$ be the contraction of $\hat{f}$ by $R(u)$, i.e., for each $Z \subseteq U - R(u)$

$$\hat{f}_u(Z) = \hat{f}(Z \cup R(u)) - \hat{f}(R(u)). \tag{14.47}$$

A linear ordering $(u_1, u_2, \cdots, u_l)$ of U is called *consistent with* D if $(u_i, u_j) \in F$ implies that $j \leq i$. The extreme base $\hat{x} \in \mathrm{B}(\hat{f})$ that is determined by a consistent linear ordering is also called *consistent with* D. It should be noted that such an extreme base $\hat{x}$ is an extreme base of a submodular system $(\hat{\mathcal{D}}, \hat{f})$ on U, where $\hat{\mathcal{D}}$ is the set of the (lower) ideals of the poset corresponding to D and the domain of $\hat{f}$ should be restricted to $\hat{\mathcal{D}}$. Hence,

Lemma 14.8: *Any extreme base $\hat{x} \in \mathrm{B}(\hat{f})$ consistent with D satisfies $\hat{x}(u) \leq \hat{f}(R(u)) - \hat{f}(R(u) - \{u\})$ for each $u \in U$.*

(Proof) See (3.89)$\sim$(3.91). Q.E.D.

Suppose that we are given a scaling parameter $\eta > 0$ and an extreme base $\hat{x} \in \mathrm{B}(\hat{f})$ consistent with D and that $\hat{f}(U) \geq \eta/3$ or there is a set $Y \subseteq U$ such that $\hat{f}(Y) \leq -\eta/3$. Starting with $\delta = \eta$, we repeat the scaling phase of the weakly polynomial IFF algorithm $\mathrm{O}(\log n)$ times until $\delta < \eta/(3n^3)$. Denote this procedure by $\mathsf{Fix}(\hat{f}, D, \eta)$. We can easily see the following.

(i) When $\hat{f}(U) \geq \eta/3$, at least one element $w \in U$ satisfies $\hat{x}(w) > n^2\delta$ at the end of the last scaling phase (since $\hat{x}(U) = \hat{f}(U) \geq \eta/3 > n^3\delta$). It follows from Lemma 14.7(b) that such an element w is not contained in any minimizer of $\hat{f}$. (Then we can delete w and some other possible elements from U and restrict $\hat{f}$ on a smaller domain.)

(ii) When $\hat{f}(Y) \leq -\eta/3$ for some $Y \subseteq U$, at least one element $w \in Y$ satisfies $\hat{x}(w) < -n^2\delta$ at the end of the last scaling phase (since $\hat{x}(Y) \leq \hat{f}(Y) \leq -\eta/3 < -n^3\delta$) . By Lemma 14.7(a), such an element w and hence elements in $R(w)$ are contained in every minimizer of $\hat{f}$. (Then we can restrict our attention to $U - R(w)$ by considering the contraction of $\hat{f}$ by $R(w)$. Accordingly we update set X.)

Now, define

$$\eta = \max\{\hat{f}(R(u)) - \hat{f}(R(u) - \{u\}) \mid u \in U\}. \tag{14.48}$$

It follows from Lemma 14.8 that $\hat{x}(u) \leq \eta$ $(u \in U)$ for any extreme base $\hat{x} \in \mathrm{B}(\hat{f})$ consistent with D.

(I) If $\eta \leq 0$, then any extreme base $\hat{x} \in \mathrm{B}(\hat{f})$ consistent with D satisfies $\hat{x} \leq \mathbf{0}$, which implies that U is a minimizer of $\hat{f}$. Hence, defining $\tilde{U}$ as the subset of V that is the union of sets corresponding to all $u \in U$, we have a minimizer $X \cup \tilde{U}$ of the original f.

(II) If $\eta > 0$, then let $\hat{u}$ be an element in U that attains the maximum in the right-hand side of (14.48). Since

$$\eta = \hat{f}(R(\hat{u})) - \hat{f}(R(\hat{u}) - \{\hat{u}\}) = \hat{f}(U) - \hat{f}(R(\hat{u}) - \{\hat{u}\}) + (\hat{f}(R(\hat{u})) - \hat{f}(U)), \tag{14.49}$$

we have

$$\max\{\hat{f}(U), -\hat{f}(R(\hat{u}) - \{\hat{u}\}), \hat{f}(R(\hat{u})) - \hat{f}(U)\} \geq \eta/3. \tag{14.50}$$

Hence there are the following three, not necessarily exclusive, subcases (II-1), (II-2) and (II-3) to consider.

(II-1) [$\hat{f}(U) \geq \eta/3$]
Apply $\mathsf{Fix}(\hat{f}, D, \eta)$ to find a new element $w \in U$ that is not in any minimizer of $\hat{f}$. For such an element w, any element v with $w \in R(v)$ does not belongs to any minimizer of $\hat{f}$, so that we delete $\{v \mid w \in R(v)\}$ from U.

(II-2) [$\hat{f}(R(\hat{u}) - \{\hat{u}\}) \leq -\eta/3$]
Apply $\mathsf{Fix}(\hat{f}, D, \eta)$ to find a new element $w \in R(\hat{u}) - \{\hat{u}\}$ that is contained in every minimizer of $\hat{f}$. For such an element w, $R(w)$ is also contained in every minimizer of $\hat{f}$, so that we put $U \leftarrow U - R(w)$, $\hat{f} \leftarrow \hat{f}_w$ and $X \leftarrow X \cup \tilde{Q}$ with $Q = R(w)$, where $\hat{f}_w$ is the contraction of $\hat{f}$ by $R(w)$ as in (14.47).

(II-3) [$\hat{f}_{\hat{u}}(U - R(\hat{u})) = \hat{f}(U) - \hat{f}(R(\hat{u})) \leq -\eta/3$]
Perform $\mathsf{Fix}(\hat{f}_{\hat{u}}, D_{\hat{u}}, \eta)$, where $D_{\hat{u}}$ is the subgraph of D induced by $U - R(\hat{u})$. Then we get an element $w \in U - R(\hat{u})$ that is contained in every minimizer of $\hat{f}_{\hat{u}}$. It follows that every minimizer of $\hat{f}$ containing $\hat{u}$ contains w. Hence we add a new arc $(\hat{u}, w)$ to F.
If this creates a cycle in D (i.e., $\hat{u} \in R(w)$), let $Z = \{v \mid v \in R(w),\ \hat{u} \in R(v)\}$ (i.e., Z is the vertex set of the strongly connected component of D that contains $\hat{u}$ and w). Shrink Z into a single vertex to obtain a new directed acyclic graph D and update U and $\hat{f}$ regarding Z as a singleton.

We repeat (I) and (II) by updating η by (14.48) till we get a minimizer by (I) or U becomes empty. When U becomes empty, the current X is a minimizer of f.

Procedure $\mathsf{Fix}(\hat{f}, D, \eta)$ for Cases (II-1) and (II-2) are performed $\mathrm{O}(n)$ times and $\mathsf{Fix}(\hat{f}_{\hat{u}}, D_{\hat{u}}, \eta)$ for Case (II-3), $\mathrm{O}(n^2)$ times. In each of Cases (II-1), (II-2) and (II-3) each Fix carries out $\mathrm{O}(\log n)$ scaling phases and each scaling phase requires $\mathrm{O}(n^5)$ time.

Hence we have

Theorem 14.9: *The algorithm described above finds a minimizer of f in $\mathrm{O}(n^7 \log n)$ time.*

It should be noted that since the algorithm keeps a set $X \cup \tilde{U} \subseteq V$ that includes all the minimizers of f, the finally obtained $X \cup \tilde{U}$, an output of the algorithm, is a unique maximal minimizer of f.

(c) Modification with multiple exchanges

We can modify the weakly polynomial IFF algorithm as follows ([Fuji02], [Fuji03a]).

In searching for a δ-augmenting path, the original IFF algorithm interchanges adjacent W- and non$W$-elements to shift forward each W-element

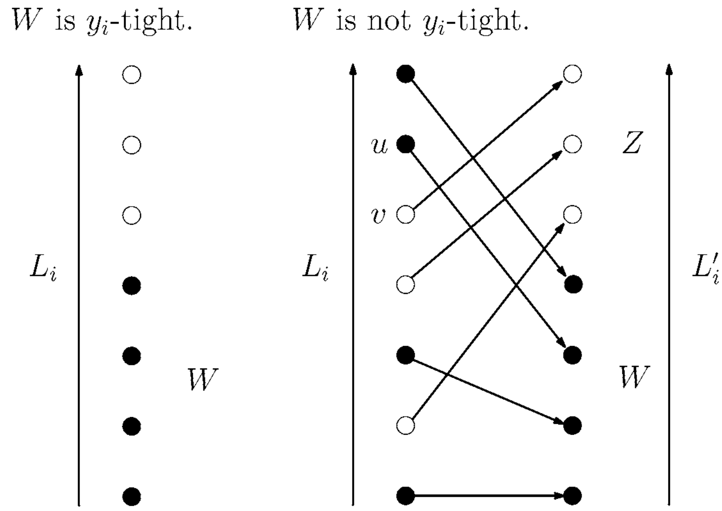

Figure 14.2: A multiple exchange.

(see Fig. 14.1). Instead of repeating such an interchanging, here we simultaneously shift all W-elements forward to make W an initial segment of a new linear ordering L'_i (see Fig. 14.2), keeping the orders among W and $V - W$.

This new linear ordering gives a new extreme base y'_i as

$$y'_i \leftarrow y_i + \sum_{u \in W} \alpha_u \chi_u - \sum_{v \in Z} \alpha_v \chi_v, \tag{14.51}$$

where $Z = V - W$. The additional part in (14.51) can be expressed as the boundary of a flow $\psi : V \times V \to \mathbf{R}_+$ in a forest:

$$\partial\psi = \sum_{u \in W} \alpha_u \chi_u - \sum_{v \in Z} \alpha_v \chi_v, \tag{14.52}$$

where $\{(u, v) \mid \psi(u, v) > 0\}$ forms a forest. Such a flow ψ can be constructed in a greedy way or by the so-called north-west corner rule (see Fig. 14.3). Then, to keep $x + \partial\varphi$ invariant, we update φ based on this ψ so that new φ cancels the additional part in (14.51) multiplied by λ_i. Here, to keep also the δ-feasibility of φ, we have a saturating push or a nonsaturating push similarly as in the IFF algorithm. In case of nonsaturating push, we need both new and old extreme bases and W gets enlarged. Moreover, while W

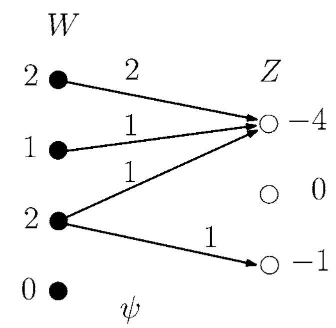

Figure 14.3: An example of a flow ψ in a forest and its boundary $\partial\psi$.

remains the same, there is at most one saturating push for each $i \in I$. This simplifies the IFF algorithm but the complexity is the same as the original IFF algorithm. It should also be noted that new extreme base y_i' computed through (14.51) is not adjacent to y_i in general.

(d) Submodular functions on distributive lattices

Let $f : \mathcal{D} \to \mathbf{Z}$ be a submodular function on a distributive lattice with $\mathcal{D} \subseteq 2^V$, $\emptyset, V \in \mathcal{D}$ and $f(\emptyset) = 0$. We cannot directly apply the IFF algorithm to such a submodular function f by formally defining $f(X) = +\infty$ for $X \in 2^V - \mathcal{D}$ since M in Theorem 14.6 becomes $+\infty$. We describe a modification of the IFF algorithm indicated in [IFF01].

Suppose that $\mathcal{D}$ is the set of all (lower) ideals of a poset $\mathcal{P} = (V, \preceq)$, and let $G(\mathcal{P}) = (V, A(\mathcal{P}))$ be an acyclic graph representing the poset $\mathcal{P}$, i.e., $(u, v) \in A(\mathcal{P}) \iff v \prec u$. Any base $x \in \mathrm{B}(f)$ is expressed as

$$x = \sum_{i \in I} \lambda_i y_i + \partial\psi, \tag{14.53}$$

where the first term is the convex combination of extreme bases y_i $(i \in I)$ of $\mathrm{B}(f)$ and the second one is the boundary of a nonnegative flow ψ in graph $G(\mathcal{P})$. Note that each extreme base y_i of $\mathrm{B}(f)$ is determined by a linear extension L_i of poset $\mathcal{P} = (V, \preceq)$ and that the characteristic cone of $\mathrm{B}(f)$ is the set of boundaries of all nonnegative flows in $G(\mathcal{P})$ (see Theorem 3.26).

We keep the expression of a base x as (14.53) instead of (14.8). We shall show how to adapt the weakly polynomial IFF algorithm for minimization of the submodular function $f : \mathcal{D} \to \mathbf{Z}$. As in the original IFF algorithm, we consider a δ-feasible flow in the complete directed network $\mathcal{N}_V = (V, V \times V)$ and also consider the residual graph $G(\varphi) = (V, E(\varphi))$.

The initialization is the same as in the IFF algorithm, except that in addition we put $\psi \leftarrow \mathbf{0}$, a zero flow in $G(\mathcal{P})$.

If there exists a δ-augmenting path P in residual graph $G(\varphi)$, then augment flow φ along P. Otherwise let W be the set of vertices that are reachable from S in $G(\varphi)$. We try to enlarge W by modifying extreme bases y_i $(i \in I)$ and a nonnegative flow ψ in $G(\mathcal{P})$ as follows.

If there is an arc $(u, v) \in A(\mathcal{P})$ with $u \in W$ and $v \notin W$, then put

$$\psi(u, v) \leftarrow \psi(u, v) + \delta, \tag{14.54}$$

$$\varphi(v, u) \leftarrow \delta - \varphi(u, v), \tag{14.55}$$

$$\varphi(u, v) \leftarrow 0. \tag{14.56}$$

We call this operation a *nonsaturating push* (for ψ). This results in enlarging W. If any δ-augmenting path P appears in the updated residual graph $G(\varphi)$, we carry out a δ-augmentation along P. Hence let us assume that all such possible nonsaturating pushes have been performed for a current ψ, so that there is no arc $(u, v) \in A(\mathcal{P})$ with $u \in W$ and $v \notin W$, i.e., W is a (lower) ideal of $\mathcal{P}$ (or $W \in \mathcal{D}$). Furthermore, if there is an arc $(v, u) \in A(\mathcal{P})$ such that $u \in W$, $v \notin W$ and $\psi(v, u) > 0$, then put

$$\gamma \leftarrow \min\{\psi(v, u), \varphi(u, v)\}, \tag{14.57}$$

$$\psi(v, u) \leftarrow \psi(v, u) - \gamma, \tag{14.58}$$

$$\varphi(u, v) \leftarrow \varphi(u, v) - \gamma. \tag{14.59}$$

If $\varphi(u, v)$ becomes zero, we call this operation a *nonsaturating push* (for ψ) and W gets enlarged. Otherwise $\psi(v, u)$ becomes zero and we call this operation *saturating push* (for ψ). Hence let us further suppose that all such possible nonsaturating/saturating pushes for ψ have been made, so that $W \in \mathcal{D}$ and that for each arc $(v, u) \in A(\mathcal{P})$ with $u \in W$ and $v \notin W$ we have $\psi(v, u) = 0$. (Note that this implies $\partial\psi(W) = 0$.)

Suppose that $W \in \mathcal{D}$ and $\psi(v, u) = 0$ for all $(v, u) \in A(\mathcal{P})$. If W is an initial segment of each L_i $(i \in I)$, then we finish the δ-scaling phase. Otherwise suppose that $u \in W$ is immediately after $v \notin W$ in a list L_i (see Figure 14.1). Then we have the following.

Lemma 14.10: *Suppose as above. Interchanging u and v in L_i, we get a linear extension $L_i{}'$ of $\mathcal{P}$.*

(Proof) Since W, $L_i(u)$, $L_i(v) - \{v\} \in \mathcal{D}$ and $\{u, v\} \cap W = \{u\}$, we have

$$L_i(u) - \{v\} = (L_i(u) \cap W) \cup (L_i(v) - \{v\}) \in \mathcal{D}, \qquad (14.60)$$

where recall that L_i is a linear extension of $\mathcal{P}$ and that $L_i(u)$ is the set of elements in the initial segment of L_i till u (including u). Hence the present lemma holds. Q.E.D.

Because of this lemma we can modify extreme bases as in the original IFF algorithm when $W \in \mathcal{D}$.

To sum up, defining $\hat{M} = \max\{|f(X)| \mid X \in \mathcal{D}\}$,

(1) By the initialization we have

$$n^2\delta = \min\{|x^-(V)|, x^+(V)\} \le x^+(V) \le \bar{\alpha}^+(V), \qquad (14.61)$$

where $\bar{\alpha}^+(v) = f(D(v)) - f(D(v) - \{v\})$ $(v \in V)$ (see (3.89)). Hence we have

$$\bar{\alpha}^+(V) \le 2n\hat{M}. \qquad (14.62)$$

It follows that there are $\mathrm{O}(\log n\hat{M})$ scaling phases till $\delta < 1/n^2$.

(2) After a nonsaturating push W gets enlarged. Hence there are at most n nonsaturating pushes for extreme bases and for flow ψ before the next δ-augmentation or the end of the current scaling phase.

(3) There are $\mathrm{O}(n^2)$ saturating/nonsaturating pushes for each y_i $(i \in I)$ and ψ before the next δ-augmentation or the end of the current scaling phase.

Relaxed weak duality (Theorem 14.4) and relaxed strong duality (Theorem 14.5) hold for submodular functions on distributive lattices with a minor modification: '$X \subseteq V$' in (14.32) should read '$X \in \mathcal{D}$.' The proofs of the two theorems are also valid *mutatis mutandis*. Note that $\partial\psi(X) \le 0$ for any $X \in \mathcal{D}$ and that when we finish a δ-scaling phase without making W enlarged, we have $W \in \mathcal{D}$ and $\partial\psi(W) = 0$.

Consequently, we finish each δ-scaling phase after $\mathrm{O}(n^2)$ δ-augmentations. Since each δ-augmentation requires $\mathrm{O}(n^3)$ time, the total running time is $\mathrm{O}(n^5 \log n\hat{M})$. Furthermore, making this weakly polynomial algorithm strongly polynomial results in an $\mathrm{O}(n^7 \log n)$ algorithm, where in the beginning of the algorithm, graph $D = (U, F)$ that keeps information about the minimizers of f coincides with $G(\mathcal{P}) = (V, A(\mathcal{P}))$.

It should also be noted that the modification with multiple exchange described in Section 14.1.c can also be adapted for minimization of submodular functions on distributive lattices.

14.2. Schrijver's Algorithm

Schrijver [Schrijver00] devised a combinatorial, strongly polynomial algorithm for submodular function minimization, independently and differently from the IFF algorithm [IFF01].

Schrijver's algorithm also takes Cunningham's approach. That is, we assume that a current base $x \in \mathrm{B}(f)$ is expressed as a convex combination of extreme bases y_i $(i \in I)$ corresponding to linear orderings L_i $(i \in I)$:

$$x = \sum_{i \in I} \lambda_i y_i, \tag{14.63}$$

where $\lambda_i > 0$ $(i \in I)$ and $\sum_{i \in I} \lambda_i = 1$. For each $i \in I$ we denote by $\leq_i$ the linear order on V determined by L_i. Also define an interval $(s, t]_i$ by

$$(s, t]_i = \{v \in V \mid s <_i v \leq_i t\} \tag{14.64}$$

for $s, t \in V$ and $i \in I$.

For some $i \in I$ and $s, t \in V$ with $s <_i t$ we consider a way of computing an elementary transformation

$$x + \eta(\chi_t - \chi_s) \tag{14.65}$$

(for some $\eta \geq 0$) of the current base x by generating new extreme bases as follows. This is a key procedure of Schrijver's algorithm.

For each $u \in (s, t]_i$ let $L_i^{s,u}$ be the linear ordering of V obtained from L_i by moving u to the position immediately before s and denote by $\leq_i^{s,u}$ the linear order corresponding to $L_i^{s,u}$. Also denote by $y_i^{s,u}$ the extreme base determined by the linear ordering $L_i^{s,u}$. Then, from the submodularity of f we can easily see the following lemma.

Lemma 14.11: *For each $u \in (s,t]_i$ we have*

$$y_i^{s,u}(v) - y_i(v) = \begin{cases} - & \text{if } s \le_i v <_i u \\ + & \text{if } v = u \\ 0 & \text{otherwise,} \end{cases} \tag{14.66}$$

where $z = -$ means $z \le 0$ and $z = +$ means $z \ge 0$.

It follows from this lemma that

(i) If for some $u \in (s,t]_i$ we have $y_i^{s,u}(u) - y_i(u) = 0$, then $y_i^{s,u} = y_i$. We replace y_i and L_i by $y_i^{s,u}$ and $L_i^{s,u}$.

(ii) If $y_i^{s,u}(u) - y_i(u) > 0$ for all $u \in (s,t]_i$, then $\chi_t - \chi_s$ is uniquely expressed as a linear combination of $y_i^{s,u} - y_i$ ($u \in (s,t]_i$) with positive coefficients. Then for a (unique) $\delta > 0$, $\delta(\chi_t - \chi_s)$ is expressed as a convex combination of $y_i^{s,u} - y_i$ ($u \in (s,t]_i$), i.e.,

$$\delta(\chi_t - \chi_s) = \sum_{u \in (s,t]_i} \mu_u (y_i^{s,u} - y_i) \tag{14.67}$$

with $\mu_u > 0$ ($u \in (s,t]_i$) and $\sum_{u \in (s,t]_i} \mu_u = 1$. Adding to (14.63) the above (14.67) multiplied by λ_i yields an expression (14.65) with $\eta = \delta\lambda_i > 0$.

Note that after the operations of (i) and (ii) the extreme base y_i disappears from the expression of the transformed (new) base and that the length of the interval $(s,t]_i^{s,u}$ in Case (i) and the lengths of all the intervals $(s,t]_i^{s,u}$ ($u \in (s,t]_i$) in Case (ii) decrease by one from that of $(s,t]_i$.

For the current y_i and L_i ($i \in I$) we define a directed graph $G = (V, A)$ with a vertex set V and an arc set

$$A = \{(u,v) \mid \exists i \in I : u <_i v\}. \tag{14.68}$$

Now, Schrijver's algorithm is described as follows. We assume $V = \{1, 2, \cdots, n\}$.

Schrijver's Algorithm

Step 0: Choose a linear ordering L_1 and let y_1 be the extreme base determined by L_1. Put $I \leftarrow \{1\}$ and $\lambda_1 \leftarrow 1$. Also let $G = (V, A)$ be the directed graph associated with the current L_1.

Step 1: Define $P = \{v \in V \mid x(v) > 0\}$ and $N = \{v \in V \mid x(v) < 0\}$.
If there exists no directed path from P to N in $G = (V, A)$, then let U be the set of vertices in G from which we can reach N by a directed path, and return U (U is a minimizer of f).

Step 2: Let $d(v)$ ($v \in V$) be the distance in G from P to v, i.e., the minimum number of arcs in a directed path from P to v.
Choose $s \in V$ and $t \in N$ such that the ordered triple $(d(t), t, s)$ satisfying $d(t) < +\infty$, $(s, t) \in A$ and $d(s) + 1 = d(t)$ is lexicographically maximum, where the order in $V = \{1, 2, \cdots, n\}$ is the ordinary order for integers. Then let i be an index in I that attains the maximum of the lengths $|(s, t]_j|$ over $j \in I$. For the chosen s, t and i compute an elementary transformation of the current base x by the procedure described above. Let x' be the new base.

(2-1): If $x'(t) \leq 0$, then from the expression of x' as a convex combination of current extreme bases, compute an expression of x' as a convex combination of affinely independent extreme bases, put $x \leftarrow x'$ and update I, y_i, L_i, λ_i ($i \in I$) and $G = (V, A)$. Go to Step 1.

(2-2): If $x'(t) > 0$, then let x'' be a convex combination of x and x' such that $x''(t) = 0$. Compute an expression of x'' as a convex combination of affinely independent extreme bases chosen from among the current extreme bases, put $x \leftarrow x''$ and update I, y_i, L_i, λ_i ($i \in I$) and $G = (V, A)$. Go to Step 1.

(End)

We will show the validity and the complexity of the algorithm. The following arguments are based on [Schrijver00] and [Vygen03].

When the algorithm terminates at Step 1, it is easy to see that then obtained U is y_i-tight for each $i \in I$, so that U is x-tight. Since $x(v) \leq 0$ ($v \in U$) and $x(v) \geq 0$ ($v \in V - U$), we have $f(U) = x(U) = x^-(V)$ and hence U is a minimizer of f.

Consider an execution of Step 2. Define $\alpha = |(s, t]_i|$ for the chosen $i \in I$ and define β as the number of j's such that $|(s, t]_j| = \alpha$. Then let $x', d', G', A', P', N', t', s', \alpha', \beta'$ be the objects $x, d, G, A, P, N, t, s, \alpha, \beta$ after the execution of Step 2.

Fact 1: *If $(u, v) \in A' - A$, then $s \leq_i v <_i u \leq_i t$.*

Fact 2: *For each $v \in V$ we have $d(v) \leq d'(v)$.*

(Proof) It follows from Fact 1 that any new arc $(u, v) \in A' - A$ satisfies

$d(v) \leq d(s) + 1 = d(t) \leq d(u) + 1$. Hence, adding arc (u, v) to G does not decrease the distance from P to v. Moreover, since we have $P' \subseteq P$ and removing arcs does not decrease the distance, this completes the proof. Q.E.D.

Fact 3: *The number of consecutive iterations of* Step 1 *and* Step 2 *with the same pair* (t, s) *is* $\mathrm{O}(n^2)$.
(Proof) For each $u \in (s, t]_i$ we have $|(s, t]_i^{s,u}| < |(s, t]_i|$, so that $\alpha' \leq \alpha$. Moreover, during the consecutive iterations with the same pair (t, s) we have $x(t) < 0$. Hence, if $\alpha' = \alpha$, then $\beta' < \beta$ because y_i disappears (since $x'(t) < 0$). It follows that (α, β) decreases lexicographically, and the number of such iterations is $\mathrm{O}(n^2)$. Q.E.D.

Fact 4: *While all distances* $d(v)$ $(v \in V)$ *remain the same,* $\max\{d(v) \mid v \in N\}$ *does not increase and furthermore if* $\max\{d(v) \mid v \in N\}$ *also remains the same, the set of the maximizers remains the same or becomes smaller.*
(Proof) A vertex v becomes a new element of N only if $v = s$. Hence the new element v satisfies $d(v) = d(t) - 1$. Q.E.D.

Fact 5: *For each* $t^* \in V$ *there are* $\mathrm{O}(n^2)$ *executions of* Step 2 *with* $t = t^*$ *and* $x'(t) = 0$.
(Proof) When we get $x'(t^*) = 0$, there holds $t^* \notin N'$. When $x(t^*)$ becomes negative next time, letting d'', s'', t'', N'' be the then objects d, s, t, N, we have $s'' = t^*$ and $\max\{d(v) \mid v \in N\} = d(t^*) \leq d''(t^*) < d''(t'') = \max\{d''(v) \mid v \in N''\}$, due to Fact 2 and the definitions of $s''(= t^*)$ and t''. It follows from Fact 4 that for some $v \in V$ we have $d(v) < d''(v)$. Hence Fact 5 follows from Fact 2. Q.E.D.

Fact 6: *For any* $u, v \in V$, u *is called* v-boring *if* $(u, v) \notin A$ *or* $d(v) \leq d(u)$. *Let* $s^*, t^* \in V$. *Consider a sequence of consecutive iterations of* Step 1 *and* Step 2 *starting with* $s = s^*$ *and* $t = t^*$ *and ending with the changing of* $d(t^*)$. *Then, for any* v *with* $v > s^*$ *is* t^**-boring in these iterations. If* s^* *becomes* t^**-boring in some of these iterations, it remains* t^**-boring until* $d(t^*)$ *changes.*
(Proof) At the beginning of the sequence of the iterations, any v with $v > s^*$ is t^*-boring due to the choice of $s = s^*$. Since $d(t^*)$ remains the same, it follows from Fact 2 that t^*-boring v can become not t^*-boring only if arc (v, t^*) newly appears in A. Suppose that $v \geq s^*$ is t^*-boring and (v, t^*) newly appears in A when t and s are chosen. Then we have $s \leq_i t^* <_i v \leq_i t$, so that $d(t^*) \leq d(s) + 1 = d(t) \leq d(v) + 1$. If $s > v$, then we have $d(t^*) \leq d(s)$ because $t^* = s$ or s was t^*-boring and $(s, t^*) \in A$. If

$s < v$, then we have $d(t) \leq d(v)$ because $t = v$ or $(v, t) \in A$ by the choice of s. It follows that $d(t^*) \leq d(v)$ and that v remains t^*-boring. Q.E.D.

Fact 7: *Call the sequence of consecutive iterations described in* Fact 3 *a* block. *There are* $\mathrm{O}(n^3)$ *blocks throughout Schrijver's algorithm.*
(Proof) A block can end only in one of the following three cases:

(a) Some distance $d(v)$ for $v \in V$ changes. (This occurs $\mathrm{O}(n^2)$ times due to Fact 2.)

(b) t is removed from N. (From Fact 5, this occurs $\mathrm{O}(n^3)$ times.)

(c) (s, t) disappears from A. (This occurs $\mathrm{O}(n^3)$ times since $d(t)$ changes before the next block with the same pair (s, t) because of Fact 6.)

Q.E.D.

It follows from Fact 3 and Fact 7 that there exist $\mathrm{O}(n^5)$ iterations of Step 1 and Step 2, which requires $\mathrm{O}(n^8)$ running time with a function evaluation oracle for f. If we assume that invoking the function evaluation oracle takes time γ, then the complexity of Schrijver's algorithm is $\mathrm{O}(n^8 + \gamma n^7)$. Note that the weakly polynomial IFF algorithm runs in $\mathrm{O}(\gamma n^5 \log M)$ time and its strongly polynomial version in $\mathrm{O}(\gamma n^7 \log n)$ time.

Remark: In Schrijver's algorithm, updating the expression of a current base as a convex combination of affinely independent extreme bases is inevitable to achieve polynomiality of the algorithm, since without such a reduction of the size of the set of extreme bases the number of extreme bases to express a current base becomes exponential before the algorithm terminates. Recall that the IFF algorithm also performs such a reduction operation but that the reduction is performed only to make the algorithm faster; without such a reduction operation the IFF algorithm remains polynomial.

Another feature of Schrijver's algorithm is that the algorithm finds not only a minimizer of f but also a maximizer, a base $x = \sum_{i \in I} \lambda_i y_i$, of $\max\{x^-(V) \mid x \in \mathrm{B}(f)\}$ expressed as a convex combination of extreme bases y_i $(i \in I)$. By the same procedure given by (7.24)$\sim$(7.27) we can efficiently construct the poset that expresses the distributive lattice $\mathcal{D}(x)$ of all the x-tight sets. Then x-tight sets X satisfying $\{v \mid v \in V,\ x(v) < 0\} \subseteq X \subseteq \{v \mid v \in V,\ x(v) \leq 0\}$ are exactly the minimizers of f. It should also be noted that the obtained maximizer x may not be integral when f is integer-valued.

14.3. Further Progress in Submodular Function Minimization

In this subsection we give a short description of further progress in submodular function minimization.

After the IFF algorithm and Schrijver's one appeared, Iwata [Iwata02] devised a fully combinatorial strongly polynomial algorithm for submodular function minimization based on the IFF algorithm. Iwata's fully combinatorial algorithm performs arithmetic operations of addition, subtraction and comparison only. In the IFF algorithm we need multiplications and divisions to compute coefficients appearing in a convex combination of extreme bases in the representation of a current base. His algorithm takes advantage of some flexibility in determining the values of saturating and nonsaturating pushes in the IFF weakly polynomial algorithm while keeping polynomiality of the algorithm. This is a key property of the IFF algorithm. In the kth scaling phase we treat only rational numbers that are integral multiples of $1/2^k$ for the positive integer k, where actual computations in [Iwata02] are performed for such numbers multiplied by 2^k so that only integers appear. The property that the IFF algorithm is still polynomial without Gaussian eliminations is another crucial key to get the fully combinatorial algorithm. Because of this we can avoid multiplications and divisions required in the Gaussian eliminations.

The problem of finding a fully combinatorial (strongly) polynomial algorithm for submodular function minimization was thus solved by [Iwata02]. However, since the algorithm in [Iwata02] parallels the IFF algorithm, it treats integers of polynomial size in n. Queyranne's algorithm for symmetric submodular function minimization, for example, is a fully combinatorial polynomial algorithm that treats only numbers given as inputs without worrying about the lengths of the numbers. It is still open to devise a 'fully' combinatorial polynomial algorithm for submodular function minimization in this sense.

Concerning the progress in complexity of submodular function minimization algorithms, Fleischer and Iwata [Fleischer+Iwata03] improved Schrijver's algorithm by using the push-relabel technique of Goldberg and Tarjan [Goldberg+Tarjan88], whose complexity, however, turned out to be the same as Schrijver's, which is due to Vygen [Vygen03]. Iwata [Iwata03] also improved the IFF algorithm by combining the scaling algorithm of IFF and the push-relabel framework of [Fleischer+Iwata03] to get an $O((n^4\gamma + n^5)\log M)$ weakly polynomial algorithm, an $O((n^6\gamma + n^7)\log n)$ strongly polynomial algorithm, and an $O((n^8\gamma + n^9)\log^2 n)$ fully combinatorial algorithm, where γ is the time required for an oracle call for evaluating $f(X)$ for any specified $X \subseteq V$ and M is equal to $\max\{|f(X)| \mid X \subseteq V\}$. These are currently the best. The multiple-exchange technique described in Section 14.1.c was also adopted in [Iwata03]. Moreover, assuming an oracle for membership in base polyhedra, Fujishige and Iwata [Fuji+Iwata02] showed an $O(n^2)$ algorithm for submodular function minimization. Practically, the minimum-norm-point algorithm for submodular function minimization given in Section 7.1.b performs well (see [Isotani03]). The behavior of the algorithm seems to be worth investigating.

Moreover, bisubmodular functions discussed in Section 3.5.b are generalization of submodular functions. Combinatorial polynomial algorithms for minimizing bisubmodular functions are given in [Fuji+Iwata01] and [McCormick+Fuji05].

Chapter VII.
Discrete Convex Analysis

Recently K. Murota has developed a theory of discrete convex analysis (see [Murota01, 03a]). In this chapter we describe the essence of discrete convex analysis in a compact way by means of the theory of submodular functions developed in this monograph and the ordinary convex analysis of [Rockafellar70]. See [Murota03a] for more details and other related subjects. We consider only polyhedral (more precisely, locally polyhedral) convex functions. We do not claim originality on the results given here but the way of the presentation of discrete convex analysis seems to be new (though it is quite a natural one).

Historical notes about developments in discrete convex analysis will be given in Section 22.

15. Locally Polyhedral Convex Functions and Conjugacy

First, we review the theory of ordinary convex analysis [Rockafellar70], especially focusing on locally polyhedral convex functions, and also give definitions of some concepts about discrete convexity. Though we treat only locally polyhedral convex functions, propositions given below hold for more general convex functions (see [Rockafellar70]).

Let V be a finite nonempty set and consider a convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, where $\mathbf{R}$ is the set of reals. Define the *effective domain* of f by $\mathrm{dom} f = \{x \mid f(x) < +\infty\}$. We also define $\mathrm{argmin} f$ as the set of minimizers of f. A function g such that $-g$ is a convex function is called a *concave function.* The effective domain $\mathrm{dom} g$ of g is defined to be the effective domain of $-g$. For any convex/concave functions we assume that their effective domains are nonempty in the sequel. We call a convex function f on $\mathbf{R}^V$ a *locally polyhedral convex function* if for every bounded box $[a, b] \subset \mathbf{R}^V$ with $[a, b] \cap \mathrm{dom} f \neq \emptyset$ the function obtained by restricting f on $[a, b]$ is a polyhedral convex function. A convex set $P \subseteq \mathbf{R}^V$ is called a *locally polyhedral convex set* if for every bounded box $[a, b] \subset \mathbf{R}^V$ with $P \cap [a, b] \neq \emptyset$, $P \cap [a, b]$ is a polyhedron.

The *epigraph* epi$f$ of a convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is the unbounded convex set

$$\mathrm{epi}f = \{(x, \alpha) \mid x \in \mathrm{dom}f,\ \alpha \geq f(x)\}, \tag{15.1}$$

which is a locally polyhedral convex set if and only if f is a locally polyhedral convex function.

We denote the dual vector space of $\mathbf{R}^V$ by $(\mathbf{R}^V)^*$. For a vector $p \in (\mathbf{R}^V)^*$ we sometimes regard p as a linear function $\langle p, x\rangle$ in $x \in \mathbf{R}^V$, i.e., $p(x) = \langle p, x\rangle$, where $\langle p, x\rangle$ is the canonical inner product of p and x defined by $\langle p, x\rangle = \sum_{v \in V} p(v)x(v)$. (Note that the canonical inner product $\langle p, x\rangle$ is written as (p, x) in Part I.)

The *convex conjugate function* $f^\bullet : (\mathbf{R}^V)^* \to \mathbf{R} \cup \{+\infty\}$ of a convex function f on $\mathbf{R}^V$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^V\} \qquad (p \in (\mathbf{R}^V)^*). \tag{15.2}$$

Similarly we define the *concave conjugate function* g° of a concave function g on $\mathbf{R}^V$ by

$$g^\circ(p) = \inf\{\langle p, x\rangle - g(x) \mid x \in \mathbf{R}^V\} \qquad (p \in (\mathbf{R}^V)^*). \tag{15.3}$$

(Note that the conjugacy unary operators $^\bullet$ and $^\circ$ are denoted by * in Part I.)

It should be noted that for a locally polyhedral convex function f we have $(f^\bullet)^\bullet = f$. It should also be noted that the convex conjugate function of a locally polyhedral convex function is not a locally polyhedral convex function in general. As will be seen later, due to this fact the convex conjugate function of an L$^\natural$-convex function[1] (or an M$^\natural$-convex function) is not always M$^\natural$-convex (or L$^\natural$-convex), by the definition here, but this is only a technical nonessential problem. Note that the convex conjugate function of any polyhedral convex function is again polyhedral.

It is useful to see what happens to the convex conjugate function of f when we translate f by a vector $z \in \mathbf{R}^V$ and add to f an affine function $\langle q, x\rangle + \beta$. Putting $f_0(x) = f(x - z) + \langle q, x\rangle + \beta$ $(x \in \mathbf{R}^V)$, we can easily show

$$f_0^\bullet(p) = f^\bullet(p - q) + \langle p, z\rangle - \langle q, z\rangle - \beta \qquad (p \in (\mathbf{R}^V)^*). \tag{15.4}$$

[1]Symbol $\natural$ should be read as 'natural.'

We see that (i) when $q = \mathbf{0}$ and $\beta = 0$, the translation by z results in the addition of a linear function $\langle p, z\rangle$ in p, (ii) when $z = \mathbf{0}$ and $\beta = 0$, the addition of the linear function $\langle q, x\rangle$ yields a translation by q, and (iii) the addition of a constant β gives the addition of $-\beta$.

We call a set $P \subseteq \mathbf{R}^V$ a *linearity domain* of a locally polyhedral convex function f on $\mathbf{R}^V$ if there exists a vector $p \in (\mathbf{R}^V)^*$ such that $P = \mathrm{argmin}(f - p)$. Note that for such a linearity domain P of f we have $f(x) = \langle p, x\rangle + \beta$ $(x \in P)$ for some $\beta \in \mathbf{R}$, i.e., $\beta = \min\{(f - p)(x) \mid x \in \mathbf{R}^V\}$. Because of the locally polyhedral convexity of f there exists a collection of finitely many (maximal) linearity domains within any bounded box $[a, b]$ in $\mathbf{R}^V$. Recall that a polyhedral convex function is a convex function having finitely many linearity domains that are polyhedra. If each linearity domain of f is an integral polyhedron, we call f *domain-integral.* If f is domain-integral, the supremum in (15.2) is equal to the supremum over $\mathbf{Z}^V$.

If the convex conjugate function $f^\bullet$ is domain-integral, then we call f *co-domain-integral.*

For any locally polyhedral concave function we define the concepts of linearity domain, domain-integrality and co-domain-integrality *mutatis mutandis.*

The following theorem, called the Fenchel duality theorem, plays an essential rôle in the convex analysis (see [Rockafellar70]). Here we also consider some integrality property of the Fenchel duality.

Theorem 15.1 (The Fenchel duality): *For any locally polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and any locally polyhedral concave function $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbf{R}^V\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in (\mathbf{R}^V)^*\}. \tag{15.5}$$

Here, if $f - g$ is domain-integral, then the infimum over $\mathbf{R}^V$ is equal to that over $\mathbf{Z}^V$, and if $g^\circ - f^\bullet$ is domain-integral, then the supremum over $(\mathbf{R}^V)^*$ is equal to that over $(\mathbf{Z}^V)^*$.

(Proof) The relation (15.5) is a well-known fact in the ordinary convex analysis ([Rockafellar70]). The latter integrality immediately follows from the definition of domain-integrality. Q.E.D.

If both $f - g$ and $g^\circ - f^\bullet$ are domain-integral, then we can replace $\mathbf{R}^V$ by $\mathbf{Z}^V$ and $(\mathbf{R}^V)^*$ by $(\mathbf{Z}^V)^*$ in Theorem 15.1. This is the case for domain-integral L-/L$^\natural$-convex functions and domain-integral M-/M$^\natural$-convex functions in Murota's discrete convex analysis, which will be seen in Section 18.

We also have the following fundamental theorem related to conjugacy.

Theorem 15.2: *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a locally polyhedral convex function. For any $x \in \mathbf{R}^V$ and $p \in (\mathbf{R}^V)^*$ the following five statements are equivalent:*

(1) $f(x) + f^\bullet(p) = \langle p, x\rangle$.

(2) $p \in \partial f(x)$.

(3) $x \in \partial f^\bullet(p)$.

(4) $x \in \mathrm{argmin}(f - p)$.

(5) $p \in \mathrm{argmin}(f^\bullet - x)$.

Moreover, the subdifferential $\partial f(x)$ (or $\partial f^\bullet(p)$), if nonempty, is equal to the linearity domain of $f^\bullet$ (or f) given by $\mathrm{argmin}(f^\bullet - x)$ *(or* $\mathrm{argmin}(f - p)$*).*

(Proof) We have (1) if and only if

$$\forall y \in \mathbf{R}^V : f(y) - f(x) \geq \langle p, y - x\rangle, \tag{15.6}$$

which is equivalent to (2). By the symmetry, (1) is equivalent to (3), where note that $f^{\bullet\bullet} = f$. Also we can easily see the equivalence between (1) (or (15.6)) and (4), and similarly, the equivalence between (1) and (5).

The second half follows from (2) and (5) (or (3) and (4)). Q.E.D.

For locally polyhedral convex functions $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$ their (*infimal*) *convolution* $f_1 \circ f_2 : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f_1 \circ f_2(x) = \inf\{f_1(y) + f_2(x - y) \mid y \in \mathbf{R}^V\} \quad (x \in \mathbf{R}^V). \tag{15.7}$$

Note that $f_1 \circ f_2(x) = \inf\{\alpha \mid (x, \alpha) \in \mathrm{epi}f_1 + \mathrm{epi}f_2\}$ for each $x \in \mathbf{R}^V$, where the sum means the Minkowski sum (vector sum). If $f_1 \circ f_2(x) > -\infty$ for all $x \in \mathbf{R}^V$, then the convolution $f_1 \circ f_2$ is also a locally polyhedral convex function and each linearity domain of $f_1 \circ f_2$ is the Minkowski sum of a linearity domain of f_1 and that of f_2.

Concerning the convolution and the conjugacy we have

Theorem 15.3: *Let $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ $(i = 1, 2)$ be locally polyhedral convex functions. If $f_1 \circ f_2(x) > -\infty$ for all $x \in \mathbf{R}^V$, then we have*

$$(f_1 \circ f_2)^\bullet(p) = f_1^\bullet(p) + f_2^\bullet(p) \quad (p \in (\mathbf{R}^V)^*), \tag{15.8}$$

and if $\mathrm{dom} f_1 \cap \mathrm{dom} f_2 \neq \emptyset$, *then*

$$(f_1 + f_2)^\bullet(p) = f_1^\bullet \circ f_2^\bullet(p) \quad (p \in (\mathbf{R}^V)^*). \tag{15.9}$$

Moreover, for any $p \in \mathrm{dom} f_1^\bullet \cap \mathrm{dom} f_2^\bullet$ we have

$$\partial (f_1 \circ f_2)^\bullet(p) = \partial f_1^\bullet(p) + \partial f_2^\bullet(p), \tag{15.10}$$

where the sum + means the Minkowski sum.

16. L- and L$^\natural$-convex Functions

In this section we consider a class of well-behaved discrete convex functions, called L- and L$^\natural$-convex functions.

16.1. L- and L$^\natural$-convex Sets

First we define L- and L$^\natural$-convex sets. A (convex) polyhedron P in $\mathbf{R}^V$ is called an *L-convex set* if it is expressed by a system of inequalities

$$x(u) - x(v) \leq b_{uv} \quad ((u, v) \in A) \tag{16.1}$$

for some $A \subseteq V \times V$ and $b_{uv} \in \mathbf{R}$ $((u, v) \in A)$. Note that for any $x \in P$ and any $\alpha \in \mathbf{R}$ we have $x + \alpha \mathbf{1} \in P$. (Recall that $\mathbf{1}$ is the vector in $\mathbf{R}^V$ with all the components being equal to one.)

For an L-convex set P in $\mathbf{R}^V$ and an element $w \in V$ define a polyhedron $P^\swarrow = \{x \mid x \in P,\ x(w) = 0\}$. Such a polyhedron $P^\swarrow$ is called an *L$^\natural$-convex set* in $\mathbf{R}^{V-\{w\}}$. We also denote the L-convex set P in $\mathbf{R}^V$ by $Q^\nearrow$ for the L$^\natural$-convex set $Q = P^\swarrow$ in $\mathbf{R}^{V-\{w\}}$. It follows that an L$^\natural$-convex set in $\mathbf{R}^V$ (not in $\mathbf{R}^{V-\{w\}}$) is expressed by a system of linear inequalities

$$\begin{aligned} x(u) \geq b_u^- \quad (u \in W^-), \quad & x(u) \leq b_u^+ \quad (u \in W^+), \\ x(u) - x(v) \leq b_{uv} & \quad ((u, v) \in A) \end{aligned} \tag{16.2}$$

for some $W^-, W^+ \subseteq V$, $A \subseteq V \times V$ and b_u^- $(u \in W^-)$, b_u^+ $(u \in W^+)$, b_{uv} $((u, v) \in A)$ in $\mathbf{R}$ (see Fig. 16.1). It should be noted that an L-convex set in $\mathbf{R}^V$ is an L$^\natural$-convex set in $\mathbf{R}^V$.

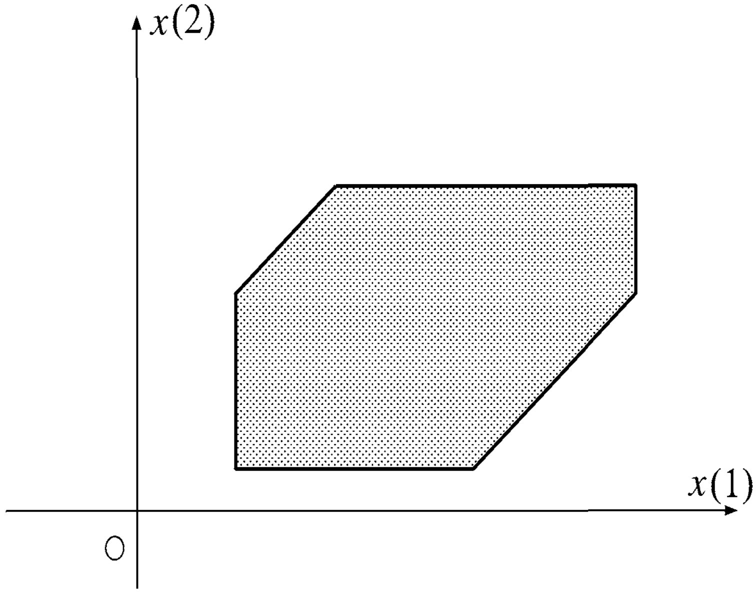

Figure 16.1: An L$^\natural$-convex set in $\mathbf{R}^{\{1,2\}}$.

Lemma 16.1: *Let P be an $L^\natural$-convex set in $\mathbf{R}^V$. Then, for any $x, y \in P$ we have $x \vee y, x \wedge y \in P$, where recall that $x \vee y = (\max\{x(v), y(v)\} \mid v \in V)$ and $x \wedge y = (\min\{x(v), y(v)\} \mid v \in V)$.*

(Proof) We show the last inequality in (16.2). Others can be shown similarly. For any $(u, v) \in A$, assuming that $x(u) \leq y(u)$, we have

$$x \vee y(u) - x \vee y(v) = y(u) - x \vee y(v) \leq y(u) - y(v) \leq b_{uv}. \qquad (16.3)$$

Hence $x \vee y \in P$. Similarly we can show $x \wedge y \in P$. Q.E.D.

Theorem 16.2: *A polyhedron P in $\mathbf{R}^V$ is an L-convex set if and only if*

(a) *P is closed with respect to the operations $\vee$ and $\wedge$ and*

(b) *for every $x \in P$ and $\alpha \in \mathbf{R}$ we have $x + \alpha\mathbf{1} \in P$.*

(Proof) The only-if part follows from Lemma 16.1 and the definition of an L-convex set. So, we show the if part.

Suppose (a) and (b). Define for each $u, v \in V$

$$b_{uv} = \sup\{x(u) - x(v) \mid x \in P\} \tag{16.4}$$

and, using these b_{uv} $(u, v \in V)$, consider

$$P' = \{x \mid x \in \mathbf{R}^V,\ \forall u, v \in V : x(u) - x(v) \leq b_{uv}\}. \tag{16.5}$$

It suffices to show $P' \subseteq P$. It follows from (16.4) and (b) that for any $z \in P'$ and any $u, v \in V$ with $u \neq v$ there exists $x_{uv} \in P$ such that $z(u) = x_{uv}(u)$ and $z(v) \geq x_{uv}(v)$. Hence we have from (a)

$$z = \bigvee_{u \in V} \bigwedge_{v \in V} x_{uv} \in P. \tag{16.6}$$

Q.E.D.

Let us examine conditions for the integrality of an L$^\natural$-convex set expressed by (16.2). Consider a graph $G = (V_0, A_0)$ with the vertex set $V_0 = V \cup \{0\}$ and the arc set A_0 given by

$$A_0 = A \cup \{(u, 0) \mid u \in W^-\} \cup \{(0, u) \mid u \in W^+\}, \tag{16.7}$$

where 0 is a new vertex not in V. Define a length function $l : A_0 \to \mathbf{R}$ by

$$l(a) = \begin{cases} \ \ b_{vu} & (a = (u, v) \in A) \\ -b_u^- & (a = (u, 0), u \in W^-) \\ \ \ b_u^+ & (a = (0, u), u \in W^+). \end{cases} \tag{16.8}$$

Recall that $p \in \mathbf{R}^{V_0}$ is a feasible potential in the network $\mathcal{N} = (G = (V_0, A_0), l)$ if and only if p satisfies

$$l(u, v) + p(u) - p(v) \geq 0 \qquad ((u, v) \in A_0). \tag{16.9}$$

We can easily see that if $p \in \mathbf{R}^{V_0}$ is a feasible potential in $\mathcal{N}$, then $(p(u) - p(0) \mid u \in V)$ satisfies (16.2) and, conversely, that if $q \in \mathbf{R}^V$ satisfies (16.2), then $p \in \mathbf{R}^{V_0}$ defined by $p(u) = q(u)$ $(u \in V)$ and $p(0) = 0$ is a feasible potential in $\mathcal{N}$. Hence, the L$^\natural$-convex set expressed by (16.2) is an integral polyhedron if b_u^-, b_u^+, b_{uv} appearing in (16.2) are integers. We can also show that if the L$^\natural$-convex set expressed by (16.2) is an integral polyhedron, the right-hand side of any non-redundant inequality in (16.2) is an integer. Hence, for any integral L$^\natural$-convex set expressed by (16.2) we can assume without loss of generality that the values of the right-hand side of (16.2) are integers. It should also be noted that there exists a feasible potential in $\mathcal{N}$ if and only if $\mathcal{N}$ has no directed cycle of negative length.

Remark: As can be seen from the above arguments, L-convex sets are exactly the dual network-flow polyhedra (see, e.g., [Berge73], [Iri69a] and [Rockafellar84]). Also Lemma 16.1 follows from the distributive-lattice property of a class of dual pre-Leontief substitution polyhedra shown in [Veinott89] (also see [Hochbaum+Naor94]).

Now, we show the following. For any $x \in \mathbf{R}^V$ we define $\lfloor x \rfloor = (\lfloor x(v) \rfloor \mid v \in V)$ and $\lceil x \rceil = (\lceil x(v) \rceil \mid v \in V)$.

Lemma 16.3: *We assume that all* b_u^-, b_u^+, b_{uv} *appearing in* (16.2) *are integers. For any* $x \in \mathbf{R}^V$ *satisfying* (16.2) *let* $(\emptyset \subset) S_1 \subset \cdots \subset S_k (\subseteq V)$ *be a (possibly empty) chain of subsets of* V *that uniquely expresses*

$$x - \lfloor x \rfloor = \sum_{i=1}^{k} \lambda_i \chi_{S_i} \tag{16.10}$$

with $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$. *Then, for each* $i = 0, 1, \cdots, k$ *the vector* $\lfloor x \rfloor + \chi_{S_i}$ *also satisfies* (16.2), *where we put* $S_0 = \emptyset$.

(Proof) Suppose that x is not integral, so that $k \geq 1$. For any i with $1 \leq i \leq k$ we can easily see that $\lfloor x \rfloor + \chi_{S_i}$ satisfies the first two inequalities in (16.2). Moreover, by the definition of S_i, if $(x - \lfloor x \rfloor)(u) > (x - \lfloor x \rfloor)(v)$ and $v \in S_i$, then $u \in S_i$. From this we see that the third inequality in (16.2) holds for $x \leftarrow \lfloor x \rfloor + \chi_{S_i}$. Q.E.D.

16.2. L- and L$^\natural$-convex Functions

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with dom$f \neq \emptyset$ that satisfies the following conditions:

(L$^\natural$1) f is a locally polyhedral convex function.

(L$^\natural$2) Each linearity domain of f is an L$^\natural$-convex set in $\mathbf{R}^V$.

Then we call f an *L$^\natural$-convex function.* (If $-g$ is an L$^\natural$-convex function, then g is called an *L$^\natural$-concave function.*) When f is a domain-integral L$^\natural$-convex function, we also call the function obtained by restricting f to $\mathbf{Z}^V$ an *L$^\natural$-convex function on* $\mathbf{Z}^V$. Moreover, if an L$^\natural$-convex function f satisfies

$$f(x + \alpha \mathbf{1}) = f(x) + \alpha r \qquad (x \in \mathrm{dom} f,\ \alpha \in \mathbf{R}) \tag{16.11}$$

for some $r \in \mathbf{R}$, then f is called an *L-convex function.*

For an L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ consider any $\underline{l} : V \to \mathbf{R} \cup \{-\infty\}$ and $\bar{l} : V \to \mathbf{R} \cup \{+\infty\}$ such that $(\mathrm{dom} f) \cap [\underline{l}, \bar{l}] \neq \emptyset$. Define a function $f_{\underline{l}}^{\bar{l}}$ by

$$f_{\underline{l}}^{\bar{l}}(x) = \begin{cases} f(x) & \text{if } x \in (\mathrm{dom} f) \cap [\underline{l}, \bar{l}] \\ +\infty & \text{otherwise} \end{cases} \qquad (x \in \mathbf{R}^V). \tag{16.12}$$

We call $f_{\underline{l}}^{\bar{l}}$ a *box reduction* of f. Note that a box $[\underline{l}, \bar{l}]$ is an L$^\natural$-convex set.

Theorem 16.4: *Any box reduction $f_{\underline{l}}^{\bar{l}}$ of an L$^\natural$-convex function is also an L$^\natural$-convex function.*

(Proof) We can easily see that any linearity domain of $f_{\underline{l}}^{\bar{l}}$ is an L$^\natural$-convex set, due to the L$^\natural$-convexity of f. Hence $f_{\underline{l}}^{\bar{l}}$ is also an L$^\natural$-convex function. Q.E.D.

The following two lemmas easily follow from the definitions of L$^\natural$- and L-convex functions.

Lemma 16.5: *For any L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ the effective domain* $\mathrm{dom} f$ *is an L$^\natural$-convex set.*

(Proof) Since $\mathrm{dom} f$ is a locally polyhedral convex set given as the union of L$^\natural$-convex sets, it is expressed by a system of inequalities of types as in (16.2). Q.E.D.

Lemma 16.6: *Let* 0 *be a new element not in* V. *For an L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and a scalar $r \in \mathbf{R}$, define a function $f^{\nearrow} : \mathbf{R}^{\{0\} \cup V} \to \mathbf{R} \cup \{+\infty\}$ by*

$$f^{\nearrow}(\alpha, x) = f(x - \alpha \mathbf{1}) + \alpha r \tag{16.13}$$

for each $\alpha \in \mathbf{R}$ and $x \in \mathbf{R}^V$, where α is the value of the 0*th coordinate. Then $f^{\nearrow}$ is an L-convex function.*

Conversely, if $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is an L-convex function, then for any $w \in V$, denoting by $x^{V-\{w\}}$ the restriction of x to $V - \{w\}$, we have an L$^\natural$-convex function $f^{\swarrow} : \mathbf{R}^{V-\{w\}} \to \mathbf{R} \cup \{+\infty\}$ defined by $f^{\swarrow}(x^{V-\{w\}}) = f(x)$ for $x \in \mathbf{R}^V$ with $x(w) = 0$.

(Proof) Suppose that $f^{\nearrow}$ is given by (16.13). Let Q be a linearity domain of $f^{\nearrow}$. Then, from (16.13) there exists a linearity domain $Q^{\swarrow}$ of f such that

$$Q = \{(\alpha, x) \mid x - \alpha\mathbf{1} \in Q^{\swarrow}\}. \tag{16.14}$$

If $Q^{\swarrow}$ is expressed as (16.2), we have $(\alpha, x) \in Q$, or $x - \alpha\mathbf{1} \in Q^{\swarrow}$, if and only if

$$\begin{gathered} x(u) - \alpha \geq b_u^- \quad (u \in W^-), \quad x(u) - \alpha \leq b_u^+ \quad (u \in W^+), \\ x(u) - x(v) \leq b_{uv} \quad ((u, v) \in A). \end{gathered} \tag{16.15}$$

Hence Q is an L-convex set, so that $f^{\nearrow}$ is an L-convex function.

Conversely, the second half follows from the fact that any box reduction of f is an L$^\natural$-convex function. Q.E.D.

Moreover, we have the following theorem, Theorem 16.8. We first show a special case in the two-dimensional space.

Lemma 16.7: *When $V = \{1, 2\}$, an L$^\natural$-convex function on $\mathbf{R}^V$ is submodular, i.e., $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ $(x, y \in \mathbf{R}^V)$.*

(Proof) Suppose that $x, y \in \mathrm{dom} f$ and $x(1) < y(1)$ and $x(2) > y(2)$. From Lemmas 16.1 and 16.5 we have $x \vee y, x \wedge y \in \mathrm{dom} f$. Put $z = x \wedge y$ and $w = x \vee y$. Define $p_\alpha = (1-\alpha)z + \alpha y$ and $q_\alpha = (1-\alpha)x + \alpha w$ for $0 \leq \alpha \leq 1$ (see Fig. 16.2). Then for any $\alpha \in [0, 1]$, moving along the line segment $\overline{p_\alpha q_\alpha}$ from p_α to q_α and crossing a boundary of a linearity domain P_0 of f, we go through an inequality of type (I) $x(2) \leq b_2^+$ or (II) $x(2) - x(1) \leq b_{21}$ (or both in a degenerate case). When crossing an inequality of type (I) but not type (II), the slope along $x(1)$-axis does not change, while when crossing an inequality of type (II), the slope along $x(1)$-axis decreases because of the convexity of f. Therefore, the slope along $x(1)$-axis at q_α is smaller than or equal to that at p_α. Hence we have $f(y) - f(z) \geq f(w) - f(x)$, where note that $z = x \wedge y$ and $w = x \vee y$. Q.E.D.

For any $x \in \mathbf{R}^V$ define

$$\mathrm{supp}^+(x) = \{v \mid v \in V,\ x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \mid v \in V,\ x(v) < 0\}. \tag{16.16}$$

Theorem 16.8: *Let f be an L$^\natural$-convex function on $\mathbf{R}^V$. Then f is a submodular function on $\mathbf{R}^V$, i.e., for each $x, y \in \mathbf{R}^V$*

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y). \tag{16.17}$$

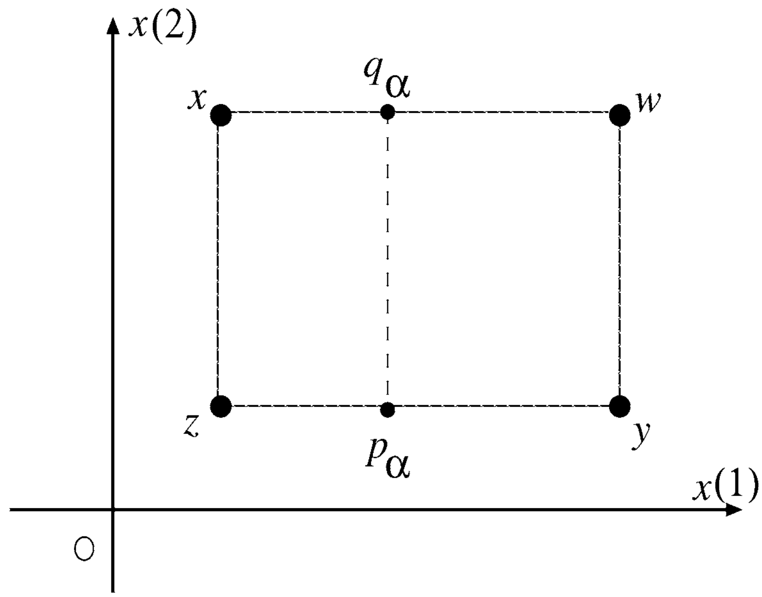

Figure 16.2: Submodularity of an L$^\natural$-convex function on $\mathbf{R}^{\{1,2\}}$.

(Proof) For any $x, y \in \mathrm{dom}f$ define a set function $f_0 : 2^V \to \mathbf{R}$ as follows.

$$f_0(X) = f(x \wedge y + \sum_{v \in X} (x \vee y(v) - x \wedge y(v))\chi_v) \quad (X \subseteq V). \tag{16.18}$$

Since the inequality in (16.17) is equivalent to $f_0(S^+) + f_0(S^-) \geq f_0(S^+ \cup S^-) + f_0(S^+ \cap S^-)$ for $S^+ = \mathrm{supp}^+(x-y)$ and $S^- = \mathrm{supp}^-(x-y)$, it suffices to prove that f_0 is a submodular set function on 2^V, or equivalently, for any $X \subset V$ and distinct $u, v \in V - X$ we have $f_0(X \cup \{u\}) + f_0(X \cup \{v\}) \geq f_0(X \cup \{u, v\}) + f_0(X)$. Hence we have only to show that for any $x \in \mathrm{dom}f$, any distinct $u, v \in V$, and any $\alpha, \beta > 0$ such that $x + \alpha\chi_u, x + \beta\chi_v \in \mathrm{dom}f$, we have

$$f(x + \alpha\chi_u) + f(x + \beta\chi_v) \geq f(x + \alpha\chi_u + \beta\chi_v) + f(x). \tag{16.19}$$

This follows from Lemma 16.7 and Theorem 16.4. Q.E.D.

A characterization of L-convex functions is given as follows. This property is adopted as the defining axiom of L-convex functions in Murota's discrete convex analysis ([Murota03a]).

Theorem 16.9: *A locally polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is an L-convex function on $\mathbf{R}^V$ if and only if*

(1) *f is a submodular function on $\mathbf{R}^V$ with join $\vee$ and meet $\wedge$ and*

(2) *there exists a scalar $r \in \mathbf{R}$ such that for any $x \in \mathrm{dom} f$ and any $\alpha \in \mathbf{R}$ we have*

$$f(x + \alpha \mathbf{1}) = f(x) + \alpha r. \tag{16.20}$$

(Proof) The only-if part holds, due to Theorem 16.8 and the definition of an L-convex function. So, suppose that f satisfies (1) and (2) above. Then we can easily see that any linearity domain P of f satisfies Conditions (a) and (b) in Theorem 16.2, and hence is an L-convex set. Q.E.D.

The relationship between L$^\natural$-/L-convex functions and M$^\natural$-/M-convex functions will be made clear later when we consider M$^\natural$-/M-convex functions and reveal the conjugacy between a class of L$^\natural$-/L-convex functions and a class of M$^\natural$-/M-convex functions.

16.3. Domain-integral L- and L$^\natural$-convex Functions

It follows from Lemma 16.3 that domain-integral L$^\natural$-convex functions are characterized by the following (L$^\natural$1$'$) and (L$^\natural$2$'$).

(L$^\natural$1$'$) f is a locally polyhedral convex function.

(L$^\natural$2$'$) For any $x \in \mathrm{dom} f$ let $(\emptyset \subset) S_1 \subset \cdots \subset S_k (\subseteq V)$ be a (possibly empty) chain of subsets of V that uniquely expresses

$$x - \lfloor x \rfloor = \sum_{i=1}^{k} \lambda_i \chi_{S_i} \tag{16.21}$$

with $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$. Then, $\lfloor x \rfloor + \chi_{S_i} \in \mathrm{dom} f$ $(i = 0, 1, \cdots, k)$, where $S_0 = \emptyset$. Moreover, we have

$$f(x) = \lambda_0 f(\lfloor x \rfloor) + \sum_{i=1}^{k} \lambda_i f(\lfloor x \rfloor + \chi_{S_i}), \tag{16.22}$$

where $\lambda_0 = 1 - \sum_{i=1}^{k} \lambda_i$ (≥ 0).

Remark: An integral vector $z \in \mathbf{Z}^V$ and a maximal chain $\mathcal{C} : S_0(=\emptyset) \subset S_1 \subset \cdots \subset S_n(=V)$ of the Boolean lattice 2^V define an n-dimensional simplex given by the convex hull of $z + \chi_{S_i}$ $(i = 0, 1, \cdots, n)$. The collection of such simplices for all integral vectors z and for all maximal chains $\mathcal{C}$ forms a simplicial division of $\mathbf{R}^V$ due to Freudentahl (Fig. 16.3) (see, e.g., [Todd76] and [Yang99]). We call any face of such a simplex in Freudentahl's simplicial division *Freudentahl's simplex cell.*

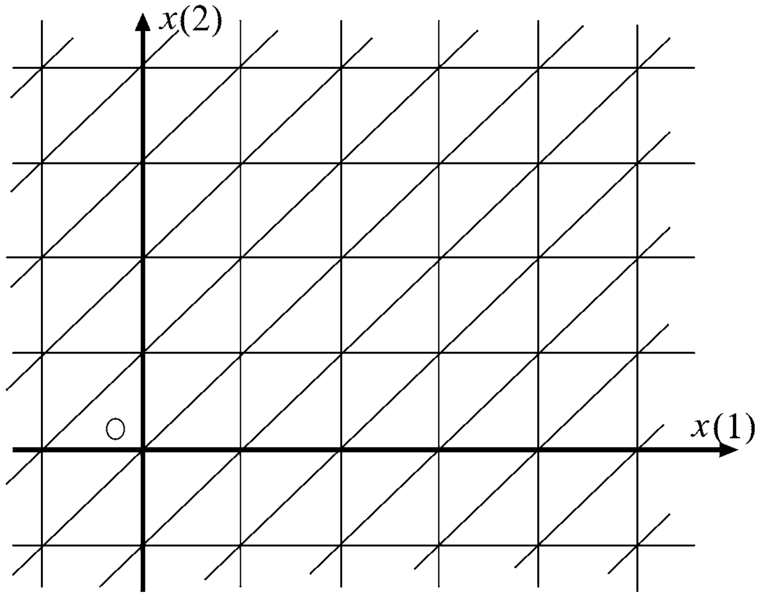

Figure 16.3: Freudentahl's simplicial division of $\mathbf{R}^{\{1,2\}}$.

Note that the truncated Lovász extensions of submodular (set) functions defined by (6.76) are exactly L$^\natural$-convex functions with their effective domains being contained in the unit hypercube $[0, 1]^V$. Hence, we have

Lemma 16.10: *A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is a domain-integral L$^\natural$-convex function if and only if*

(a) *f is a locally polyhedral convex function and*

(b) *for each integral vector $z \in \mathbf{Z}^V$ and each set $W \subseteq V$ such that $z, z + \chi_W \in \mathrm{dom} f$, the restriction of $f(x) - f(z)$ in x on the interval*

$[z, z+\chi_W]$ is the truncated Lovász extension (on $\mathbf{R}^W$) of a submodular (set) function whose domain is imbedded in $\mathbf{R}^V$ and translated by z.

For a function $h : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} h \neq \emptyset$, if a locally polyhedral (not necessarily convex) function $\hat{h} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is obtained by ($\mathrm{L}^\natural 2'$) with f being replaced by h, then we call $\hat{h}$ the *Lovász-Freudentahl extension* of h. From ($\mathrm{L}^\natural 1'$) and ($\mathrm{L}^\natural 2'$) we have

Theorem 16.11: *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is an $L^\natural$-convex function on $\mathbf{Z}^V$ if and only if the Lovász-Freudentahl extension of f is a convex function.*

The concept of a domain-integral $\mathrm{L}^\natural$-convex function with its domain being a box was considered by P. Favati and F. Tardella [Favati+Tardella90], who called it a *submodular integrally convex function.* It should be noted that $\mathrm{L}^\natural$- and L-convex functions of [Favati+Tardella90] and [Murota98b] are originally defined on integral lattice points in $\mathbf{Z}^V$, while we are here considering locally polyhedral convex functions defined on $\mathbf{R}^V$ of real (or rational) vectors that are uniquely determined from the values on $\mathbf{Z}^V$ by the scheme of ($\mathrm{L}^\natural 2'$) given above (also see [Murota03a, Sections 6.11 and 7.8]).

The origin of the following characterization is found in [Favati+Tardella 90] (also see [Fuji+Murota00]).

Theorem 16.12: *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is an $L^\natural$-convex function on $\mathbf{Z}^V$ if and only if for each $p, q \in \mathbf{Z}^V$*

$$f(p) + f(q) \geq f(\lceil (p+q)/2 \rceil) + f(\lfloor (p+q)/2 \rfloor). \tag{16.23}$$

(Proof) (The if part): We assume without loss of generality that $\mathrm{dom} f$ is full-dimensional. Suppose that (16.23) holds for each $p, q \in \mathbf{Z}^V$. It suffices to prove the convexity of the Lovász-Freudentahl extension $\hat{f}$ of f on the union of two adjacent full-dimensional Freudentahl's simplex cells. We have adjacent simplex cells of the following two types. For an integral vector z in $\mathrm{dom} f$ and a linear ordering $(v_1, v_2, \cdots, v_n)$, defining

$$S_i = \{v_1, v_2, \cdots, v_i\} \qquad (i = 1, 2, \cdots, n) \tag{16.24}$$

and $S_0 = \emptyset$, consider

(I) two simplices formed by

$$z + \chi_{S_i} \quad (i = 0, 1, \cdots, n) \tag{16.25}$$

and by

$$z - \chi_{v_n}, \quad z + \chi_{S_i} \ (i = 0, 1, \cdots, n-1), \tag{16.26}$$

where the common face of the two simplices is formed by the n points $z + \chi_{S_i}$ $(i = 0, 1, \cdots, n-1)$, and $p = z + \mathbf{1}$ and $q = z - \chi_{v_n}$ are the points of the two simplices outside the common face, and

(II) two simplices formed by

$$z + \chi_{S_i} \quad (i = 0, 1, \cdots, n) \tag{16.27}$$

and by

$$z + \chi_{S_i} \ (i = 0, 1, \cdots, k, k+2, \cdots, n), \quad z + \chi_{S_k \cup \{v_{k+2}\}} \tag{16.28}$$

for some k with $0 \leq k \leq n-2$, where the common face of the two simplices is formed by the n points $z+\chi_{S_i}$ $(i = 0, 1, \cdots, k, k+2, \cdots, n)$, and $p = z + \chi_{S_{k+1}}$ and $q = z + \chi_{S_k \cup \{v_{k+2}\}}$ are the points of the two simplices outside the common face.

Since we have

$$p + q = \lceil (p+q)/2 \rceil + \lfloor (p+q)/2 \rfloor, \tag{16.29}$$

it follows from (16.23) and the definition of the Lovász-Freudentahl extension $\hat{f}$ that

$$\begin{aligned} \frac{1}{2}\{\hat{f}(p) + \hat{f}(q)\} &= \frac{1}{2}\{f(p) + f(q)\} \\ &\geq \frac{1}{2}\{f(\lceil (p+q)/2 \rceil) + f(\lfloor (p+q)/2 \rfloor)\} \\ &= \hat{f}((p+q)/2). \end{aligned} \tag{16.30}$$

Here note that for (I) we have $\lceil (p+q)/2 \rceil = z + \chi_{S_{n-1}}$ and $\lfloor (p+q)/2 \rfloor = z$ and for (II) $\lceil (p+q)/2 \rceil = z + \chi_{S_{k+2}}$ and $\lfloor (p+q)/2 \rfloor = z + \chi_{S_k}$. Since these two points $\lceil (p+q)/2 \rceil$ and $\lfloor (p+q)/2 \rfloor$ are vertices of the common face, $(p+q)/2$ belongs to the common face. Hence, it follows from (16.30) that the Lovász-Freudentahl extension $\hat{f}$ of f restricted to the union of the two adjacent simplex cells is convex.

(The only-if part): Suppose that f is an L$^\natural$-convex function on $\mathbf{Z}^V$. Consider the Lovász-Freudentahl extension $\hat{f}$ of f. Then, because of the convexity of $\hat{f}$ and the definition of the Lovász-Freudentahl extension, we have for each $p, q \in \mathbf{Z}^V$

$$\begin{aligned} \hat{f}(p) + \hat{f}(q) &\geq 2\hat{f}((p+q)/2) \\ &= 2\hat{f}((\lceil (p+q)/2 \rceil + \lfloor (p+q)/2 \rfloor)/2) \\ &= \hat{f}(\lceil (p+q)/2 \rceil) + \hat{f}(\lfloor (p+q)/2 \rfloor). \end{aligned} \tag{16.31}$$

Q.E.D.

We give characterizations of integral L$^\natural$-convex sets as follows. (Recall that by a polyhedron we mean a convex polyhedron.)

Theorem 16.13: *For a polyhedron P in $\mathbf{R}^V$ the following four statements are equivalent:*

(1) *P is an integral L$^\natural$-convex set.*

(2) *P is a polyhedron formed by the union of Freudentahl's simplex cells.*

(3) *P is an integral polyhedron and for any integral points $p, q \in P$ we have $\lfloor (p+q)/2 \rfloor,\ \lceil (p+q)/2 \rceil \in P$.*

(4) *P is an integral polyhedron and for each integral point $p \in P$,*

$$\mathcal{D}_p^+ \equiv \{X \mid X \subseteq V,\ p + \chi_X \in P\}, \quad \mathcal{D}_p^- \equiv \{X \mid X \subseteq V,\ p - \chi_X \in P\} \tag{16.32}$$

are distributive lattices with join $\cup$ and meet $\cap$, and $P \cap [p, p + \mathbf{1}]$ and $P \cap [p - \mathbf{1}, p]$ are, respectively, equal to the convex hull of χ_X ($X \in \mathcal{D}_p^+$) and that of χ_X ($X \in \mathcal{D}_p^-$).

(Proof) By (L$^\natural$2$'$) in the characterization of domain-integral L$^\natural$-convex functions and Theorem 16.12 we see that (1), (2) and (3) are equivalent. So, we show the equivalence between (4) and $\{(1), (2), (3)\}$. Suppose (3). Then for each integral point $p \in P$ and $X, Y \subseteq V$ such that $p + \chi_X, p + \chi_Y \in P$, we have $\lfloor (p + \chi_X + p + \chi_Y)/2 \rfloor = p + \chi_{X \cap Y}$ and $\lceil (p + \chi_X + p + \chi_Y)/2 \rceil = p + \chi_{X \cup Y}$. Similarly, for $X, Y \subseteq V$ such that $p - \chi_X, p - \chi_Y \in P$ we have $\lfloor (p - \chi_X + p - \chi_Y)/2 \rfloor = p - \chi_{X \cup Y}$ and $\lceil (p - \chi_X + p - \chi_Y)/2 \rceil = p - \chi_{X \cap Y}$. Hence (4) holds. Conversely, suppose (4). Then it follows that for each integral point $p \in P$ sets $P \cap [p, p + \mathbf{1}]$ and $P \cap [p - \mathbf{1}, p]$ are the unions of Freudentahl's simplex cells. Hence (2) holds. Q.E.D.

Examples of an L-/L$^\natural$-convex function

(a) Any submodular set function $f : \mathcal{D} \to \mathbf{R}$ on a distributive lattice $\mathcal{D} \subseteq 2^V$ is identified with the L$^\natural$-convex function $\bar{f}$ on $\mathbf{Z}^V$ defined by $\bar{f}(z) = f(X)$ if $z = \chi_X$ for $X \in \mathcal{D}$ and $\bar{f}(z) = +\infty$ otherwise.

(b) Consider a real symmetric matrix $M = (\mu(u, v) \mid u, v \in V)$ such that

$$\mu(u, v) \le 0 \quad (u \ne v), \qquad \sum_{v \in V} \mu(u, v) \ge 0 \quad (u \in V). \tag{16.33}$$

(Such a matrix M is a diagonally dominant symmetric M-matrix.) Then the quadratic function

$$f(z) = \sum_{u,v \in V} \mu(u, v) z(u) z(v) \qquad (z \in \mathbf{Z}^V) \tag{16.34}$$

is an L$^\natural$-convex function on $\mathbf{Z}^V$. Conversely, if f defined by (16.34) is an L$^\natural$-convex function on $\mathbf{Z}^V$, then $M = (\mu(u, v) \mid u, v \in V)$ satisfies (16.33). Note that an L$^\natural$-convex function f defined by (16.34) is an L-convex function if and only if $\sum_{v \in V} \mu(u, v) = 0$ $(u \in V)$. (See [Murota01, 03a] and [Murota+Shioura04b] for more details.)

17. M- and M$^\natural$-convex Functions

We first show the following theorem, due to Tomizawa (recall that 'g-polymatroid' stands for 'generalized polymatroid' (see Section 3.5.a)).

Theorem 17.1: *A nonempty polyhedron B in $\mathbf{R}^V$ is a base polyhedron if and only if for every point x in B the tangent cone of B at x is generated by vectors chosen from $\chi_u - \chi_v$ $(u, v \in V)$.*

Also, a nonempty polyhedron Q in $\mathbf{R}^V$ is a g-polymatroid if and only if for every point x in Q the tangent cone of Q at x is generated by vectors chosen from χ_u, $-\chi_u$ $(u \in V)$ and $\chi_u - \chi_v$ $(u, v \in V)$.

(Proof) Because of Theorem 3.58 the former statement is equivalent to the latter. The only-if part of the former statement follows from Theorem 3.28. Hence we show its if part.

Suppose that for every point x in B the tangent cone T_x of B at x is generated by vectors $\chi_u - \chi_v$ $((u, v) \in A_x)$ for some $A_x \subseteq V \times V$. Consider a graph $G_x = (V, A_x)$ with vertex set V and arc set A_x and let $\mathcal{I}(G_x)$ be the collection of closed sets of G_x (where $U \subseteq V$ is a closed set of G_x if and only if there is no arc $a \in A_x$ such that $\partial^+ a \in U$ and $\partial^- a \notin U$). Then, by adapting Theorem 3.26, we see that the tangent cone is expressed by

$$x(X) \leq 0 \quad (X \in \mathcal{I}(G_x)), \qquad x(V) = 0. \tag{17.1}$$

Put $\mathcal{F} = \cup\{\mathcal{I}(G_x) \mid x \in B\}$, where note that $\mathcal{F}$ is a finite set since $\mathcal{F} \subseteq 2^V$. It follows from (17.1) that for some function $f : \mathcal{F} \to \mathbf{R}$ the polyhedron B is expressed by

$$x(X) \leq f(X) \quad (X \in \mathcal{F}), \qquad x(V) = f(V). \tag{17.2}$$

We can assume that for each $x \in B$ we have $x(X) = f(X)$ $(X \in \mathcal{I}(G_x))$ and in particular $f(\emptyset) = 0$.

Now, suppose that we are given two sets $X, Y \in \mathcal{F}$ with $X - Y \neq \emptyset$ and $Y - X \neq \emptyset$. Since $\max\{\langle \chi_X + \chi_Y, x\rangle \mid x \in B\} \leq f(X) + f(Y) < +\infty$, there exists a point $y \in B$ that attains the maximum of the linear function $\langle \chi_X + \chi_Y, x\rangle$ over B. Note that $\mathcal{I}(G_y)$ is closed with respect to set union and intersection, i.e., it is a distributive lattice, and hence the normal vector $\chi_X + \chi_Y$ is a positive linear combination of characteristic vectors of sets in $\mathcal{I}(G_y)$ that form a chain $\mathcal{C}$ of $\mathcal{I}(G_y)$. Since $\chi_X + \chi_Y = \chi_{X\cup Y} + \chi_{X\cap Y}$, the chain of $X \cap Y \subset X \cup Y$ must be the $\mathcal{C}$, so that $X \cap Y, X \cup Y \in \mathcal{I}(G_y) \subseteq \mathcal{F}$. It follows from (17.2) that

$$f(X)+f(Y) \geq y(X)+y(Y) = y(X\cup Y)+y(X\cap Y) = f(X\cup Y)+f(X\cap Y). \tag{17.3}$$

This implies that f is a submodular function on the distributive lattice $\mathcal{F}$ with $\emptyset, V \in \mathcal{F}$ and $f(\emptyset) = 0$. Hence the polyhedron B expressed by (17.2) is the base polyhedron associated with the submodular system $(\mathcal{F}, f)$ on V. Q.E.D.

The first half of Theorem 17.1 was announced by Tomizawa [Tomi83] without a proof (see [Fuji+Yang03, Appendix] and [Danilov+Koshevoy+Lang03]; also see [Gelfand+Goresky+MacPherson+Serganova87]).

Suppose that a function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ satisfies the following conditions:

(M$^\natural$1) f is a locally polyhedral convex function on $\mathbf{R}^V$.

(M$^\natural$2) Each linearity domain of f is a g-polymatroid.

Then we call f an *M$^\natural$-convex function* on $\mathbf{R}^V$. Note that each linearity domain of f is given by $\mathrm{argmin}(f-p)$ for some $p \in (\mathbf{R}^V)^*$. Since $\mathrm{dom}f$ is convex and is the union of g-polymatroids, each being a linearity domain of f, it follows from Theorem 17.1 that $\mathrm{dom}f$ is also a g-polymatroid. If $\mathrm{dom}f$ is a base polyhedron, then we call f an *M-convex function* on $\mathbf{R}^V$. In [Murota03a] a g-polymatroid is called an *M$^\natural$-convex set*, and a base polyhedron an *M-convex set*. When $-g$ is an M$^\natural$-convex function, g is called an *M$^\natural$-concave function*. Figure 17.4 shows an example of an M$^\natural$-concave function.

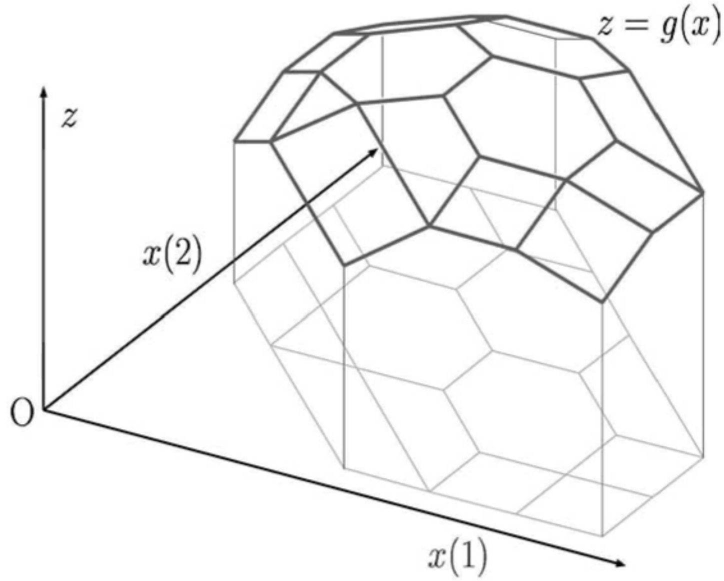

Figure 17.4: An M$^\natural$-concave function g.

Remark: Rephrasing the above conditions (M$^\natural$1) and (M$^\natural$2), we can say that a locally polyhedral convex function f on $\mathbf{R}^V$ with $\mathrm{dom}f \neq \emptyset$ is an M$^\natural$-convex function if and only if the effective domain of f is divided into g-polymatroids Q_i $(i \in I)$ (where distinct Q_i and $Q_{i'}$ $(i, i' \in I)$ may have common boundary points but do not have any common relative interior points) and f is an affine function on each g-polymatroid Q_i $(i \in I)$. When Q_i $(i \in I)$ are integral g-polymatroids, f is domain-integral. A domain-integral M$^\natural$-convex function (having finitely many linearity domains) is called an *integral polyhedral M$^\natural$-convex function* in [Murota03a].

Examples of an M-/M$^\natural$-convex function:

(a) For a submodular system $(\mathcal{D}, f)$ on V the vector rank function $\mathrm{r}_f : \mathbf{R}^V \to \mathbf{R}$ is an M$^\natural$-concave function (see Lemma 6.2).

(b) Let $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c}, \gamma)$ be a flow network, where $G = (V, A)$ is an underlying graph with a vertex set V and an arc set A, functions $\underline{c} : A \to \mathbf{R} \cup \{-\infty\}$ and $\bar{c} : A \to \mathbf{R} \cup \{+\infty\}$ are, respectively, a lower-capacity function and an upper-capacity function such that $\underline{c} \le \bar{c}$, and $\gamma : A \to \mathbf{R}$ is a cost function. For a given vector $b \in \mathbf{R}^V$ consider the following minimum-cost flow problem:

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{a \in A} \gamma(a)\varphi(a) & \\ \text{subject to} & \underline{c}(a) \le \varphi(a) \le \bar{c}(a) \quad (a \in A), & \\ & \partial\varphi(v) = b(v) \quad (v \in V). & \qquad (17.4) \end{array}$$

Define $f(b)$ to be the minimum objective function value of (17.4), where we put $f(b) = +\infty$ if (17.4) is infeasible for b. Then thus defined f is an M-convex function ([Murota98b, 99]), which can be seen from the following fact. For any $p \in \mathbf{R}^V$ such that $\mathrm{argmin}(f - p) \ne \emptyset$, $\mathrm{argmin}(f - p)$ is equal to the boundary base polyhedron consisting of the boundaries of minimum-cost flows for the above problem with the objective function modified as $\sum_{a \in A} \gamma(a)\varphi(a) - \sum_{v \in V} p(v)b(v)$ with b regarded as another variable vector. (Also see [Murota03a, Section 2.2 and Chapter 9] for more details.)

(c) Let $\mathcal{F}$ be a laminar family of subsets of V, and for each $X \in \mathcal{F}$ let f_X be a convex function on $\mathbf{R}$. Then, a function f on $\mathbf{Z}^V$ defined by

$$f(x) = \sum_{X \in \mathcal{F}} f_X(x(X)) \qquad (17.5)$$

is called *laminar convex*. Laminar convex functions on $\mathbf{Z}^V$ are M$^\natural$-convex ([Danilov+ Koshevoy+Murota01], [Murota01, 03a]). Moreover, based on results on quadratic M-convex functions of [Murota+Shioura04b], Hirai and Murota [Hirai+Murota04] showed: (1) a quadratic form defined on $\mathbf{Z}^V$ is M$^\natural$-convex if and only if it is a laminar convex function as in (17.5) with convex quadratic functions f_X $(X \in \mathcal{F})$ and (2) a quadratic form $f(x)$ with its effective domain restricted to $\{x \mid x \in \mathbf{Z}^V,\ x(V) = 0\}$ is an M-convex function on $\mathbf{Z}^V$ if and only if $f(x)$ is expressed as

$$f(x) = -\frac{1}{2} \sum_{u,v \in V} d_T(u,v)x(u)x(v), \tag{17.6}$$

where $d_T : V \times V \to \mathbf{R}_+$ is a tree metric induced by a tree T having a leaf set V and a set of edges with nonnegative lengths. It follows from the well-known relationship between tree metrics and cut metrics (see, e.g., [Deza+Laurent97]) that (17.6) can be rewritten as

$$f(x) = \sum_{X \in \mathcal{F}} c_X x(X)^2, \tag{17.7}$$

where $\mathcal{F}$ is a cross-free family of subsets of V and c_X $(X \in \mathcal{F})$ are nonnegative reals. Here, the set $\mathcal{K}$ of pairs $\{X, V - X\}$ $(X \in \mathcal{F})$ is uniquely determined by f, and c_X $(X \in \mathcal{F})$ regarded as a function on $\mathcal{K}$ is also unique (note that $x(X)^2 = x(V - X)^2$ on the effective domain of f since $x(V) = 0$).

By definition, an M-convex function is an M$^\natural$-convex function. Furthermore, since the projection of any base polyhedron along an axis $v_0 \in V$ on the hyperplane $x(v_0) = 0$ is a g-polymatroid and any g-polymatroid is obtained by such a projection (see Theorem 3.58), we can easily show the following.

Theorem 17.2: *Let* 0 *be a new element not in* V. *For any M-convex function* f *on* $\mathbf{R}^{V \cup \{0\}}$, *defining a function* $f^\downarrow$ *on* $\mathbf{R}^V$ *by*

$$f^\downarrow(y) = \begin{cases} f(x) & \text{if } \exists x \in \mathrm{dom} f : y(v) = x(v)\ (v \in V) \\ +\infty & \text{otherwise} \end{cases} \tag{17.8}$$

for each $y \in \mathbf{R}^V$, *we get an M$^\natural$-convex function* $f^\downarrow$ *on* $\mathbf{R}^V$. *Conversely, for any M$^\natural$-convex function* f *on* $\mathbf{R}^V$ *define a function* $f^\uparrow$ *on* $\mathbf{R}^{V \cup \{0\}}$ *by*

$$f^\uparrow(y) = \begin{cases} f(x) & \text{if } \exists x \in \mathrm{dom} f : y(v) = x(v)\ (v \in V),\ y(0) = -x(V) \\ +\infty & \text{otherwise} \end{cases} \tag{17.9}$$

for each $y \in \mathbf{R}^{V \cup \{0\}}$. *Then* $f^\uparrow$ *is an M-convex function on* $\mathbf{R}^{V \cup \{0\}}$.

Note that (17.8) and (17.9) give one-to-one mappings, one being the inverse of the other, between the set of M$^\natural$-convex functions on $\mathbf{R}^V$ and that of M-convex functions on $\mathbf{R}^{V\cup\{0\}}$ whose effective domains lie on the hyperplane $x(V)+x(0)=0$.

For an M-convex (or M$^\natural$-convex) function f on $\mathbf{R}^V$ and vectors $\underline{l}: V \to \mathbf{R}\cup\{-\infty\}$ and $\bar{l}: V \to \mathbf{R}\cup\{+\infty\}$ such that $(\mathrm{dom} f)\cap[\underline{l},\bar{l}] \neq \emptyset$, define the function $f_{\underline{l}}^{\bar{l}}$ by

$$f_{\underline{l}}^{\bar{l}}(x) = \begin{cases} f(x) & \text{if } x \in (\mathrm{dom} f)\cap[\underline{l},\bar{l}] \\ +\infty & \text{otherwise} \end{cases} \qquad (x \in \mathbf{R}^V). \tag{17.10}$$

We call $f_{\underline{l}}^{\bar{l}}$ a *box reduction* of f. Note that any box $[\underline{l},\bar{l}]$ is a g-polymatroid.

Lemma 17.3: *Any box reduction $f_{\underline{l}}^{\bar{l}}$ of an M-convex (or M$^\natural$-convex) function f on $\mathbf{R}^V$ is also an M-convex (or M$^\natural$-convex) function on $\mathbf{R}^V$.*

(Proof) The present lemma follows from the fact that for any base polyhedron B (or g-polymatroid Q) $B\cap[\underline{l},\bar{l}]$ (or $Q\cap[\underline{l},\bar{l}]$), if nonempty, is again a base polyhedron (or a g-polymatroid). Q.E.D.

As shown in Chapter II, there are several operations on base polyhedra (or g-polymatroids) that are closed within the class of base polyhedra (or g-polymatroids). The above defined $f_{\underline{l}}^{\bar{l}}$ corresponds to the vector minor of a base polyhedron (or a g-polymatroid) by the vector reduction by $\bar{l}$ and the vector contraction by $\underline{l}$. Similar operations such as a truncation and its dual (an elongation) can be adapted to define the corresponding operations on M-convex (or M$^\natural$-convex) functions. (Examining such possible operations on M-convex (or M$^\natural$-convex) functions is left to readers.)

When $V=\{1,2\}$, we have the following (communicated by K. Murota).

Lemma 17.4: *A locally polyhedral convex function f on $\mathbf{R}^{\{1,2\}}$ is L$^\natural$-convex if and only if the function g on $\mathbf{R}^{\{1,2\}}$ defined by $g(x(1),x(2)) = f(-x(1),x(2))$ for $x \in \mathbf{R}^{\{1,2\}}$ is M$^\natural$-convex.*

(Proof) By the definition g is a locally polyhedral convex function. Note that f is an L$^\natural$-convex function if and only if each linearity domain of f is expressed by a system of inequalities of type

$$b_i^- \le x(i) \le b_i^+ \ (i=1,2), \quad -b_{21} \le x(1)-x(2) \le b_{12}, \tag{17.11}$$

where we allow the parameters to take values of $\pm\infty$. By the reflection of the $x(1)$-axis, i.e., by putting $x(1) \leftarrow -x(1)$, (17.11) becomes

$$a_i^- \le x(i) \le a_i^+ \ (i = 1, 2), \quad -a_{21} \le x(1) + x(2) \le a_{12}, \tag{17.12}$$

where a_i^-, a_i^+ $(i = 1, 2)$, a_{21} and a_{12} are appropriately defined in terms of the parameters in (17.11). In the two-dimensional space $\mathbf{R}^{\{1,2\}}$, if (17.12) gives a nonempty polyhedron, it is a g-polymatroid (see Fig. 16.1). (This can be seen from the fact that there is no crossing pair of X, Y in $2^{\{1,2,3\}}$ so that any set function on $2^{\{1,2,3\}}$ is a crossing-submodular function and defines a base polyhedron if nonempty (see Theorems 2.5 and 3.58).) Conversely, any g-polymatroid in $\mathbf{R}^{\{1,2\}}$ is expressed as (17.12) and by the reflection of the $x(1)$-axis we get an L$^\natural$-convex set expressed by (17.11). Q.E.D.

Note that for a submodular system $(\mathcal{D}, f)$ on V its vector rank function $\mathrm{r}_f : \mathbf{R}^V \to \mathbf{R}$ is M$^\natural$-concave and is submodular on $\mathbf{R}^V$ (see (6.9)). In general, we have

Theorem 17.5: *Any M$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies*

$$f(x) + f(y) \le f(x \vee y) + f(x \wedge y) \tag{17.13}$$

for any $x, y \in \mathbf{R}^V$ such that $x \vee y, x \wedge y \in \mathrm{dom} f$.

(Proof) Consider vectors $x, y \in \mathbf{R}^V$ such that $x \vee y, x \wedge y \in \mathrm{dom} f$. Note that since $\mathrm{dom} f$ is a g-polymatroid, we have $[x \wedge y, x \vee y] \subseteq \mathrm{dom} f$ and hence $x, y \in \mathrm{dom} f$. Put $S^+ = \mathrm{supp}^+(x - y)$, $S^- = \mathrm{supp}^-(x - y)$ and $S = S^+ \cup S^-$. Also put $d = x \vee y - x \wedge y$. Then define a function f_0 on 2^S by

$$f_0(X) = f(x \wedge y + \sum_{v \in X} d(v)\chi_v) \qquad (X \subseteq S), \tag{17.14}$$

where note that $x \wedge y + \sum_{v \in X} d(v)\chi_v$ is in $\mathrm{dom} f$. In order to show (17.13) it suffices to prove that f_0 is a supermodular set function on 2^S, or for any $X \subseteq S$ and distinct $u, v \in S - X$

$$f_0(X \cup \{u\}) + f_0(X \cup \{v\}) \le f_0(X \cup \{u, v\}) + f_0(X). \tag{17.15}$$

Therefore, we see from Lemma 17.3 that it suffices to prove (17.13) when $V = \{1, 2\}$. It follows from Lemmas 17.4 and 16.7 that we have inequalities of (17.13) in the two-dimensional space $\mathbf{R}^{\{1,2\}}$, where note that for any $x, y \in \mathbf{R}^{\{1,2\}}$ with $x(1) < y(1)$ and $x(2) > y(2)$ the pair of $x \wedge y$ and $x \vee y$ uniquely determines the pair of x and y. Q.E.D.

It is interesting to see that the supermodularity of M$^\natural$-convex functions comes from the submodularity of L$^\natural$-convex functions through the reflection in the two-dimensional space shown in Lemma 17.4.

It should be noted that although the inequalities of (17.13) show the supermodularity of f, the range of f is $\mathbf{R} \cup \{+\infty\}$ unlike the ordinary supermodular functions that take on values from $\mathbf{R} \cup \{-\infty\}$. Hence the effective domain dom$f$ is not closed with respect to join $\vee$ and meet $\wedge$ in general. When dom$f = \mathbf{R}^V$, M$^\natural$-convex function f is a supermodular function on $\mathbf{R}^V$ in the ordinary sense, like a vector rank function.

Let f be a separable convex function on $\mathbf{R}^V$ given as

$$f(x) = \sum_{v \in V} f_v(x(v)) \quad (x \in \mathbf{R}^V) \tag{17.16}$$

for locally polyhedral (piecewise linear) convex functions $f_v : \mathbf{R} \to \mathbf{R}$ $(v \in V)$. We can easily see that such a separable convex function f is an M$^\natural$-convex and, at the same time, L$^\natural$-convex function on $\mathbf{R}^V$ (note that any linearity domain of such a function f is a box $[a, b] \subseteq \mathbf{R}^V$ and that any such box is an M$^\natural$-convex and L$^\natural$-convex set). Hence from Theorems 16.8 and 17.5 f is a modular function on $\mathbf{R}^V$, i.e., $f(x) + f(y) = f(x \vee y) + f(x \wedge y)$ for $x, y \in \mathbf{R}^V$. We can also see that the converse holds, i.e., any M$^\natural$-convex and L$^\natural$-convex function is a separable convex function given as above ([Murota+Shioura01]).

18. Conjugacy between L-/L$^\natural$-convex Functions and M-/M$^\natural$-convex Functions

We show a one-to-one conjugacy correspondence between the set of integer-valued domain-integral L-convex functions (L$^\natural$-convex functions) and that of integer-valued domain-integral M-convex functions (M$^\natural$-convex functions).

Theorem 18.1: *For any L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ the convex conjugate function $f^\bullet$, if locally polyhedral, is an M$^\natural$-convex function. Also the convex conjugate function of any L-convex function is an M-convex function if it is locally polyhedral.*

(Proof) Let f be an L$^\natural$-convex function on $\mathbf{R}^V$ such that its convex conjugate function is locally polyhedral. Suppose that for a vector $p \in (\mathbf{R}^V)^*$

we have $\mathrm{argmin}(f-p) \neq \emptyset$. Because of the inequalities that express the L$^\natural$-convex set $\mathrm{argmin}(f-p)$ as in (16.2) (where we assume that each inequality in (16.2) is tight at a feasible point), the tangent cone of the epigraph $\mathrm{epi}f^\bullet$ at $(p, f^\bullet(p))$ is generated by vector $(\mathbf{0}, 1)$ and vectors of type

$$\begin{aligned}&(-\alpha\chi_w, f^\bullet(p-\alpha\chi_w) - f^\bullet(p)), \quad (\alpha\chi_w, f^\bullet(p+\alpha\chi_w) - f^\bullet(p)),\\ &(\alpha(\chi_u - \chi_v), f^\bullet(p+\alpha(\chi_u-\chi_v)) - f^\bullet(p))\end{aligned} \tag{18.1}$$

for a sufficiently small $\alpha > 0$ and $u, v, w \in V$ with $u \neq v$. It follows from (18.1) and Theorems 17.1 and 15.2 that each linearity domain of $f^\bullet$ is a g-polymatroid.

The latter part can be shown similarly. Q.E.D.

It should be noted that if f is a domain-integral L$^\natural$-convex function and is integer-valued on $(\mathrm{dom}f)\cap\mathbf{Z}^V$, then p appearing in the proof of Theorem 18.1 can be an integral vector.

Theorem 18.2: *For a domain-integral L$^\natural$-convex (L-convex) function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, *if f is integer-valued on* $(\mathrm{dom}f) \cap \mathbf{Z}^V$, *then the convex conjugate function $f^\bullet$ is a domain-integral M$^\natural$-convex (M-convex) function and is also integer-valued on* $(\mathrm{dom}f^\bullet) \cap (\mathbf{Z}^V)^*$.

(Proof) For any maximal linearity domain P of f there exists a vector $p \in (\mathbf{R}^V)^*$ such that $P = \mathrm{argmin}(f-p)$. We assume without loss of generality that $\mathrm{dom}f$ is full-dimensional. Then, from the domain-integrality of f and Theorem 16.13 (2), there exist an integral vector $z \in \mathrm{dom}f$ and a maximal chain $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = V$ such that $z + \chi_{S_i} \in \mathrm{dom}f$ and

$$\langle p, z+\chi_{S_i}\rangle + \beta = f(z+\chi_{S_i}) \qquad (i = 0, 1, \cdots, n) \tag{18.2}$$

with $\beta \in \mathbf{R}$. From (18.2) we have the system of equations

$$\langle p, \chi_{S_i}\rangle = f(z+\chi_{S_i}) - f(z) \qquad (i = 1, \cdots, n) \tag{18.3}$$

with p regarded as a variable vector, which has a unimodular (triangular) coefficient matrix and an integral right-hand side vector. Hence p is an integral vector. Since each maximal linearity domain of f corresponds, one to one, to an extreme point of $\mathrm{epi}f^\bullet$, it follows that $f^\bullet$ is domain-integral (note that $(p, f^\bullet(p))$ is an extreme point of $\mathrm{epi}f^\bullet$ and that a pointed g-polymatroid or base polyhedron is integral if and only if its vertices are integral). Moreover, since for p appearing above

$$-\beta = \langle p, z\rangle - f(z) = f^\bullet(p), \tag{18.4}$$

we see that $f^\bullet(p)$ is an integer. For each maximal linearity domain Q of $f^\bullet$ such that $p \in Q$ there uniquely exists a minimal face F of a (maximal) linearity domain of f such that for an integral vector $x \in F$ we have $\mathrm{argmin}(f^\bullet - x) = Q$. Note that such an integral vector x exists due to the assumption that f is domain-integral. Hence we have $f^\bullet(q) = \langle q-p, x\rangle + f^\bullet(p)$ for all $q \in Q$. It follows that $f^\bullet$ is a domain-integral M$^\natural$-convex function and is integer-valued on $(\mathrm{dom} f^\bullet) \cap (\mathbf{Z}^V)^*$. Q.E.D.

Conversely, we have

Theorem 18.3: *The convex conjugate function of any M$^\natural$-convex (or M-convex) function is an L$^\natural$-convex (or L-convex) function if it is locally polyhedral.*

(Proof) For an M$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ any linearity domain of f is a g-polymatroid. It follows from Theorem 17.1 that the tangent cone of the epigraph $\mathrm{epi} f$ at $(p, f(p))$ is generated by vectors of type given by (18.1) with $f^\bullet$ being replaced by f. Hence any linearity domain P of the conjugate function $f^\bullet$ is expressed as (16.2), i.e., P is an L$^\natural$-convex set. The statement for M and L can be shown to hold similarly. Q.E.D.

Since the convex conjugate function of a polyhedral convex function is polyhedral, it follows from Theorem 18.3 (or Theorem 18.1) that the conjugacy relation gives a bijection between the set of polyhedral L$^\natural$-convex functions on $\mathbf{R}^V$ and that of polyhedral M$^\natural$-convex functions on $(\mathbf{R}^V)^*$.

For an M-convex function f whose effective domain lies on a hyperplane $x(V) = r$ for an $r \in \mathbf{R}$ the convex conjugate function $f^\bullet$ satisfies

$$f^\bullet(x + \alpha\mathbf{1}) = f^\bullet(x) + \alpha r \tag{18.5}$$

for any $x \in \mathrm{dom} f$ and $\alpha \in \mathbf{R}$ (see (16.11)).

Theorem 18.4: *For a domain-integral M$^\natural$-convex (M-convex) function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, if f is integer-valued on $\mathrm{dom} f \cap \mathbf{Z}^V$, then the convex conjugate function $f^\bullet$ is a domain-integral L$^\natural$-convex (L-convex) function and is also integer-valued on $\mathrm{dom} f^\bullet \cap (\mathbf{Z}^V)^*$.*

(Proof) Let f be a domain-integral M$^\natural$-convex function. Without loss of generality we assume that $\mathrm{dom} f$ is full-dimensional. Let Q be any maximal linearity domain of f such that $Q = \mathrm{argmin}(f - p)$ for some $p \in (\mathbf{R}^V)^*$. For any integral point x of Q there exist $W^+ \subseteq V$, $W^- \subseteq V$ and $A \subseteq V \times V$

such that vectors χ_v ($v \in W^+$), $-\chi_v$ ($v \in W^-$) and $\chi_u - \chi_v$ ($(u,v) \in A$) generate the tangent cone of Q at x. Hence we have

$$\begin{aligned} \langle p, \chi_v \rangle &= f(x+\chi_v) - f(x) \quad (v \in W^+), \\ \langle p, -\chi_v \rangle &= f(x-\chi_v) - f(x) \quad (v \in W^-), \\ \langle p, \chi_u - \chi_v \rangle &= f(x+\chi_u-\chi_v) - f(x) \quad ((u,v) \in A). \end{aligned} \tag{18.6}$$

With p regarded as a variable vector the system of equations (18.6) has a totally unimodular coefficient matrix of rank $|V|$ (since Q is full-dimensional) and an integral right-hand side vector. Hence p is an integral vector. Similarly as we proved Theorem 18.2, we can show that $f^\bullet$ is a domain-integral L$^\natural$-convex function and is also integer-valued on $\mathrm{dom} f^\bullet \cap (\mathbf{Z}^V)^*$. Q.E.D.

From Theorems 18.2 and 18.4 we have

Corollary 18.5: *The conjugacy relation gives a bijection between the set of domain-integral and co-domain-integral L$^\natural$-convex functions on $\mathbf{R}^V$ and that of domain-integral and co-domain-integral M$^\natural$-convex functions on $(\mathbf{R}^V)^*$, and similarly for the set of domain-integral and co-domain-integral L-convex functions on $\mathbf{R}^V$ and that of domain-integral and co-domain-integral M-convex functions on $(\mathbf{R}^V)^*$.*

Here note that a domain-integral L$^\natural$-convex function f is co-domain-integral if and only if it is integer-valued on $\mathrm{dom} f \cap \mathbf{Z}^V$ and that a domain-integral M$^\natural$-convex function g is co-domain-integral if and only if it is integer-valued on $\mathrm{dom}\, g \cap \mathbf{Z}^V$.

19. The Discrete Fenchel-Duality Theorem

Concerning the discrete structure of L-/L$^\natural$-convex functions and M-/M$^\natural$-convex functions relevant to the discrete Fenchel-duality theorem, we have the following two lemmas.

Lemma 19.1: *For two L$^\natural$-convex functions f_1 and f_2 on $\mathbf{R}^V$ with $\mathrm{dom} f_1 \cap \mathrm{dom} f_2 \neq \emptyset$, the sum $f_1 + f_2$ is also an L$^\natural$-convex function. Moreover, if both f_1 and f_2 are domain-integral, then $f_1 + f_2$ is also domain-integral.*

(Proof) The sum $f_1 + f_2$ is a locally polyhedral convex function and each linearity domain of $f_1 + f_2$ is given as the intersection of a linearity domain of f_1 and that of f_2. We can easily see that a nonempty intersection of two

L$^\natural$-convex sets is again an L$^\natural$-convex set, due to the definition in (16.2). Similarly we can show the domain-integrality of $f_1 + f_2$ for domain-integral f_1 and f_2, due to the equivalence of (1) and (2) in Theorem 16.13. Q.E.D.

Lemma 19.2: *For two domain-integral M$^\natural$-convex functions g_1 and g_2 on $\mathbf{R}^V$ with $\mathrm{dom} g_1 \cap \mathrm{dom} g_2 \neq \emptyset$, the sum $g_1 + g_2$ is domain-integral. Any linearity domain of $g_1 + g_2$ is the intersection of two integral g-polymatroids.*

(Proof) The sum $g_1 + g_2$ is a locally polyhedral convex function and each linearity domain of $g_1 + g_2$ is the intersection of a linearity domain of g_1 and that of g_2. Since linearity domains of an M$^\natural$-convex function are g-polymatroids and the nonempty intersection of two integral g-polymatroids is an integral polyhedron (see Theorems 3.58 and 5.6), the present lemma follows. Q.E.D.

As shown in Lemma 19.2, the sum of two domain-integral M$^\natural$-convex functions is domain-integral, but the sum of two M$^\natural$-convex functions is not necessarily M$^\natural$-convex. From Theorem 15.3 we have

Theorem 19.3: *Let g_i $(i = 1, 2)$ be domain-integral M$^\natural$-convex functions on $\mathbf{R}^V$. If $g_1 \circ g_2(x) > -\infty$ for all $x \in \mathbf{Z}^V$, then the convolution $g_1 \circ g_2$ is again a domain-integral M$^\natural$-convex function. Moreover, we have*

$$g_1 \circ g_2(x) = \inf\{g_1(y) + g_2(x - y) \mid y \in \mathbf{Z}^V\} \quad (x \in \mathbf{Z}^V). \tag{19.1}$$

(Proof) Any linearity domain of the convolution $g_1 \circ g_2$ is the Minkowski sum of a linearity domain of g_1 and that of g_2, and under the assumption of the present theorem, linearity domains of g_1 and g_2 are integral g-polymatroids. Hence any linearity domain of $g_1 \circ g_2$ is an integral g-polymatroid, due to Theorem 3.58 and (3.33) with the underlying totally ordered group being the set of integers $\mathbf{Z}$. This implies that $g_1 \circ g_2$ is a domain-integral M$^\natural$-convex function, and that we can take the infimum over $\mathbf{Z}^V$ (instead of $\mathbf{R}^V$) in the convolution formula for $g_1 \circ g_2$ to get (19.1). Q.E.D.

It should be noted that the first half of Theorem 19.3 holds with the term, domain-integral, being suppressed and $\mathbf{Z}^V$ being replaced by $\mathbf{R}^V$.

Now, from Lemmas 19.1 and 19.2 and Theorem 15.1 we have the following.

Theorem 19.4 (The discrete Fenchel-duality theorem): *For a domain-integral $L^\natural$-convex function f on $\mathbf{R}^V$ and a domain-integral $L^\natural$-concave function g on $\mathbf{R}^V$ such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ *we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in (\mathbf{R}^V)^*\}. \tag{19.2}$$

Moreover, if f and g are also integer-valued on $\mathbf{Z}^V$, then

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in (\mathbf{Z}^V)^*\}. \tag{19.3}$$

(Proof) The first half of the present theorem follows from Lemma 19.1 and Theorem 15.1. The second half is due to Theorem 18.2 and Lemma 19.2. Q.E.D.

Furthermore, because of Theorem 18.4 and Lemma 19.2 the conjugate form of Theorem 19.4 is given as

Theorem 19.5: *For a domain-integral $M^\natural$-convex function f on $\mathbf{R}^V$ and a domain-integral $M^\natural$-concave function g on $\mathbf{R}^V$ such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ *we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in (\mathbf{R}^V)^*\}. \tag{19.4}$$

Moreover, if f and g are also integer-valued on $\mathbf{Z}^V$, then

$$\inf\{f(x) - g(x) \mid x \in \mathbf{Z}^V\} = \sup\{g^\circ(p) - f^\bullet(p) \mid p \in (\mathbf{Z}^V)^*\}. \tag{19.5}$$

We also have the following separation theorem for discrete convex functions.

Theorem 19.6: (1) *For an $L^\natural$-convex function f and an $L^\natural$-concave function g on $\mathbf{R}^V$ that are domain-integral and satisfy* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ *and* $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, *there exist $w \in (\mathbf{R}^V)^*$ and $\beta \in \mathbf{R}$ such that*

$$\forall x \in \mathbf{R}^V : \; f(x) \geq \langle w, x\rangle + \beta \geq g(x). \tag{19.6}$$

Moreover, if f and g are integer-valued on $\mathbf{Z}^V$, there exist integral such w and β.

(2) *For an $M^\natural$-convex function f and an $M^\natural$-concave function g on $\mathbf{R}^V$ that are domain-integral and satisfy* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ *and* $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, *there exist $w \in (\mathbf{R}^V)^*$ and $\beta \in \mathbf{R}$ such that*

$$\forall x \in \mathbf{R}^V : \; f(x) \geq \langle w, x\rangle + \beta \geq g(x). \tag{19.7}$$

Moreover, if f and g are integer-valued on $\mathbf{Z}^V$, there exist integral such w and β.

(Proof) (1) For a domain-integral L$^\natural$-convex function f and a domain-integral L$^\natural$-concave function g with $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$, we have $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$ if and only if $f(x) \geq g(x)$ $(x \in \mathbf{R}^V)$, since $f - g$ is a domain-integral L$^\natural$-convex function due to Lemma 19.1. Hence the existence of $w \in (\mathbf{R}^V)^*$ and $\beta \in \mathbf{R}$ satisfying (19.6) follows from a separation theorem in ordinary convex analysis [Rockafellar70]. The integrality property in the latter half can be shown from the integrality property of the intersection of two g-polymatroids as follows. Since f and g are domain-integral and $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, there exists an integral minimizer $\hat{x}$ of $f - g$. Since integer-valued domain-integral L$^\natural$-convex functions are co-domain-integral from Theorem 18.2, the subdifferential $\partial f(\hat{x})$ and the superdifferential $\partial g(\hat{x})$ are integral g-polymatroids, and since they have a nonempty intersection by the ordinary separation theorem for $f(x)$ and $g(x) + f(\hat{x}) - g(\hat{x})$, there exists an integral $w \in \partial f(\hat{x}) \cap \partial g(\hat{x})$. For such a w and any integer β such that $g(\hat{x}) - \langle w, \hat{x}\rangle \leq \beta \leq f(\hat{x}) - \langle w, \hat{x}\rangle$ we have (19.6).

(2) This can be shown similarly as (1). Note that for a domain-integral M$^\natural$-convex function f and a domain-integral M$^\natural$-concave function g with $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$, we have $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$ if and only if $f(x) \geq g(x)$ $(x \in \mathbf{R}^V)$, due to Lemma 19.2. Also, we can see the integrality part from the following facts: (a) integer-valued domain-integral M$^\natural$-convex functions are co-domain-integral, (b) subdifferentials of a co-domain-integral M$^\natural$-convex function are integral L$^\natural$-convex sets, and (c) the nonempty intersection of two integral L$^\natural$-convex sets is again an integral L$^\natural$-convex set, where (a) and (b) follows from Theorem 18.4 and the conjugacy relation, and (c) from Lemma 19.1. Q.E.D.

See [Murota03a, Chapter 8] for more details about the discrete Fenchel duality.

20. Algorithmic and Structural Properties of Discrete Convex Functions

We consider algorithmic and structural properties of L-/L$^\natural$-convex functions and M-/M$^\natural$-convex functions.

20.1. L- and L$^\natural$-convex Functions

Minimizers of a domain-integral L$^\natural$-convex function are characterized by the following.

Theorem 20.1: *For a domain-integral L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, a vector $x \in \mathbf{Z}^V$ is a minimizer of f if and only if x is a minimizer of the restriction of f to $[x - \mathbf{1}, x] \cup [x, x + \mathbf{1}]$.*

(Proof) Let $n = |V|$. Any n-dimensional Freudentahl's simplex cell that includes $x \in \mathbf{Z}^V$ is given by $n+1$ integral points $x - \chi_W$, $x - \chi_W + \chi_{\{v_1,\cdots,v_k\}}$ $(k = 1, 2, \cdots, n)$ for some linear ordering $(v_1, v_2, \cdots, v_n)$ of V with $W = \{v_1, v_2, \cdots, v_l\}$ for some l $(0 \le l \le n)$. We can easily see that these points belong to $[x - \mathbf{1}, x] \cup [x, x + \mathbf{1}]$. Hence the present theorem holds. Q.E.D.

The following property for L-/L$^\natural$-convex functions is fundamental and useful for introducing scaling techniques. For any positive integer k we say that f is an *L$^\natural$-convex function on* $(k\mathbf{Z})^V$ if $f_k : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by $f_k(z) = f(kz)$ for all $z \in \mathbf{Z}^V$ is an L$^\natural$-convex function on $\mathbf{Z}^V$.

Theorem 20.2: *An L-convex (L$^\natural$-convex) function f on $\mathbf{Z}^V$ is also an L-convex (L$^\natural$-convex) function on $(k\mathbf{Z})^V$ for any positive integer k.*

(Proof) Because of Lemma 16.6 it suffices to prove the present theorem for any L-convex function f on $\mathbf{Z}^V$. Since f is a submodular function on $(k\mathbf{Z})^V$ and satisfies $f(x + \alpha k\mathbf{1}) = f(x) + \alpha kr$ $(x \in (k\mathbf{Z})^V)$, it follows from Theorem 16.9 that f is an L-convex function on $(k\mathbf{Z})^V$. Q.E.D.

Iwata [Iwata99] pointed out that a polynomial algorithm for minimizing L-convex functions could be obtained by combining Theorem 20.2 and the proximity theorem shown in [Iwata+Shigeno02] (see Theorem 20.9 given below) by the use of any polynomial algorithm for submodular (set) function minimization. See [Murota03b] for a faster algorithm for L-convex function minimization.

20.2. M- and M$^\natural$-convex Functions

As a generalization of Theorem 3.16 we have the following characterization of minimizers of M- and M$^\natural$-convex functions.

Theorem 20.3: *For an M-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ a vector $x \in \mathrm{dom} f$ is a minimizer of f if and only if for each $u, v \in V$ and each $\alpha > 0$ we have*

$$f(x + \alpha(\chi_u - \chi_v)) \ge f(x). \tag{20.1}$$

Furthermore, if f is domain-integral, then the above statement holds with 'each $\alpha > 0$' being replaced by '$\alpha = 1$.'

(Proof) The present theorem follows from Theorem 3.16 and Remark given before Theorem 17.2. Q.E.D.

As a projected version of this theorem we have

Corollary 20.4: *For an M$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ a vector $x \in \mathrm{dom} f$ is a minimizer of f if and only if for each $u, v \in V \cup \{0\}$ and each $\alpha > 0$ we have*

$$f(x + \alpha(\chi_u - \chi_v)) \geq f(x), \tag{20.2}$$

where 0 is a new element not in V and χ_0 is defined to be the zero vector $\mathbf{0}$ in $\mathbf{R}^V$.

Furthermore, if f is domain-integral and x is an integral vector, then the above statement holds with 'each $\alpha > 0$' being replaced by '$\alpha = 1$.'

As shown above, we have a pair of equivalent statements, one for M-convex functions and the other for M$^\natural$-convex functions. In the sequel we sometimes give either of the two such statements and omit the other.

Let f be an M-convex function. For any $x \in B_f \equiv \mathrm{dom} f$ and $u, v \in V$ with $v \in \mathrm{dep}_{B_f}(x, u) - \{u\}$ (where dep_{B_f} denotes the dependence function defined for the base polyhedron B_f), we define the *directional derivative* $f'(x; u, v)$ by

$$f'(x; u, v) = \lim_{\alpha \to +0} \{f(x + \alpha(\chi_u - \chi_v)) - f(x)\}/\alpha. \tag{20.3}$$

Note that because of the locally polyhedral convexity of f we have for a sufficiently small $\alpha > 0$

$$f'(x; u, v) = \{f(x + \alpha(\chi_u - \chi_v)) - f(x)\}/\alpha. \tag{20.4}$$

Moreover, if f is domain-integral and x is an integral vector, then α appearing in (20.4) can be chosen as $\alpha = 1$. When $v \notin \mathrm{dep}_{B_f}(x, u) - \{u\}$ for $x \in B_f$, we define $f'(x; u, v) = +\infty$.

M-convex functions have the following exchange property. The exchange property is adopted as the defining axiom for M-convex functions in Murota's discrete convex analysis [Murota03a].

Theorem 20.5: *Let f be any M-convex function on $\mathbf{R}^V$. For any $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x-y)$ there exist $v \in \mathrm{supp}^-(x-y)$ and $\alpha > 0$ such that*

$$f(x) + f(y) \geq f(x + \alpha(-\chi_u + \chi_v)) + f(y + \alpha(\chi_u - \chi_v)). \tag{20.5}$$

If f is domain-integral and x and y are integral vectors, α can be chosen as $\alpha = 1$.

(Proof) Suppose to the contrary that for some $x, y \in \mathrm{dom} f$ and some $u^* \in \mathrm{supp}^+(x-y)$ we have

$$\begin{aligned}&\forall v \in \mathrm{supp}^-(x-y),\ \forall \alpha > 0:\\ &\qquad f(x) + f(y) < f(x + \alpha(-\chi_{u^*} + \chi_v)) + f(y + \alpha(\chi_{u^*} - \chi_v)).\end{aligned} \tag{20.6}$$

By considering the box reduction of f by box $[x \wedge y, x \vee y]$ if necessary, we assume that f is a polyhedral convex function so that $f^\bullet$ is L-convex. We can easily see that (20.6) is equivalent to

$$\forall v \in \mathrm{supp}^-(x-y):\ 0 < f'(x; v, u^*) + f'(y; u^*, v). \tag{20.7}$$

This implies that for each $v \in \mathrm{supp}^-(x-y)$ there exist

$$p_v \in \partial f(x), \qquad q_v \in \partial f(y) \tag{20.8}$$

such that

$$0 < (p_v(v) - p_v(u^*)) + (q_v(u^*) - q_v(v)). \tag{20.9}$$

Since $\partial f(x)$ and $\partial f(y)$ are L-convex sets, we can further assume that

$$p_v(u^*) = q_v(u^*) = 0. \tag{20.10}$$

Define

$$p^* = \bigvee\{p_v \mid v \in \mathrm{supp}^-(x-y)\}, \qquad q^* = \bigwedge\{q_v \mid v \in \mathrm{supp}^-(x-y)\}. \tag{20.11}$$

Then for the same reason we have

$$p^* \in \partial f(x), \qquad q^* \in \partial f(y), \tag{20.12}$$

$$0 < p^*(v) - q^*(v) \quad (v \in \mathrm{supp}^-(x-y)), \tag{20.13}$$

$$p^*(u^*) = q^*(u^*) = 0. \tag{20.14}$$

Putting

$$\epsilon = \min\{p^*(v) - q^*(v) \mid v \in \mathrm{supp}^-(x-y)\}, \tag{20.15}$$

$$p' = (p^* - \epsilon \mathbf{1}) \vee q^*, \qquad q' = p^* \wedge (q^* + \epsilon \mathbf{1}), \tag{20.16}$$

we have $\epsilon > 0$ and hence from Theorems 16.8

$$\begin{aligned}
&(f^\bullet - x)(p^*) + (f^\bullet - y)(q^*) - (f^\bullet - x)(p') - (f^\bullet - y)(q') \\
&= f^\bullet(p^*) + f^\bullet(q^*) - f^\bullet(p') - f^\bullet(q') + \langle p' - p^*, x\rangle + \langle q' - q^*, y\rangle \\
&\geq \langle p' - p^*, x\rangle + \langle q' - q^*, y\rangle \\
&= \epsilon \sum\{y(v) - x(v) \mid v \in \mathrm{supp}^+(p^* - q^*)\} \\
&\qquad + \sum\{(p^*(v) - q^*(v))(y(v) - x(v)) \mid v \in V - \mathrm{supp}^+(p^* - q^*)\} \\
&\geq \epsilon \sum\{y(v) - x(v) \mid v \in \mathrm{supp}^+(p^* - q^*)\} \\
&\geq \epsilon(x(u^*) - y(u^*)) \\
&> 0,
\end{aligned} \tag{20.17}$$

where note that $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p^* - q^*)$, $x(V) = y(V)$ and $x(u^*) > y(u^*)$. Because of Theorem 15.2 this contradicts (20.12), i.e., $(f^\bullet - x)(p^*) \leq (f^\bullet - x)(p')$ and $(f^\bullet - y)(q^*) \leq (f^\bullet - y)(q')$.

Moreover, the integrality follows from the domain-integrality of f. Q.E.D.

We can easily see that conversely, for a locally polyhedral convex function f (20.5) implies that any linearity domain of f is a base polyhedron and hence f is an M-convex function.

Theorem 20.5 is a generalization of a simultaneous exchange property for base polyhedra (see Corollary 20.6 given below), which supplements the arguments in Chapter II. We also give a direct proof of it here.

For a submodular system $(\mathcal{D}, f)$ on V with the associated base polyhedron $\mathrm{B}(f)$, recall that the dependence function dep is defined by

$$\mathrm{dep}(x, u) = \{v \mid v \in V,\ \exists \alpha > 0 : x + \alpha(\chi_u - \chi_v) \in \mathrm{B}(f)\} \tag{20.18}$$

for any base $x \in \mathrm{B}(f)$ and any $u \in V$. Also the dual dependence function $\mathrm{dep}^\#$ is defined by

$$\mathrm{dep}^\#(x, u) = \{v \mid v \in V,\ \exists \alpha > 0 : x + \alpha(-\chi_u + \chi_v) \in \mathrm{B}(f)\} \tag{20.19}$$

(see (8.37)).

Corollary 20.6: *Let $(\mathcal{D}, f)$ be a submodular system on V. For any bases $x, y \in \mathrm{B}(f)$ and any $u \in \mathrm{supp}^+(x-y)$ we have*

$$\mathrm{dep}^{\#}(x,u) \cap \mathrm{dep}(y,u) \cap \mathrm{supp}^-(x-y) \neq \emptyset. \tag{20.20}$$

(Proof) Suppose to the contrary that

$$\mathrm{dep}^{\#}(x,u) \cap \mathrm{dep}(y,u) \cap \mathrm{supp}^-(x-y) = \emptyset. \tag{20.21}$$

Then, putting $X = \mathrm{dep}^{\#}(x,u)$ and $Y = \mathrm{dep}(y,u)$, define $\underline{l}, \bar{l} \in \mathbf{R}^V$ by

$$\underline{l}(w) = \begin{cases} y(w) & (w \in Y) \\ x \wedge y(w) & (w \in V - Y), \end{cases} \tag{20.22}$$

$$\bar{l}(w) = \begin{cases} x(w) & (w \in X) \\ x \vee y(w) & (w \in V - X), \end{cases} \tag{20.23}$$

From (20.21)~(20.23) we have $\underline{l} \leq \bar{l}$ and from (20.22) and (20.23) we have

$$y \in \mathrm{B}(f)_{\underline{l}}, \qquad x \in \mathrm{B}(f)^{\bar{l}}. \tag{20.24}$$

(See Section 3.1.b for the definitions of $\mathrm{B}(f)_{\underline{l}}$ and $\mathrm{B}(f)^{\bar{l}}$.) Hence from Theorem 3.8 there exists a base $z \in \mathrm{B}(f)_{\underline{l}}^{\bar{l}}$. Since

$$x(X) = f^{\#}(X) = f(V) - f(V - X), \qquad y(Y) = f(Y), \tag{20.25}$$

it follows from (20.22) and (20.23) that $z(w) = x(w)$ $(w \in X)$ and $z(w) = y(w)$ $(w \in Y)$. This is a contradiction because we have $u \in X \cap Y$ and $x(u) > y(u)$. Q.E.D.

It should be noted that Corollary 20.6 is a generalization of a basic fact about matroids that for any circuit C and any cocircuit K of a matroid we have $|C \cap K| \neq 1$. Also note that Corollary 20.6 holds for any totally ordered additive group $\mathbf{R}$ though we assume in the present chapter that $\mathbf{R}$ is the set of reals.

The following theorem is also fundamental.

Theorem 20.7: *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be an M♮-convex function on $\mathbf{R}^V$. Suppose that for a vector $p \in (\mathbf{R}^V)^*$ we have a vector $x \in \mathrm{argmin}(f + p)$. For any $q \geq p$ such that $\mathrm{argmin}(f + q) \neq \emptyset$ there exists a vector $y \in \mathrm{argmin}(f+q)$ such that for any $v \in V$, $p(v) = q(v)$ implies $y(v) \geq x(v)$.*

Moreover, if f is domain-integral, then x and y appearing above can be chosen to be integral vectors.

The property of M$^\natural$-convex functions shown in Theorem 20.7 is called the *gross substitutes condition* in economic theory. We show this theorem by using the following lemma, essentially due to Shioura [Shioura98]. Here 0 is a new element not in V and χ_0 is equal to $\mathbf{0}$ in $\mathbf{R}^V$.

Lemma 20.8: *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function on $\mathbf{R}^V$ with* $\mathrm{argmin} f \neq \emptyset$. *For a vector $x \in \mathrm{dom} f$ and an element $u \in V$, if for each $v \in V \cup \{0\}$ and each $\alpha > 0$ we have*

$$f(x + \alpha(\chi_u - \chi_v)) \geq f(x), \tag{20.26}$$

then there exists a vector $y \in \mathrm{argmin} f$ such that $y(u) \leq x(u)$. Also, if for each $v \in V \cup \{0\}$ and each $\alpha > 0$ we have

$$f(x + \alpha(-\chi_u + \chi_v)) \geq f(x), \tag{20.27}$$

then there exists a vector $y \in \mathrm{argmin} f$ such that $y(u) \geq x(u)$.

Moreover, if f is domain-integral, then it suffices to require (20.26) *and* (20.27) *for $\alpha = 1$, and y can be chosen to be integral.*

(Proof) Suppose that (20.26) holds. If there is no $y \in \mathrm{argmin} f$ such that $y(u) \leq x(u)$, then let y^* be a minimizer of f that attains the minimum of $y(u) - x(u)$ over $\mathrm{argmin} f$. Note that $y^*(u) - x(u) > 0$ since $\mathrm{argmin} f$ is a g-polymatroid. Hence, from Theorem 20.5 there exists an element $v \in \mathrm{supp}^-(y^* - x) \cup \{0\}$ such that

$$f(y^*) + f(x) \geq f(y^* + \beta(-\chi_u + \chi_v)) + f(x + \beta(\chi_u - \chi_v)) \tag{20.28}$$

for some $\beta > 0$. From (20.26) we have $f(x) \leq f(x + \beta(\chi_u - \chi_v))$. Hence, we see from (20.28) that $f(y^*) \geq f(y^* + \beta(-\chi_u + \chi_v))$, i.e., $y^* + \beta(-\chi_u + \chi_v)$ is a minimizer of f. This contradicts the choice of y^*.

We can show the second statement with (20.27) in a similar way.

Moreover, the latter integrality property follows from the integrality in Theorem 20.5 and that of g-polymatroids. For an integral x we can choose an integral y^* and $\beta = 1$ in (20.28). Q.E.D.

(Proof of Theorem 20.7) Put $W = \{w \mid w \in V,\ p(w) = q(w)\}$. For any $w \in W$ and any $\alpha > 0$ we have

$$\begin{aligned}&(f + q)(x + \alpha(-\chi_w + \chi_v)) \\ &= (f + p)(x + \alpha(-\chi_w + \chi_v)) + (q - p)(x + \alpha(-\chi_w + \chi_v))\end{aligned}$$

$$\begin{aligned}
&\geq (f+p)(x) + (q-p)(x+\alpha(-\chi_w+\chi_v)) \\
&= (f+q)(x) + (q-p)(\alpha(-\chi_w+\chi_v)) \\
&\geq (f+q)(x) \qquad (20.29)
\end{aligned}$$

for all $v \in V \cup \{0\}$. Hence from (20.29) and Lemma 20.8 there exists a vector $y \in \mathrm{argmin}(f+q)$ such that $y(w) \geq x(w)$. Then, define $f' : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by $f'(z) = f(z)$ for $z \in \mathbf{R}^V$ with $z(w) \geq x(w)$ and $f'(z) = +\infty$ for $z \in \mathbf{R}^V$ with $z(w) < x(w)$. (Note that f' is a box reduction of f.) Then put $f \leftarrow f'$ and $W \leftarrow W - \{w\}$. The new f satisfies $x \in \mathrm{argmin}(f+p)$ and $\mathrm{argmin}(f+q) \neq \emptyset$. Hence for a new $w \in W$ we have (20.29) and see that there exists a vector $y \in \mathrm{argmin}(f+q)$ such that $y(w) \geq x(w)$. We can repeat this process until W becomes empty. The present theorem then follows. Q.E.D.

Remark: The gross substitutes condition captures the essential structural property of M$^\natural$-convexity. Fujishige and Yang [Fuji+Yang03] showed the equivalence between the gross substitutes condition and the M$^\natural$-convexity property for M$^\natural$-convex functions on the unit hypercube by using a result by Gul and Stacchetti [Gul+Stacchetti99]. The equivalence between the gross substitutes condition and the M-convexity property for general M-convex functions on $\mathbf{Z}^V$ was later shown by [Danilov+Koshevoy+Lang03] and [Murota+Tamura03a]. The integrality property plays an important rôle in economies with indivisible commodities.

20.3. Proximity Theorems

Proximity theorems are concerned with solutions of relaxed or restricted problems modified from an original one and show how close to an optimal solution of the original problem the approximate solutions are.

The following is a proximity theorem for L-convex functions, which is due to [Iwata+Shigeno02], and plays an important rôle in algorithms related to L-convex functions (also see [Murota03b]).

Theorem 20.9 (L-proximity theorem): *Let f be a domain-integral L-convex function on $\mathbf{R}^V$ such that $f(x) = f(x+\lambda\mathbf{1})$ for all $x \in \mathbf{R}^V$ and $\lambda \in \mathbf{R}$. Suppose that for an integral vector $y \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(y) \leq f(y+\alpha\chi_X) \qquad (X \subseteq V). \qquad (20.30)$$

Then, there exists $x^* \in \mathrm{argmin} f$ *such that*

$$y \leq x^* \leq y + (n-1)\lceil \alpha - 1 \rceil \mathbf{1}, \tag{20.31}$$

where $n = |V|$.
(Proof) By considering, if necessary, the box reduction $f_{\underline{l}}^{\bar{l}}$ with $\bar{l} = y + n\lceil \alpha \rceil \mathbf{1}$ and $\underline{l} = y - \mathbf{1}$, we can assume that $\mathrm{dom} f$ is bounded and hence $\mathrm{argmin} f \neq \emptyset$.

Let x^* be an integral minimizer of f that is minimal (with respect to $\leq$ in $\mathbf{R}^V$) with the property that $y \leq x^*$. Note that such an integral minimizer of f exists since f is domain-integral. Suppose $y \neq x^*$. There uniquely exist a chain $X_1 \subset X_2 \subset \cdots \subset X_k$ of 2^V and positive integers β_i $(i = 1, 2, \cdots, k)$ such that

$$x^* = y + \sum_{i=1}^{k} \beta_i \chi_{X_i}. \tag{20.32}$$

For each $j = 1, 2, \cdots, k$ define

$$z_j = y + \sum_{i=1}^{j} \beta_i \chi_{X_i} \tag{20.33}$$

and $z_0 = y$. Then by the L-convexity and submodularity of f and by the assumption we have for each $j = 1, 2, \cdots, k$

$$\begin{aligned} f(x^*) + f(z_{j-1}) &= f(x^*) + f(z_{j-1} + (\sum_{i=j+1}^{k} \beta_i)\mathbf{1}) \\ &\geq f(z_j + (\sum_{i=j+1}^{k} \beta_i)\mathbf{1}) + f(x^* - \beta_j \chi_{X_j}) \\ &= f(z_j) + f(x^* - \beta_j \chi_{X_j}). \end{aligned} \tag{20.34}$$

Since $f(x^*) < f(x^* - \beta_j \chi_{X_j})$ due to the choice of x^*, it follows from (20.34) that

$$f(z_{j-1}) > f(z_j) \qquad (j = 1, 2, \cdots, k). \tag{20.35}$$

Similarly we have for each $j = 1, 2, \cdots, k$

$$\begin{aligned} f(z_j) + f(y) &= f(z_j) + f(y + \beta_j \mathbf{1}) \\ &\geq f(z_{j-1} + \beta_j \mathbf{1}) + f(y + \beta_j \chi_{X_j}) \\ &= f(z_{j-1}) + f(y + \beta_j \chi_{X_j}). \end{aligned} \tag{20.36}$$

Hence, from (20.35) and (20.36) we have

$$f(y) > f(y + \beta_j \chi_{X_j}) \qquad (j = 1, 2, \cdots, k). \tag{20.37}$$

It follows from (20.30), (20.37) and the convexity of f that

$$\beta_j < \alpha \qquad (j = 1, 2, \cdots, k). \tag{20.38}$$

Consequently,

$$x^* = y + \sum_{i=1}^{k} \beta_i \chi_{X_i} \leq y + (n-1)\lceil \alpha - 1 \rceil \mathbf{1}, \tag{20.39}$$

where note that $k < n$ (or $\exists v \in V : x^*(v) = y(v)$) because of the minimality of x^* with $y \leq x^*$, and β_j $(j = 1, 2, \cdots, k)$ are integers since x^* and y are integral vectors. Q.E.D.

The L$^\natural$ version of Theorem 20.9 is given as follows.

Theorem 20.10 (L$^\natural$-proximity theorem): *Let f be a domain-integral L$^\natural$-convex function on $\mathbf{R}^V$. Suppose that for an integral $y \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(y) \leq f(y \pm \alpha \chi_X) \qquad (X \subseteq V). \tag{20.40}$$

Then, there exists $x^* \in \mathrm{argmin} f$ such that

$$y - (n-1)\lceil \alpha - 1 \rceil \mathbf{1} \leq x^* \leq y + (n-1)\lceil \alpha - 1 \rceil \mathbf{1}. \tag{20.41}$$

Without the domain-integrality of f we have the following. We omit its L$^\natural$ version.

Theorem 20.11: *Let f be an L-convex function on $\mathbf{R}^V$ such that $f(x) = f(x + \lambda \mathbf{1})$ for all $x \in \mathbf{R}^V$ and $\lambda \in \mathbf{R}$. Suppose that for a vector $y \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(y) \leq f(y + \alpha \chi_X) \qquad (X \subseteq V). \tag{20.42}$$

Then, there exists $x^* \in \mathrm{argmin} f$ such that

$$y \leq x^* \leq y + (n-1)\alpha \mathbf{1}. \tag{20.43}$$

(Proof) The proof of Theorem 20.9 can easily be adapted to the present theorem without domain-integrality. Q.E.D.

We also have a proximity theorem for M-convex functions as follows ([Moriguchi+ Murota+Shioura02]).

Theorem 20.12 (M-proximity theorem): *Let f be a domain-integral M-convex function on $\mathbf{R}^V$. Suppose that for an integral vector $z \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(z) \leq f(z + \alpha(\chi_u - \chi_v)) \qquad (u, v \in V). \tag{20.44}$$

Then, there exists an integral minimizer x^* of f such that

$$z - (n-1)\lceil \alpha - 1 \rceil \mathbf{1} \leq x^* \leq z + (n-1)\lceil \alpha - 1 \rceil \mathbf{1}. \tag{20.45}$$

(Proof) By considering, if necessary, the box reduction $f_{\underline{l}}^{\overline{l}}$ with $\overline{l} = z + n\lceil\alpha\rceil\mathbf{1}$ and $\underline{l} = z - n\lceil\alpha\rceil\mathbf{1}$, we can assume that $\mathrm{dom} f$ is bounded and hence $\mathrm{argmin} f \neq \emptyset$. Let x^* be an integral minimizer of (domain-integral) f such that $||x^* - z||_1 = \min\{||y - z||_1 \mid y \in \mathrm{argmin} f\}$. If $x^* = z$, then we are done, so that we assume $x^* \neq z$.

Choose any $w \in V$ such that $x^*(w) \neq z(w)$. We assume without loss of generality that $x^*(w) > z(w)$. Then, it follows from Theorem 20.5 and the domain-integrality of f that there exists a finite sequence of elementary transformations $y_{i+1} = y_i + (\chi_w - \chi_{u_i})$ $(i = 0, 1, \cdots, k)$ such that $f(y_{i+1}) < f(y_i)$ $(i = 0, 1, \cdots, k)$, $y_0 = z$, $y_{k+1}(w) = x^*(w)$, and $u_i \in V - \{w\}$ $(i = 0, 1, \cdots, k)$. (Starting from $y_0 = z$, while $x^*(w) > y_i(w)$, find $u_i \in \mathrm{supp}^-(x^* - y_i)$ such that $f(x^*) + f(y_i) \geq f(x^* + (-\chi_w + \chi_{u_i})) + f(y_i + (\chi_w - \chi_{u_i}))$, and put $y_{i+1} = y_i + (\chi_w - \chi_{u_i})$ and $i \leftarrow i + 1$.) Put $U = \{u_i \mid i = 0, 1, \cdots, k\}$, where note that $|U|$ can be less than $k+1$ because of possible repetition of elements u_i. For any $u^* \in U$ let u_{i^*} be $u_{i^*} = u^*$ with the maximum index i^*. We shall show $x^*(w) - z(w) \leq \lceil \alpha - 1 \rceil$.

Define a function $h(\lambda) = f(z + \lambda(\chi_w - \chi_{u^*}))$ in $\lambda \geq 0$. Since h is a piecewise linear convex function, let $[\lambda_{j-1}, \lambda_j]$ $(j = 1, 2, \cdots, k)$ be the integral intervals on each of which h is affine, where $\lambda_0 = 0$. Since $h(0) < +\infty$, suppose that $h(\lambda_j) < +\infty$ and $y_{i^*}(u^*) - 1 < z(u^*) - \lambda_j$ for some $j \geq 0$. Put $y' = y_{i^*} + (\chi_w - \chi_{u^*})$ and $z' = z + \lambda_j(\chi_w - \chi_{u^*})$. Then, since $y'(w) - z'(w) \geq z'(u^*) - y'(u^*) > 0$, we have

$$u^* \in \mathrm{supp}^+(z' - y'), \qquad \{w\} = \mathrm{supp}^-(z' - y'). \tag{20.46}$$

It follows from Theorem 20.5 that

$$f(z') + f(y') \geq f(z' + (-\chi_{u^*} + \chi_w)) + f(y' + (\chi_{u^*} - \chi_w)). \tag{20.47}$$

Since $f(y') < f(y_{i^*}) = f(y' + (\chi_{u^*} - \chi_w))$, we see from (20.47) that $f(z') > f(z' + (-\chi_{u^*} + \chi_w))$. This means that $f(z + \lambda_{j+1}(\chi_w - \chi_{u^*})) < +\infty$. We can repeat this argument by putting $j \leftarrow j+1$ until we eventually get $z(u^*) - \lambda_j \leq y'(u^*)$. Hence we have

$$f(z+(z(u^*)-y_{i^*+1}(u^*))(\chi_w-\chi_{u^*})) < f(z) \leq f(z+\alpha(\chi_w-\chi_{u^*})). \quad (20.48)$$

It follows from (20.48) that

$$z(u^*) - y_{i^*+1}(u^*) = z(u^*) - y_{k+1}(u^*) \leq \lceil \alpha - 1 \rceil. \quad (20.49)$$

Since u^* is arbitrarily chosen from U, we have from (20.49)

$$\begin{aligned} x^*(w) - z(w) &= y_{k+1}(w) - z(w) \\ &= \sum_{u \in U} (z(u) - y_{k+1}(u)) \\ &\leq (n-1)\lceil \alpha - 1 \rceil. \end{aligned} \quad (20.50)$$

Q.E.D.

The M$^\natural$ version of Theorem 20.12 is given by

Theorem 20.13 (M$^\natural$-proximity theorem): *Let f be a domain-integral M$^\natural$-convex function on $\mathbf{R}^V$. Suppose that for an integral $z \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(z) \leq f(z + \alpha(\chi_u - \chi_v)) \qquad (u, v \in V \cup \{0\}). \quad (20.51)$$

Then, there exists an integral minimizer x^* of f such that

$$z - n\lceil \alpha - 1 \rceil \mathbf{1} \leq x^* \leq z + n\lceil \alpha - 1 \rceil \mathbf{1}. \quad (20.52)$$

Without the domain-integrality of f we also have

Theorem 20.14: *Let f be an M-convex function on $\mathbf{R}^V$. Suppose that for a vector $z \in \mathrm{dom} f$ and an $\alpha > 0$ we have*

$$f(z) \leq f(z + \alpha(\chi_u - \chi_v)) \qquad (u, v \in V). \quad (20.53)$$

Then, there exists a minimizer x^* of f such that

$$z - (n-1)\alpha \mathbf{1} \leq x^* \leq z + (n-1)\alpha \mathbf{1}. \quad (20.54)$$

The M-/M$^\natural$-proximity theorems are essential for M-/M$^\natural$-convex function minimization (see [Moriguchi+Murota+Shioura02] and [Shioura04]).

21. Other Related Topics

In this section we briefly describe recent results related to discrete convex functions.

21.1. The M-convex Submodular Flow Problem

The submodular flow problem described in Chapter III is extended by the use of M-convex functions as follows [Murota99] (also see [Murota03a]).

Let $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c}, \gamma)$ be an ordinary flow network, where G is an underlying graph with a vertex set V and an arc set A, $\underline{c} : A \to \mathbf{R} \cup \{-\infty\}$ and $\bar{c} : A \to \mathbf{R} \cup \{+\infty\}$ are, respectively, a lower capacity function and an upper capacity function, and $\gamma : A \to \mathbf{R}$ is a cost function. We assume that we are given an M-convex function g on $\mathbf{R}^V$. Then consider the following problem.

$$\begin{aligned} \text{(MSF)}\quad \text{Minimize}\quad & \sum_{a \in A} \gamma(a)\varphi(a) + g(\partial\varphi) \\ \text{subject to}\quad & \underline{c}(a) \le \varphi(a) \le \bar{c}(a) \quad (a \in A). \end{aligned} \tag{21.1}$$

We call this problem an *M-convex submodular flow problem*. We can easily see that the M-convex submodular flow problem is a generalization of the submodular flow problem (and hence the neoflow problem) considered in Chapter III.

The optimality conditions given in Sections 5 and 12 for submodular flows can naturally be extended to M-convex submodular flows (see [Murota99], [Murota03a, Theorems 9.14 and 9.15]; also [Murota+Shioura00] and [Iwata+Shigeno02]).

Theorem 21.1: *A feasible flow φ of Problem* (MSF) *is optimal if and only if there exists a vector $p \in (\mathbf{R}^V)^*$ such that we have for each $a \in A$*

$$\gamma_p(a) > 0 \implies \varphi(a) = \underline{c}(a), \tag{21.2}$$

$$\gamma_p(a) < 0 \implies \varphi(a) = \bar{c}(a) \tag{21.3}$$

and

$$\partial\varphi \in \operatorname{argmin}(g - p), \tag{21.4}$$

where $\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a)$.

Moreover, if $\underline{c}$ *and* $\bar{c}$ *are integer-valued,* g *is domain-integral, and there exists an optimal flow, then there exists an integral optimal flow. Furthermore, if* g *is co-domain-integral and* γ *is integer-valued, then* p *appearing above can be restricted to an integral vector.*

When g is co-domain-integral and p is integral, relation (21.4) is equivalent to

$$\partial\varphi \in \mathrm{B}_p(g^\bullet), \tag{21.5}$$

where

$$\mathrm{B}_p(g^\bullet) = \{x \mid x \in \mathbf{R}^V,\ x(V) = 0,\ \forall X \subseteq V : x(X) \le g^\bullet(p + \chi_X) - g^\bullet(p)\} \tag{21.6}$$

(see [Iwata+Shigeno02]). This follows from the co-domain-integrality and the fact that $x \in \operatorname{argmin}(g - p)$ is equivalent to $p \in \operatorname{argmin}(g^\bullet - x)$ (see Theorem 15.2).

Polynomial algorithms for solving the M-convex submodular flow problem (MSF) were proposed in [Iwata+Shigeno02] and [Iwata+Moriguchi+Murota04]. The basis for the polynomial algorithms is given by submodular function minimization algorithms, the L-convexity preserving scaling (Theorem 20.2) and the L-proximity theorem (Theorem 20.9). Also see [Murota96b] for cycle-cancelling and augmenting algorithms for valuated matroids and [Murota00] for a cycle-cancelling algorithm for integral M-convex submodular flows.

21.2. A Two-sided Discrete-Concave Market Model

The stable matching problem [Gale+Shapley62] and the stable assignment problem [Shapley+Shubik72] have recently been extended to a two-sided market model with M$^\natural$-concave functions by Fujishige and Tamura [Fuji+Tamura03, 04].

Let P and Q be finite nonempty sets with $P \cap Q = \emptyset$. We may consider P as a set of workers and Q as a set of firms. Define

$$E = P \times Q, \quad E_{(i)} = \{i\} \times Q \quad (i \in P), \quad E_{(j)} = P \times \{j\} \quad (i \in Q). \tag{21.7}$$

Suppose that we are given lower and upper bounds of salaries per hour $\underline{\pi} : E \to \mathbf{R} \cup \{-\infty\}$ and $\overline{\pi} : E \to \mathbf{R} \cup \{+\infty\}$ such that $\underline{\pi} \leq \overline{\pi}$. For each worker $i \in P$ and firm $j \in Q$ we denote by $f_i : \mathbf{Z}^{E(i)} \to \mathbf{R} \cup \{-\infty\}$ the utility function of worker i on his/her working hours allocated to Q, and by $f_j : \mathbf{Z}^{E(j)} \to \mathbf{R} \cup \{-\infty\}$ the profit function of firm j on working hours of employees in P. For a vector $x \in \mathbf{Z}^E$ and $k \in P$ (or $k \in Q$) we define $x_{(k)}$ as the restriction of x to $\{k\} \times Q$ (or $P \times \{k\}$).

We assume that each function f_k ($k \in P \cup Q$) is an M$^\natural$-concave function (with $\mathrm{dom} f_k \neq \emptyset$) and satisfies

(A1) $\mathrm{dom} f_k \subseteq [\mathbf{0}, b_k]$ for some $b_k \in \mathbf{Z}^{E(k)}$.

(A2) $\mathbf{0} \leq y' \leq y \in \mathrm{dom} f_k$ implies $y' \in \mathrm{dom} f_k$.

In other words, $\mathrm{dom} f_k$ is bounded and hereditary, and has $\mathbf{0}$ as the minimum point.

For a nonnegative vector $x \in \mathbf{Z}^E$ each $x(i,j)$ ($(i,j) \in E$) represents the working hour that worker i allocates to firm j. A nonnegative vector $x \in \mathbf{Z}^E$ is called a *feasible allocation* if $x_{(k)} \in \mathrm{dom} f_k$ for each $k \in P \cup Q$. By the assumption we always have a feasible allocation $x = \mathbf{0}$. We call a vector $s \in (\mathbf{R}^E)^*$ a *feasible salary vector* if $\underline{\pi}(i,j) \leq s(i,j) \leq \overline{\pi}(i,j)$ for each $(i,j) \in E$. Note that $s(i,j)$ denotes the amount of money per hour paid by firm j to worker i, which may be negative. A pair (x,s) of a feasible allocation $x \in \mathbf{Z}^E$ and a feasible salary vector $s \in (\mathbf{R}^E)^*$ is called an *outcome*. An outcome (x,s) is said to satisfy *incentive constraints* if

$$(f_i + s_{(i)})(x_{(i)}) = \max\{(f_i + s_{(i)})(y) \mid y \leq x_{(i)}\} \qquad (i \in P), \quad (21.8)$$

$$(f_j - s_{(j)})(x_{(j)}) = \max\{(f_j - s_{(j)})(y) \mid y \leq x_{(j)}\} \qquad (j \in Q). \quad (21.9)$$

For any $s \in (\mathbf{R}^E)^*$, $i \in P$, $j \in Q$ and $\alpha \in \mathbf{R}$, define $(s_{(i)}^{-j}, \alpha)$ as the vector obtained from $s_{(i)}$ by replacing the jth component by α, and $(s_{(j)}^{-i}, \alpha)$ as the vector obtained from $s_{(j)}$ by replacing the ith component by α. We call an outcome (x,s) *pairwise unstable* if it does not satisfy incentive constraints or there exist $i \in P$, $j \in Q$, $\alpha \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E(i)}$ and $y'' \in \mathbf{Z}^{E(j)}$ such that

$$y'(i,j') \leq x(i,j') \qquad (j' \in Q - \{j\}), \qquad (21.10)$$

$$y''(i',j) \leq x(i',j) \qquad (i' \in P - \{i\}), \qquad (21.11)$$

$$y'(i,j) = y''(i,j), \qquad (21.12)$$

$$(f_i + s_{(i)})(x_{(i)}) < (f_i + (s_{(i)}^{-j}, \alpha))(y'), \tag{21.13}$$

$$(f_j - s_{(j)})(x_{(j)}) < (f_j - (s_{(j)}^{-i}, \alpha))(y''). \tag{21.14}$$

Conditions (21.10)$\sim$(21.14) mean that worker i and firm j can get better off without increasing working hours $x(i', j')$ $((i', j') \in (E_{(i)} \cup E_{(j)}) - \{(i, j)\})$ and without changing salaries $s(i', j')$ $((i', j') \in (E_{(i)} \cup E_{(j)}) - \{(i, j)\})$ per hour. We call an outcome (x, s) *pairwise stable* if it is not pairwise unstable.

Remark: When $\underline{\pi} = \overline{\pi}$, the present model includes the stable marriage model of Gale and Shapley. The other extremal case is when $\underline{\pi}(i, j) = -\infty$ and $\overline{\pi}(i, j) = +\infty$ for all $(i, j) \in E$, which includes the assignment game of Shapley and Shubik.

The following theorem has been shown in [Fuji+Tamura04].

Theorem 21.2: *Under the above assumption there exists a pairwise stable outcome.*

More specifically we have the following. For any $x \in \mathbf{Z}^E$ put

$$f_P(x) = \sum_{i \in P} f_i(x_{(i)}), \qquad f_Q(x) = \sum_{j \in Q} f_j(x_{(j)}). \tag{21.15}$$

Theorem 21.3: *Under the above assumption there exist $x \in \mathbf{Z}^E$, $p \in (\mathbf{R}^E)^*$ and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ such that*

$$x \in \operatorname{argmax}\{(f_P + p)(y) \mid y \leq z_P\}, \tag{21.16}$$

$$x \in \operatorname{argmax}\{(f_Q - p)(y) \mid y \leq z_Q\}, \tag{21.17}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{21.18}$$

$$\forall e \in E : (z_P(e) < +\infty \Longrightarrow p(e) = \underline{\pi}(e),\ z_Q(e) = +\infty), \tag{21.19}$$

$$\forall e \in E : (z_Q(e) < +\infty \Longrightarrow p(e) = \overline{\pi}(e),\ z_P(e) = +\infty). \tag{21.20}$$

Moreover, if f_P, f_Q, $\underline{\pi}$ and $\overline{\pi}$ are integer-valued, then the p appearing above can be restricted to an integral vector.

Theorem 21.2 follows from Theorem 21.3.

Remark: The pairwise stability concept can fit the framework of discrete convex analysis as shown above. Fleiner [Fleiner01] first introduced the matroidal structure into the Gale-Shapley stable matching model, which was further extended to two-sided discrete concave models in [Eguchi+Fuji02], [Eguchi+Fuji+Tamura03] and [Fuji+Tamura03, 04].

Related developments in the analysis of competitive equilibria in economies with indivisible commodities and money have been made by [Danilov+Koshevoy+Murota01], [Fuji+Yang02] and [Murota+Tamura03a, 03b] (also see [Murota03a, Chapter 11], [Koshevoy03] and a survey [Tamura04] for more details and references). Also for the interface between auction theory and discrete convexity see [Gul+Stacchetti00], [Lehmann+Lehmann+Nisan 04] and [Bing+Lehmann+Milgrom04].

22. Historical Notes

In Sections 15∼21 we have described discrete convex analysis viewed from the theory of submodular functions and ordinary convex analysis. The definitions of L-/L$^\natural$-convex functions and M-/M$^\natural$-convex functions given in this chapter are different from the original ones given by Murota et al. In this section we give some notes on historical developments in discrete convex analysis.

The concept equivalent to L$^\natural$-convex function was introduced by Favati and Tardella [Favati+Tardella90], who called it a submodular integrally convex function. On the other hand, the concept of M-concave function on the unit hypercube was considered by Dress and Wenzel [Dress+Wenzel90, 92], who called it a *valuated matroid.* Concerning the discrete Fenchel duality, the concept of convex conjugate function of an L$^\natural$-convex function on the unit hypercube (i.e., a submodular set function) was considered in [Fuji84f] and the discrete Fenchel-duality theorem for such a restricted class of L$^\natural$-convex and M$^\natural$-convex functions was shown (see Chapter IV, Section 6.1). Also the discrete separation theorem for an L$^\natural$-convex function and an L$^\natural$-concave function on the unit hypercube was given by Frank [Frank82b] (see Chapter III, Section 4.2). Here the intersection theorem for submodular systems (Theorem 4.9) and Lovász's characterization of submodular set functions [Lovász82] (Theorem 6.13) play an essential rôle.

Murota's research in discrete convex analysis started from investigation on Dress and Wenzel's work on valuated matroids [Dress+Wenzel90, 92] (also see [Dress+ Terhalle95]). See his earlier results on valuated matroids in [Murota95, 96a, 96b, 97, 98a, 00a], which include most of the essence of discrete convex analysis on M-convex functions and their conjugate L-convex functions.

Murota originally defined an L-convex function and an M-convex function as follows (see [Murota96c, 98b]). A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called an L-convex function if f satisfies

(SBF[Z]) $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \quad (x, y \in \mathbf{Z}^V)$,

(TRF[Z]) $\exists r \in \mathbf{R}$ such that $f(x + \mathbf{1}) = f(x) + r \ (x \in \mathbf{Z}^V)$.

(Here, acronyms **(SBF[Z])** and **(TRF[Z])** are taken from [Murota03a], and similarly in the sequel.)

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called an M-convex function if f satisfies

(M-EXC[Z]) For any $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x-y)$ such that $f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v)$.

Moreover, an L$^\natural$-convex function on $\mathbf{Z}^{V-\{w\}}$ for an element $w \in V$ is defined in [Fuji+Murota00] from an L-convex function f on $\mathbf{Z}^V$ through the correspondence shown in the second half of Lemma 16.6 as $f^{\swarrow}$. It turned out that L$^\natural$-convex functions are the same as submodular integrally convex functions introduced by Favati and Tardella [Favati+Tardella90].

Also, an M$^\natural$-convex function f on $\mathbf{Z}^V$ was introduced by Murota and Shioura [Murota+Shioura99] as a function $f^{\downarrow}$ defined by (17.8) from an M-convex function f. They showed that an M$^\natural$-convex function on $\mathbf{Z}^V$ is exactly a function f satisfying

(M$^\natural$-EXC[Z]) For any $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that $f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$, where 0 is a new element not in V and $\chi_0 \equiv \mathbf{0} \in \mathbf{Z}^V$.

Furthermore, a polyhedral L-convex function and a polyhedral M-convex function are defined by Murota and Shioura [Murota+Shioura00] as follows. A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called L-convex if f satisfies

(SBF[R]) $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \quad (x, y \in \mathbf{R}^V)$,

(TRF[R]) $\exists r \in \mathbf{R}$ such that $f(x + \alpha\mathbf{1}) = f(x) + \alpha r \quad (x \in \mathbf{R}^V,\ \alpha \in \mathbf{R})$.

A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called M-convex if f satisfies

(M-EXC[R]) For any $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x-y)$ there exists $v \in \mathrm{supp}^-(x-y)$ and $\alpha > 0$ such that $f(x) + f(y) \geq f(x + \alpha(-\chi_u + \chi_v)) + f(y + \alpha(\chi_u - \chi_v))$.

A polyhedral L$^\natural$-convex function and a polyhedral M$^\natural$-convex function are, respectively, defined from a polyhedral L-convex function and a polyhedral M-convex function by the same construction as an L$^\natural$-convex function and an M$^\natural$-convex function on $\mathbf{Z}^V$ are, respectively, defined from an L-convex function and an M-convex function ([Murota+Shioura00]). Moreover, non-polyhedral L-/L$^\natural$-convex functions and M-/M$^\natural$-convex functions are also considered in [Murota+Shioura04a].

Locally polyhedral L-/L$^\natural$-convex functions and M-/M$^\natural$-convex functions have not been defined in the literature but such extensions are easy as shown in this chapter.

The facts shown in Sections 16~20 are given in the literature as follows.

Theorem 16.2: [Murota98b] in $\mathbf{Z}^V$ and [Murota+Shioura00];
Lemma 16.6: The definition of L$^\natural$-convexity in [Fuji+Murota00];
Theorem 16.8: Part of the definition in [Fuji+Murota00];
Theorem 16.9: [Murota03a], [Murota+Shioura00];
Theorem 16.11: [Favati+Tardella90], [Fuji+Murota00];
Theorem 16.12: [Favati+Tardella90], [Fuji+Murota00];
Theorem 17.2: Definitions in [Murota+Shioura99];
Theorem 17.5: [Dress+Terhalle95] and [Murota97] for valuated matroids, [Murota+Shioura99] in $\mathbf{Z}^V$, [Murota+Shioura00];
Theorem 18.1: [Murota98b, 03a], [Murota+Shioura00, 04a];
Theorem 18.2: [Murota98b];
Theorem 18.3: [Murota+Shioura00];
Theorem 18.4: [Murota98b];
Corollary 18.5: [Murota98b];
Theorem 19.3: [Murota96c], [Murota-Shioura99];
Theorem 19.4: [Murota98b];
Theorem 19.5: [Murota96c, 98a, 98b];
Theorem 19.6: [Murota96c, 98b];
Theorem 20.1: [Murota00b];
Theorem 20.3: [Dress+Wenzel92] for valuated matroids, [Murota96c], [Murota+Shioura00];
Corollary 20.4: [Murota+Shioura99] in $\mathbf{Z}^V$, [Murota+Shioura00];
Theorem 20.5: Definitions in [Murota96c] in $\mathbf{Z}^V$ and [Murota+Shioura00];
Theorem 20.7: [Fuji+Yang03] for valuated matroids, [Danilov+Koshevoy+

Lang03], [Murota+Tamura03a];
Theorem 20.9: [Iwata+Shigeno02], [Murota03a] for L$^\natural$-proximity;
Theorem 20.12: [Moriguchi+Murota+Shioura02], [Murota03a] for M$^\natural$-proximity.

Variants of these results can be obtained by unimodular transformations. One of such results is found in [Altman+Gaujal+Hordijk00, 03] about multimodular functions (see [Murota03a, 04]).

Slightly more general discrete convex functions are considered by Danilov and Koshevoy ([Danilov+Koshevoy04] and [Koshevoy03]). They have also shown connection between discrete convex analysis and representation theory [Danilov+Koshevoy03a, 03b].

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Index

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