NCERT-9.3.11

EE24BTECH11023 - RASAGNA

Question:
Find the solution of the following differential equation:

$$(x^3 + x^2 + x + 1)\frac{d}{dx}(y) = 2x^2 + x$$

Given that $x = 0$ when $y = 1$.

Solution: From the question, after simplification:

$$\frac{dy}{dx} = \frac{2x^2 + x}{(1 + x)(1 + x^2)}$$

Let,

$$\frac{2x^2 + x}{(1 + x)(1 + x^2)} = \frac{A}{1 + x} + \frac{Bx + C}{1 + x^2}$$

When $x = 0$:

$$A + C = 0 \quad \text{(eq 1)}$$

When $x = 1$:

$$A + B + C = \frac{3}{2} \quad \text{(eq 2)}$$

When $x = \frac{-1}{2}$:

$$5A - B + 2C = 0 \quad \text{(eq 3)}$$

Solving these equations,

$$A = \frac{1}{2}, \quad B = \frac{3}{2}, \quad C = -\frac{1}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(1 + x)} + \frac{3x - 1}{2(1 + x^2)}$$

Integrating both sides:

$$y = \int \left(\frac{1}{2(1 + x)} + \frac{3x}{2(1 + x^2)} - \frac{1}{2(1 + x^2)} \right) dx$$

The solution is,

$$y = \frac{1}{2}\ln(1 + x) + \frac{3}{4}\ln(1 + x^2) - \frac{1}{2}\tan^{-1}(x) + C$$

Given $x = 0$ when $y = 1$, solve for C;

$$C = 1 - \left(\frac{1}{2}\ln(1 + 0) + \frac{3}{4}\ln(1 + 0^2) - \frac{1}{2}\tan^{-1}(0) \right)$$

$$\therefore C = 1$$

Thus, the final required equation is,

$$y = \frac{1}{2}\ln(1 + x) + \frac{3}{4}\ln(1 + x^2) - \frac{1}{2}\tan^{-1}(x) + 1$$