

Belief Propagation for Min-cost Network Flow: Convergence & Correctness *

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Abstract

Distributed, iterative algorithms operating with minimal data structure while performing little computation per iteration are popularly known as *message-passing* in the recent literature. Belief Propagation (BP), a prototypical message-passing algorithm, has gained a lot of attention across disciplines including communications, statistics, signal processing and machine learning as an attractive scalable, general purpose heuristic for a wide class of optimization and statistical inference problems. Despite its empirical success, the theoretical understanding of BP is far from complete.

With the goal of advancing the state-of-art of our understanding of BP, we study the performance of BP in the context of the capacitated minimum-cost network flow problem – a cornerstone in the development of theory of polynomial time algorithms for optimization problems as well as widely used in practice of operations research. As the main result of this paper, we prove that BP converges to the optimal solution in the pseudo-polynomial-time, provided that the optimal solution of the underlying network flow problem instance is unique and the problem parameters are integral. We further provide a simple modification of the BP to obtain a fully polynomial-time randomized approximation scheme (FPRAS) without requiring uniqueness of the optimal solution. This is the first instance where BP is proved to have fully-polynomial running time. Our results thus provide a theoretical justification for the viability of BP as an attractive method to solve an important class of optimization problems.

1 Introduction

Message-passing has emerged as canonical algorithmic architecture to deal with the scale of the optimization and inference problems arising in the context of variety of disciplines including communications, networks, machine learning, image processing and computer vision, signal processing and statistics. The Belief Propagation (BP) is a message-passing heuristic for solving optimization and inference problems in the context of graphical model. The graphical model or a Markov random field provides a succinct representation for capturing the dependency structure between a

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collection of random variables. In the recent years, the need for large scale statistical inference and optimization has made graphical models the representation of choice in a variety of applications. There are two key problems for a graphical model of interest. The first problem is the computation of marginal distribution of a random variable. This problem is (computationally) equivalent to the computation of the so-called partition function and can be thought of as a weighted combinatorial counting problem (e.g., counting the number of independent sets of a graph is a special case of this problem). The second problem is that of finding the mode of a distribution, i.e., an assignment with the maximum likelihood (ML). For a constrained optimization (maximization) problem, when the constraints are modeled through a graphical model and probability is proportional to the cost of the assignment, an ML assignment is an optimal solution to the optimization problem. Both of these questions, in general, are computationally hard either in the $\#P$ or NP-complete sense.

Belief Propagation (BP) is an “umbrella” message-passing heuristic designed for these two problems. Its version for the first problem is known as the “sum-product algorithm” and for the second problem is known as the “max-product” or “min-sum algorithm”. Both versions of the BP algorithm are iterative, easy to implement and distributed in nature. When the underlying graph is a tree, the BP algorithm essentially performs the dynamic programming recursion [10], [33], [24], and, as a result, leads to a correct solution both for the optimization and inference problems. Specifically, BP provides a natural parallel iterative version of the dynamic programming in which variable nodes pass messages between each other along edges of the graphical model. Somewhat surprisingly, this seemingly naive BP heuristic has become quite popular in practice even for graphical models which do not have the tree structure [3], [14], [17], [25]. In our opinion, there are two primary reasons for the popularity of BP. First, it is generically applicable, easy to understand and implementation-friendly due to its iterative, simple and message-passing nature. Second, in many practical scenarios, the performance of BP is surprisingly good [32],[33]. On one hand, for an optimist, this unexpected success of BP provides a hope for it being a genuinely much more powerful algorithm than what we know thus far (e.g., better than primal-dual methods). On the other hand, a skeptic would demand a systematic understanding of the limitations (and strengths) of BP, in order to caution a practitioner. Thus, irrespective of the perspective of an algorithmic theorist, rigorous understanding of BP is very important.

Despite the apparent empirical success of the BP algorithm for solving a variety of problems, theoretical understanding of BP is far from complete. In this paper, primarily our interest lies in the correctness and convergence properties of the min-sum version of BP when applied to the minimum-cost network flow problems (or simply min-cost flow) - an important class of linear (or more generally convex) optimization problems. As a secondary interest, we wish to bring BP to the attention of researchers in the Operations Research (OR) community and thereby improving the current state in which BP has remained elusive in OR.

1.1 Contributions

As the main contribution of this paper, we establish that BP converges to the optimal solution of a min-cost network flow problem in the pseudo-polynomial time, provided that the optimal solution of the underlying problem is unique and the problem input is integral. At the same time, it is known [29] that BP fails to converge for general linear programming (LP) problem by means of a counter-example. Thus our results extend, in an important way, the scope of the problems that are provably solvable by the BP algorithm. We also point out that identifying the broadest class of optimization problems solvable using the BP algorithm is an interesting open problem. Indeed,

resolution of it will lead to the precise understanding of the structure of optimization problems that are solvable by BP.

The contributions of this paper, in detail are as follows. First, we show that an exact version of BP can be implemented for the min-cost flow problems, by encoding each message in BP as a piece-wise linear convex function. This is significant because the natural formulation of BP requires maintaining a vector of real-valued functions which may require an infinite amount of memory to store and computation to update. Then, we provide a proof to show that BP finds the optimal solution in pseudo-polynomial time, provided that the optimal solution is unique. Next, we present a simple modification of the BP algorithm which gives a fully polynomial-time randomized approximation scheme (FPRAS) for the same problem, which no longer requires the uniqueness of the optimal solution. This is the first instance where BP is proved to have fully-polynomial running time, except for the case when the underlying graph is a tree and BP solves the problem exactly. The modification of BP is obtained by applying a novel lemma; it is a natural generalization of the so-called Isolation Lemma found in [21]. Unlike the Isolation Lemma, our lemma can be used for generic LP. In essence, we show that it is possible to perturb the cost of any LP using *little* randomness so that the resulting modified LP has unique solution which is a good approximation to the original LP, and its gap to the next optimal solution is *large enough*. Indeed this is a general method and can be useful in a variety of applications including improving performance of distributed algorithms; it is no surprise that it is already used in a subsequent work [15].

1.2 Prior work on BP

Despite compelling reasons explained earlier, only recently we have witnessed an explosion of research for theoretical understanding of the performance of the BP algorithm in the context of various combinatorial optimization problems, both tractable and intractable (NP-hard) versions. In the earlier work, Weiss and Freeman [32] identified certain local optimality properties of the BP (max-product) for arbitrary graphs. It implies that when graph has a single-cycle then the fixed point of max-product corresponds to the correct answer. However they do not provide any guarantee on the convergence of max-product. Bayati, Shah and Sharma [5] considered the performance of BP for finding the maximum weight matching in a bipartite graph. They established that BP converges in pseudo-polynomial time to the optimal solution when the optimal solution is unique [5]. Bayati et al. [4] as well as Sanghavi et al. [28] generalized this result by establishing correctness and convergence of the BP algorithm for b-matching problem when the linear programming relaxation corresponding to the node constraints has a unique integral optimal solution. Note that the LP relaxation corresponding to the node constraints is not tight in general, as inclusion of the odd-cycle elimination constraints [30] is essential. Furthermore, [4] and [28] established that the BP does not converge if this LP relaxation does have a non-integral solution. Thus, for a b-matching problem BP finds an optimal answer when the LP relaxation can find an optimal solution. In the context of maximum weight independent set problem, a one-sided relation between LP relaxation and BP is established [29]; if BP converges then it is correct and LP relaxation is tight. In [29], a counter-example was produced that shows that BP does not converge to the optimal solution of an LP. This seem to suggest that BP is unlikely to solve all forms of LP.

Beyond LP, the performance of BP for quadratic optimization problems (QP) and more generally convex optimization problems (CP) are recently studied. The conditions for correctness and convergence of BP in the context of inference in Gaussian graphical models such as those established by Malioutov, Johnson and Willsky [16] lead to sufficient conditions for when BP can solve

(a certain class of) QP. More recently, in a sequence of works, Moallemi and Van Roy [18, 19] have identified sufficient conditions under which BP converges to correct solution for convex optimization problems. It is worth identifying the differences between results of this paper and that of Moallemi and Van Roy [18, 19]. To start with, our work applies to constrained min-cost network flow LP while that of [18, 19] applies to unconstrained convex optimization problem. While constrained min-cost network flow LP can be seen as an unconstrained convex optimization problem (e.g. via Lagrangian relaxation), the resulting convex optimization is not a *strictly* convex and hence sufficient conditions (the diagonal dominance of Hessian) of [18, 19] is not applicable. Indeed, the proof methods are different, and results of this paper provide ‘implementation’ of BP unlike results of [18, 19]. We also take note of a work by Ruozzi and Tatikonda [27] that utilizes BP to find source-sink paths in the network.

1.3 Prior work on min-cost network flow

The min-cost network flow problem (\mathcal{MCF}) has been fundamental in the development of theory of polynomial time algorithms for optimization problems. The first polynomial-time algorithm for \mathcal{MCF} was developed by Edmonds and Karp [8] with a running time of $O(m(\log U)(m + n \log n))$, where m represents the number of edges, n represents the number of nodes and U the largest capacity of an arc. Subsequently the first strongly polynomial time algorithm was proposed by Tardos [31]. Since \mathcal{MCF} has been central to the development of algorithmic theory, a wide variety of efficient algorithms have been proposed over years with different virtues such as [26],[22],[23],[9],[6],[12],[13], [1]. Among these, the fastest polynomial time algorithm runs (evaluated in the centralized computation model) in essentially $O(n^3 \log(nC))$ time [6], [13], [1], where C is the largest cost of an arc. On the other hand, the fastest strongly polynomial time algorithm for \mathcal{MCF} runs (again, evaluated in the centralized computation model) in $O(m \log n(m + n \log n))$ [23].

It is worth comparing the running time of the BP algorithm that we have obtained for \mathcal{MCF} . The basic version of BP takes (evaluated under decentralized computation model) $O(C^3 mn^4 \log n)$ computation (C represents the largest cost) in total. The modified FPRAS version of BP algorithm requires $O(\varepsilon^{-3} n^8 m^7 \log n)$ computation in total on average (w.r.t. decentralized computation model) for obtaining $(1 + \varepsilon)$ approximation. It should be noted that the number of iterations required by the algorithm scales as nL where L is the maximal cost of a directed path.

It is clear from the comparison that the bounds implied by our results for BP are not competitive with respect to the best known results for \mathcal{MCF} . BP’s performance is evaluated for the decentralized model while the above reported computation time analysis for other algorithms is for centralized model. Indeed, some of the known algorithms can be implemented in decentralized model such as that of [6] and [12] (see [2, Chapters 10-12] for further details). The analysis of BP for \mathcal{MCF} , when specialized to specific instances of \mathcal{MCF} like the bipartite matching problem, leads to tighter performance bounds that are competitive with respect to the best known results (see Theorem 4.14 in Section 4.2). But the important thing is that BP is a general purpose algorithm, not specialized for the problem at hand like the best known algorithm for \mathcal{MCF} . For this reason, BP is highly desirable from an implementor’s perspective as it does not require specific modifications for the problem of interest. Finally, it should be noted that the BP algorithm can operate in asynchronous decentralized environment unlike most known algorithms.

1.4 Organization

The rest of the paper is organized as follows. In Section 2, we introduce the BP algorithm as an iterative heuristic for a generic optimization problem. We provide an intuitive explanation by means of an example of how BP is derived as an iterative heuristic for generic problem inspired by parallel implementation of dynamic programming on tree-like problem structure. In Section 3, we specialize BP for linear programming (LP). We recall a (counter-)example of an LP for which BP cannot find its optimal solution. In Section 4, we further specialize BP algorithm for the capacitated min-cost network flow problem (\mathcal{MCF}). We state the main result that establishes pseudo-polynomial time convergence of BP to the optimal solution of \mathcal{MCF} , when the optimal solution is unique. Specifically, Section 4.1 explains how each message function in the BP algorithm can be computed leading to an efficient implementation of BP. In Section 4.2, we consider a subclass \mathcal{MCF}^o of \mathcal{MCF} that includes the problems of min-cost path as well as bipartite matching or more generally b -matching. For this subclass of \mathcal{MCF} , it turns out that BP has very simple message functions and this subsequently leads to a tighter bound on the running time. In Section 5, the proof of the main result about convergence of BP for \mathcal{MCF} is provided. Section 6 presents an extension of our result for min-cost flow problems with piece-wise linear convex cost functions. In Section 7, we provide the running time analysis of BP for \mathcal{MCF} and \mathcal{MCF}^o . From the analysis, we show that BP for the min-cost flow problem is a pseudo-polynomial-time algorithm when the data input is integral. In Section 8, we present a randomized approximation scheme for the min-cost flow problem which uses the standard BP as a subroutine. We prove that for any $\varepsilon \in (0, 1)$, the approximation scheme finds a solution that is within $1 + \varepsilon$ of the optimal solution, while its expected running time is polynomial in m , n , and $\frac{1}{\varepsilon}$. In doing so, we introduce a variation of the Isolation Lemma for LP in Section 8.1. Finally, Section 9 presents conclusions and directions for future work.

2 Belief Propagation for optimization problem

Here we introduce the min-sum version of BP as a heuristic for optimization problem in the general form. We shall utilize the notations similar to those used in [18],[19]. In the remainder of the paper, by BP we mean it's min-sum version for solving optimization problem. To this end, consider the optimization problem

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} \phi_i(x_i) + \sum_{C \in \mathcal{C}} \psi_C(x_C) && (\mathcal{P}) \\ & \text{subject to} && x_i \in \mathbb{R}, \forall i \in V, \end{aligned}$$

where V is a finite set of *variables* and \mathcal{C} is a finite collection of subsets of V representing *constraints*. Here $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$, $\forall i \in V$ and $\psi_C : \mathbb{R}^{|C|} \rightarrow \mathbb{R}$, $\forall C \in \mathcal{C}$ are extended real-valued functions where \mathbb{R} represents extended real-numbers $\mathbb{R} \cup \{\infty\}$. We call each ϕ_i a *variable function*, each ψ_C a *factor function* and (\mathcal{P}) a *factorized optimization problem*.

It is not difficult to see that essentially any constrained optimization problem of interest can be represented as a factorized optimization problem. For example, consider the well-known maximum-size independent set problem on a simple undirected graph $G = (V, E)$ which requires selecting subset V of maximal cardinality so that no two vertices of the chosen subset are neighbor of each

other as per E . The factorized form of the maximum weight independent set is given by

$$\begin{aligned} & \text{minimize} \quad \sum_{i \in V} \phi_i(x_i) + \sum_{(i,j) \in E} \psi_{ij}(x_i, x_j) \\ & \text{subject to} \quad x_i \in \mathbb{R}, \forall i \in V, \end{aligned}$$

where

$$\phi_i(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ -1 & \text{if } x_i = 1 \\ \infty & \text{otherwise} \end{cases}$$

$$\psi_{ij}(x_i, x_j) = \begin{cases} 0 & \text{if } x_i + x_j \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

In above, $x_i = 1$ if and only if node i is selected in the independent set. Finally, we introduce the notion of factor graph of a factorized optimization problem. A factor graph $F_{\mathcal{P}}$ of (\mathcal{P}) is a bipartite graph with one partition containing *variable* nodes V and the other partition containing *factor* nodes \mathcal{C} corresponding to the constraints. There is an edge $(v, C) \in V \times \mathcal{C}$ if and only if $v \in C$. For example, the graph shown in Figure 1, is the factor graph for optimization problem:

$$\begin{aligned} & \text{minimize} \quad \left(\sum_{i=1}^5 \phi_i(x_i) \right) + \psi_{1,2,3}(x_1, x_2, x_3) + \psi_{1,4,5}(x_1, x_4, x_5) + \psi_{1,5}(x_1, x_5) \\ & \text{subject to} \quad x_i \in \mathbb{R}, \forall 1 \leq i \leq 5. \end{aligned} \quad (\mathcal{P}')$$

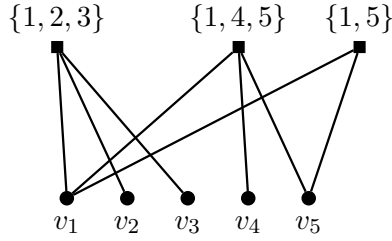


Figure 1: An example of a factor graph

Now we introduce BP. To start with, suppose the factor graph $F_{\mathcal{P}}$ of \mathcal{P} is a tree (note that factor graph in Figure 1 is *not* a tree because there is a cycle $(v_1, \{1, 4, 5\}, v_5, \{1, 5\}, v_1)$). In this case, let us consider the dynamic programming algorithm. The dynamic programming algorithm would suggest computation of the value or assignment of a given variable node $i \in V$ in the optimal solution as follows: fix a specific value $z \in \mathbb{R}$ of variable x_i corresponding to the variable $i \in V$. Subject to $x_i = z$ compute the cost of optimal assignment for the rest of the problem, say $b_i(z)$. Then the optimal assignment of variable node i is in $\arg \min_{z \in \mathbb{R}} b_i(z)$. Now to compute $b_i(z)$ for all $z \in \mathbb{R}$, the dynamic programming would recurse the same approach on the problem

$$\begin{aligned} & \text{minimize} \quad \phi_i(z) + \sum_{j \in V \setminus \{i\}} \phi_j(x_j) + \sum_{C \in \mathcal{C}} \psi_C(x_C), \\ & \text{subject to} \quad x_i = z, \quad x_j \in \mathbb{R}, \quad \forall j. \end{aligned} \quad (1)$$

Now implementation of this recursion of dynamic programming in general is not straightforward and can be computationally expensive. However, when the factor graph $F_{\mathcal{P}}$ is a tree, it is quite simple because the problem decomposes into sub-problems on disconnected trees. It is the dynamic programming implementation for tree factor graph which leads to the derivation of BP. To that end, given a node i consider any constraint C such that $i \in C$, i.e. (i, C) is an edge in $F_{\mathcal{P}}$. Since $F_{\mathcal{P}}$ is a tree, $F_{\mathcal{P}} \setminus (i, C)$ has two disjoint components, say T_1 and T_2 . Without loss of generality, we assume i is contained in T_1 and C is contained in T_2 . Due to this division of the problem structure, $b_i(z)$ for $z \in \mathbb{R}$ or equivalently solution of optimization problem (1), can be computed recursively as follows. For edge (i, C) , define ‘messages’ $m_{i \rightarrow C}(z)$ and $m_{C \rightarrow i}(z)$ as

$$\begin{aligned} m_{i \rightarrow C}(z) = & \text{minimize} \quad \sum_{j \in V \cap T_1} \phi_j(x_j) + \sum_{D \in \mathcal{C} \cap T_1} \psi_D(x_D), \\ & \text{subject to } x_i = z, \quad x_j \in \mathbb{R}, \quad \forall j. \end{aligned}$$

$$\begin{aligned} m_{C \rightarrow i}(z) = & \text{minimize} \quad \sum_{j \in V \cap T_2} \phi_j(x_j) + \sum_{D \in \mathcal{C} \cap T_2} \psi_D(x_D), \\ & \text{subject to } x_j \in \mathbb{R}, \quad \forall j. \end{aligned}$$

Note that such two directional ‘messages’ can be defined for any edge in $F_{\mathcal{P}}$ in a similar manner since it is a tree. Again, invoking the tree structure of $F_{\mathcal{P}}$ and definition of ‘messages’, the solution of (1) can be re-written as

$$b_i(z) = \phi_i(z) + \sum_{C \in \mathcal{C}_i} m_{C \rightarrow i}(z), \quad \forall z \in \mathbb{R}, \quad (2)$$

where \mathcal{C}_i is the set of all factor nodes (or constraints) that contain i , i.e.

$$\mathcal{C}_i \triangleq \{C \in \mathcal{C} : i \in C\}.$$

That is, if the graph underlying $F_{\mathcal{P}}$ is a tree, then in order to compute $b_i(z)$ it is sufficient to have knowledge of the ‘messages’ coming towards node i from the factor nodes to which it is connected to. For the tree $F_{\mathcal{P}}$, such messages can be recursively defined as follows: for any edge (i, C) in $F_{\mathcal{P}}$, for any $z \in \mathbb{R}$

$$m_{i \rightarrow C}(z) = \phi_i(z) + \sum_{K \in \mathcal{C}_i \setminus C} m_{K \rightarrow i}(z), \quad (3)$$

$$m_{C \rightarrow i}(z) = \min_{\substack{y \in \mathbb{R}^{|C|} \\ y_i = z}} \psi_C(y) + \sum_{j \in C \setminus i} m_{j \rightarrow C}(y_j). \quad (4)$$

For tree structured $F_{\mathcal{P}}$, starting from leaf nodes using (3)-(4) the ‘messages’ $m_{i \rightarrow C}(z)$ and $m_{C \rightarrow i}(z)$ for all edges (i, C) can be computed. A parallel implementation of this recursive procedure is as follows. Initially, for $t = 0$ we set $m_{C \rightarrow i}^0(z) = m_{i \rightarrow C}^0(z) = 0$ for all edges (i, C) of $F_{\mathcal{P}}$. For $t \geq 1$, update messages for each edge (i, C) of $F_{\mathcal{P}}$ as

$$m_{i \rightarrow C}^t(z) = \phi_i(z) + \sum_{K \in \mathcal{C}_i \setminus C} m_{K \rightarrow i}^{t-1}(z), \quad (5)$$

$$m_{C \rightarrow i}^t(z) = \min_{\substack{y \in \mathbb{R}^{|C|} \\ y_i = z}} \psi_C(y) + \sum_{j \in C \setminus i} m_{j \rightarrow C}^t(y_j). \quad (6)$$

The estimation of $b_i(z)$ at the end of iteration t for each $i \in V$ and $z \in \mathbb{R}$ is given by

$$b_i^t(z) = \phi_i(z) + \sum_{C \in \mathcal{C}_i} m_{C \rightarrow i}^t(z). \quad (7)$$

It is easy to show by induction that if the graph underlying $F_{\mathcal{P}}$ is a tree, then for t larger than the diameter of the tree, $b_i^t(\cdot)$ equals to the value produced by the dynamic programming problem, therefore resulting in the optimal assignment of x_i .

The parallelized implementation of the dynamic programming problem described by (5) and (6) can be applied to any factor graph in general. This is precisely the BP min-sum heuristic. The algorithm is described in detail next. For the non-tree graphs the convergence and/or correctness of such a heuristic is, by no means guaranteed in general.

Algorithm 1 min-sum BP

- 1: Given a factorized optimization problem (\mathcal{P}) , construct factor graph $F_{\mathcal{P}}$.
- 2: Set N to be the number of iterations for BP.
- 3: Initialize $t = 0$, and for each edge (i, C) in $F_{\mathcal{P}}$, initialize $m_{C \rightarrow i}^0(z) = 0 = m_{i \rightarrow C}^0(z)$ for all $z \in \mathbb{R}$.
- 4: **for** $t = 1, 2, \dots, N$ **do**
- 5: For any edge (i, C) in $F_{\mathcal{P}}$ and $z \in \mathbb{R}$, update

$$m_{i \rightarrow C}^t(z) = \phi_i(z) + \sum_{K \in \mathcal{C}_i \setminus C} m_{K \rightarrow i}^{t-1}(z), \quad (8)$$

$$m_{C \rightarrow i}^t(z) = \min_{y \in \mathbb{R}^{|C|}, y_i = z} \psi_C(y) + \sum_{j \in C \setminus i} m_{j \rightarrow C}^t(y_j). \quad (9)$$

- 6: $t := t + 1$
 - 7: **end for**
 - 8: Set the belief function as $b_i^N(z) = \phi_i(z) + \sum_{C \in \mathcal{C}_i} m_{C \rightarrow i}^N(z)$, $\forall 1 \leq i \leq n$.
 - 9: Estimate the optimal assignment as $\hat{x}_i^N \in \arg \min b_i^N(z)$ for each $i \in V$.
 - 10: Return \hat{x}^N .
-

3 BP for Linear Programming

The linear programming (LP) problem in the standard form is given by

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax = g, \\ & && x \geq 0, \quad x \in \mathbb{R}^n, \end{aligned} \quad (\mathcal{LP})$$

where $A \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. In the notation of factorized optimization problem introduced earlier, variable nodes are $V = \{1, \dots, n\}$ with associated variables x_i , $i \in V$; rows of A correspond to constraint nodes $\mathcal{C} = \{C_j : 1 \leq j \leq m\}$ where $C_j = \{i \in V : a_{ji} \neq 0\}$; and $\mathcal{C}_i = \{C_j : a_{ji} \neq 0\}$, $\forall i \in V$. Define factor function $\psi_j : \mathbb{R}^{|C_j|} \rightarrow \bar{\mathbb{R}}$ for $1 \leq j \leq m$ as:

$$\psi_j(z) = \begin{cases} 0 & \text{if } \sum_{i \in C_j} a_{ji} z_i = g_j \\ \infty & \text{otherwise.} \end{cases}$$

And define variable function $\phi_i : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ for $i \in V$ as:

$$\phi_i(z) = \begin{cases} c_i z & \text{if } z \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Then, (\mathcal{LP}) is equivalent to following the factorized optimization problem:

$$\begin{aligned} & \text{minimize} \quad \sum_{i=1}^n \phi_i(x_i) + \sum_{j=1}^m \psi_{C_j}(x_{C_j}), \\ & \text{subject to} \quad x_i \in \mathbb{R}, \quad \forall i \in V. \end{aligned} \tag{\mathcal{P}_{LP}}$$

Then BP for this factorized optimization problem becomes the BP heuristic for LP. BP described earlier requires computing message functions of the form $m_{i \rightarrow C}^t$ and $m_{C \rightarrow i}^t$. In general, it is not clear if such message functions can be stored and updated efficiently. For LP, however it can be shown that every message function is a piece-wise linear convex function, which allows efficient encoding of them in terms of a finite vector describing the break points and the slopes of its linear pieces. In Section 4.1, we will do this in the context of min-cost network flow problem and we will explain the associated computation procedure in detail.

Now BP being a distributed algorithm, it is unlikely to work well when the (\mathcal{LP}) does not have a unique optimal solution. Yet, even with the assumption that (\mathcal{LP}) has a unique optimal solution, in general the estimation of BP may not converge to the unique optimal solution. One such instance is an LP-relaxation of the maximum-weight independent set problem on a complete bipartite graph [29]:

$$\begin{aligned} & \text{minimize} \quad - \sum_{i=1}^3 2x_i - \sum_{j=1}^3 3y_j \\ & \text{subject to} \quad x_i + y_j + z_{ij} = 1, \quad \forall 1 \leq i, j \leq 3, \\ & \quad \quad \quad x, y, z \geq 0. \end{aligned} \tag{\mathcal{P}_I}$$

Although BP in [29] was stated in a somewhat different manner, it can be checked that it is equivalent to the description presented here. It turns out that although this problem has a unique optimal solution, the BP algorithm does not converge at all, let alone to the optimal solution. Specifically, the messages \hat{x}^N oscillate between two different values as the number of iterations N oscillates between odd and even values.

4 BP Algorithm for Min-Cost Network Flow Problem

In this section, we formulate BP for the capacitated min-cost network flow problem (\mathcal{MCF}) , and state our main result about the convergence of BP for \mathcal{MCF} . As mentioned earlier, each message of BP for \mathcal{MCF} is a function, and we describe how these messages can be efficiently updated and stored as vectors in Section 4.1. In Section 4.2, we consider a subclass of \mathcal{MCF} , it includes bipartite matching, for which BP can take advantage of its special structure to obtain much faster running time.

Let us define the capacitated min-cost network flow problem (\mathcal{MCF}) . Given a directed graph $G = (V, E)$, let V, E denote the set of vertices and arcs or directed edges respectively with $|V| = n$

and $|E| = m$. For any vertex $v \in V$, let E_v be the set of arcs incident to v , and for any $e \in E_v$, let $\Delta(v, e) = 1$ if e is an *out-arc* of v (i.e. arc $e = (v, w)$, for some $w \in V$), and $\Delta(v, e) = -1$ if e is an *in-arc* of v (i.e. arc $e = (w, v)$, for some $w \in V$). The \mathcal{MCF} on G is formulated as follows [2, 7]:

$$\begin{aligned}
& \text{minimize } \sum_{e \in E} c_e x_e && (\mathcal{MCF}) \\
& \text{subject to } \sum_{e \in E_v} \Delta(v, e) x_e = f_v, \forall v \in V && (\text{demand/supply constraints}) \\
& 0 \leq x_e \leq u_e, \forall e \in E && (\text{flow constraints})
\end{aligned}$$

where $c_e \geq 0$, $u_e \geq 0$, $c_e \in \mathbb{R}$, $u_e \in \bar{\mathbb{R}}$, for each $e \in E$, and $f_v \in \mathbb{R}$ for each $v \in V$. The variables x_e represent flow value assigned to each arc $e \in E$; the first type of constraints state that the difference of in-flow and out-flow at each node $v \in V$ equals the node demand f_v (could be positive or negative); and the second type of constraints state that flow on each arc $e \in E$ is non-negative and can not be larger than its capacity u_e . We shall assume the instance of network flow is feasible. Without loss of generality, let each node $v \in V$ be such that $|E_v| \geq 2$; or else either $E_v = \emptyset$ in which case we ignore such v or $|E_v| = 1$ in which case the flow on $e \in E_v$ is determined by f_v . For the \mathcal{MCF} , define factor and variable functions ψ , ϕ as follows: for $v \in V$, $e \in E$

$$\begin{aligned}
\psi_v(z) &= \begin{cases} 0 & \text{if } \sum_{e \in E_v} \Delta(v, e) z_e = f_v, \\ \infty & \text{otherwise,} \end{cases} \\
\phi_e(z) &= \begin{cases} c_e z & \text{if } 0 \leq z \leq u_e, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Then, solving \mathcal{MCF} is equivalent to solving $\min_{x \in \mathbb{R}^{|E|}} \{\sum_{v \in V} \psi_v(x_{E_v}) + \sum_{e \in E} \phi_e(x_e)\}$. Therefore, the BP algorithm can be applied for \mathcal{MCF} in this standard form. Because of the special structure of \mathcal{MCF} that each variable node is adjacent to exactly two factor nodes, it is indeed possible to skip the message update step $m_{v \rightarrow e}^t$ and resulting into a simplified Algorithm 2 stated next.

Intuitively, in Algorithm 2 each arc can be thought of as an agent, who is trying to figure out its own flow while meeting the conservation constraints at its endpoints. Each link maintains an estimate of its “local cost” as a function of its flow (thus this estimate is a function, not a single number). At each time step an arc updates its function as follows: the cost of assigning x units of flow to link e is the cost of pushing x units of flow through e plus the minimum-cost way of assigning flow to neighboring edges (with respect to the functions computed at the previous iteration) to restore flow conservation at the endpoints of e .

Similar to BP for LP, the message functions in BP for \mathcal{MCF} , $m_{e \rightarrow v}^t$ for suitable pairs of e and v , are also piece-wise linear convex functions. In Section 4.1, we establish this fact and present an explicit procedure for computing $m_{e \rightarrow v}^t$. Hence, Algorithm 2 is indeed a procedure that can be implemented on a computer. Next, we state conditions under which the estimates of BP converge to the optimal solution of \mathcal{MCF} . Before formally stating the result, we first give the definition of a residual network [2]. Define $G(x)$ to be the residual network of G with respect to flow x as follows: $G(x)$ has the same vertex set as G , $\forall e = (v, w) \in E$ if $x_e < u_e$ then e is an arc in $G(x)$ with cost

Algorithm 2 BP for \mathcal{MCF}

- 1: Initialize $t = 0$, messages $m_{e \rightarrow v}^0(z) = 0$, $m_{e \rightarrow w}^0(z) = 0$, $\forall z \in \mathbb{R}$ for each $e = (v, w) \in E$.
- 2: **for** $t = 1, 2, 3, \dots, N$ **do**
- 3: For each $e = (v, w) \in E$ update messages as follows:

$$m_{e \rightarrow v}^t(z) = \phi_e(z) + \min_{\bar{z} \in \mathbb{R}^{|E_w|}, \bar{z}_e = z} \left\{ \psi_w(\bar{z}) + \sum_{\tilde{e} \in E_w \setminus e} m_{\tilde{e} \rightarrow w}^{t-1}(\bar{z}_{\tilde{e}}) \right\}, \quad \forall z \in \mathbb{R}$$

$$m_{e \rightarrow w}^t(z) = \phi_e(z) + \min_{\bar{z} \in \mathbb{R}^{|E_v|}, \bar{z}_e = z} \left\{ \psi_v(\bar{z}) + \sum_{\tilde{e} \in E_v \setminus e} m_{\tilde{e} \rightarrow v}^{t-1}(\bar{z}_{\tilde{e}}) \right\}, \quad \forall z \in \mathbb{R}$$

- 4: $t := t + 1$
- 5: **end for**
- 6: For each $e = (v, w) \in E$, set the belief function as

$$b_e^N(z) = \phi_e(z) + \sum_{\tilde{e} \in E_v \setminus e} m_{\tilde{e} \rightarrow v}^{N-1}(z) + \sum_{\tilde{e} \in E_w \setminus e} m_{\tilde{e} \rightarrow w}^{N-1}(z)$$

- 7: Calculate the belief estimate by finding $\hat{x}_e^N \in \arg \min b_e^N(z)$ for each $e \in E$.
 - 8: Return \hat{x}^N as an estimation of the optimal solution of \mathcal{MCF} .
-

$c_e^x = c_e$. Finally, if $x_e > 0$ then there is an arc $e' = (w, v)$ in $G(x)$ with cost $c_{e'}^x = -c_e$. Let

$$\delta(x) = \min_{C \in \mathcal{C}} \{c^x(C) = \sum_{e \in C} c_e^x\}, \quad (10)$$

where \mathcal{C} is the set of directed cycles in $G(x)$. Note that if x^* is the unique optimal solution of \mathcal{MCF} with directed graph G , then it must be that $\delta(x^*) > 0$ in $G(x^*)$ or else we can change flow x^* along the minimal cost cycle in (10) without increasing its cost.

Theorem 4.1. *Suppose \mathcal{MCF} has a unique optimal solution x^* . Define L to be the maximum cost of a simple directed path in $G(x^*)$. Then for any $N \geq (\lfloor \frac{L}{2\delta(x^*)} \rfloor + 1)n$, $\hat{x}^N = x^*$.*

The proof of Theorem 4.1 is presented in Section 5. The above stated theorem claims that the BP algorithm finds the unique optimal solution of \mathcal{MCF} in at most $(\lfloor \frac{L}{2\delta(x^*)} \rfloor + 1)n$ iterations: this convergence is exact in the sense that BP finds the optimal solution exactly in finite number of iterations. This is in contrast with the asymptotic convergence established for many iterative algorithms in the theory of continuous optimization. We note that this result is similar in flavor to those established in the context of BP's convergence for combinatorial optimization [5, 4, 29]. However, it differs from the convergence results in [18, 19] where the estimates converge to the optimal solution with an exponential rate, but are not established to reach exact optimal in finitely many steps. Next we state the total computation performed by Algorithm 2 to find the optimal solution when the parameters (capacities and costs) are integral in the \mathcal{MCF} .

Theorem 4.2. *Given an \mathcal{MCF} with a unique optimal solution x^* and integral data, BP algorithm finds the unique optimal solution of \mathcal{MCF} in $O(c_{\max}^3 mn^4 \log n)$ operations, where $c_{\max} = \max_e c_e$.*

Theorem 4.2 follows by utilizing Theorem 4.1 to bound the number of iterations along with a bound on the number of operations required for updating message functions $m_{e \rightarrow v}^t$ up to those many iterations. The formal proof of this statement is presented in Section 7.

4.1 Computing/encoding message functions

Here we provide a procedure for constructing message function $m_{e \rightarrow v}^t$ in BP for \mathcal{MCF} . This construction procedure shows that each message function $m_{e \rightarrow v}^t$ is a piece-wise linear convex function. Moreover, we provide a bound for the number of operations required for this construction procedure, which will help in bounding the running time of Algorithm 2. First, we formally define piece-wise linear convex function:

Definition 4.3. A function f is called piece-wise linear convex if for some finite set of reals, $a_0 < a_1 < \dots < a_n$, (allowing $a_0 = -\infty$ and $a_n = \infty$),

$$f(z) = \begin{cases} c_1(z - a_1) + f(a_1) & \text{if } z \in [a_0, a_1] \\ c_{i+1}(z - a_i) + f(a_i) & \text{if } z \in (a_i, a_{i+1}], 1 \leq i \leq n \\ \infty & \text{otherwise} \end{cases}$$

where $f(a_1) \in \mathbb{R}$ and $c_1 < c_2 < \dots < c_n$ satisfy $c_{i+1}(a_{i+1} - a_i) + f(a_i) = f(a_{i+1})$ for $1 \leq i \leq n - 1$.

We define a_0, a_1, \dots, a_n as the *vertices* of f . We define n to be the number of pieces of f , denoted by $p(f)$. We call $c_i(z - a_{i-1}) + f(a_{i-1})$ for $z \in [a_{i-1}, a_i]$ as the i th linear piece of f . Clearly, if f is a piece-wise linear convex function, then all relevant information about f can be stored using a finite vector of size $O(p(f))$. We make the following observation that will be useful for efficient update of messages of BP.

Observation 4.4. Suppose f_1, f_2 are piece-wise linear convex functions. Then, $f_1(ax+b), cf_1(x) + df_2(x)$ are also convex piecewise-linear functions, for any real numbers a, b, c and d , where $c \geq 0, d \geq 0$.

Definition 4.5. Let $S = \{f_1, f_2, \dots, f_k\}$ be a set of piece-wise linear convex functions, and let $\Psi_t : \mathbb{R}^k \rightarrow \mathbb{R}$ be

$$\Psi_t(x) = \begin{cases} 0 & \text{if } \sum_{i=1}^k x_i = t \\ \infty & \text{otherwise} \end{cases}$$

Then the interpolation of f_1, \dots, f_k or S , denoted by $I_S(\cdot)$ is defined as

$$I_S(t) = \min_{x \in \mathbb{R}^k} \left\{ \Psi_t(x) + \sum_{i=1}^k f_i(x_i) \right\}, \quad \forall t \in \mathbb{R}.$$

Lemma 4.6. Suppose f_1, f_2 are piece-wise linear convex functions. Then for $S = \{f_1, f_2\}$ the $I_S(t)$ is a piece-wise linear convex function and it can be computed in $O(p(f_1) + p(f_2))$ operations.

Proof. We shall provide a constructive proof of this result by describing a procedure to construct $I_S(t)$. The idea behind construction of $I_S(t)$ is essentially to “stitch” together the linear pieces of f_1 and f_2 . To this end, let z_1^*, z_2^* be vertices of f_1, f_2 such that $z_1^* = \arg \min f_1(z), z_2^* = \arg \min f_2(z)$.

Let $S = \{f_1, f_2\}$. In case the case of ties, we select z_i^* to be the smallest point in the arg min set. Let $g(t)$ be the function that is defined only at $z_1^* + z_2^*$ with $g(z_1^* + z_2^*) = f_1(z_1^*) + f_2(z_2^*)$. Let $L_1 = U_1 = z_1^*$ and $L_2 = U_2 = z_2^*$. We shall construct g iteratively for all $t \in \mathbb{R}$ so that we shall end up with $g(t) = I_S(t)$. The construction is described as follows. At every iteration, let X_1 (and X_2) be the linear piece of f_1 (and f_2) at the left side of L_1 (and L_2). Choose the linear piece with the larger slope from $\{X_1, X_2\}$, and “stitch” this piece onto the left side of the left endpoints of g . If piece, say P_i , of function f_i is chosen then update L_i to the vertex which is on the left end of P_i for $i = 1, 2$. As an example, consider f_1 and f_2 shown in the Figure 2. Here $z_1^* = 1$ and $z_2^* = 0$ are vertices of f_1 and f_2 such that $z_1^* = \arg \min f_1(z)$, $z_2^* = \arg \min f_2(z)$. Note that the linear piece X_1 in the procedure is labeled as $P1$ on the graph, while X_2 does not exist (since there is no linear piece for f_2 on the right side of z_2). Hence, we “stitch” $P1$ to the left side of g , and update L_1 to 0. In a similar manner, let Y_1 (Y_2) be the linear piece of f_1 (f_2) to the right side of U_1 (U_2). Then choose the linear piece with the smaller slope and “stitch” this piece onto the right side of the right endpoint of g . If Q_i is the chosen piece, update U_i to the vertex which is on the right side of Q_i for $i = 1, 2$. Again, we use f_1 and f_2 in Figure 2 as an illustration. The linear piece Y_1 in the procedure is labeled as $P2$, while Y_2 is labeled as $P3$. As $P2$ has a lower slope than $P3$, we “stitch” $P2$ to the right side of g and update U_1 to 2.

Repeat this procedure until both L_1 (and L_2) and U_1 (and U_2) are the left most (and right most) endpoints of f_1 (and f_2), or both endpoints of g are infinity. See Figure 2 and Figure 3 as an illustration of resulting interpolation of the two functions.

Note that the total number of iterations is bounded by $O(p(f_1) + p(f_2))$ and each iteration takes at most constant number of operations. Thus total computation performed to obtain g is $O(p(f_1) + p(f_2))$. By construction, it is clear that g is a piece-wise linear convex function. Also $g(z_1^* + z_2^*) = f_1(z_1^*) + f_2(z_2^*)$ and by the way we have constructed g , we must have $g(t) \leq \{\Psi_t(x) + f_1(x_1) + f_2(x_2)\}$ for any $t \in \mathbb{R}$. Therefore, it follows that $g = I_S$. This completes the proof of Lemma 4.6. \square

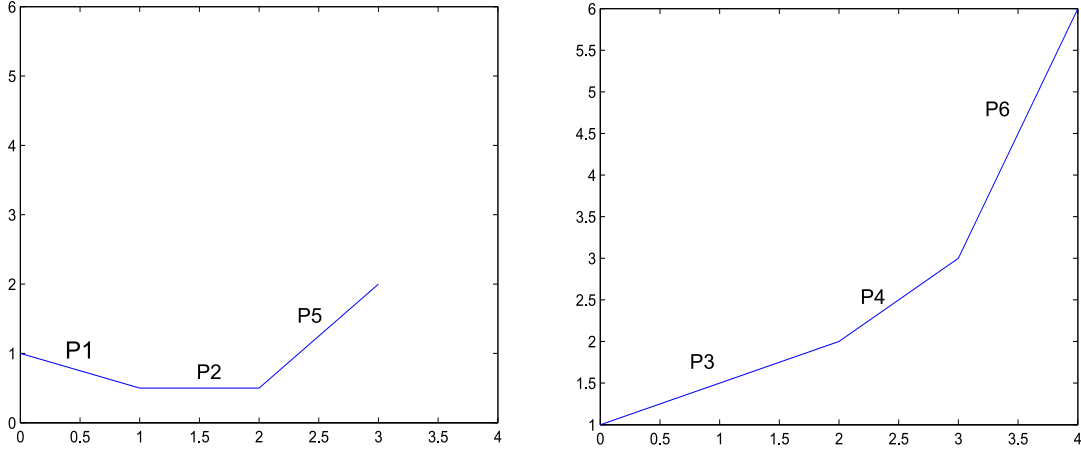


Figure 2: Functions f_1 and f_2

Theorem 4.7. *Given a set $S\{f_1, \dots, f_k\}$ of piece-wise linear convex functions, $I_S(t)$ is also a piece-wise linear convex function. Let $P = \sum_{f \in S} p(f)$. Then $I_S(t)$ can be computed in $O(P \log k)$*

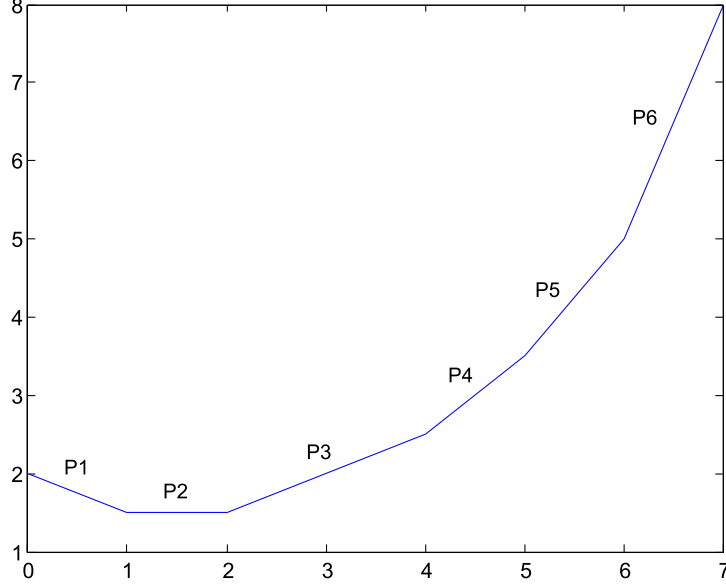


Figure 3: Interpolation of f_1 and f_2

operations.

Proof. Without the loss of generality we may assume that k is divisible by 2. Let $S_1 = \{f_1, f_2\}$, $S_2 = \{f_3, f_4\}, \dots, S_{\frac{k}{2}} = \{f_{k-1}, f_k\}$ and $S' = \{I_{S_1}, I_{S_2}, \dots, I_{S_{\frac{k}{2}}}\}$. Then one can observe that $I_{S'} = I_S$ by the definition of I_S . By Lemma 4.6 each function in S' is piece-wise linear convex and S' can be computed in $O(P)$ operations. Consider changing S to S' as a procedure of decreasing the number of piece-wise linear convex functions. This procedure reduces the number by a factor of 2 each time while it consumes $O(P)$ operations. Hence, it takes $O(\log k)$ procedures to reduce set S into a single piece-wise linear convex function. And hence computing $I_S(t)$ takes $O(P \log k)$ operations. \square

Definition 4.8. Let $S = \{f_1, f_2, \dots, f_k\}$ be a set of convex piecewise-linear functions, $a \in \mathbb{R}^k$, and let $\Psi_t : \mathbb{R}^k \rightarrow \mathbb{R}$ be:

$$\Psi_t(x) = \begin{cases} 0 & \text{if } \sum_{i=1}^k a_i x_i = t \\ \infty & \text{otherwise} \end{cases}, \quad \forall v \in V$$

We call $I_S^a(t) = \min_{x \in \mathbb{R}^k} \{\psi_t(x) + \sum_{i=1}^k f_i(x_i)\}$ the scaled interpolation of S .

Theorem 4.9. Given a set of piece-wise linear convex functions $S = \{f_1, \dots, f_k\}$, $I_S^a(t)$ is also a piece-wise linear convex function. Let $P = \sum_{f \in S} p(f)$. Then $I_S(t)$ can be computed in $O(P \log k)$ operations.

Proof. Let $S = \{f_1, \dots, f_k\}$ and $S' = \{f'_1, \dots, f'_k\}$ with $f'_i(x) = f_i(a_i x)$ for $1 \leq i \leq k$. If f_i is a piece-wise linear convex function, then it can be easily checked that so is f'_i for $1 \leq i \leq k$. Therefore, Theorem 4.9 follows immediately by an application of Theorem 4.7 to S' . \square

Now recall that for any $t \geq 1$, the message update in the BP for \mathcal{MCF} problem has the following

form:

$$m_{e \rightarrow v}^t(z) = \phi_e(z) + \min_{\bar{z} \in \mathbb{R}^{|E_w|}, \bar{z}_e = z} \left\{ \psi_w(\bar{z}) + \sum_{\tilde{e} \in E_w \setminus e} m_{\tilde{e} \rightarrow w}^{t-1}(\bar{z}_{\tilde{e}}) \right\} \quad \text{for } z \in \mathbb{R}.$$

Therefore, the message update can be performed using the scaled interpolation. Specifically, we make the following observation.

Observation 4.10. *Let $S = \{m_{\tilde{e} \rightarrow w}^{t-1}, \tilde{e} \in E_w \setminus e\}$ and $a_{\tilde{e}} = \Delta(w, \tilde{e})$ for any $\tilde{e} \in E_w \setminus e$. Then the function $\tilde{m}_{e \rightarrow v}^t(z) = m_{e \rightarrow v}^t(z) - \phi_e(z)$ is equal to $I_S^a(-\Delta(w, e)z + f_w)$.*

From above Observation 4.10, the following Corollaries are immediate.

Corollary 4.11. *For $t \geq 1$ and $e \in E$ with $e = (v, w)$, the message functions $m_{e \rightarrow v}^t, m_{e \rightarrow w}^t$ of BP algorithm for \mathcal{MCF} are piece-wise linear convex functions.*

Proof. The proof follows by induction on t . Initially, $t = 0$ and $m_{e \rightarrow v}^0$ is constant function (equal to 0). Therefore, it is a piece-wise linear convex function by definition. For $t \geq 1$, by Corollary 4.9 and Observation 4.10, $m_{e \rightarrow v}^t(z) - \phi_e(z)$ is a piece-wise linear convex. Now ϕ_e is a piece-wise linear convex function. Therefore, $m_{e \rightarrow v}^t$ is a summation of two piece-wise linear convex functions which is piece-wise linear convex as well. \square

Corollary 4.12. *Suppose the components of cost vector c in \mathcal{MCF} are integers. At iteration t , for piece-wise linear convex message function $m_{e \rightarrow v}^t(z)$ of BP algorithm for \mathcal{MCF} , let $\{s_1, s_2, \dots, s_k\}$ be the slopes of its pieces. Then $-tc_{\max} \leq s_i \leq tc_{\max}$ and s_i is integral for each $1 \leq i \leq k$, where $c_{\max} = \max_e c_e$.*

Proof. The proof follows by induction on t . Initially, $t = 0$ and the statement is immediate. For $t \geq 1$, since $\Delta(w, e) = \pm 1$ for any $e \in E_w$, by Observation 4.10 it follows that the absolute values of the slopes for the linear pieces of $m_{e \rightarrow v}^t - \phi_e$ is the same as the absolute values of the slopes for the linear pieces of message functions $m_{\tilde{e} \rightarrow w}^{t-1}$. By induction hypothesis, the absolute values of the slopes of $m_{\tilde{e} \rightarrow w}^{t-1} - \phi_e$ are integral and bounded by $(t-1)c_{\max}$. The slope of pieces in ϕ_e is c_e and therefore, the absolute values of slopes of $m_{e \rightarrow v}^t$ are integral and bounded by tc_{\max} . \square

Corollary 4.13. *Suppose components of vectors f and u take integer values in \mathcal{MCF} . Then at iteration $t \geq 1$, for any message function $m_{e \rightarrow v}^t$, the vertices of $m_{e \rightarrow v}^t$ are integral as well.*

Proof. Again, the proof is by induction on t . Initially, $t = 0$ and the statement trivially holds. For $t \geq 1$, first observe that since u has integral components, all of its vertices of ϕ_e are integral as well. By Observation 4.10 and induction hypothesis, all vertices of $m_{e \rightarrow v}^t - \phi_e$ are integral. Therefore, all vertices of $m_{e \rightarrow v}^t$ are integral. \square

Corollaries 4.9 and 4.11 shows that at every iteration, each message function can be encoded in terms of a finite vector describing the corners and slopes of its linear pieces in finite number of iterations. These arguments extend easily to the form of linear program considered earlier. That is, BP for LP can be truly implemented on a computer.

The Corollary 4.12 provides a bound for the number of linear pieces in $m_{e \rightarrow v}^t$. This bound will help us bound the running time of BP algorithm for \mathcal{MCF} . We shall discuss this in detail in Section 7. Finally, we would like to note that the result that message functions $m_{e \rightarrow v}^t$ are piece-wise linear convex functions can be also shown by sensitivity analysis of LP, cf. [7, Chapter 5].

4.2 BP for a sub-class of \mathcal{MCF}

The Section 4.1 established that each message function is a piece-wise linear convex function. However, as per the bounds established, the number of pieces increase linearly with iterations and this requires more computation for message update as iterations grow. Now for an instance of \mathcal{MCF} with integral components of vector b and u , the message function $m_{e \rightarrow v}^t$ is a piece-wise linear convex function with integral vertices as per Corollary 4.13. Therefore, it has at most u_e linear pieces. Thus, if u_e is bounded by some constant for all e , the message functions at every iteration is piece-wise linear convex function with a bounded number of pieces. This results in a computationally efficient update of messages. Next, we present a sub-class of \mathcal{MCF} , denoted by \mathcal{MCF}^o , for which such property holds and which contains important classes of network flow problems.

To this end, given a directed graph $G = (V, E)$, consider the following sub-class of problem: with notation $\text{in}(v) = \{(u, v) \in E\}$

$$\begin{aligned}
& \text{minimize} && \sum_{e \in E} c_e x_e && (\mathcal{MCF}^o) \\
& \text{subject to} && \sum_{e \in E_v} \Delta(v, e) x_e = f_v, && \forall v \in V \quad (\text{demand/supply constraints}) \\
& && \sum_{e \in \text{in}(v)} x_e \leq \tilde{u}_v, && \forall v \in V \\
& && 0 \leq x_e \leq u_e. && \forall e \in E \quad (\text{flow constraints})
\end{aligned}$$

In above, c , u , and \tilde{u} are all integral. To see \mathcal{MCF}^o is indeed an instance of \mathcal{MCF} consider the following. Split each $v \in V$ into two vertices v_{in} and v_{out} , where v_{in} is incident to all in-arcs of v with $f_{v_{in}} = 0$ and v_{out} is incident to all out-arcs of v with $f_{v_{out}} = f_v$. Create an arc from v_{in} to v_{out} with capacity \tilde{u}_v and cost equal to 0. Denote thus created new graph as G^o . Then the \mathcal{MCF} on G^o is equivalent to \mathcal{MCF}^o . Instead of using the Algorithm 2 to solve the \mathcal{MCF} on G^o , we shall use it on G with the following functions ψ , ϕ :

$$\begin{aligned}
\psi_v(x) &= \begin{cases} 0 & \text{if } \sum_{e \in E_v} \Delta(v, e) x_e = f_v \text{ and } \sum_{e \in \text{in}(v)} x_e \leq \tilde{u}_v \\ \infty & \text{otherwise} \end{cases} \quad \forall v \in V, \\
\phi_e(x) &= \begin{cases} c_e x & \text{if } 0 \leq x \leq u_e \\ \infty & \text{otherwise} \end{cases} \quad \forall e \in E.
\end{aligned}$$

Now to update message functions $m_{e \rightarrow v}^t$ for all $e \in E_w$, the inequality $\sum_{e \in \text{in}(w)} x_e \leq \tilde{u}_w$ implies that it is sufficient to check \tilde{u}_w linear pieces from message functions $m_{\tilde{e} \rightarrow w}^{t-1}$ for all but constant number of $e \in E_w$. This leads to efficient implementation of BP for \mathcal{MCF}^o . Specifically, we state the following result.

Theorem 4.14. *Suppose the \mathcal{MCF}^o as described above has a unique optimal solution with*

$$\max_v \left(\tilde{u}_v, u_v, |f_v| \right) \leq K, \quad \max_e c_e \leq K.$$

Then Algorithm 2 for \mathcal{MCF}^o finds the unique optimal solution using $O(K^2 m n^2 \log n)$, which is $O(K^2 n^4 \log n)$, operations in total. As a result, Algorithm 2 is polynomial time when K is a constant.

The proof of Theorem 4.14 is presented in Section 7.1. It is worth taking note of the fact that both the shortest-path problem and maximum weight matching in a bipartite graph belong to the \mathcal{MCF}^o class of problems with all components of f, u being bounded by 2. For these two classes of problems we do not need the extra constraint $\sum_{e \in \text{in}(v)} x_e \leq \tilde{u}_v$, but we do need this constraint to make a general statement of the theorem. We see that under the uniqueness assumptions, BP solves these problems in polynomial (as opposed to just pseudo-polynomial) time.

5 Convergence of BP for \mathcal{MCF}

This section is devoted to establishing the convergence of BP to the optimal solution of the \mathcal{MCF} under the assumption of the uniqueness of the optimal solution, namely we shall prove Theorem 4.1. The outline of the proof is as follows. First, we define the notion of a computation tree T_e^N that is associated with each variable node x_e of \mathcal{MCF} for iteration N . We show that in fact the estimation \hat{x}_e^N under BP is the optimal solution of an appropriately defined \mathcal{MCF} problem on T_e^N (Lemma 5.1). Next, we show that the optimal assignment to x_e under the min-cost flow problem on the computation tree T_e^N is the same as the optimal assignment to x_e under the original \mathcal{MCF} as long as N is large enough (see Section 5.2). This immediately implies that BP finds the correct optimal solution for \mathcal{MCF} for large enough N leading to Theorem 4.1. We note that this strategy is similar to that of [5]. However, the technical details are quite different.

5.1 Computation Tree and BP

We start with the definition of computation tree. The N -level computation tree associated with arc $e = (v, w) \in E$ is denoted by T_e^N . It is essentially the breadth first search tree of G (with repetition of nodes allowed) starting from e up to depth N . Formally, computation tree T_e^N is defined inductively as follows. $T_e^0 = (V(T_e^0), E(T_e^0))$ is a tree with vertex set $V(T_e^0) = \{v', w'\}$ and arc set $E(T_e^0) = \{e' = (v', w')\}$. The v', w' are considered replicas of $v, w \in V$ and this is represented by a mapping $\Gamma_e^0 : V(T_e^0) \rightarrow V$ with $\Gamma_e^0(v') = v$ and $\Gamma_e^0(w') = w$. The arc e' is considered the “root” of T_e^0 and vertices v', w' are considered to be at level 0. Define w' (resp. v') as parent of v' (resp. w') denoted as $P(v') = w'$ (resp. $P(w') = v'$). Inductively, let us suppose that tree $T_e^N = (V(T_e^N), E(T_e^N))$ is defined with corresponding $\Gamma_e^N : V(T_e^N) \rightarrow V$ such that for $u'_1, u'_2 \in V(T_e^N)$, $(u'_1, u'_2) \in E(T_e^N)$ only if $(\Gamma_e^N(u'_1), \Gamma_e^N(u'_2)) \in E$. Let $P : V(T_e^N) \rightarrow V(T_e^N)$ represent the parent relation in T_e^N . Let $L(T_e^N)$ be the set of leaves¹ of T_e^N . Now we shall define $T_e^{N+1} = (V(T_e^{N+1}), E(T_e^{N+1}))$ which contains T_e^N as a sub-tree. Specifically, $V(T_e^{N+1})$ and $E(T_e^{N+1})$ are obtained by adding vertices to $V(T_e^N)$ and arcs to $E(T_e^N)$ as follows. For each leaf node $u' \in L(T_e^N)$, add node \tilde{u}' to expand $V(T_e^N)$ and add arc (u', \tilde{u}') or (\tilde{u}', u') to expand $E(T_e^N)$ if (a) there is a node $\tilde{u} \in V$ so that (u, \tilde{u}) or (\tilde{u}, u) is in E with $\Gamma_e^N(u') = u$, and (b) $\Gamma_e^N(P(u')) \neq \tilde{u}$. In this case, define $P(\tilde{u}') = u'$, the map $\Gamma_e^{N+1}(\tilde{u}') = \tilde{u}$ and level of \tilde{u}' as $N+1$. Indeed, Γ_e^{N+1} is identical to Γ_e^N for nodes $V(T_e^N) \subset V(T_e^{N+1})$. In what follows, we shall drop reference to e, N in notation of Γ_e^N when clear from context and abuse notation by denoting $\Gamma(e' = (u'_1, u'_2)) = (\Gamma(u'_1), \Gamma(u'_2))$.

Sometimes T_e^N is also called ‘unwrapped tree’ of G rooted at e . Figure 4 gives an example of a computation tree. It should be noted that the definition of computation tree may appear slightly different compared to that in related works such as [4], [5], [28] (arc is root here in contrast to a vertex as root). However, the utility of the computation trees is very similar.

¹A vertex v' is called leaf if it is connected to exactly one other vertex.

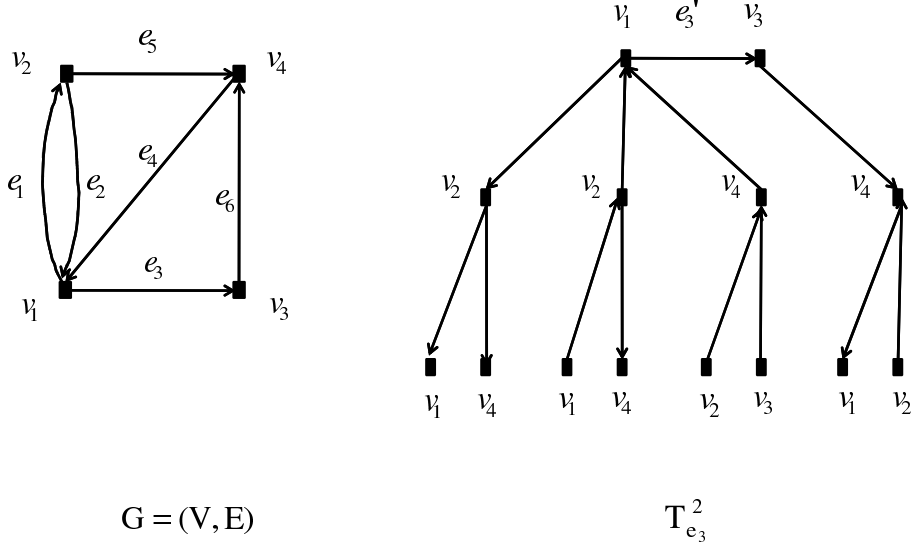


Figure 4: Computation tree of G rooted at $e_3 = (1, 3)$

Now we are ready to relate the computation tree with the BP. Let $V^o(T_e^N) \subset V(T_e^N)$ denote the set of all the vertices which are not on the N -th level of T_e^N . Consider the problem

$$\begin{aligned}
 & \text{minimize} && \sum_{\tilde{e} \in E(T_e^N)} c_{\Gamma(\tilde{e})} x_{\tilde{e}} && (\mathcal{MCF}_e^N) \\
 & \text{subject to} && \sum_{\tilde{e} \in E_{u'}} \Delta(u', \tilde{e}) x_{\tilde{e}} = f_{\Gamma(u')}, \forall u' \in V^o(T_e^N) \\
 & && 0 \leq x_{\tilde{e}} \leq u_{\Gamma(f)}, \forall f \in E(T_e^N).
 \end{aligned}$$

In above, $E_{u'} \subset E(T_e^N)$ is the set of arcs incident on $u' \in V^o(T_e^N)$ in T_e^N and $\Delta(u', \tilde{e})$ for $\tilde{e} \in E_{u'}$ is defined as -1 or $+1$ depending upon whether e' is in-arc or out-arc for node u' . Loosely speaking, \mathcal{MCF}_e^N is essentially an \mathcal{MCF} on T_e^N : there is a flow constraint for every arc $\tilde{e} \in E(T_e^N)$ and a demand/supply constraint for every node, except for the nodes on the N th level. Now, we state the following well known result which exhibits the connection between BP and the computation trees.

Lemma 5.1. *Let \hat{x}_e^N be the value produced by BP at the end of iteration N for the flow value on edge $e \in E$. Then there exists an optimal solution y^* of \mathcal{MCF}_e^N such that $y_{e'}^* = \hat{x}_e^N$ where e' is the root of T_e^N (and $\Gamma(e') = e$).*

Proof. Let $e' = (v', w')$ be the root arc of computation tree T_e^N with $e = (v, w)$ such that $\Gamma(e') = e, \Gamma(v') = v$ and $\Gamma(w') = w$. By definition, T_e^N has two components connected via the root arc e' . Let C be the component containing w' and $T_{e' \rightarrow v'}^N$ denote the C with edge e' ; indeed $T_{e' \rightarrow v'}^N$ is a

tree. As before, let $V^o(T_{e' \rightarrow v'}^N)$ be the set of all nodes excluding those at the N th level. Define

$$\begin{aligned}
& \text{minimize} && \sum_{\tilde{e} \in E(T_{e' \rightarrow v'}^N)} c_{\Gamma(\tilde{e})} x_{\tilde{e}} && (\mathcal{MCF}_{e' \rightarrow v'}^N(z)) \\
& \text{subject to} && \sum_{\tilde{e} \in E_{q'}} \Delta(q', \tilde{e}) x_{\tilde{e}} = f_{\Gamma(q')}, \forall q' \in V^o(T_{e' \rightarrow v'}^N) \\
& && x_{e'} = z, \\
& && 0 \leq x_{\tilde{e}} \leq u_{\Gamma(\tilde{e})}, \forall \tilde{e} \in E(T_{e' \rightarrow v'}^N).
\end{aligned}$$

Now, we shall establish that under the BP algorithm (running on G) the value of message function from $e \rightarrow v$ evaluated at z , that is $m_{e \rightarrow v}^N(z)$, is the same as the cost of the optimal assignment for $\mathcal{MCF}_{e' \rightarrow v'}^N(z)$. This can be established inductively. To start with, for $N = 1$, the statement can be checked to be true trivially. For $N > 1$, let $E_{w'}$ denote the edges incident on w' in T_e^N where recall $e' = (v', w')$ is it's root arc. Then for each $g' \in E_{w'} \setminus e'$ with $g' = (u', w')$ (or (w', u')), let $T_{g' \rightarrow w'}^{N-1}$ be the subtree of $T_{e' \rightarrow v'}^N$ that includes g' and everything in $T_{e' \rightarrow v'}^N$ that is part of it's component that does not include w' . Define optimization problem

$$\begin{aligned}
& \text{minimize} && \sum_{\tilde{e} \in E(T_{g' \rightarrow w'}^{N-1})} c_{\Gamma(\tilde{e})} x_{\tilde{e}} && (\mathcal{MCF}_{g' \rightarrow w'}^{N-1}(z)) \\
& \text{subject to} && \sum_{\tilde{e} \in E_{q'}} \Delta(q', \tilde{e}) x_{\tilde{e}} = f_{\Gamma(q')}, \forall q' \in V^o(T_{g' \rightarrow w'}^{N-1}) \\
& && x_{g'} = z, \\
& && 0 \leq x_{\tilde{e}} \leq u_{\Gamma(\tilde{e})}, \forall \tilde{e} \in E(T_{g' \rightarrow w'}^{N-1}).
\end{aligned}$$

By induction hypothesis, it must be that $m_{g' \rightarrow w'}^{N-1}(z)$ equals the cost of the solution of $\mathcal{MCF}_{g' \rightarrow w'}^{N-1}(z)$. Given this hypothesis and the relation of sub-tree $T_{g' \rightarrow w'}^{N-1}$ for all $g' \in E_{w'} \setminus e'$ with $T_{e' \rightarrow v'}^N$, it follows that the optimization problem $\mathcal{MCF}_{e' \rightarrow v'}^N(z)$ is equivalent to

$$\begin{aligned}
& \text{minimize} && c_e z + \sum_{g' \in E_{w'} \setminus e'} m_{\Gamma(g') \rightarrow \Gamma(w')}^{N-1}(x_{g'}) \\
& \text{subject to} && \Delta(w', e') z + \sum_{g' \in E_{w'} \setminus e'} \Delta(w', g') x_{g'} = f_{\Gamma(w')} \\
& && 0 \leq x_{g'} \leq u_{\Gamma(g')}, \forall g' \in E_{w'} \setminus e'.
\end{aligned}$$

This is exactly the same as the relation between $m_{e \rightarrow v}^N(z)$ and message function $m_{g \rightarrow w}^{N-1}(\cdot)$ for $g \in E_w \setminus e$ as defined by BP. That is, $m_{e \rightarrow v}^N(z)$ is exactly the same as the cost of optimal assignment of $\mathcal{MCF}_{e' \rightarrow v'}^N$. We shall use this equivalence, to complete the proof of Lemma 5.1.

To that end, for given $e = (v, w)$ with $0 \leq z \leq u_e$, the optimization problem $\mathcal{MCF}_e^N(z)$ is

equivalent to

$$\begin{aligned}
& \text{minimize } c_e z + \sum_{\tilde{e} \in E(T_{e' \rightarrow v'}^N)} c_{\Gamma(\tilde{e})} x_{\tilde{e}} + \sum_{\tilde{e} \in E(T_{e' \rightarrow w'}^N)} c_{\Gamma(\tilde{e})} x_{\tilde{e}} \\
& \text{subject to } \sum_{\tilde{e} \in E_{q'}} \Delta(q', \tilde{e}) x_{\tilde{e}} = f_{\Gamma(q')}, \quad \forall q' \in V^o(T_e^N) \cap \left(V(T_{e' \rightarrow v'}^N) \cup V(T_{e' \rightarrow w'}^N) \right) \\
& \quad 0 \leq x_{\tilde{e}} \leq u_{\Gamma(\tilde{e})}, \quad \tilde{e} \in E(T_{e' \rightarrow v'}^N) \cup E(T_{e' \rightarrow w'}^N).
\end{aligned}$$

That is, the cost of an optimal assignment of $\mathcal{MCF}_e^N(z)$ equals $m_{e \rightarrow u}^N(z) + m_{e \rightarrow v}^N(z) + c_e z$ for any $0 \leq z \leq u_e$. Now the claim of Lemma 5.1 follows immediately. \square

5.2 Proof of theorem 4.1

Now we are ready to establish Theorem 4.1. Suppose to the contrary that there exists $e_0 = (v_\alpha, v_\beta) \in E$ and $N \geq (\lfloor \frac{L}{2\delta(x^*)} \rfloor + 1)n$ such that $\hat{x}_{e_0}^N \neq x_{e_0}^*$. By Lemma 5.1, there exists an optimal solution y^* of $\mathcal{MCF}_{e_0}^N$ such that $y_{e'_0}^* = \hat{x}_{e_0}^N$. Without loss of generality, assume $y_{e'_0}^* > x_{e_0}^*$. Using the optimality of x^* , we will show that it is possible to modify y^* to obtain a feasible solution of $\mathcal{MCF}_{e_0}^N$ with cost strictly lower than that of y^* . This will lead to contradiction to the assumption that $\hat{x}_{e_0}^N \neq x_{e_0}^*$ and establish the result.

To that end, let $e'_0 = (v'_\alpha, v'_\beta)$ be the root edge of the computation tree $T_{e_0}^N$ as discussed earlier. Because y^* is a feasible solution of $\mathcal{MCF}_{e_0}^N$ and x^* is a feasible solution of \mathcal{MCF} ,

$$\begin{aligned}
f_{\Gamma(v'_\alpha)} &= \sum_{\tilde{e} \in E_{v'_\alpha}} \Delta(v'_\alpha, \tilde{e}) y_{\tilde{e}}^* = y_{e'_0}^* + \sum_{\tilde{e} \in E_{v'_\alpha} \setminus e'_0} \Delta(v'_\alpha, \tilde{e}) y_{\tilde{e}}^* \quad (\text{constraint at } v'_\alpha \text{ in } \mathcal{MCF}_{e_0}^N) \\
f_{\Gamma(v'_\alpha)} &= \sum_{\tilde{e} \in E_{\Gamma(v'_\alpha)}} \Delta(\Gamma(v'_\alpha), \tilde{e}) x_{\tilde{e}}^* = x_{e_0}^* + \sum_{\tilde{e} \in E_{\Gamma(v'_\alpha)} \setminus e_0} \Delta(\Gamma(v'_\alpha), \tilde{e}) x_{\tilde{e}}^* \quad (\text{constraint at } \Gamma(v'_\alpha) \text{ in } \mathcal{MCF}).
\end{aligned}$$

Note that the edges in $E_{v'_\alpha}$ in the computation tree $T_{e_0}^N$ are copies of edges in E_{v_α} in G where $v_\alpha = \Gamma(v'_\alpha)$. Therefore, $\Delta(v'_\alpha, \tilde{e}) = \Delta(\Gamma(v_\alpha), \Gamma(\tilde{e}))$ for $\tilde{e} \in E_{v'_\alpha}$. Therefore, from above inequalities, it follows that since $y_{e'_0}^* > x_{e_0}^*$, there exists arc $e'_1 \neq e'_0$ incident on v'_α in $T_{e_0}^N$ such that $\Delta(v'_\alpha, e'_1)(x_{\Gamma(e'_1)}^* - y_{e'_1}^*)$ is strictly positive. Therefore, if $\Delta(v'_\alpha, e'_1) = 1$ then $x_{\Gamma(e'_1)}^* > y_{e'_1}^*$ else $x_{\Gamma(e'_1)}^* < y_{e'_1}^*$. That is, if edge e'_1 has the opposite orientation with respect to e'_0 at node v'_α (both are outgoing from v'_α and hence opposite orientation), then $x_{\Gamma(e'_1)}^* > y_{e'_1}^*$ else $x_{\Gamma(e'_1)}^* < y_{e'_1}^*$. The Figure 5 explains this by means of a simple example.

More generally, using similar argument we can find arc $e'_{-1} \neq e'_0$ incident to v'_β satisfying similar condition. Let $v'_{\alpha_1}, v'_{\alpha_{-1}}$ be the other end points of e'_1, e'_{-1} respectively. A recursive application of similar argument utilizing the feasibility condition of x^* and y^* and the inequalities between value of components of x^* and y^* at edges e'_1 and e'_{-1} , leads to existence of arcs e'_2, e'_{-2} incident on $v'_{\alpha_1}, v'_{\alpha_{-1}}$ respectively so that $x_{e'_2}^* \neq y_{e'_2}^*$ and $x_{e'_{-2}}^* \neq y_{e'_{-2}}^*$ with inequalities being $<$ or $>$ depending upon the orientation of the edges with respect to e_0 . Continuing further in this manner all the way down to the leaves, it is possible to find arcs $\{e'_{-N}, e'_{-N+1}, \dots, e'_{-1}, e'_1, \dots, e'_N\}$ such that for $-N \leq i \leq N$,

$$\begin{aligned}
y_{e'_i}^* &> x_{\Gamma(e'_i)}^* \iff e'_i \text{ has the same orientation as } e_0, \\
y_{e'_i}^* &< x_{\Gamma(e'_i)}^* \iff e'_i \text{ has the opposite orientation as } e_0.
\end{aligned}$$

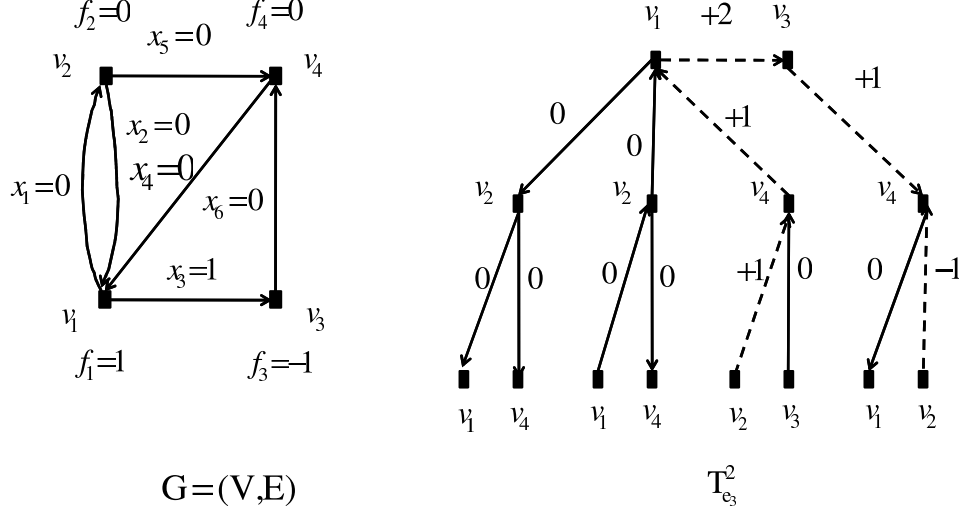


Figure 5: An example of Augmenting path between the flow assignment on computation tree $T_{e_3}^2$ and the flow assignment on G . The dashed edges represent the edges belonging to the augmenting path. Root edge and edge from v_4 to v_1 have same orientation.

Let us denote the path containing these edges as $X = \{e'_{-N}, e'_{-N+1}, \dots, e'_{-1}, e'_0, e'_1, \dots, e'_N\}$. For any $e' = (v'_p, v'_q) \in X$, define $Aug(e') = (v'_p, v'_q)$ if $y_{e'}^* > x_{\Gamma(e')}^*$, and $Aug(e') = (v'_q, v'_p)$ if $y_{e'}^* < x_{\Gamma(e')}^*$. Given the feasibility conditions of y^* and definition of $Aug(e')$, it can be checked that $\Gamma(Aug(e'))$ is an arc in the residual graph $G(x^*)$. The directed path $W = (Aug(e'_{-N}), \dots, Aug(e'_0), \dots, Aug(e'_N))$ on $T_{e_0}^N$ will be called the *augmenting* path of y^* with respect to x^* . Also, $\Gamma(W)$ is a directed walk on $G(x^*)$. Now we can decompose $\Gamma(W)$ into a simple directed path P and a collection of simple directed cycles C_1, \dots, C_k . Now each simple directed cycle or path on $G(x^*)$ can have at most n edges. Since W has $2N + 1$ arcs and $N \geq (\lfloor \frac{L}{2\delta(x^*)} \rfloor + 1)n$, it follows that $k > \frac{L}{\delta(x^*)}$. Now the cost of path P , denoted by $c^*(P)$, with respect to the residual graph $G(x^*)$ is at least $-L$ (and at most L) by definition of L . Since each C_i is a simple cycle in $G(x^*)$, by definition it's cost, denoted by $c^*(C_i)$ with respect to $G(x^*)$ is at least $\delta(x^*)$; $\delta(x^*) > 0$ since x^* is the unique optimal solution. Therefore, as explained below we obtain that the cost of W is strictly positive:

$$\begin{aligned}
\sum_{i=-N}^N c_{\Gamma(e'_i)}^* &= c^*(W) \\
&= c^*(P) + \sum_{j=1}^k c^*(C_j) \\
&\geq -L + k\delta(x^*) \\
&> -L + \frac{L}{\delta(x^*)}\delta(x^*) = 0.
\end{aligned}$$

Let $FWD = \{e \in X : y_e^* > x_{\Gamma(e)}^*\}$, $BCK = \{e \in X : y_e^* < x_{\Gamma(e)}^*\}$. Since both FWD and BCK are finite, there exists $\lambda > 0$ such that $y_e^* - \lambda \geq x_{\Gamma(e)}^*$, $\forall e \in FWD$ and $y_e^* + \lambda \leq x_{\Gamma(e)}^*$, $\forall e \in BCK$. Define

$\tilde{y} \in \mathbb{R}^{|E(T_{e_0}^N)|}$ as

$$\tilde{y}_e = \begin{cases} y_e^* - \lambda & e \in \text{FWD} \\ y_e^* + \lambda & e \in \text{BCK} \\ 0 & \text{otherwise.} \end{cases}$$

The \tilde{y} can be thought of as flow that is obtained by pushing λ units of additional flow along path W over the existing flow y^* in $T_{e_0}^N$. Since for each $e \in \text{FWD}$, $y_e^* - \lambda \geq x_{\Gamma(e)}^* \geq 0$ and for each $e \in \text{BCK}$, $y_e^* + \lambda \leq x_{\Gamma(e)}^* \leq u_{\Gamma(e)}$, \tilde{y} satisfies all the flow constraints. Further since all edges in FWD have the same orientation as e_0 and those in BCK have the opposite orientation compared to e_0 , we have that for any $v' \in V^o(T_{e_0}^N)$,

$$\begin{aligned} \sum_{e' \in E_{v'}} \Delta(v', e') \tilde{y}_{e'} &= \sum_{e' \in E_{v'}} \Delta(v', e') y_{e'}^* \\ &= f_{\Gamma(e')}, \end{aligned}$$

which implies that \tilde{y} satisfies all the demand/supply constraints. Therefore, \tilde{y} is a feasible solution of $\mathcal{MCF}_{e_0}^N$. Now

$$\begin{aligned} \sum_{e' \in E(T_{e_0}^N)} c_{\Gamma(e')} y_{e'}^* - \sum_{e' \in E(T_{e_0}^N)} c_{\Gamma(e')} \tilde{y}_{e'} &= \sum_{e' \in E(T_{e_0}^N)} c_{\Gamma(e')} (y_{e'}^* - \tilde{y}_{e'}) \\ &= \sum_{e' \in \text{FWD}} c_{\Gamma(e')} \lambda - \sum_{e' \in \text{BCK}} c_{\Gamma(e')} \lambda \\ &= c^*(W) \lambda \\ &> 0. \end{aligned}$$

In above we have used the fact that $c_{\Gamma(e')}^* = c_{\Gamma(e')}$ for $e' \in \text{FWD}$ and $c_{\Gamma(e')}^* = -c_{\Gamma(e')}$ for $e' \in \text{BCK}$. The above contradicts the optimality of y^* . Therefore, the assumption about BP estimate not converging is false. This completes the proof of Theorem 4.1.

5.3 Detection of uniqueness of optimal solution using BP

In this section, we establish an unusual property of BP in terms of its ability to detect the uniqueness of optimal solution in the \mathcal{MCF} in distributed manner as long as the input parameters c , f and u are integral. We state this as the following Corollary of Theorem 4.1.

Corollary 5.2. *Consider an instance of \mathcal{MCF} with integral c , f and u . Suppose $c_{\max} = \max_{e \in E} c_e$. Suppose the BP Algorithm 2 runs for $N = n^2 c_{\max} + n$ iterations. Let $z_e^* \in \arg \min b_e^N(z)$. Then*

$$\forall e \in E, \min(b_e^N(z_e^* - 1), b_e^N(z_e^* + 1)) > n c_{\max} + b_e^N(z_e^*) \quad (11)$$

if and only if the \mathcal{MCF} instance has a unique solution.

Proof. We first establish the implication that if \mathcal{MCF} has a unique optimal solution then (11) holds. To that end, let us suppose that the instance of \mathcal{MCF} of interest has a unique solution.

Consider any edge $e \in E$ and its computation tree T_e^N . Then from Lemma 5.1 it follows that z_e^* is an optimal assignment of the root edge e' of T_e^N with respect to the associated optimization problem \mathcal{MCF}_e^N . Now suppose y is an optimal solution of \mathcal{MCF}_e^N with the additional constraint that flow on the root edge e' of T_e^N , denoted by $y_{e'}$ is fixed to value $z_e^* - 1$. Then, using arguments similar to those used in the proof of Theorem 4.1, it can be shown that there exists an augmenting path W of y with respect to z^* of length $2n^2c_{\max}$ in $T_{e_0}^N$. As before, W can be decomposed into at least $2nc_{\max}$ disjoint simple cycles and a simple path. Now each cycle has a cost of at least $\delta(x^*)$, which is at least 1 as \mathcal{MCF} has integral data. Since the \mathcal{MCF} and \mathcal{MCF}_e^N have integral parameters, the y and x^* can be restricted to be integral. Therefore, the augmenting path W must allow for pushing at least unit amount of flow to modify y to result in the decrease of its cost by at least nc_{\max} . This is because (a) the increase, due to pushing unit amount of flow on the simple path, could be at most nc_{\max} , and (b) decrease along (at least) $2nc_{\max}$ cycles is at least $2nc_{\max}$. In summary, the modified solution is feasible for \mathcal{MCF}_e^N on T_e^N with cost decreased by at least nc_{\max} . Therefore, it would follow that the optimal cost $b_e^N(z_e^*)$ for \mathcal{MCF}_e^N is less than $b_e^N(z_e^* - 1) - nc_{\max}$. In a very similar manner, it can be argued that $b_e^N(z_e^*) < b_e^N(z_e^* + 1) - nc_{\max}$. This concludes that $\min(b_e^N(z_e^* + 1), b_e^N(z_e^* - 1))$ is at least $b_e^N(z_e^*) + nc_{\max}$.

To establish the other side of the equivalence, suppose \mathcal{MCF} does not have a unique optimal solution. Consider any arc $e \in E$, corresponding computation tree T_e^N and optimization problem \mathcal{MCF}_e^N . Let e' be the root arc of T_e^N as before. Let y be the optimal assignment of \mathcal{MCF}_e^N with the assignment for root arc e' being $y_{e'} = z_e^*$. Now since \mathcal{MCF} has multiple optimal solution, there exists another optimal assignment x^* of \mathcal{MCF} so that $x_e^* \neq z_e^*$. Indeed given that both \mathcal{MCF}_e^N and \mathcal{MCF} are integral, we can restrict our attention to z^* , x^* and y having integral components. Since $x_e^* \neq z_e^*$, using arguments similar to those used in the proof of Theorem 4.1, it is indeed possible to find an augmenting path W , of length $2N$, on T_e^N with respect to y and x^* . This augmenting path decomposes into one simple path P of length at most $n - 1$ and at least $2nc_{\max}$ simple cycles. Since x^* is an optimal solution, the cost of each of the cycles with respect to the residual graph $G(x^*)$ is non-positive (it is not strictly negative like the proof of Theorem 4.1 since the x^* is not unique). The cost of the path, however is between $-(n - 1)c_{\max}$ and $(n - 1)c_{\max}$. Therefore, by pushing unit amount of flow (which is possible along this augmenting path W due to integrality of x^* and y), the resulting flow \tilde{y} on T_e^N is such that its total cost is at most $(n - 1)c_{\max}$ more than the cost of y . Now either $\tilde{y}_{e'} = z_e^* - 1$ or $z_e^* + 1$. Suppose $\tilde{y}_{e'} = z_e^* - 1$. In that case, the \tilde{y} is a feasible solution of \mathcal{MCF}_e^N with additional constraint that the root arc e' has flow $z_e^* - 1$. This cost is no less than the cost of an optimal solution of \mathcal{MCF}_e^N with additional constraint that the root arc e' has flow $z_e^* - 1$, which is defined as $b_e^N(z_e^* - 1)$. Putting all together, we obtain

$$b_e^N(z_e^* - 1) \leq b_e^N(z_e^*) + nc_{\max}.$$

In a similar manner, if $\tilde{y}_{e'} = z_e^* + 1$ then we would conclude that

$$b_e^N(z_e^* + 1) \leq b_e^N(z_e^*) + nc_{\max}.$$

That is, we have established that if \mathcal{MCF} does not have a unique optimal solution then

$$\min(b_e^N(z_e^* - 1), b_e^N(z_e^* + 1)) \leq b_e^N(z_e^*) + nc_{\max}.$$

This completes the proof of the other side of equivalence and hence the proof of Corollary 5.2. \square

6 Network Flow: Piece-wise Linear Convex Objective

This section describes the extension of Theorem 4.1 for network flow problem with piece-wise linear convex objective or cost function. Specifically, given a graph $G = (V, E)$ as before, consider

$$\begin{aligned} & \text{minimize} \quad \sum_{e \in E} c_e(x_e) & (\mathcal{CP}) \\ & \text{subject to} \quad \sum_{e \in E_v} \Delta(v, e)x_e = f_v, \quad \forall v \in V \quad (\text{demand/supply constraints}) \\ & \quad \quad \quad 0 \leq x_e \leq u_e, \quad \forall e \in E \quad (\text{non-negativity constraints}), \end{aligned}$$

where $c_e : \mathbb{R} \rightarrow \mathbb{R}$ is a piece-wise linear convex function for each $e \in E$. As before, we shall assume that the \mathcal{CP} is feasible. Let ψ be the same as before and define

$$\phi_e(z) = \begin{cases} c_e(z) & \text{if } 0 \leq z \leq u_e \\ \infty & \text{otherwise.} \end{cases}$$

The Algorithm 2 on G with functions ψ and ϕ thus defined is the BP for this problem instance. Before we state our result, we need to define the corresponding residual graph. Suppose x is a feasible solution for \mathcal{CP} . Define the residual graph of G and x , denoted by $G(x)$ as follows: $\forall e = (v_\alpha, v_\beta) \in E$, if $x_e < u_e$, then e is an arc in $G(x)$ with cost $c_e^x = \lim_{t \downarrow 0} \frac{c(x_e+t) - c(x_e)}{t}$; if $x_e > 0$, then there is an arc $e' = (v_\beta, v_\alpha)$ in $G(x)$ with cost $c_{e'}^x = \lim_{t \downarrow 0} \frac{c(x_e) - c(x_e-t)}{t}$. Finally, let

$$\delta(x) = \min_{C \in \mathcal{C}} \left\{ \sum_{e \in C} c_e^x \right\},$$

where \mathcal{C} is the set of all directed simple cycles in $G(x)$. We state result about convergence property of BP.

Theorem 6.1. *Suppose x^* is the unique optimal solution for \mathcal{CP} and hence $\delta(x^*) > 0$. Let L to be the maximum cost of a simple directed path in $G(x^*)$. Then, for any $N \geq (\lfloor \frac{L}{2\delta(x^*)} \rfloor + 1)n$, $\hat{x}^N = x^*$.*

The proof of Theorem 6.1 is identical to that of Theorem 4.1 with the above defined notions. Therefore, we shall skip it.

7 Integral \mathcal{MCF} : Run-time analysis of BP

In the next two sections, we shall consider \mathcal{MCF} with integral components for c , u and f . Our goal is to analyze the run-time of BP for such integral \mathcal{MCF} .

Lemma 7.1. *For an integral \mathcal{MCF} , the total number of operations performed by Algorithm 2 to update all the messages at iteration t is $O(tc_{\max} m \log n)$.*

Proof. Recall that, for edge $e \in E$ with v as one of its end point (and w at the other), message function is updated as

$$m_{e \rightarrow v}^t(z) = \phi_e(z) + \min_{\bar{z} \in \mathbb{R}^{|E_w|}, \bar{z}_e = z} \left\{ \psi_w(\bar{z}) + \sum_{\bar{e} \in E_w \setminus e} m_{\bar{e} \rightarrow w}^{t-1}(\bar{z}_{\bar{e}}) \right\}.$$

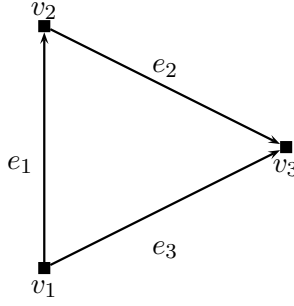


Figure 6:

From Corollary 4.12, all the message functions have integral slopes for an instance of \mathcal{MCF} with integral components. The absolute values of these slopes are bounded by $(t-1)c_{\max}$. This implies that each (convex piece-wise linear) message (function) has at most $2(t-1)c_{\max}$ linear pieces. By Corollary 4.9 and Observation 4.10 it follows that $g(z)$ can be computed in $O(tc_{\max}|E_w|\log|E_w|) = O(tc_{\max}|E_w|\log n)$ total operations since $|E_w| \leq n$. Here

$$g(z) = \min_{\bar{z} \in \mathbb{R}^{|E_w|}, \bar{z}_e = z} \left\{ \psi_w(\bar{z}) + \sum_{\tilde{e} \in E_w \setminus e} m_{\tilde{e} \rightarrow w}^{t-1}(\bar{z}_{\tilde{e}}) \right\}.$$

Now computing $g(z) + \phi_e(z)$ is a simple procedure which requires increasing the slopes of linear pieces of $g(z)$ by a constant. Since $g(\cdot)$ has at most $2tc_{\max}$ linear pieces, computing $g(z) + \phi_e(z)$ takes further $O(tc_{\max})$ operations. In summary, it follows that all message updates can be performed in total of $O(tc_{\max}m \log n)$ operations since $\sum_w |E_w| = \Theta(m)$. \square

We now complete the proof of Theorem 4.2.

Proof of Theorem 4.2. The integral instance of \mathcal{MCF} with unique optimal solution has $\delta(x^*) \geq 1$. Therefore by Theorem 4.1, the BP Algorithm 2 converges after at most $O(nL)$ iterations. By Lemma 7.1, the total computation performed up to iteration t is $O(m \log nc_{\max} t^2)$. Therefore, the total computation performed till convergence is $O(m \log nc_{\max} n^2 L^2)$. The L can be bounded as $L = O(nc_{\max})$. Therefore, it follows that the overall cost is at most $O(mn^4 c_{\max}^3 \log n)$. \square

The bound of Theorem 4.2 is pseudo-polynomial time. In fact qualitatively this is the best bound one can hope for. To see this, consider an example of \mathcal{MCF} defined on a directed graph G as shown in Figure 6. Given large integer D , set the costs of edges as $c_{e_1} = c_{e_2} = D$, $c_{e_3} = 2D - 1$; demands as $b_{v_1} = 1$, $b_{v_2} = 0$ and $b_{v_3} = -1$. It can be checked that \hat{x}_1^N alternates between 1 and -1 when $2N + 1 < \frac{2D}{3}$. This means that BP algorithm takes at least $\Omega(D)$ iterations to converge. Since the input size is $\Theta(\log D)$, we have that Algorithm 2 for \mathcal{MCF} does not converge to the unique optimal solution in polynomial-time in the size of the input.

7.1 Runtime of BP for integral \mathcal{MCF}^o

Here we analyze the run time of BP for integral \mathcal{MCF}^o , the subclass of \mathcal{MCF} defined in Section 4.2 and prove Theorem 4.14.

Proof of Theorem 4.14. Since \mathcal{MCF}^o is an instance of \mathcal{MCF} with integral components and unique optimal solution, Theorem 4.1 it follows that the BP Algorithm 2 converges to the optimal solution within $O(Ln)$ iterations. To bound computation performed in each iteration and subsequently

bound overall computation cost, without loss of generality we shall assume that the piece-wise linear convex message function is such that each linear piece is of unit length. This assumption is without loss of generality, as each linear piece has integral vertices from Corollary 4.13 and hence assumption of each piece being unit length only leads to upper bound on computation. Now each message function is defined on a uniformly bounded interval due to uniform bound K on capacity of each edge in \mathcal{MCF}^0 . Therefore, the number of pieces in each piece-wise linear convex message function is bounded by $K + 1$. Recall that for $t \geq 1$,

$$m_{e \rightarrow v}^t(z) = \phi_e(z) + \min_{\bar{z} \in \mathbb{R}^{|E_w|}, \bar{z}_e = z} \left\{ \psi_w(\bar{z}) + \sum_{\tilde{e} \in E_w \setminus e} m_{\tilde{e} \rightarrow w}^{t-1}(\bar{z}_{\tilde{e}}) \right\}.$$

As explained in detail in Section 4.1, specifically Lemma 4.6 and Theorem 4.9, computing $m_{e \rightarrow v}^t$ takes at most $O(K \log |E_w|)$ which is $O(K \log n)$ as $|E_w| \leq n$ for all w . Since there are at most $O(m)$ messages, total computation per iteration is $O(Km \log n)$. As discussed earlier, it takes $O(Ln)$ iterations for the algorithm to converge. Therefore, overall computation scales $O(KLmn \log n)$. Finally, due to uniform bound of K on cost of edges, $L = O(nc_{\max}) = O(nK)$. In summary, the total computation cost is bounded above by $O(K^2 mn^2 \log n)$. \square

8 FPRAS for \mathcal{MCF} using BP

In this section, we provide a fully polynomial-time randomized approximation scheme (FPRAS) for \mathcal{MCF} using BP as a subroutine. As mentioned earlier, we shall assume integral \mathcal{MCF} . We start by describing the insights behind the algorithm followed by precise description in Section 8.2. To this end, recall that the key hurdles in making BP fully polynomial-time as indicated by Theorem 4.2 are the following:

1. The convergence of BP requires \mathcal{MCF} to have a unique optimal solution.
2. The running time of BP is polynomial in m, n and c_{\max} .

Therefore, to find FPRAS for any given instance of \mathcal{MCF} we need to overcome the requirement of uniqueness and dependence over c_{\max} of running time. To do so, we shall utilize appropriate randomized modification of cost vector so that the resulting problem with modified cost vector \bar{c} has the following properties:

1. The modified problem has a unique optimal solution with high probability.
2. The modified cost vector has \bar{c}_{\max} polynomial in m, n and $\frac{1}{\varepsilon}$.
3. The optimal solution of the modified problem provides $1 + \varepsilon$ multiplicative approximation to the optimal solution of \mathcal{MCF} .

It seems intuitive that by adding enough randomness to cost vector, the modified problem will have unique solution with high probability. However, requiring the resulting cost vector to be polynomially small in m, n and $1/\varepsilon$ as well as having small approximation error is challenging and a priori not clear if it is even feasible. The so called Isolation Lemma introduced in [21] helps to address precisely this question for a specific class of combinatorial problems including matching. It

is not directly applicable to our setup primarily because the Isolation Lemma requires the feasible set of optimization problem to be a monotone subset of $\{0,1\}^M$ (for appropriate M) while the feasible set of interest here is a polytope derived from a linear programming problem. For this reason we state and prove a variation of Isolation Lemma for our setup next.

8.1 Variation of the Isolation Lemma

Theorem 8.1. *Let $\overline{\mathcal{MCF}}$ be an instance of min-cost flow problem with underlying graph $G = (V, E)$, demand vector b , constraint vector u . Let its cost vector \bar{c} be generated as follows: for each $e \in E$, \bar{c}_e is chosen independently and uniformly over N_e , where N_e is a discrete set of $4m$ positive numbers ($m = |E|$). Then, the probability that $\overline{\mathcal{MCF}}$ has a unique optimal solution is at least $\frac{1}{2}$.*

Proof. Fix an arc $e_1 \in E$ and fix \bar{c}_e for all $e \in E \setminus e_1$. First suppose there exists a value $\alpha \geq 0$ such that when $\bar{c}_{e_1} = \alpha$, $\overline{\mathcal{MCF}}$ has two optimal solutions x^* , x^{**} and, moreover, $x_{e_1}^* = 0$ and $x_{e_1}^{**} > 0$. Then, if $\bar{c}_{e_1} > \alpha$, for any feasible solution x of $\overline{\mathcal{MCF}}$ with $x_{e_1} > 0$,

$$\begin{aligned} \sum_{e \in E} \bar{c}_e x_e^* &= \sum_{e \in E, e \neq e_1} \bar{c}_e x_e^* \\ &\stackrel{(a)}{\leq} \sum_{e \in E, e \neq e_1} \bar{c}_e x_e + x_{e_1} \alpha \\ &\stackrel{(b)}{<} \sum_{e \in E} \bar{c}_e x_e. \end{aligned}$$

In above, (a) follows from the fact that x^* is optimal with $\bar{c}_{e_1} = \alpha$; (b) follows $\bar{c}_{e_1} > \alpha$ and $x_{e_1} > 0$. On the other hand, if $\bar{c}_{e_1} < \alpha$, then for any feasible solution x of $\overline{\mathcal{MCF}}$ where $x_{e_1} = 0$, we have

$$\begin{aligned} \sum_{e \in E} \bar{c}_e x_e^{**} &\stackrel{(a)}{<} \sum_{e \in E, e \neq e_1} \bar{c}_e x_e^{**} + \alpha x_{e_1}^{**} \\ &\stackrel{(b)}{\leq} \sum_{e \in E, e \neq e_1} \bar{c}_e x_e + \alpha x_{e_1} \\ &= \sum_{e \in E} \bar{c}_e x_e. \end{aligned}$$

In above (a) follows from $x_{e_1}^{**} > 0$ and $\bar{c}_{e_1} < \alpha$; (b) follows from x^{**} being an optimal solution with $\bar{c}_{e_1} = \alpha$. In summary, there exists at most one value for α such that when $\bar{c}_{e_1} = \alpha$, $\overline{\mathcal{MCF}}$ has two solutions x^* , x^{**} with $x_{e_1}^* = 0$ and $x_{e_1}^{**} > 0$. In a similar manner, it can be established that there exists at most one value β such that with $\bar{c}_{e_1} = \beta$, $\overline{\mathcal{MCF}}$ has two optimal solutions x^* , x^{**} with $x_{e_1}^* < u_{e_1}$ and $x_{e_1}^{**} = u_{e_1}$.

Let \mathcal{O} be the set of all optimal solutions of $\overline{\mathcal{MCF}}$. From above discussion, it follows that for a given arc e , if \bar{c}_e is chosen uniformly at random from $4m$ distinct positive integers, then the probability that there exists two solutions x^* , x^{**} in \mathcal{O} that satisfy either $x_e^* = 0, x_e^{**} > 0$ or $x_e^* < u_e, x_e^{**} = u_e$ is at most $1/(2m)$. Therefore, with probability at least $1 - 1/(2m)$ all solutions x in \mathcal{O} satisfy either $x_e = 0$ or $0 < x_e < u_e$ or $x_e = u_e$. Denote this event by $D(e)$. By union bound $\cap_{e \in E} D(e)$ holds with probability at least $1/2$. Now to conclude the proof of Theorem 8.1, we state the following Lemma.

Lemma 8.2. *Under event $\cap_{e \in E} D(e)$, the $\overline{\mathcal{MCF}}$ has a unique optimal solution.*

Proof. Suppose to the contrary that under event $\cap_{e \in E} D(e)$, $\overline{\mathcal{MCF}}$ has two distinct optimal solutions x^* and x^{**} . Let $d = x^{**} - x^*$, then $x^* + \lambda d$ is an optimal solution of $\overline{\mathcal{MCF}}$ iff $0 \leq (x^* + \lambda d)_e \leq u_e$, $\forall e \in E$. Since $\bar{c}_e > 0$ for any $e \in E$ and $\bar{c}^T d = \bar{c}^T x^{**} - \bar{c}^T x^* = 0$, there exists some $e' \in E$ such that $d_{e'} < 0$. Let

$$\lambda^* = \sup\{\lambda \geq 0 : x^* + \lambda d \text{ is a feasible solution of } \overline{\mathcal{MCF}}\}.$$

Since $d_{e'} < 0$, λ^* is bounded and since $x^* + d = x^{**}$, $\lambda^* \geq 1$. Further, the supremum λ^* is achieved, that is $x^* + \lambda^* d$ is a feasible solution of $\overline{\mathcal{MCF}}$ since the feasible space of $\overline{\mathcal{MCF}}$ is a closed set. By definition of λ^* , there must exist some e'' such that $x_{e''}^* \neq x_{e''}^{**}$ and either $(x^* + \lambda^* d)_{e''} = 0$ or $u_{e''}$. Since $\lambda^* > 0$, $x_{e''}^* \neq (x^* + \lambda^* d)_{e''}$. That is, we have two solutions x^* and $x^* + \lambda^* d$ that do not satisfy $D(e'')$. This contradicts the hypothesis and hence $\overline{\mathcal{MCF}}$ must have a unique optimal solution. \square

\square

We note that Theorem 8.1 can be easily modified for LP in the standard form.

Corollary 8.3. *Let $\overline{\mathcal{LP}}$ be an LP problem with constraint $Ax = b$, where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$. The cost vector \bar{c} of $\overline{\mathcal{LP}}$ is generated as follows: for each $e \in E$, \bar{c}_e is chosen independently and uniformly over N_e , where N_e is a discrete set of $2n$ elements. Then, the probability that $\overline{\mathcal{LP}}$ has a unique optimal solution is at least $\frac{1}{2}$.*

8.2 Finding the correct modified cost vector \bar{c}

Next, we construct a randomly generated cost vector \bar{c} with the desired properties stated in the beginning of this section. Let $X : E \rightarrow \{1, 2, \dots, 4m\}$ be a random function where for each $e \in E$, $X(e)$ is chosen independently and uniformly over the range. Let $t = \frac{c_{\max} \varepsilon}{4mn}$ and generate \bar{c} as follows: for each $e \in E$, let $\bar{c}_e = 4m \lfloor \frac{c_e}{t} \rfloor + X(e)$. Then, \bar{c}_{\max} is polynomial in m, n and $\frac{1}{\varepsilon}$. By Theorem 8.1, the probability of $\overline{\mathcal{MCF}}$ having a unique optimal solution is greater than $\frac{1}{2}$.

Now, we introduce algorithm $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$ as follows. Select a random \bar{c} ; try to solve $\overline{\mathcal{MCF}}$ using BP. If BP discovers that $\overline{\mathcal{MCF}}$ has no unique optimal solution (using Corollary 5.2), then restart the procedure by selecting another \bar{c} at random, otherwise, return the unique optimal solution found by BP. Formally, we present $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$ as Algorithm 3.

Corollary 8.4. *The $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$ runs in $O\left(\frac{n^8 m^7 \log n}{\varepsilon^3}\right)$ expected time.*

Proof. Theorem 8.1 implies that on average $O(1)$ instances of $\overline{\mathcal{MCF}}$ are required to be solved by the BP. Each such instance requires running Algorithm 2 for $O(n^2 \bar{c}_{\max})$ iterations. Therefore, the total cost scales as $O(\bar{c}_{\max}^3 m n^4 \log n)$ on average by Lemma 7.1. Since $\bar{c}_{\max} = O\left(\frac{m^2 n}{\varepsilon}\right)$, it is bounded as $O(\varepsilon^{-3} m^7 n^7 \log n)$. \square

Now let \bar{c} be the randomly chosen vector as per above described procedure such that $\overline{\mathcal{MCF}}$ has a unique optimal solution, say $x^{(2)}$. Next, we show that $x^{(2)}$ is a “near optimal” solution of \mathcal{MCF} .

Algorithm 3 APRXMT($\mathcal{MCF}, \varepsilon$)

- 1: Let $t = \frac{c_{\max} \varepsilon}{4mn}$, for any $e \in E$, assign $\bar{c}_e = 4m \cdot \lfloor \frac{c_e}{t} \rfloor + p_e$, where p_e is an integer chosen independently, uniformly random from $\{1, 2, \dots, 4m\}$
 - 2: Let $\overline{\mathcal{MCF}}$ be the problem with modified cost \bar{c} .
 - 3: Run Algorithm 2 on $\overline{\mathcal{MCF}}$ for $N = 2\bar{c}_{\max} n^2$ iterations.
 - 4: Use Corollary 5.2 to determine if $\overline{\mathcal{MCF}}$ has a unique solution.
 - 5: **if** $\overline{\mathcal{MCF}}$ does not have a unique solution **then**
 - 6: Restart the procedure APRXMT($\mathcal{MCF}, \varepsilon$).
 - 7: **else**
 - 8: Terminate and return $x^{(2)} = \hat{x}^N$, where \hat{x}^N is the estimate of optimal flow assignments found in Algorithm 2.
 - 9: **end if**
-

To accomplish this, let $e' = \arg \max c_e$, ties broken arbitrarily, and define a new optimization problem $\underline{\mathcal{MCF}}$ as follows:

$$\begin{aligned} & \text{minimize} \quad \sum_{e \in E} c_e x_e && (\underline{\mathcal{MCF}}) \\ & \text{subject to} \quad \sum_{e \in E_v} \Delta(v, e) x_e = b_v, && \forall v \in V \quad (\text{demand/supply constraints}) \\ & \quad \quad \quad x_{e'} = x_{e'}^{(2)} \\ & \quad \quad \quad 0 \leq x_e \leq u_e, && \forall e \in E \quad (\text{flow constraints}). \end{aligned}$$

Lemma 8.5. Suppose $x^{(3)}$ is an optimal solution for $(\underline{\mathcal{MCF}})$ and $x^{(1)}$ is an optimal solution of \mathcal{MCF} . Then

$$c^T x^{(3)} - c^T x^{(1)} \leq |x_{e'}^{(2)} - x_{e'}^{(1)}| nt.$$

Proof. Let $d = x^{(2)} - x^{(1)}$. Call $\gamma \in \{-1, 0, 1\}^{|E|}$ as a *synchronous cycle vector* of d if for any $e \in E$, $\gamma_e = 1$ only if $d_e > 0$, $\gamma_e = -1$ only if $d_e < 0$ and the set $\{e \in E : \gamma_e = 1 \text{ or } \gamma_e = -1\}$ forms exactly one directed cycle in G . Now d is an integral vector of circulation (i.e., d send 0 unit amount of flow to every vertex $v \in V$) since it is difference of two feasible solution of the same network flow problem. Therefore, d can be decomposed as $\sum_{\gamma \in \mathcal{K}'} \gamma = d$ with $\mathcal{K}' \subset \mathcal{K}$ and \mathcal{K} being a finite set of synchronous cycle vectors of G (cf. see [2]). For any $\gamma \in \mathcal{K}'$, observe that $x^{(2)} - \gamma$ is a feasible solution for $\overline{\mathcal{MCF}}$. Now since $x^{(2)}$ is an optimal solution for $\overline{\mathcal{MCF}}$, it follows that $\bar{c}^T \gamma \leq 0$. Now for any $e \in E$,

$$\begin{aligned} & \bar{c}_e = 4m \left\lfloor \frac{c_e}{t} \right\rfloor + p_e, \quad 1 \leq p_e \leq 4m, \\ \implies & \bar{c}_e, \frac{4mc_e}{t} \in \left[4m \left\lfloor \frac{c_e}{t} \right\rfloor, 4m \left(\left\lfloor \frac{c_e}{t} \right\rfloor + 1 \right) \right], \\ \implies & \left| \bar{c}_e - \frac{4mc_e}{t} \right| \leq 4m, \\ \implies & \sum_e \left| \frac{4mc_e}{t} - \bar{c}_e \right| |\gamma_e| \leq 4m \sum_e |\gamma_e| \leq 4mn. \end{aligned}$$

Using this and fact that $\bar{c}^T \gamma \leq 0$, we have

$$\begin{aligned} \frac{4m}{t} c^T \gamma &\leq \frac{4m}{t} c^T \gamma - \bar{c}^T \gamma \\ &\leq \sum_e \left| \frac{4mc_e}{t} - \bar{c}_e \right| |\gamma_e| \\ &\leq 4mn. \end{aligned}$$

Therefore, we have $c^T \gamma \leq nt$. By definition of \mathcal{K}' , $x^{(2)} = x^{(1)} + \sum_{\gamma \in \mathcal{K}'} \gamma$. Therefore, for all $e \in E$

$$\min\{x_e^{(1)}, x_e^{(2)}\} \leq x_e^{(1)} + \sum_{\gamma \in \mathcal{K}'} \gamma_e \leq \max\{x_e^{(1)}, x_e^{(2)}\}.$$

Therefore, it follows that $x^{(1)} + \sum_{\gamma \in \mathcal{K}'} \gamma$ is a feasible solution for $\underline{\mathcal{MCF}}$. Since $x^{(3)}$ is the optimal solution of $\underline{\mathcal{MCF}}$,

$$\begin{aligned} c^T x^{(3)} &\leq c^T x^{(1)} + \sum_{\gamma \in \mathcal{K}'} c^T \gamma \\ &\leq c^T x^{(1)} + |\mathcal{K}'| nt. \end{aligned}$$

Since $|\mathcal{K}'| \leq |x_{e'}^{(2)} - x_{e'}^{(1)}|$, it follows that

$$c^T x^{(3)} - c^T x^{(1)} \leq |x_{e'}^{(2)} - x_{e'}^{(1)}| nt.$$

□

Corollary 8.6. *For any $\varepsilon \in (0, 1)$,*

$$c^T x^{(3)} \leq \left(1 + \frac{\varepsilon}{2m}\right) c^T x^{(1)}.$$

Proof. By Lemma 8.5 we may assume without the loss of generality that $x_{e'}^{(2)} \neq x_{e'}^{(1)}$. Also by Lemma 8.5,

$$\begin{aligned} \frac{c^T x^{(3)} - c^T x^{(1)}}{c^T x^{(3)}} &\leq \frac{|x_{e'}^{(2)} - x_{e'}^{(1)}| nt}{c^T x^{(3)}} \\ &\leq \frac{|x_{e'}^{(2)} - x_{e'}^{(1)}| nt}{|x_{e'}^{(2)} - x_{e'}^{(1)}| c_{e'}} = \frac{nt}{c_{e'}}, \end{aligned} \tag{12}$$

where the last inequality follows because of $c^T x^{(3)} \geq |x_{e'}^{(2)} - x_{e'}^{(1)}| c_{e'}$ justified as follows: using $x_{e'}^{(3)} = x_{e'}^{(2)}$ by definition,

$$c^T x^{(3)} \geq x_{e'}^{(2)} c_{e'} \geq (x_{e'}^{(2)} - x_{e'}^{(1)}) c_{e'};$$

the optimal solution $x^{(3)}$ of $\underline{\mathcal{MCF}}$ is a feasible solution for \mathcal{MCF} , $x^{(1)}$ is optimal solution for \mathcal{MCF} and therefore

$$c^T x^{(3)} \geq c^T x^{(1)} \geq x_{e'}^{(1)} c_{e'} \geq (x_{e'}^{(1)} - x_{e'}^{(2)}) c_{e'}.$$

That is, $c^T x^{(3)} \geq |x_{e'}^{(2)} - x_{e'}^{(1)}| c_{e'}$.

Using $t = \frac{c_{e'} \varepsilon}{4mn}$, from (12) it follows that

$$\frac{c^T x^{(3)} - c^T x^{(1)}}{c^T x^{(3)}} \leq \frac{\varepsilon}{4m}.$$

Therefore

$$c^T x^{(3)} \leq \left(1 - \frac{\varepsilon}{4m}\right)^{-1} c^T x^{(1)} \leq \left(1 + \frac{\varepsilon}{2m}\right) c^T x^{(1)},$$

where the last inequality holds because $\varepsilon \in (0, 1)$. □

8.3 The FPRAS

Loosely speaking, Corollary 8.6 shows that $x^{(2)}$ at arc e' is “near optimal”, since fixing the flow at arc e' to $x_{e'}^{(2)}$ helps us in finding a feasible solution of \mathcal{MCF} which is close to optimal. This leads us to an approximation algorithm $\text{AS}(\mathcal{MCF}, \varepsilon)$ (Algorithm 4) below. This algorithm at every iteration uses APRXMT (Algorithm 3), and iteratively fixes the flow values at the arc with the largest cost. Theorem 8.7 establishes that this algorithm $\text{AS}(\mathcal{MCF}, \varepsilon)$ is indeed an FPRAS.

Algorithm 4 $\text{AS}(\mathcal{MCF}, \varepsilon)$

- 1: Let $G = (V, E)$ be the underlying directed graph of \mathcal{MCF} with $m = |E|$, $n = |V|$.
 - 2: **while** \mathcal{MCF} flows for all arcs are not assigned **do**
 - 3: Run $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$, let $x^{(2)}$ be the solution returned.
 - 4: Find $e' = \arg \max_{e \in E} c_e$ and modify \mathcal{MCF} by fixing the flow on arc e' by $x_{e'}^{(2)}$; change the demands/supply on node v', w' with $e' = (v', w')$.
 - 5: **end while**
-

Theorem 8.7. *Given $\varepsilon \in (0, 1)$, algorithm $\text{AS}(\mathcal{MCF}, \varepsilon)$ takes $O(\varepsilon^{-3} n^7 m^8 \log n)$ operations on average. Let x^* be the solution produced by $\text{AS}(\mathcal{MCF}, \varepsilon)$. Then*

$$c^T x^* \leq (1 + \varepsilon) c^T x^{(1)}.$$

Proof. By Corollary 8.4, $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$ takes $O(\varepsilon^{-3} n^7 m^7 \log n)$ operations on average. Since $\text{AS}(\mathcal{MCF}, \varepsilon)$ invokes the method $\text{APRXMT}(\mathcal{MCF}, \varepsilon)$ m times, $\text{AS}(\mathcal{MCF}, \varepsilon)$ performs on average total operations bounded as $O(\varepsilon^{-3} n^7 m^8 \log n)$. By successive application of Corollary 8.6,

$$\begin{aligned} c^T x^* &\leq \left(1 + \frac{\varepsilon}{2m}\right)^m c^T x^{(1)} \\ &\leq e^{\frac{\varepsilon}{2}} c^T x^{(1)} \\ &\leq (1 + \varepsilon) c^T x^{(1)} \end{aligned}$$

where the last two inequalities follows for $\varepsilon \in (0, 1)$ and $m \geq 1$. □

9 Conclusions

In this paper, we formulated and analyzed the Belief Propagation (BP) algorithm for the capacitated min-cost network flow problem \mathcal{MCF} . We proved that the BP solves \mathcal{MCF} exactly in pseudo-polynomial time when the optimal solution is unique. This result generalizes an earlier result from [5], and provides new insights for understanding BP as an optimization solver. Although the running time of BP for \mathcal{MCF} is slower than other existing algorithms for \mathcal{MCF} , the advantage of BP is that it is a general purpose distributed heuristic which is widely applicable and which is easy to formulate and implement for a broad class of constrained optimization problems. We also showed that a similar result holds for the network flow problem with the piece-wise linear convex cost function. A salient feature of the BP established in this work is ability to detect uniqueness of the optimal solution in an entirely distributed manner.

We showed that the BP algorithm, in its original form, at best leads to a pseudo-polynomial time algorithmic complexity. To address this problem we have introduced a randomized variant of BP and showed that this variant provides FPRAS. This is the first FPRAS result for the BP type algorithms. Our variant of BP is based on fixing the values of flow variables one-by-one in a sequential manner. Such methodology, used commonly in practice, is known as the “decimation” procedure (see [20]). To the best of our knowledge, this is the first disciplined, provable instance of the decimation procedure in the context of BP algorithms.

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