NCERT-10.4.4.1.2

EE24BTECH11023 - RASAGNA

Question: Find the nature of the roots of the following quadratic equation. If the real roots exist,find them:

$$3x^2 - 4\sqrt{3}x + 4 = 0$$

Theoritical Solution: The standard quadratic equation is:

$$ax^2 + bx + c = 0 \tag{0.1}$$

Here, $a = 3, b = -4\sqrt{3}, c = 4$. Computing the value of determinant,

$$\triangle = b^2 - 4ac \tag{0.2}$$

$$\triangle = (-4\sqrt{3})^2 - 4(3)(4) \tag{0.3}$$

$$\therefore \triangle = 0 \tag{0.4}$$

$\therefore$ The given equation has two equal and real roots.
The roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.5}$$

Substitute the values of a, b, and c:

$$x = \frac{-(-4\sqrt{3}) \pm \sqrt{(-4\sqrt{3})^2 - 4(3)(4)}}{2(3)} \tag{0.6}$$

$$= \frac{2}{\sqrt{3}} \tag{0.7}$$

Thus, the roots of the equation are:

$$x_1 = x_2 = \frac{2}{\sqrt{3}} \tag{0.8}$$

Solution using Matrix Approach by finding eigen values
Matrix-Based Method
For a polynomial equation of the form:

$$x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0 \tag{0.9}$$

we construct companion matrix, which is defined as:

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \tag{0.10}$$

The eigenvalues of this matrix are the roots of the given polynomial equation.

For the quadratic equation $x^2 - 4\sqrt{3}x + 4 = 0$, we can write it as:

$$x^2 + (-4\sqrt{3})x + (4) = 0 \tag{0.11}$$

The coefficients are:

$$b_1 = -4\sqrt{3}, \quad b_0 = 4$$

The companion matrix for this equation is:

$$\Lambda = \begin{bmatrix} 0 & 1 \\ -(4) & -(-4\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 4\sqrt{3} \end{bmatrix} \tag{0.12}$$

The eigenvalues of Λ are obtained by solving:

$$\det(\Lambda - \lambda I) = 0 \tag{0.13}$$

Substitute Λ:

$$\begin{vmatrix} 0 - \lambda & 1 \\ -4 & 4\sqrt{3} - \lambda \end{vmatrix} = 0 \tag{0.14}$$

Simplify the determinant,we get

$$(-\lambda)(4\sqrt{3} - \lambda) - (-4)(1) = 0 \tag{0.15}$$

$$\lambda^2 - 4\sqrt{3}\lambda + 4 = 0 \tag{0.16}$$

This is the original quadratic equation, the eigen values are equal to the roots ;

$$\lambda_1 = \frac{2}{\sqrt{3}}, \quad \lambda_2 = \frac{2}{\sqrt{3}} \tag{0.17}$$

Computational Solution

We use fixed point iteration method to find the roots,
From the equation we can reformulate it as

$$g(x) = \frac{\sqrt{4\sqrt{3}x - 4}}{\sqrt{3}} \tag{0.18}$$

Now let us assume the initial guess of x as $x_0 = 2$ Use the iterative formula,

$$x_{n+1} = g(x_n) \tag{0.19}$$

Repeat the iterations until the sequence x_{n+1} converges to a fixed point. Convergence is often checked using A predefined tolerance ϵ where $|x_{n+1} - x_n| < \epsilon$ or maximum number of iterations. The result obtained by computational approach is :

Root 1 = 1.154701 + 0.000000i

Root 2 = 1.154701 - 0.000000i