2024-ST-'53-65'

EE24BTECH11023

1) Let $\{X_n\}_{n\geq 1}$ be a time homogeneous discrete-time Markov chain with state space $\{0,1,2\}$ and transition probability matrix
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$
Which of the following statements is/are true?
   a) 0 and 1 are recurrent states.
   b) 2 is a transient state.
   c) The Markov chain has a unique stationary distribution.
   d) The Markov chain is irreducible.
2) Let $X_1, X_2, \dots, X_n$ be a random sample of size $n(\geq 2)$ from a population having Poisson distribution with mean λ, where $\lambda > 0$ is an unknown parameter. If $T_1 = \bar{X}$ and $T_2 = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then which of the following statements is/are true?
   a) T_1 is an unbiased estimator of λ.
   b) T_2 is an unbiased estimator of $\sqrt{\lambda}$.
   c) T_2^2 is an unbiased estimator of λ.
   d) Both T_1 and T_2 are estimators of λ as well as λ^2.
3) Let X_1, X_2, X_3 be a random sample of size 3 from a population having Bernoulli distribution with parameter p, where $p \in (0,1)$ is unknown. Define
$$T_1(X_i, X_j, X_k) = X_i - X_j(1 - X_k), \quad T_2(X_i, X_j, X_k) = \frac{1}{2}(X_iX_j + X_jX_k),$$
for $i, j, k = 1, 2, 3$ with $i \neq j \neq k$. Let x_1, x_2, x_3 denote realizations from the random sample. Then which of the following statements is/are true?
   a) $T_1(X_1, X_2, X_3)$ has the same distribution as $T_1(X_2, X_3, X_1)$, but $T_1(x_1, x_2, x_3) \neq T_1(x_2, x_3, x_1)$ for some realization x_1, x_2, x_3.
   b) $T_2(X_1, X_2, X_3)$ and $T_2(X_3, X_2, X_1)$ are both unbiased estimators for p^2.
   c) $T_1(X_1, X_2, X_3)$ and $T_1(X_2, X_3, X_1)$ are both unbiased estimators for p^2, and $T_1(x_1, x_2, x_3) = T_1(x_2, x_3, x_1)$ for all realizations x_1, x_2, x_3.
   d) $T_2(x_1, x_2, x_3) = T_2(x_2, x_3, x_1)$ for all realizations x_1, x_2, x_3.
4) Let $X_1, X_2, \dots, X_n$ be a random sample of size $n(\geq 2)$ from a population having probability density function
$$f(x;\lambda) = \begin{cases} \frac{1}{\lambda}e^{-x/\lambda}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$
where $\lambda > 0$ is an unknown parameter. Let $T_1 = \sum_{i=1}^{n} X_i$ and $T_2 = \left(\sum_{i=1}^{n} X_i\right)^{-1}$. For any positive integer v and any $\alpha \in (0,1)$, let $X^2_{v,\alpha}$ denote the $(1-\alpha)$-th quantile of the central

chi-square distribution with ν degrees of freedom. Consider testing $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda > \lambda_0$. Then which of the following tests is/are uniformly most powerful at level α?

a) H_0 is rejected if $\frac{2}{\lambda_0}T_1 > X^2_{2n,\alpha}$.
b) H_0 is rejected if $\frac{2}{\lambda_0}T_1 > X^2_{2n,1-\alpha}$.
c) H_0 is rejected if $\frac{\lambda_0}{2}T_1 > X^2_{2n,\alpha}$.
d) H_0 is rejected if $\frac{\lambda_0}{2}T_1 > X^2_{2n,1-\alpha}$.

5) Let $\{1, 6, 5, 3\}$ and $\{11, 7, 15, 4\}$ be realizations of two independent random samples of size 4 from two separate populations having cumulative distribution functions $F(\cdot)$ and $G(\cdot)$, respectively, and probability density functions $f(\cdot)$ and $g(\cdot)$, respectively. To test $H_0 : F(t) = G(t)$ for all t against $H_1 : F(t) \geq G(t)$ with strict inequality for some t, let U_{MW} denote the Mann-Whitney U-test statistic. Let, under H_0, $P(U_{MW} > 12) \leq 0.10$, $P(U_{MW} > 14) \leq 0.05$, $P(U_{MW} > 15) \leq 0.025$, and $P(U_{MW} > 16) \leq 0.01$. Then,based on the given data, which of the following statements is/are true?

a) H_0 is rejected at level 0.10.
b) H_0 is rejected at level 0.05.
c) H_0 is rejected at level 0.025.
d) H_0 is rejected at level 0.01.

6) Let (X_1, X_2, X_3) have $N_3(\mu, \Sigma)$ distribution with $\mu = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}$. For which of the following vectors a, X_2 and $X_2 - a^T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ are independent?

a) $a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
b) $a = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$
c) $a = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
d) $a = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

7) Let A be a 2×2 real matrix such that the trace of A is 5 and the determinant of A is 6. If the characteristic polynomial of $(A + I_2)^{-1}$ is $x^2 - bx + c$, where I_2 is the 2×2 identity matrix, then $\frac{b}{c}$ equals __________ (in integer).

8) Let $\{X_n\}_{n \geq 1}$ be a time homogeneous discrete-time Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

If $P(X_0 = 0) = P(X_0 = 1) = \frac{1}{4}$, then $32E(X_2)$ equals __________ (integer).

9) Let (X, Y) be a random vector having joint moment generating function given by

$$M_{X,Y}(u, v) = \frac{e^{\frac{u^2}{2}}}{(1 - 2v)^3}, \quad -\infty < u < \infty, -\infty < v < \frac{1}{2}.$$

Then $E(\frac{6X^2}{Y})$ equals __________ (rounded off to two decimal places).

10) Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate λ, where $\lambda > 0$ is an unknown parameter. Starting from the origin, an intercity road has $N(t)$ number of potholes up to a distance of t kilometers. Starting from the origin, potholes are found at the following distances (in kilometers):

$$0.9, 1.3, 1.8, 2.7, 3.4, 4.1, 4.7, 5.5, 6.2, 6.8, 7.4, 8.1, 8.9, 9.2, 9.7.$$

Based on the above data, the method of moment estimate of λ equals __________ (rounded off to two decimal places).

11) Let X_1, X_2, X_3, X_4 be a random sample of size 4 from a population having uniform distribution over the interval $(0, \theta)$, where $\theta > 0$ is an unknown parameter. Let $X_{(4)} = \max\{X_1, X_2, X_3, X_4\}$. To test $H_0 : \theta = 1$ against $H_1 : \theta = 0.1$, consider the critical region that rejects H_0 if $X_{(4)} < 0.3$. Let p be the probability of the Type-I error. Then $100p$ equals __________ (rounded off to two decimal places).

12) Let a random sample of size 4 from a normal population with unknown mean μ and variance 1 yield the sample mean of 0.16. Let $\Phi(\cdot)$ be the cumulative distribution function of the standard normal random variable. It is given that $\Phi(2.28) = 0.989$, $\Phi(1.96) = 0.975$, and $\Phi(1.64) = 0.95$. If the likelihood ratio test of size 0.05 is used to test $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$, then the power of the test at the sample mean equals __________ (rounded off to three decimal places).

13) Consider the multiple linear regression model

$$y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i, \quad i = 1, 2, \ldots, 25$$

where β_0, β_1, and β_2 are unknown parameters, and ϵ_i's are uncorrelated random errors with mean 0 and finite variance $\sigma^2 > 0$. Let $F_{\alpha,m,n}$ be such that $P(Y > F_{\alpha,m,n}) = \alpha$, where Y is a random variable with an F-distribution with m and n degrees of freedom. Suppose that testing

$$H_0 : \beta_1 = \beta_2 = 0 \quad \text{against} \quad H_1 : \text{At least one of } \beta_1, \beta_2 \text{ is not } 0$$

involves computing $F_0 = 11\frac{R^2}{1-R^2}$ and rejecting H_0 if the computed value F_0 exceeds $F_{\alpha,2,22}$. Given that $F_{0.025,2,22} = 4.38$ and $F_{0.05,2,22} = 3.44$, the smallest value of R^2 that would lead to rejection of H_0 for $\alpha = 0.05$ equals __________ (rounded off to two decimal places).