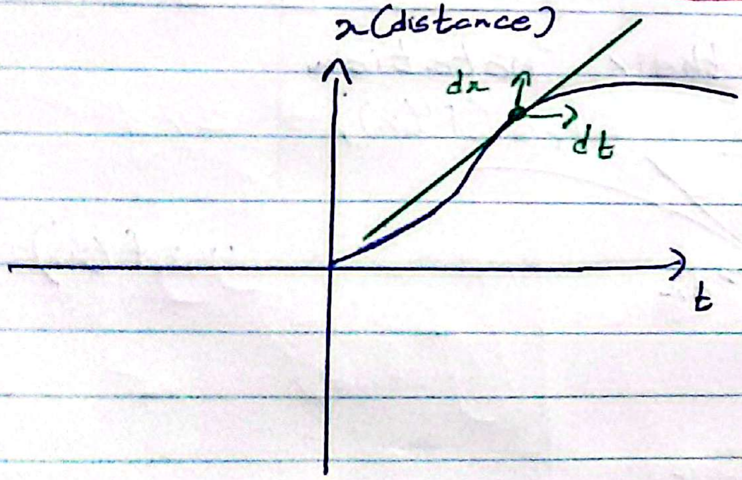
Week 01

1) Derivatives and tangents

* Derivative is the rate of change of a function at a particular point.

* Tangent line is a straight line that just touches a curve at one point without crossing it.

The slope of this tangent line at that point	=	The derivative of the function there

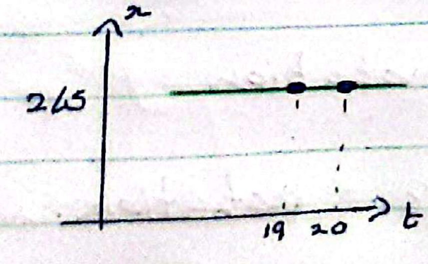


$$\frac{\Delta x}{\Delta t} \longrightarrow \frac{dx}{dt}$$

2) Slopes, maxima, minima

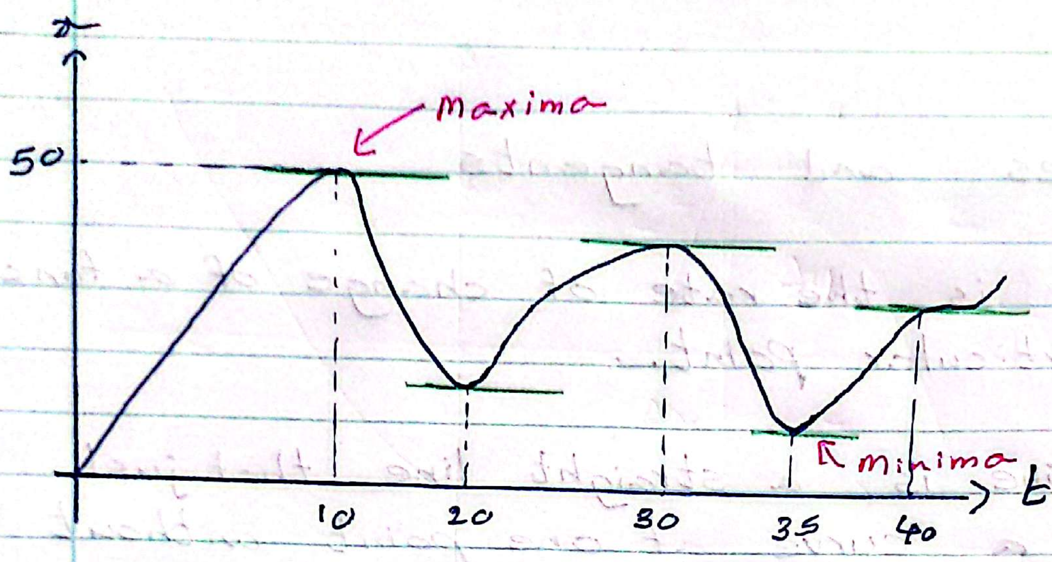
Zero Slope

$$\text{Slope} = \frac{\Delta x}{\Delta t} = \frac{265 - 265}{1s}$$



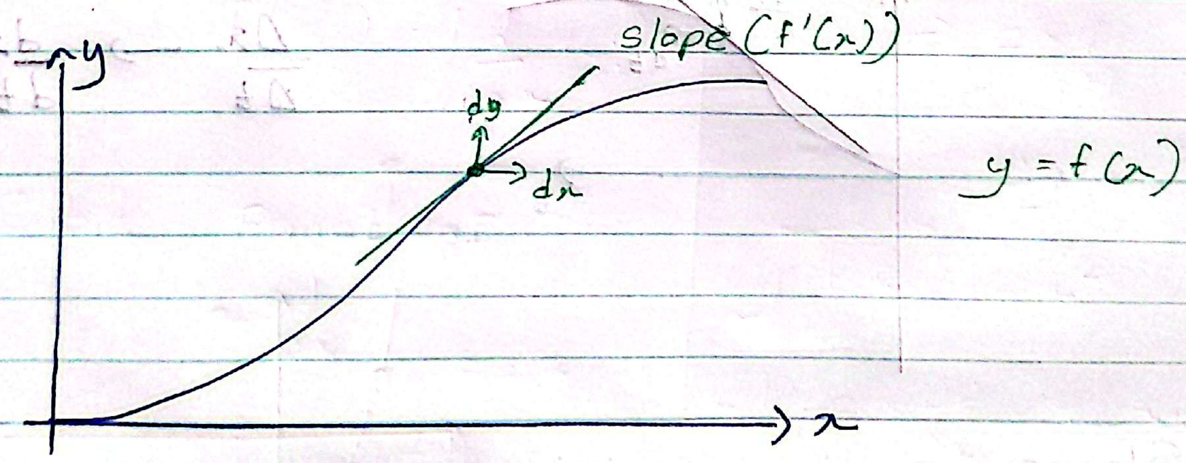
Slope = 0
(No rise in distance)

Maxima and Minima



*) Maximum and minimum in a functions occurs at one of the points where the slope (derivative) is zero. (where the tangent line is horizontal

3) Derivatives and their notation



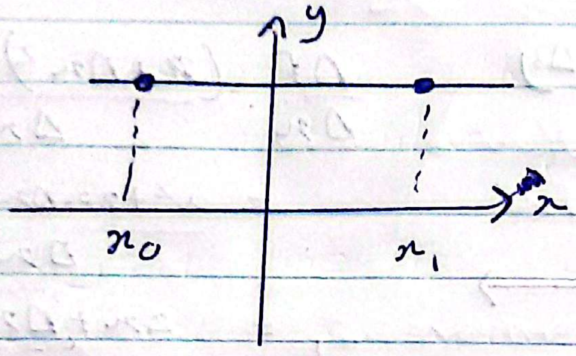
$$\text{Slope} = \frac{\text{vertical rate of change}}{\text{horizontal rate of change}} = \frac{dy}{dx}$$

1] $f'(x)$ Lagrange's notation

2] $\frac{dy}{dx} = \frac{d}{dx} f(x)$ Leibniz's notation

4] Lines

Derivative of a constant

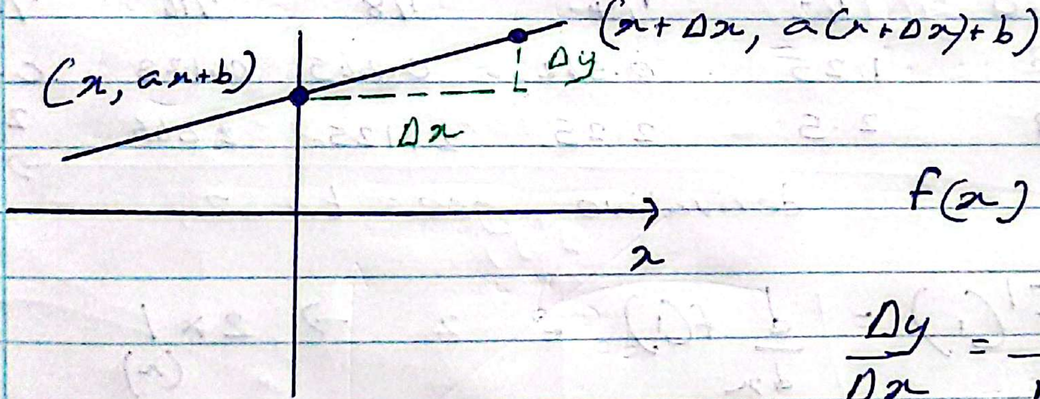


$$y = f(x) = c$$

$$\frac{\Delta y}{\Delta x} = \frac{c - c}{x_1 - x_0} = 0$$

$$\boxed{f'(x) = 0}$$

Derivative of a line



$$f(x) = ax+b$$

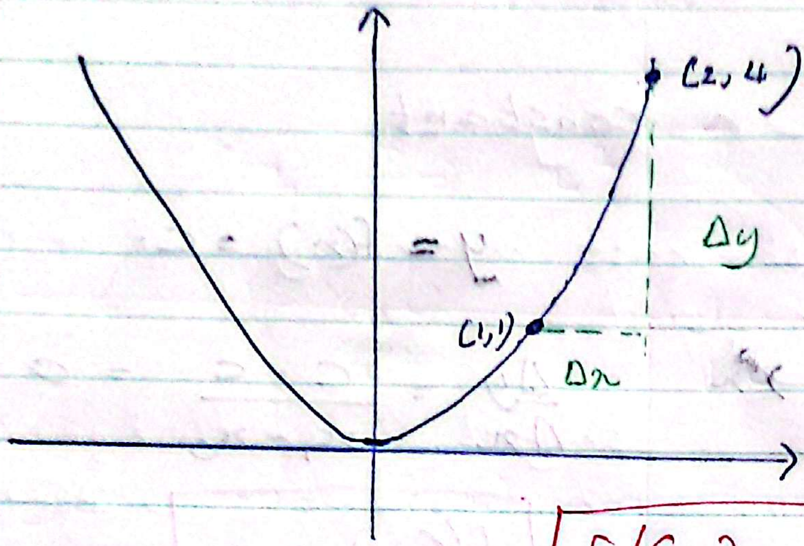
$$\frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = a$$

$$\frac{\Delta y}{\Delta x} = \frac{a(x+\Delta x)+b - (ax+b)}{\Delta x}$$

$$= a\frac{\Delta x}{\Delta x} = a$$

$$\boxed{f(x) = a}$$

5) Quadratics



$y = f(x) = x^2$

$$\frac{\Delta f}{\Delta x} = \frac{(x + \Delta x^2) - x^2}{\Delta x}$$

$$= \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= 2x + \underbrace{\Delta x}_{} \approx 0$$

$$\boxed{f'(x) = 2x}$$

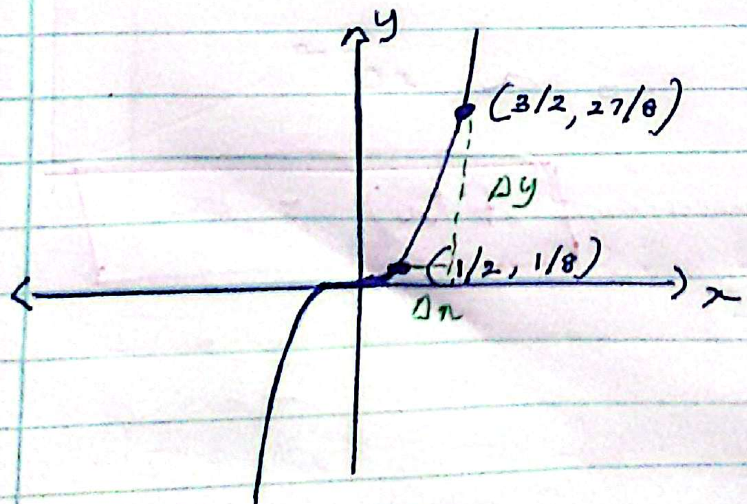
when Δx moves to 0

$\Delta x \rightarrow$	1.0	1/2	1/4	1/8	1/16	1/1000
$\Delta f \rightarrow$	3	1.25	0.562	0.265	0.128	0.002
Slope $\rightarrow$	3	2.5	2.25	2.125	2.065	2.001

derivative goes to 2

$$f'(1) = \frac{d}{dx} f(1) = 2 \rightarrow 2 \times \underset{(x)}{1}$$

6) Cubic Functions



$y = f(x) = x^3$

$$\text{Slope} = \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

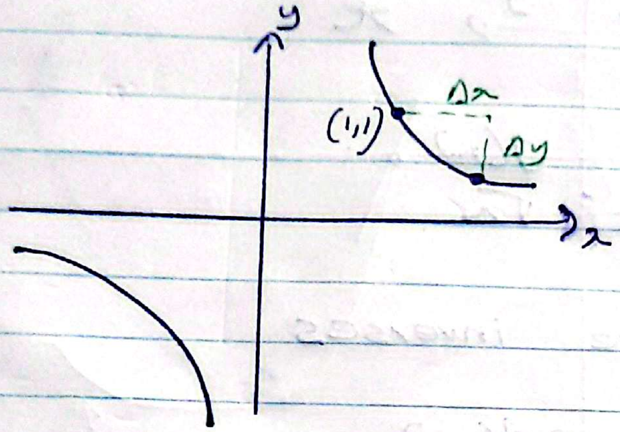
$$= \frac{x^3 + 3x(\Delta x)^2 + 3x^2\Delta x + (\Delta x)^3 - x^3}{\Delta x}$$

$$= 3x\underbrace{\Delta x}_{0} + 3x^2 + \underbrace{\Delta x^2}_{0}$$

$$\boxed{f'(x) = 3x^2}$$

7] Other power functions

Derivative of $\frac{1}{x}$ (x^{-1})



$y = f(x) = x^{-1} = \frac{1}{x}$

Slope: $\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$\frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^{-1} - x^{-1}}{\Delta x}$

$= \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)x}}{\Delta x}$

$= -\frac{1}{x^2 + x\underbrace{\Delta x}_{=0}}$

$\boxed{f'(x) = -x^{-2}}$

Pattern of derivative of power functions

$f(x) = x^2$ $\downarrow$ $f'(x) = 2x^1$

$f(x) = x^3$ $\downarrow$ $f'(x) = 3x^2$

$f(x) = x^{-1}$ $f'(x) = (-1)x^{-2}$

$f(x) = x^n \Longrightarrow \boxed{f'(x) = \frac{d}{dx} f(x) = nx^{n-1}}$

8] The inverse function and its derivative

$$x \xrightarrow{f} x^2 \xrightarrow{g} x$$

$f(x)$ $\qquad$ $g(x)$

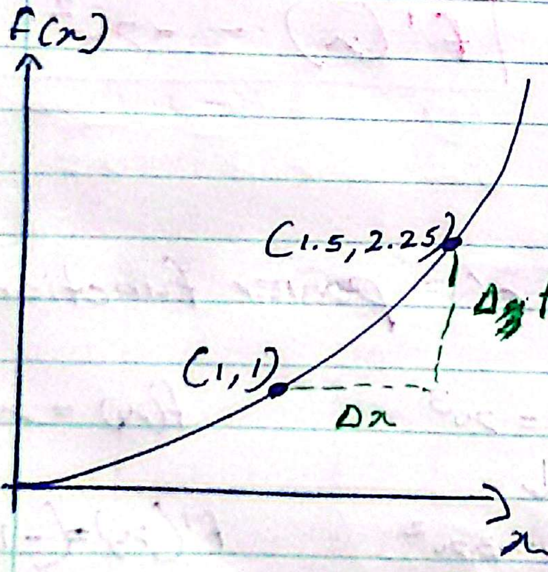
x^2 $\qquad$ $\sqrt{x}$

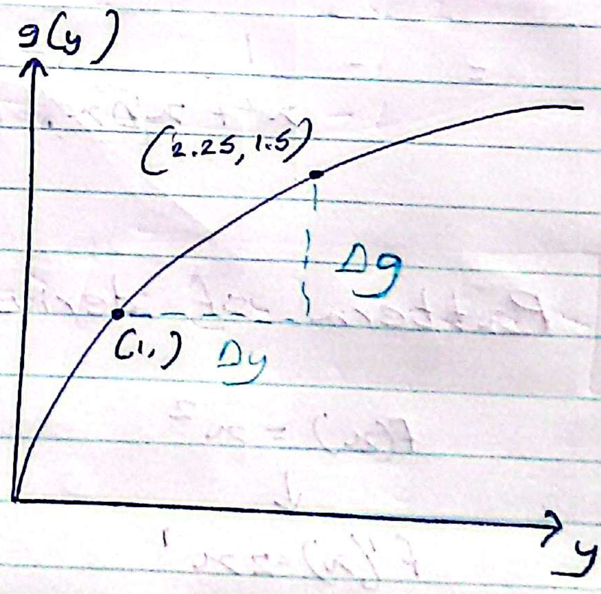
$g(x)$ and $f(x)$ are inverses

$$g(x) = f^{-1}(x)$$

$$g(f(x)) = x$$

$$\sqrt{x^2} = x \quad (\text{for } x > 0)$$





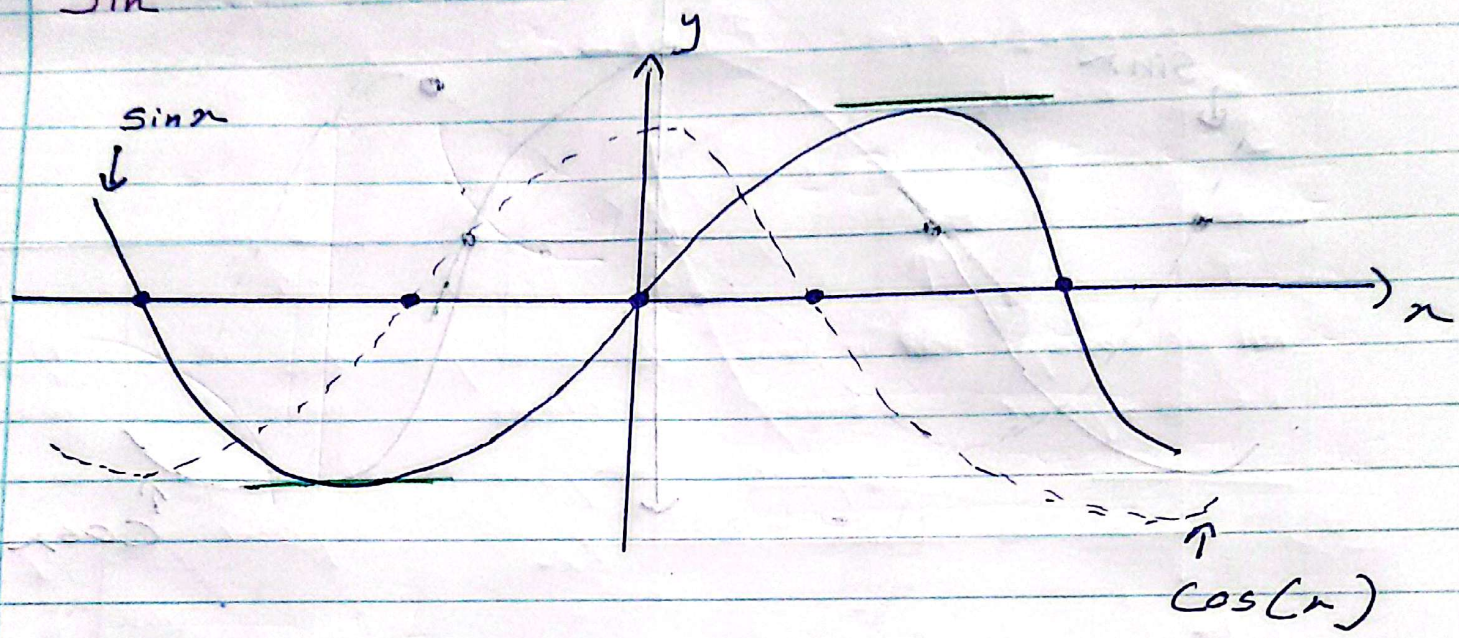
Slope: $\dfrac{\Delta g}{\Delta y} = \dfrac{\Delta x}{\Delta f}$

$$\frac{\Delta f}{\Delta x} = f'(x) \rightarrow \frac{\Delta g}{\Delta y} = g'(y)$$

$$\boxed{g'(y) = \frac{1}{f'(x)}}$$

9] Derivative of trignometric functions

Sin

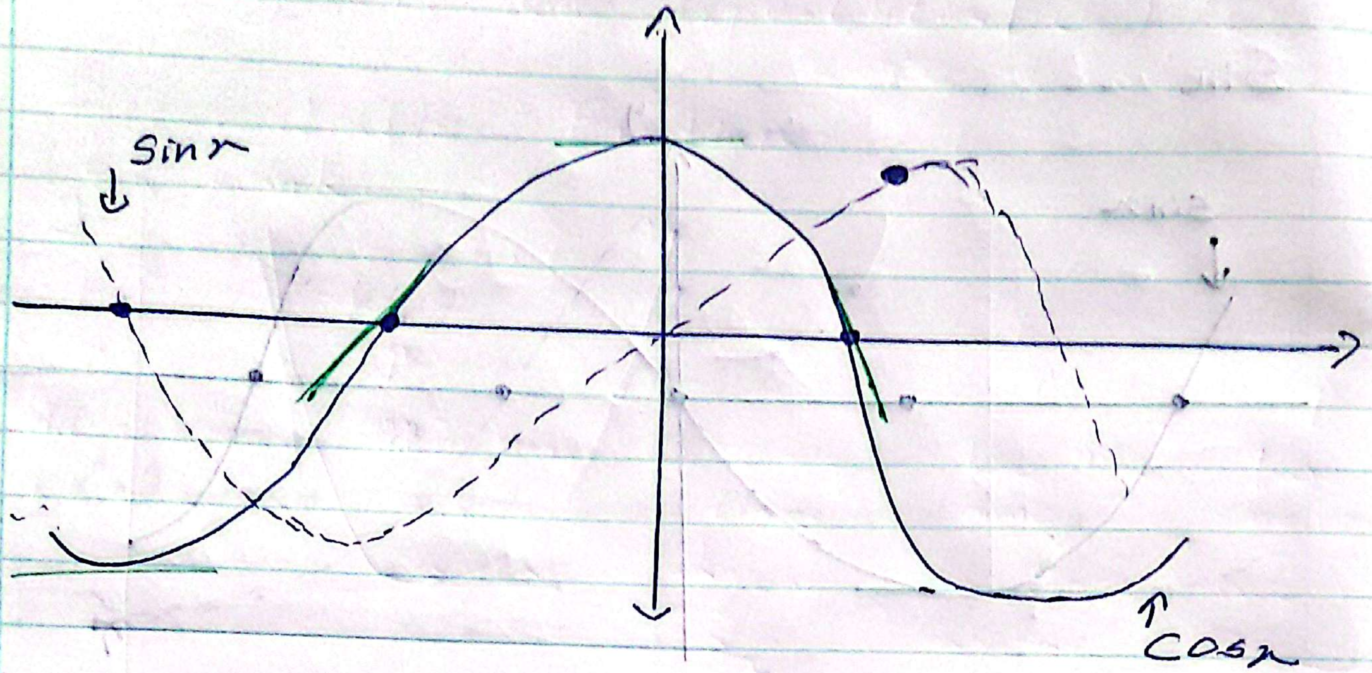


$\sin y = f(x) = \sin(x)$

$x \rightarrow$	$\pi/2$	$-\pi/2$	0	π
Slope $\rightarrow$	0	0	1	-1
$\cos(x) \rightarrow$	0	0	1	-1

$f(x) = \sin(x) \Longrightarrow f'(x) = \cos(x)$

Cos

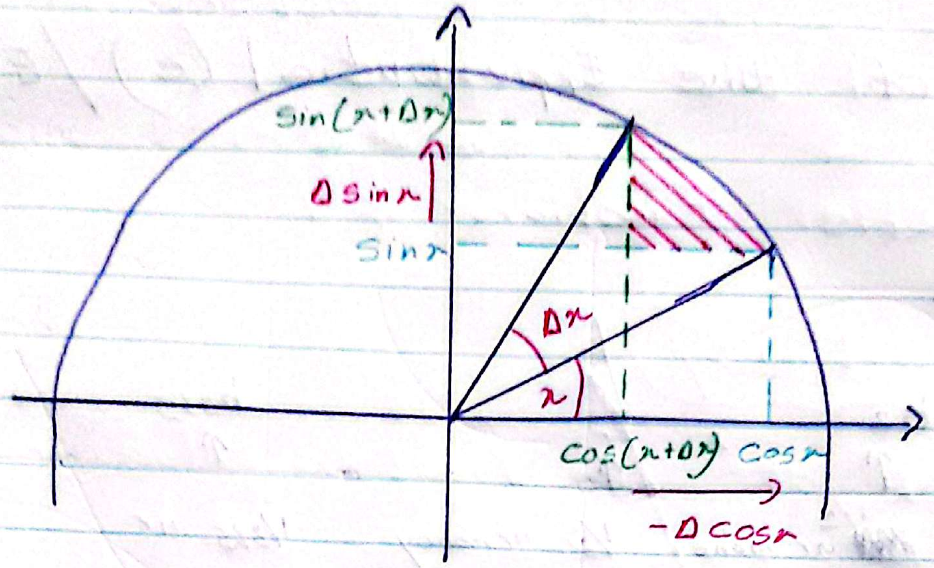


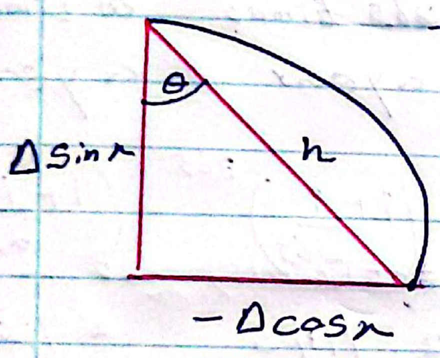
$$\cos.\ y = f(x) = \cos(x)$$

$x \rightarrow$	0	$-\pi$	$\pi/2$	$-\pi/2$
Slope $\rightarrow$	0	0	-1	1
$\sin x \rightarrow$	0	0	1	-1

$$f(x) = \cos x \implies \boxed{f'(x) = -\sin x}$$

Proof





$$\cos\theta = \frac{adj}{hyp} = \frac{\Delta(\sin x)}{h}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{-\Delta\cos x}{h}$$

$\Delta\sin(x) = h\cos\theta$

$-\Delta\cos x = h\sin\theta$

when $\Delta x \to 0$ h also goes to 0

$$\Delta\sin x = \Delta x\cos(x)$$
$$-\Delta\cos x = \Delta x\sin x$$

$f(x) = \sin(x) \longrightarrow f'(x) = \cos(x)$

$g(x) = \cos(x) \longrightarrow g'(x) = -\sin x$

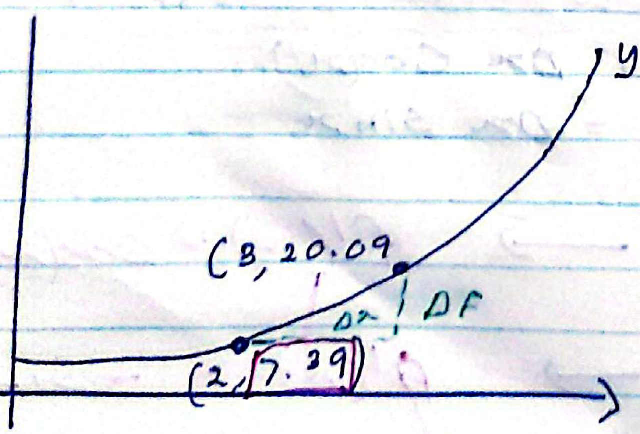
10] Meaning of the Exponential (e) / Eular

b = bank

b_1	b_2	b_3	... b_{365}	... ∞
All ur money 1 time a year	1/2 ur money 2 times a year	1/3 ur money 3 times a year	1/365 ur money 365 times a year	$1/\infty$ ur money ∞ times a year
$(1+1)^1$	$\left(1+\frac{1}{2}\right)^2$	$\left(1+\frac{1}{3}\right)^3$	... $\left(1+\frac{1}{365}\right)^{365}$	$\left(1+\frac{1}{\infty}\right)^{\infty}$
2	2.25	2.37	2.7145	2.71828 182...

$e \leftarrow$

11] The derivative of e^x



Slope: $\frac{\Delta F}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x}$

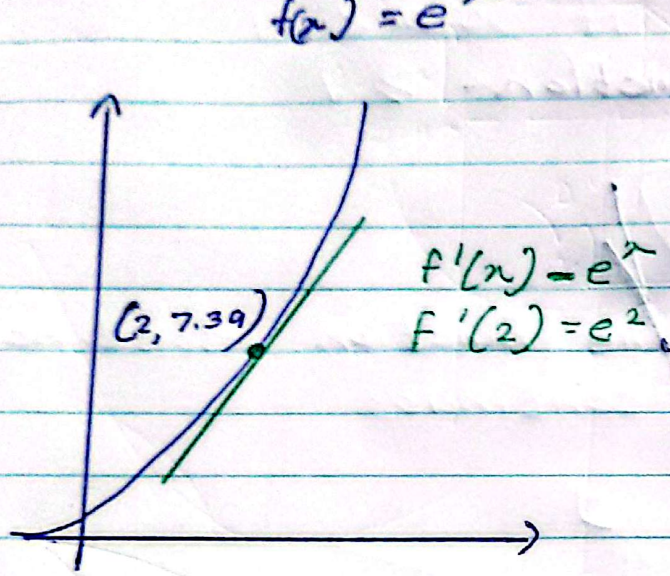
$x = 2$,

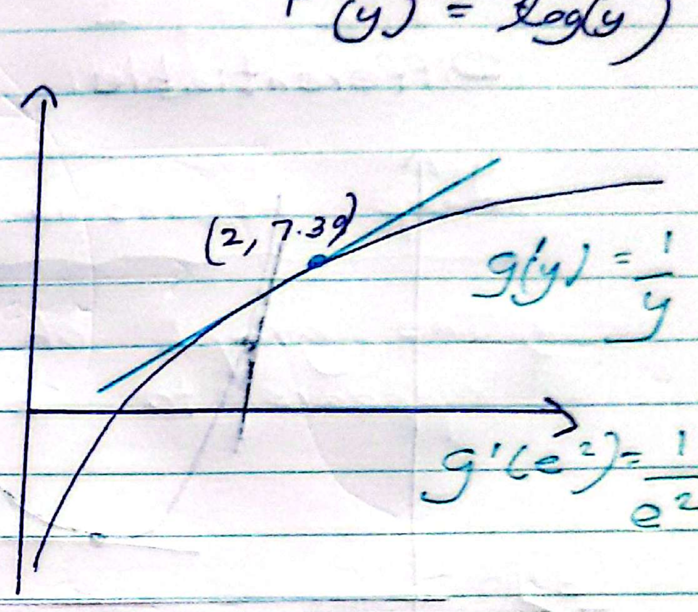
Δx	1.0	1/2	1/4	1/1000
ΔF	12.70	4.79	2.10	0.007
Slope	12.70	9.59	8.39	$\boxed{7.39}$

e^2

$$\boxed{f'(x) = e^x}$$

12] The derivative of log(x)





$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

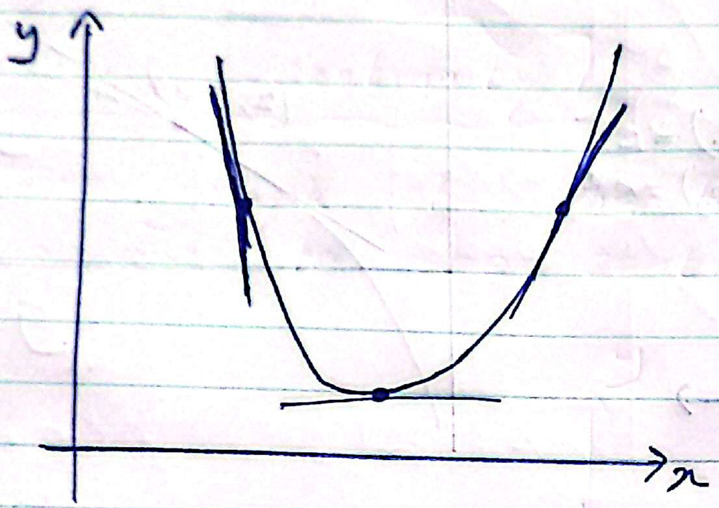
$$\frac{d}{dy} \log(y) = \frac{1}{e^{\log(y)}}$$

$$= \frac{1}{y}$$

$$\boxed{\frac{d}{dy}(\log y) = \frac{1}{y}}$$

13] Existence of the derivative

Differentiable Function



*] For a function to be differentiable at a point:

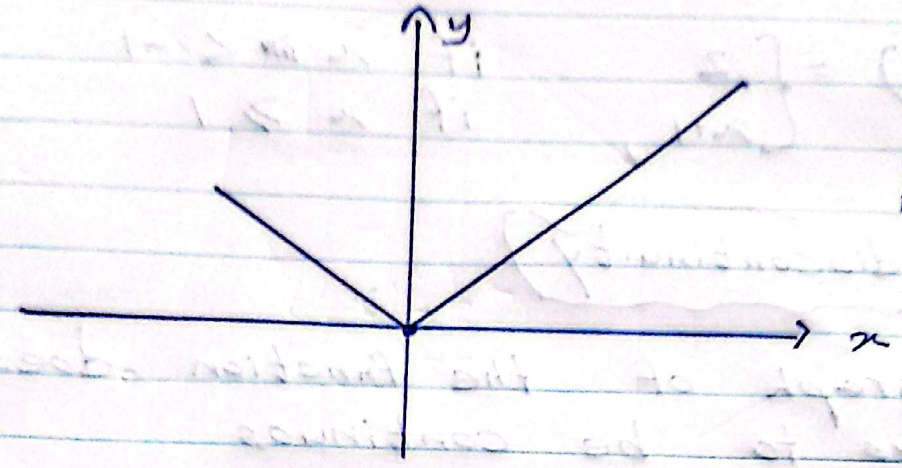
The derivative has to exist for that point

*] For a function to be differentiable at an interval:

The derivative has to exist for every point in the interval

Non Differentiable functions

[1]



$f(x) = |x|$

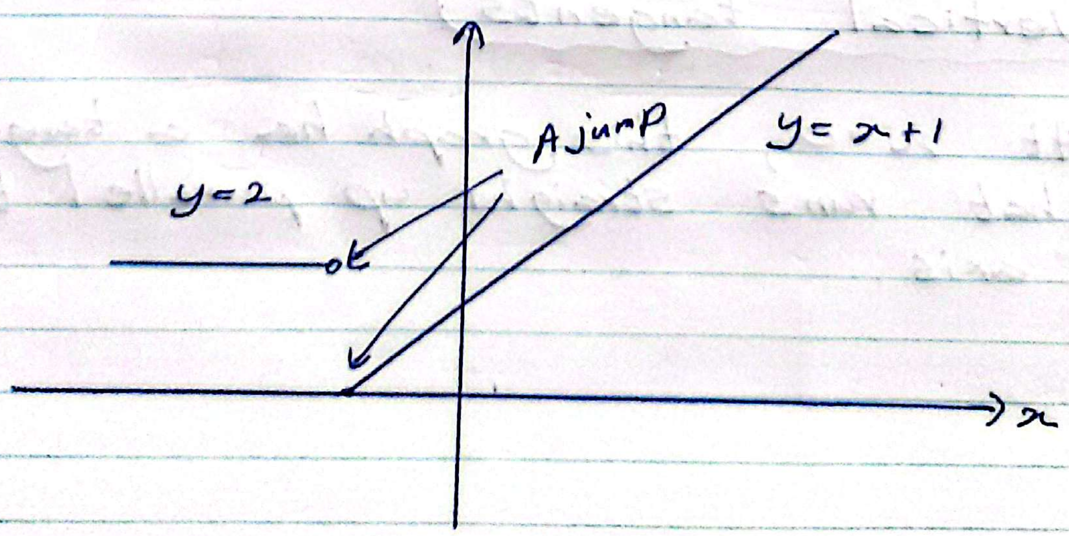
The absolute value function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

At $x = 0$, the derivative does not exist

*] Generally when a function has a corner or a cusp, the function is not differentiable at that point.

[2]



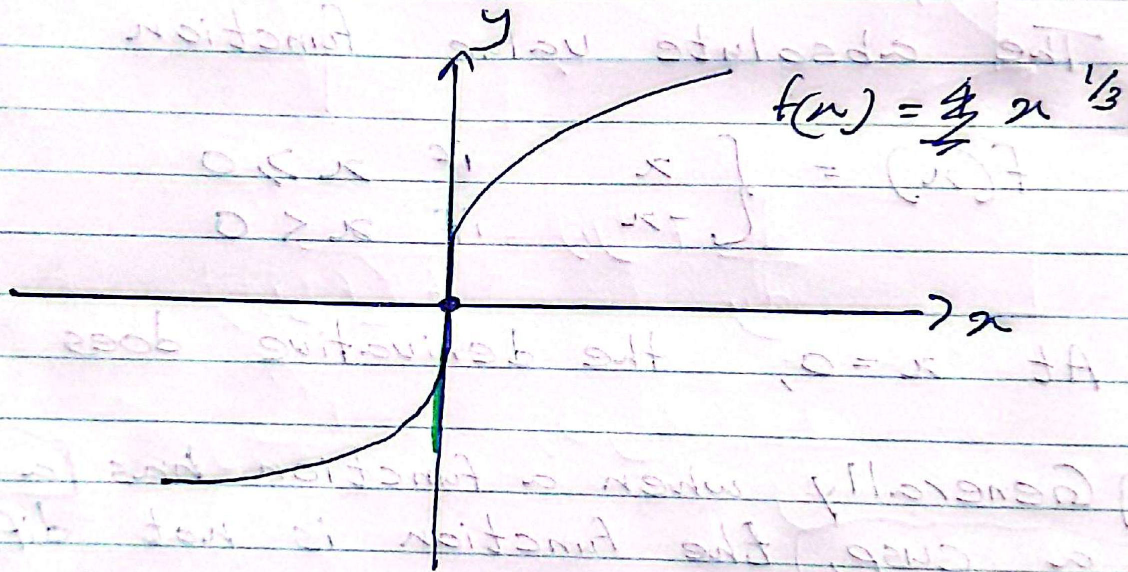
This is a piece-wise function

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ x+1, & \text{if } x \geq 1 \end{cases}$$

Jump discontinuity

*) The graph of the function does not appears to be continuos

[3]



Vertical tangents

*) At $x=0$, this graph ha a tangent line that run's straight up parallel to the y axis.