NCERT-8.1.10

EE24BTECH11023 - RASAGNA

Question:

Find the are bound by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution: The area bounded by two curves $f(x)$ and $g(x)$ over a specific interval $[a, b]$ is given by:

$$A = \int_a^b |f(x) - g(x)| dx. \tag{0.1}$$

For the given problem:

$$l(x) = \frac{x+2}{4} (\text{line}) \tag{0.2}$$

$$c(x) = \frac{x^2}{4} (\text{parabola}) \tag{0.3}$$

The points of intersection of the curves are found by solving:

$$\frac{x^2}{4} = \frac{x+2}{4}. \tag{0.4}$$

Simplifying:

$$x^2 - x - 2 = 0 \tag{0.5}$$

$$(x-2)(x+1) = 0 \tag{0.6}$$

Thus, the points of intersection are $x = -1$ and $x = 2$, so the interval is $[-1, 2]$.

The area is then:

$$A = \int_{-1}^{2} \left| \frac{x+2}{4} - \frac{x^2}{4} \right| dx \tag{0.7}$$

$$= \frac{1}{4} \int_{-1}^{2} \left(x + 2 - x^2 \right) dx. \tag{0.8}$$

Expanding and integrating:

$$A = \frac{1}{4} \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}. \tag{0.9}$$

Substituting the limits:

$$A = \frac{1}{4}\left[\left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right)\right] \tag{0.10}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] \tag{0.11}$$

$$= \frac{9}{8} \tag{0.12}$$

Thus, the area bounded by the curves $x^2 = 4y$ and $x = 4y - 2$ is:

$$A = 1.125\,\text{sq. units.} \tag{0.13}$$

Numerical Integration Using the Trapezoidal Rule(simulation):
Trapezoidal Rule is evaluates the area under the curves by dividing the area under the curve into smaller trapezoids instead of rectangles.This integration method approximates the region under a function's graph as a trapezoid and computes its area.
The final integral;

$$\frac{1}{4}\int_{-1}^{2}(x + 2 - x^2)\,dx \tag{0.14}$$

let us assume $f(x) = (x + 2 - x^2)$.
THE TRAPEZOIDAL RULE The trapezoidal rule is a numerical method used to approximate the value of a definite integral. It works by approximating the region under the curve as a series of trapezoids and calculating their areas.
To implement this, we discretize the range of x-coordinates with a uniform step size $h \to 0$, such that the points are $x_0, x_1, x_2, \dots, x_n$ and $x_{n+1} = x_n + h$.
Let A_n represent the sum of all trapezoidal areas up to x_n. Given $f(x) = \frac{1}{4}(x + 2 - x^2)$, the difference equation can then be expressed as

$$A_n = \frac{h}{2}\left(f(x_0) + f(x_1)\right) + \frac{h}{2}\left(f(x_1) + f(x_2)\right) + \cdots + \frac{h}{2}\left(f(x_{n-1}) + f(x_n)\right) \tag{0.15}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(f(x_{n+1}) + f(x_n)\right) \tag{0.16}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(\left(x_{n+1} + 2 - x_{n+1}^2\right) + \left(x_n + 2 - x_n^2\right)\right) \tag{0.17}$$

The obtained theoritical solution is 1.125 sq.units.
The computational solution is 1.1249988750000004sq.units.