



DeepLearning.AI

Optimization in Neural Networks and Newton's Method

Gradient Descent and Backpropagation

Back Propagation Introduction

Back Propagation Introduction



Back Propagation Introduction



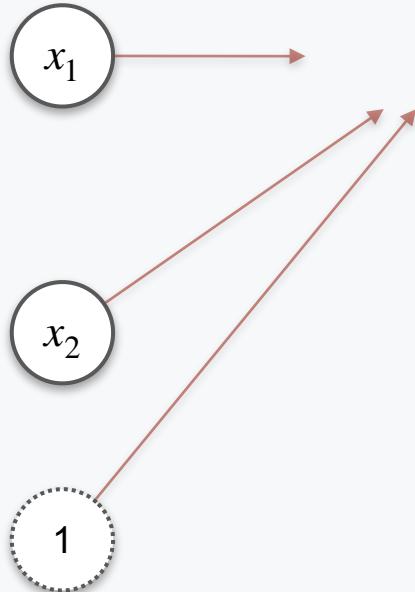
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x_1

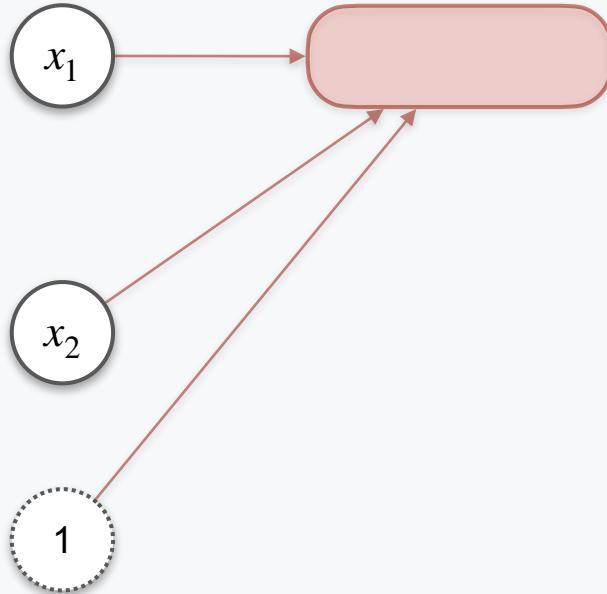
x_2

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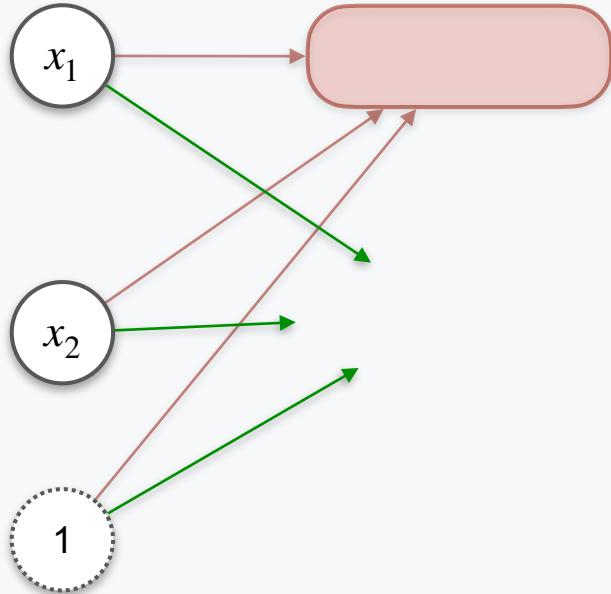
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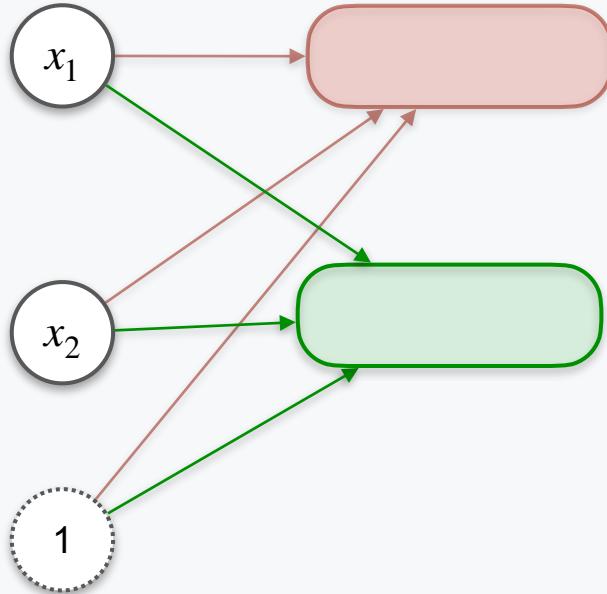
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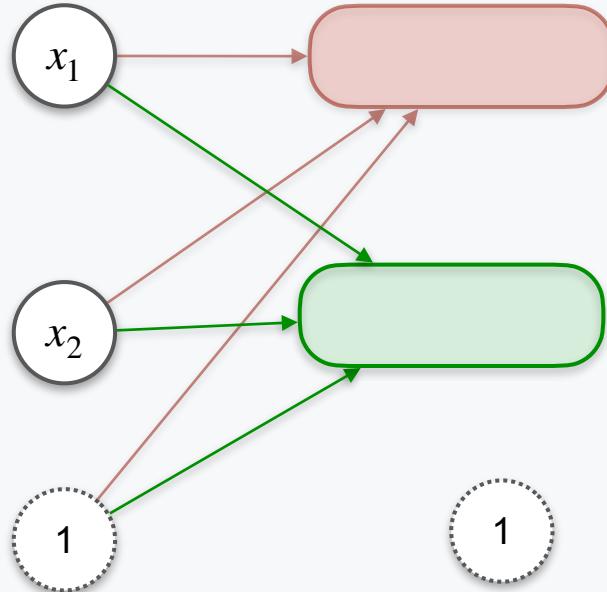
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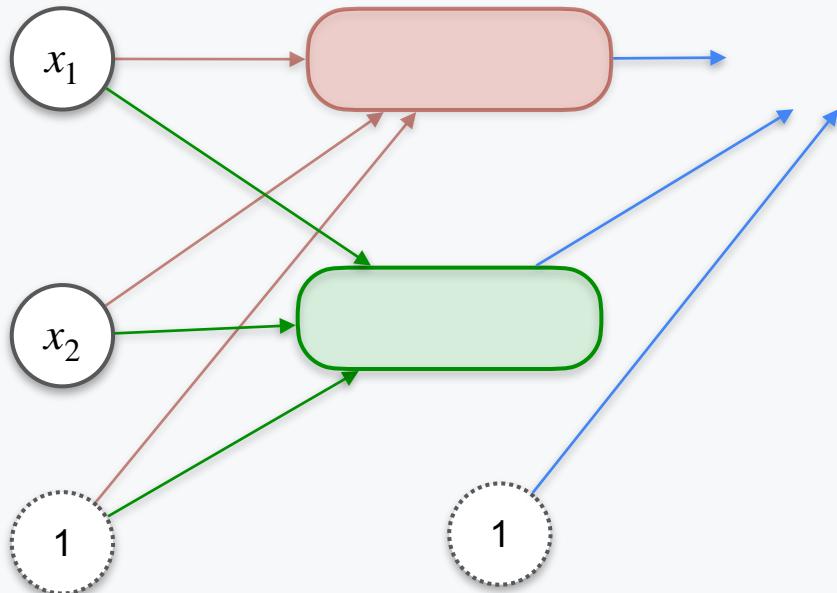
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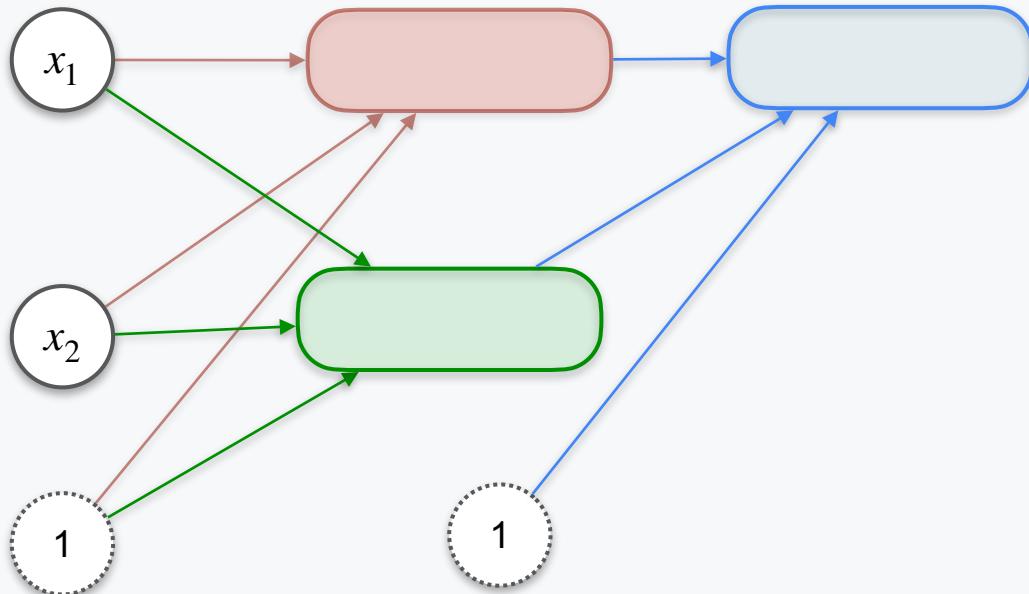
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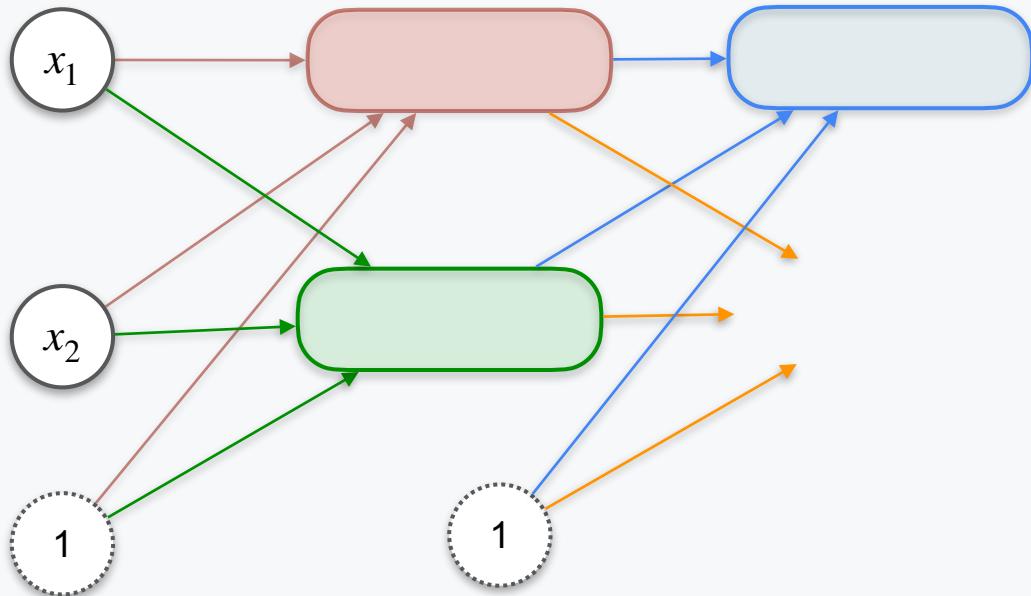
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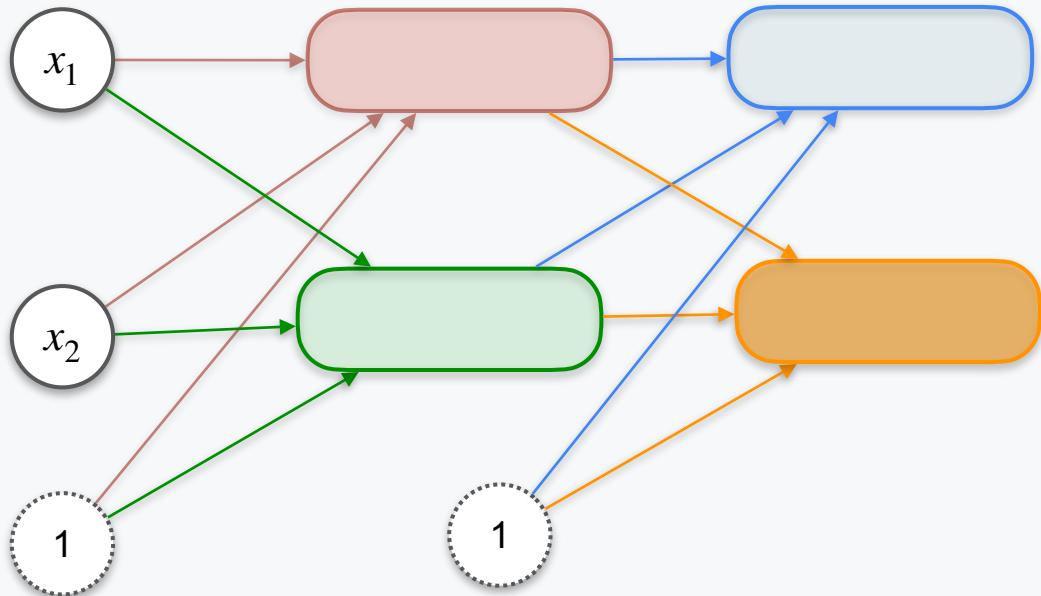
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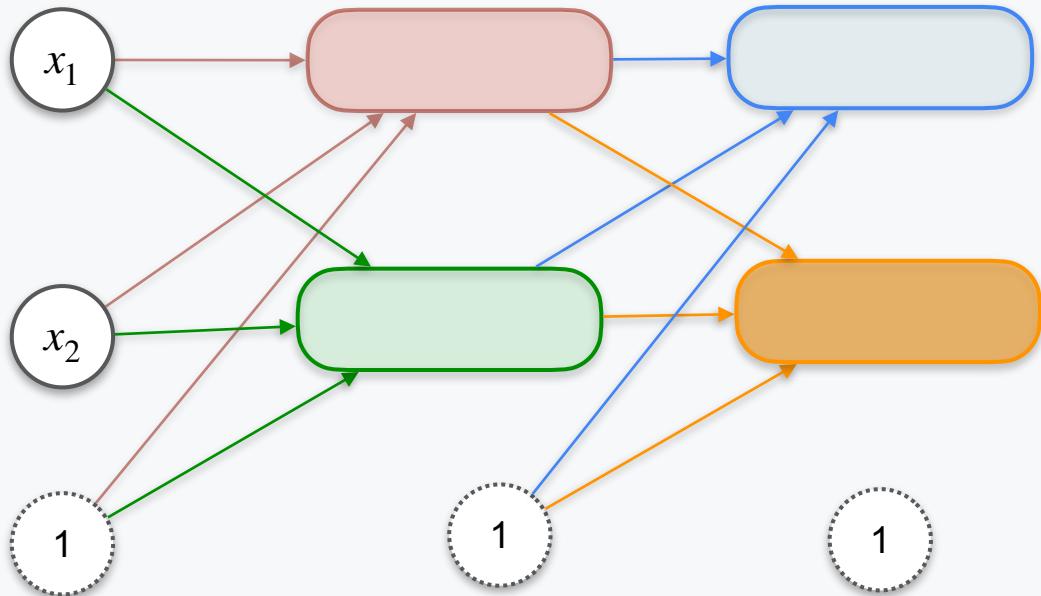
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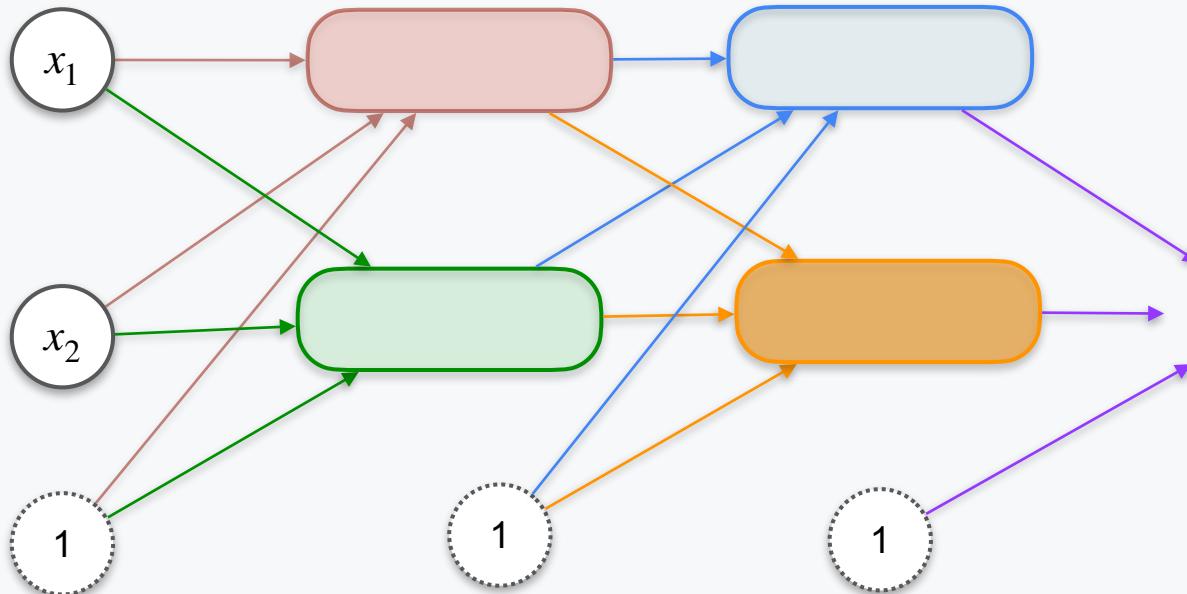
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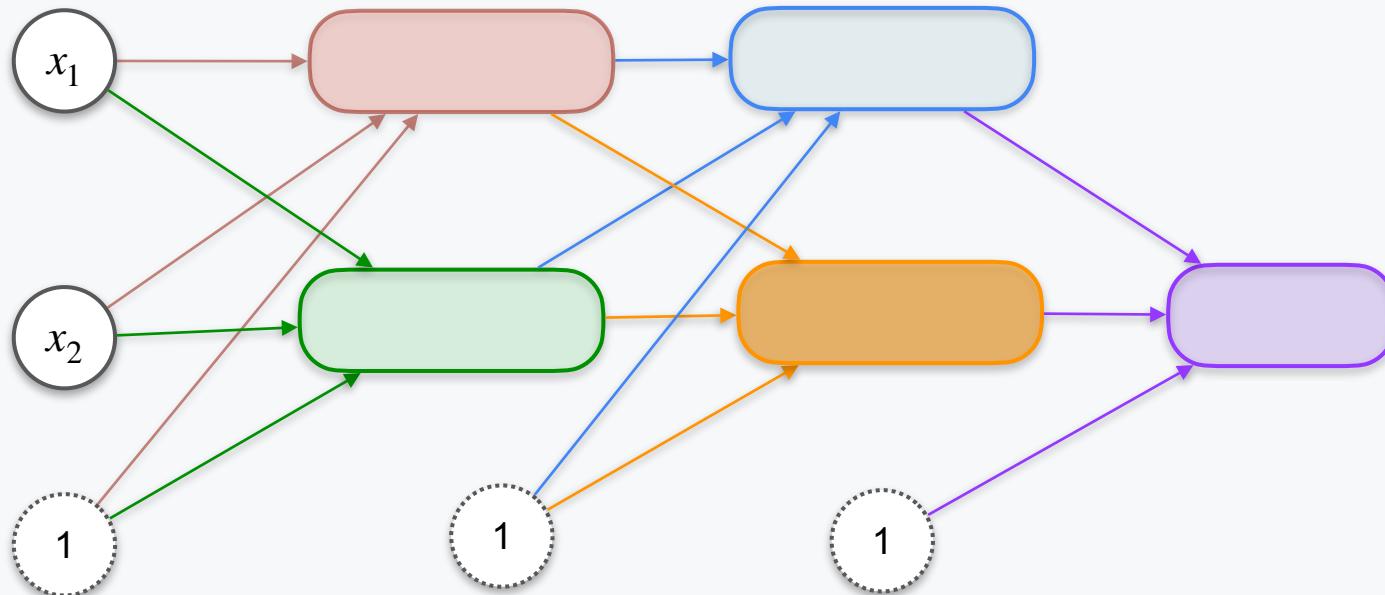
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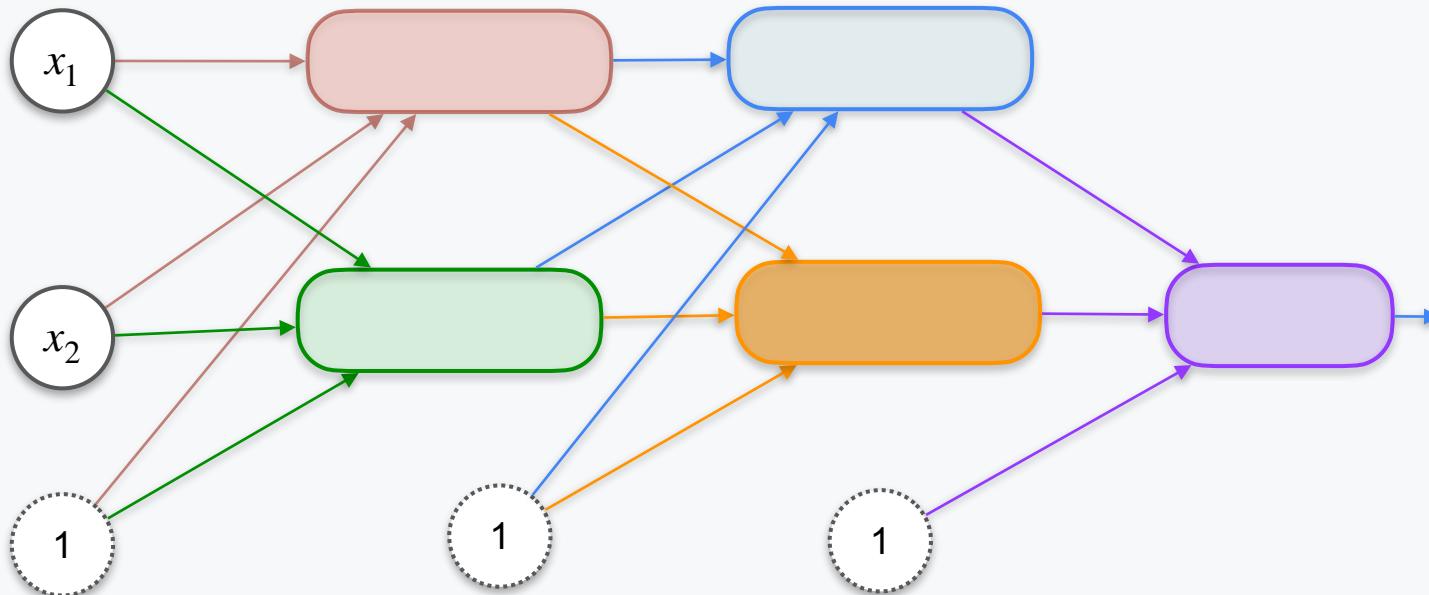
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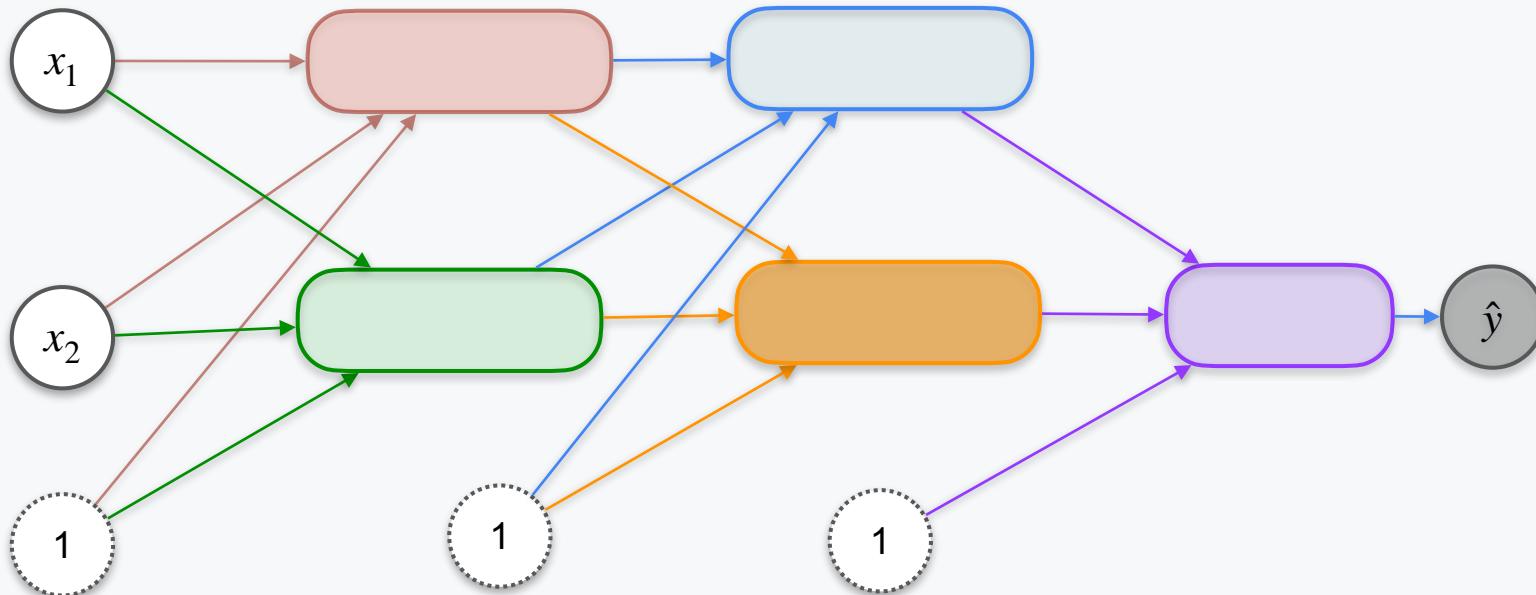
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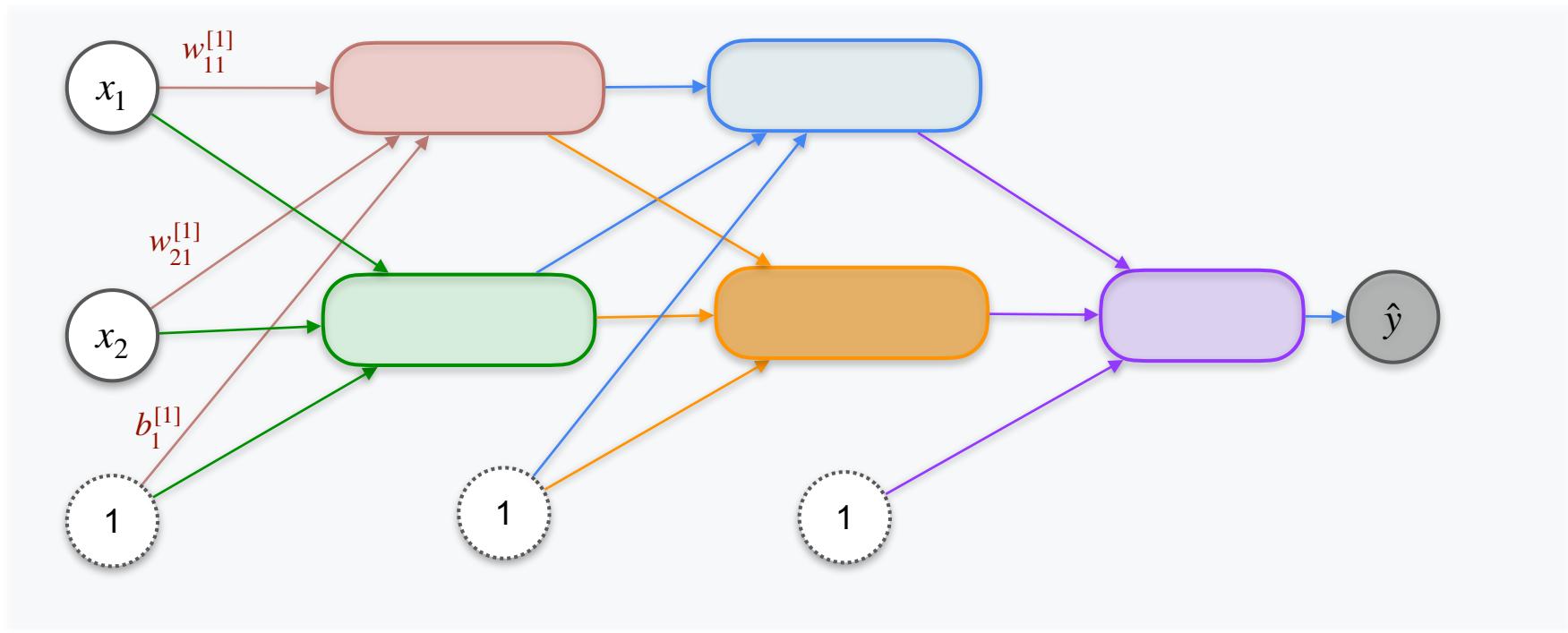
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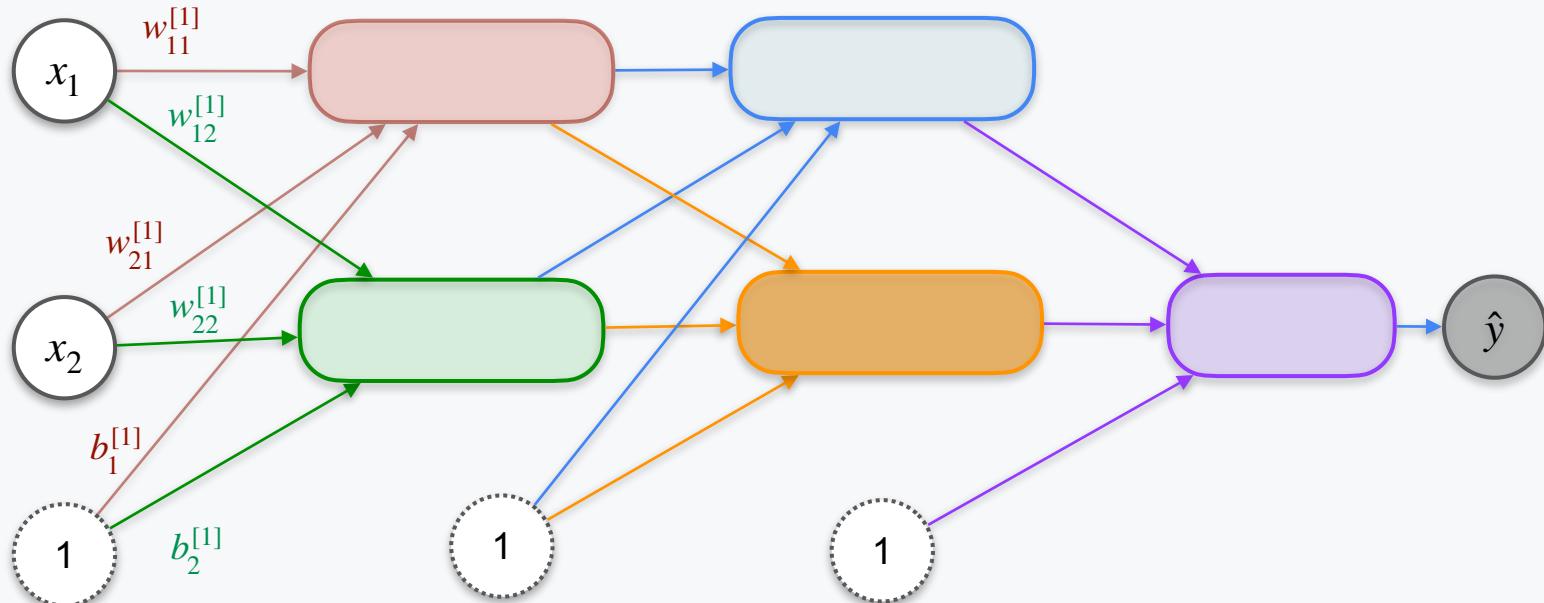
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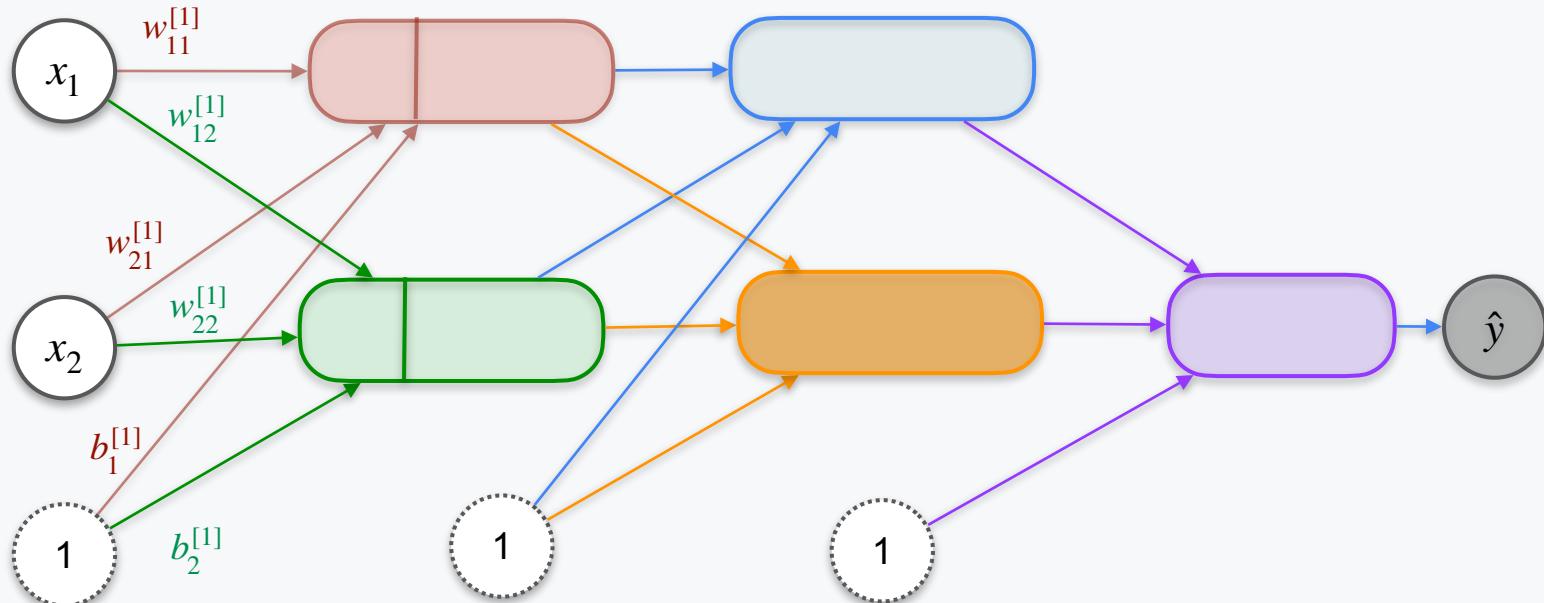
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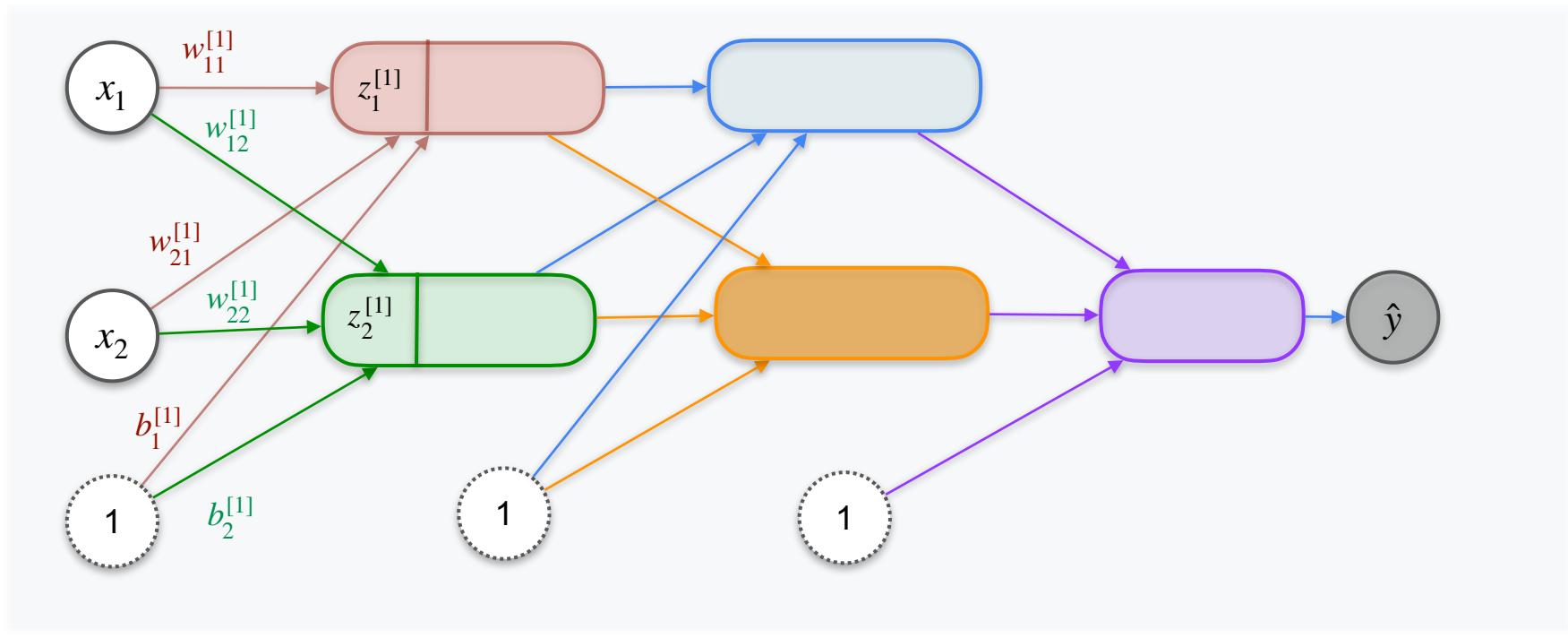
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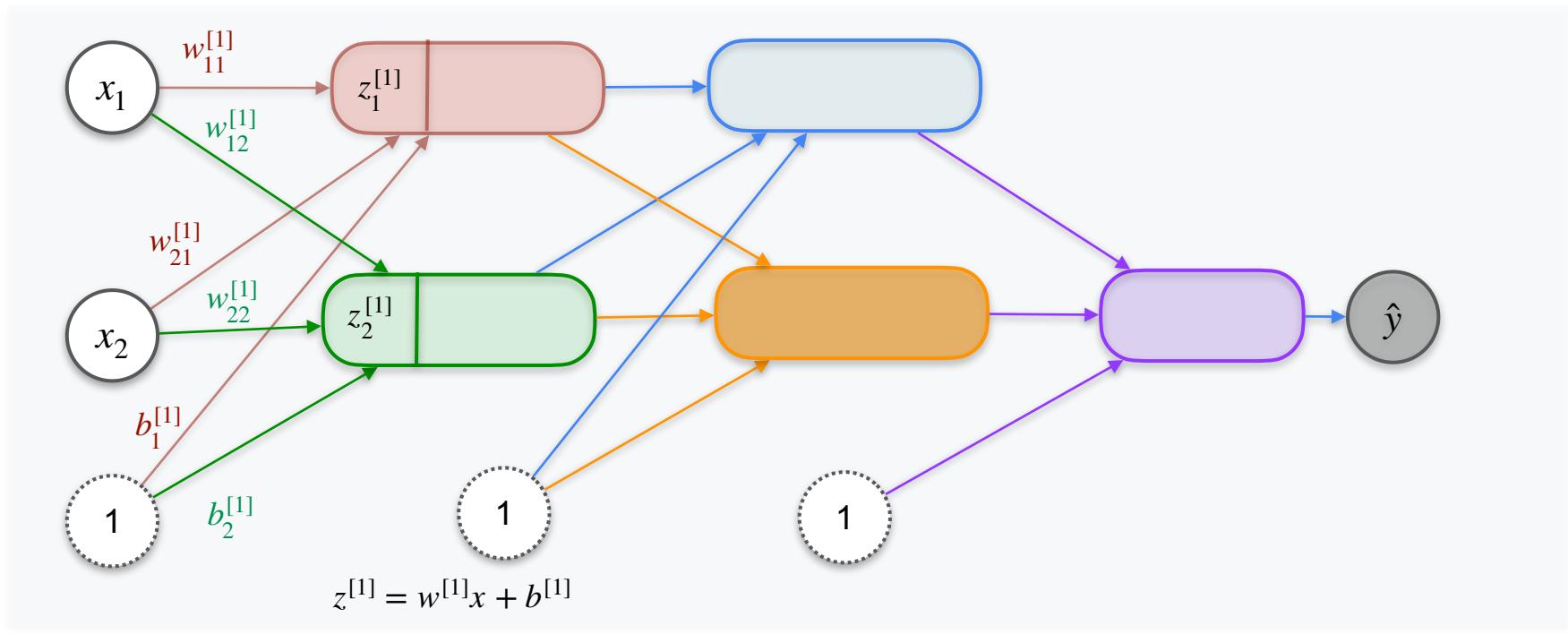
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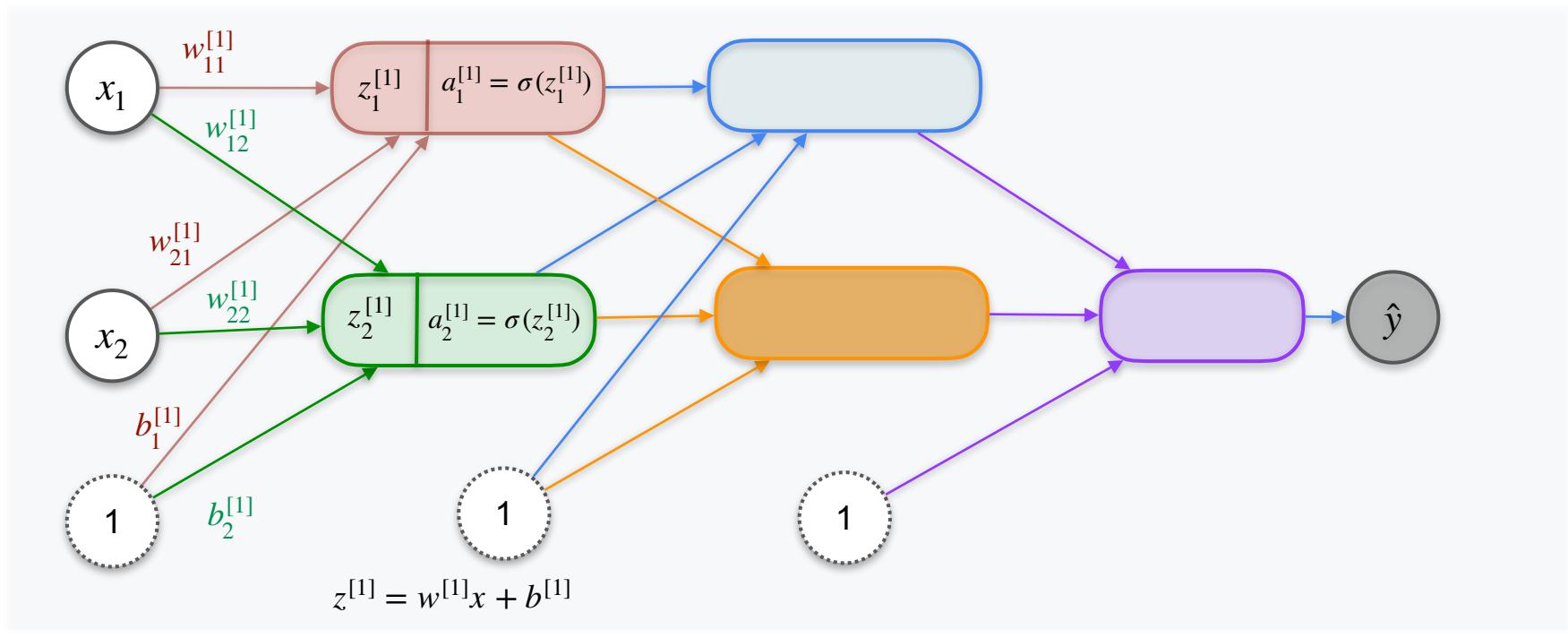
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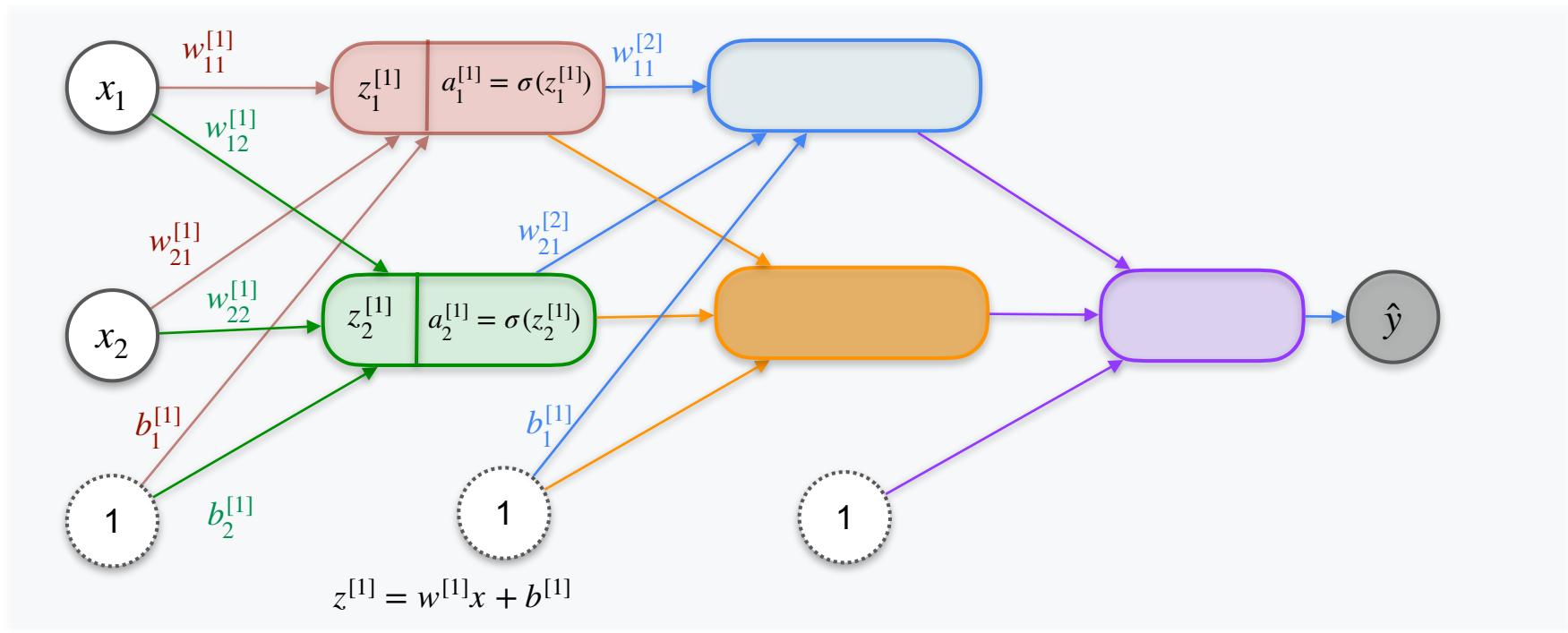
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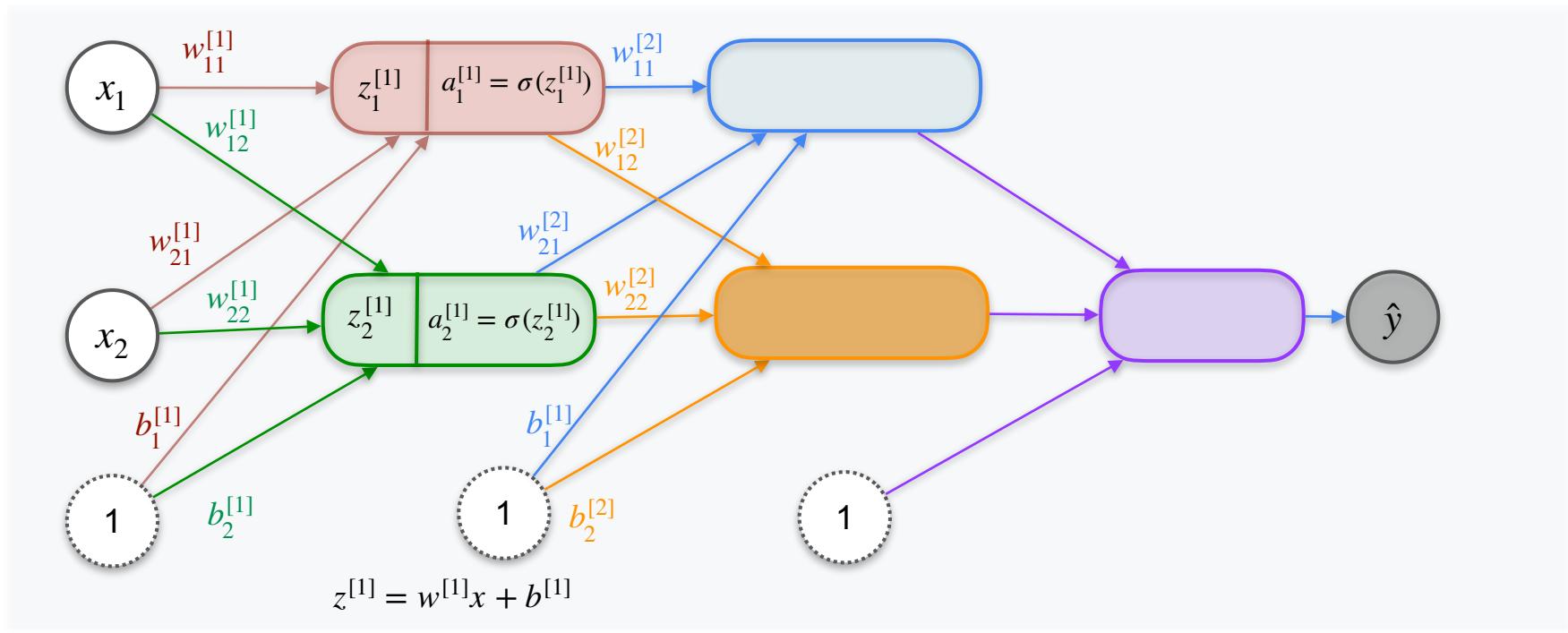
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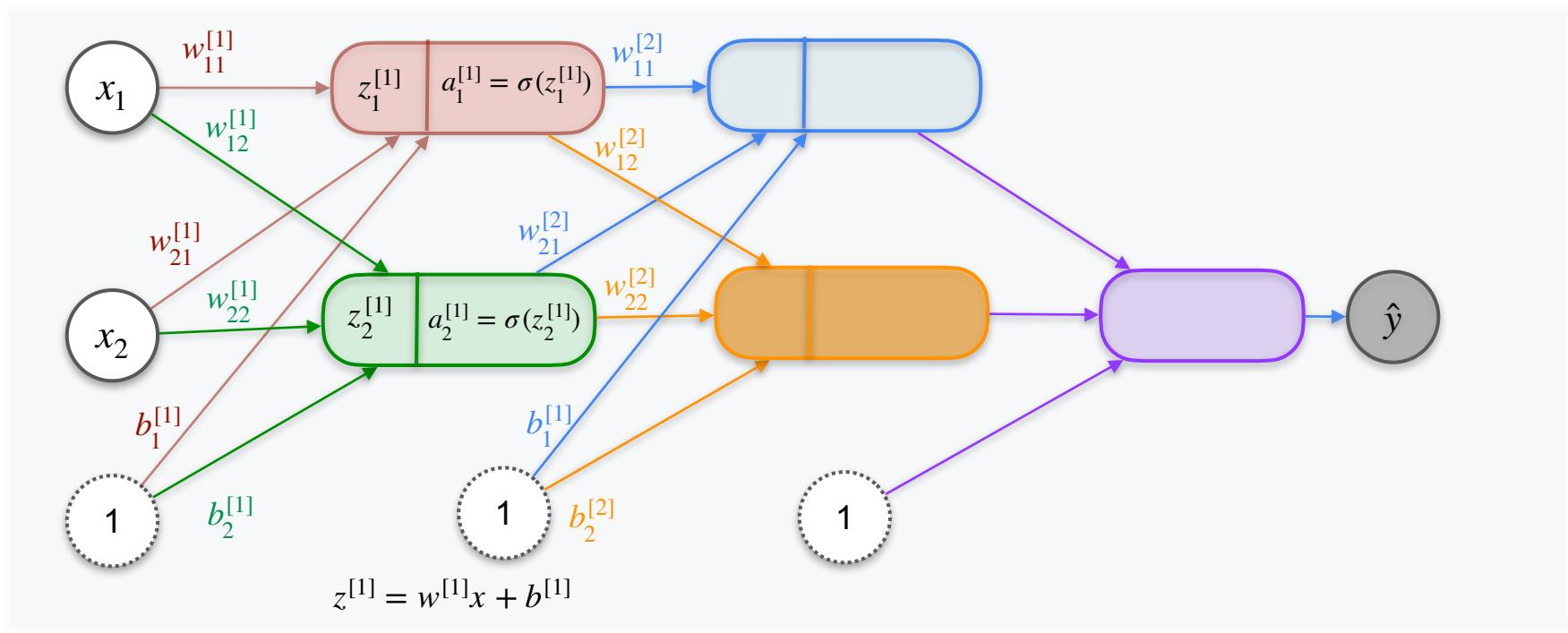
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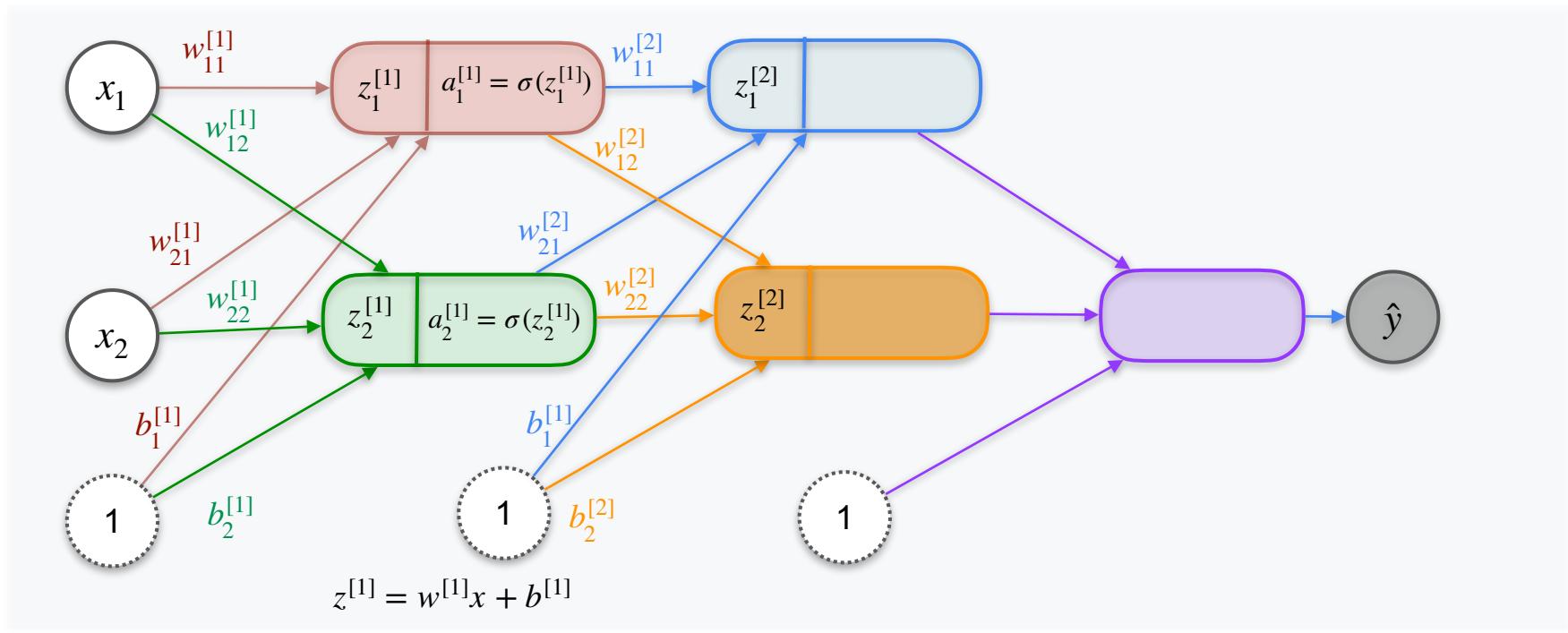
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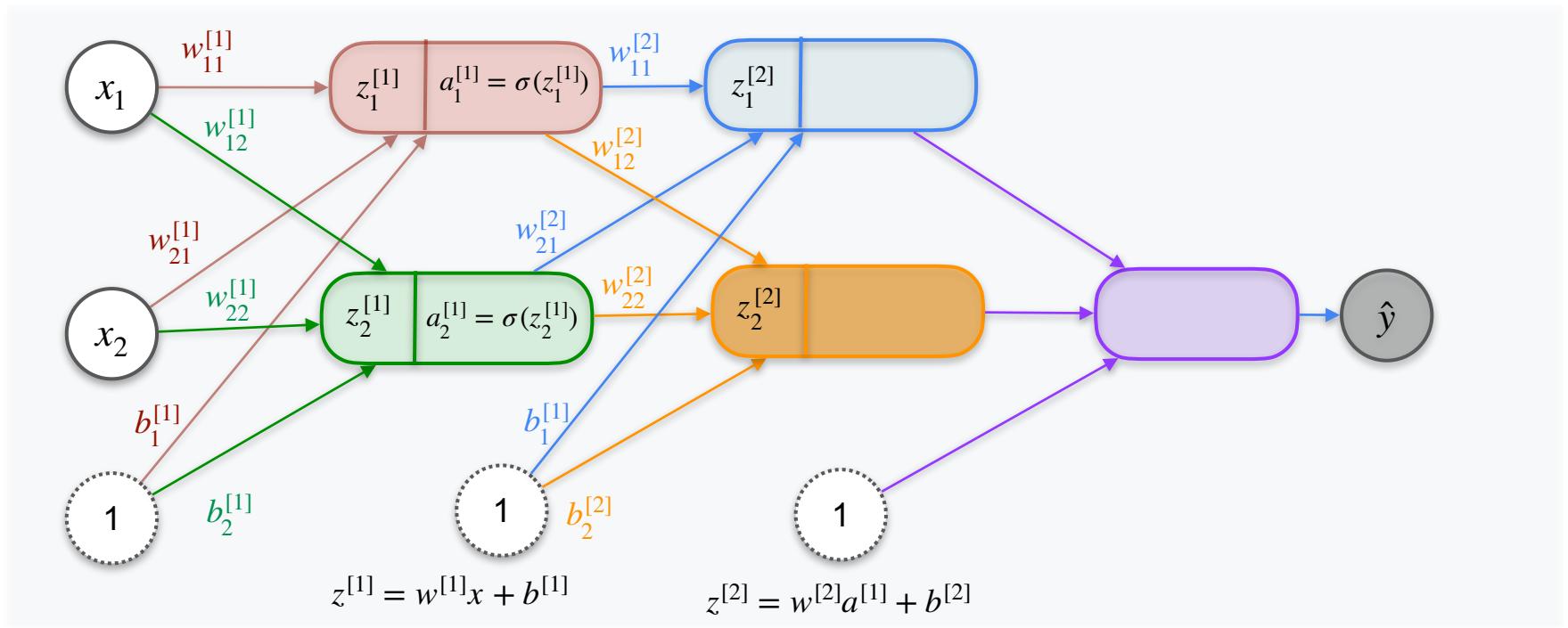
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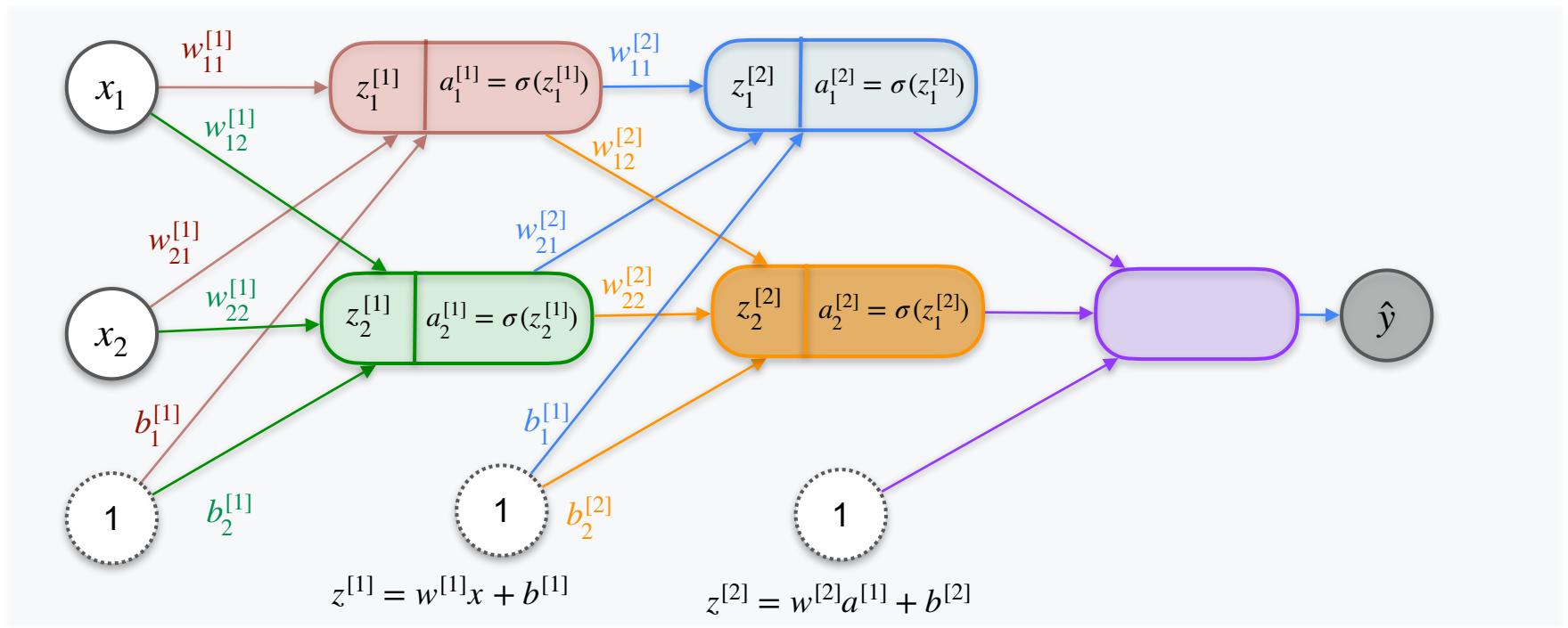
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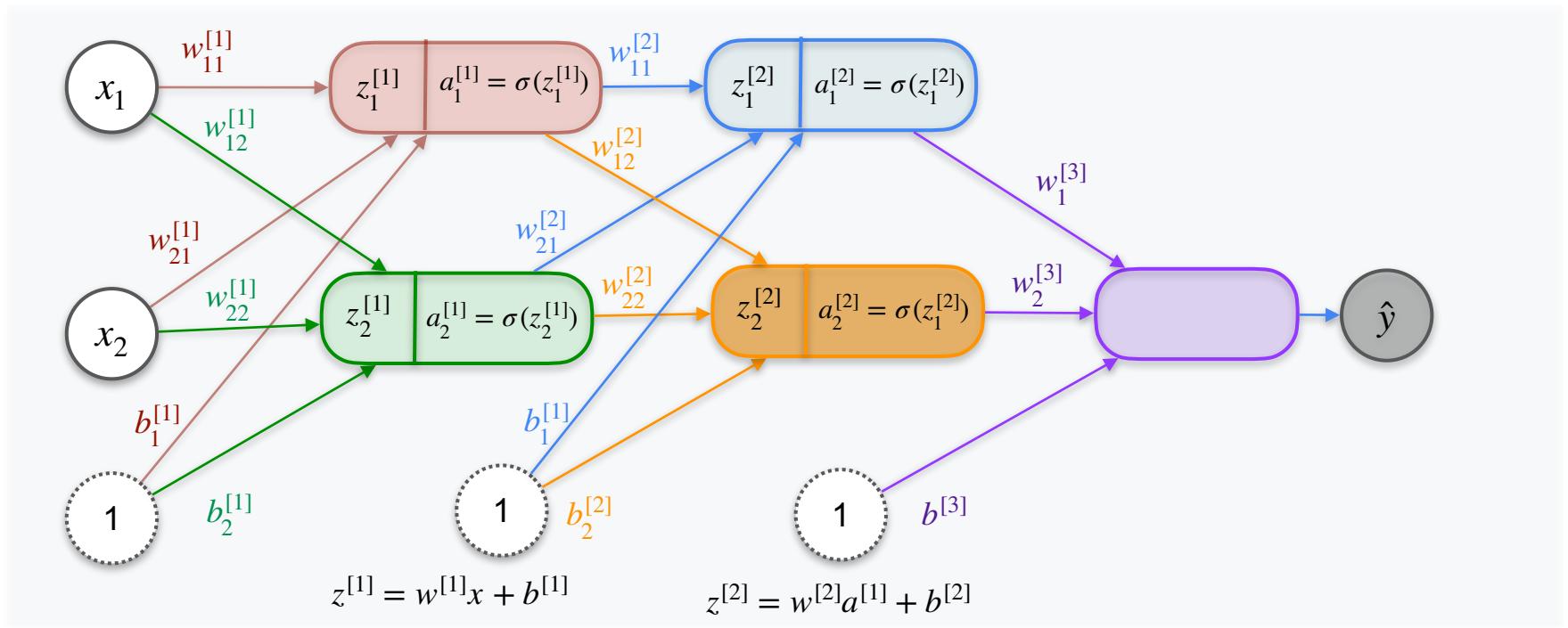
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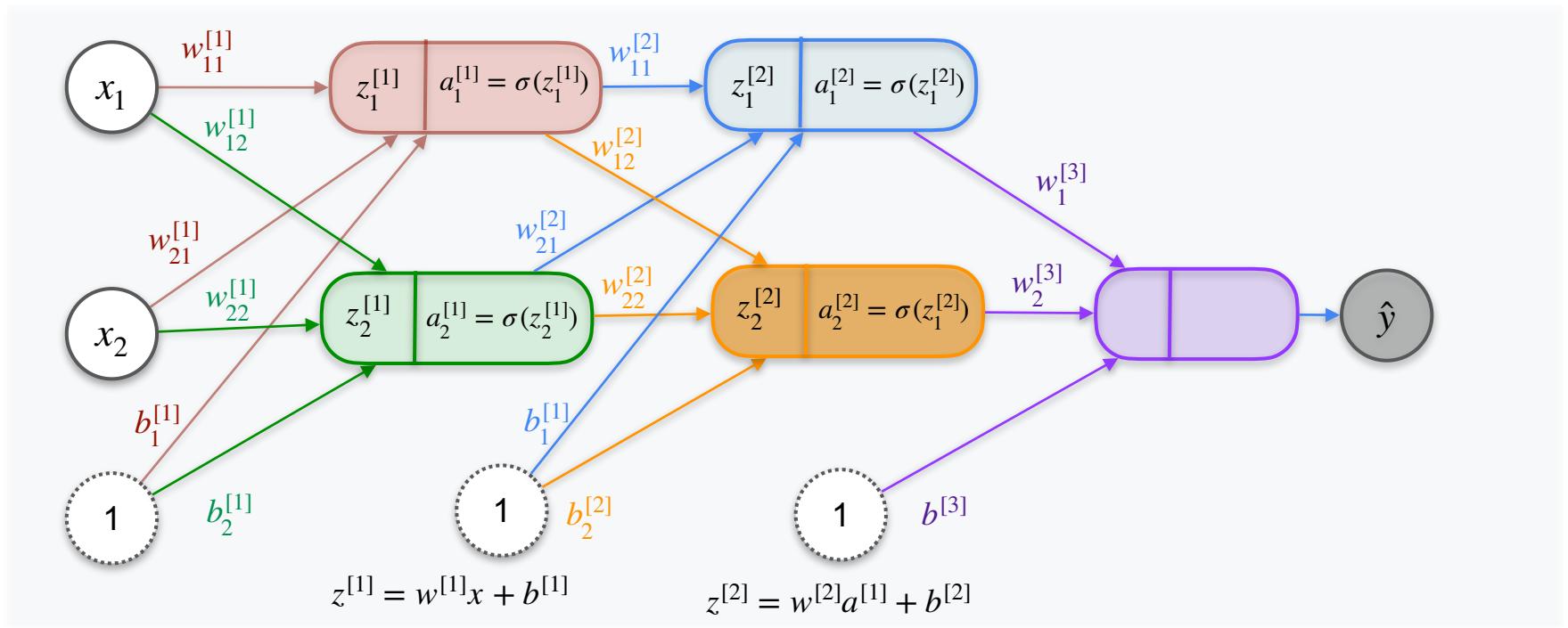
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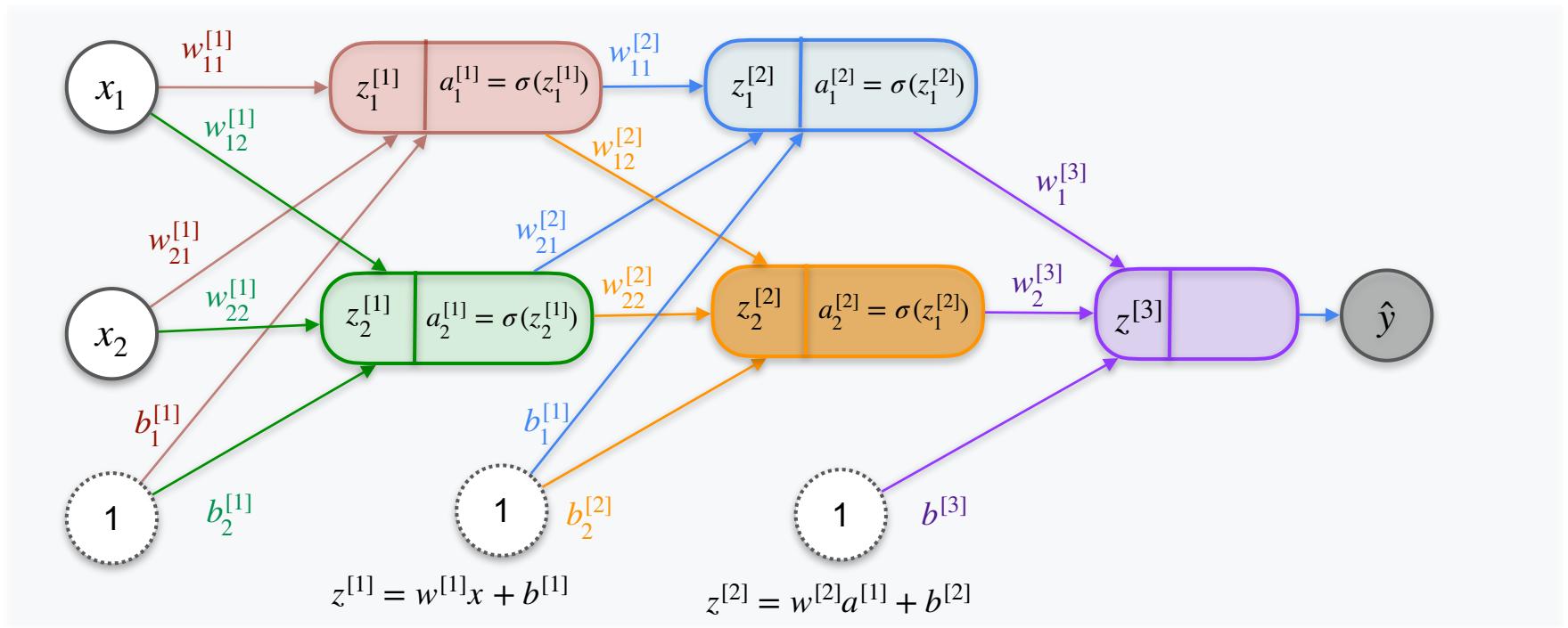
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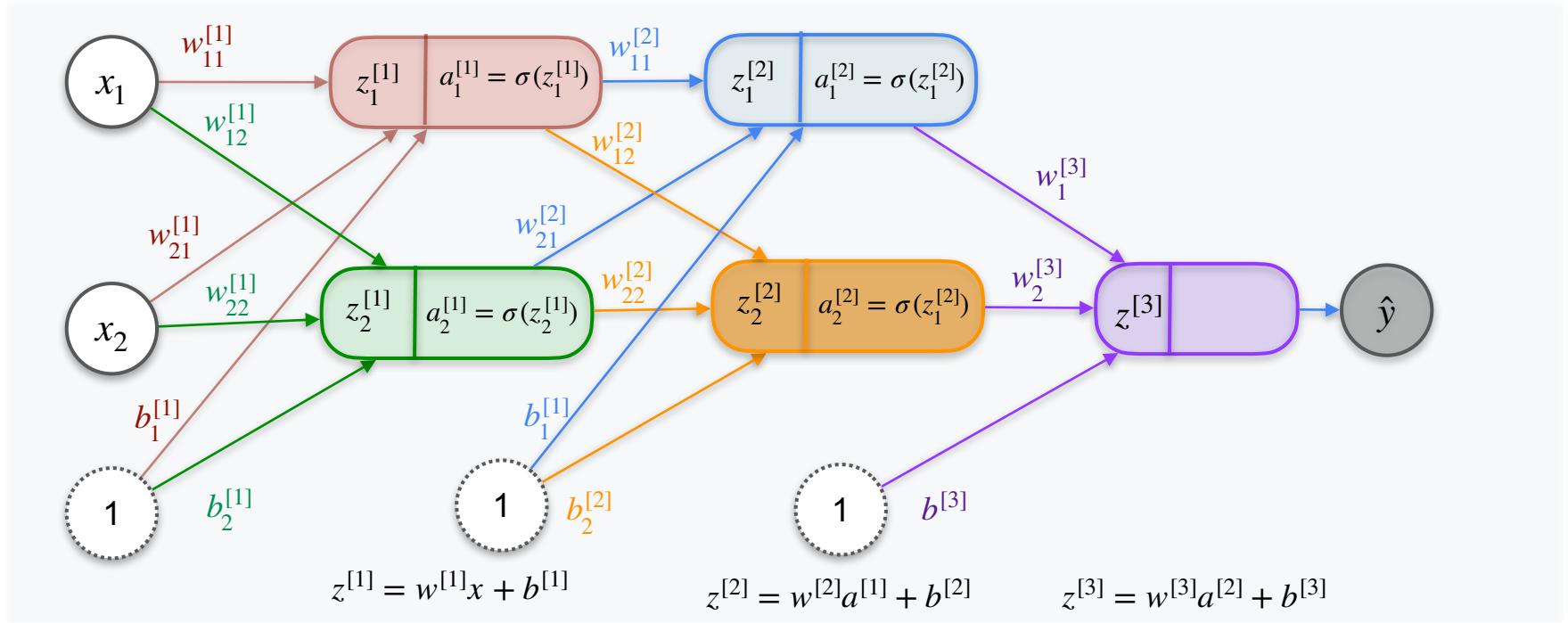
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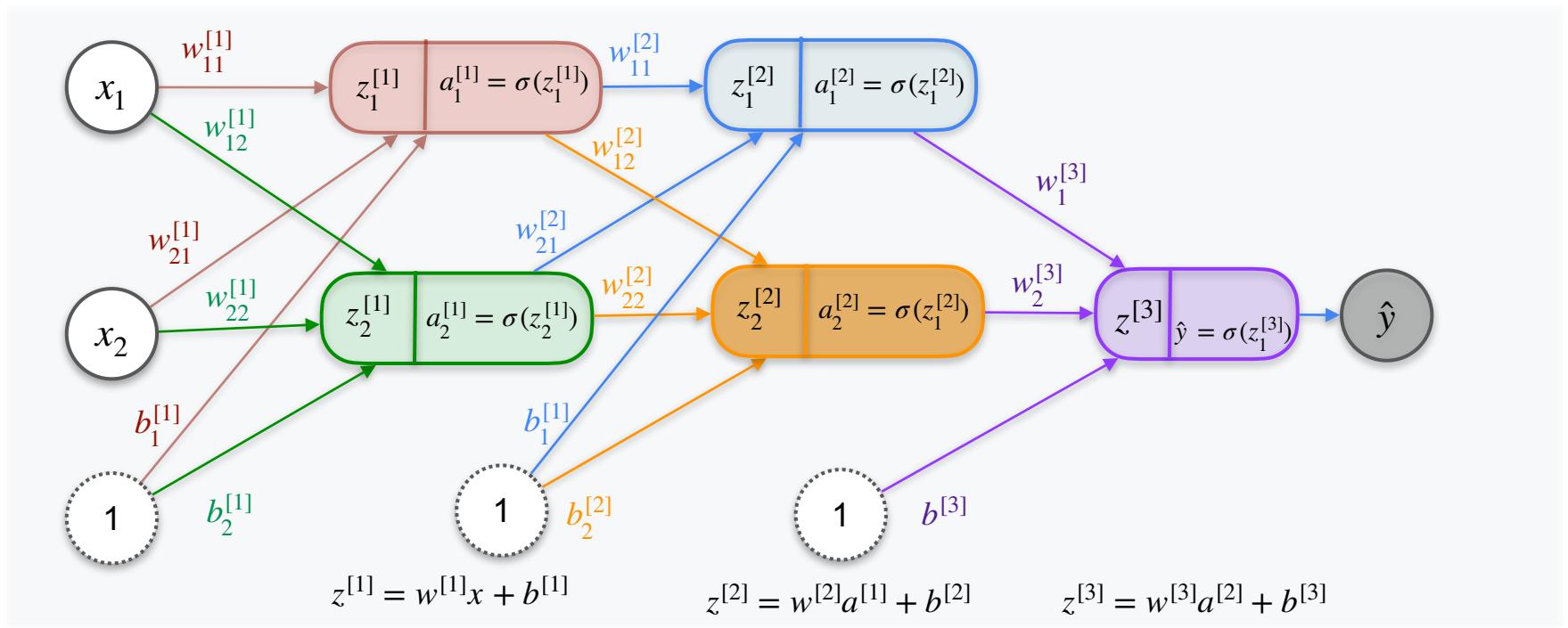
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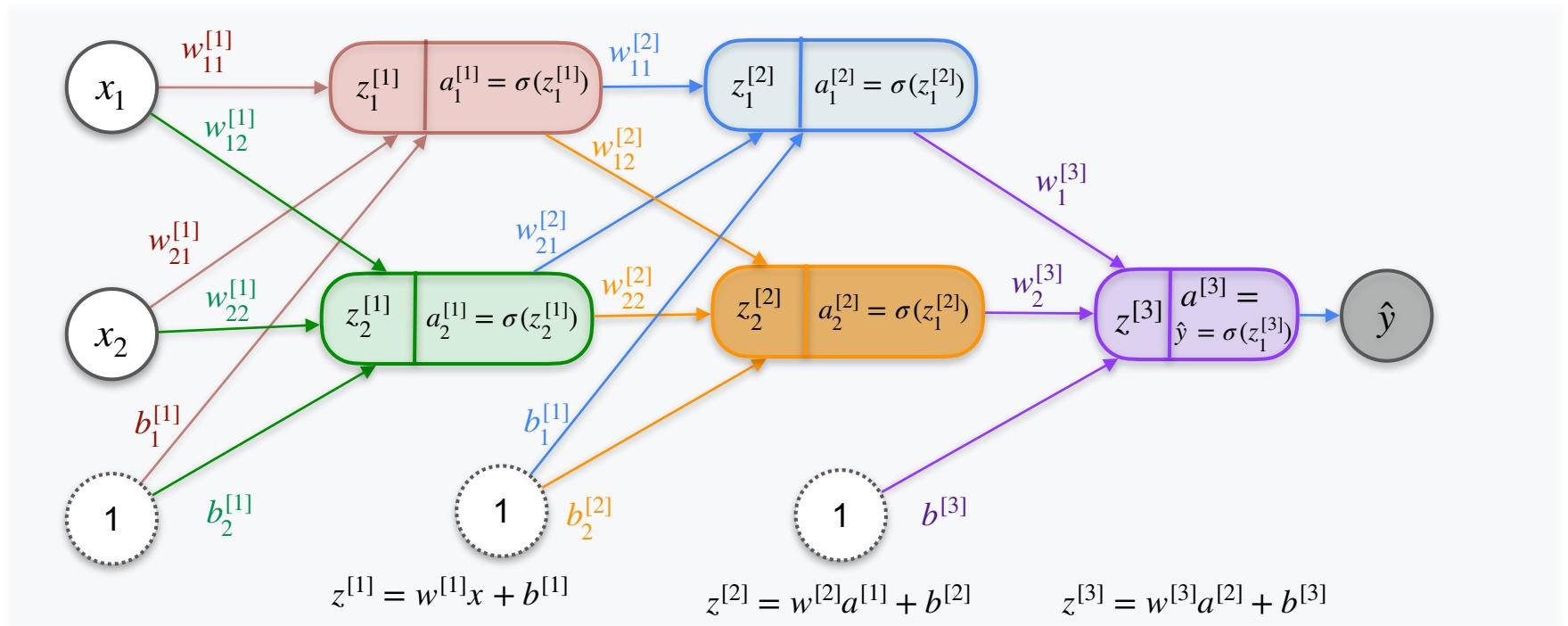
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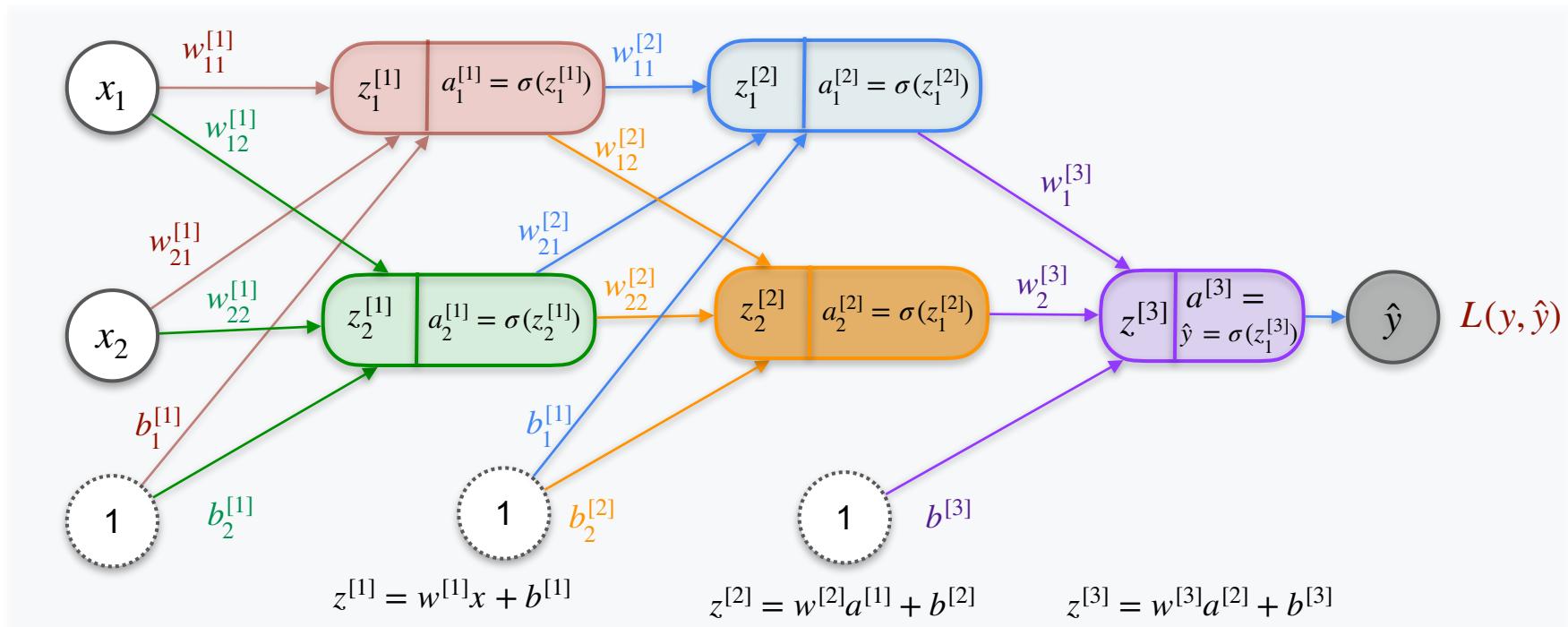
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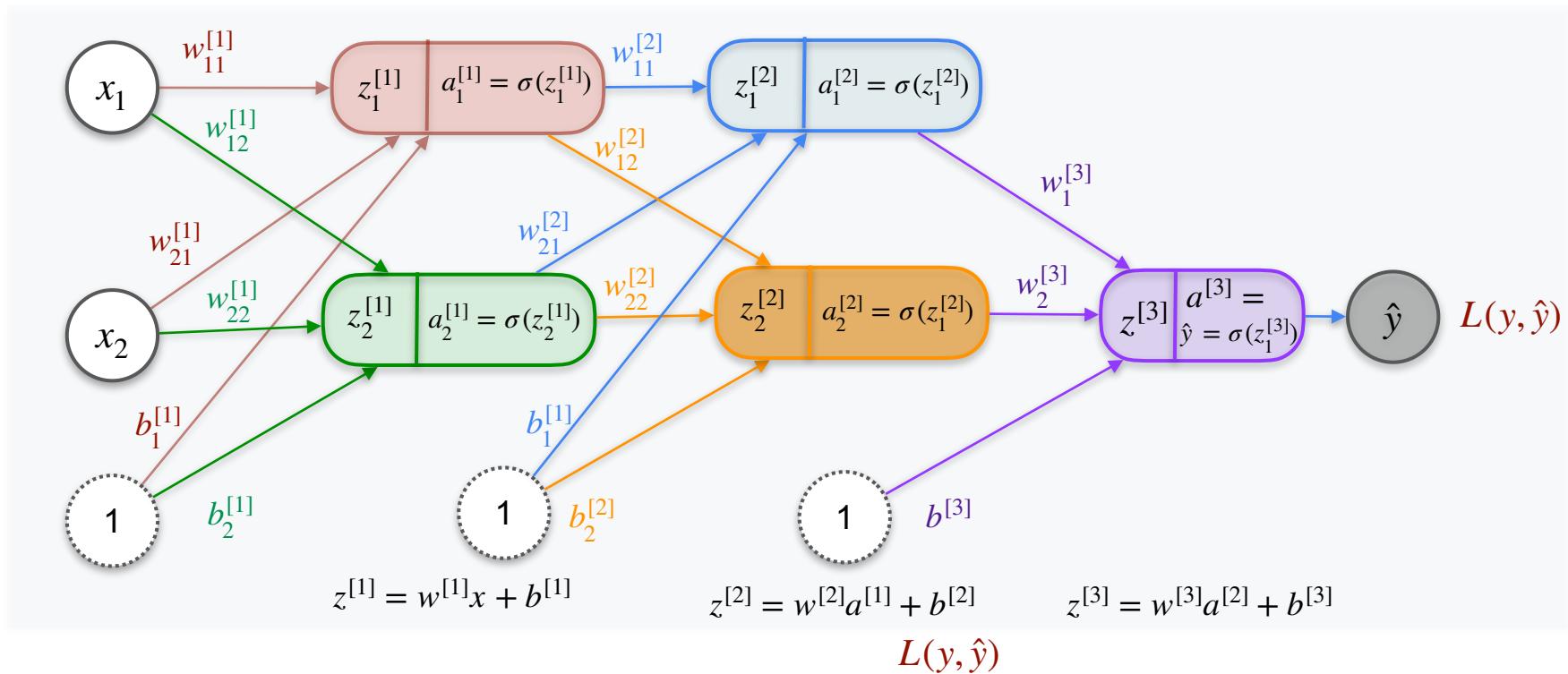
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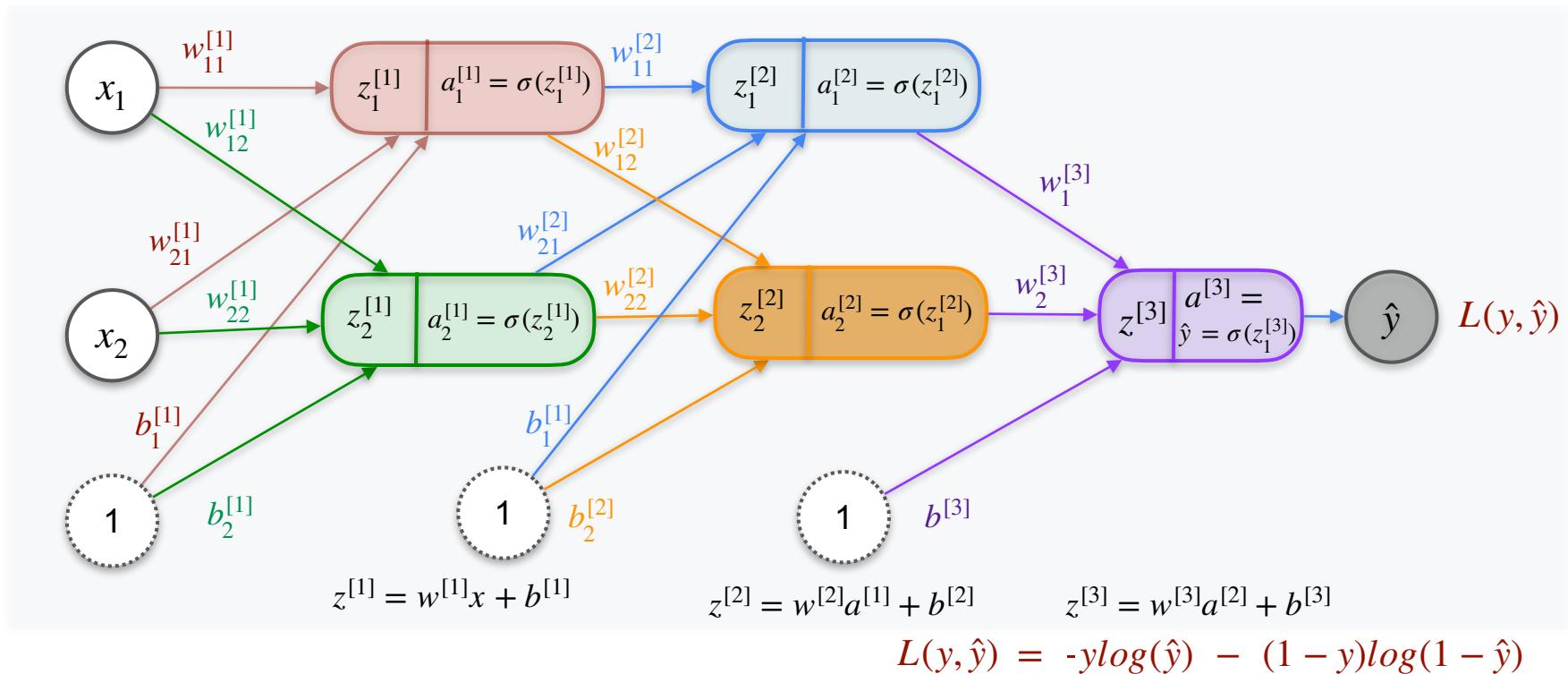
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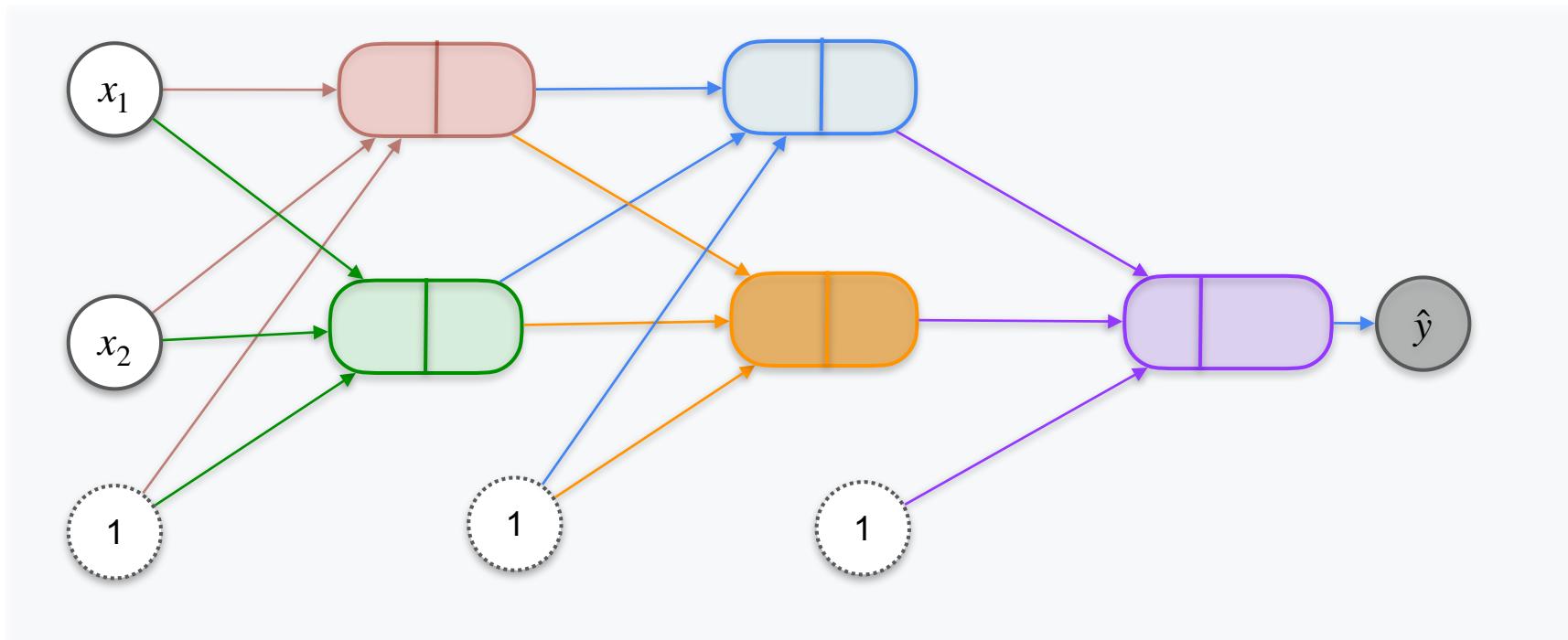
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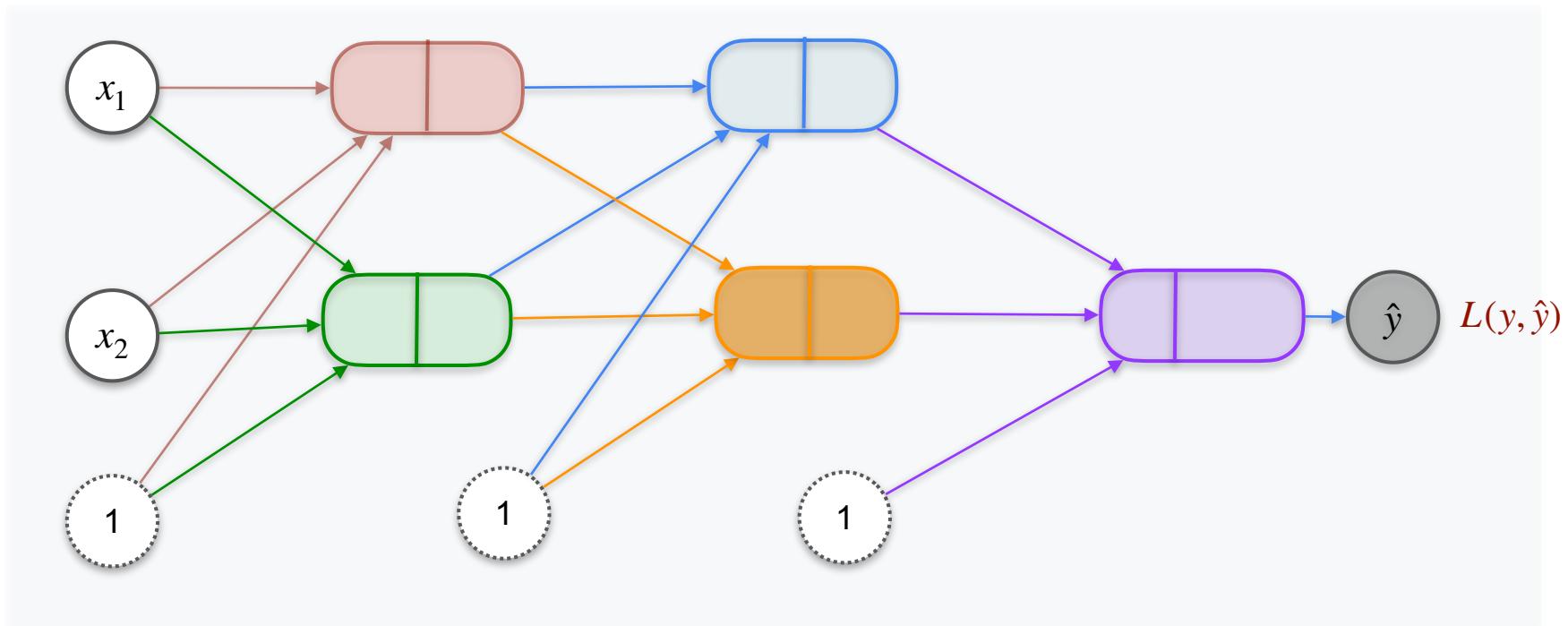
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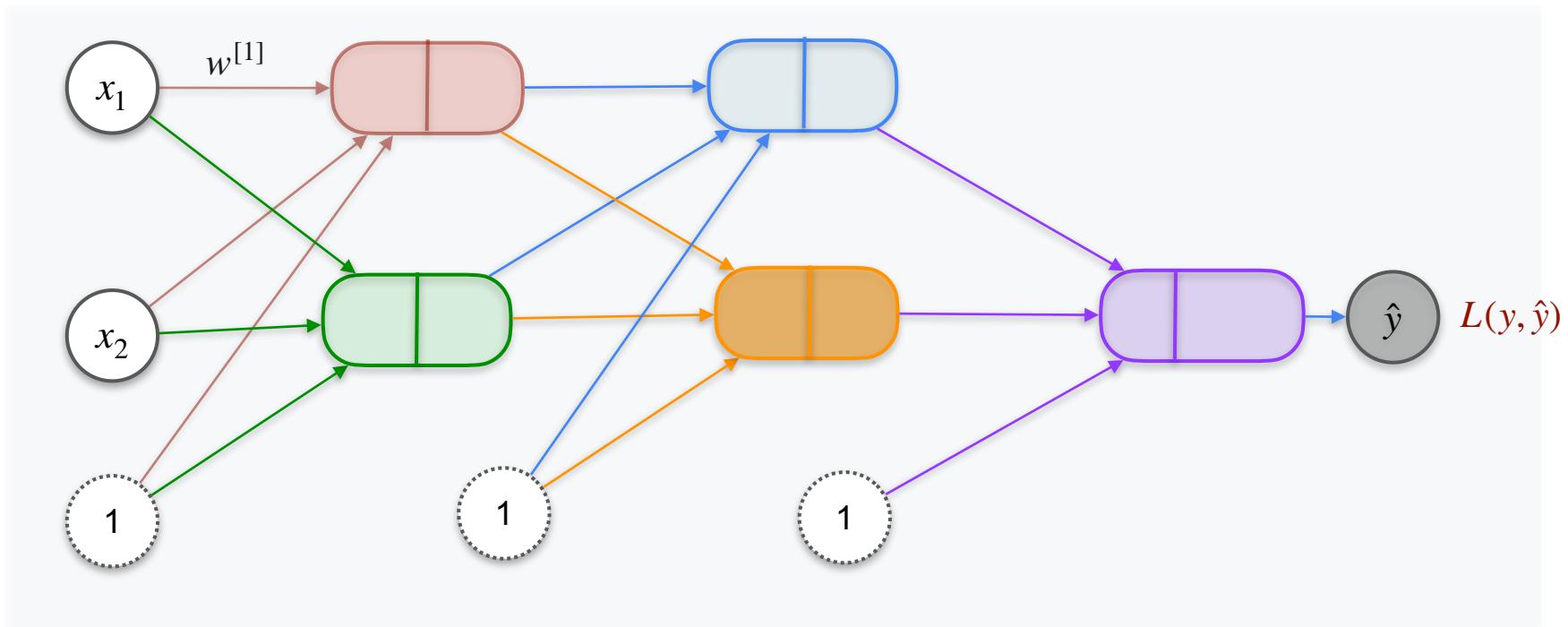
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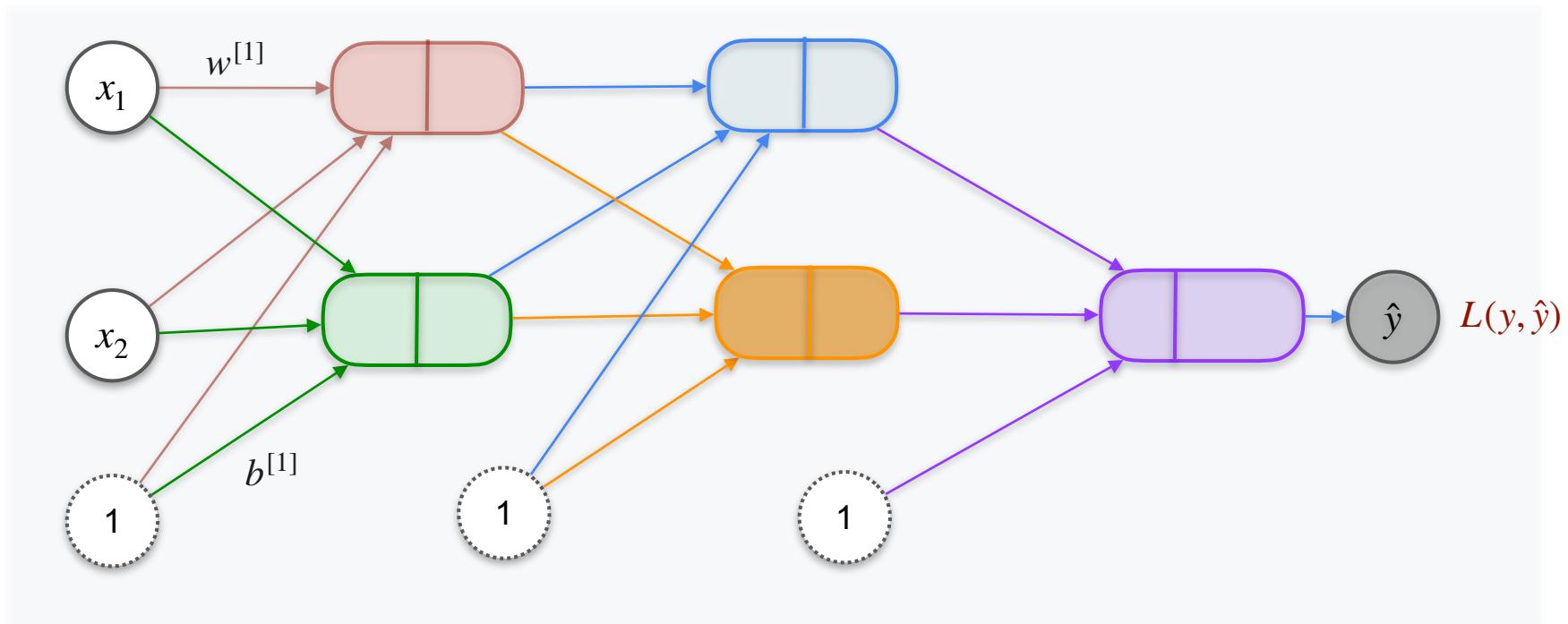
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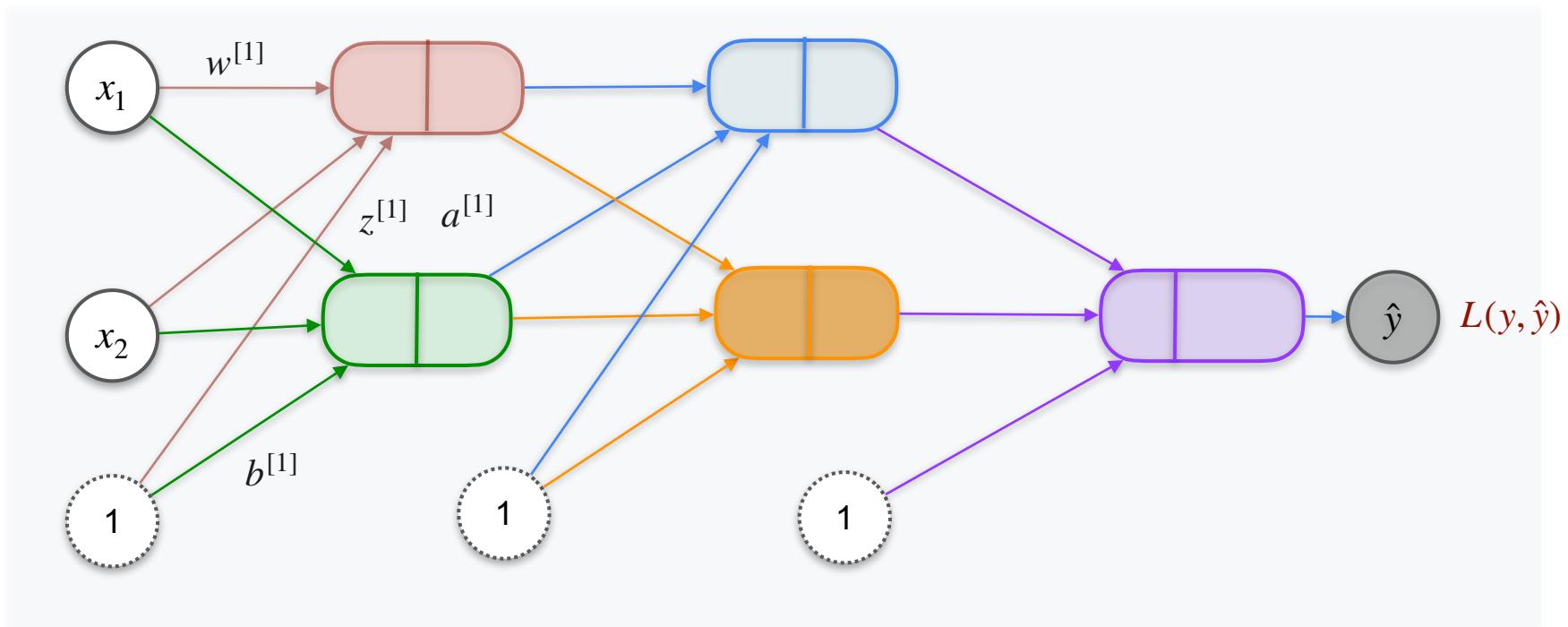
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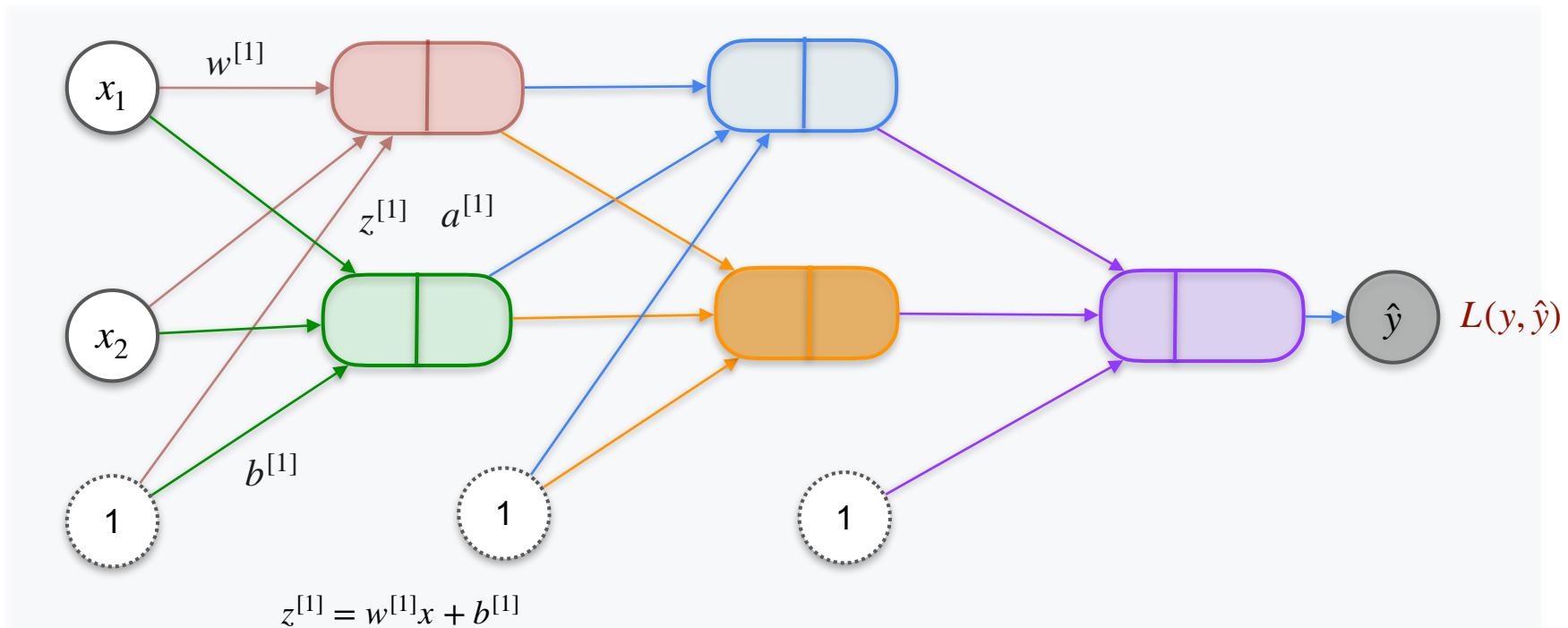
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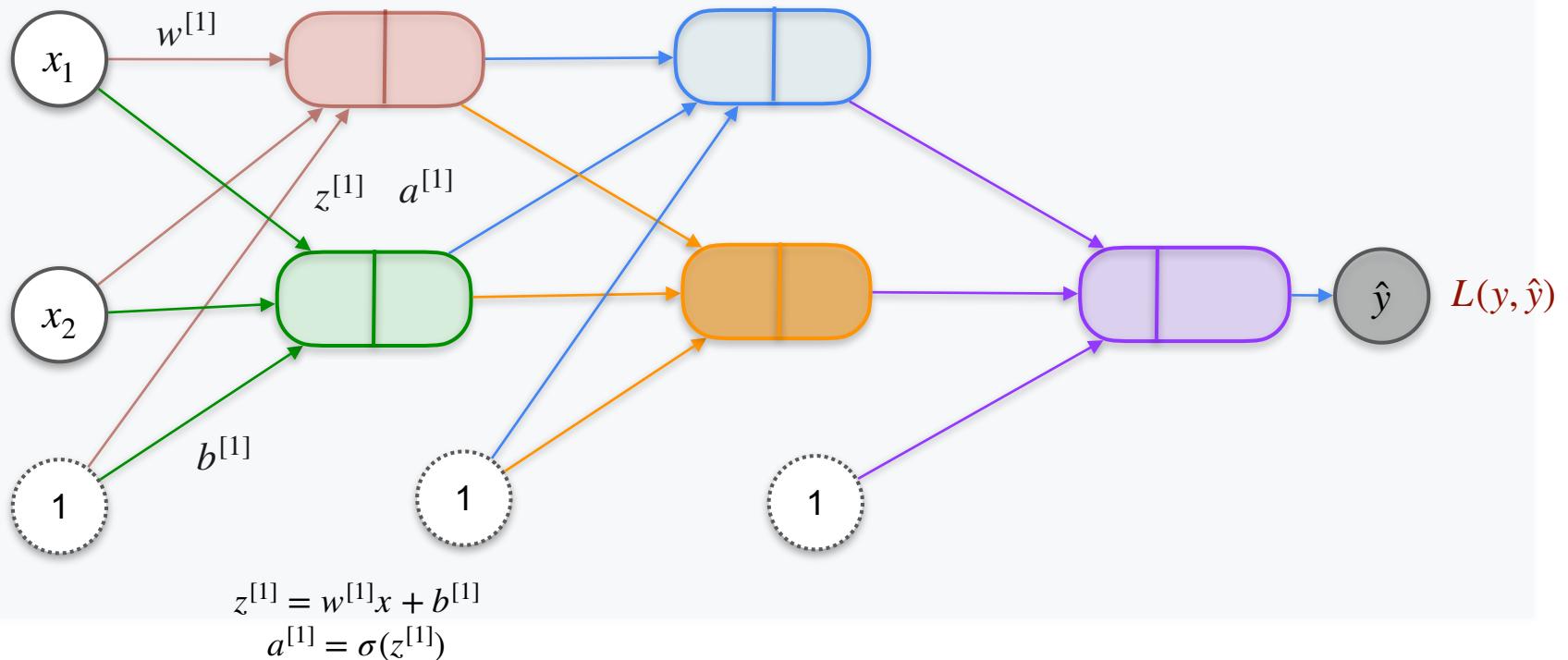
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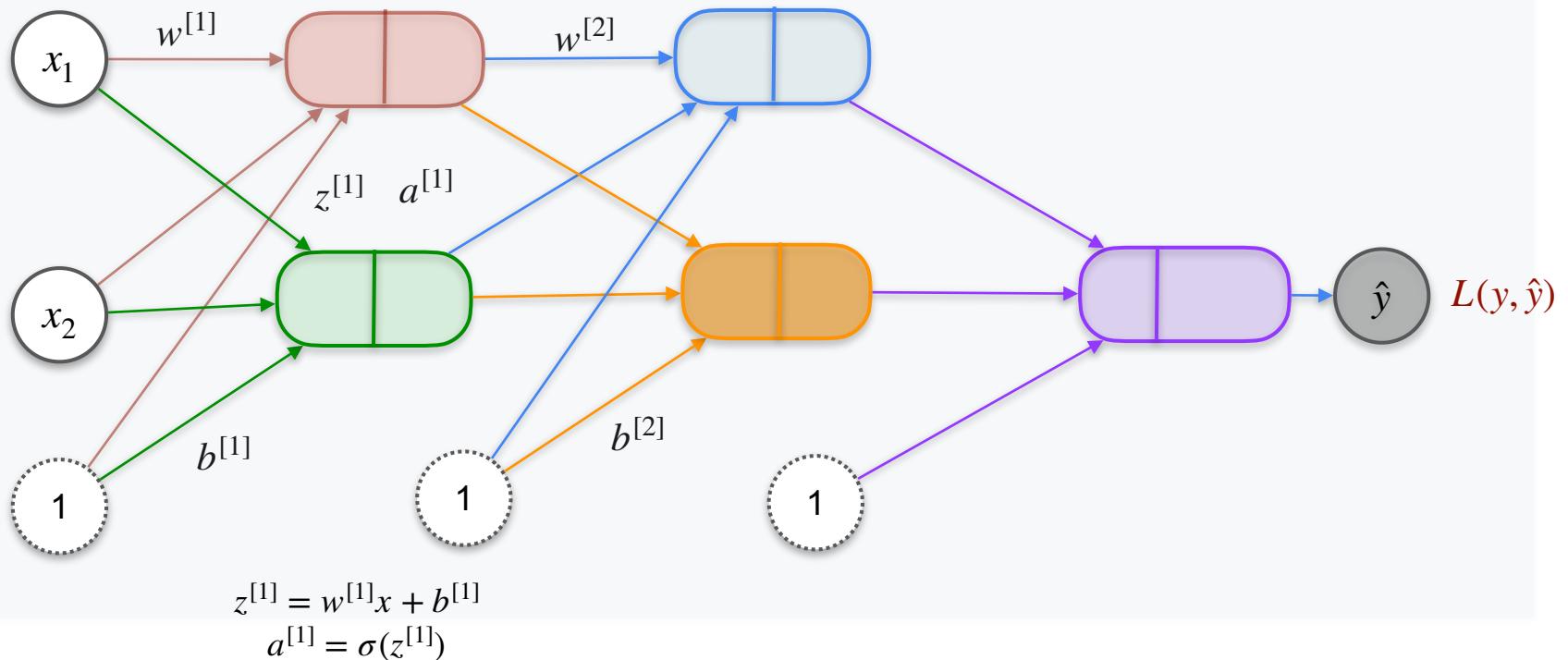
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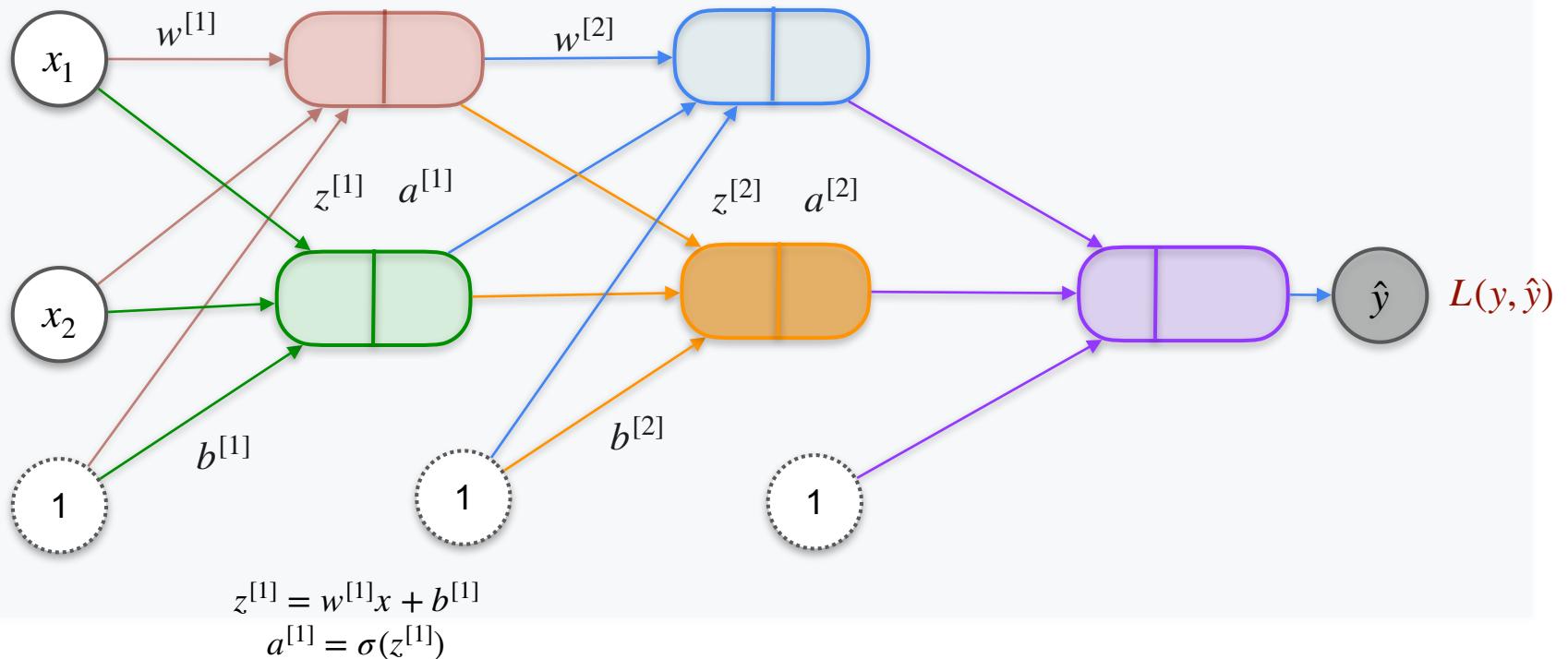
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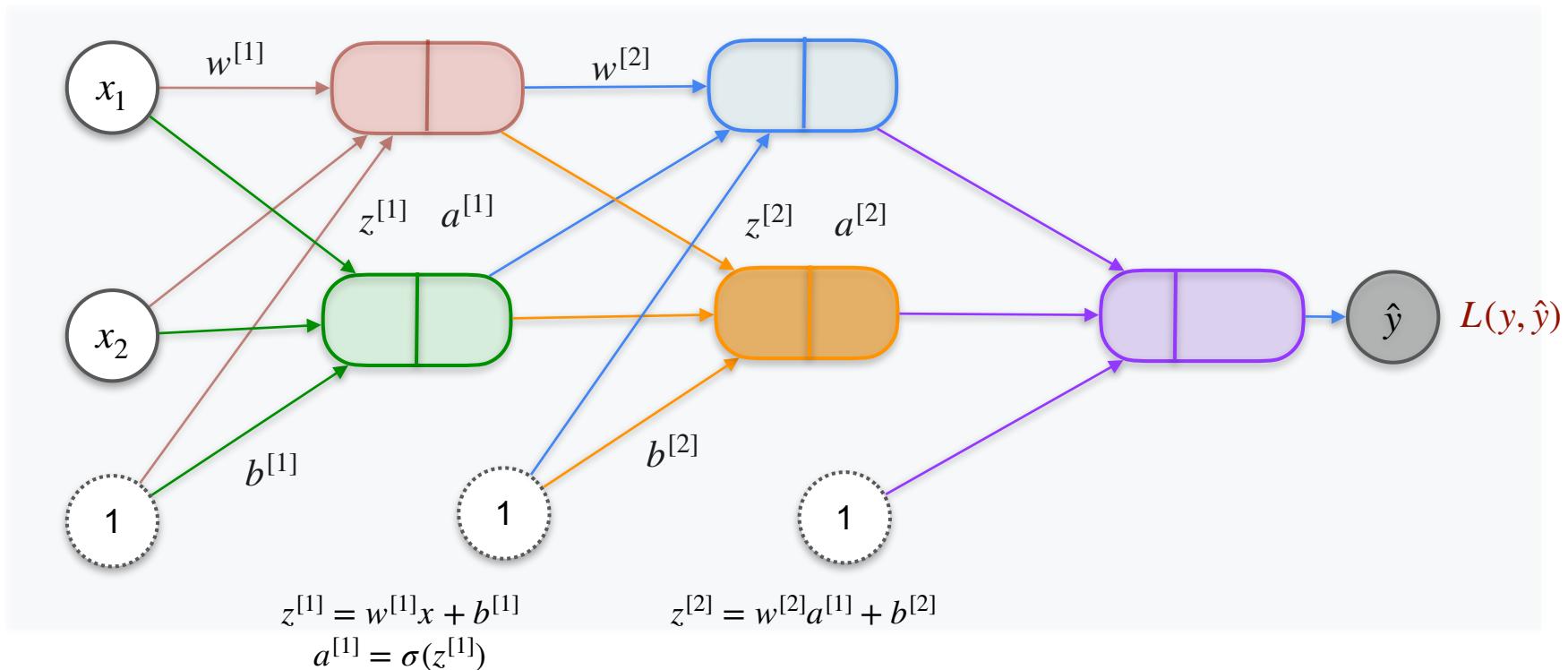
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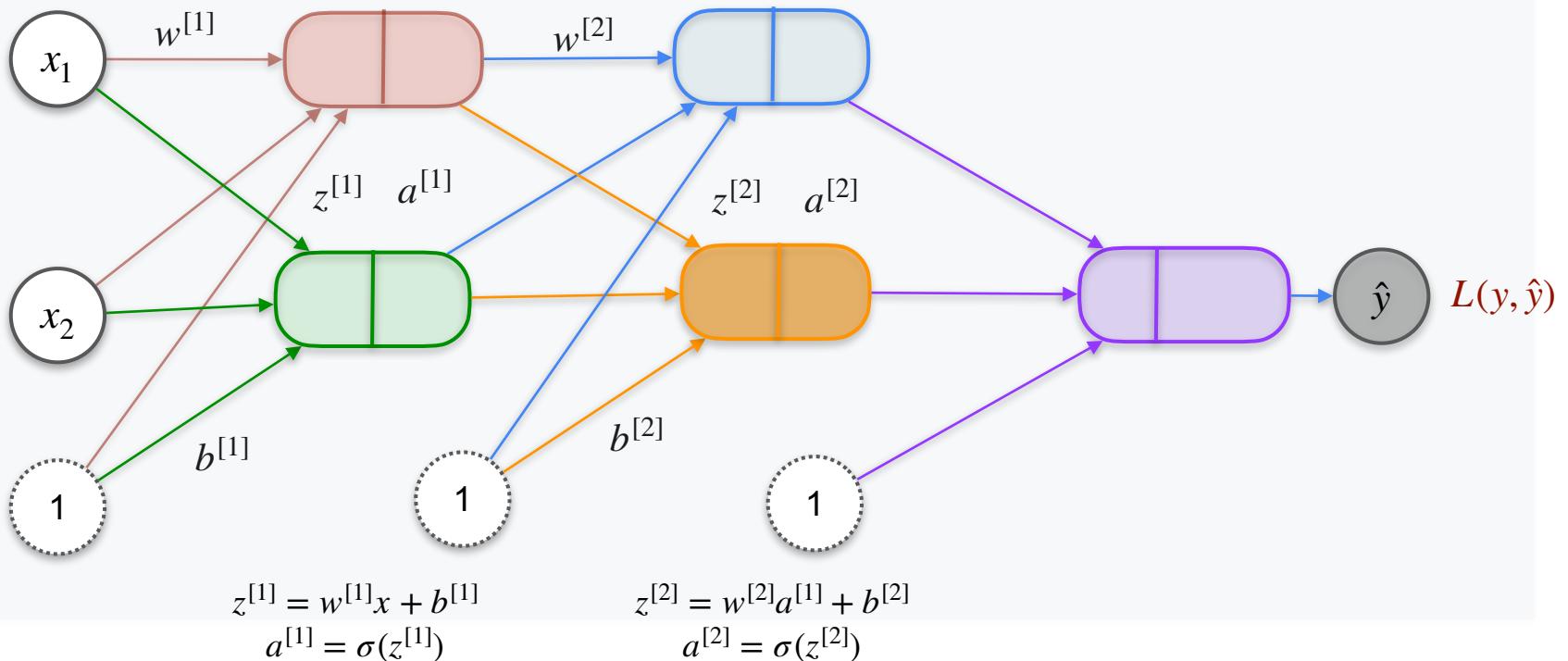
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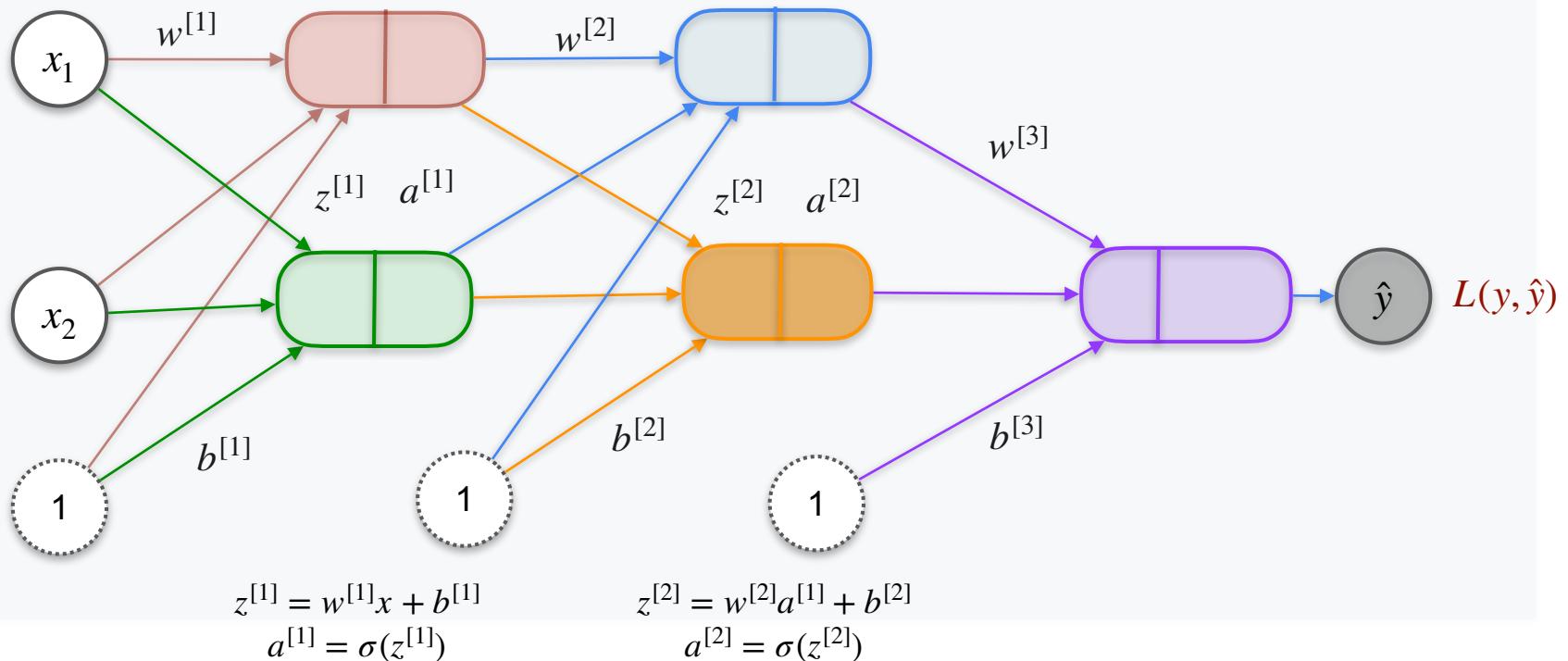
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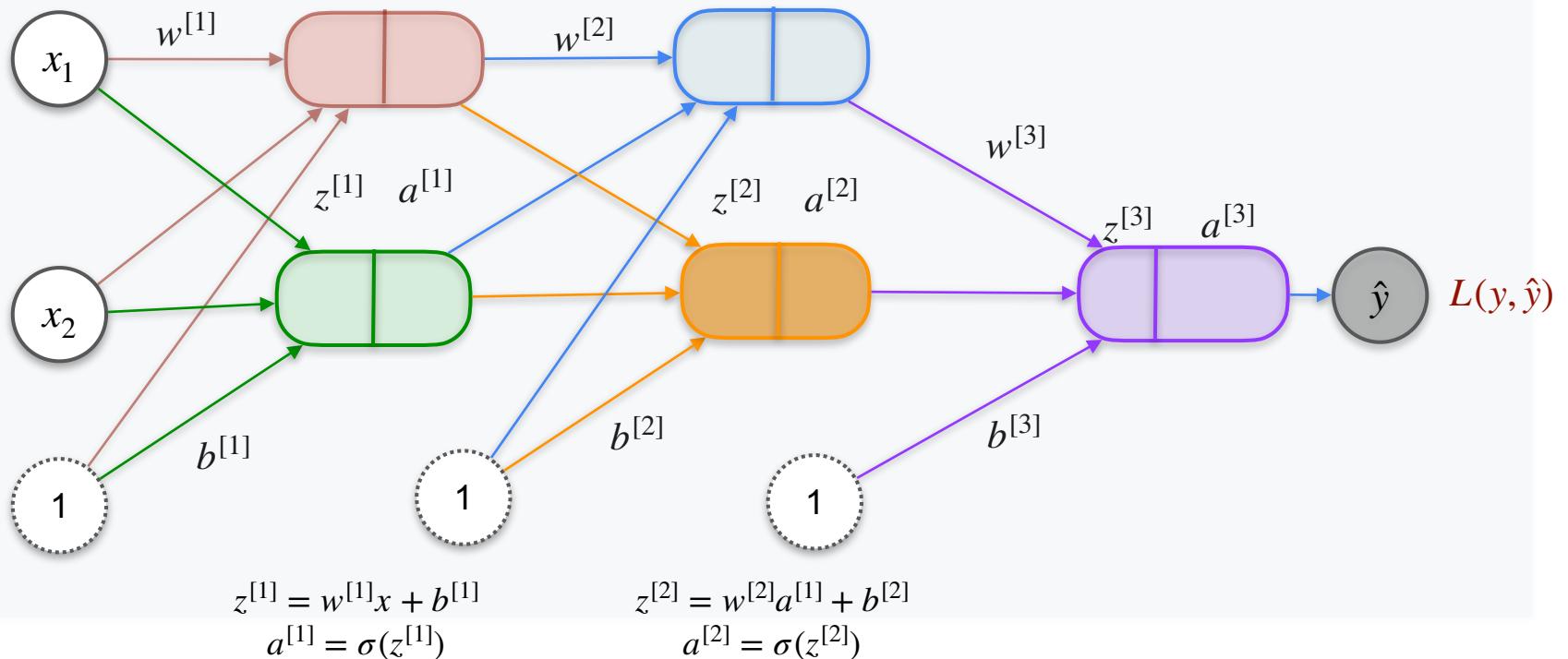
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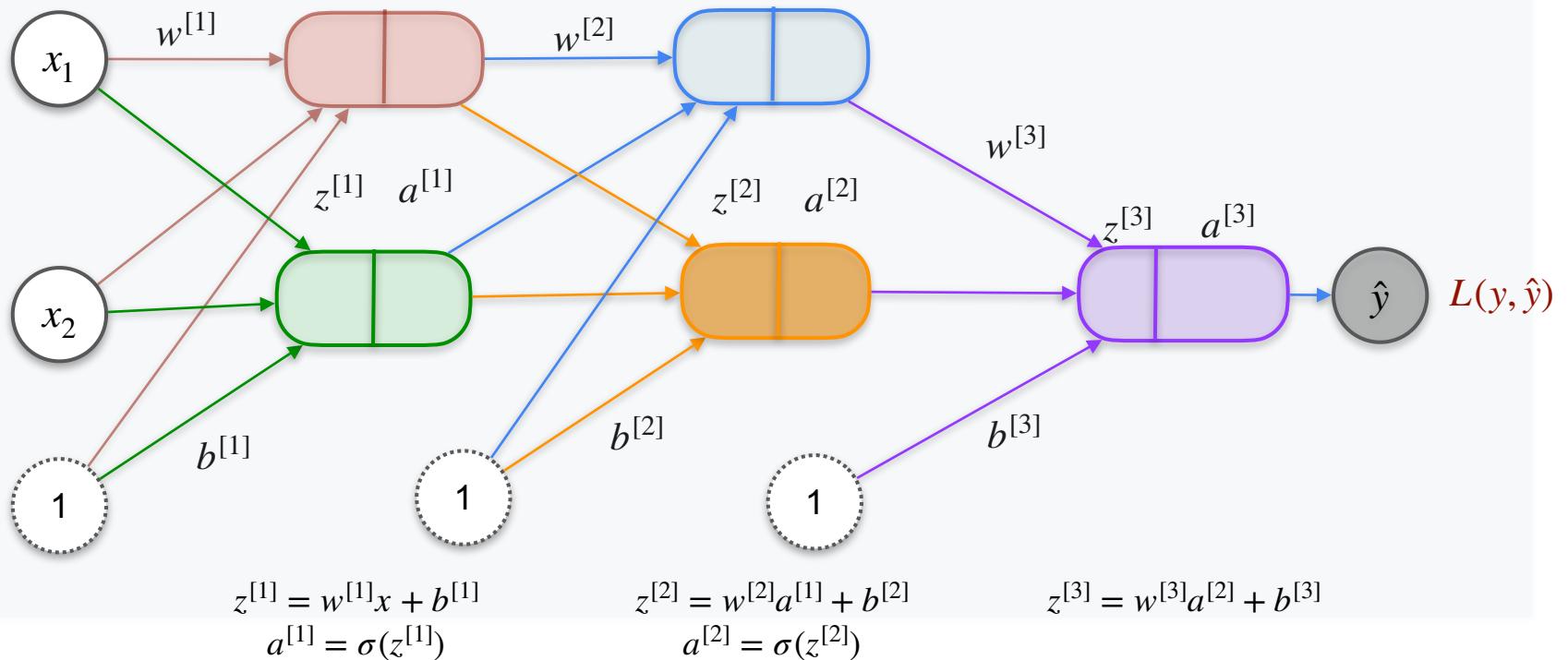


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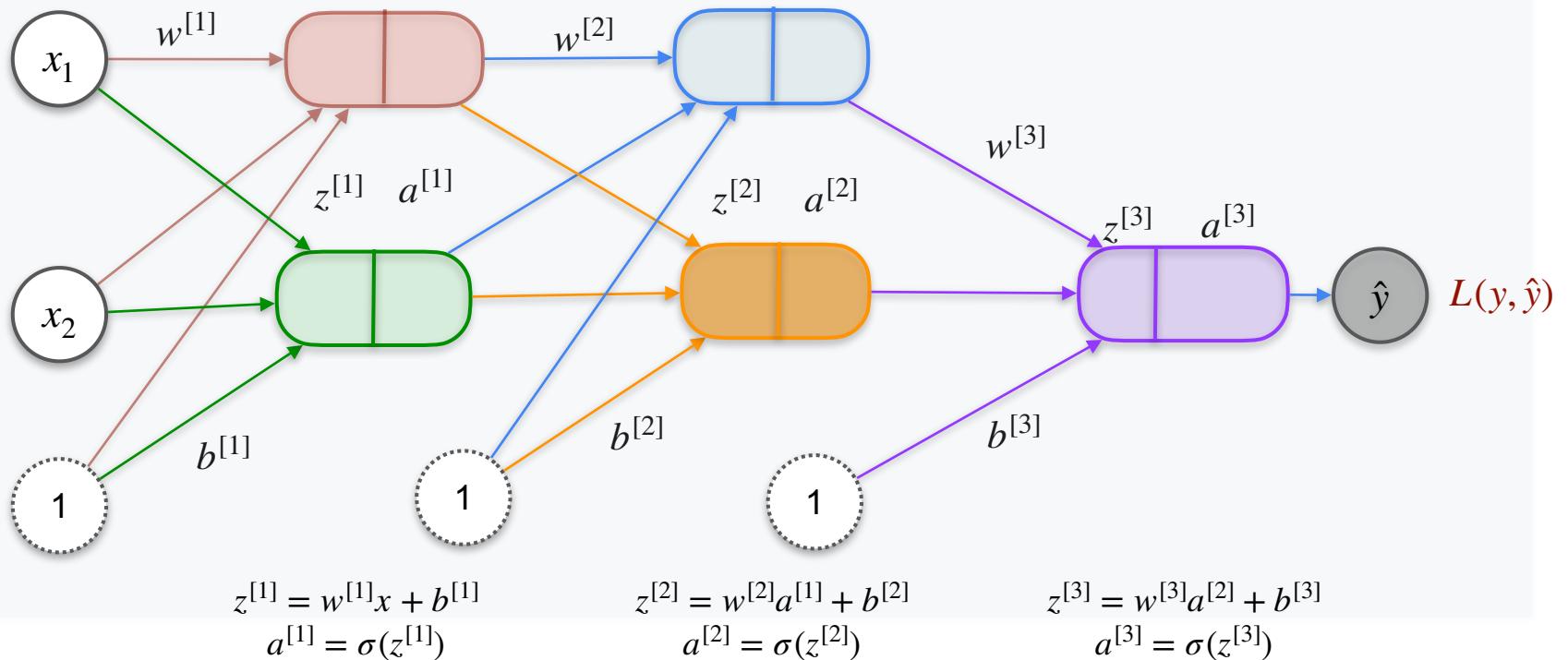


$$\begin{aligned} z^{[1]} &= w^{[1]}x + b^{[1]} \\ a^{[1]} &= \sigma(z^{[1]}) \end{aligned}$$
$$\begin{aligned} z^{[2]} &= w^{[2]}a^{[1]} + b^{[2]} \\ a^{[2]} &= \sigma(z^{[2]}) \end{aligned}$$

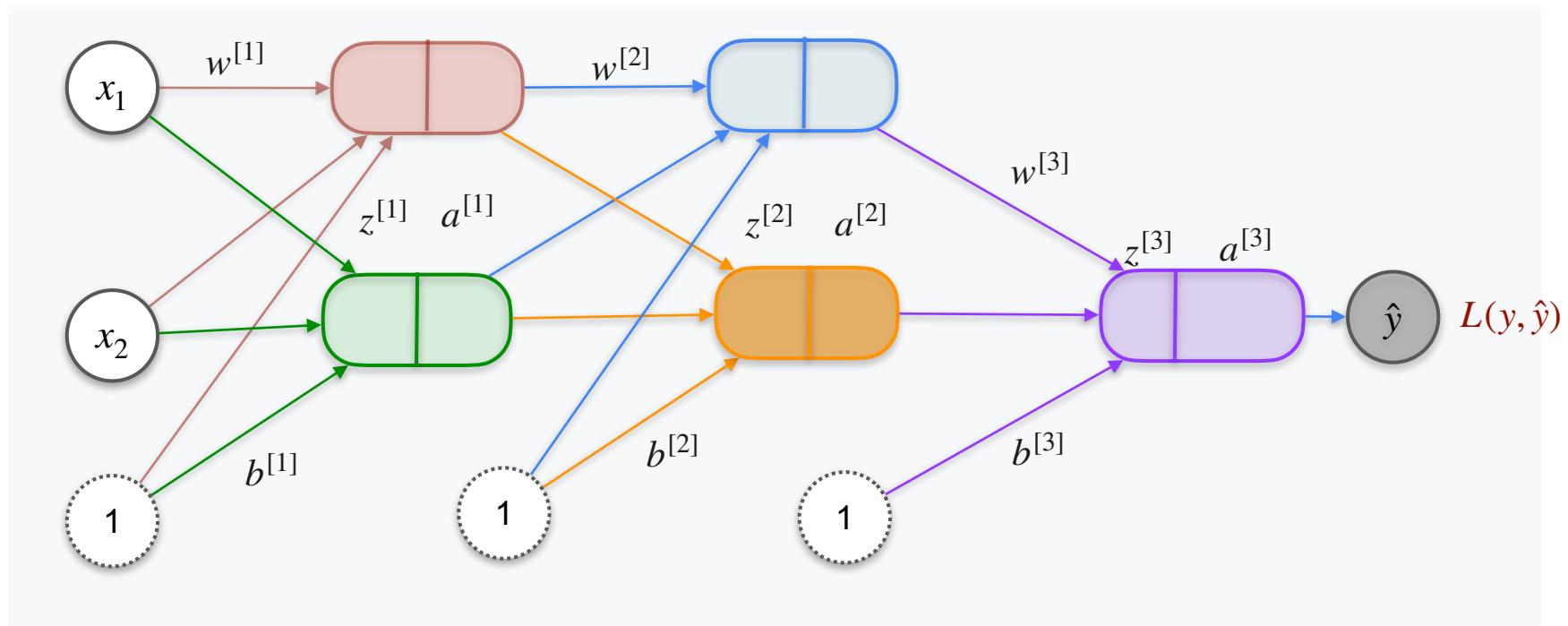
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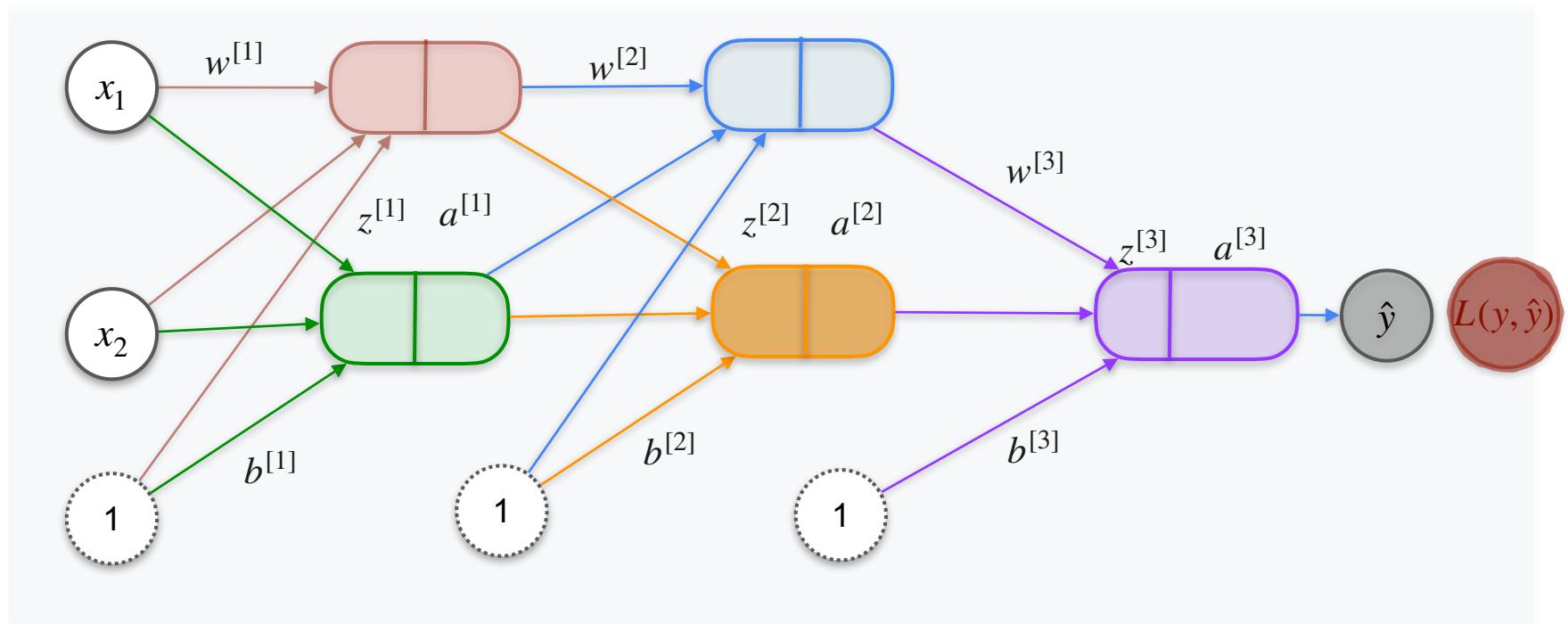
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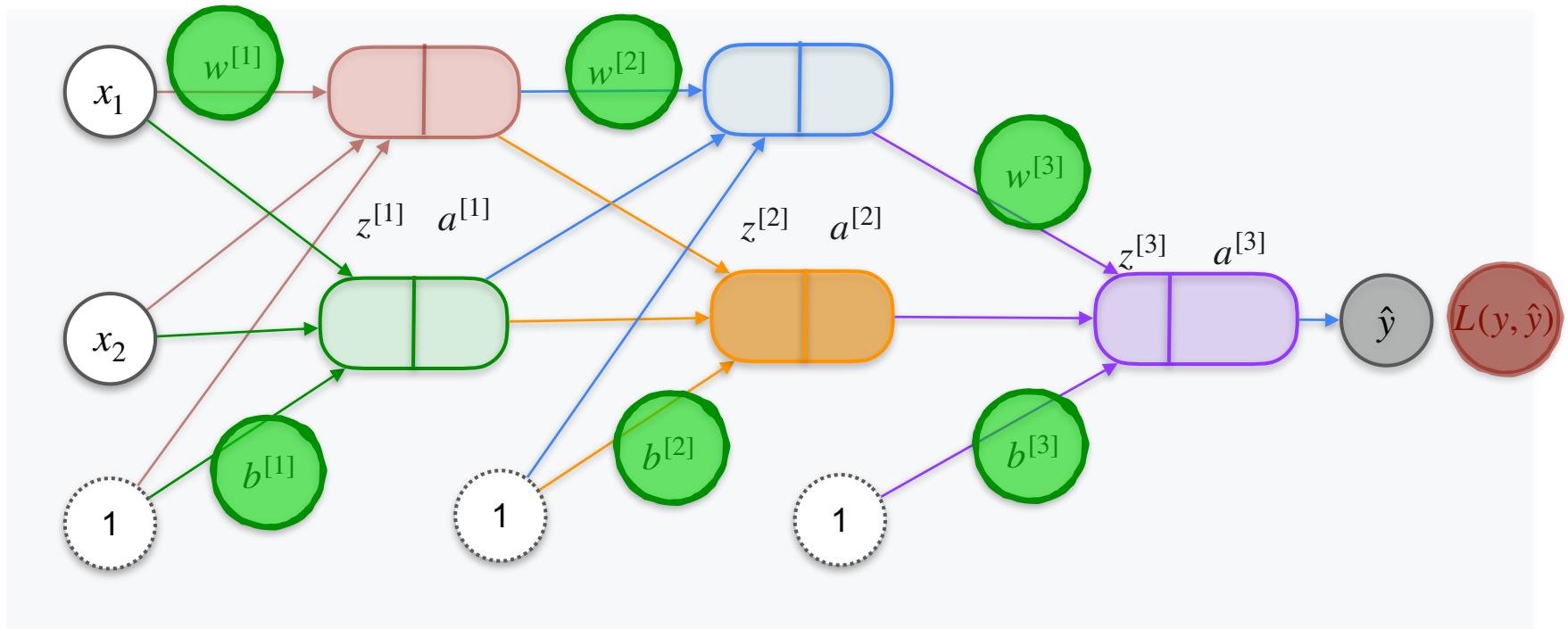
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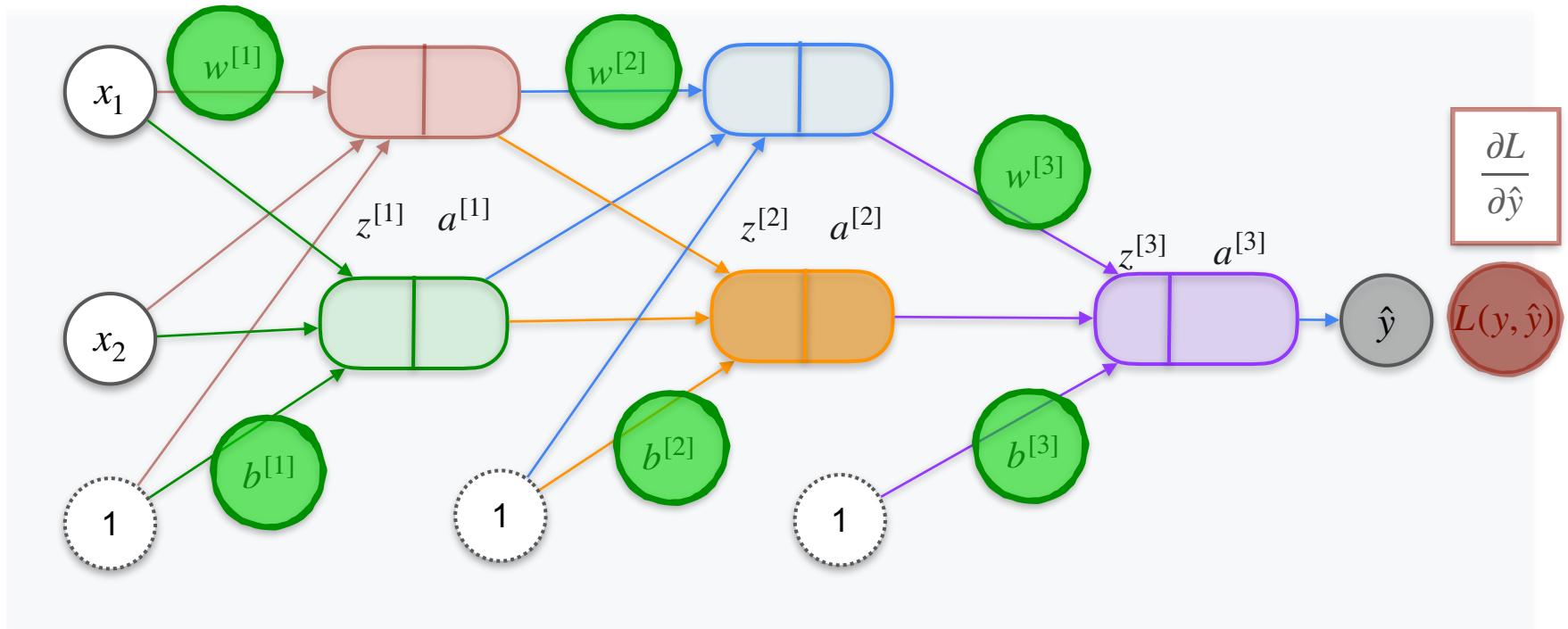
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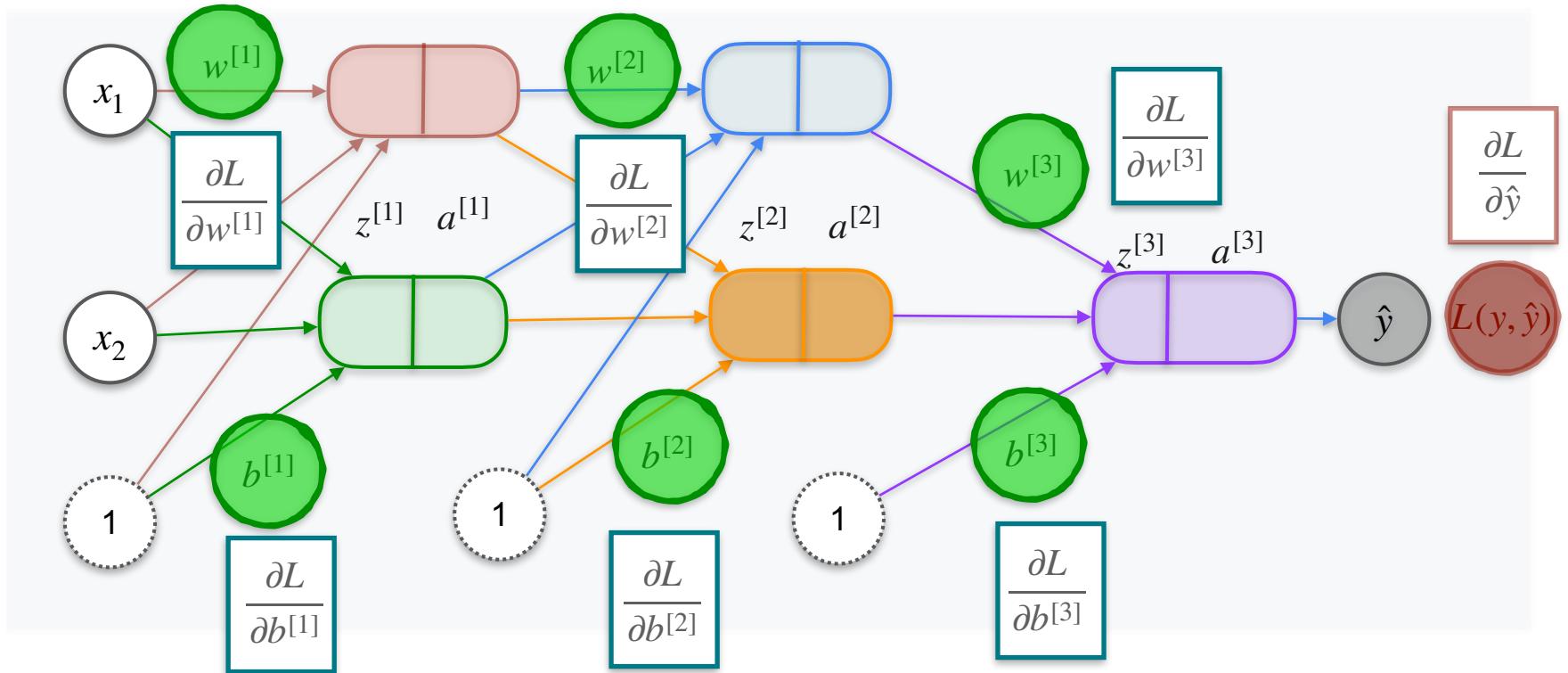
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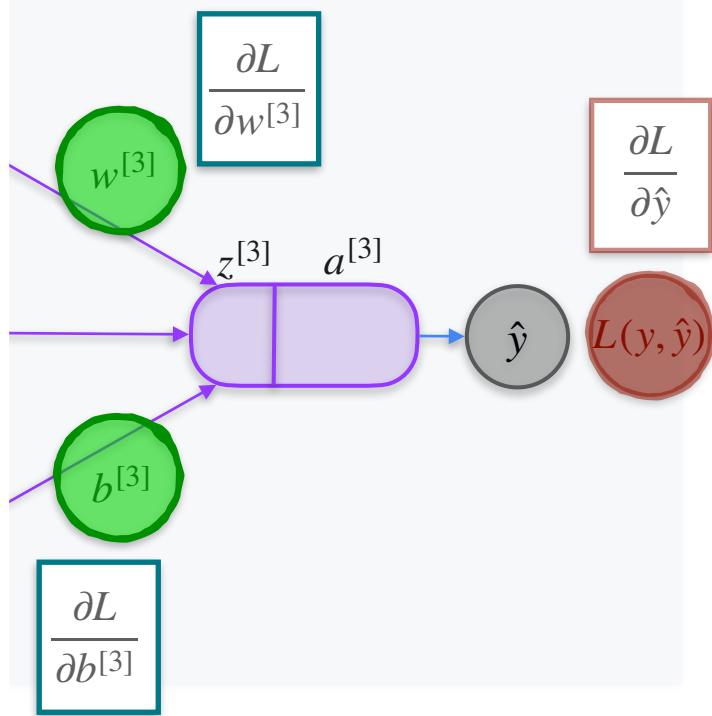
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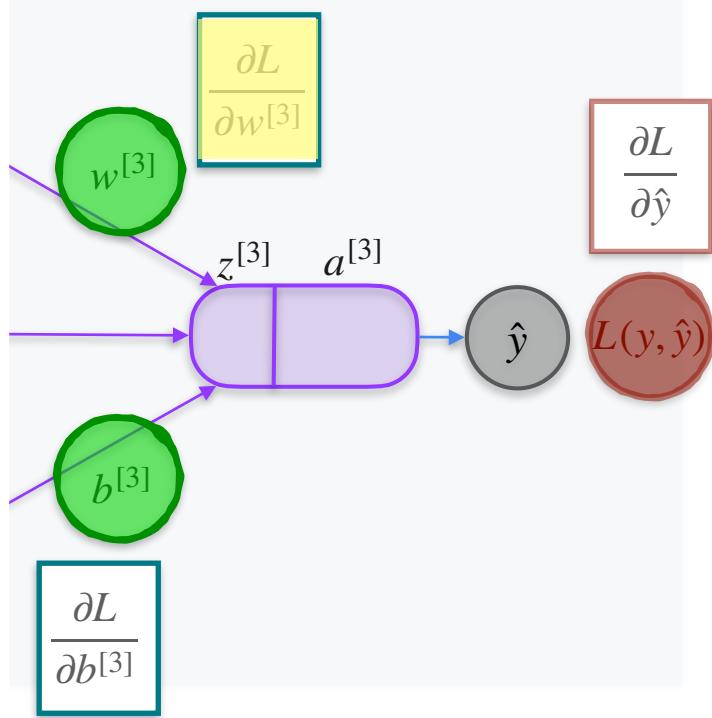
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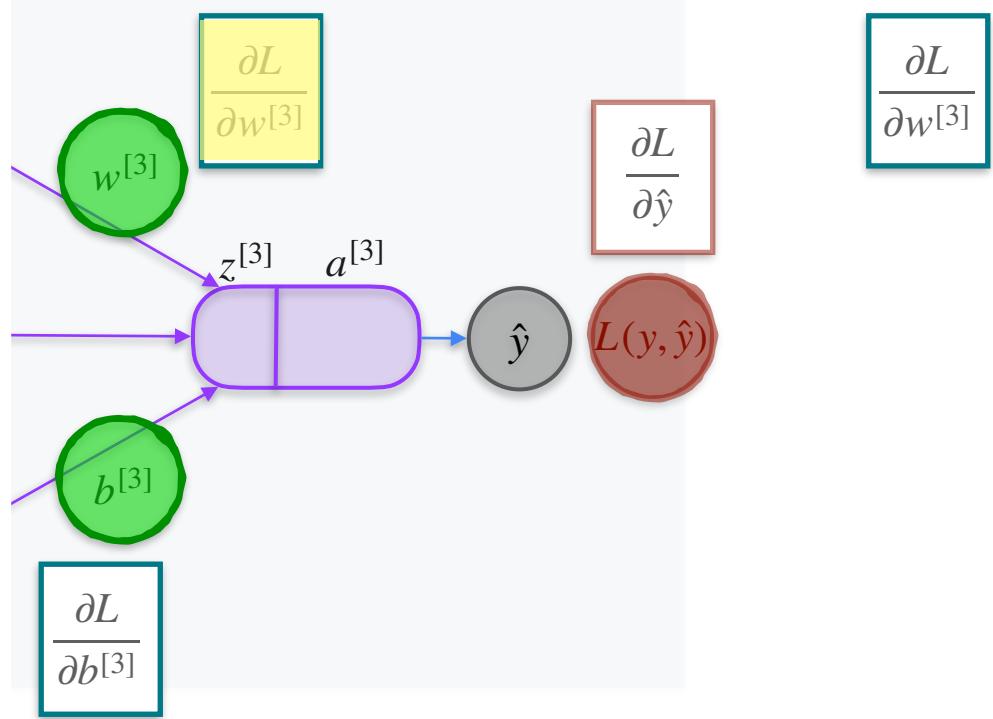
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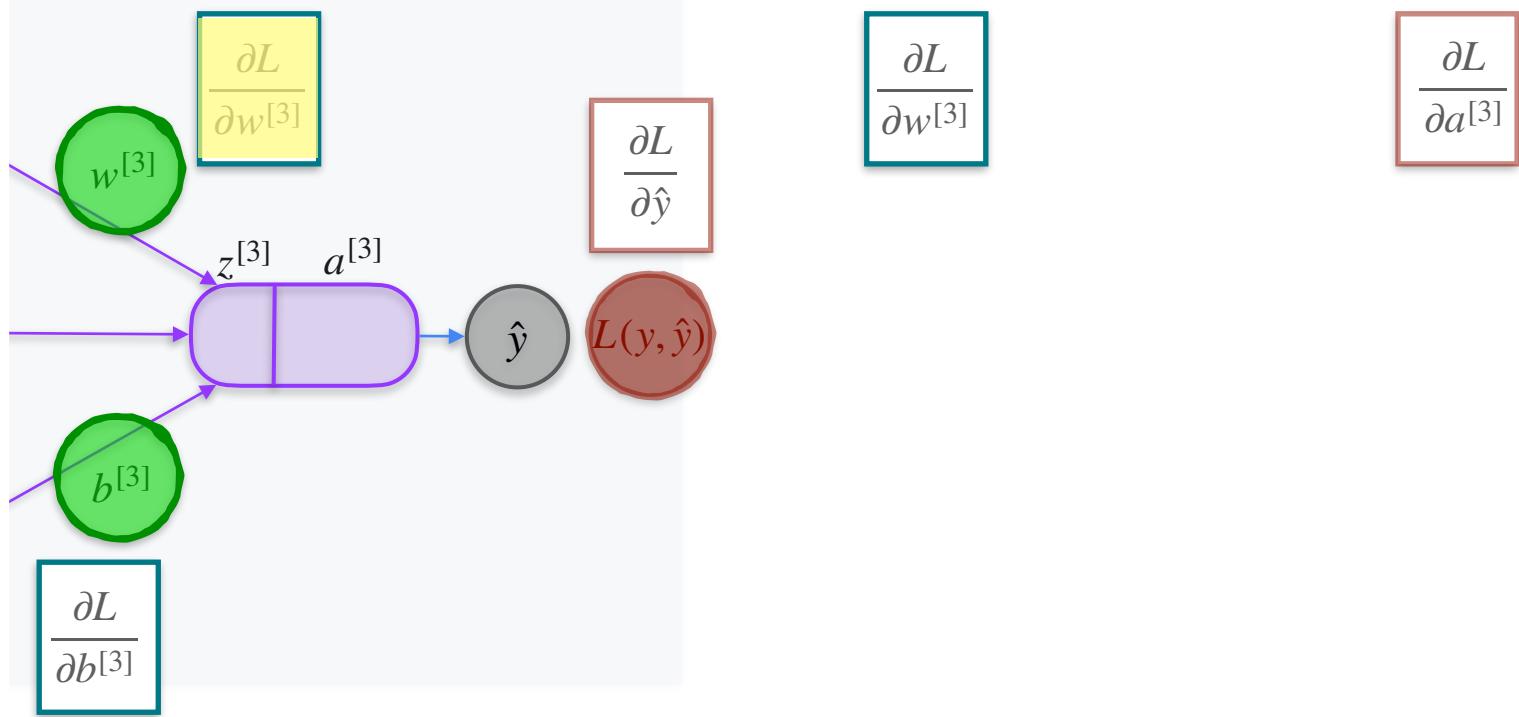
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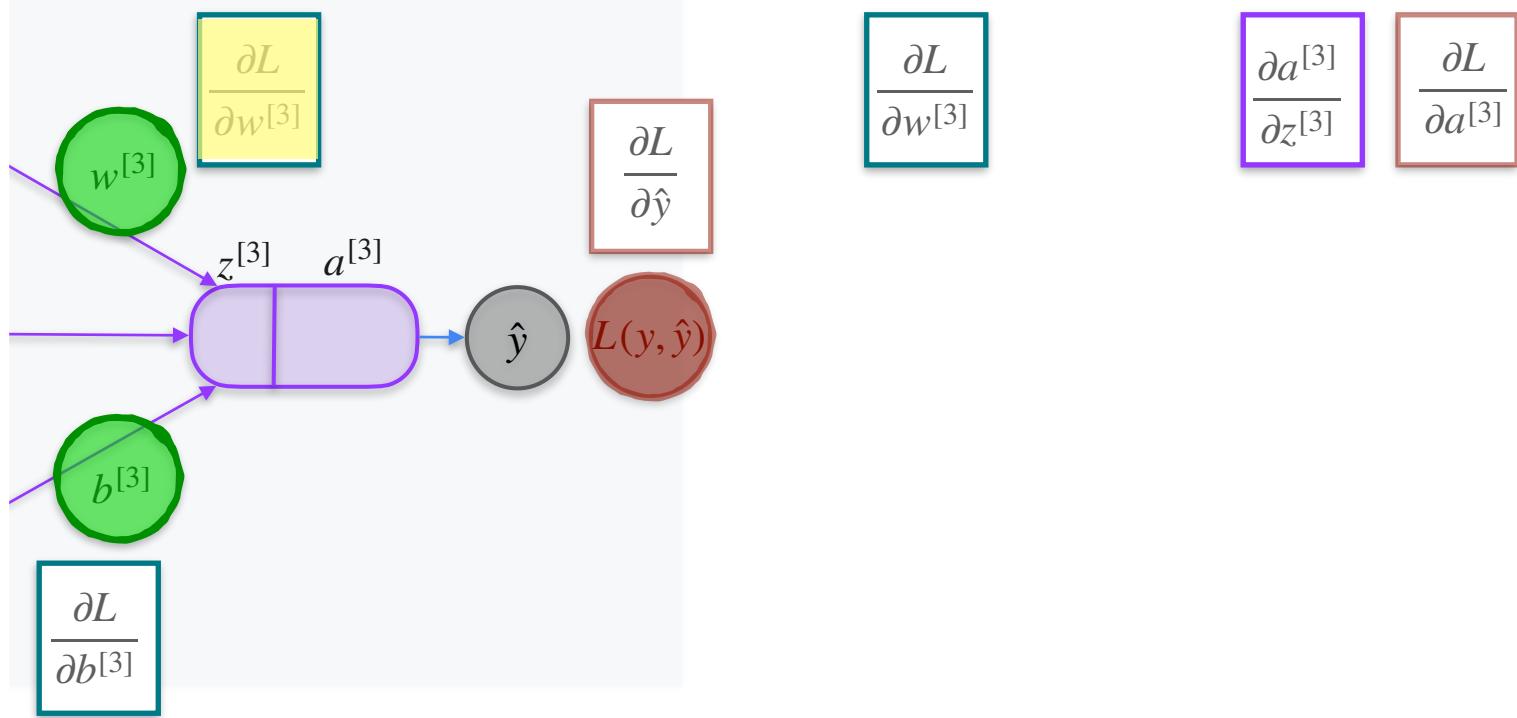
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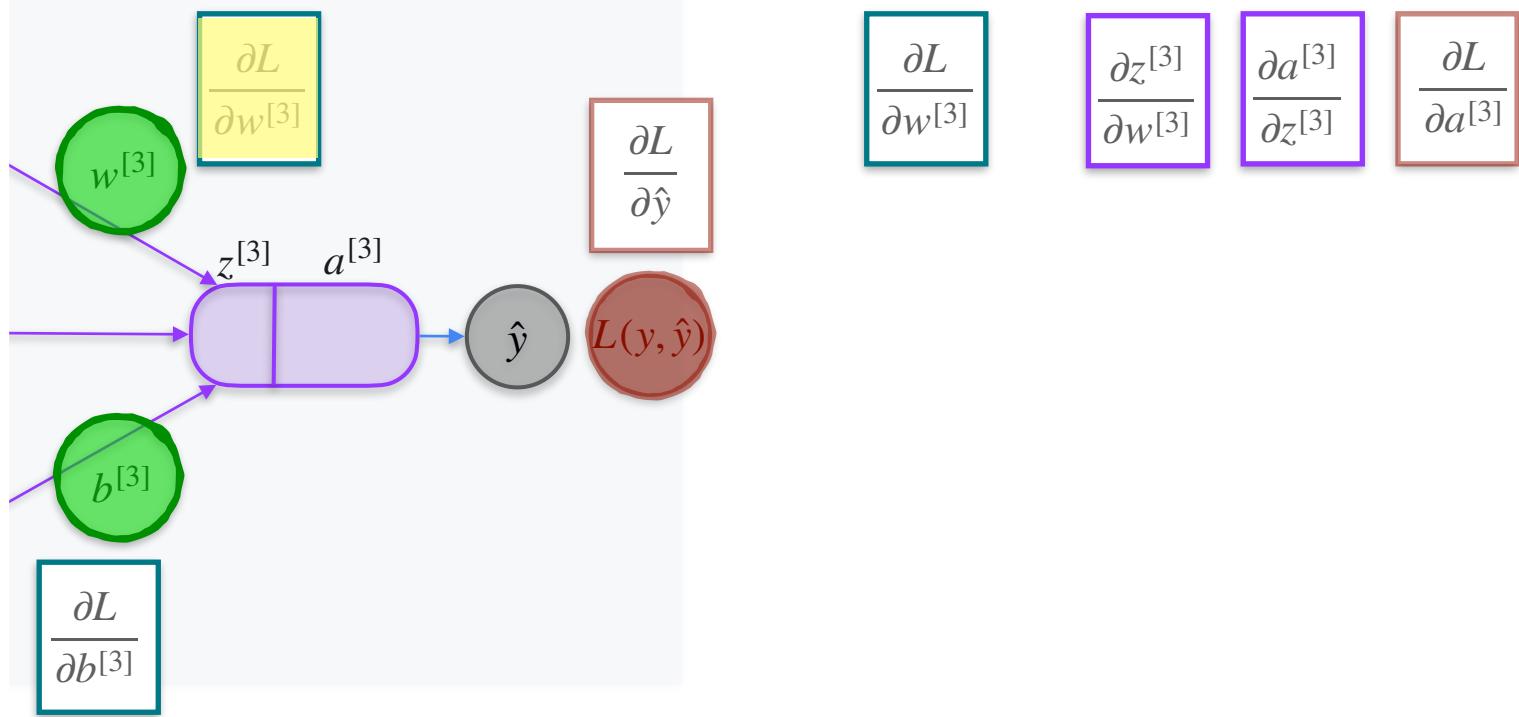
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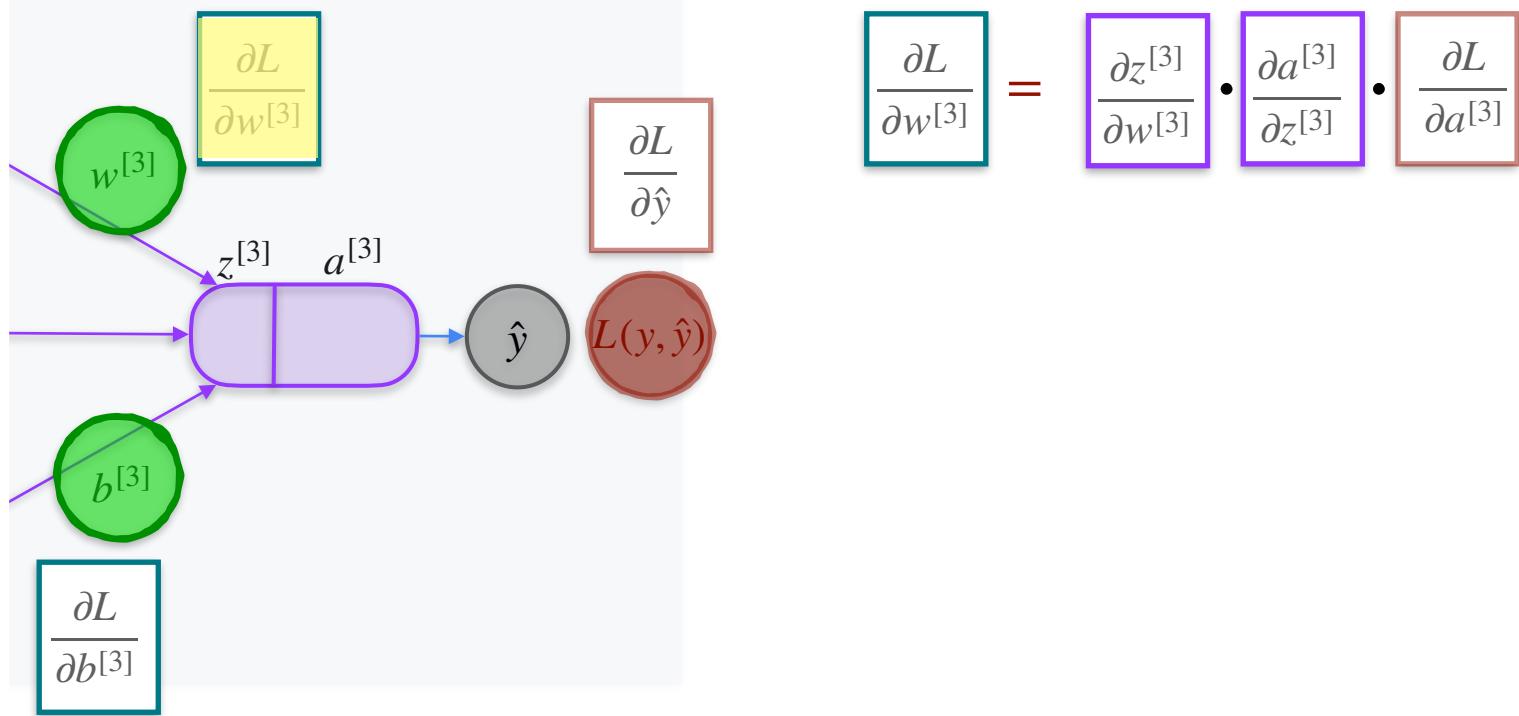
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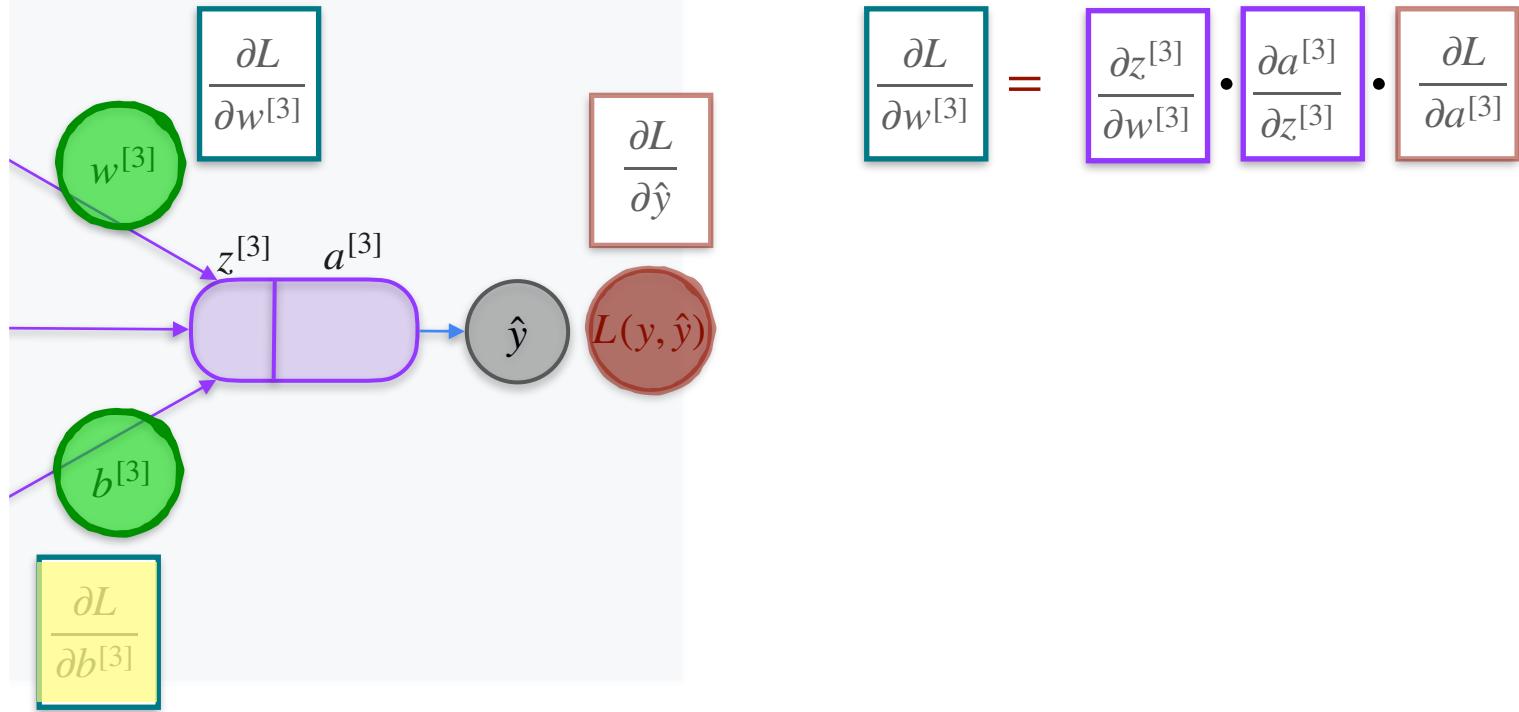
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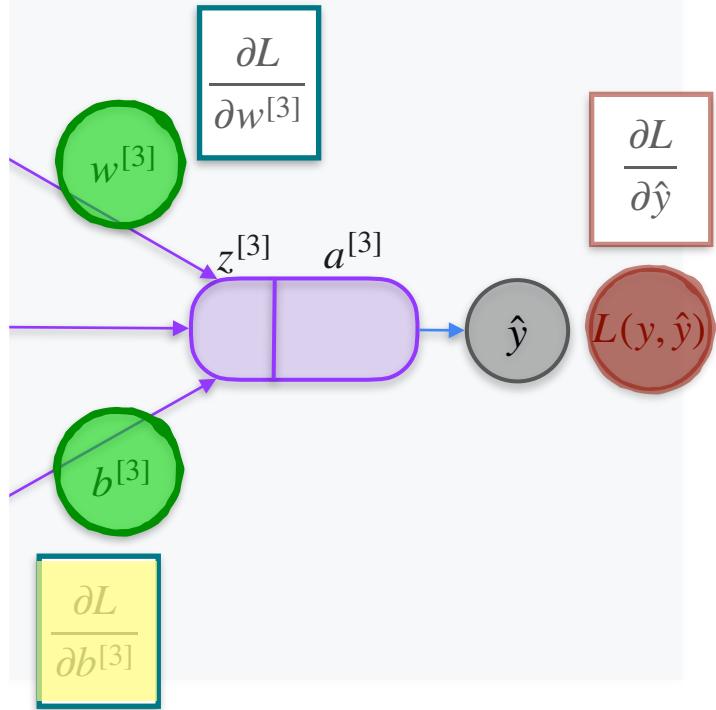
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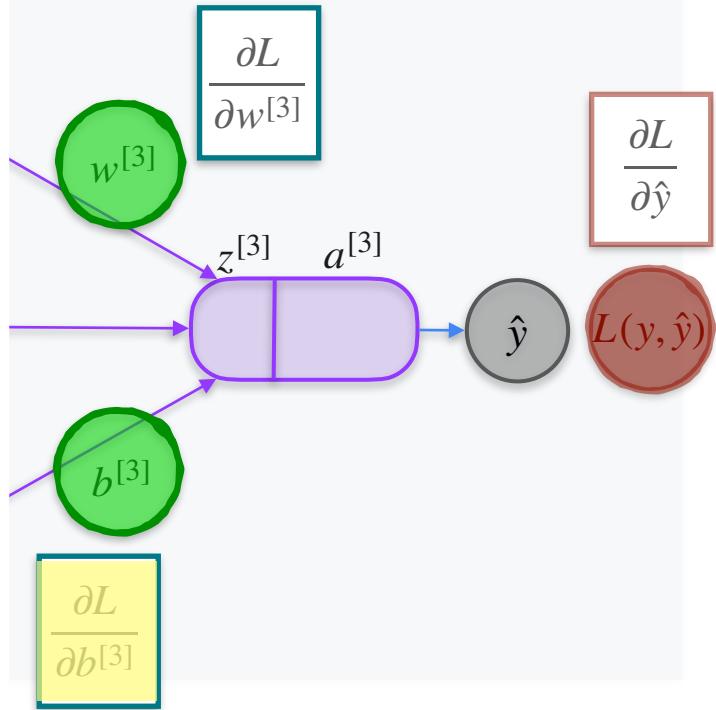
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$$\frac{\partial L}{\partial w^{[3]}} = \frac{\partial z^{[3]}}{\partial w^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial L}{\partial a^{[3]}}$$

$$\frac{\partial L}{\partial b^{[3]}}$$

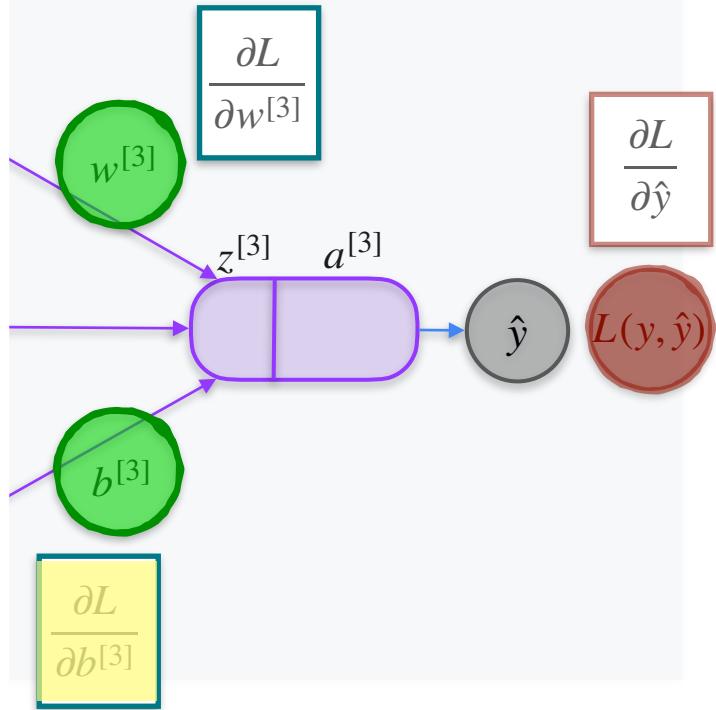
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$$\frac{\partial L}{\partial w^{[3]}} = \frac{\partial z^{[3]}}{\partial w^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial L}{\partial a^{[3]}}$$

$$\begin{array}{c} \frac{\partial L}{\partial b^{[3]}} \\ \frac{\partial z^{[3]}}{\partial b^{[3]}} \\ \frac{\partial a^{[3]}}{\partial z^{[3]}} \\ \frac{\partial L}{\partial a^{[3]}} \end{array}$$

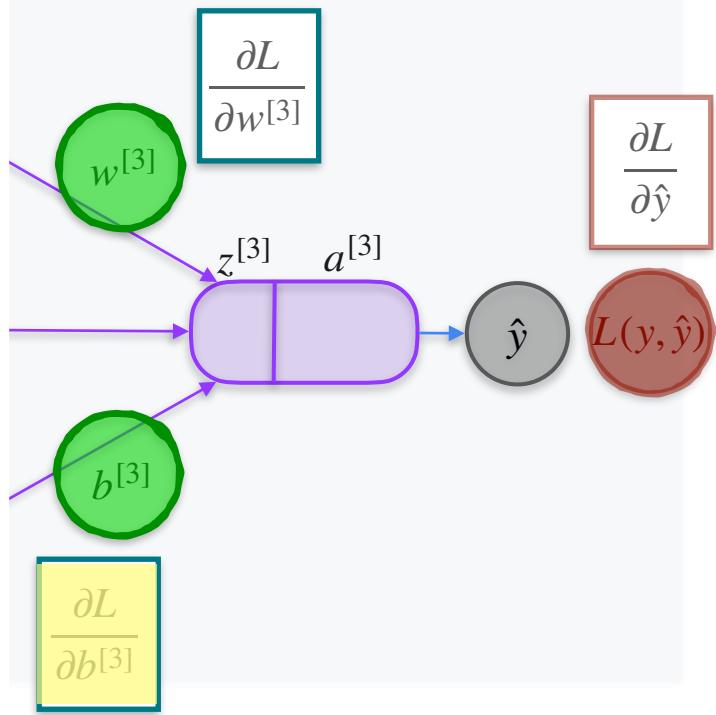
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$$\frac{\partial L}{\partial w^{[3]}} = \frac{\partial z^{[3]}}{\partial w^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial L}{\partial a^{[3]}}$$

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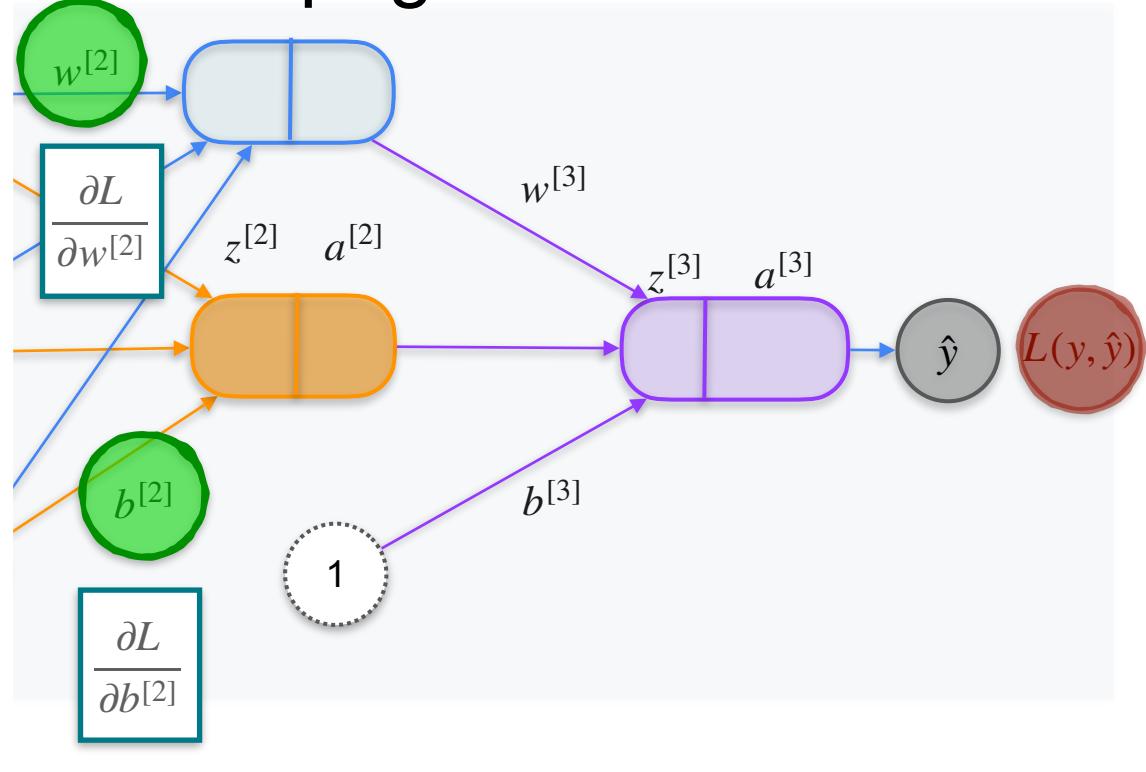


$$\frac{\partial L}{\partial w^{[3]}} = \frac{\partial L}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial a^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial z^{[3]}}{\partial w^{[3]}}$$

$$\frac{\partial L}{\partial b^{[3]}} = \frac{\partial L}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial a^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial z^{[3]}}{\partial b^{[3]}}$$

$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

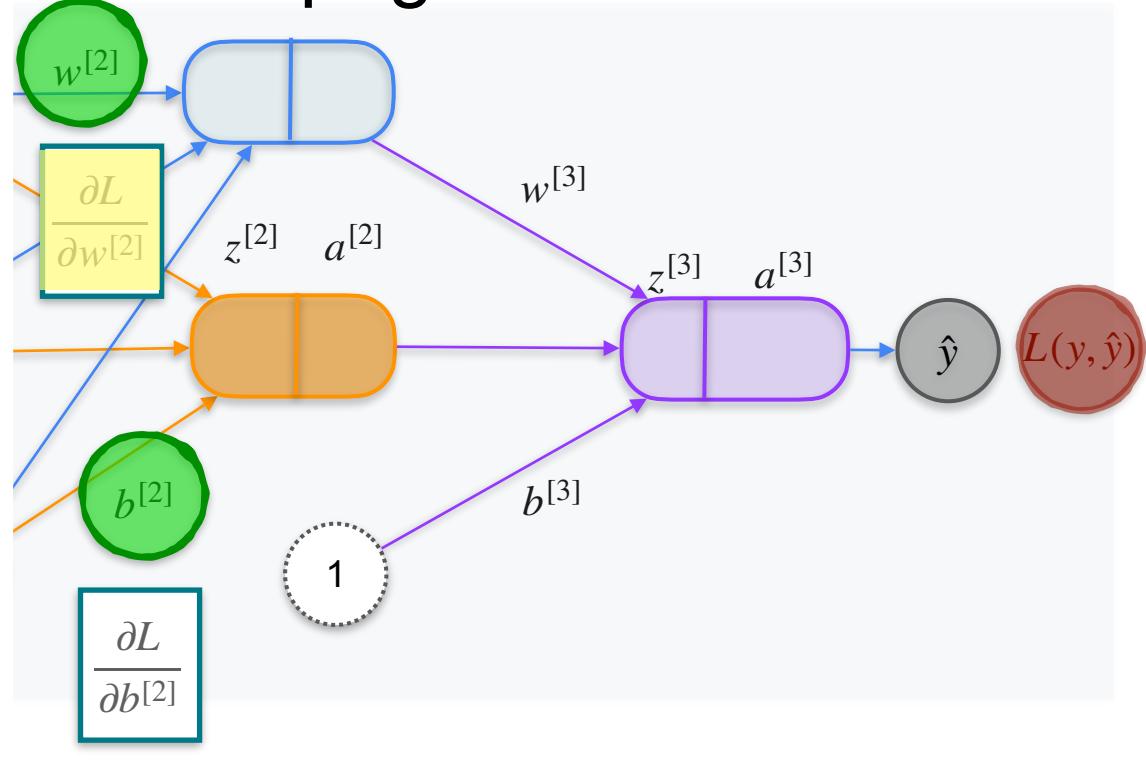
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$$\frac{\partial a^{[3]}}{\partial z^{[3]}}$$

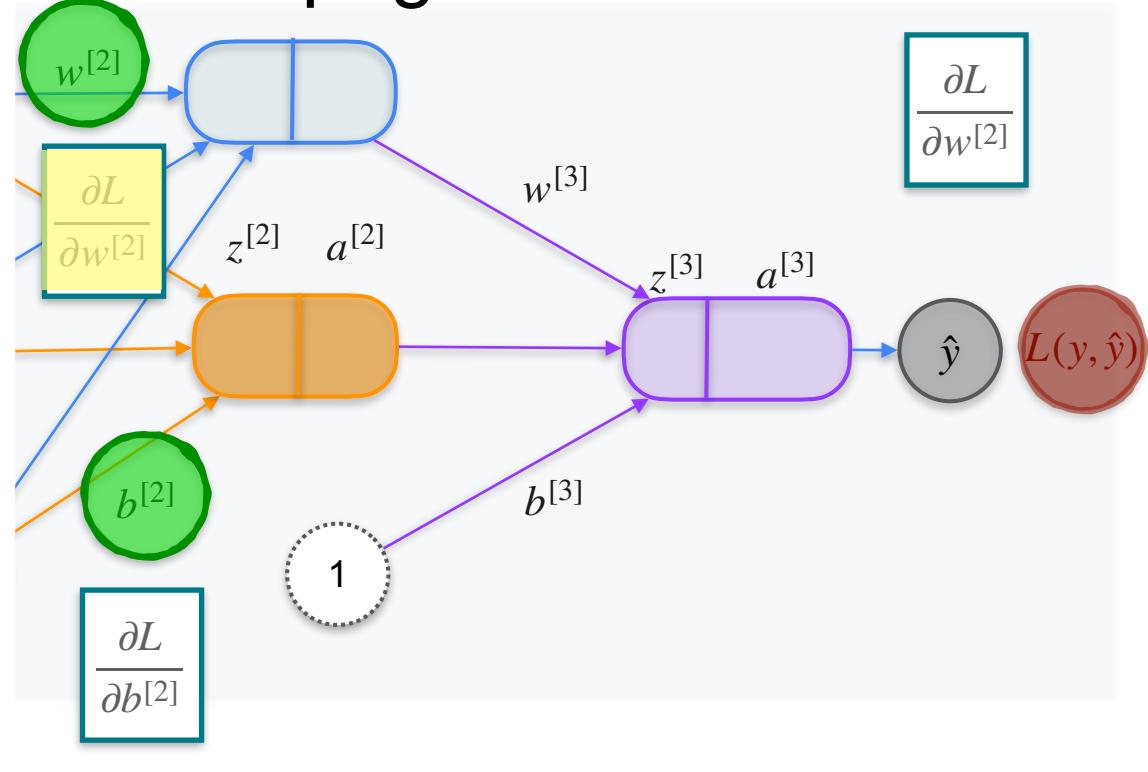
$$\frac{\partial L}{\partial a^{[3]}}$$

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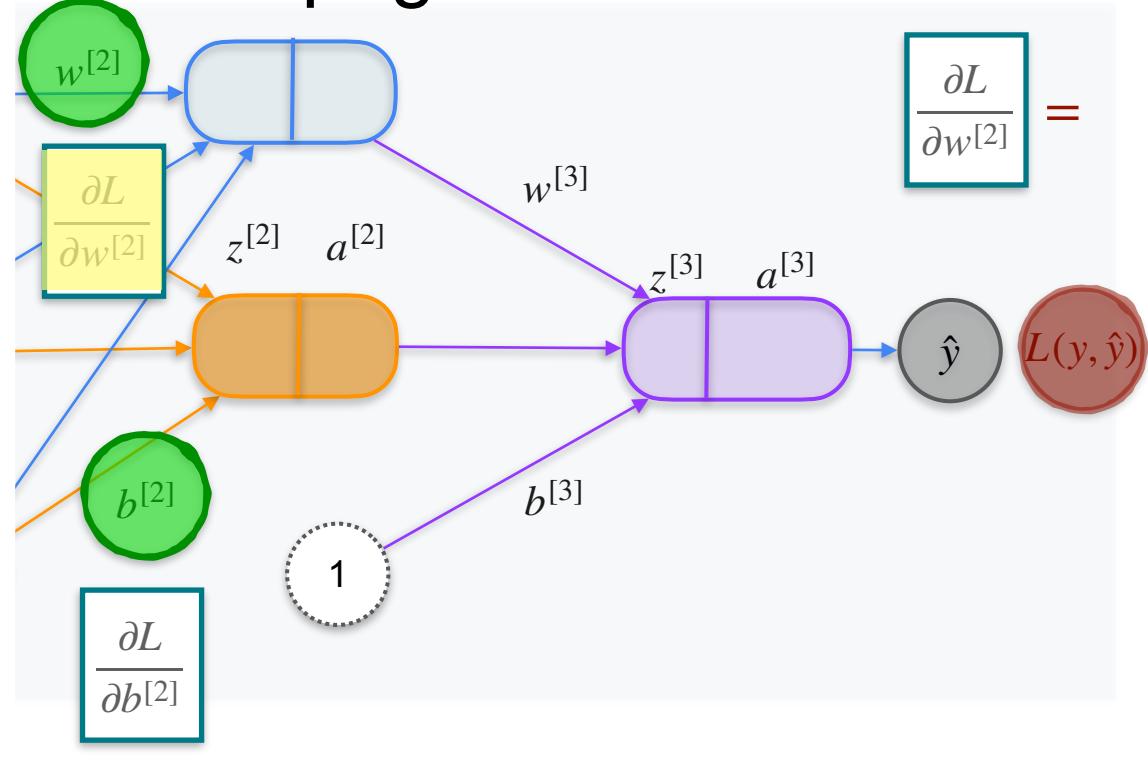


$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

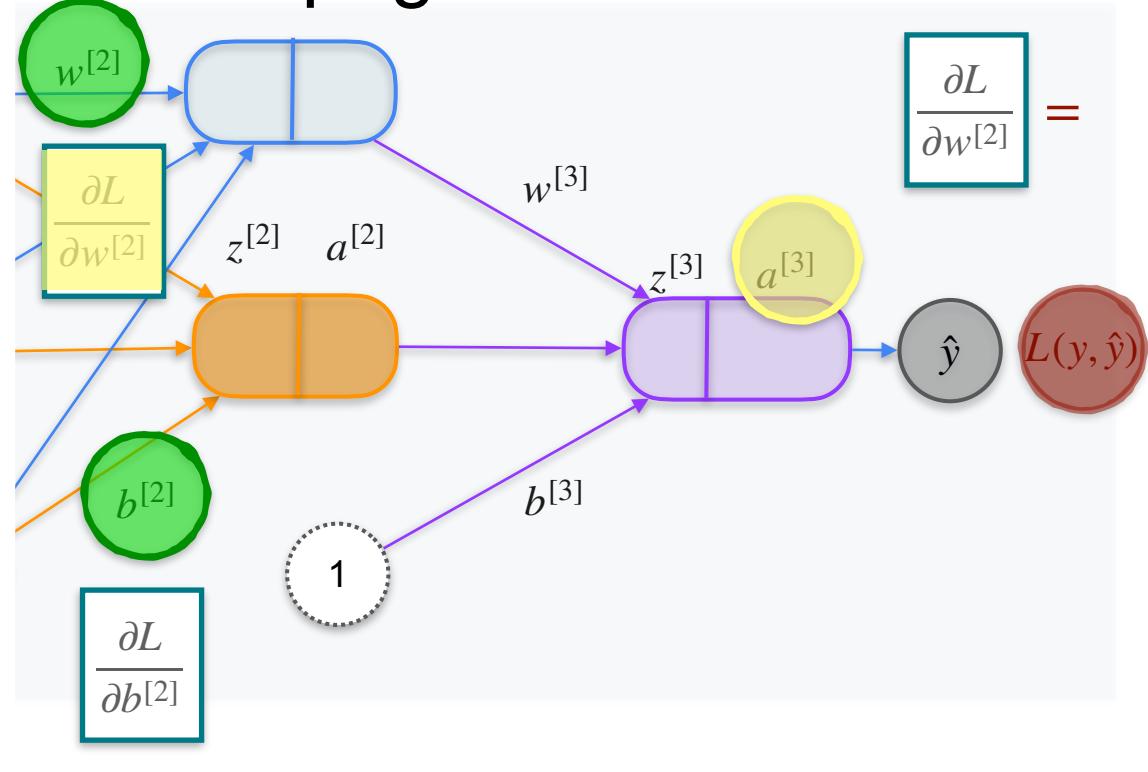
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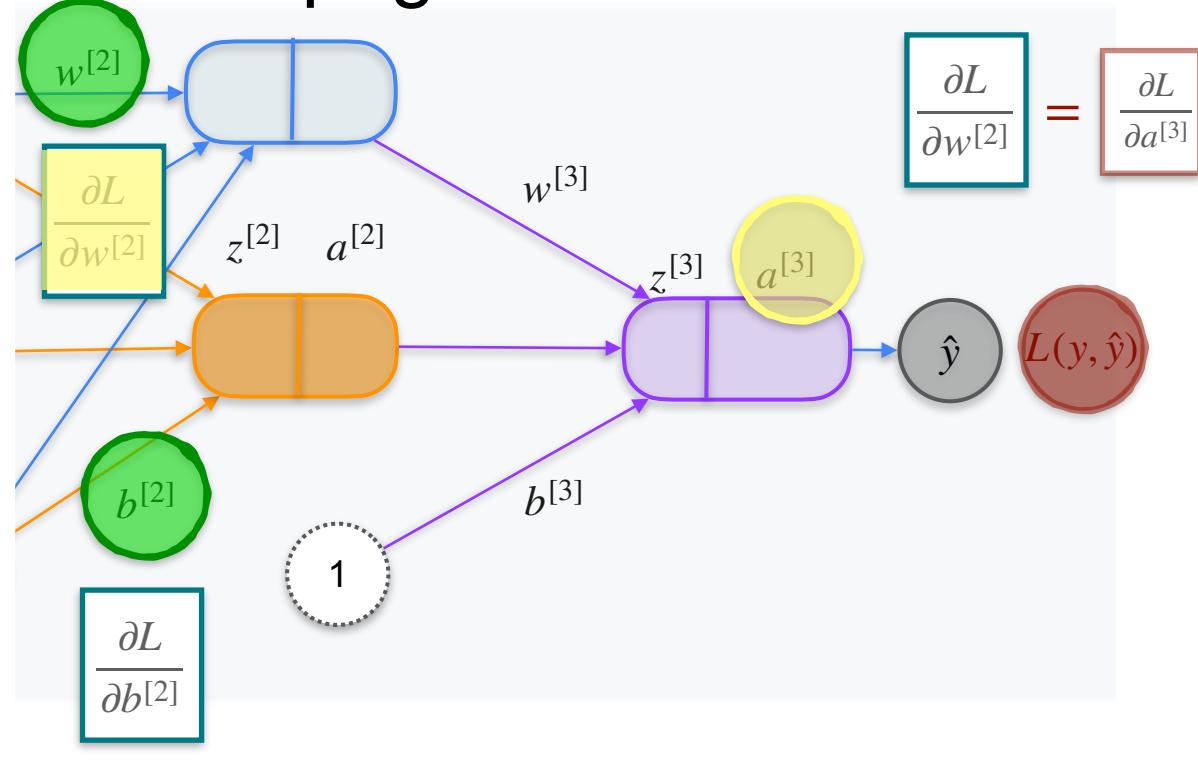
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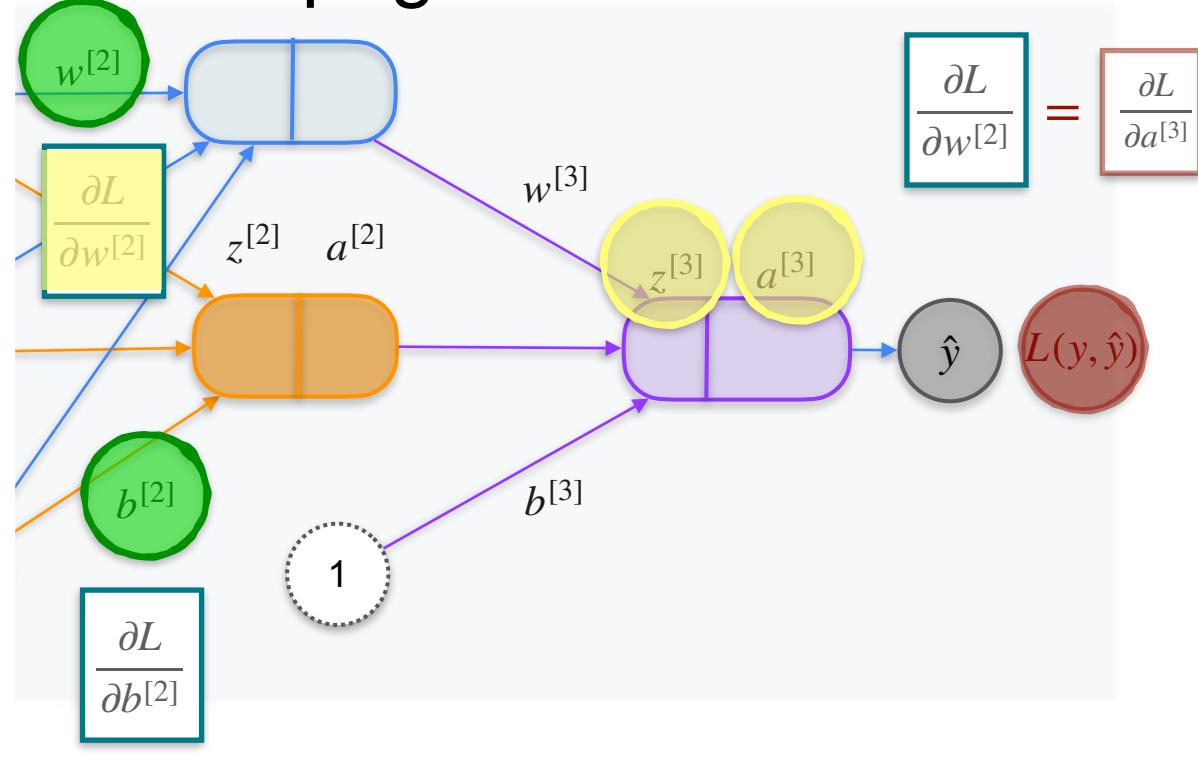


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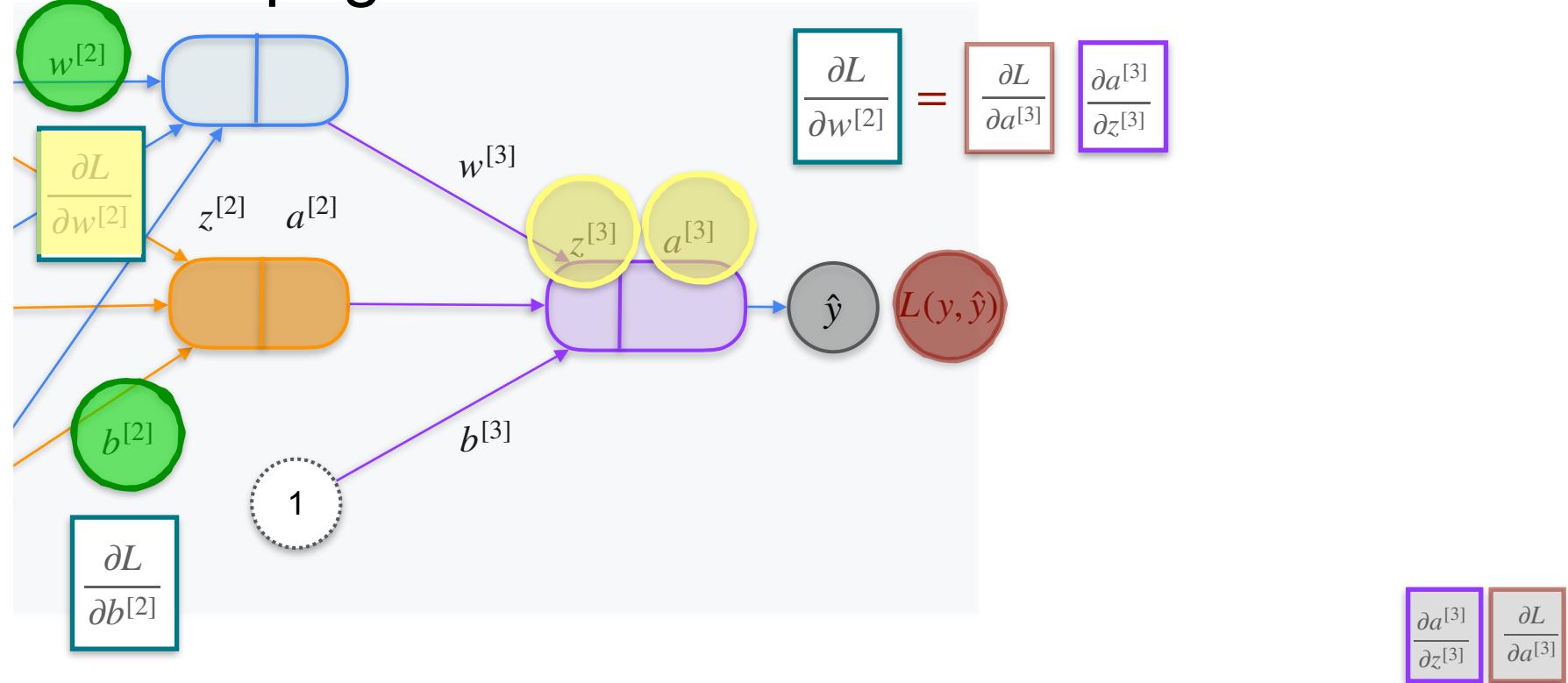


$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

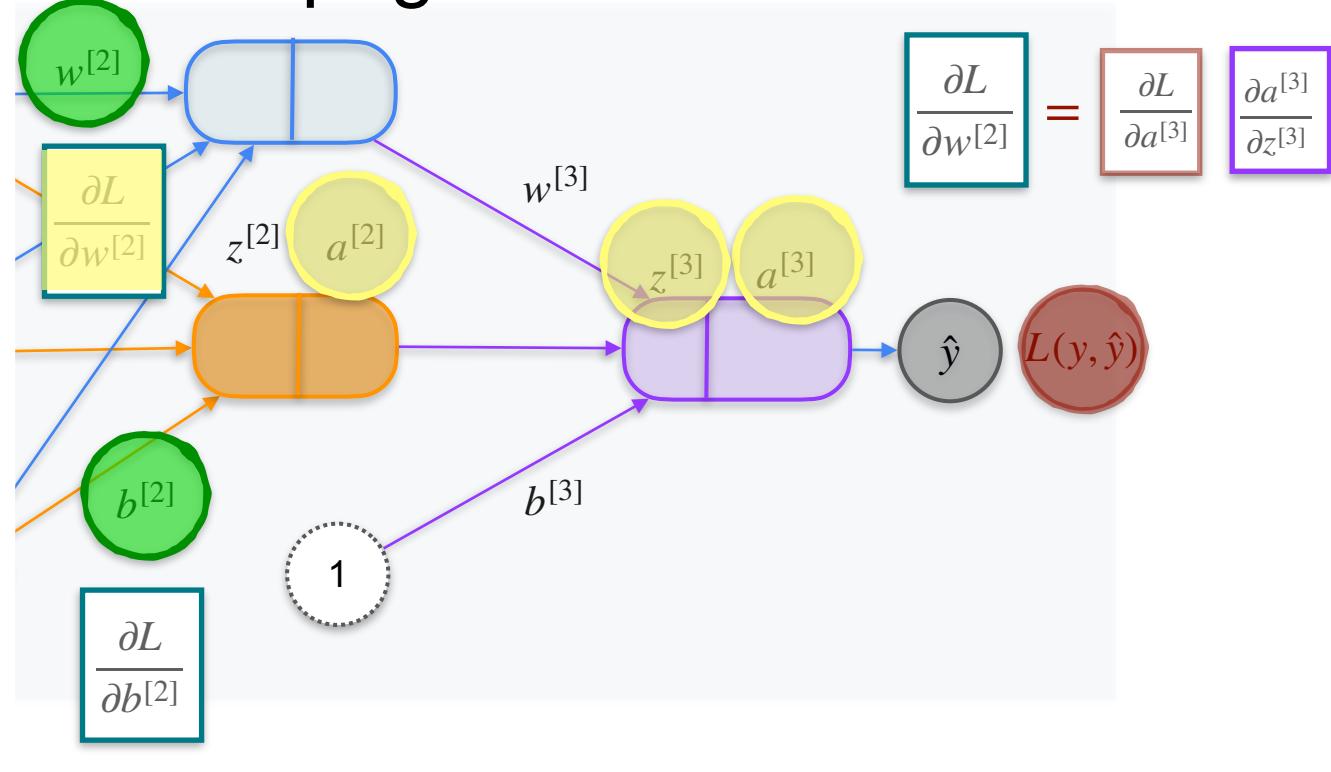
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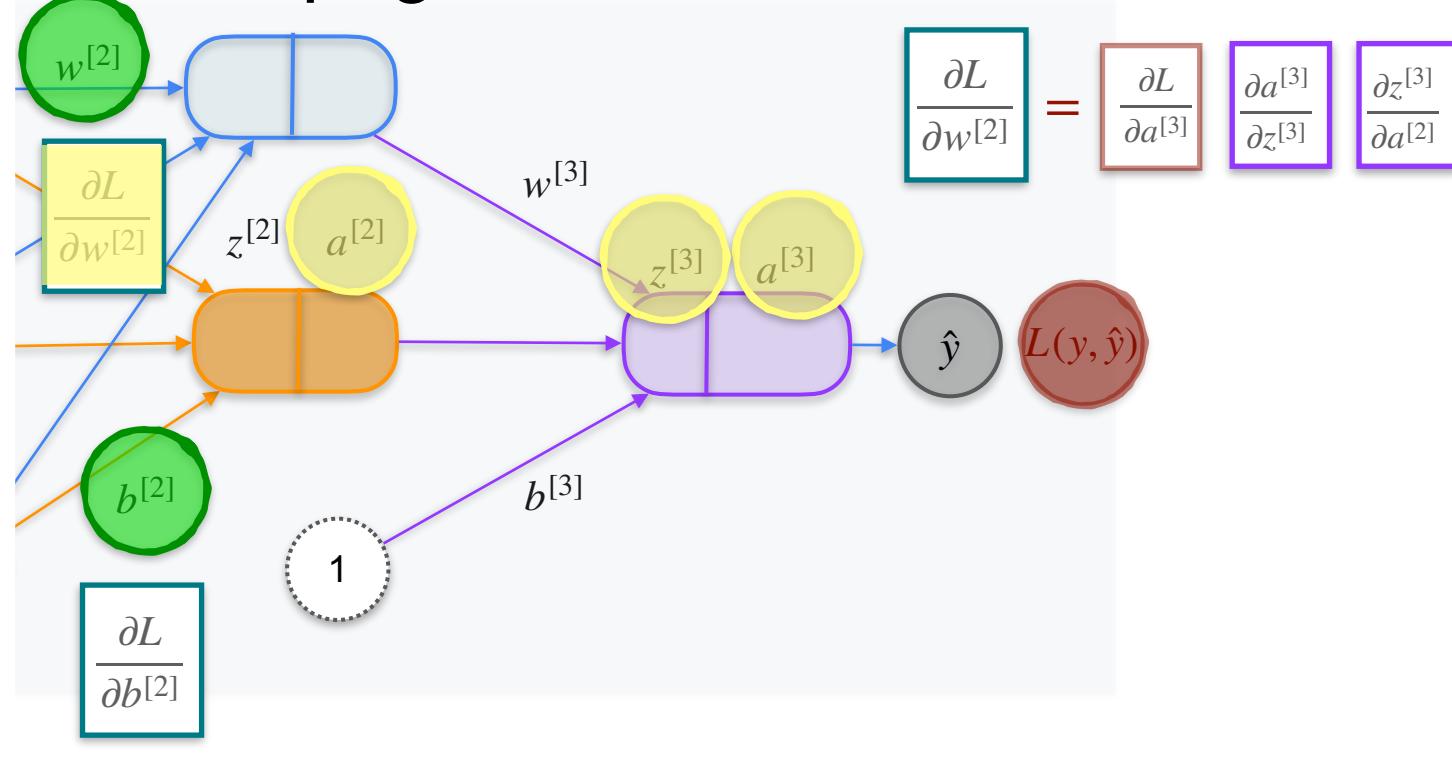
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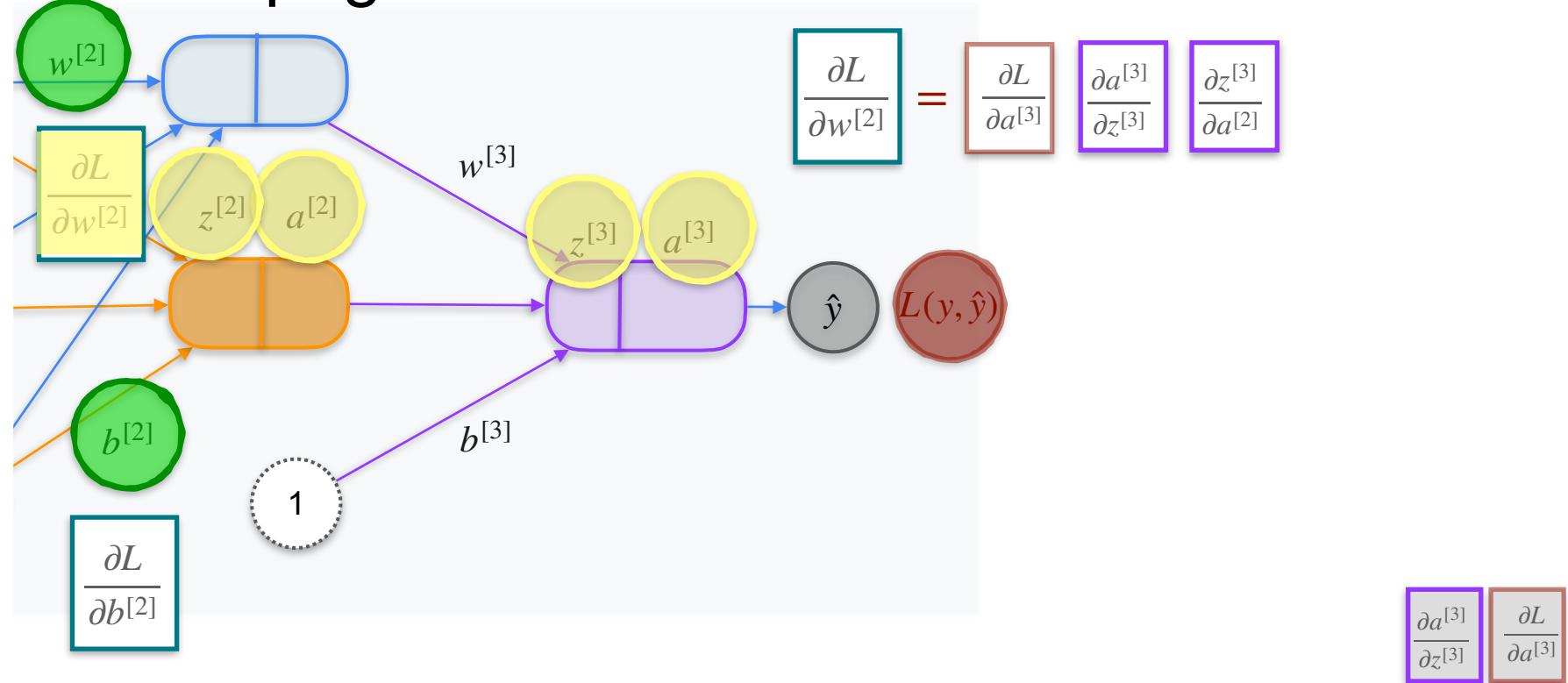
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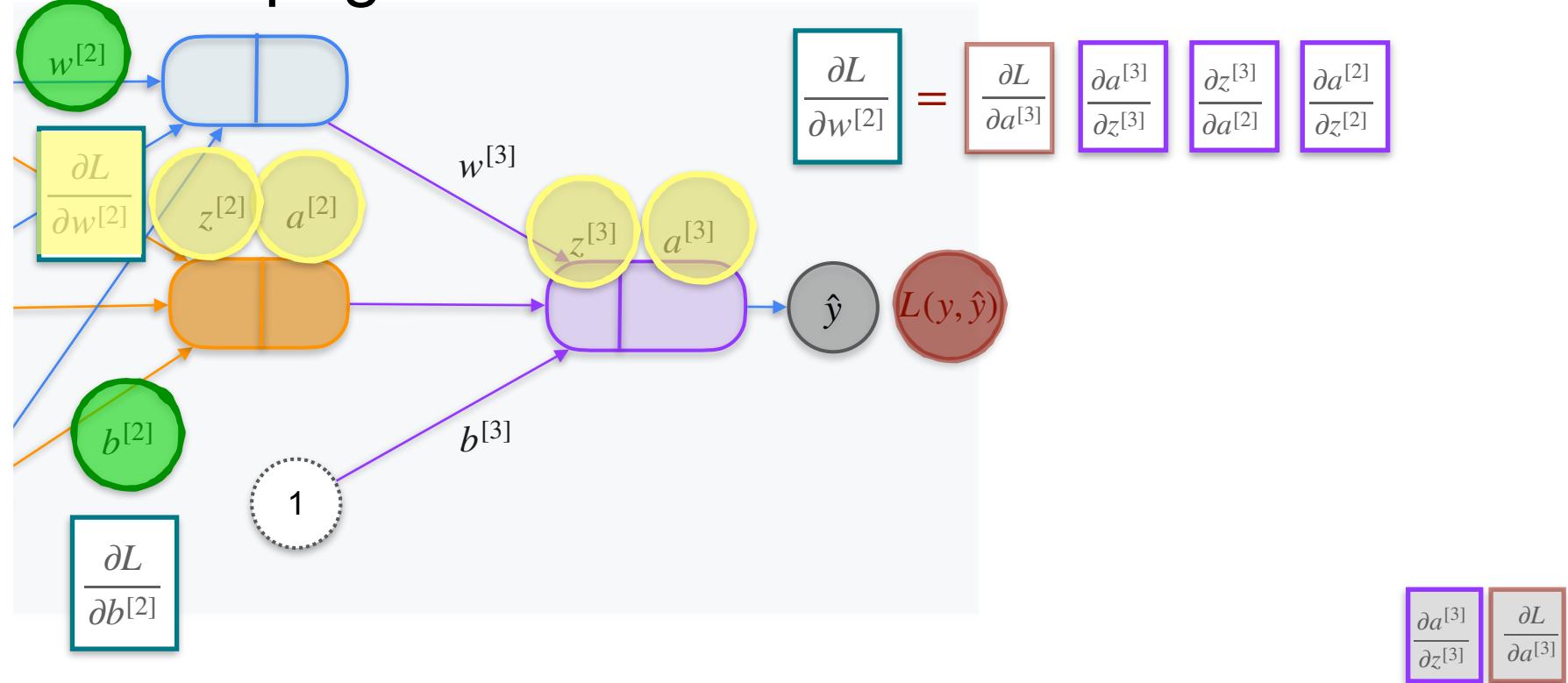
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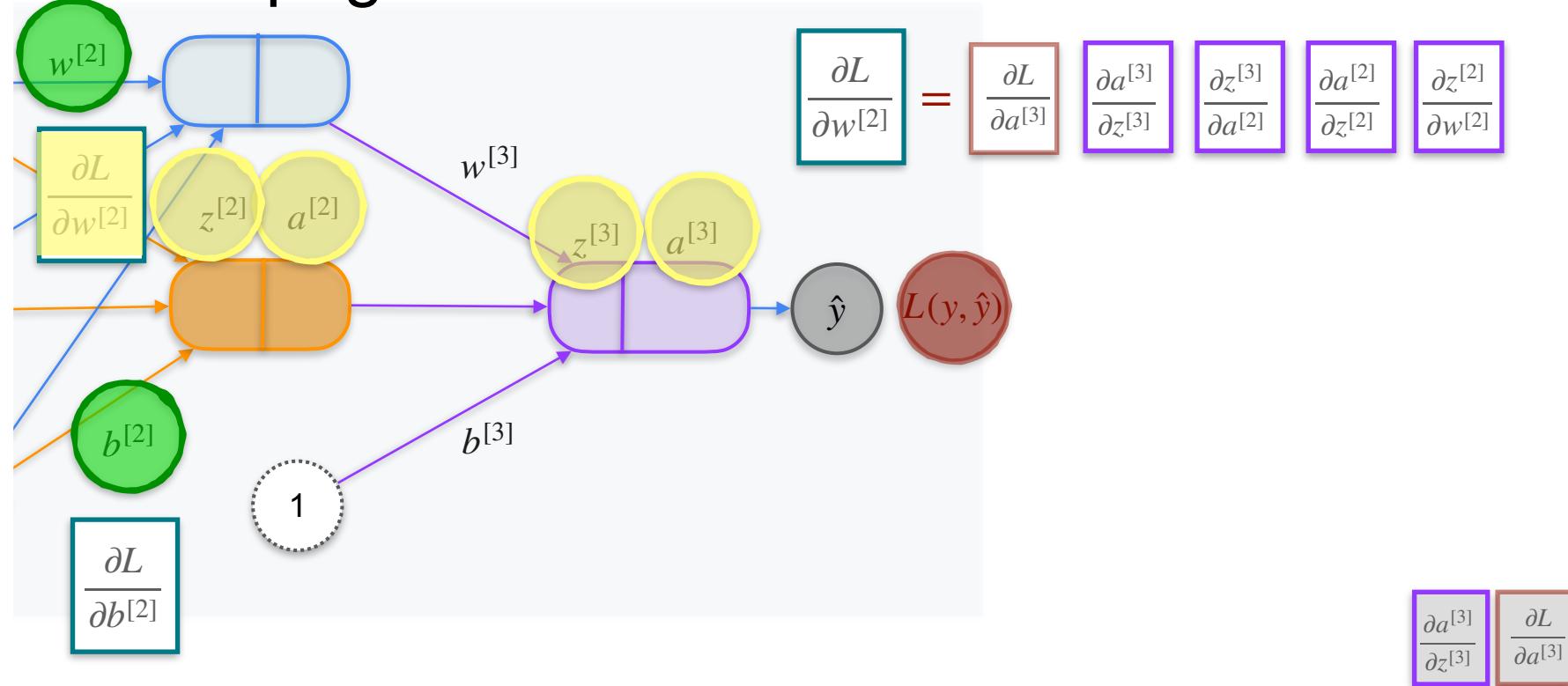
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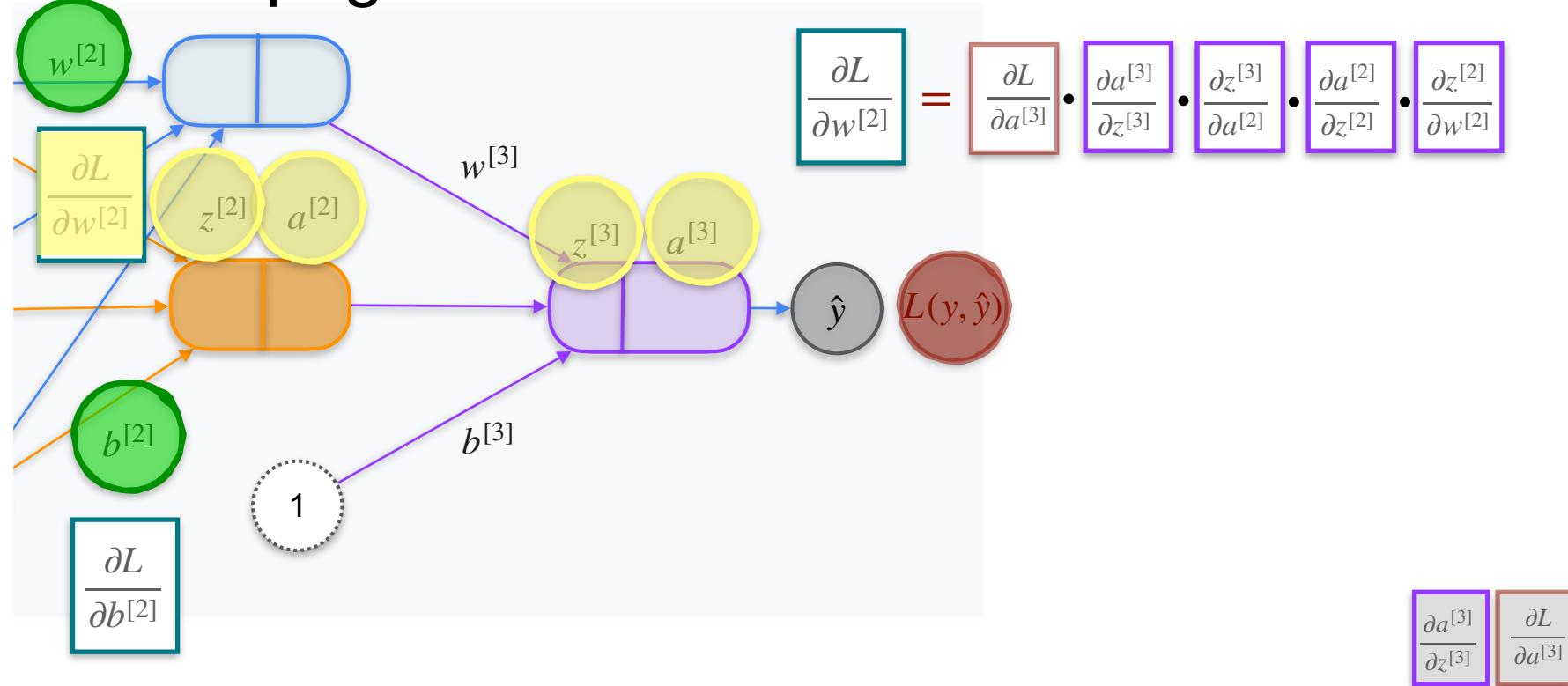
Back Propagation Introduction



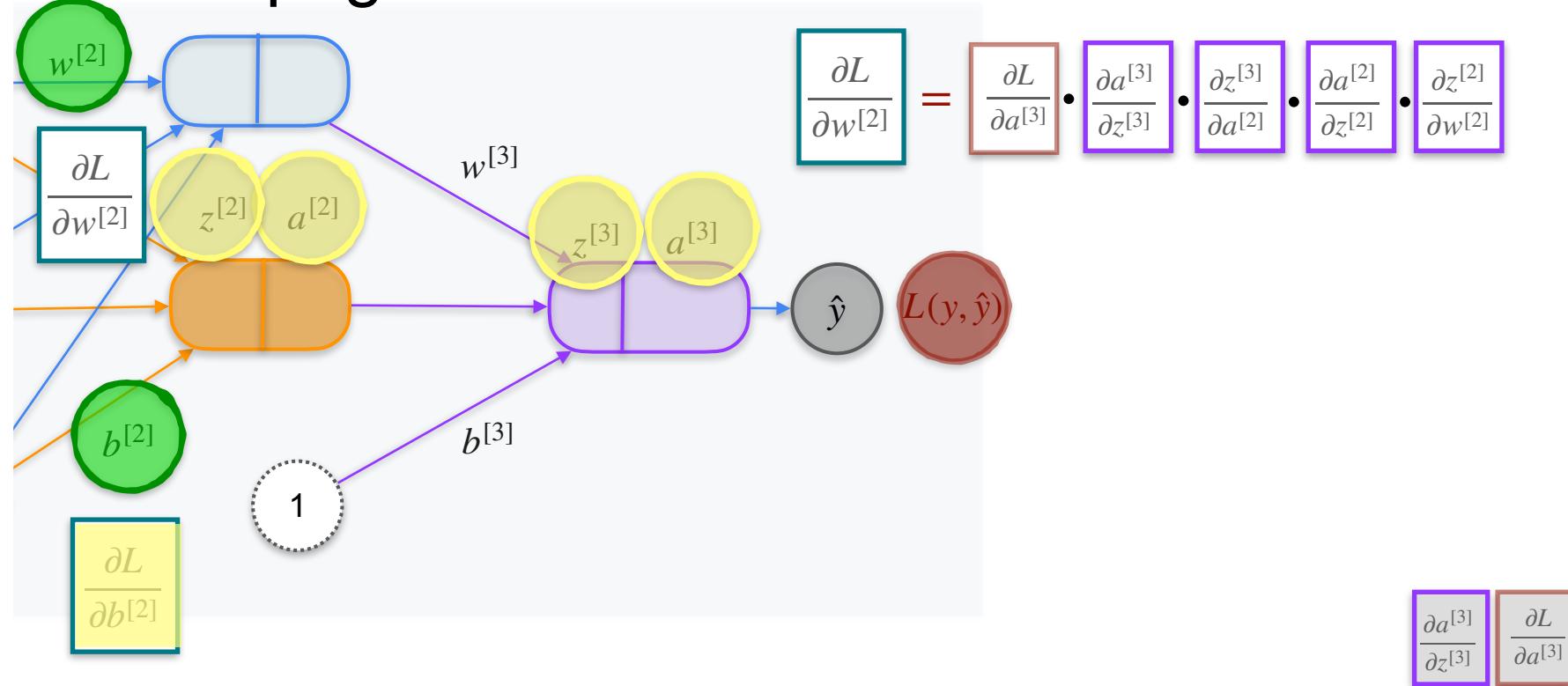
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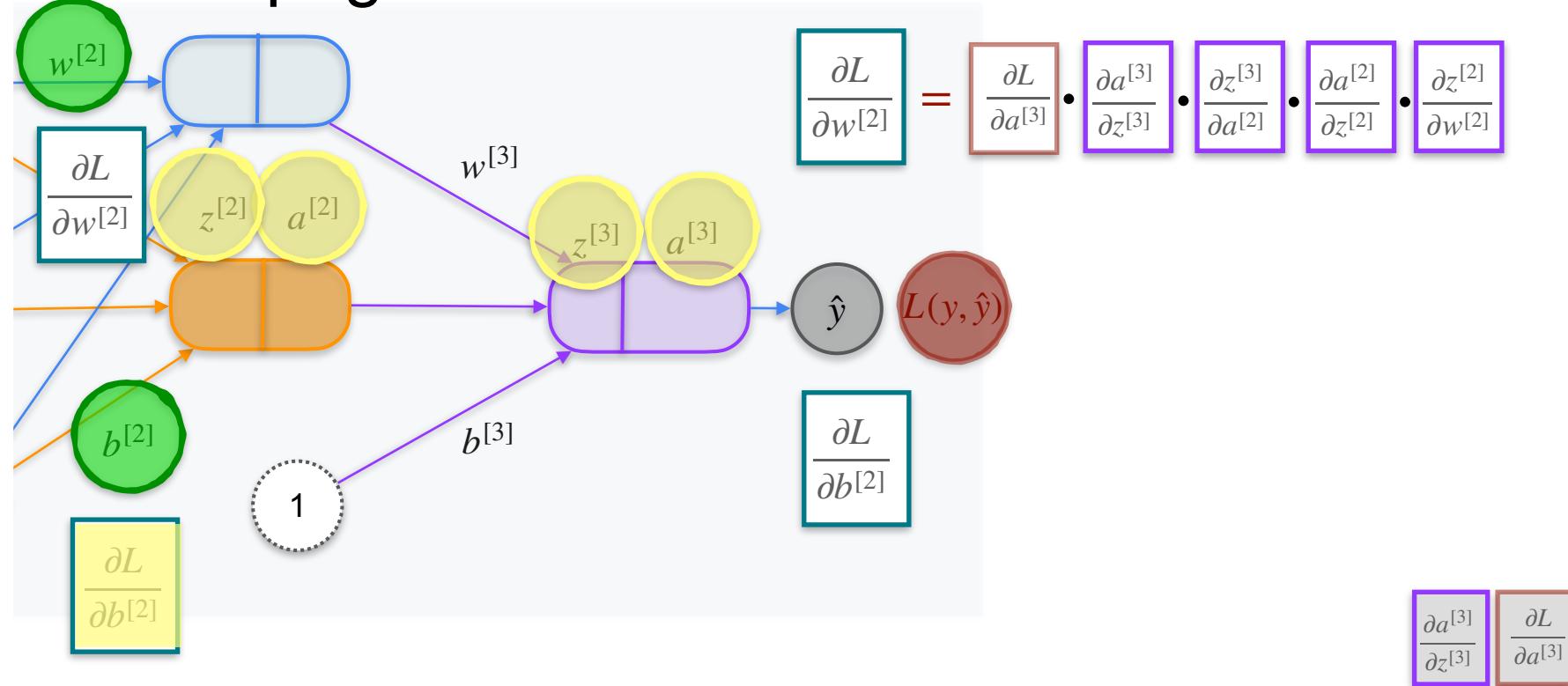
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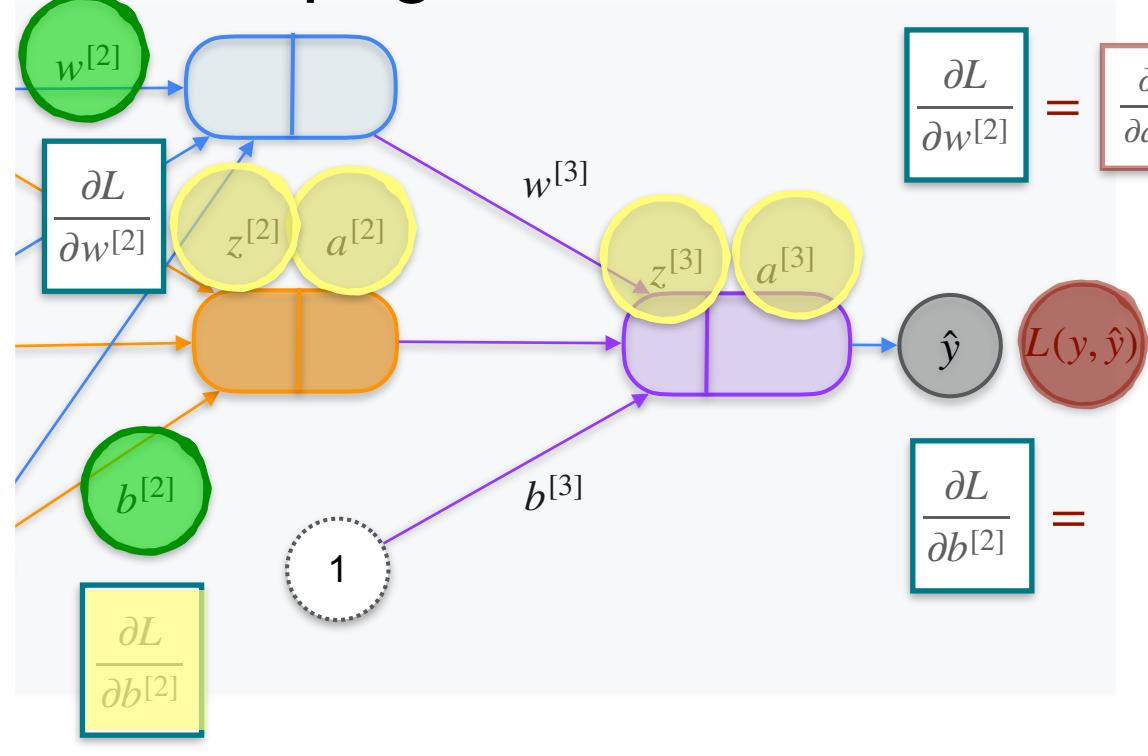
Back Propagation Introduction



Back Propagation Introduction



Back Propagation Introduction

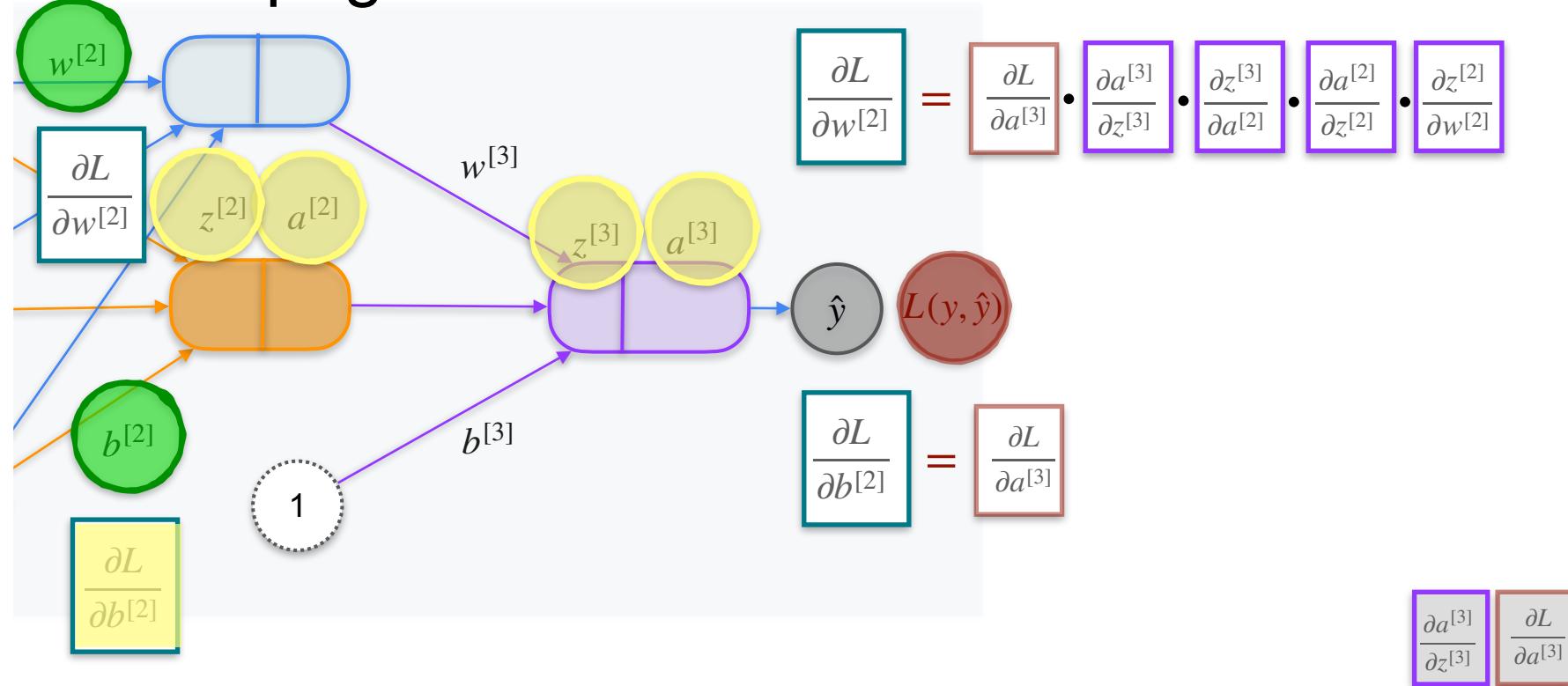


$$\frac{\partial L}{\partial w^{[2]}} = \frac{\partial L}{\partial a^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial z^{[3]}}{\partial a^{[2]}} \cdot \frac{\partial a^{[2]}}{\partial z^{[2]}} \cdot \frac{\partial z^{[2]}}{\partial w^{[2]}}$$

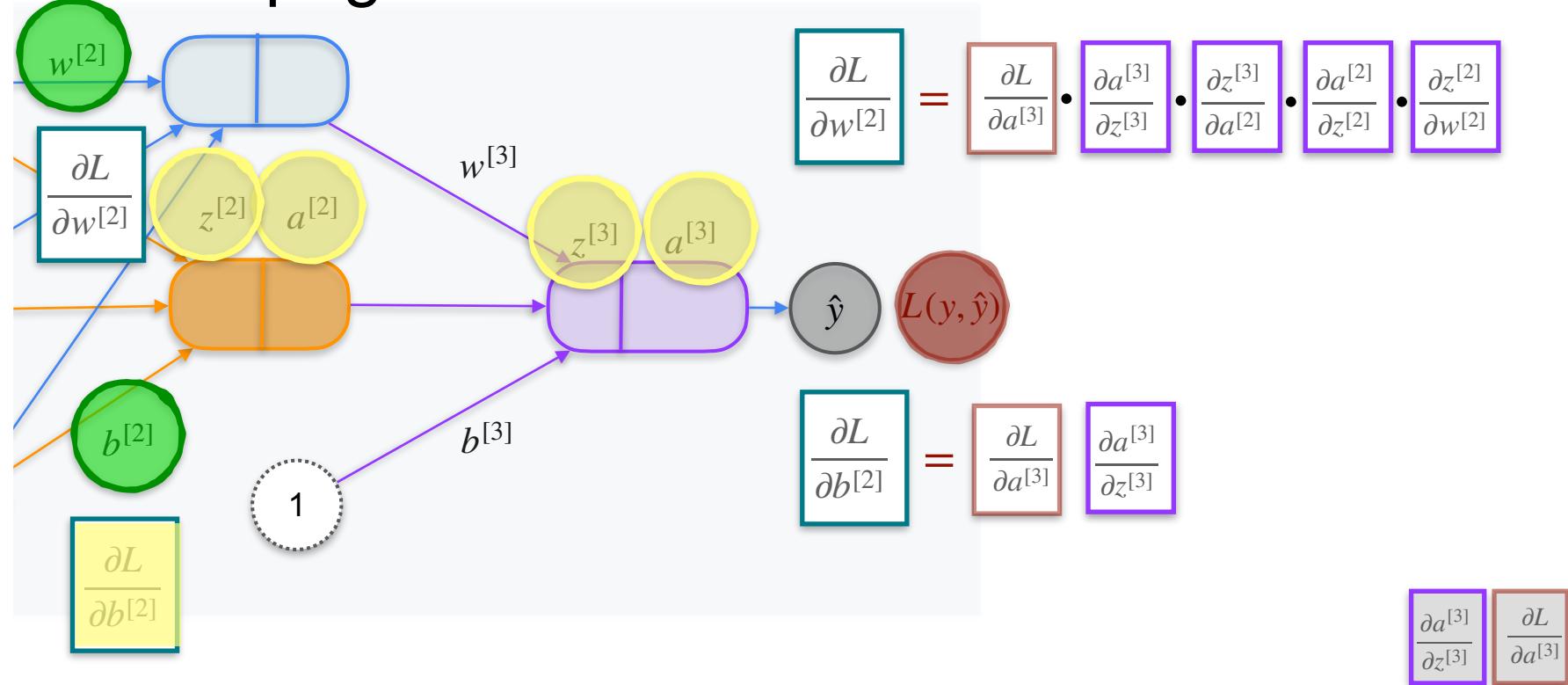
$$\frac{\partial L}{\partial b^{[2]}} =$$

$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

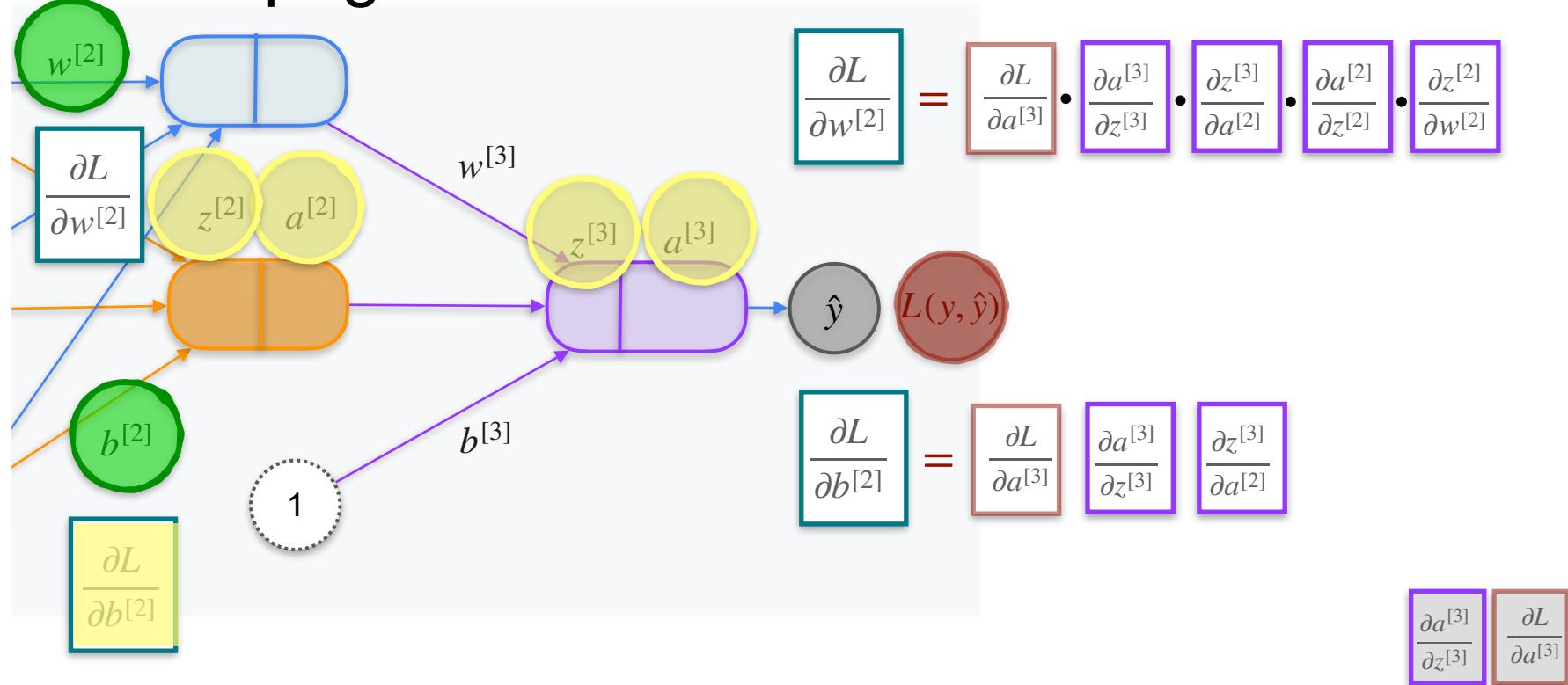
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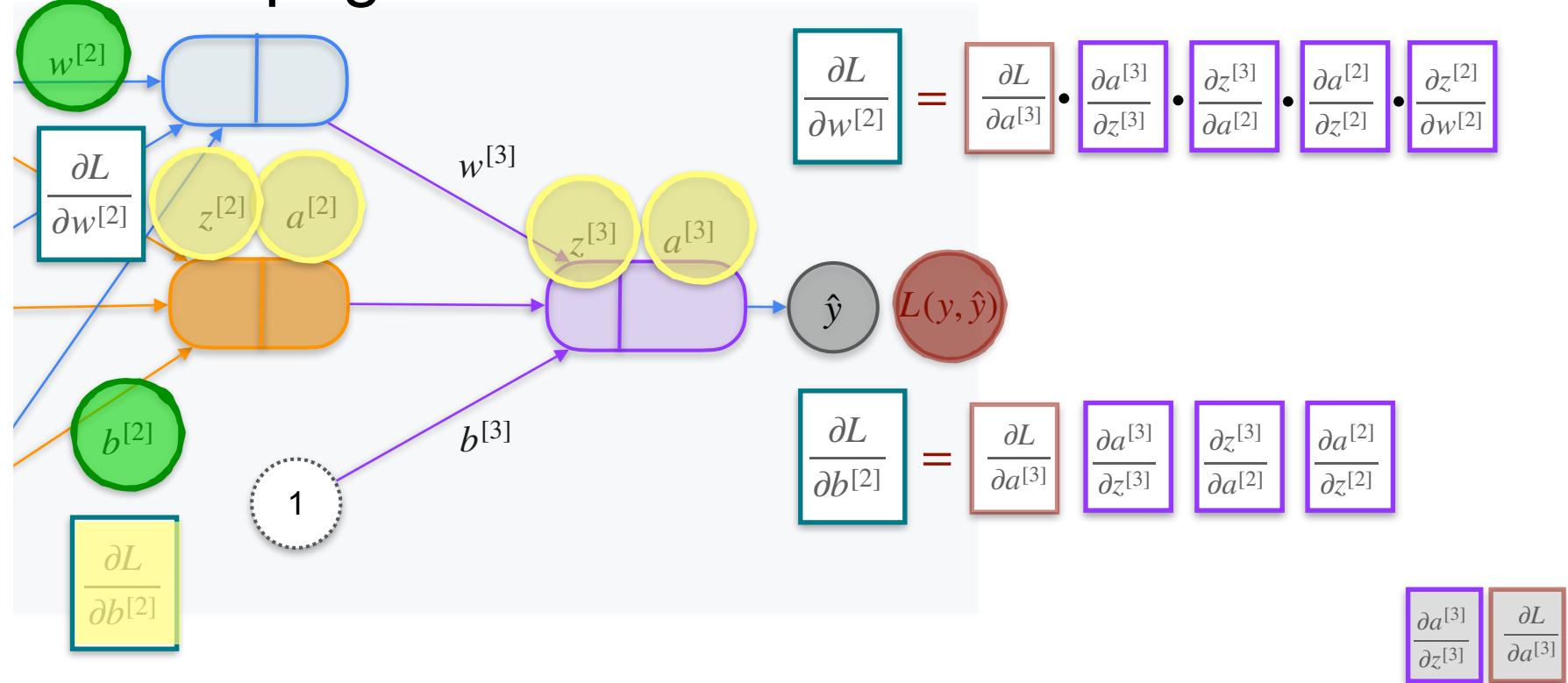
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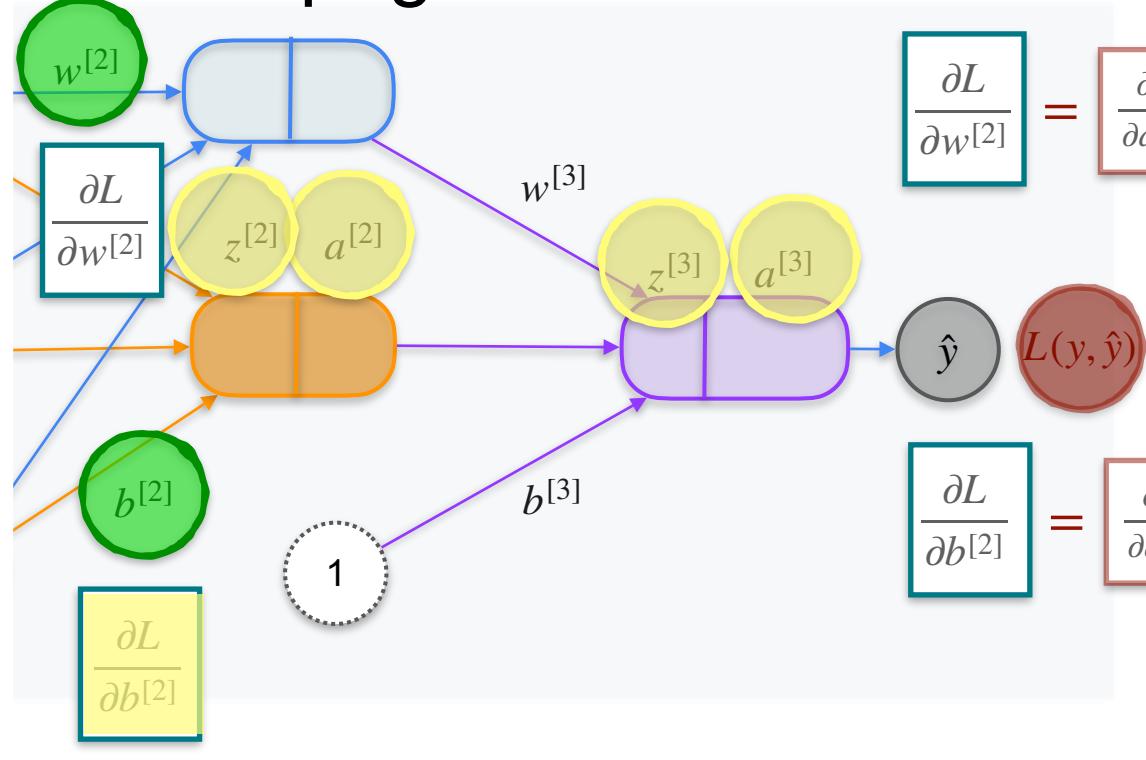
Back Propagation Introduction



Back Propagation Introduction



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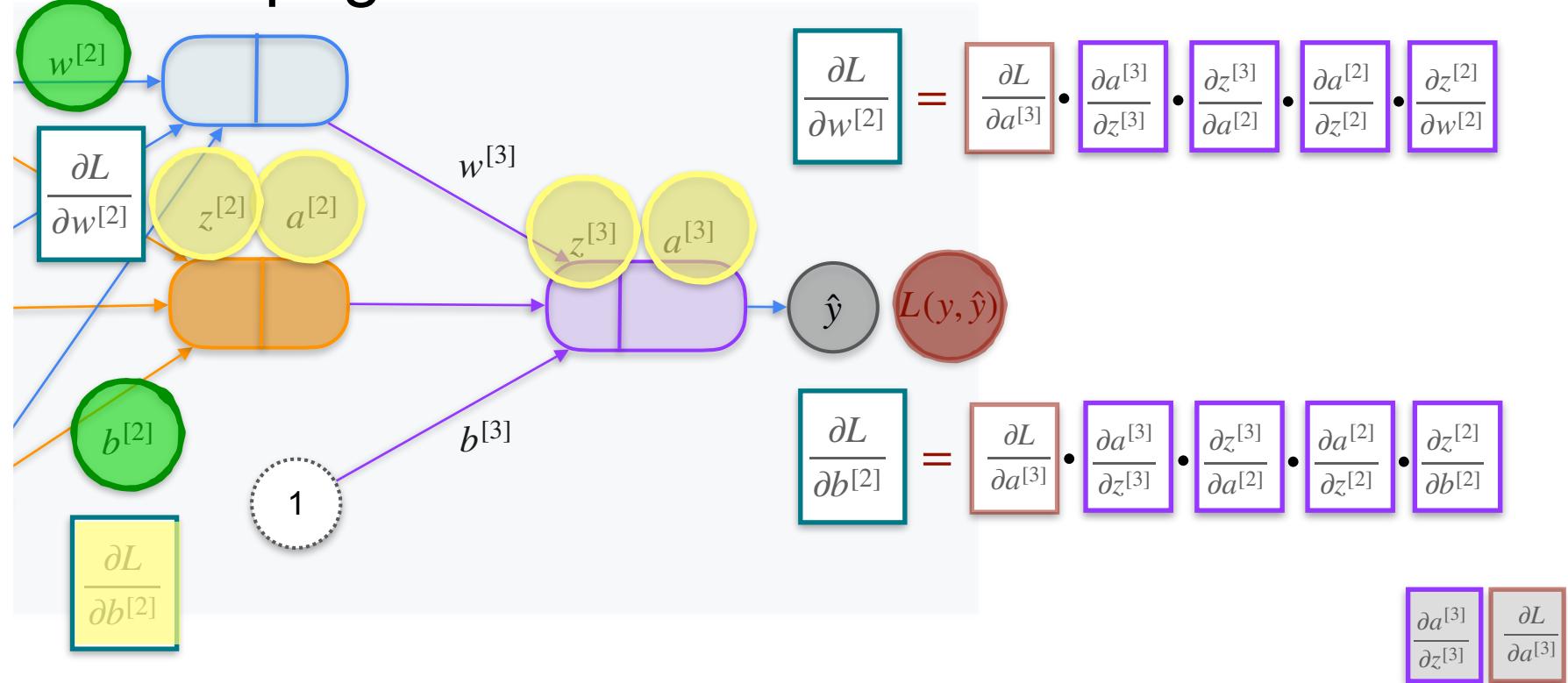


$$\frac{\partial L}{\partial w^{[2]}} = \frac{\partial L}{\partial a^{[3]}} \cdot \frac{\partial a^{[3]}}{\partial z^{[3]}} \cdot \frac{\partial z^{[3]}}{\partial a^{[2]}} \cdot \frac{\partial a^{[2]}}{\partial z^{[2]}} \cdot \frac{\partial z^{[2]}}{\partial w^{[2]}}$$

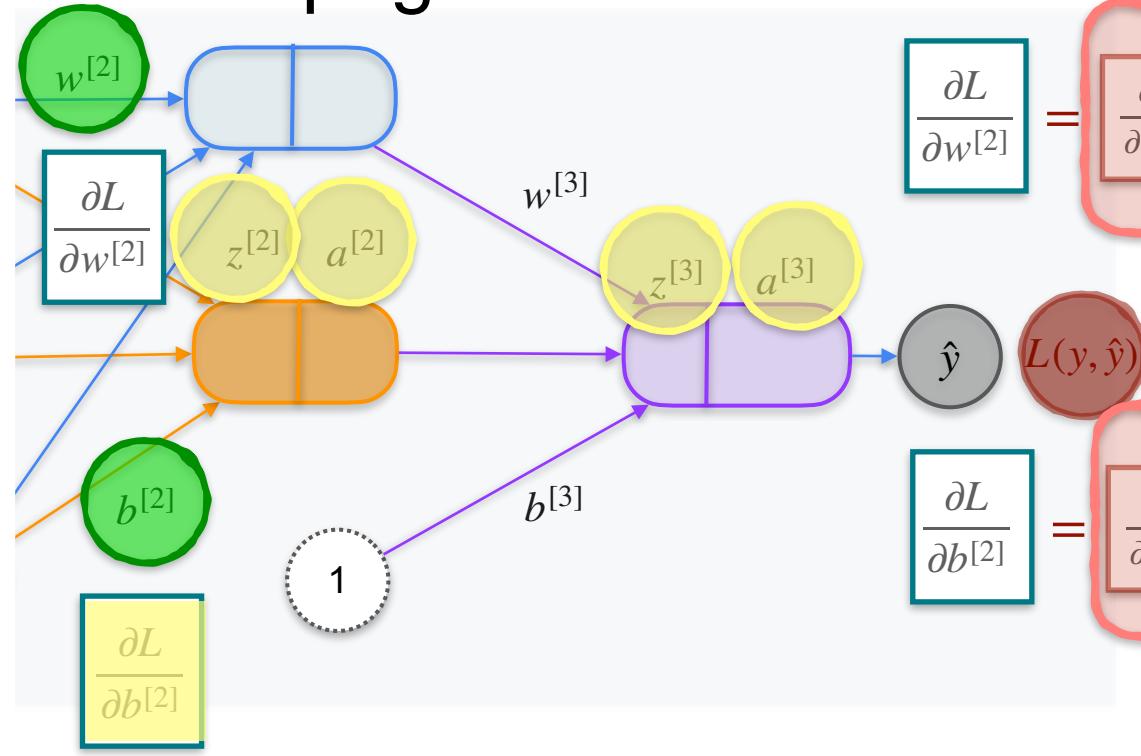
$$\frac{\partial L}{\partial b^{[2]}} = \frac{\partial L}{\partial a^{[3]}} \frac{\partial a^{[3]}}{\partial z^{[3]}} \frac{\partial z^{[3]}}{\partial a^{[2]}} \frac{\partial a^{[2]}}{\partial z^{[2]}} \frac{\partial z^{[2]}}{\partial b^{[2]}}$$

$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

Back Propagation Introduction



Back Propagation Introduction

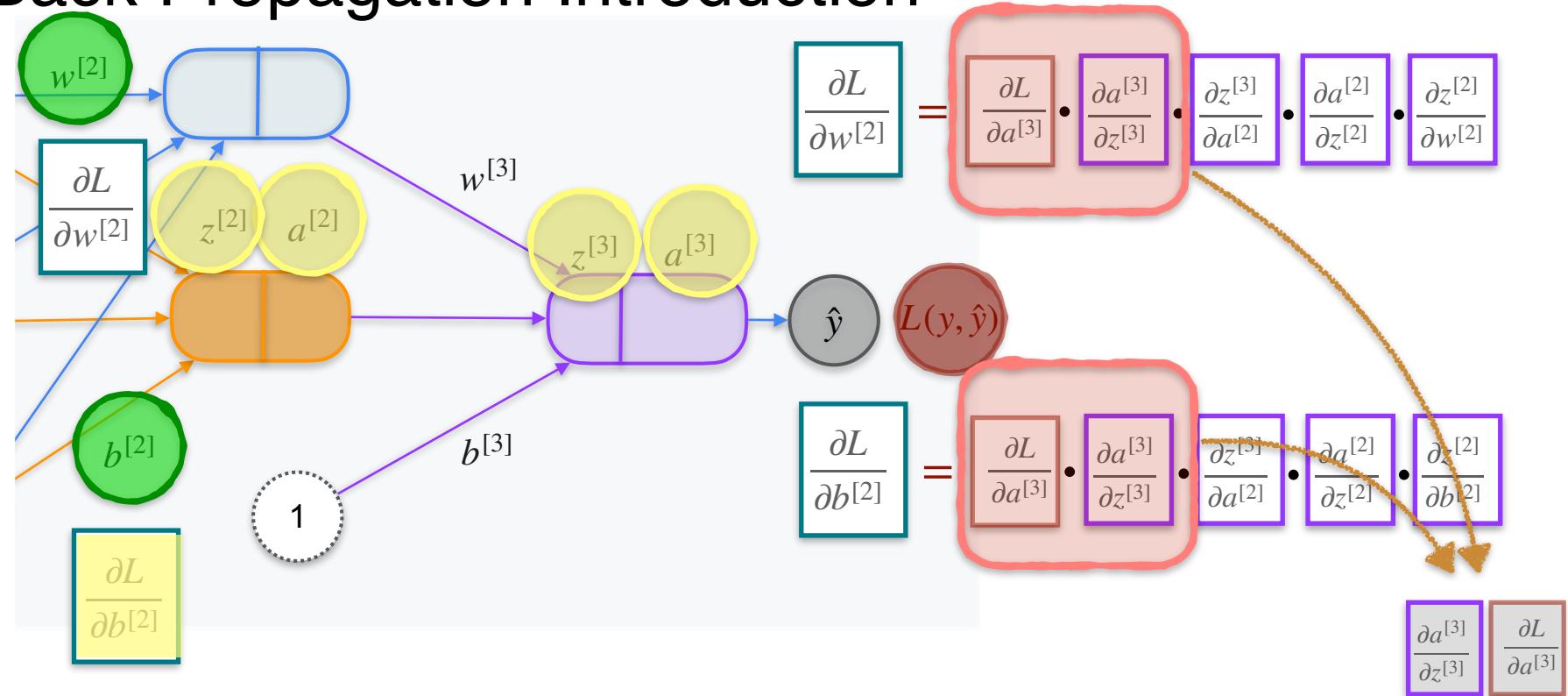


$$\frac{\partial L}{\partial w^{[2]}} = \boxed{\frac{\partial L}{\partial a^{[3]}}} \cdot \boxed{\frac{\partial a^{[3]}}{\partial z^{[3]}}} \cdot \boxed{\frac{\partial z^{[3]}}{\partial a^{[2]}}} \cdot \boxed{\frac{\partial a^{[2]}}{\partial z^{[2]}}} \cdot \boxed{\frac{\partial z^{[2]}}{\partial w^{[2]}}}$$

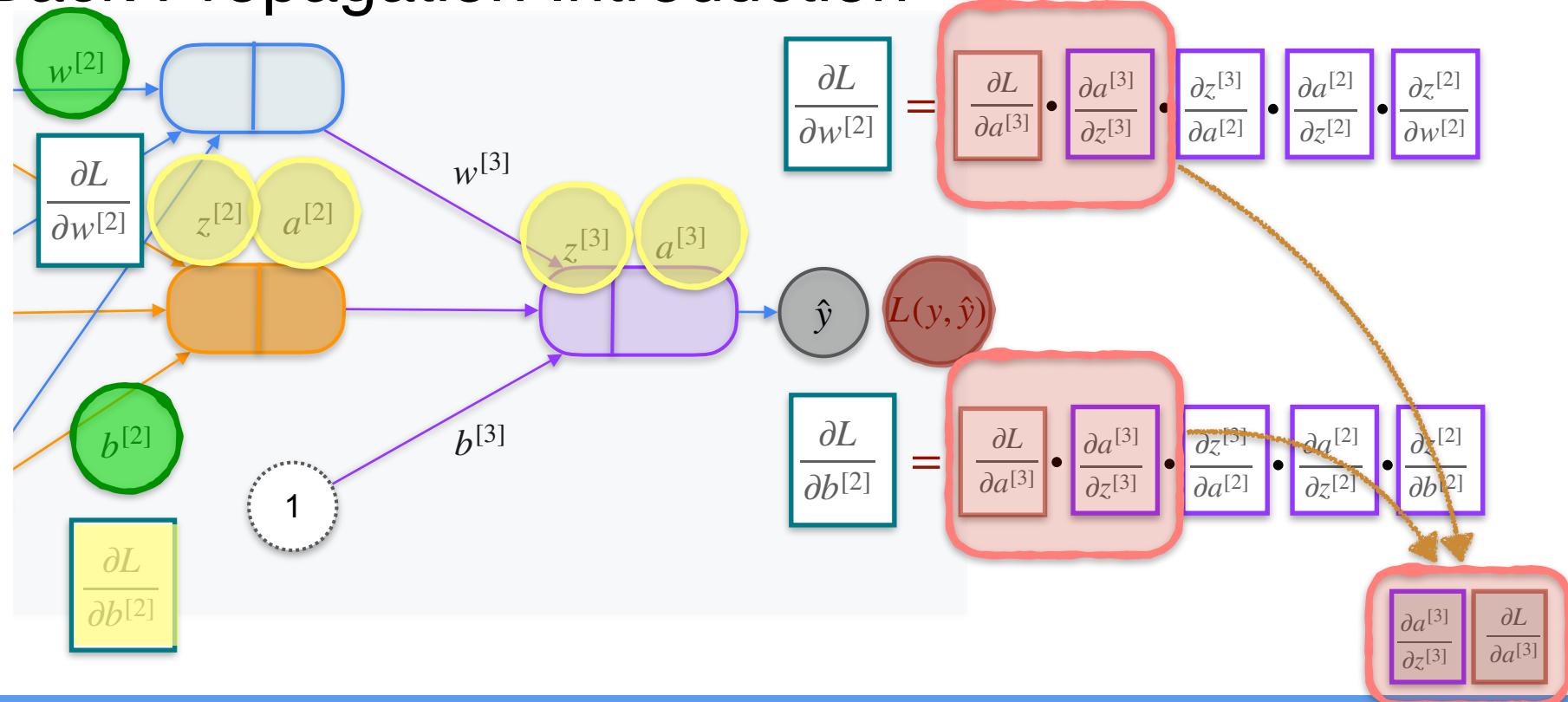
$$\frac{\partial L}{\partial b^{[2]}} = \boxed{\frac{\partial L}{\partial a^{[3]}}} \cdot \boxed{\frac{\partial a^{[3]}}{\partial z^{[3]}}} \cdot \boxed{\frac{\partial z^{[3]}}{\partial a^{[2]}}} \cdot \boxed{\frac{\partial a^{[2]}}{\partial z^{[2]}}} \cdot \boxed{\frac{\partial z^{[2]}}{\partial b^{[2]}}}$$

$$\frac{\partial a^{[3]}}{\partial z^{[3]}} \quad \frac{\partial L}{\partial a^{[3]}}$$

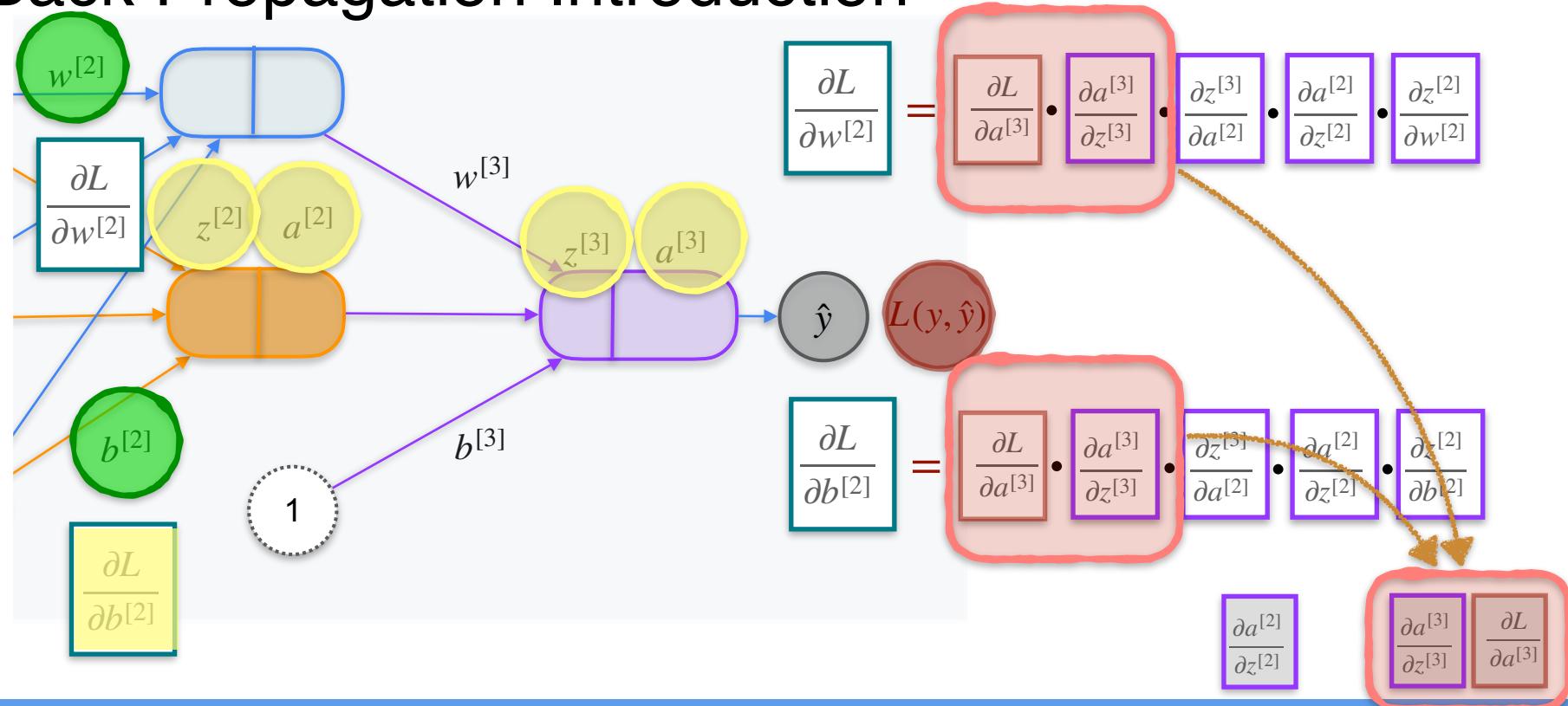
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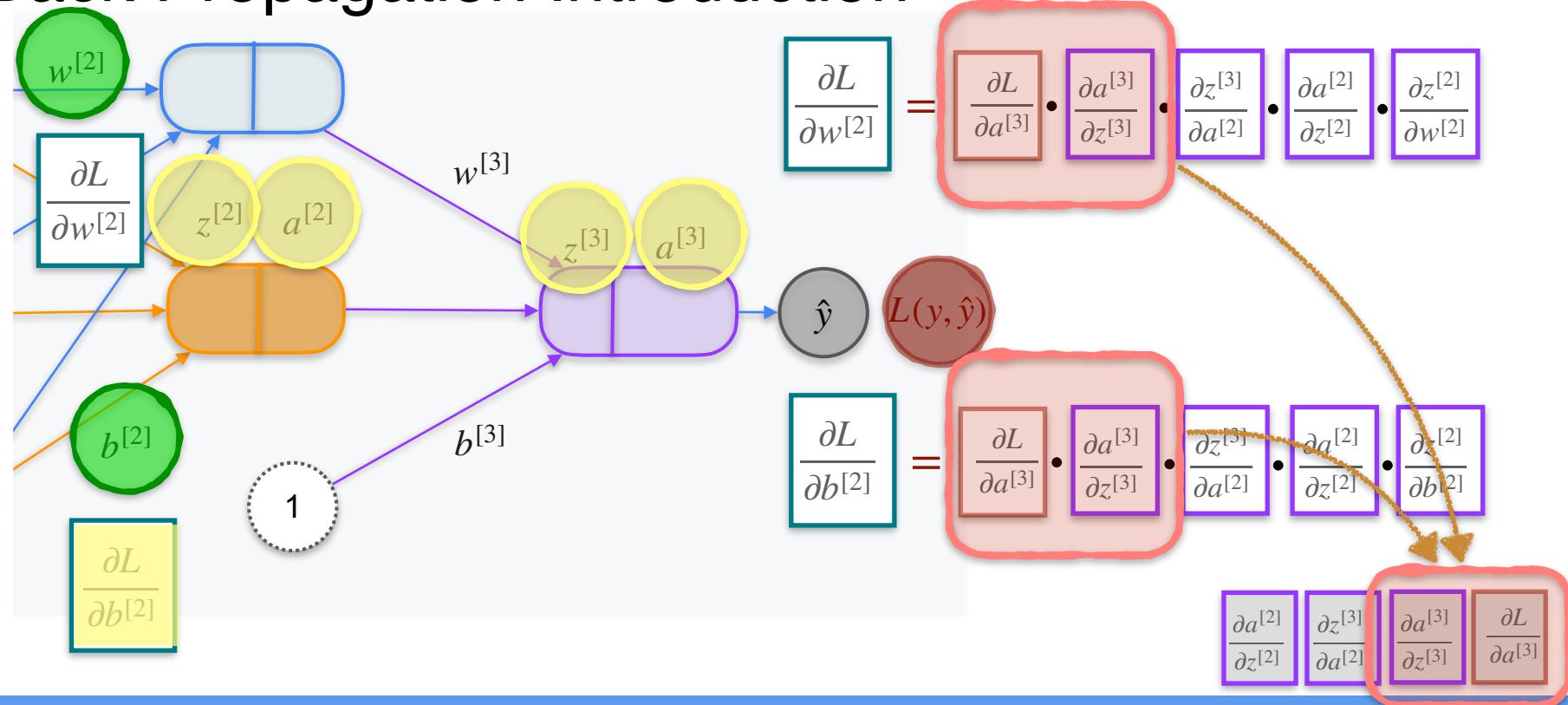
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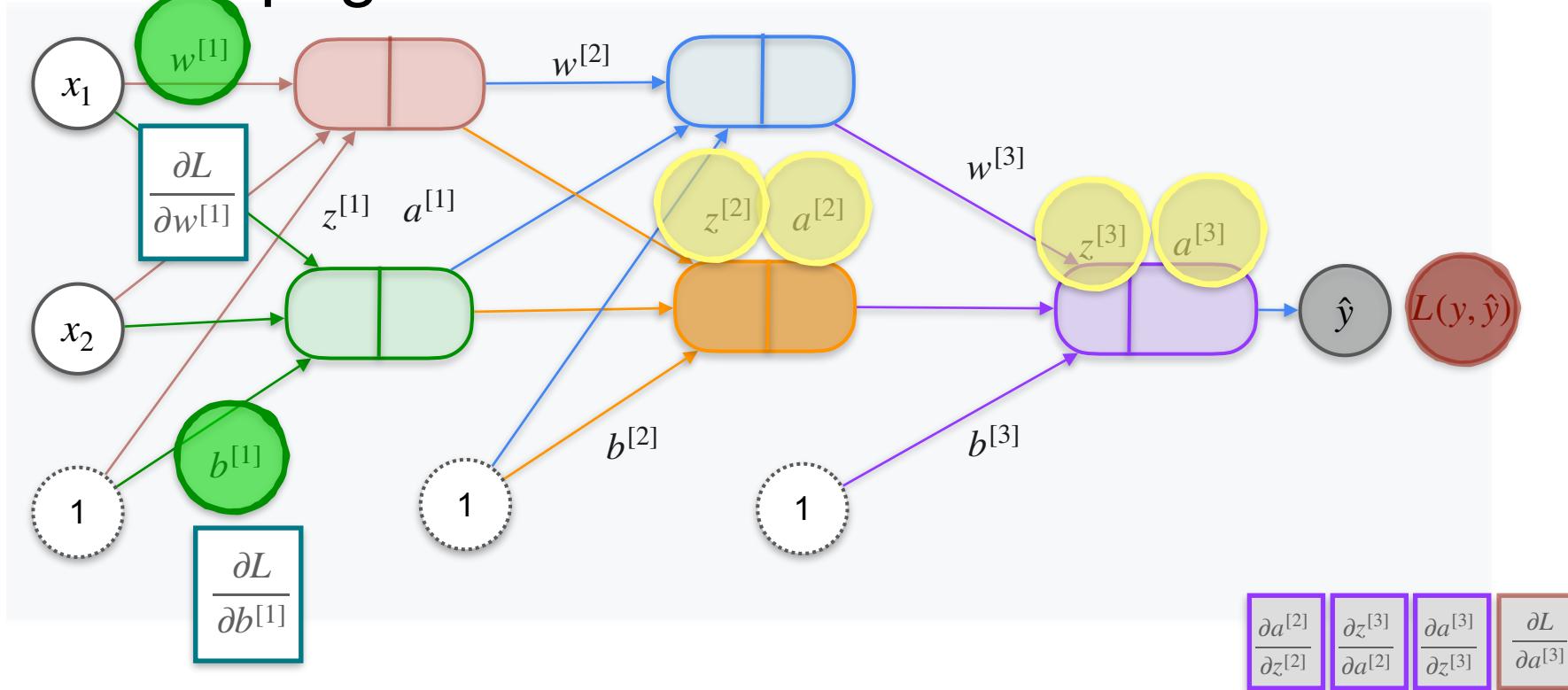
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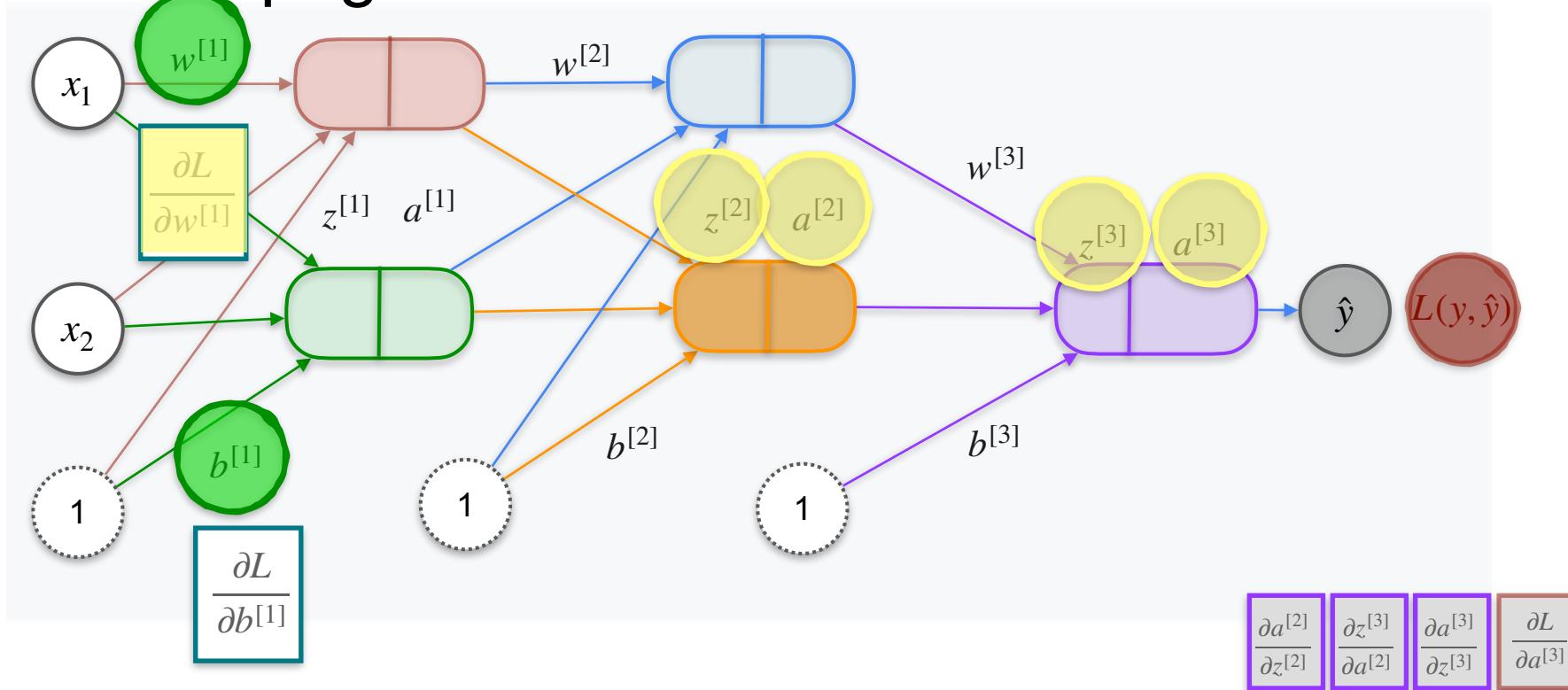
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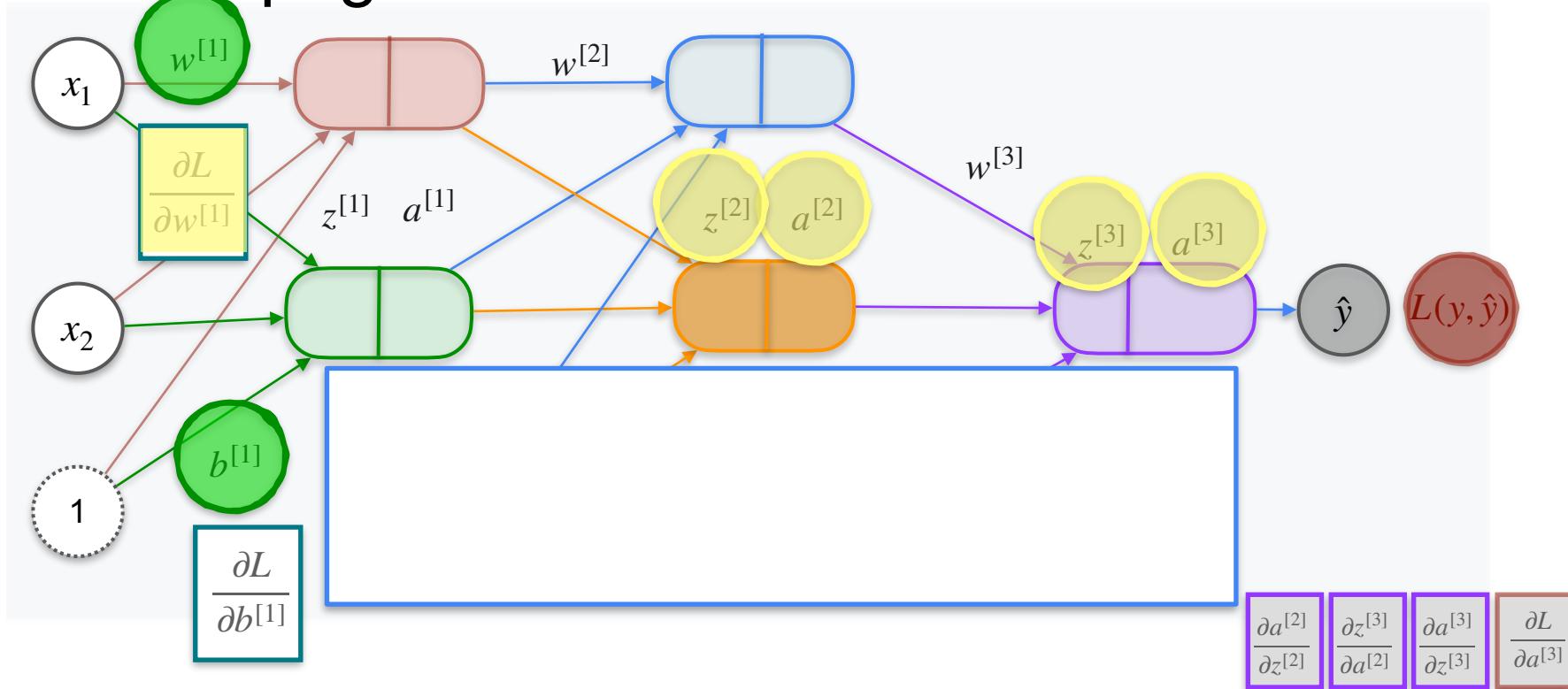
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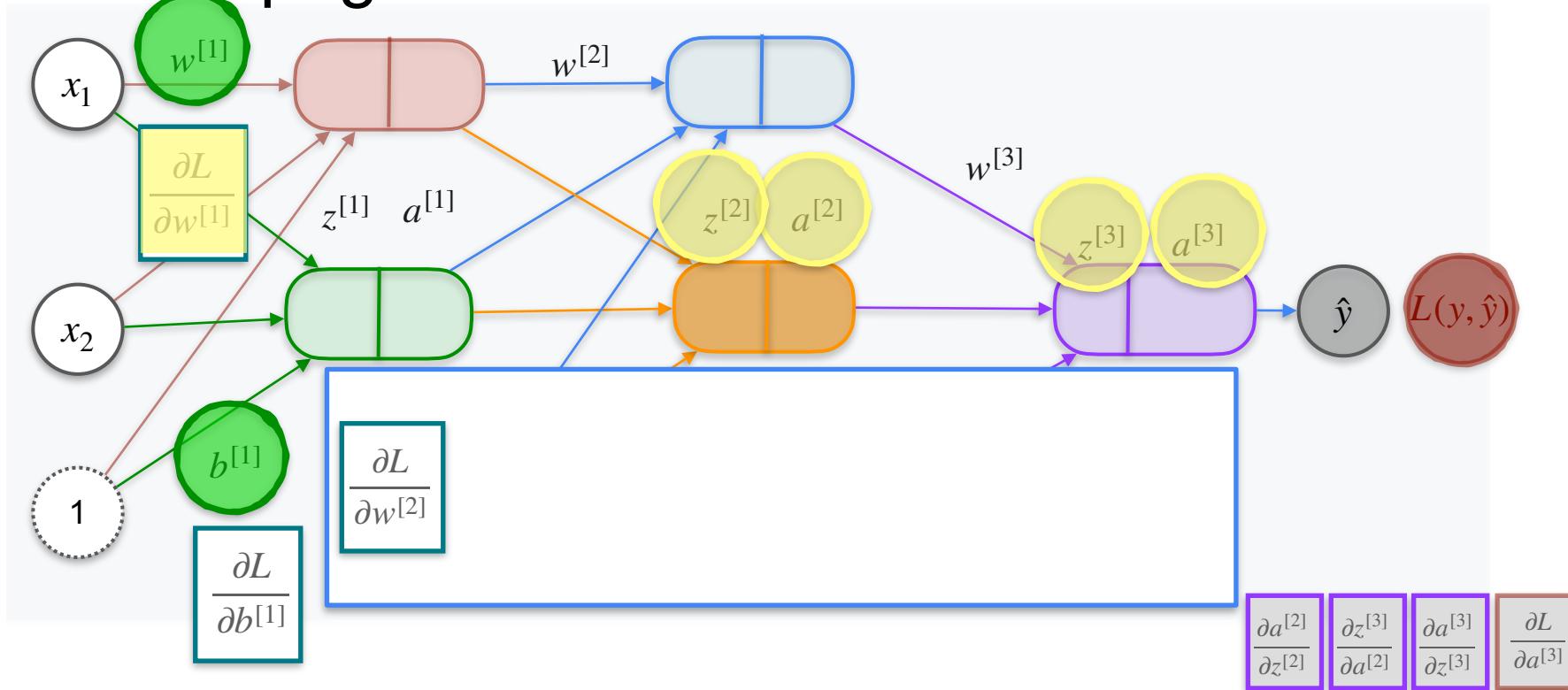
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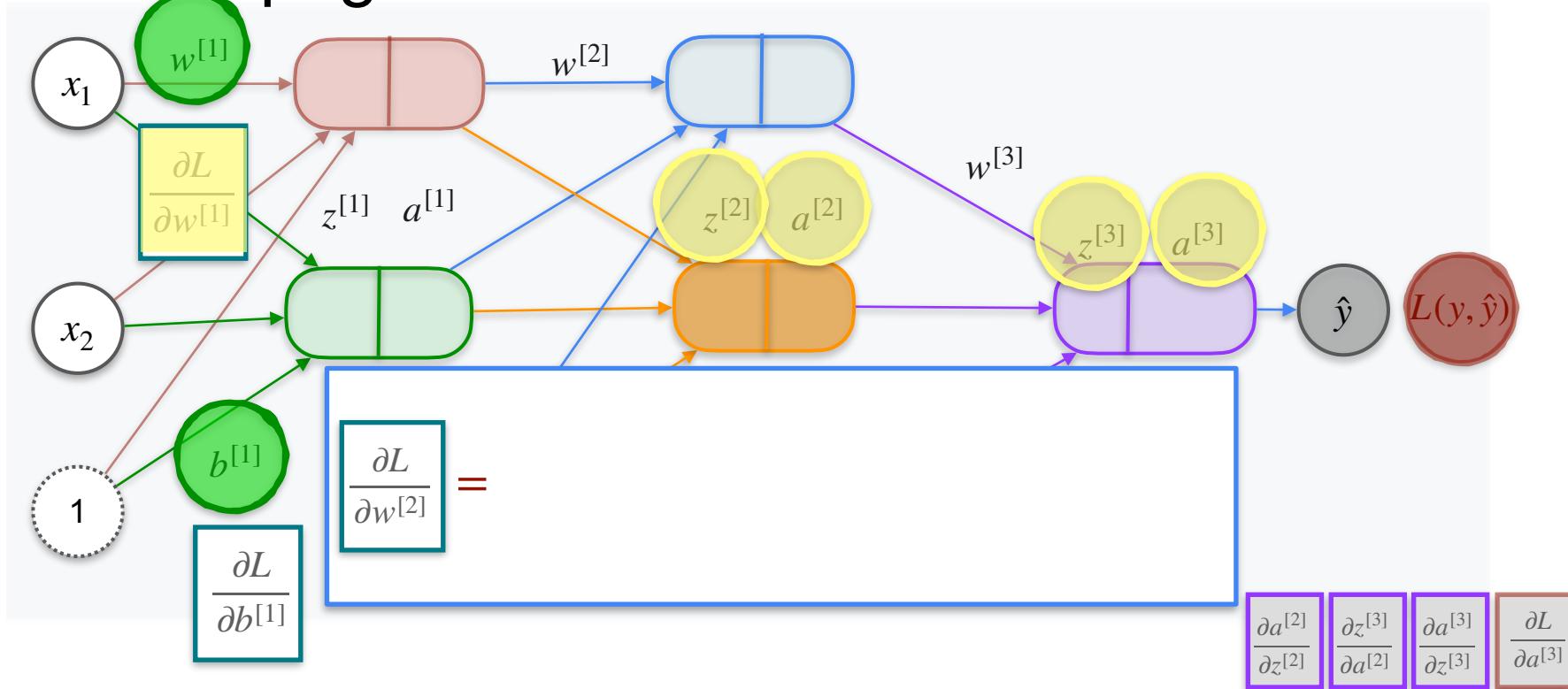
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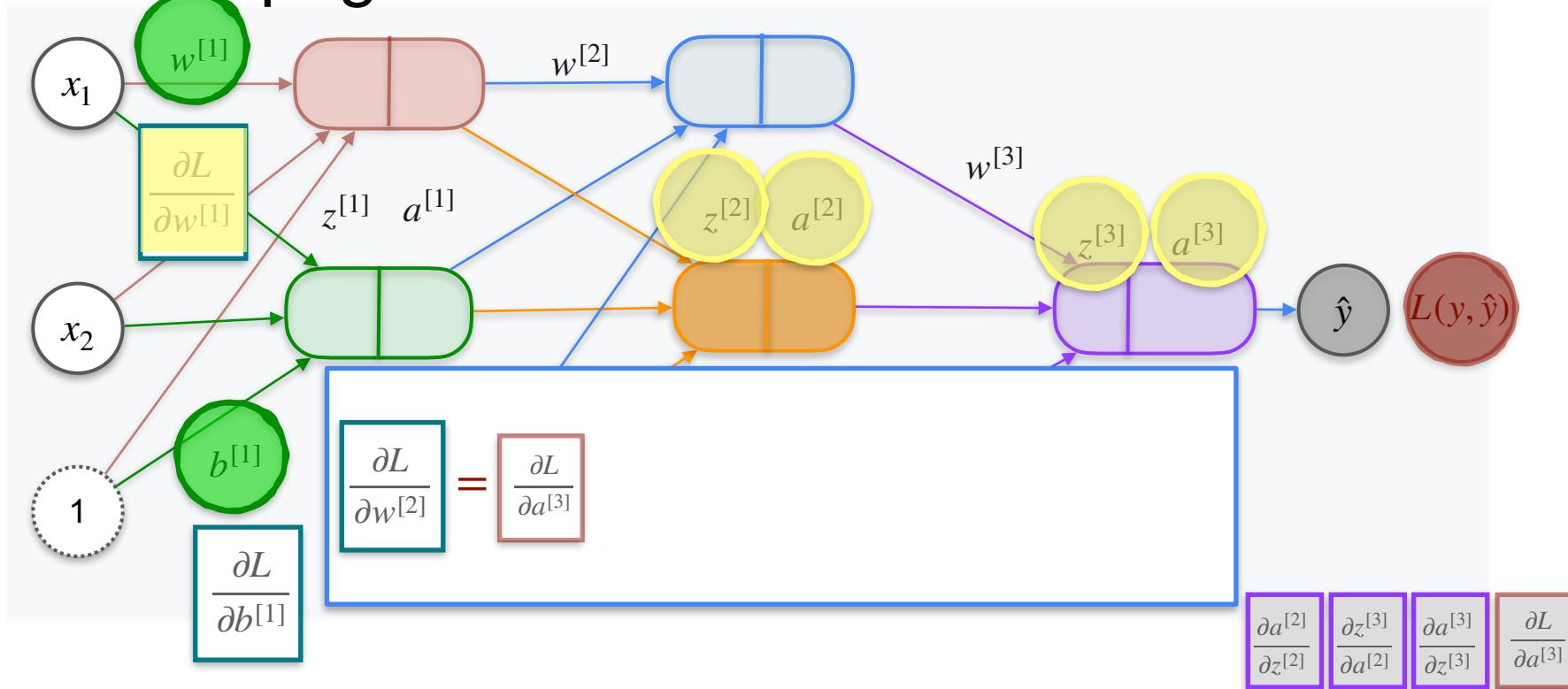
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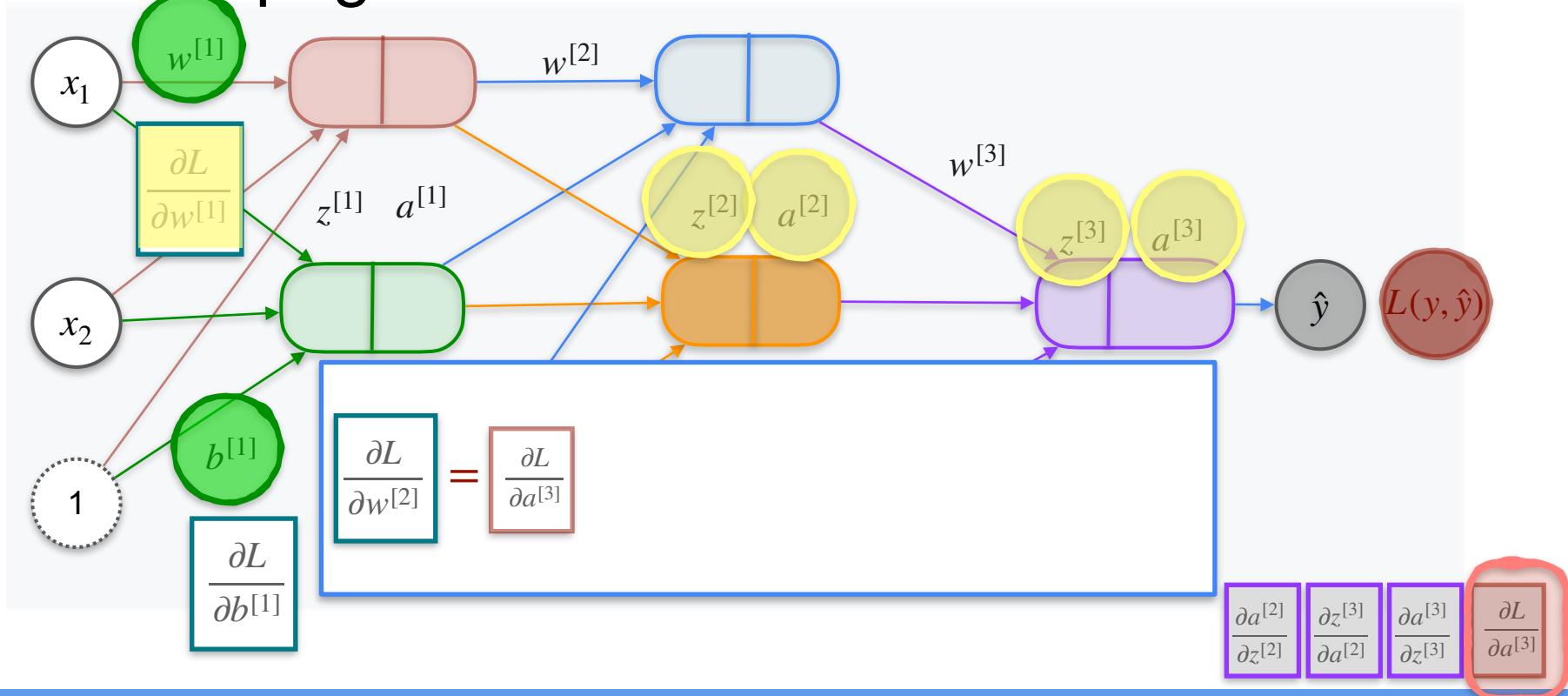
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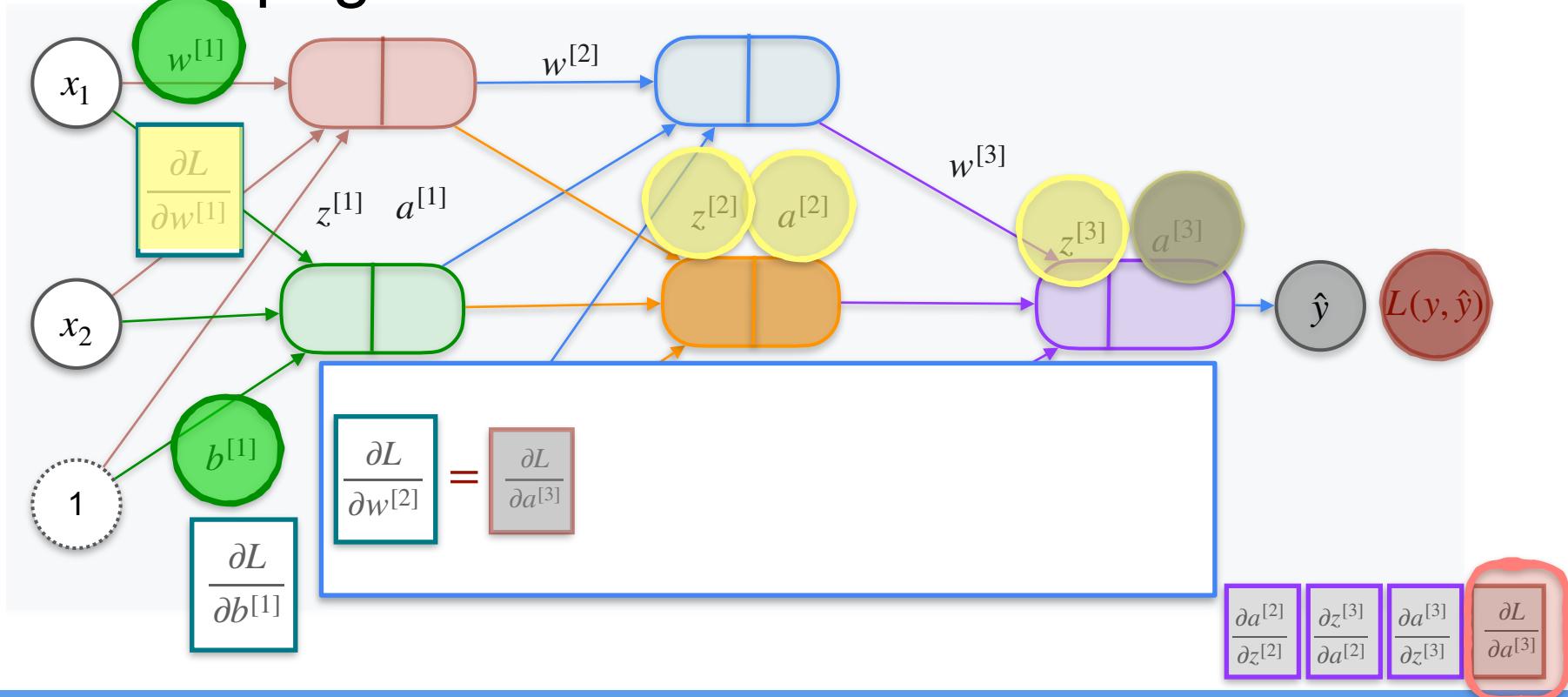
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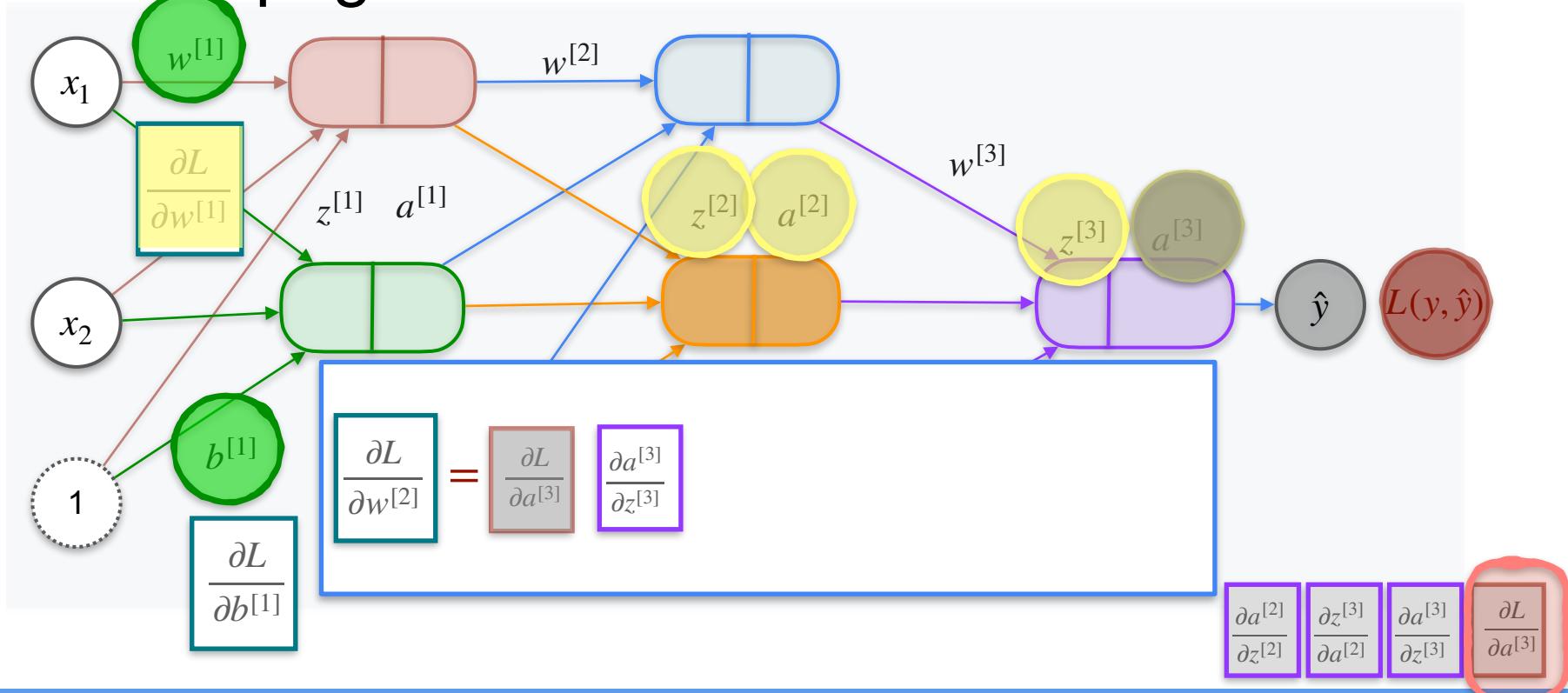
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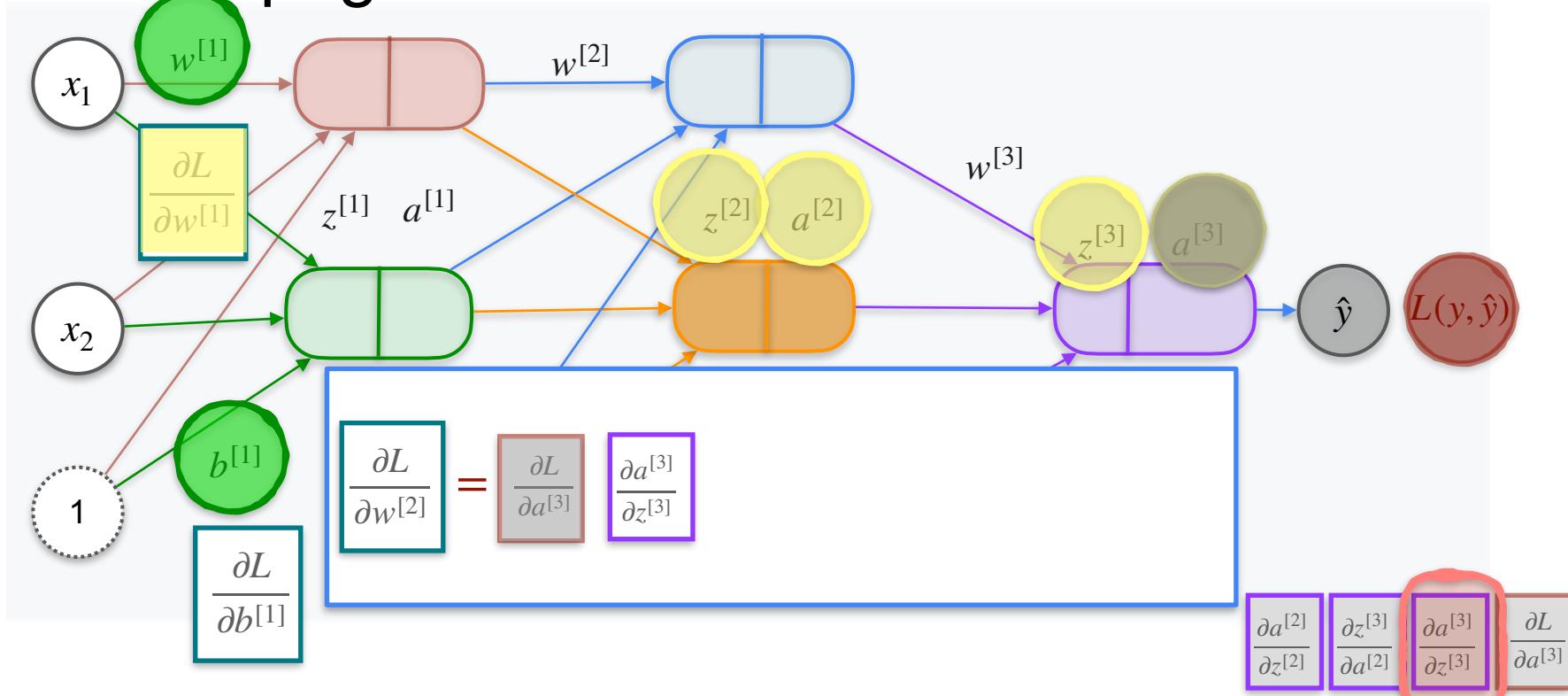
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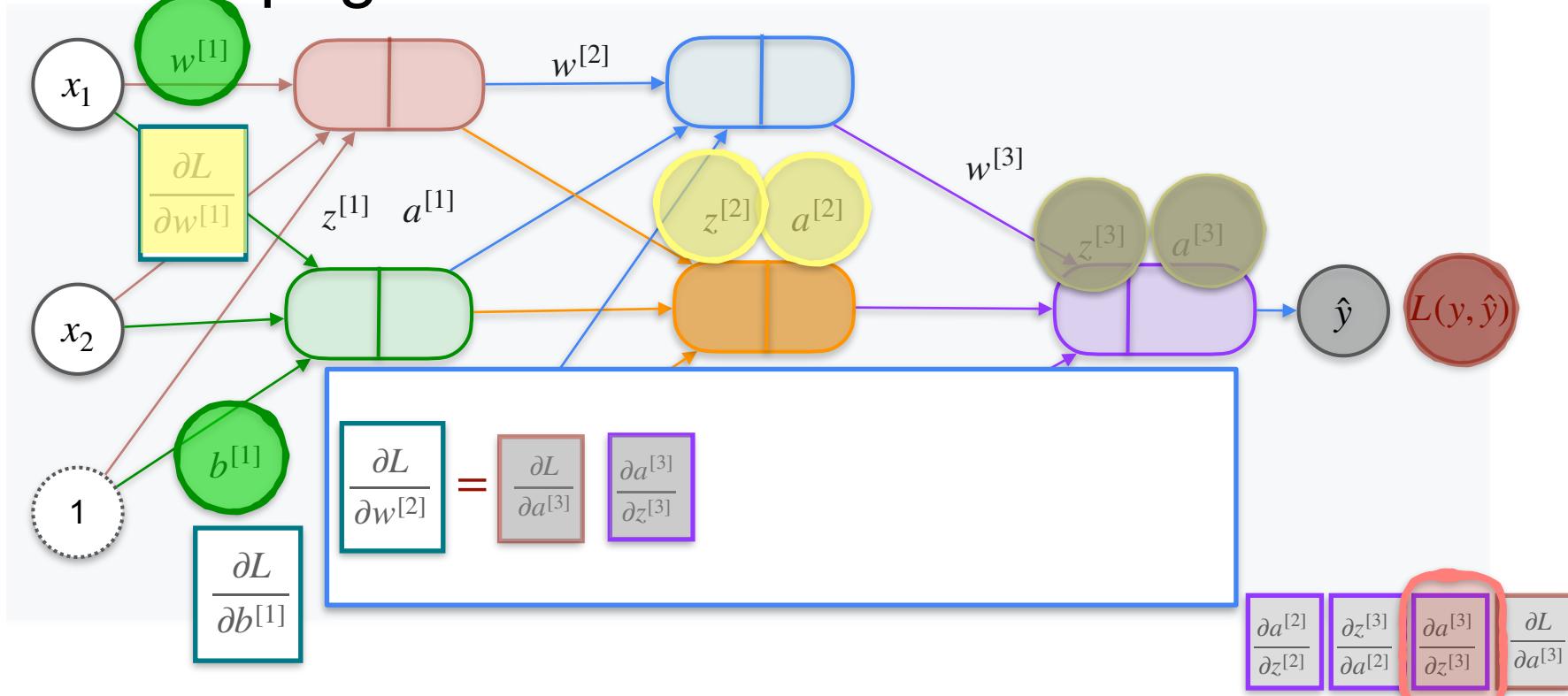
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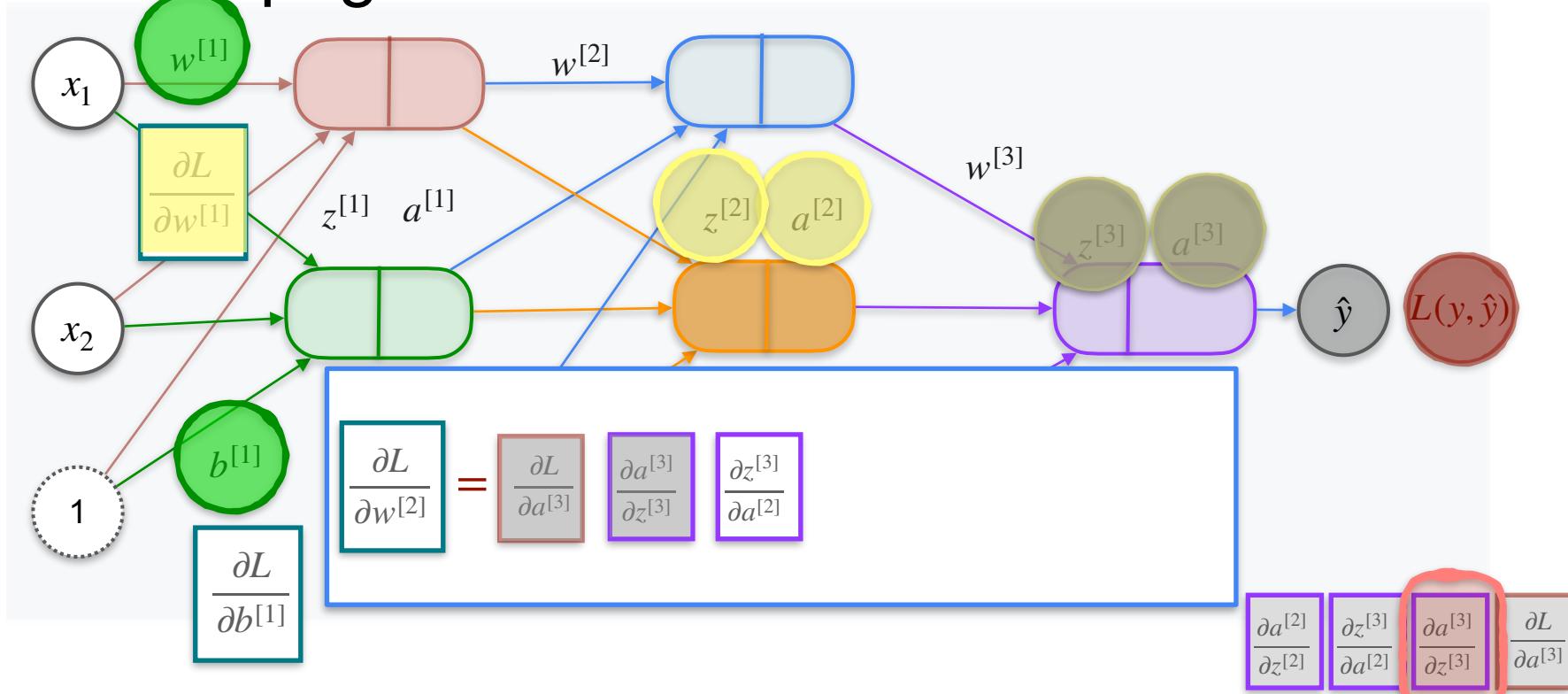
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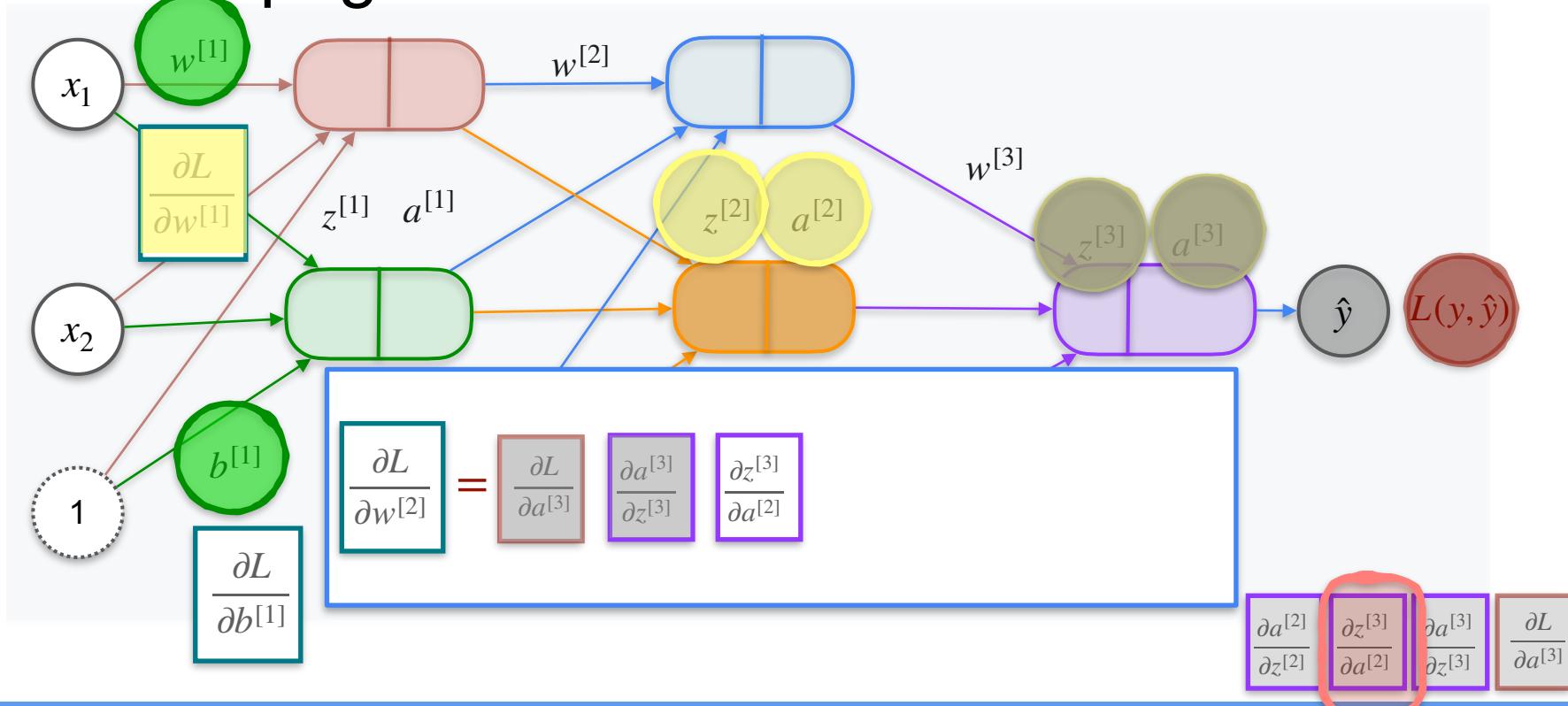
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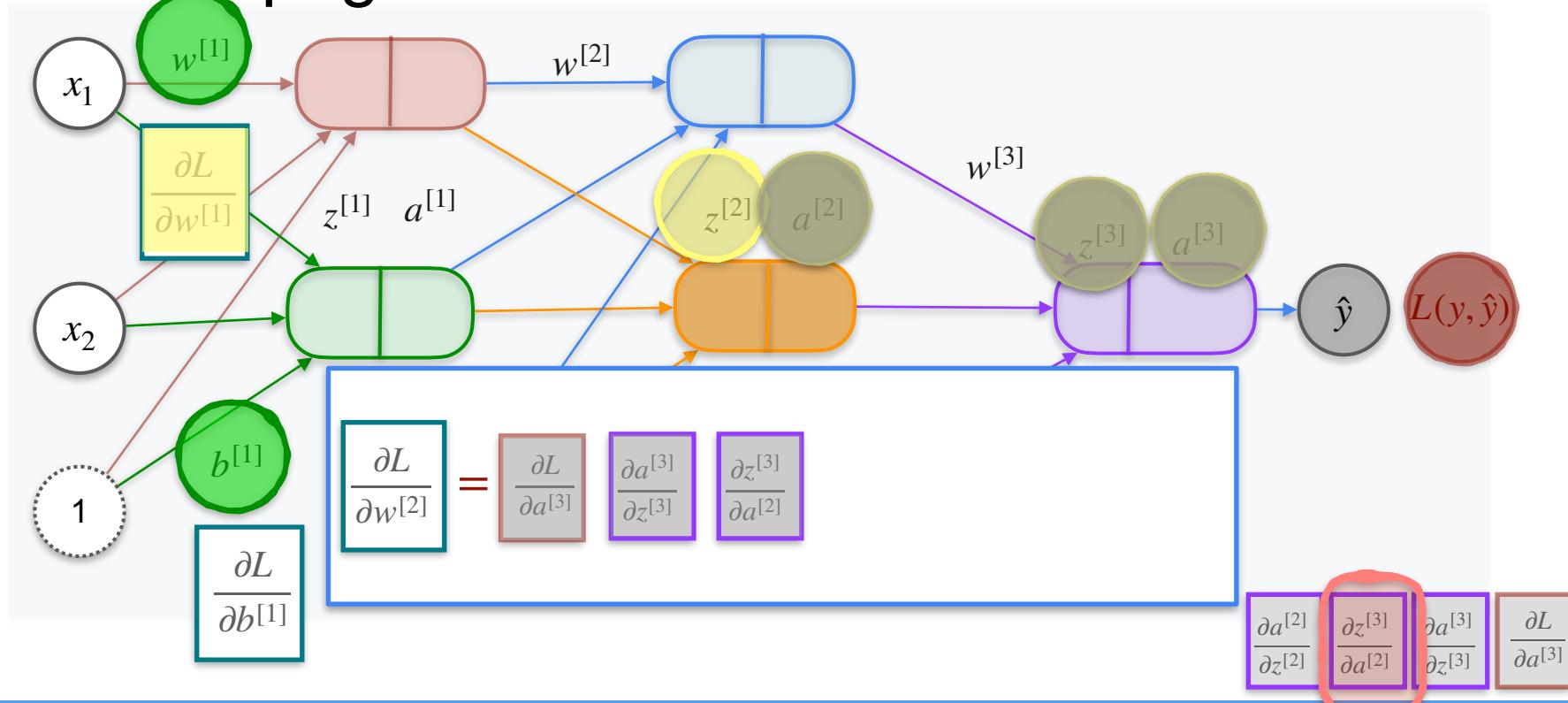
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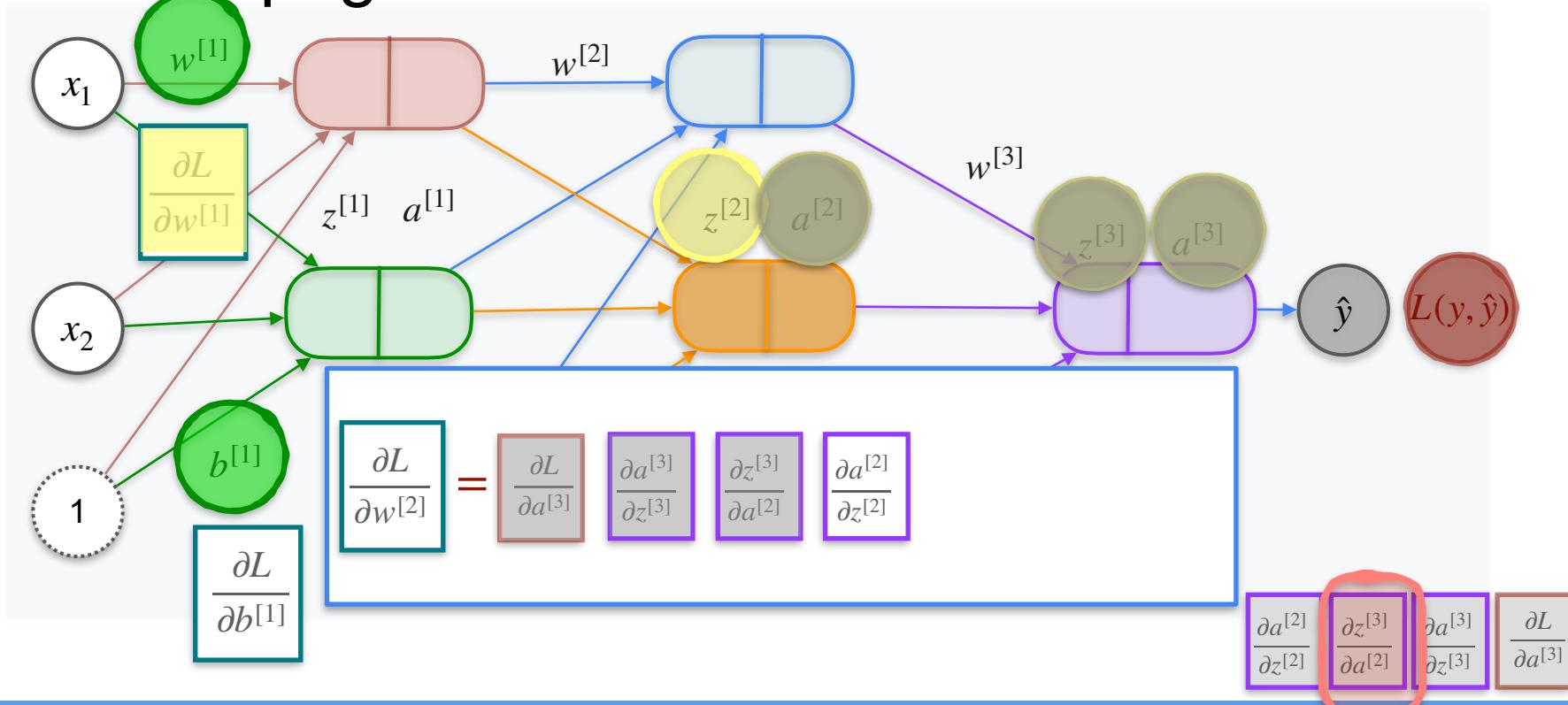
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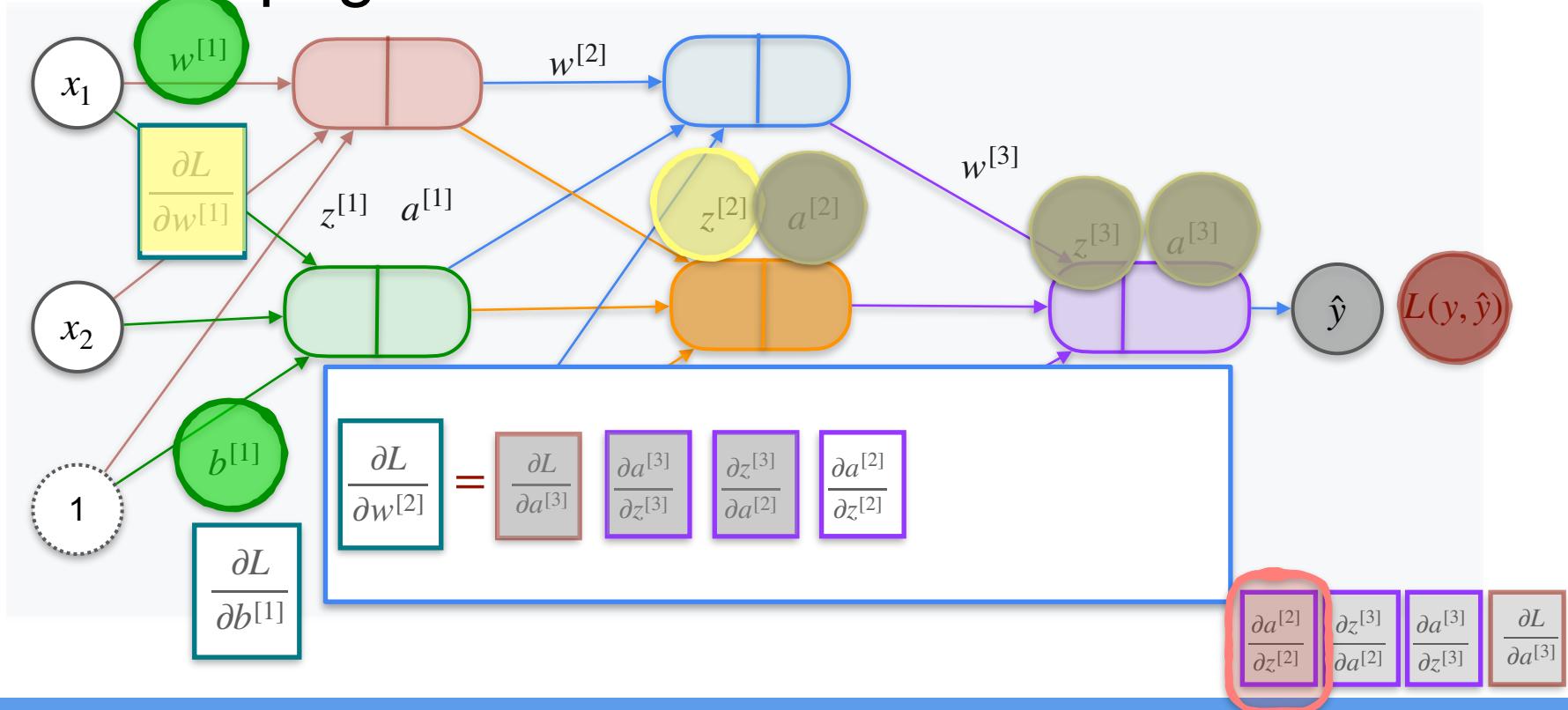
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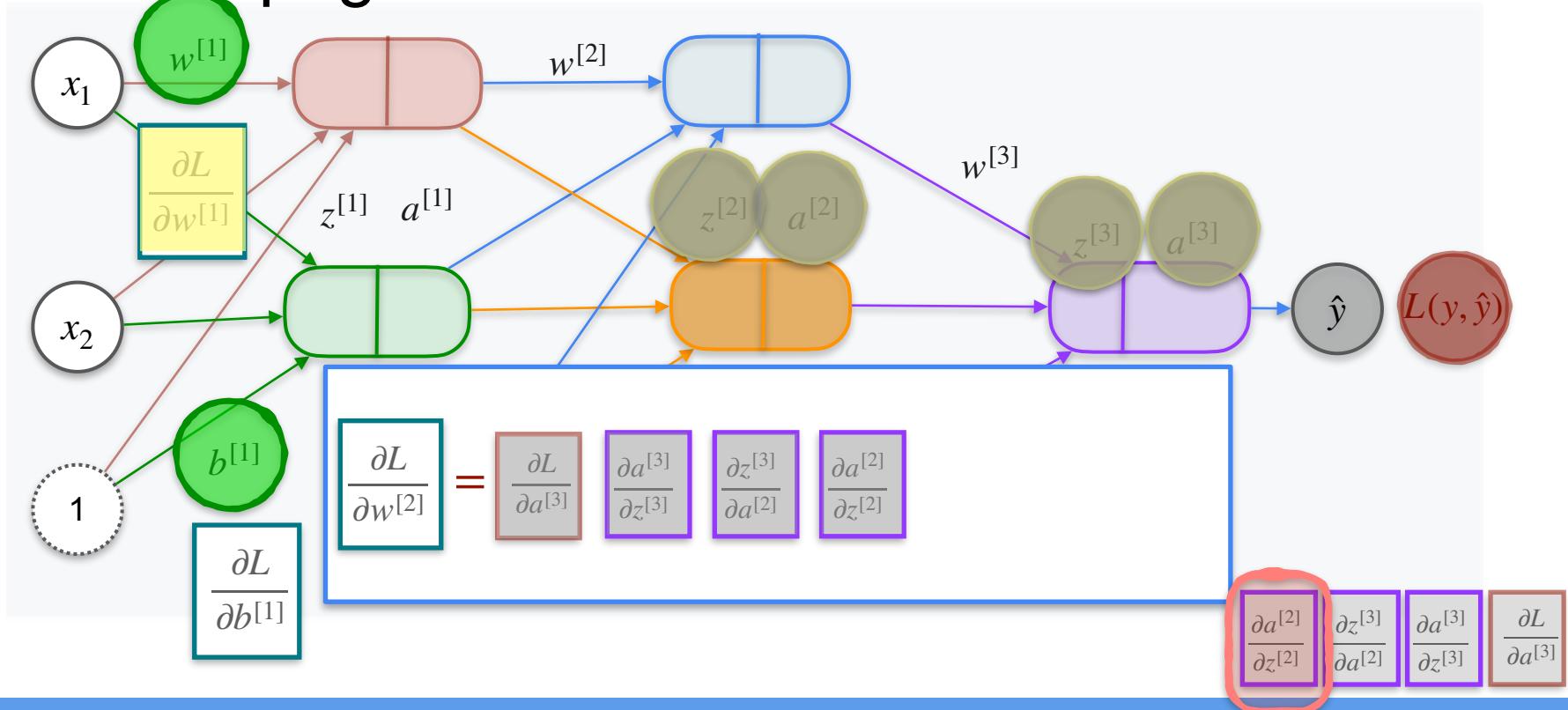
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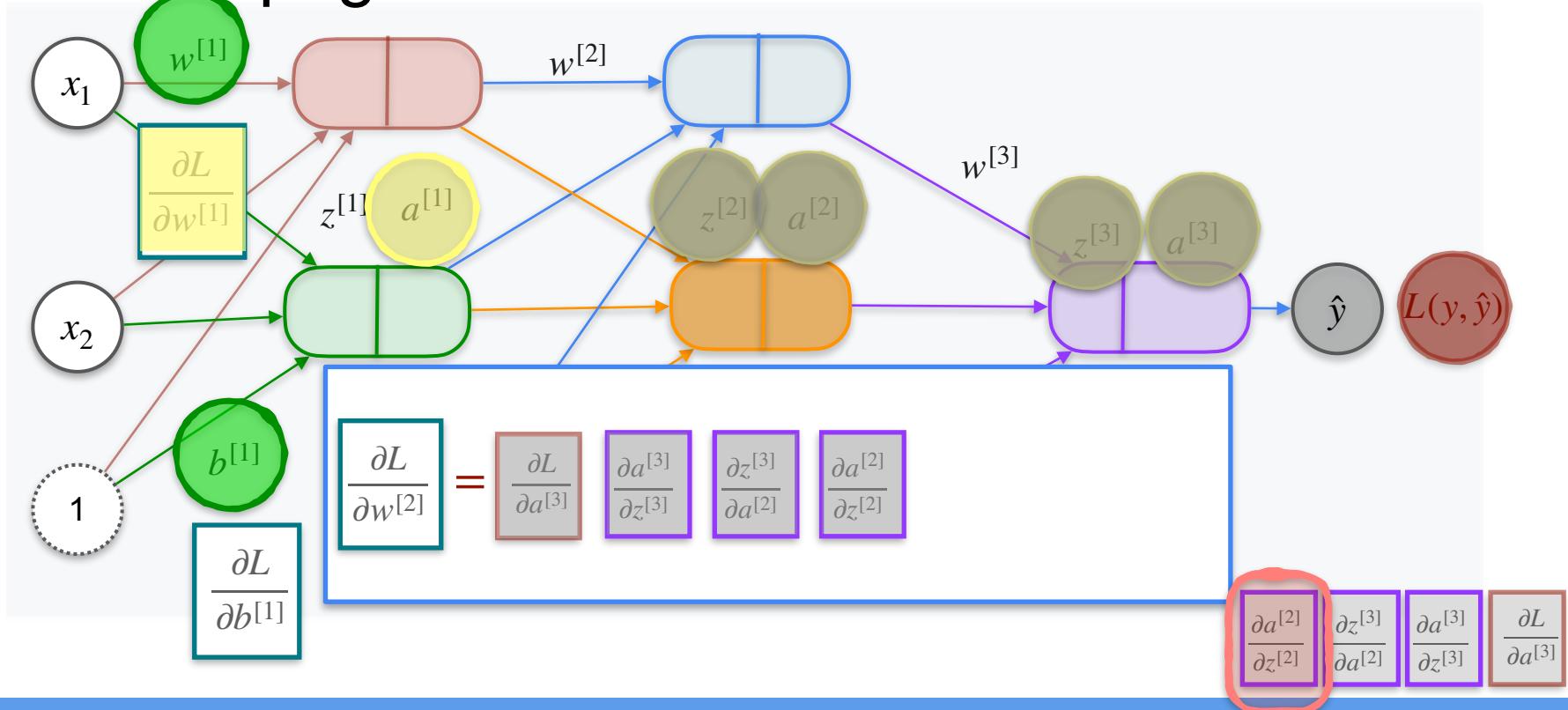
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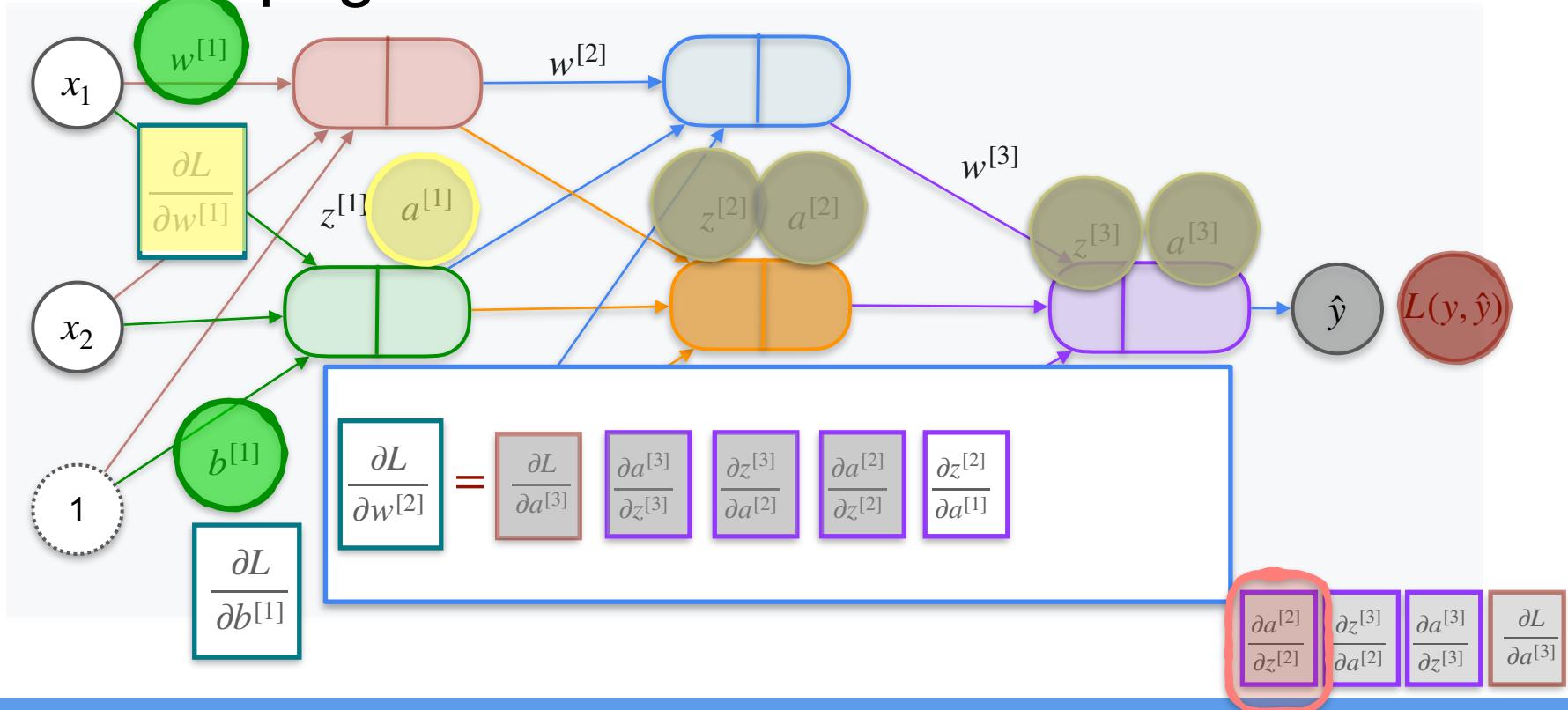
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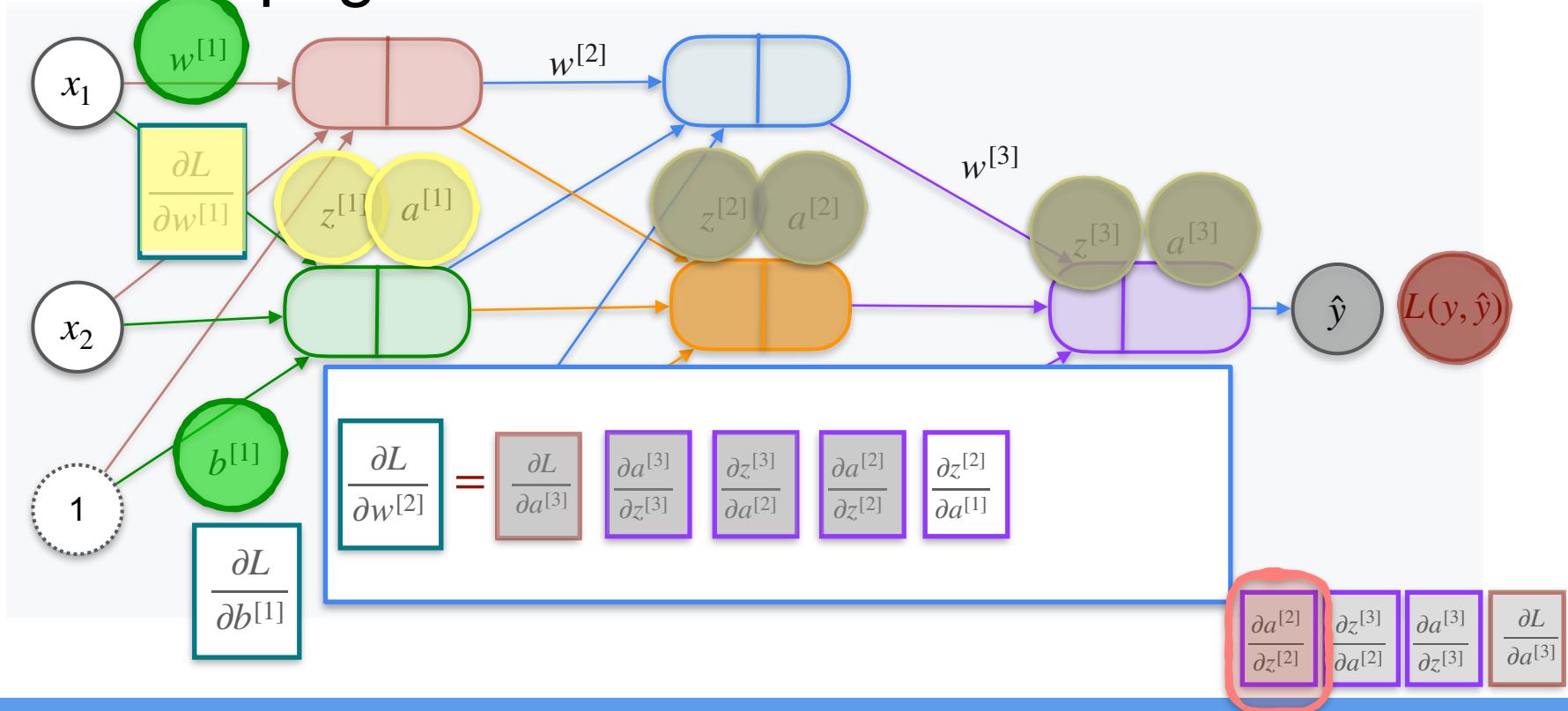
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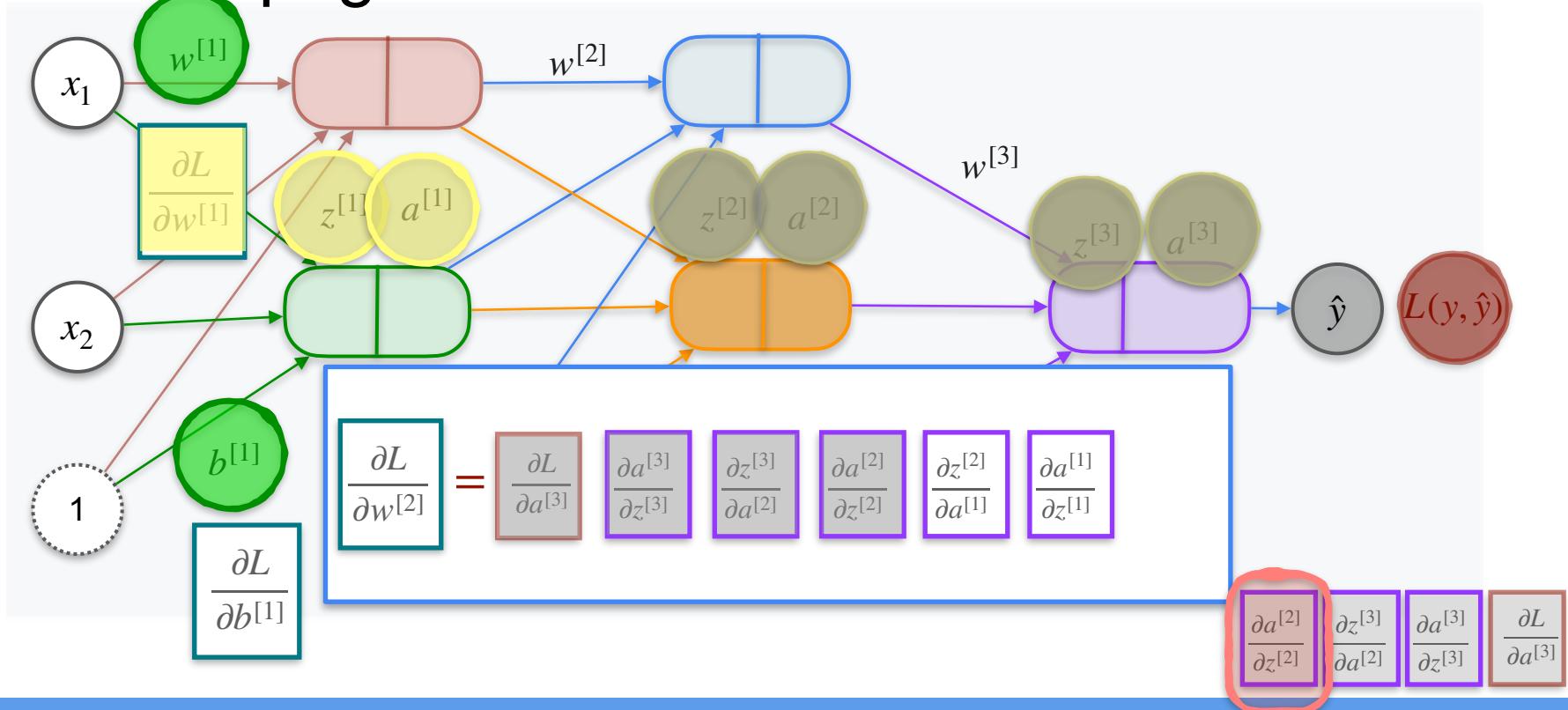
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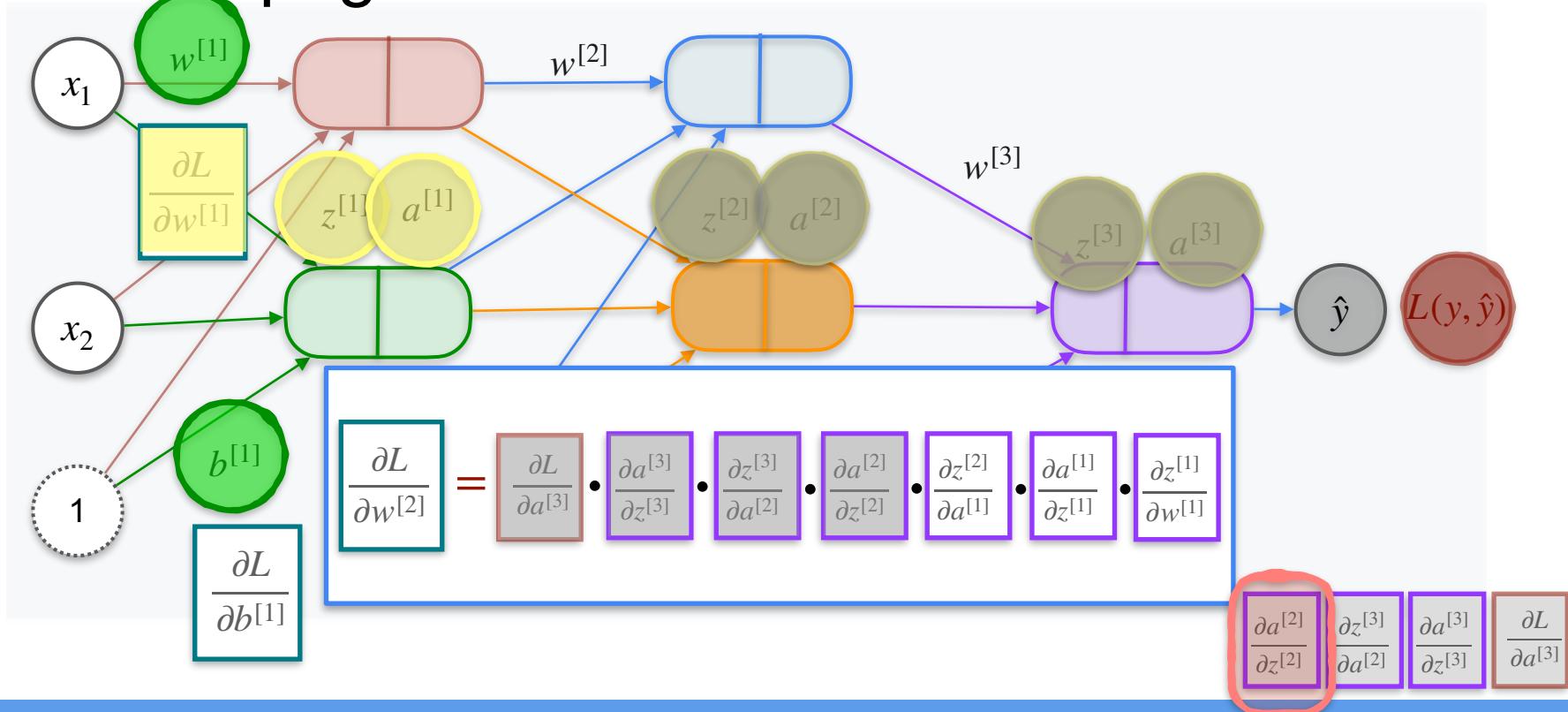
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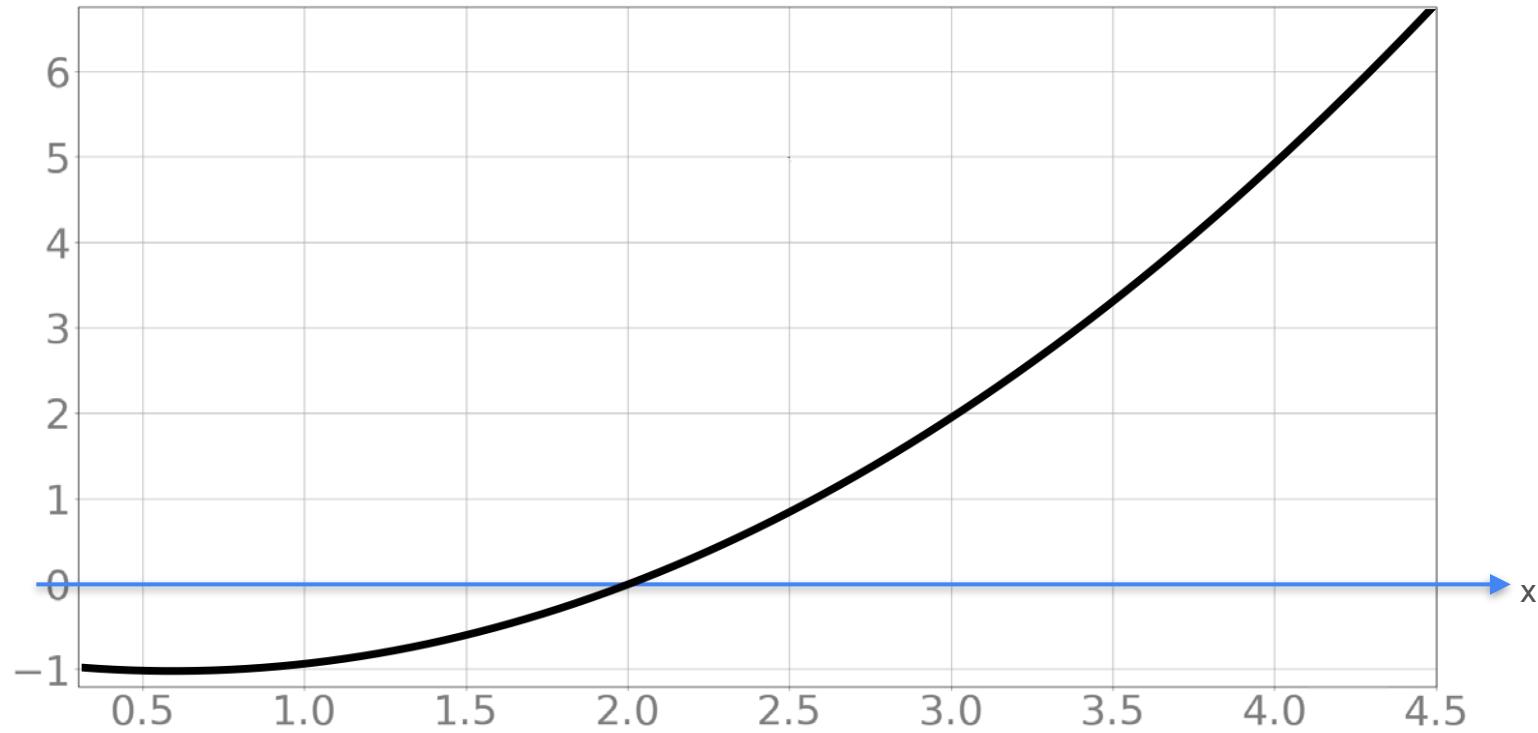
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Optimization in Neural Networks and Newton's Method

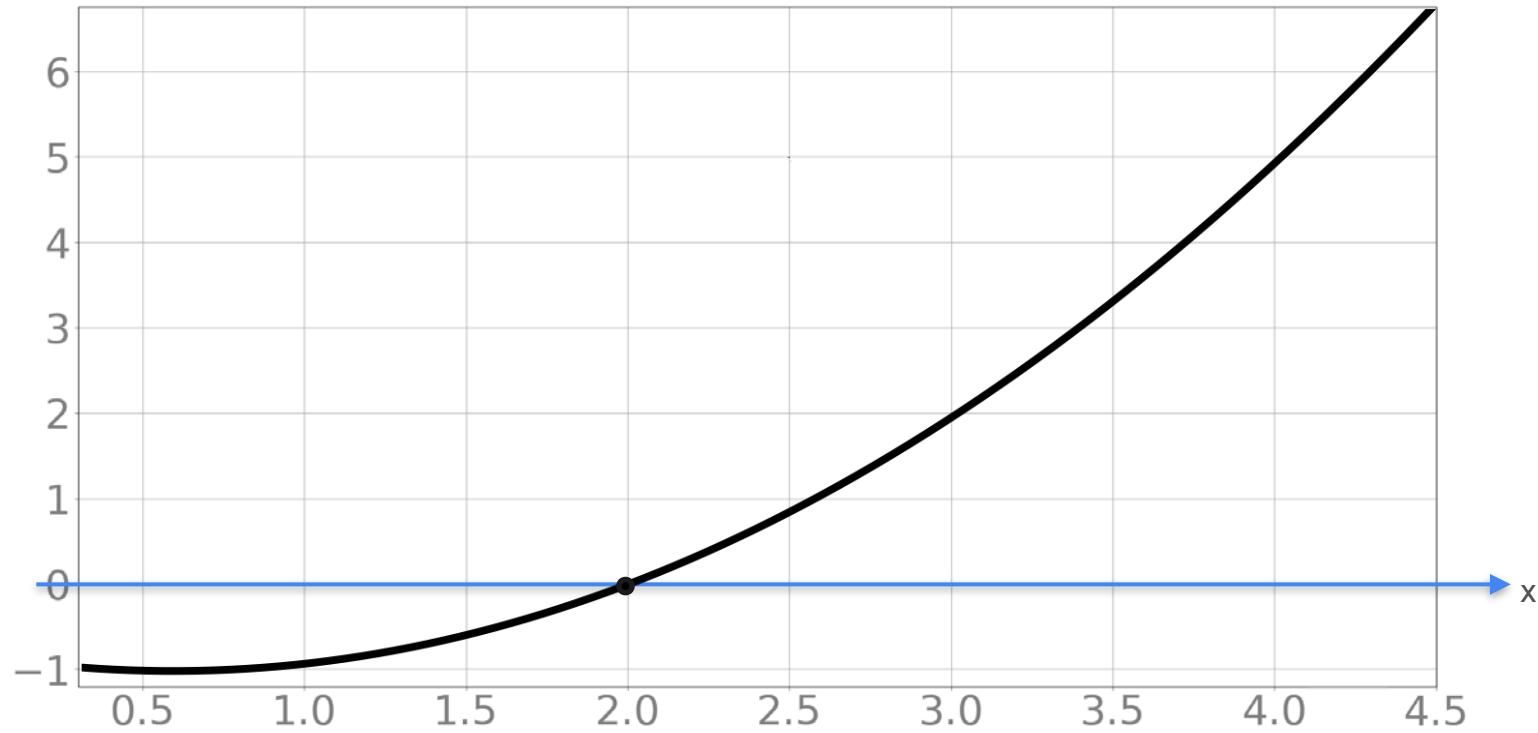
Newton's method

Newton's Method

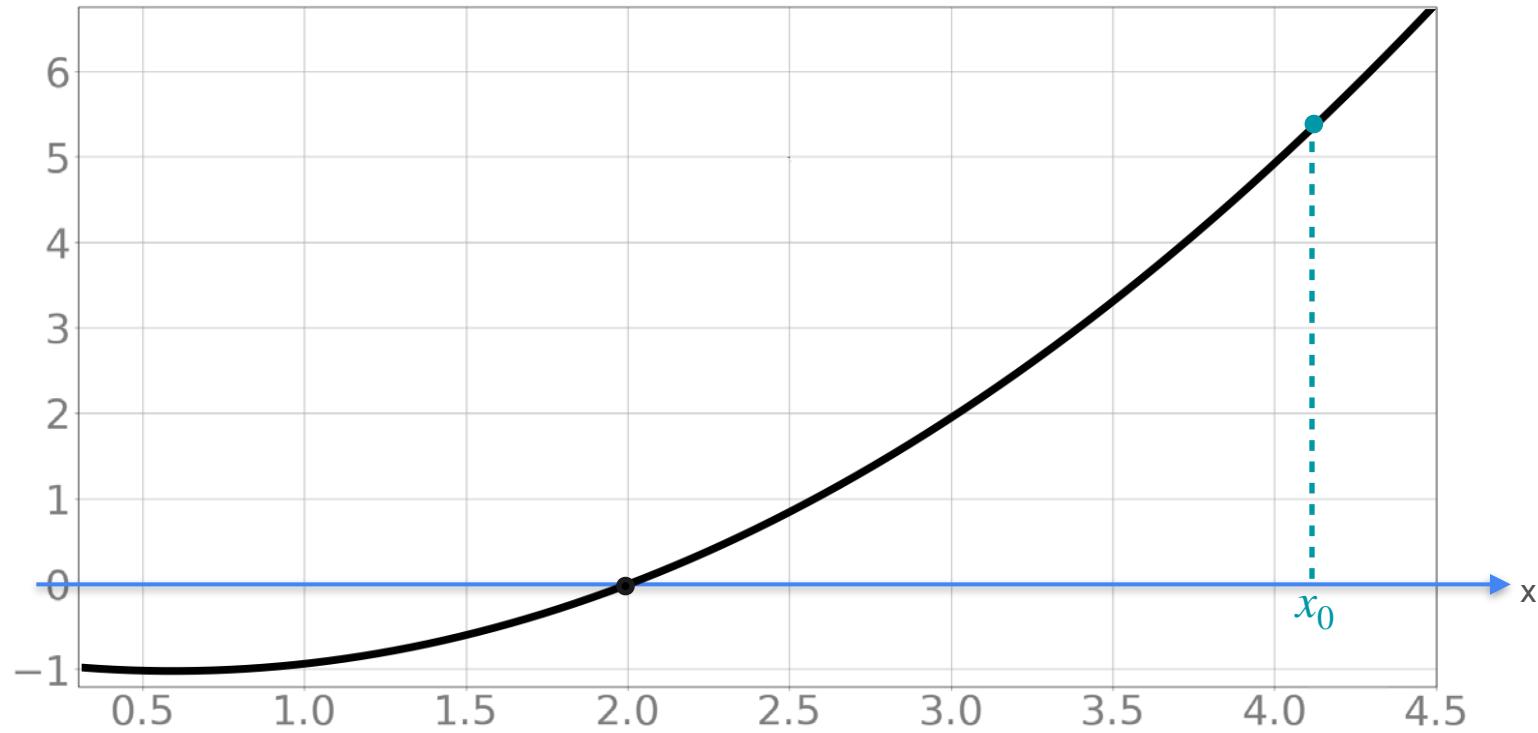
Newton's Method



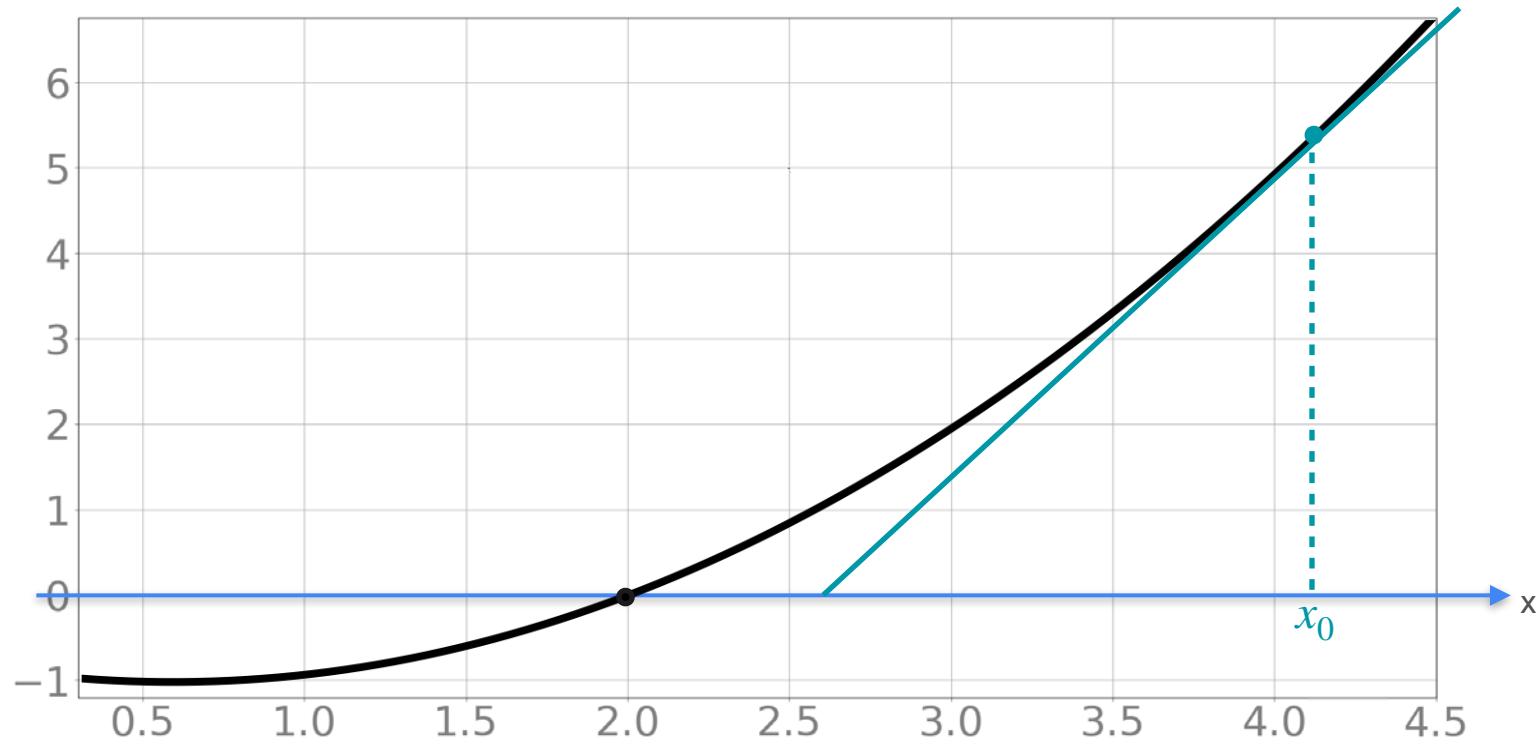
Newton's Method



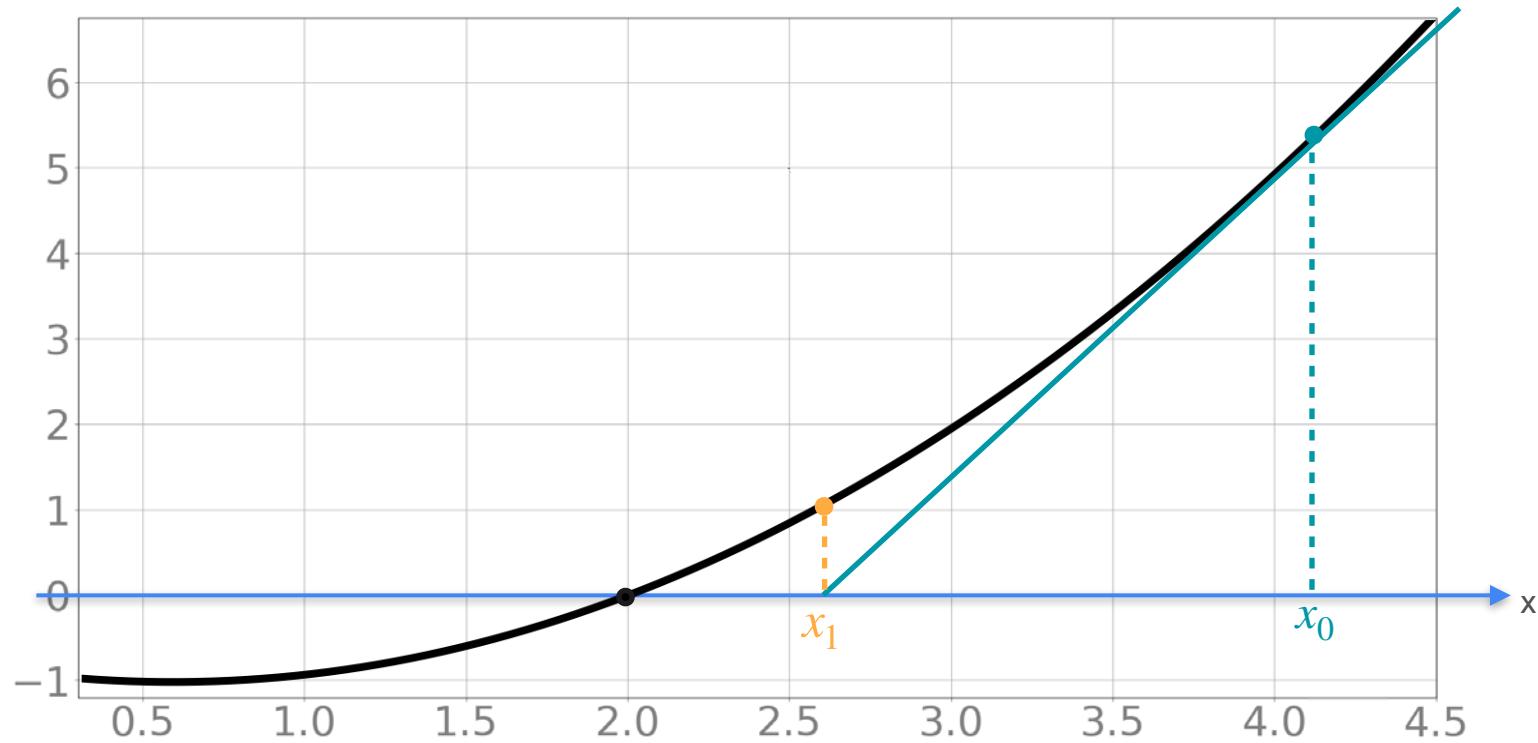
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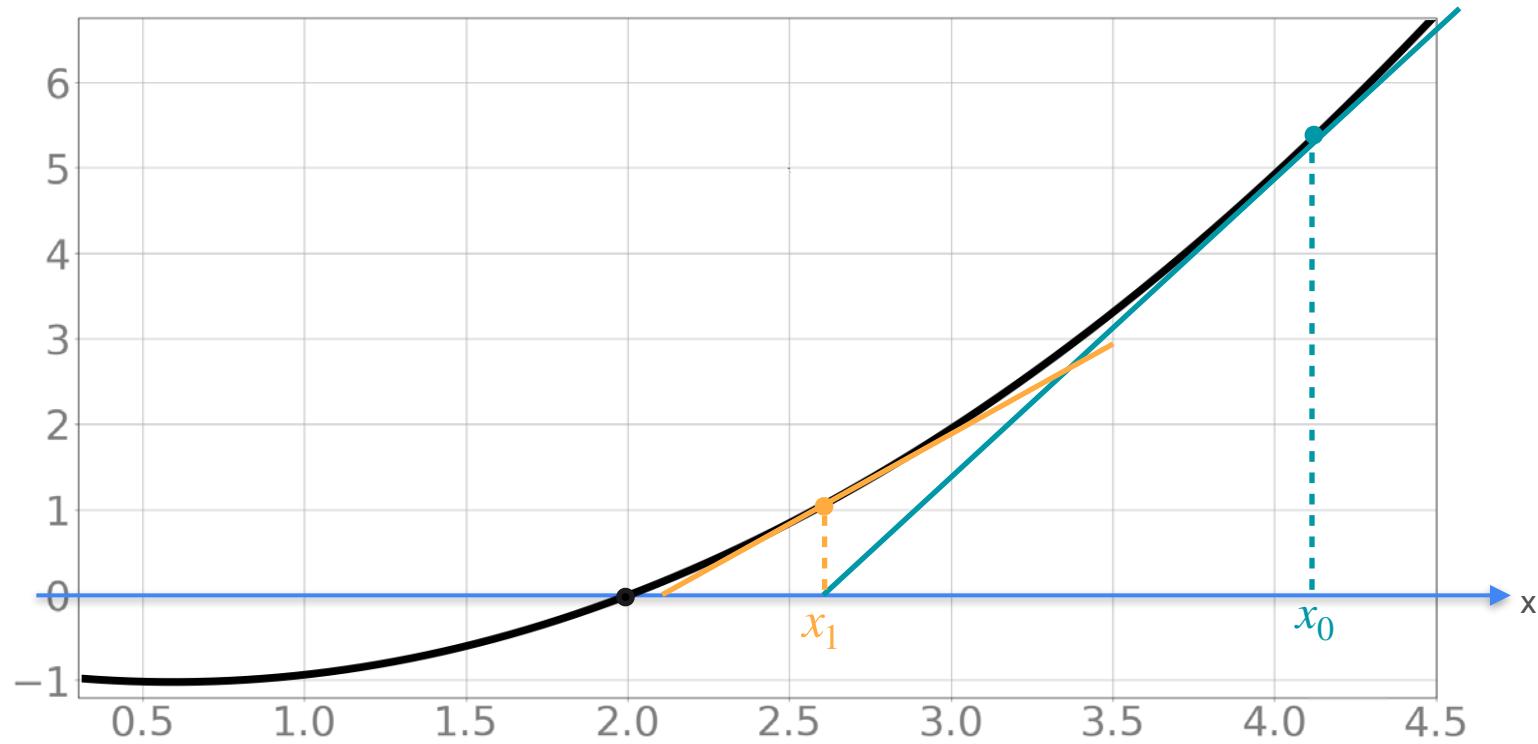
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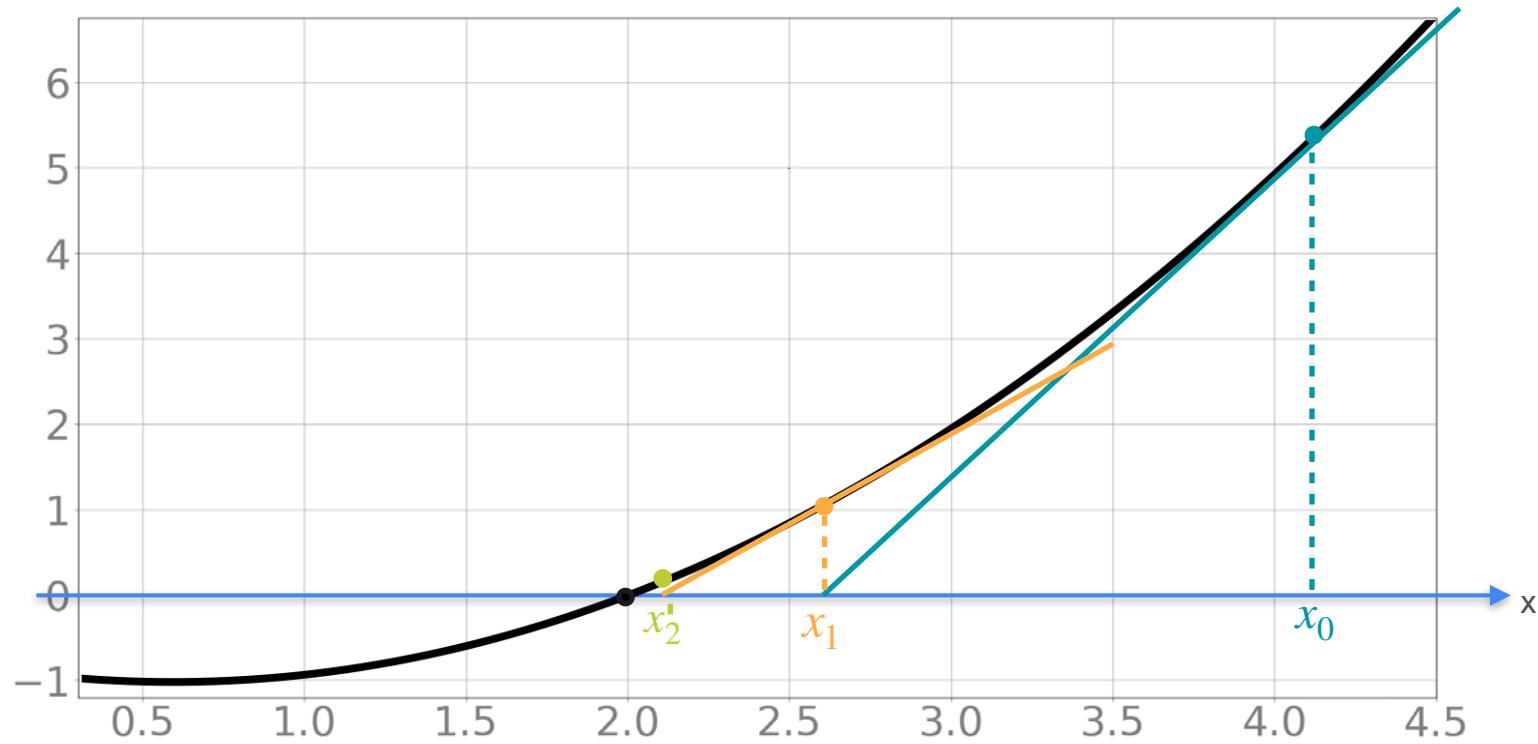
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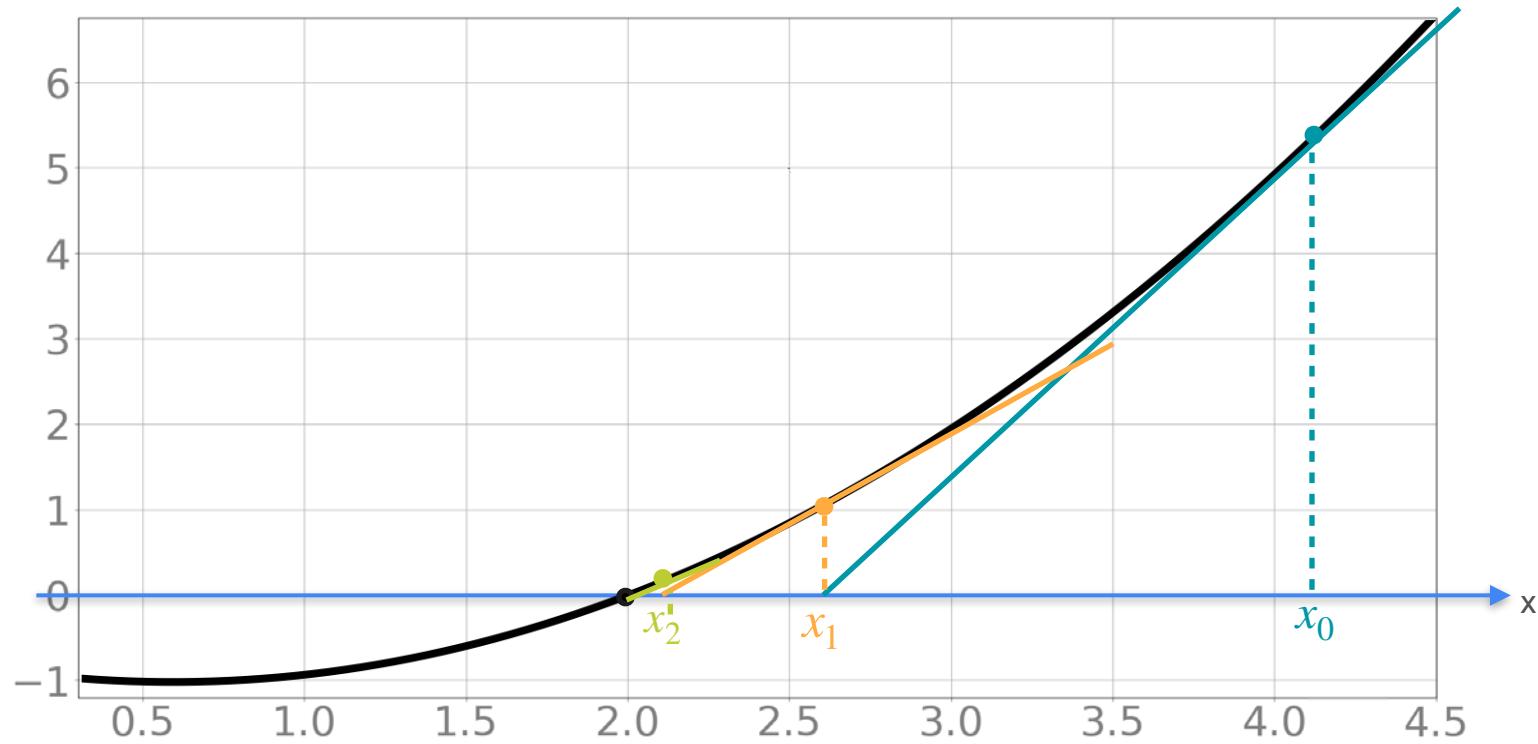
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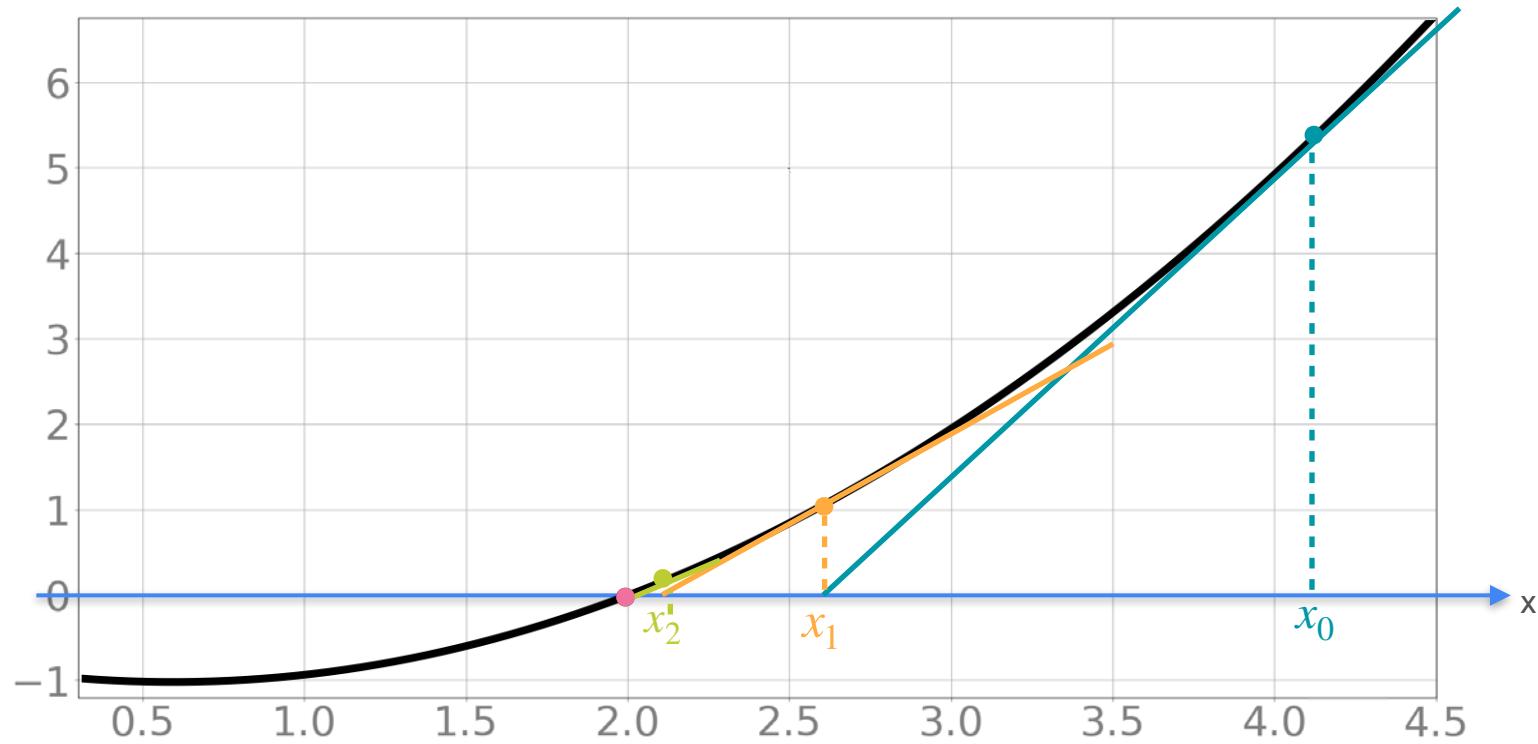
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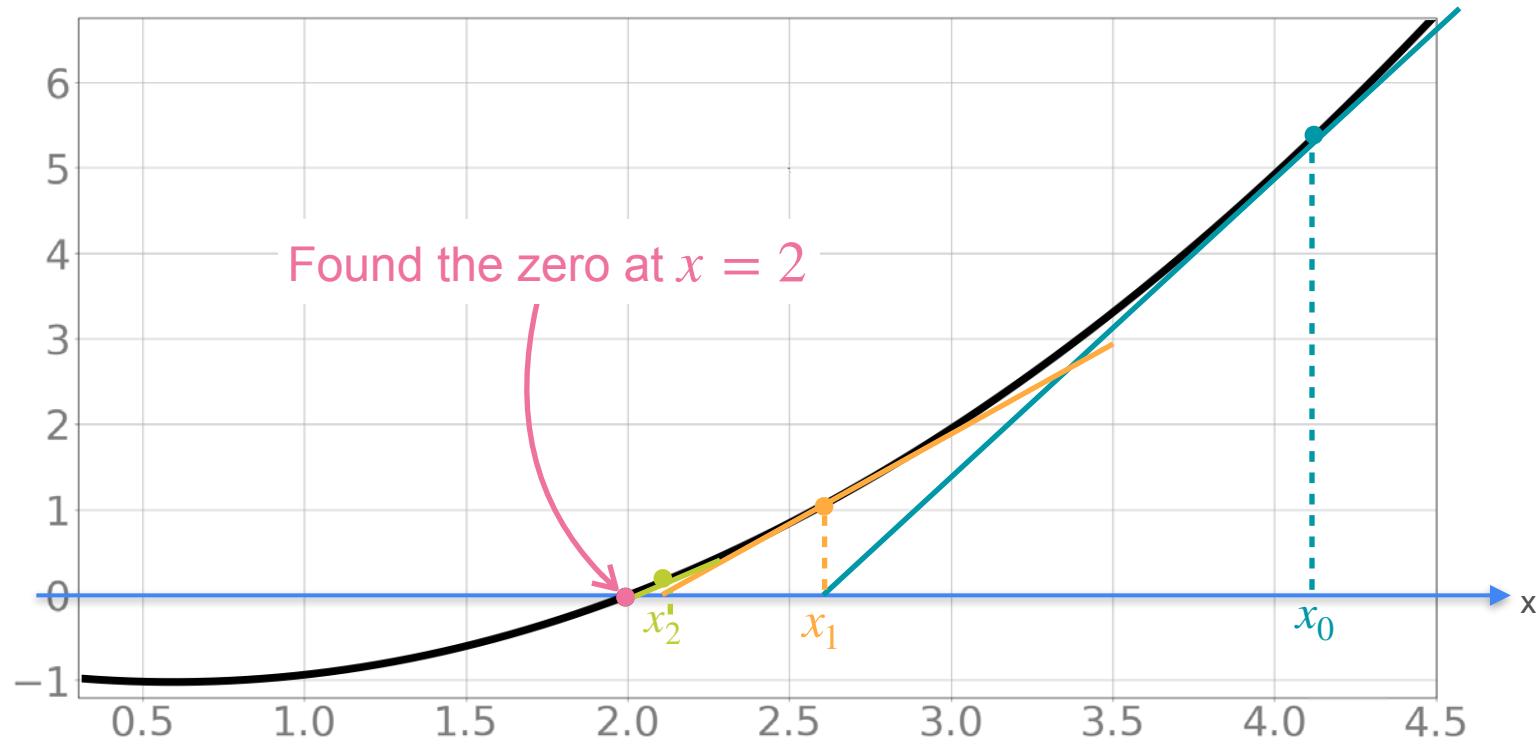
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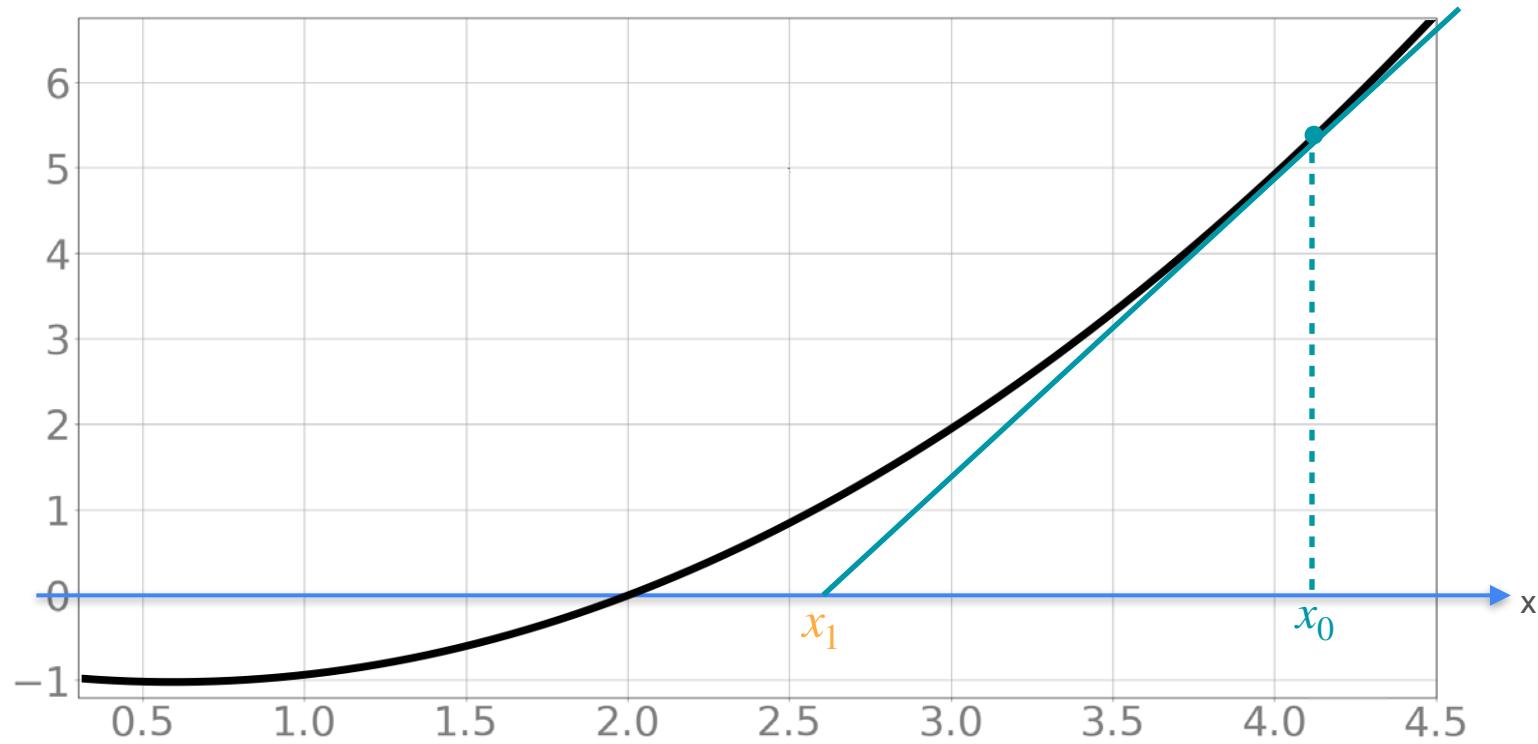
Newton's Method



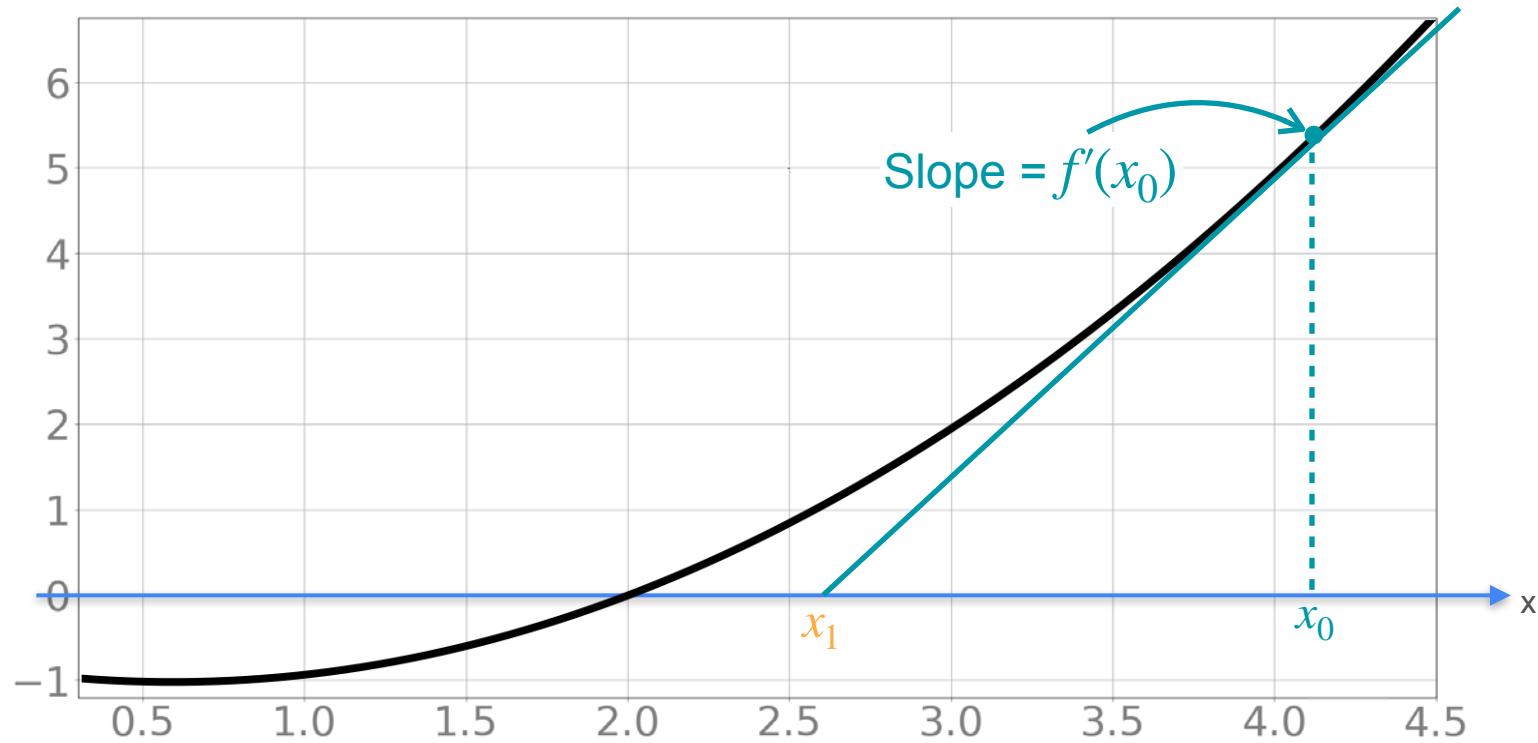
Newton's Method



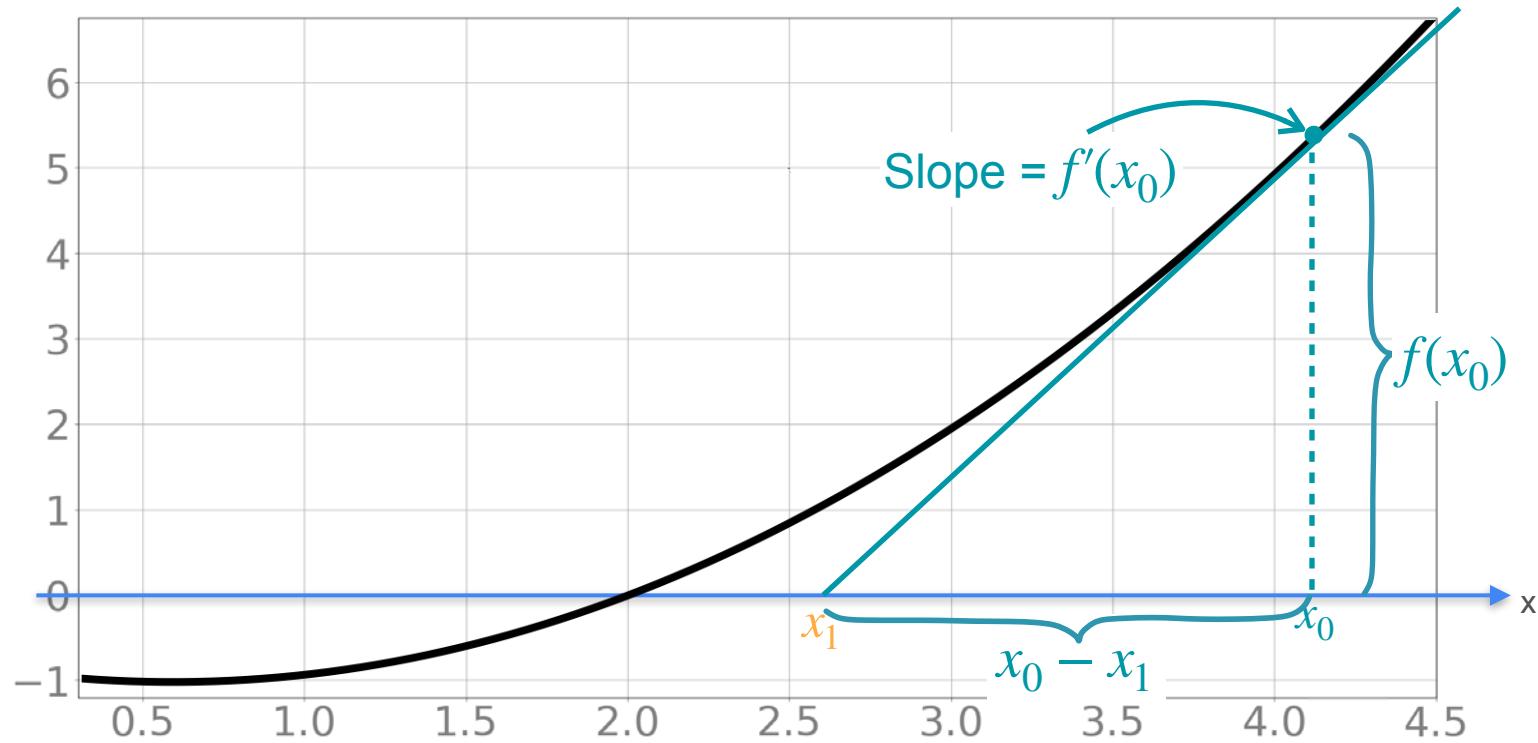
Update Approximation



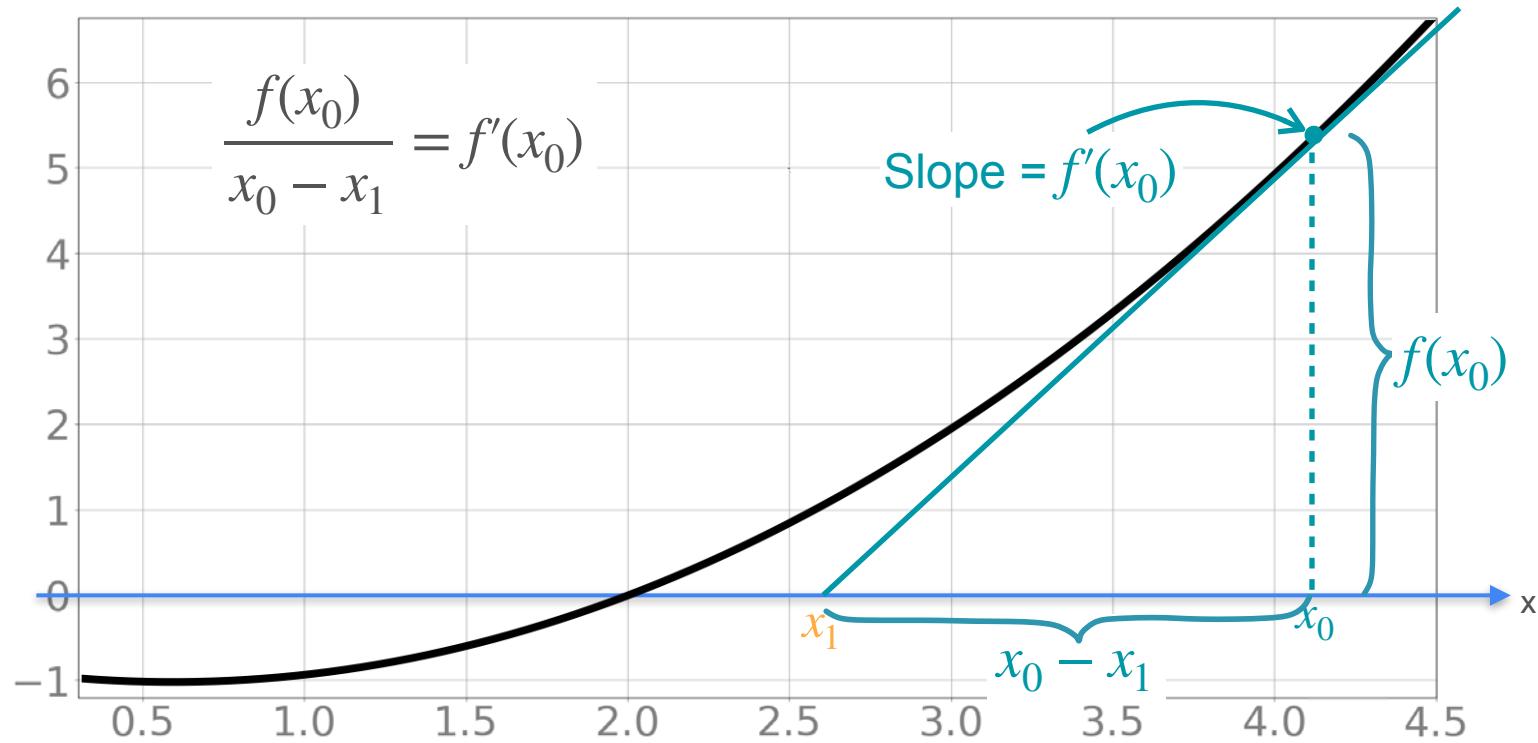
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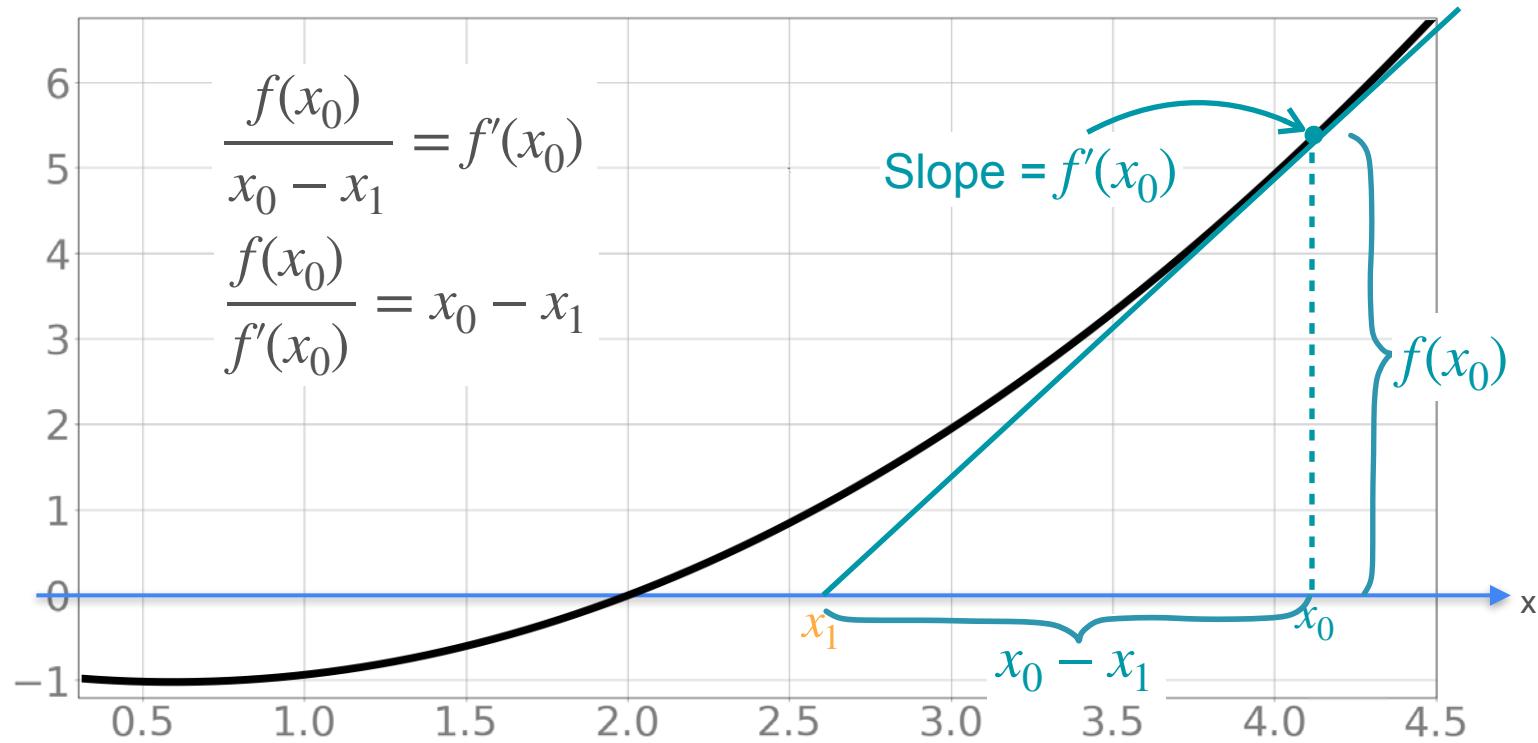
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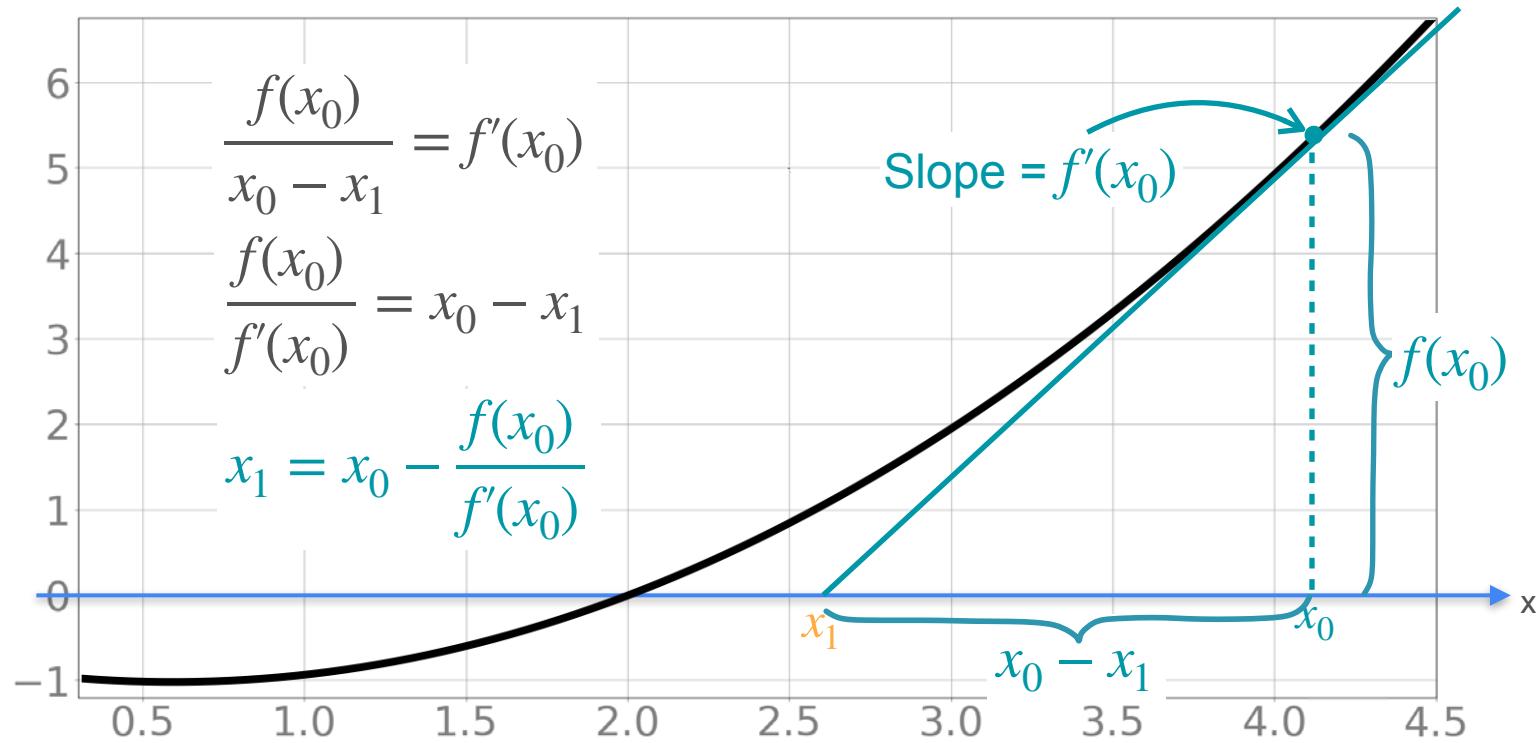
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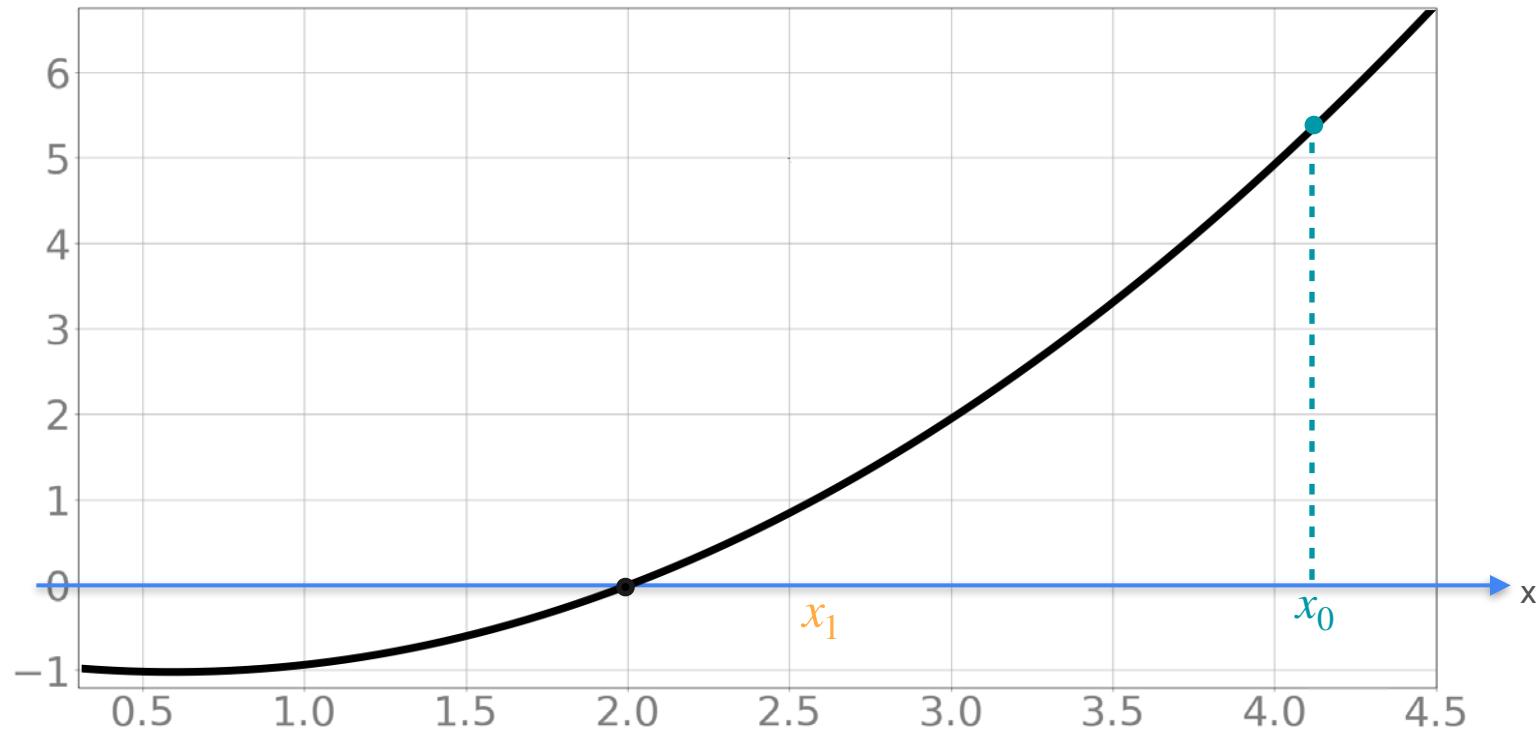
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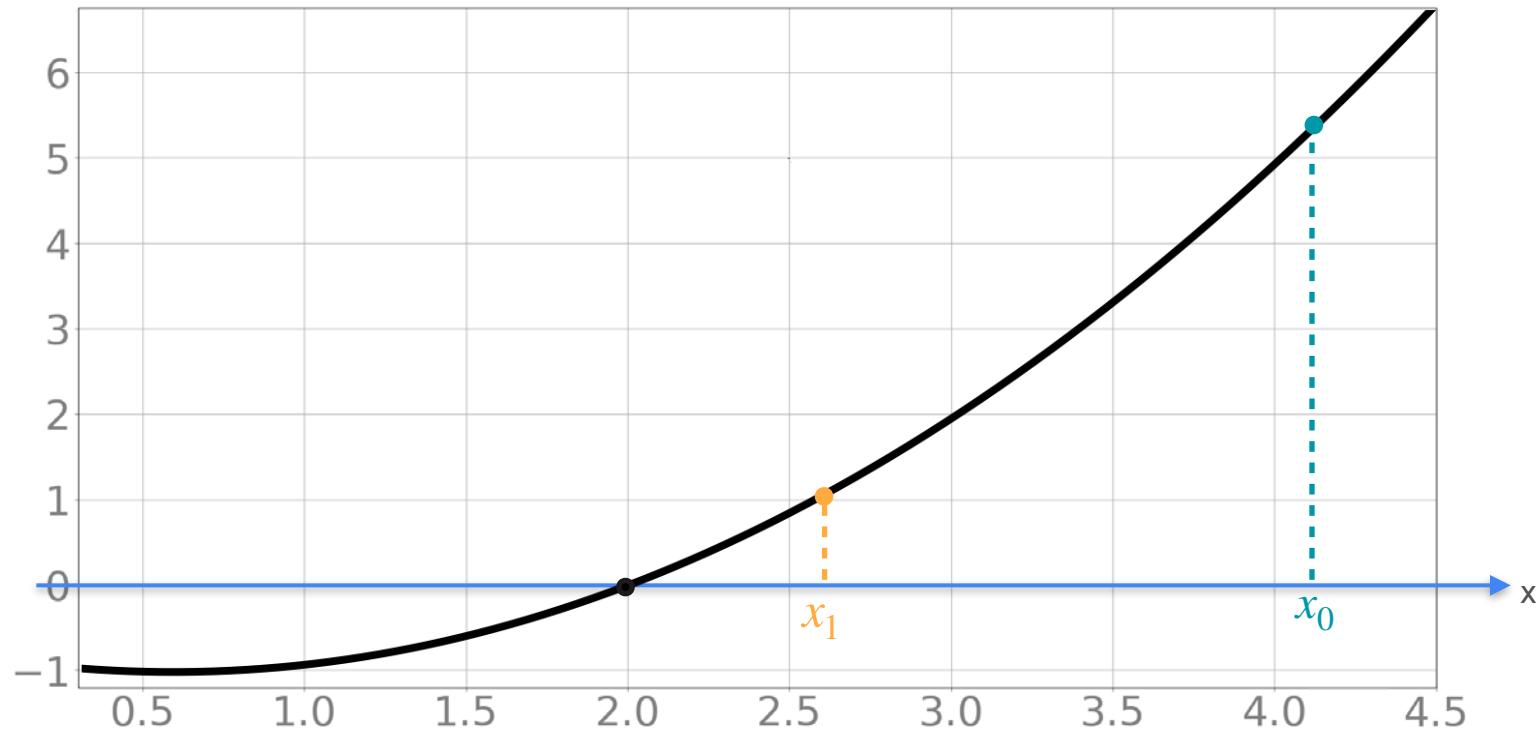
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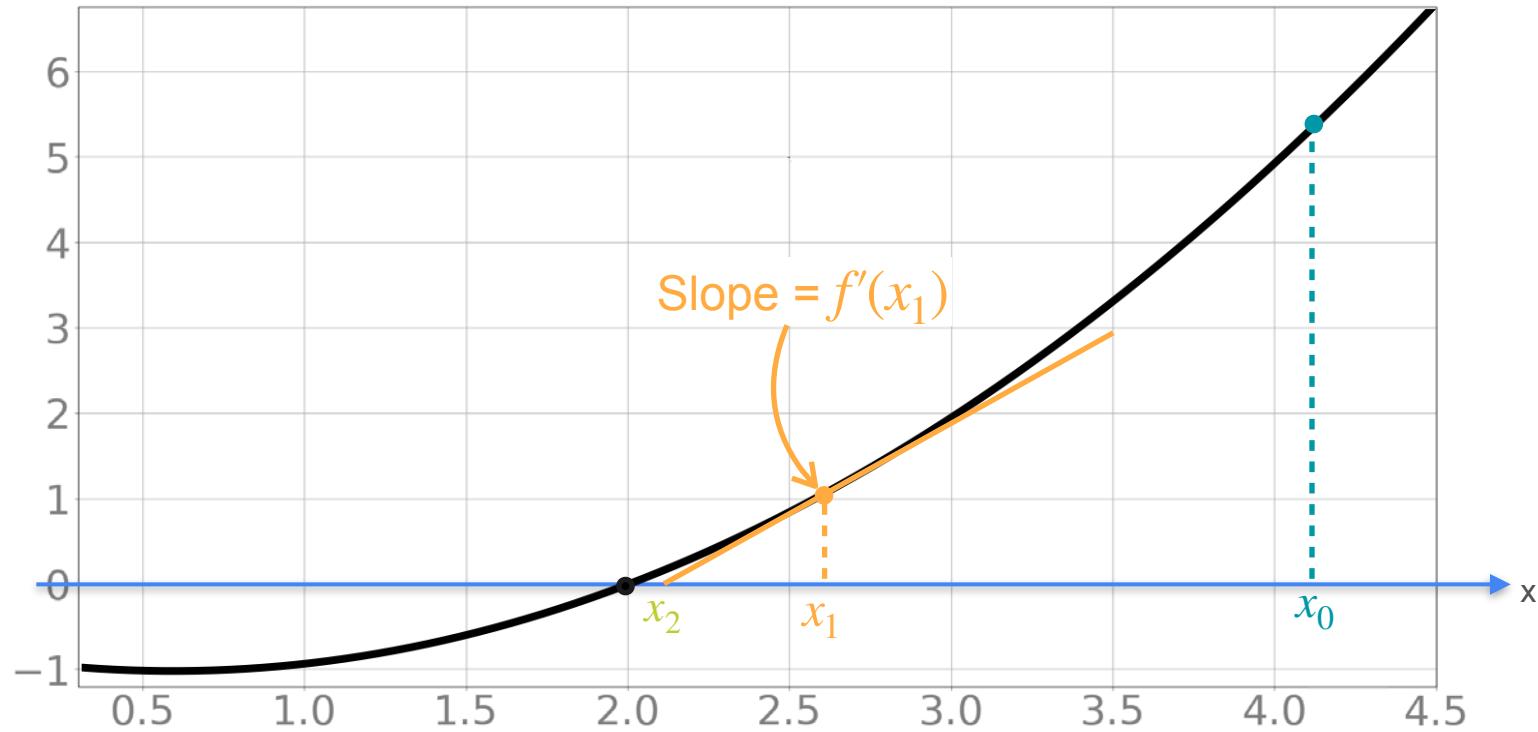
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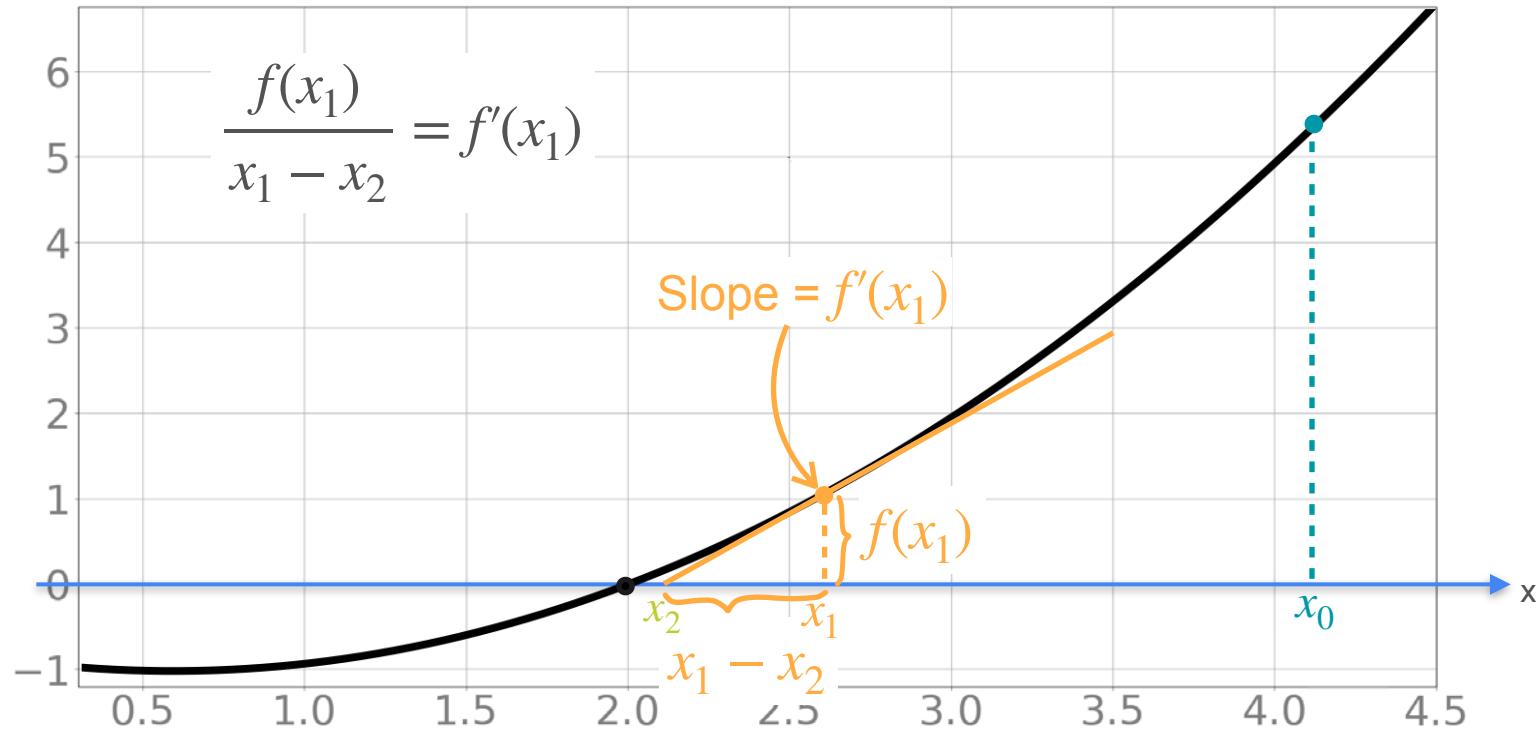
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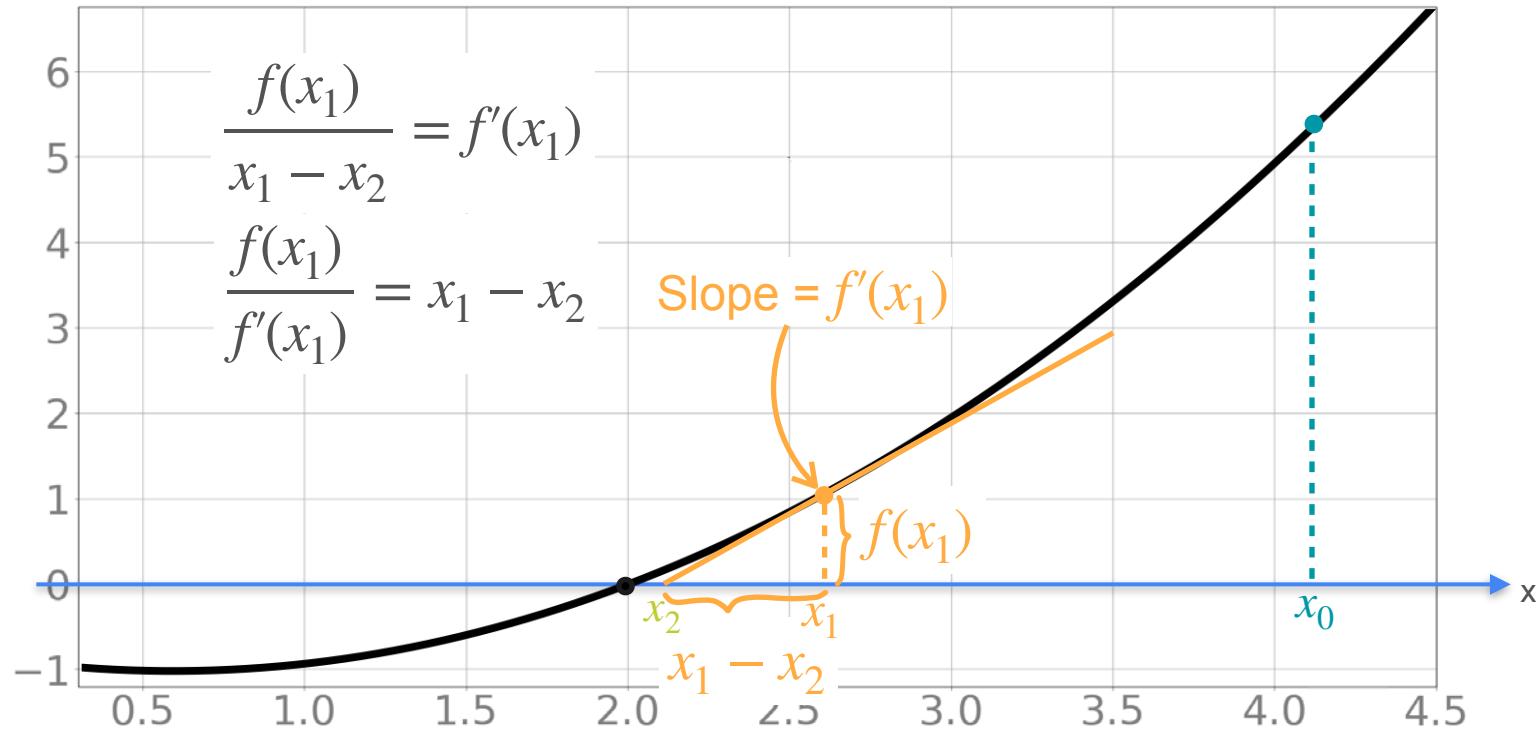
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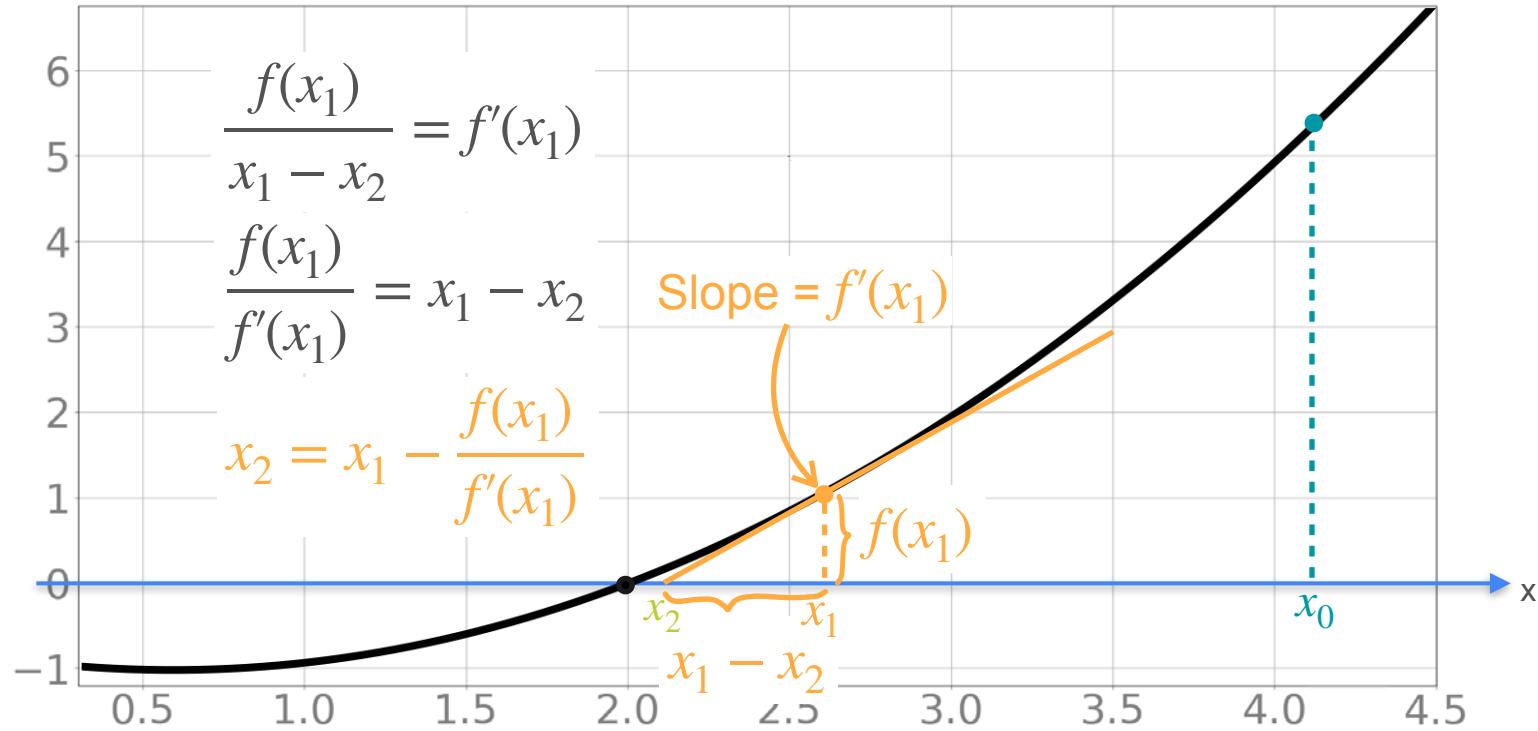
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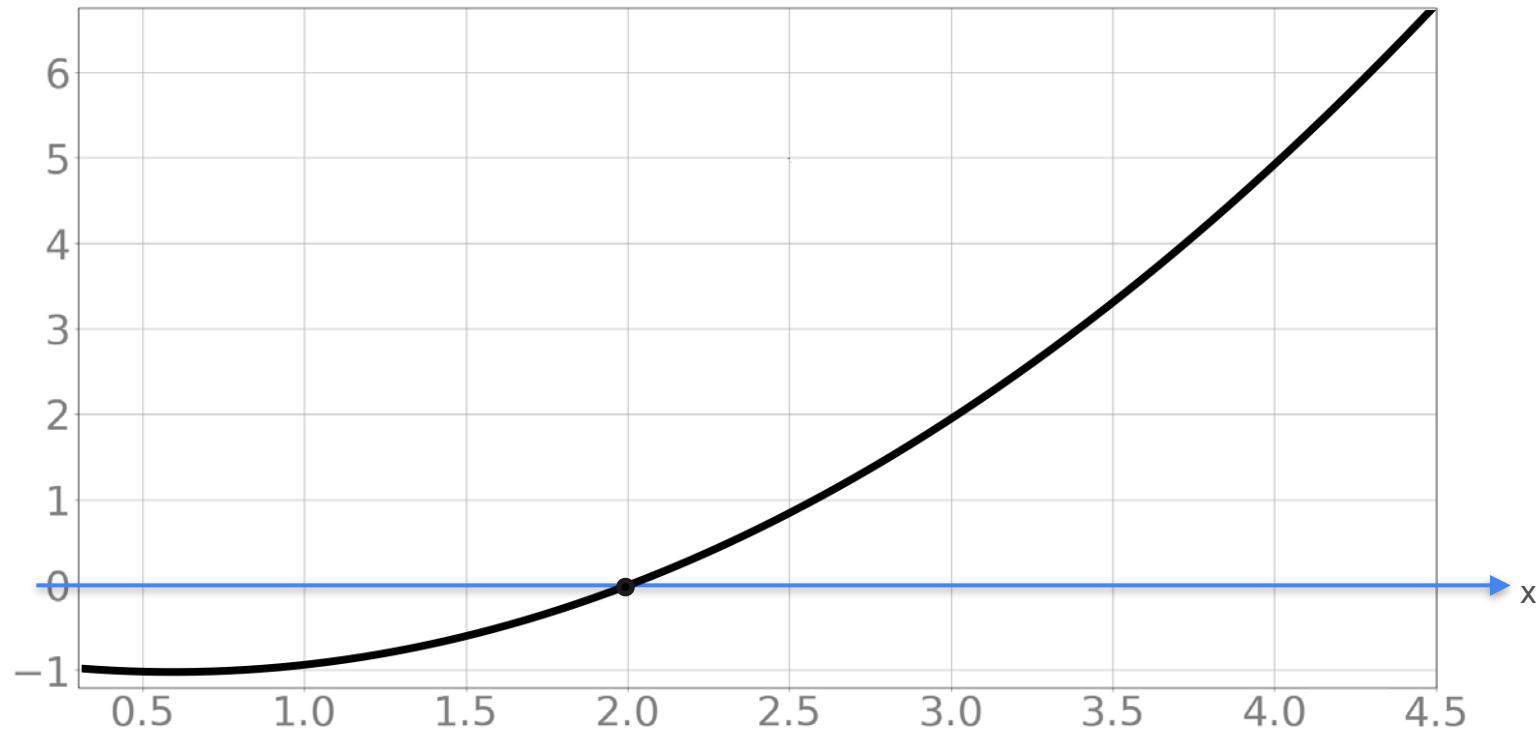
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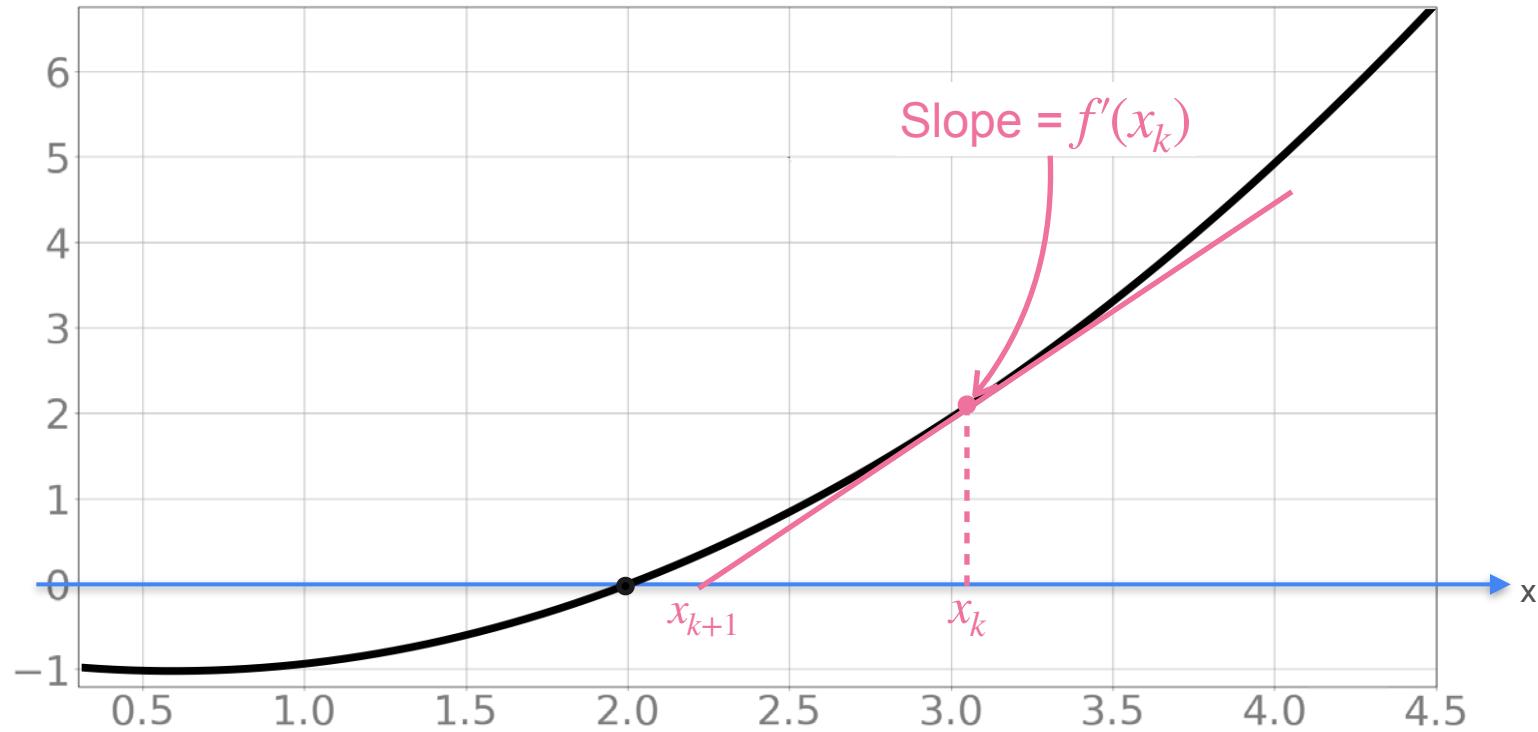
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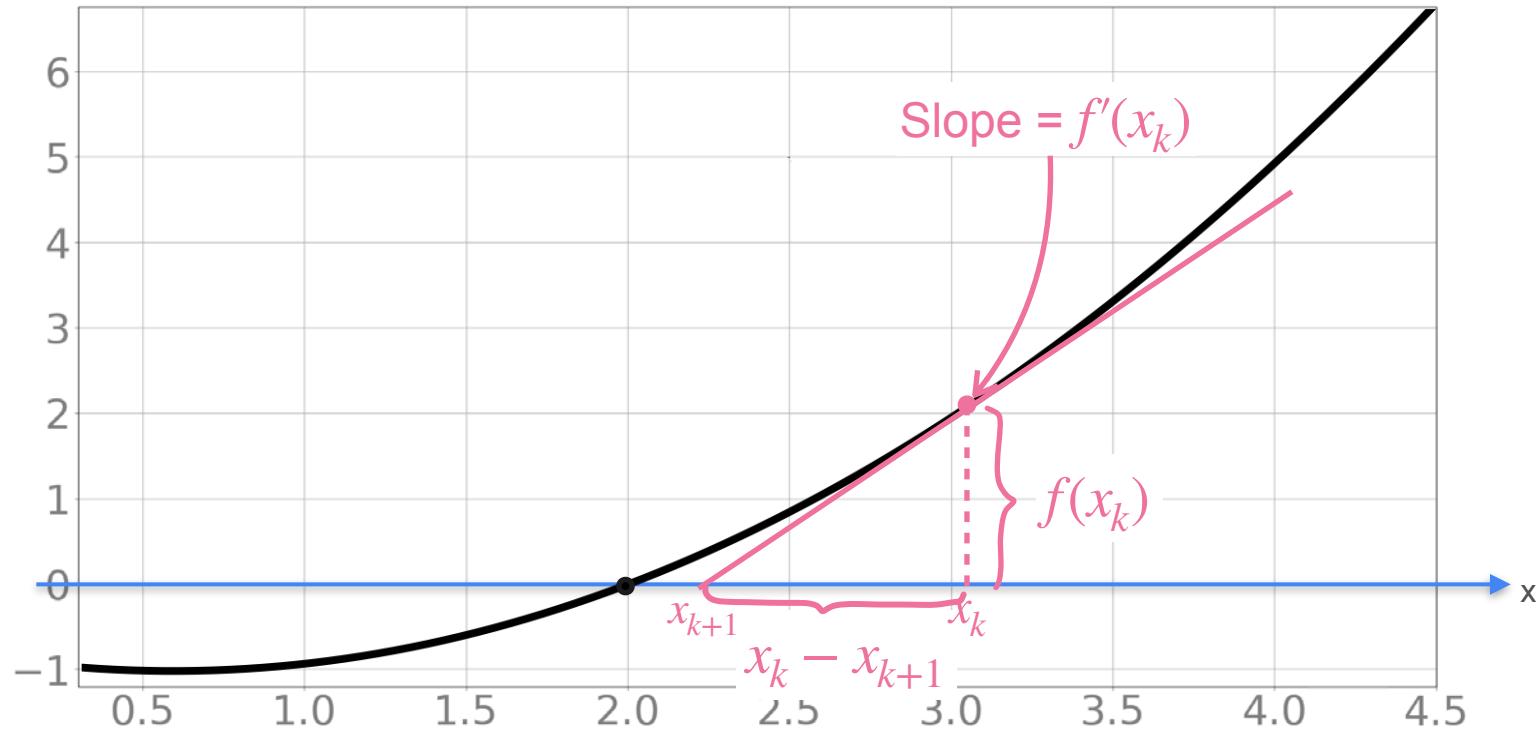
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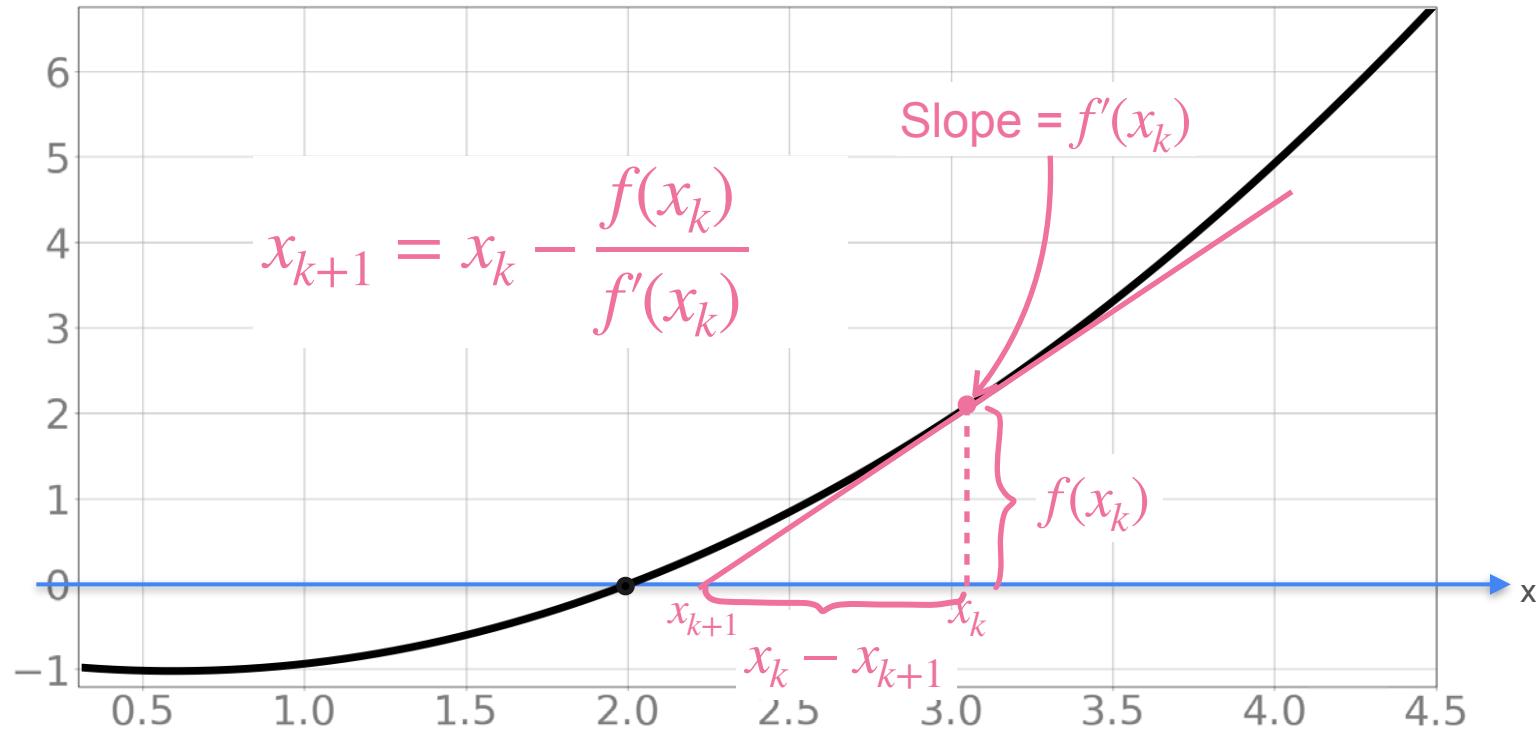
Update Approximation



Update Approximation



Update Approximation



Newton's Method for Optimization

Newton's Method for Optimization



Newton's Method for Optimization

Newton's method

Goal: find a zero of $f(x)$



Newton's Method for Optimization

Newton's method

Goal: find a zero of $f(x)$



NM for Optimization

Goal: minimize $g(x)$ find zeros of $g'(x)$

Newton's Method for Optimization

Newton's method

Goal: find a zero of $f(x)$



NM for Optimization

Goal: minimize $g(x)$ → find zeros of $g'(x)$

$$f(x) \mapsto g'(x)$$

$$f'(x) \mapsto (g'(x))'$$

Newton's Method for Optimization

Newton's method

Goal: find a zero of $f(x)$

1) Start with some x_0



NM for Optimization

Goal: minimize $g(x)$ find zeros of $g'(x)$

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Newton's Method for Optimization

Newton's method

Goal: find a zero of $f(x)$

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2) Update:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



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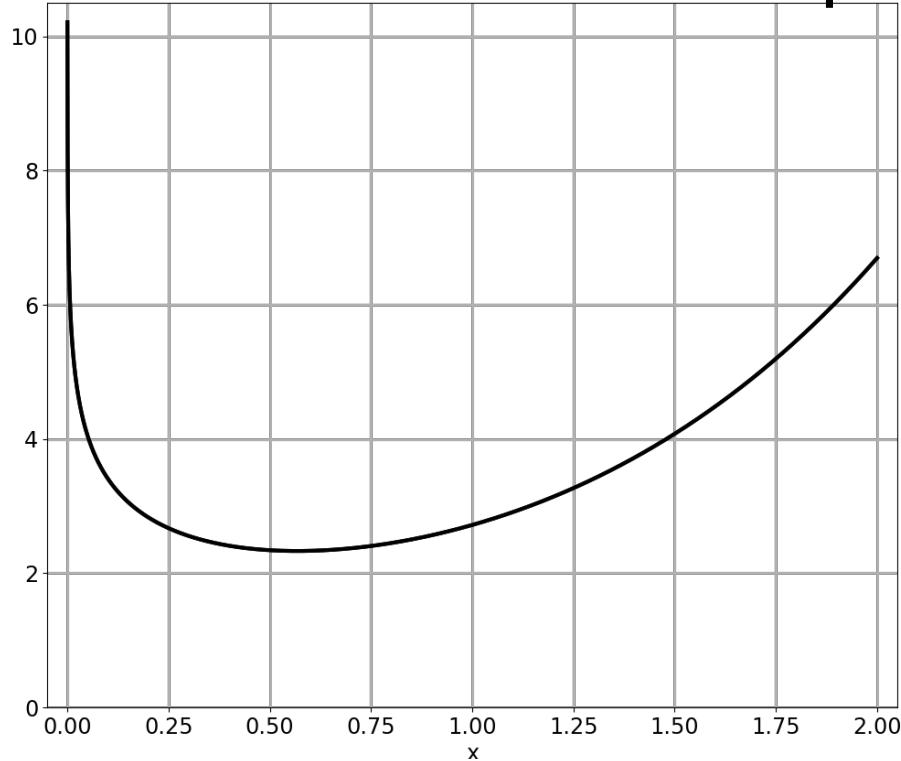
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Optimization in Neural Networks and Newton's Method

**Newton's method:
An example**

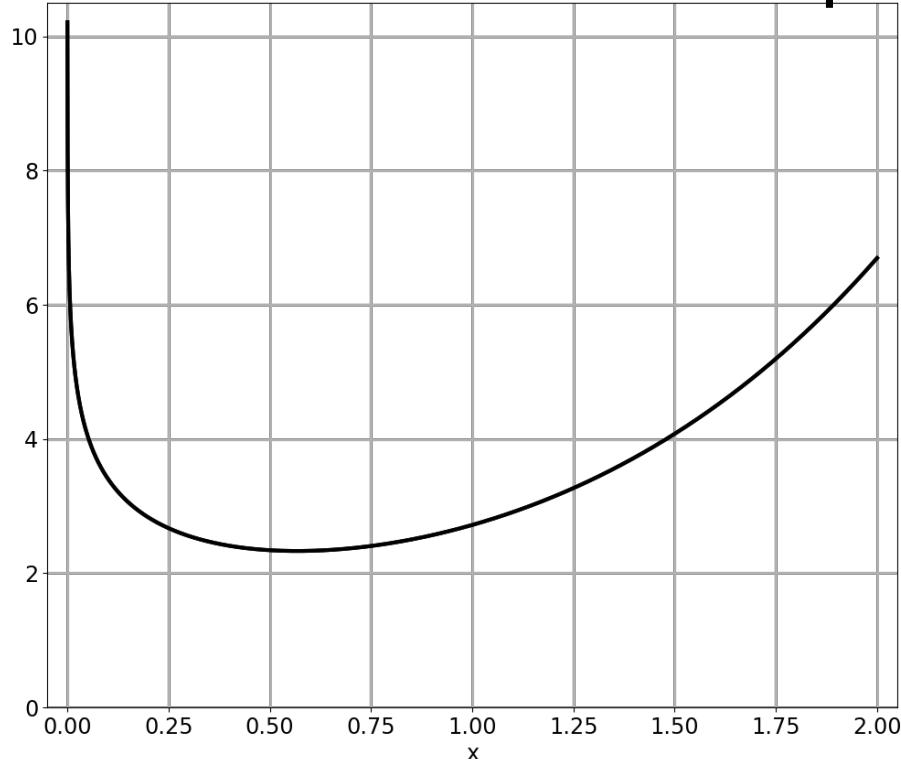
Newton's Method for Optimization

Newton's Method for Optimization



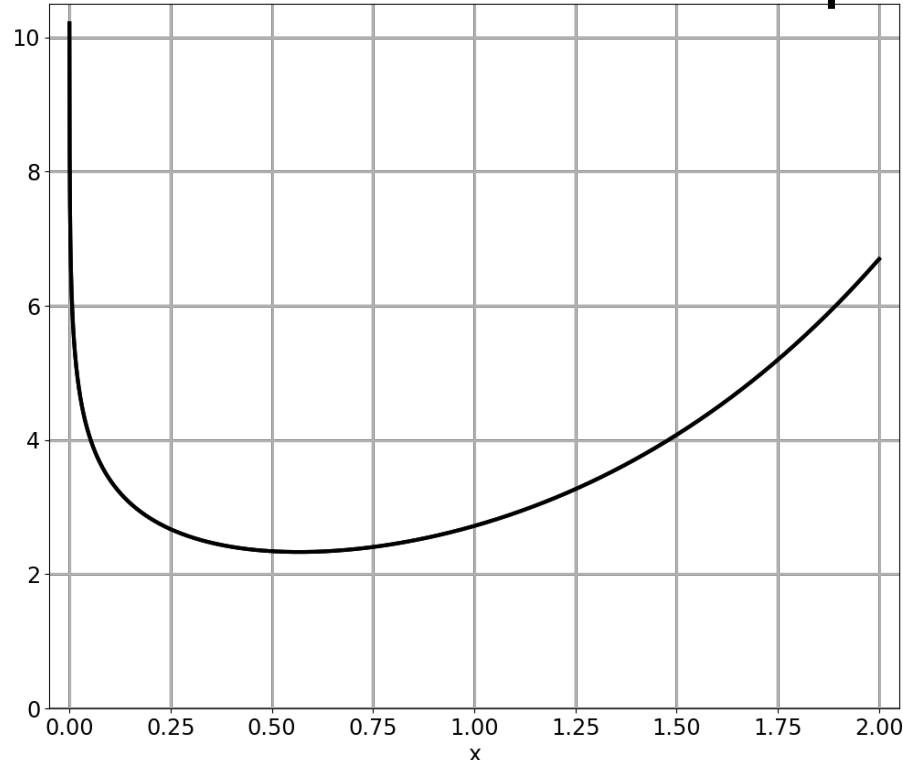
$$g(x) = e^x - \log(x)$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

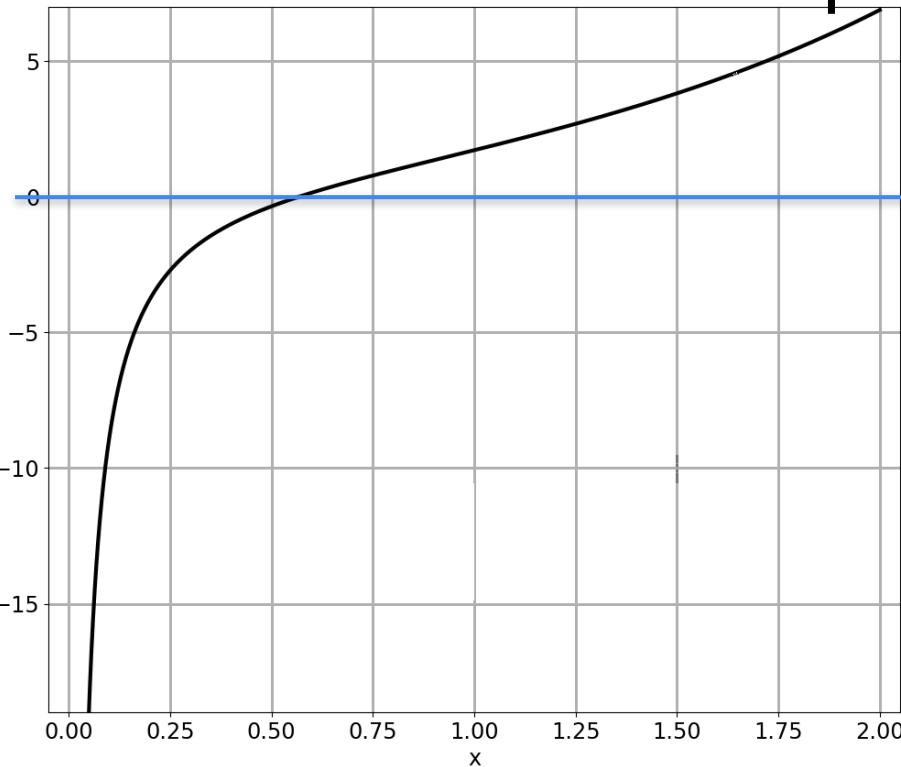
Newton's Method for Optimization



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Minimum: $x^* = 0.5671$

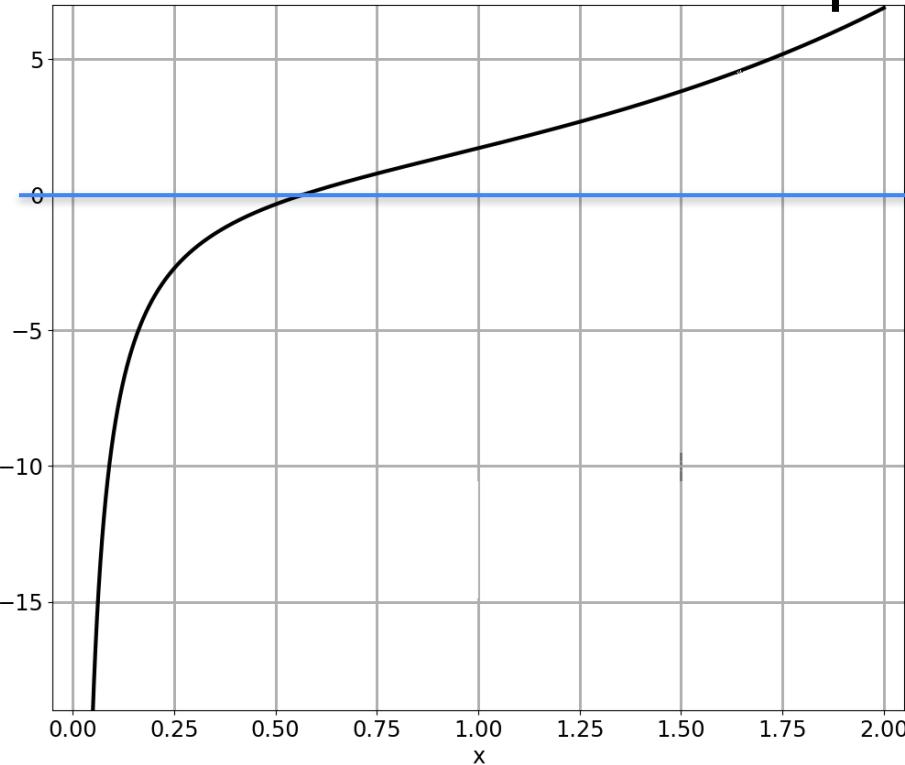
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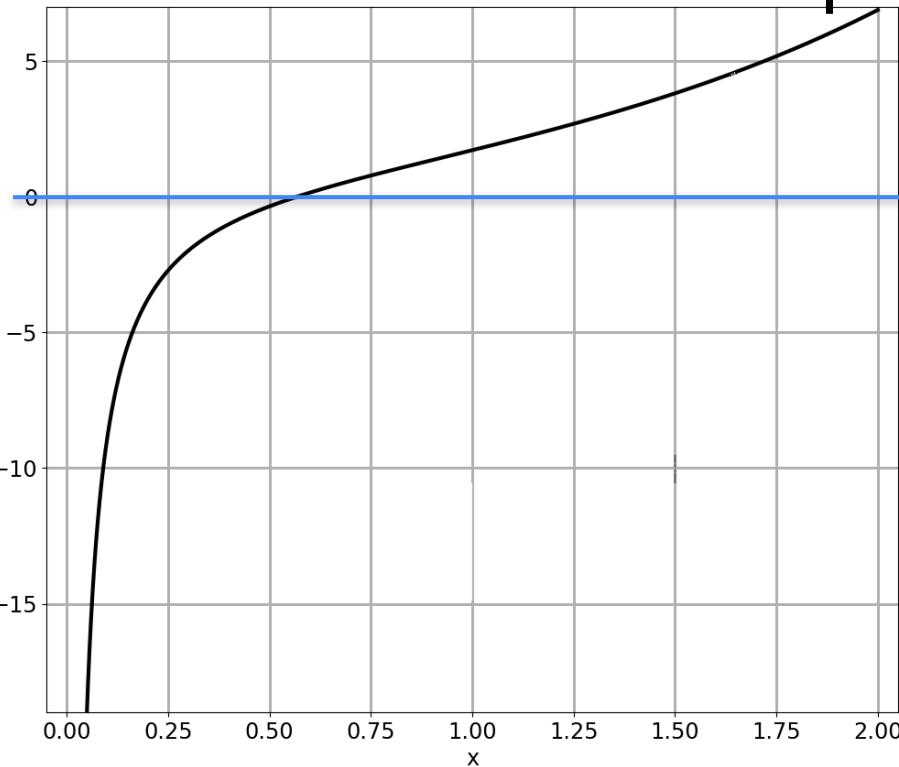


$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

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Newton's Method for Optimization

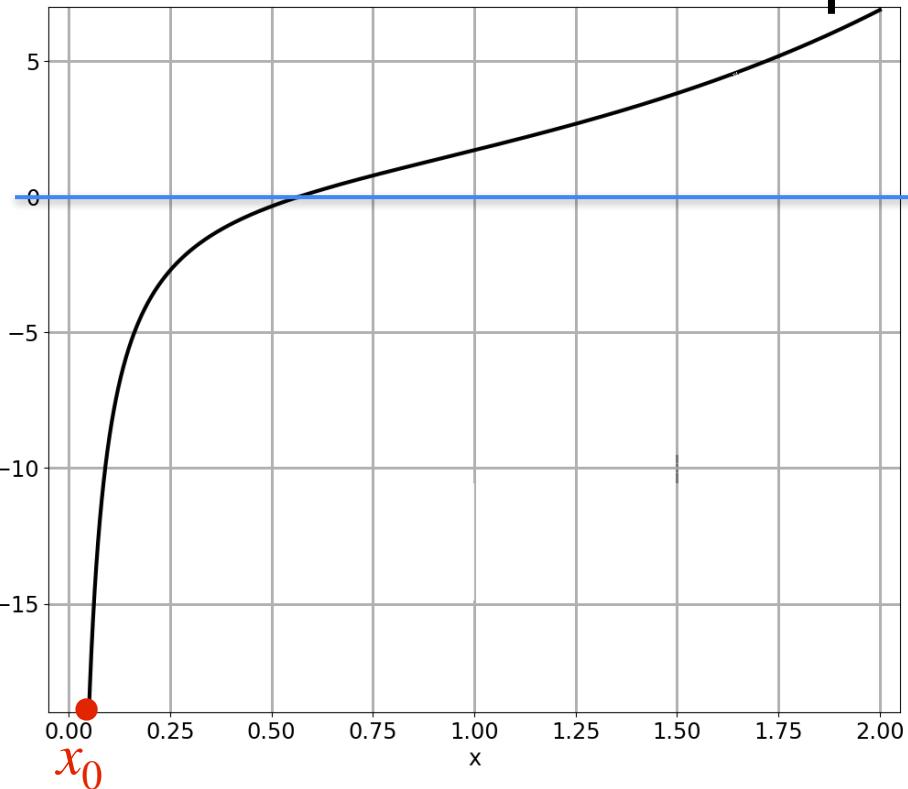


$$g(x) = e^x - \log(x) \quad \underbrace{g'(x) = e^x - 1/x}_{f(x)}$$

Minimum: $x^* = 0.5671$

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Newton's Method for Optimization

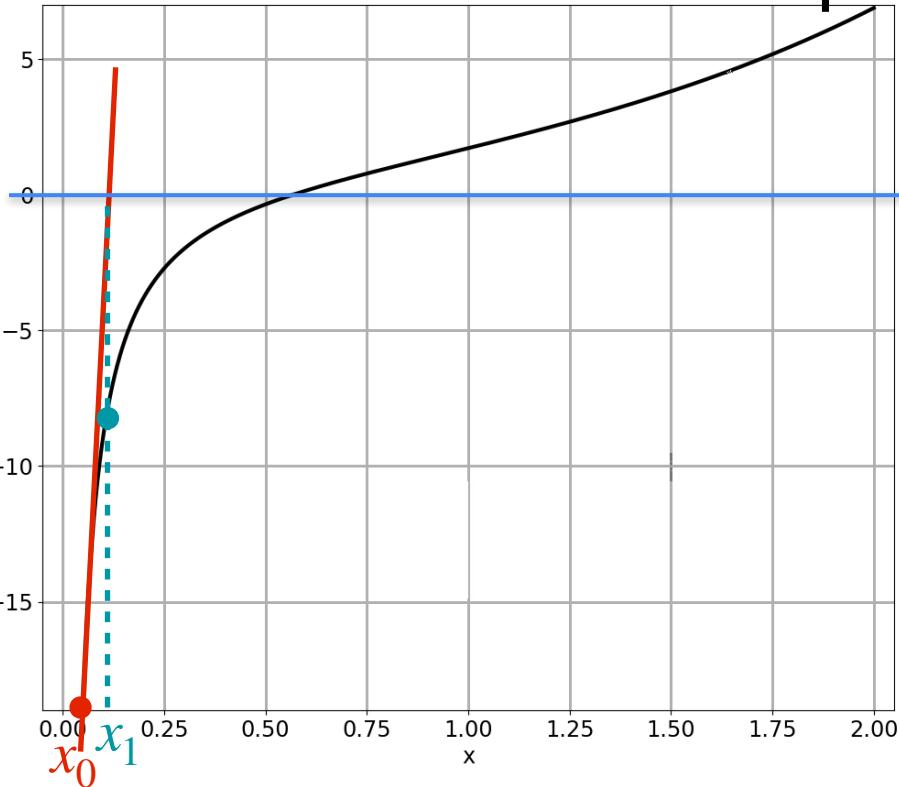


$$g(x) = e^x - \log(x) \quad \overbrace{g'(x) = e^x - 1/x}^{f(x)}$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$
$$x_0 = 0.05 \quad \overbrace{f'(x)}^{g'(x)}$$

Newton's Method for Optimization

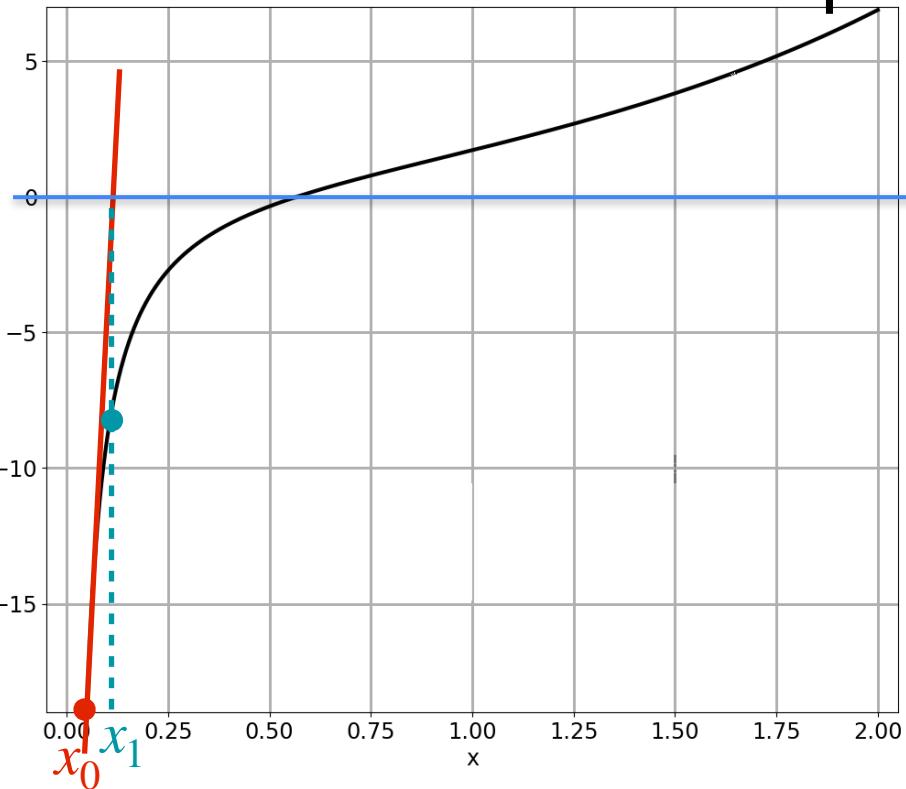


$$g(x) = e^x - \log(x)$$
$$g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$
$$f'(x)$$
$$x_0 = 0.05$$

Newton's Method for Optimization

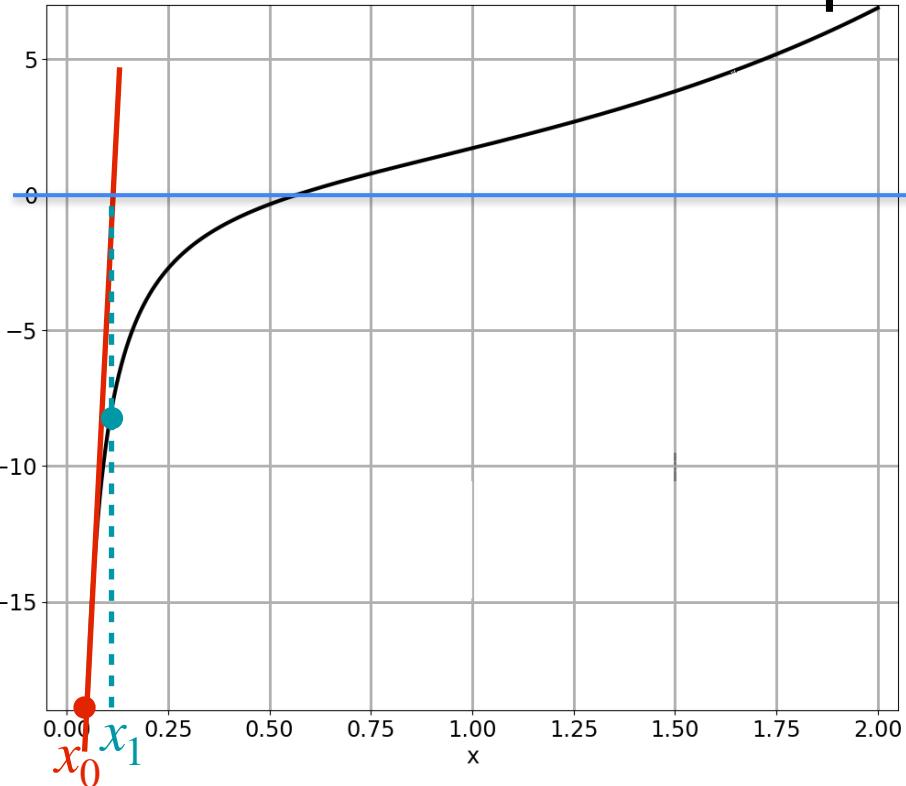


$$g(x) = e^x - \log(x) \quad \overbrace{g'(x)}^{f(x)} = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$
$$x_0 = 0.05$$
$$x_1 = x_0 - \frac{g'(x_0)}{(g'(x_0))'}$$
$$= 0.05 - \frac{\left(e^{0.05} - \frac{1}{0.05}\right)}{\left(e^{0.05} + \frac{1}{0.05^2}\right)}$$

Newton's Method for Optimization

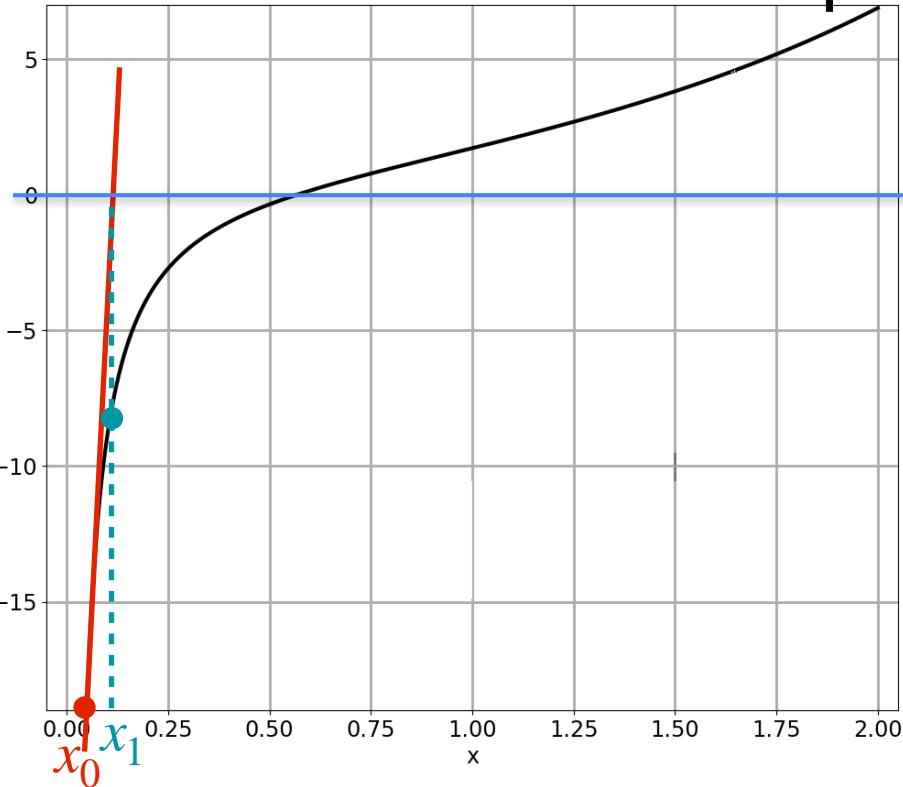


$$g(x) = e^x - \log(x) \quad \overbrace{g'(x)}^{f'(x)} = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$
$$x_0 = 0.05$$
$$x_1 = x_0 - \frac{g'(x_0)}{(g'(x_0))'} \quad \overbrace{f'(x)}^{(g'(x))'}$$
$$= 0.05 - \frac{\left(e^{0.05} - \frac{1}{0.05}\right)}{\left(e^{0.05} + \frac{1}{0.05^2}\right)} = 0.097$$

Newton's Method for Optimization

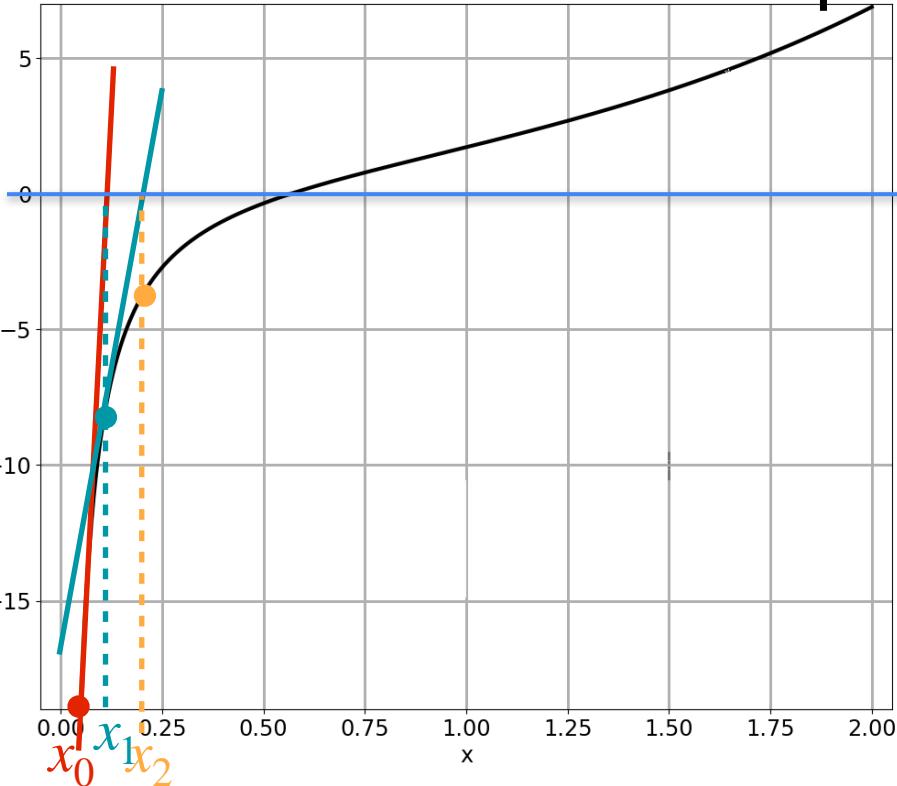


$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

Newton's Method for Optimization



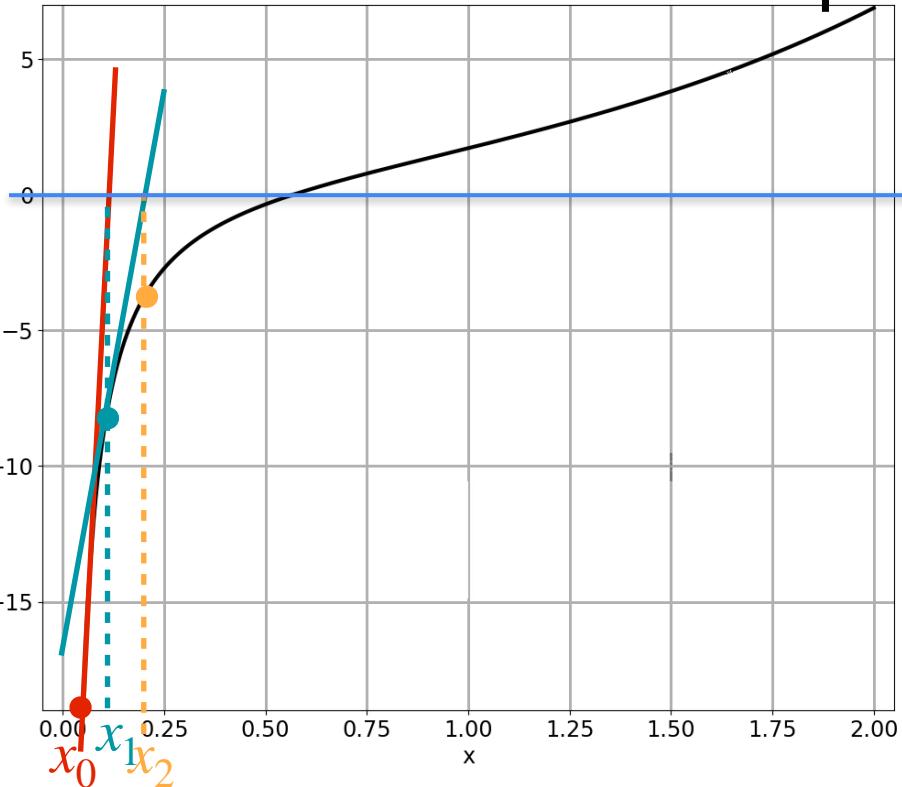
$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_1 = 0.097$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

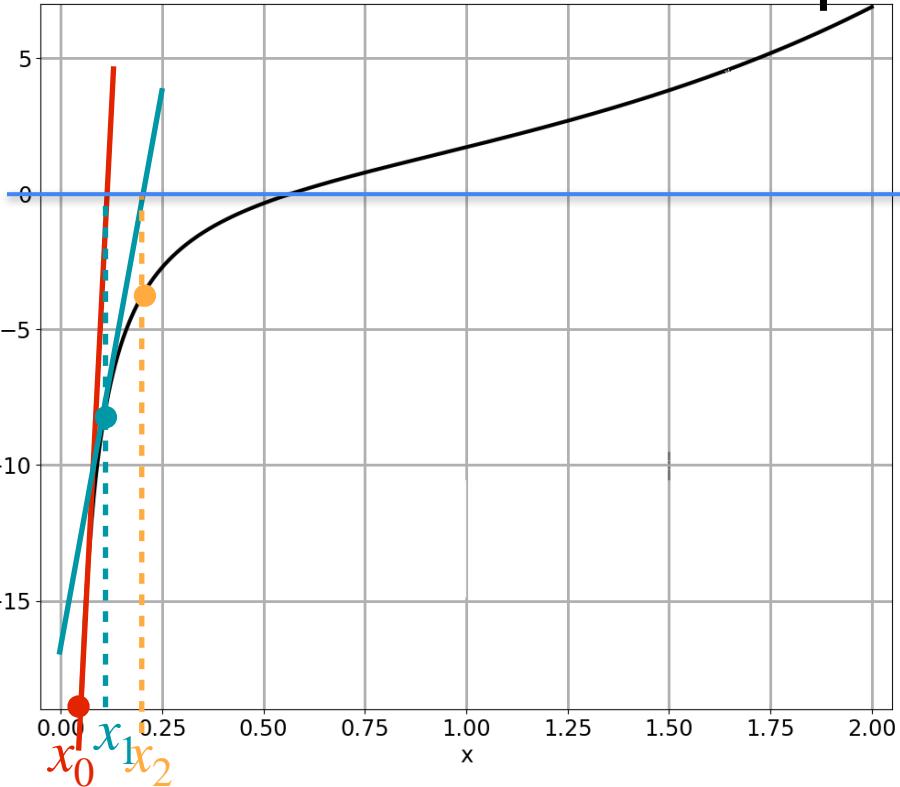
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_1 = 0.097$$

$$x_2 = x_1 - \frac{g'(x_1)}{(g'(x_1))'}$$

$$= 0.097 - \frac{\left(e^{0.097} - \frac{1}{0.097}\right)}{\left(e^{0.097} + \frac{1}{0.097^2}\right)} = 0.183$$

Newton's Method for Optimization

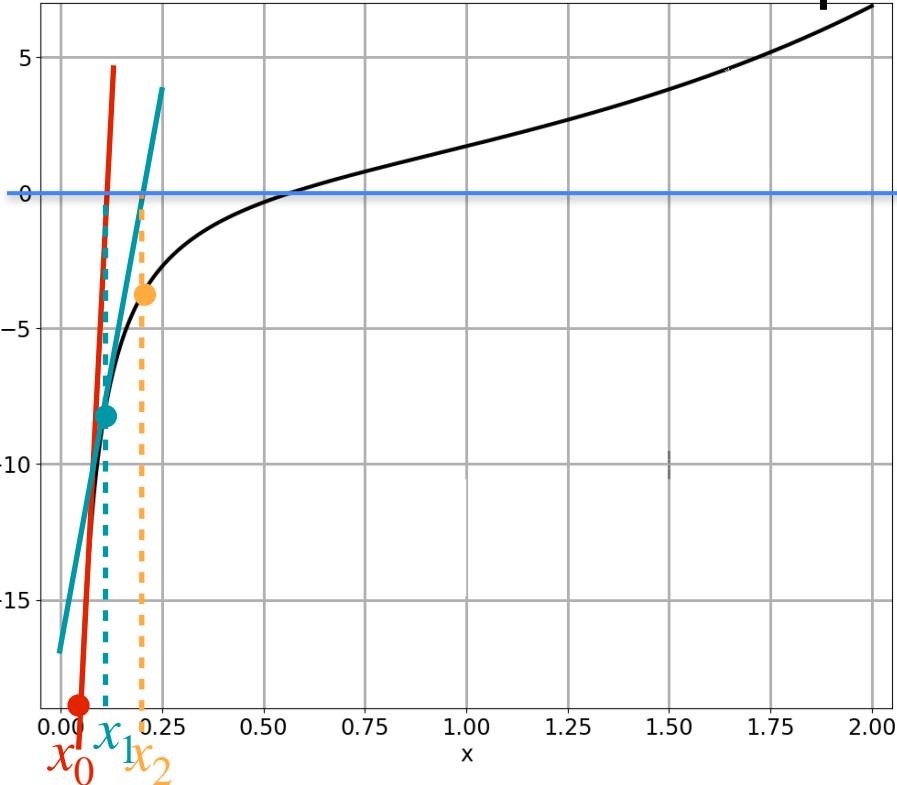


$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

Newton's Method for Optimization



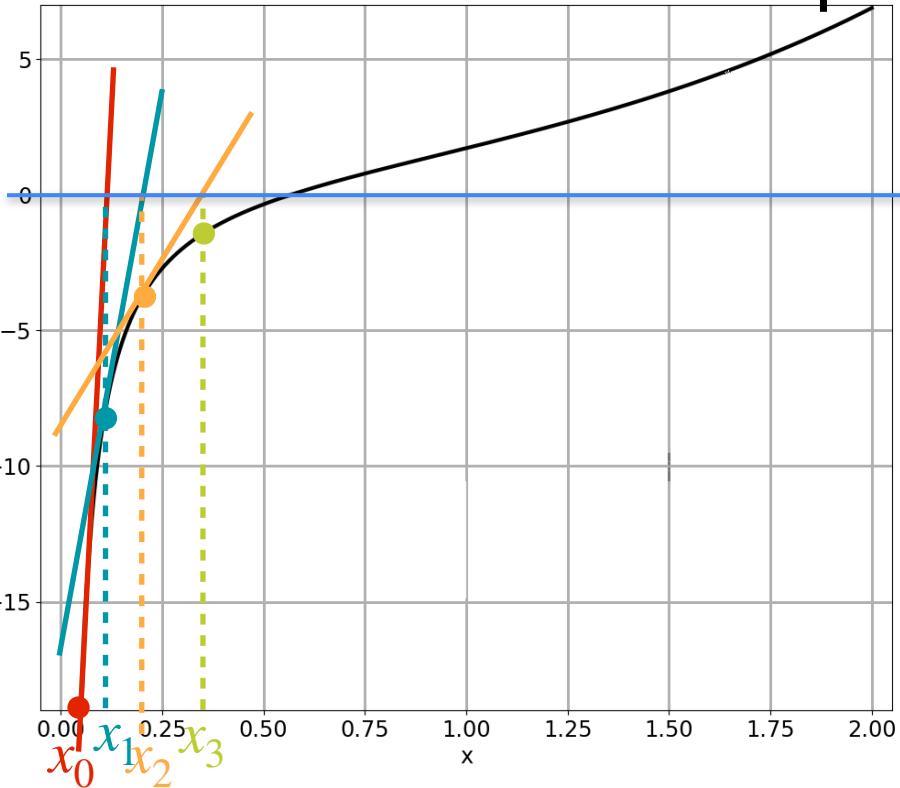
$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_2 = 0.183$$

Newton's Method for Optimization



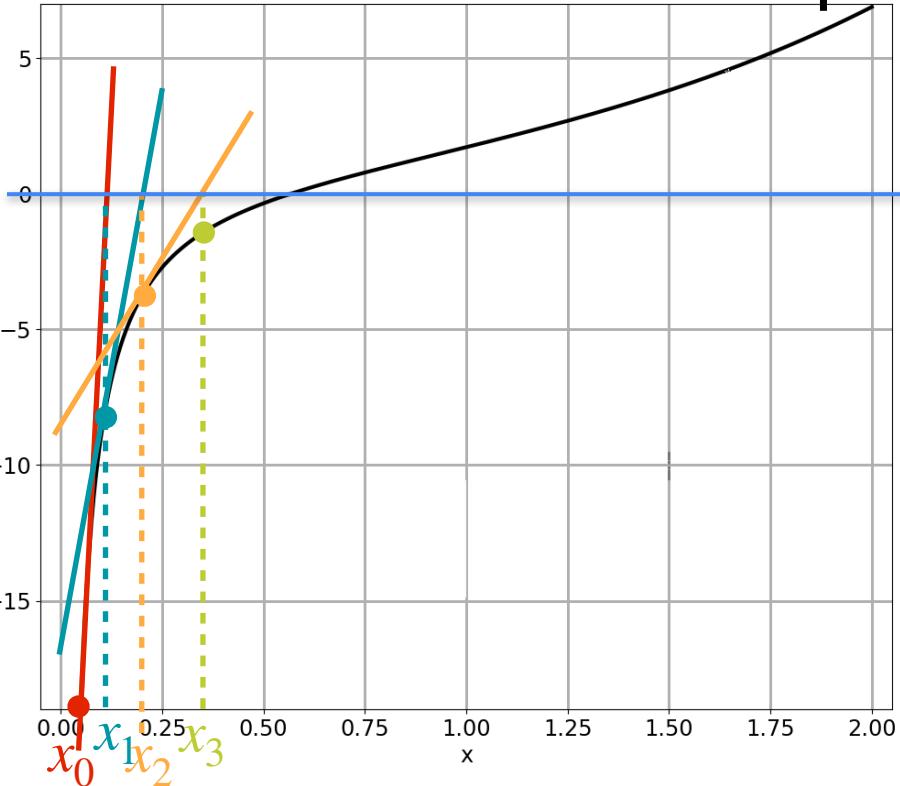
$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

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$$x_2 = 0.183$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

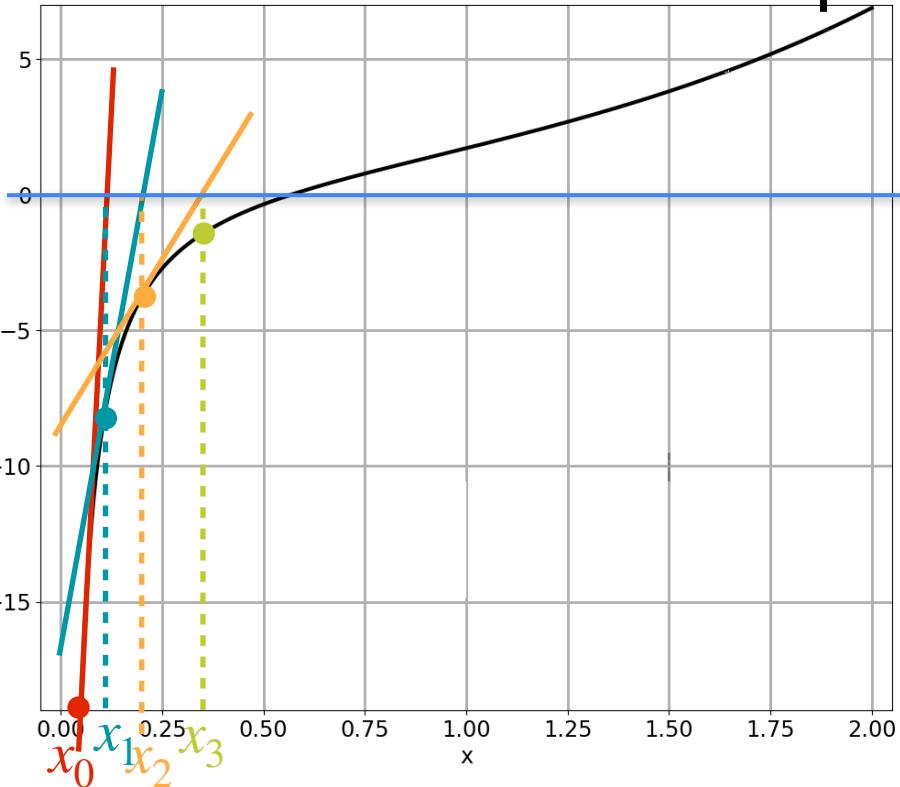
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_2 = 0.183$$

$$x_3 = x_2 - \frac{g'(x_2)}{(g'(x_2))'}$$

$$= 0.183 - \frac{\left(e^{0.183} - \frac{1}{0.183}\right)}{\left(e^{0.183} + \frac{1}{0.183^2}\right)}$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

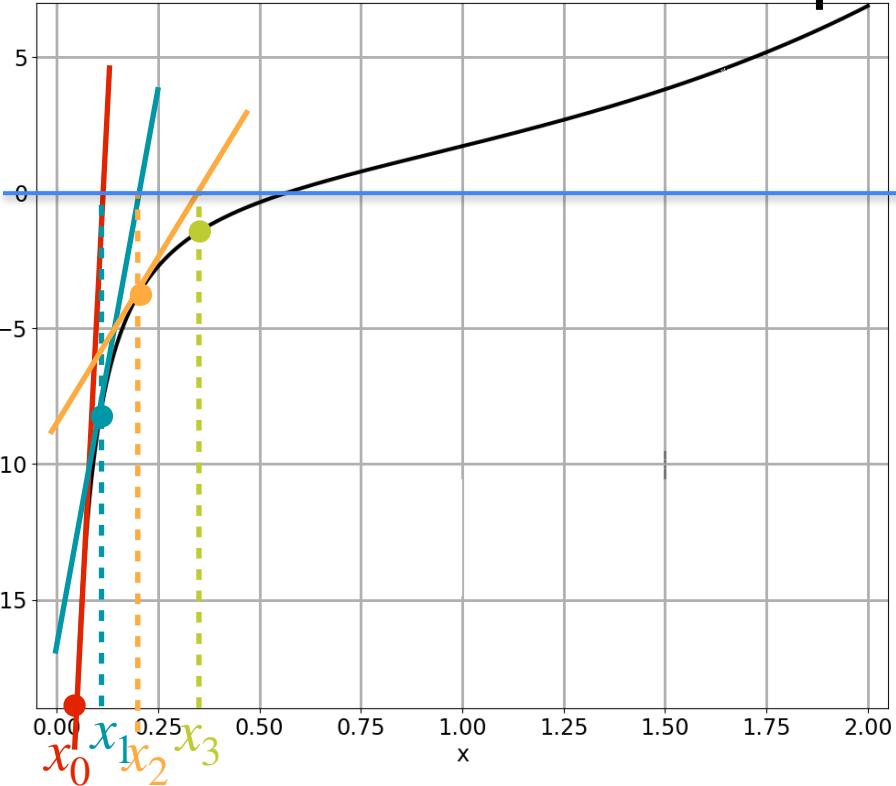
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_2 = 0.183$$

$$x_3 = x_2 - \frac{g'(x_2)}{(g'(x_2))'}$$

$$= 0.183 - \frac{\left(e^{0.183} - \frac{1}{0.183}\right)}{\left(e^{0.183} + \frac{1}{0.183^2}\right)} = 0.320$$

Newton's Method for Optimization



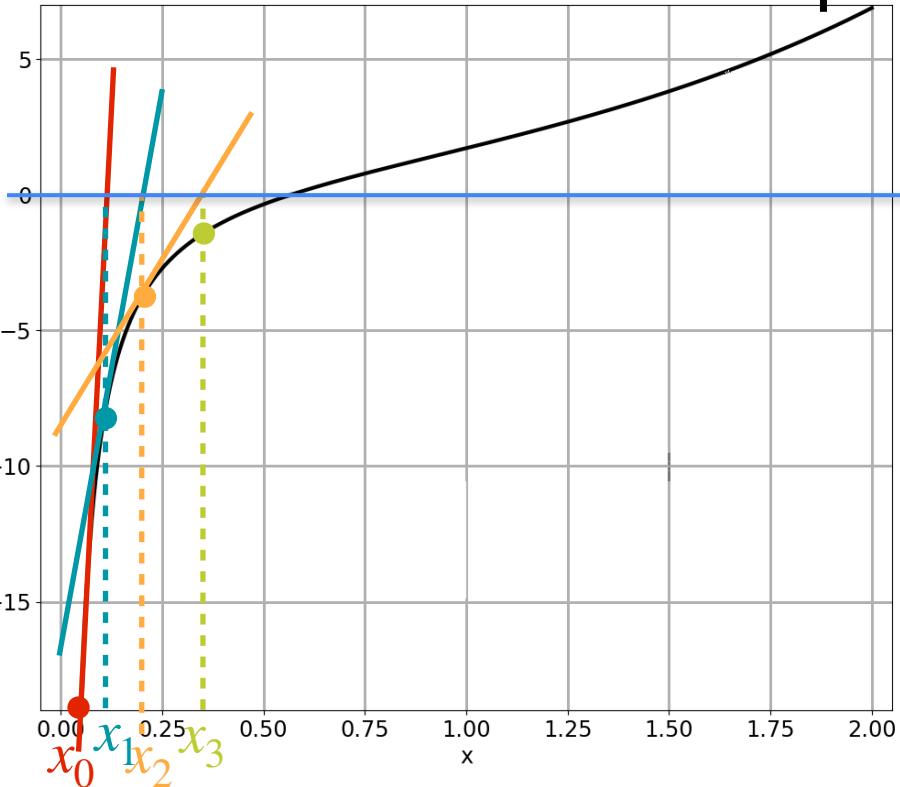
$$g(x) = e^x - \log(x)$$

$$g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

Newton's Method for Optimization



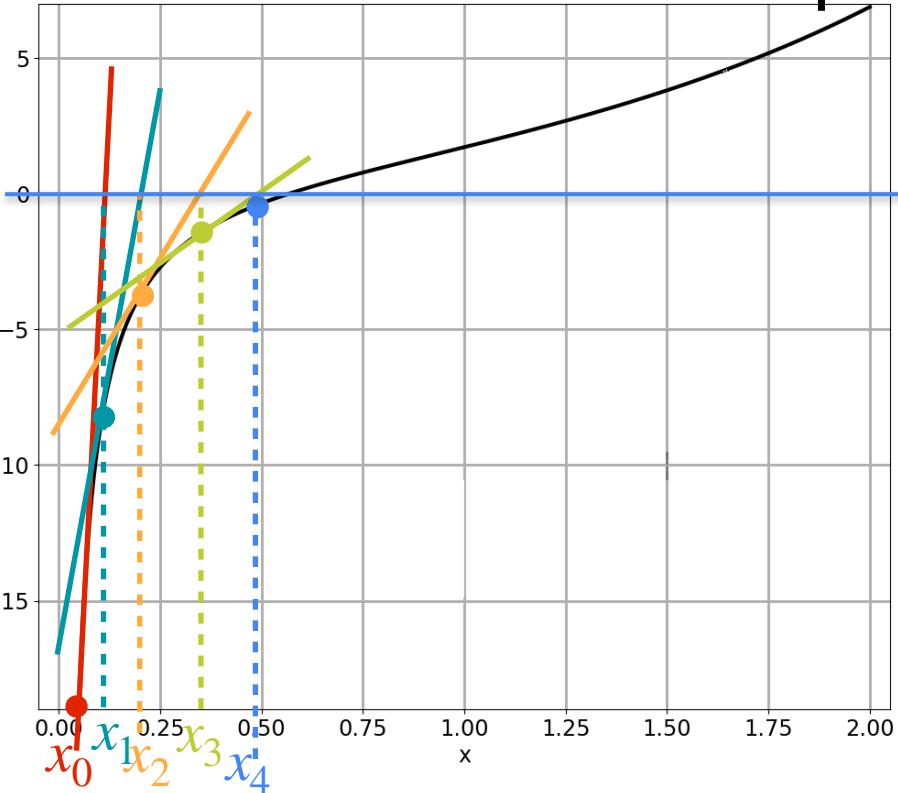
$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_3 = 0.320$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

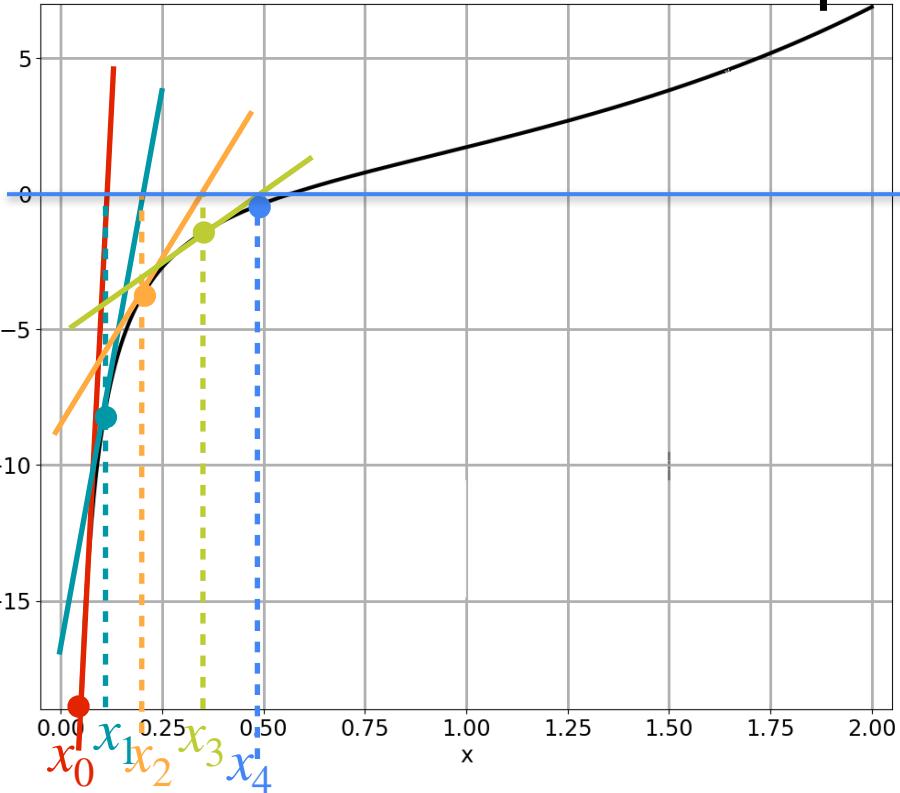
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_3 = 0.320$$

$$x_4 = x_3 - \frac{g'(x_3)}{(g'(x_3))'}$$

$$= 0.320 - \frac{\left(e^{0.320} - \frac{1}{0.320}\right)}{\left(e^{0.320} + \frac{1}{0.320^2}\right)}$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

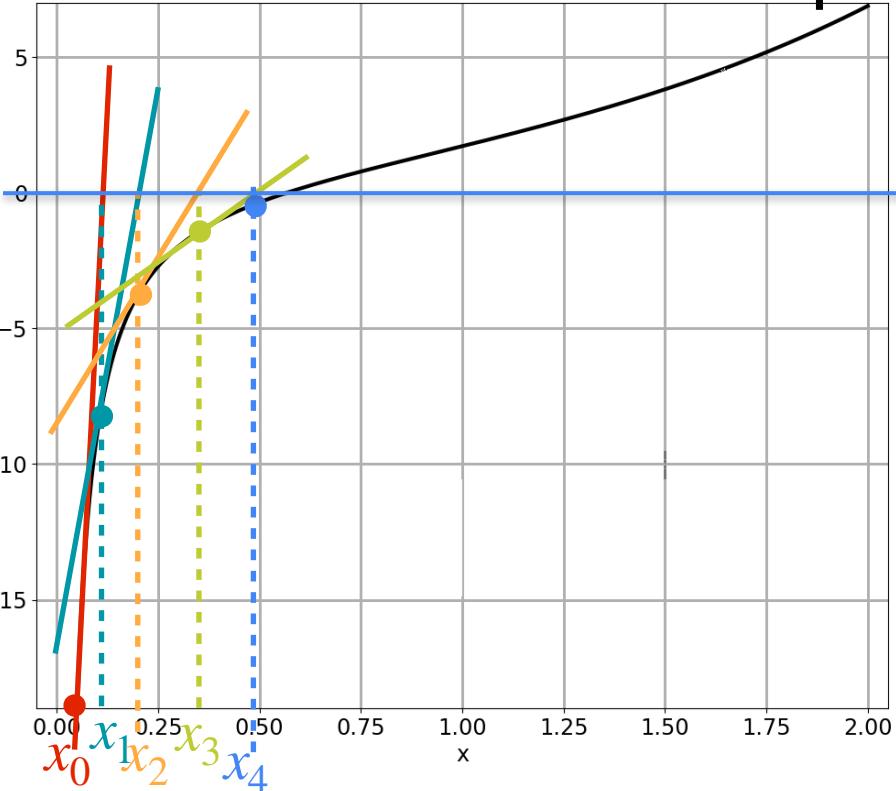
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_3 = 0.320$$

$$x_4 = x_3 - \frac{g'(x_3)}{(g'(x_3))'}$$

$$= 0.320 - \frac{\left(e^{0.320} - \frac{1}{0.320}\right)}{\left(e^{0.320} + \frac{1}{0.320^2}\right)} = 0.477$$

Newton's Method for Optimization



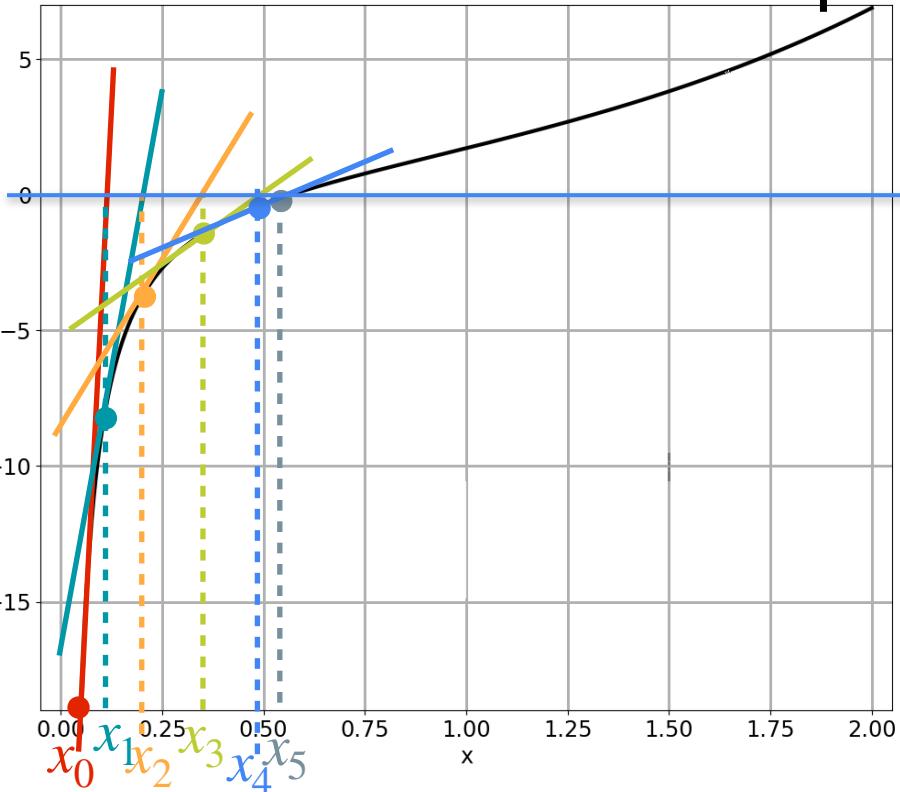
$$g(x) = e^x - \log(x)$$

$$g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.5671$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.5671$$

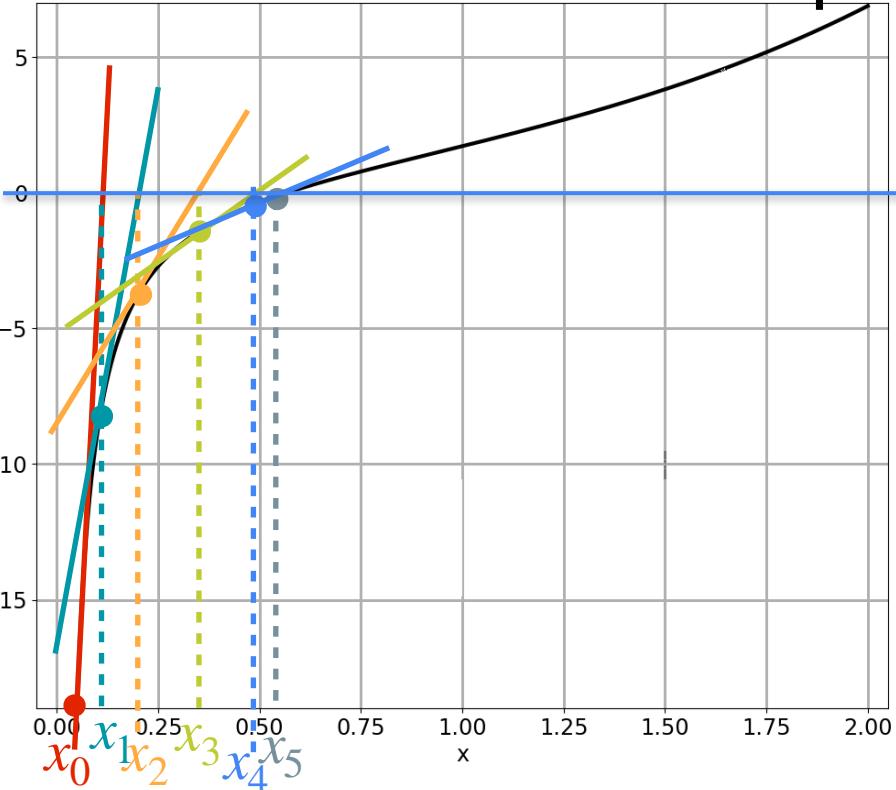
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_4 = 0.477$$

$$x_5 = x_4 - \frac{g'(x_4)}{(g'(x_4))'}$$

$$= 0.447 - \frac{\left(e^{0.447} - \frac{1}{0.447}\right)}{\left(e^{0.447} + \frac{1}{0.447^2}\right)} = 0.558$$

Newton's Method for Optimization

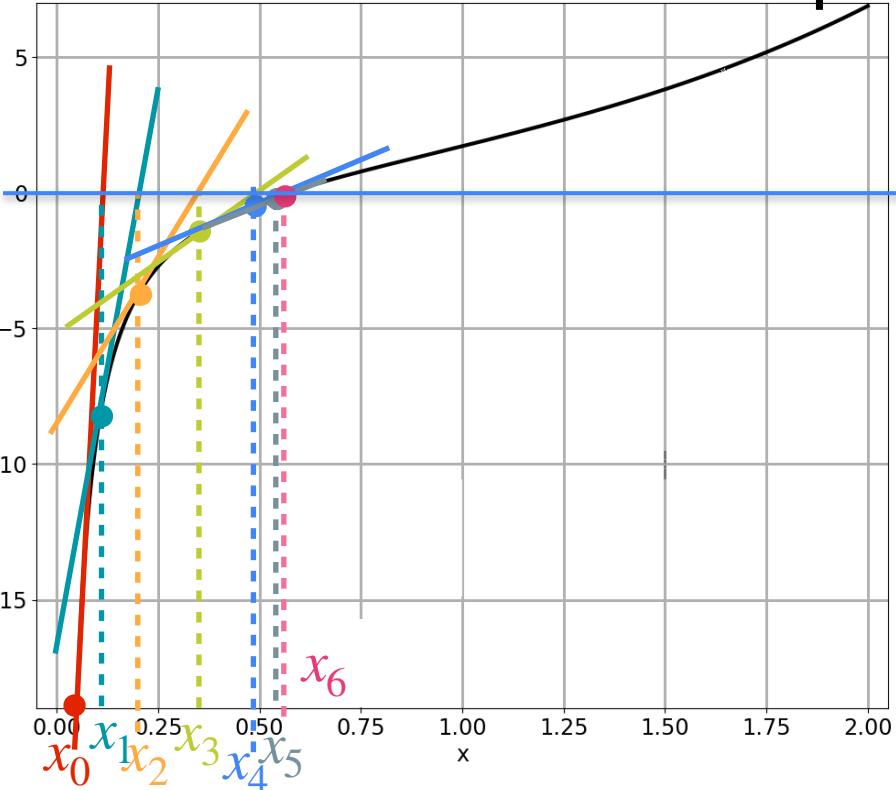


$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.567$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

Newton's Method for Optimization

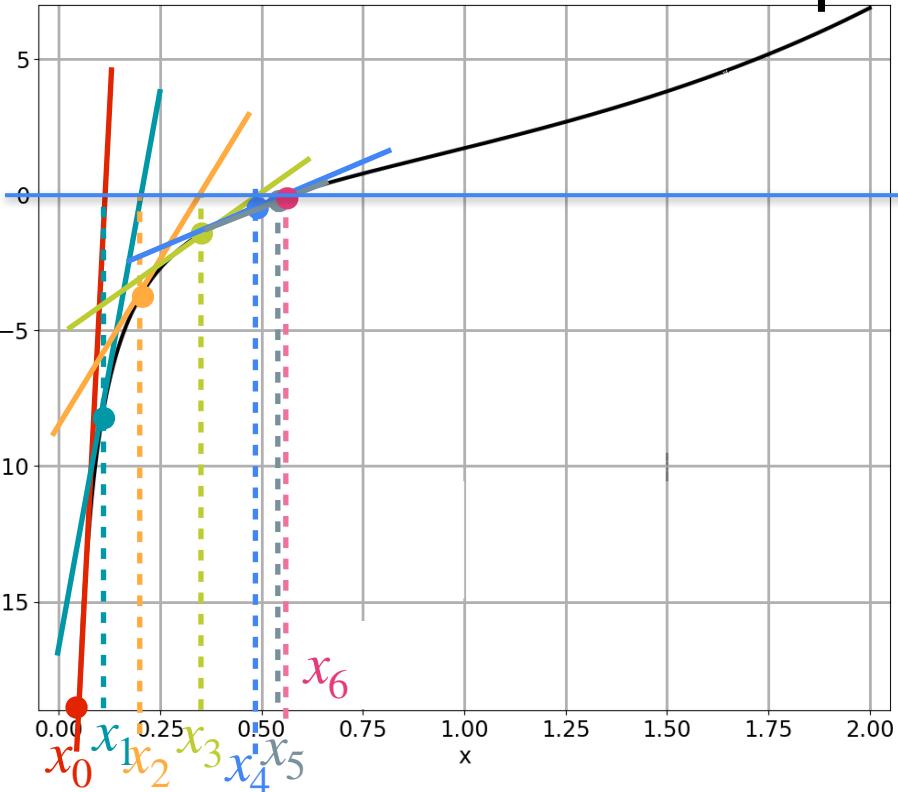


$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.567$

$$(g'(x))' = e^x + \frac{1}{x^2}$$
$$x_5 = 0.558$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

$$\text{Minimum: } x^* = 0.567$$

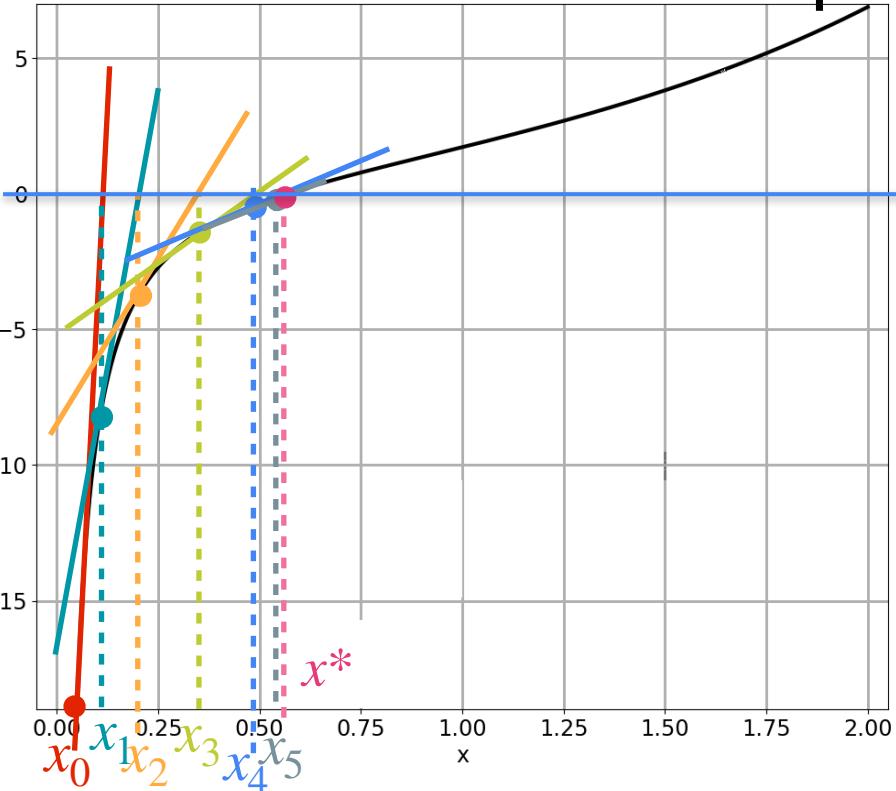
$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_5 = 0.558$$

$$x_6 = x_5 - \frac{g'(x_5)}{(g'(x_5))'}$$

$$= 0.558 - \frac{\left(e^{0.558} - \frac{1}{0.558}\right)}{\left(e^{0.558} + \frac{1}{0.558^2}\right)}$$

Newton's Method for Optimization



$$g(x) = e^x - \log(x) \quad g'(x) = e^x - 1/x$$

Minimum: $x^* = 0.567$

$$(g'(x))' = e^x + \frac{1}{x^2}$$

$$x_5 = 0.558$$

$$x^* = x_5 - \frac{g'(x_5)}{(g'(x_5))'}$$

$$= 0.558 - \frac{\left(e^{0.558} - \frac{1}{0.558}\right)}{\left(e^{0.558} + \frac{1}{0.558^2}\right)} = 0.567$$



DeepLearning.AI

Optimization in Neural Networks and Newton's Method

The second derivative

Second Derivative

Second Derivative

Newton's method:

Second Derivative

Newton's method: $x_{k+1} = x_k - \frac{g'(x_k)}{(g'(x_k))'}$

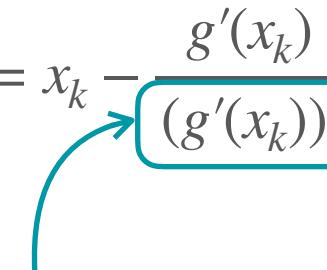
Second Derivative

Newton's method: $x_{k+1} = x_k - \frac{g'(x_k)}{(g'(x_k))'} ??$

Second Derivative

Newton's method: $x_{k+1} = x_k - \frac{g'(x_k)}{(g'(x_k))'}$??

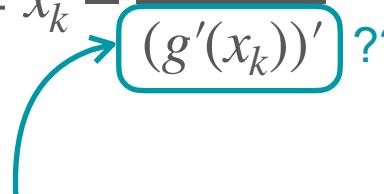
Second derivative



Second Derivative

Newton's method: $x_{k+1} = x_k - \frac{g'(x_k)}{(g'(x_k))'}$??

Second derivative



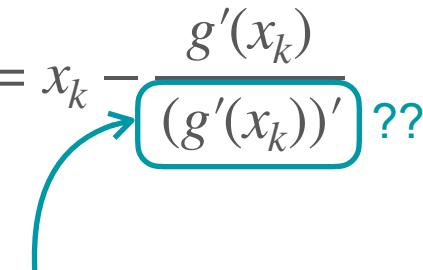
Leibniz notation:

$$\frac{d^2f(x)}{dx^2} = \frac{d}{dx} \left(\frac{df(x)}{dx} \right)$$

Second Derivative

Newton's method: $x_{k+1} = x_k - \frac{g'(x_k)}{(g'(x_k))'}$??

Second derivative



Leibniz notation:

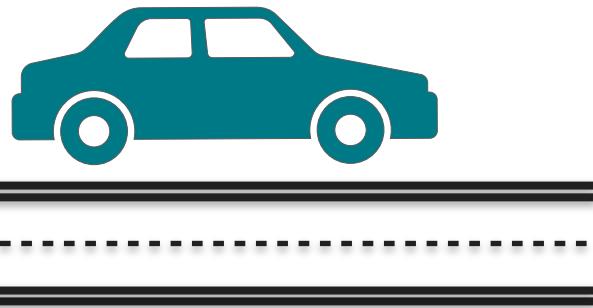
$$\frac{d^2f(x)}{dx^2} = \frac{d}{dx} \left(\frac{df(x)}{dx} \right)$$

Lagrange notation:

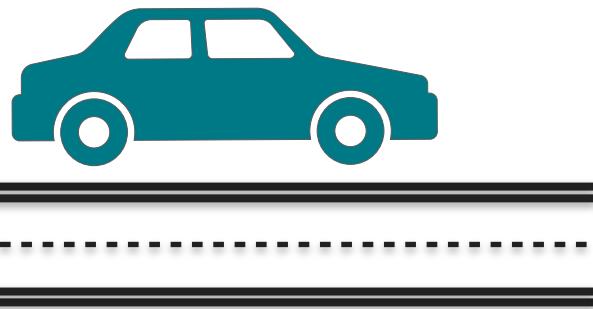
$$f''(x)$$

Understanding Second Derivative

Understanding Second Derivative

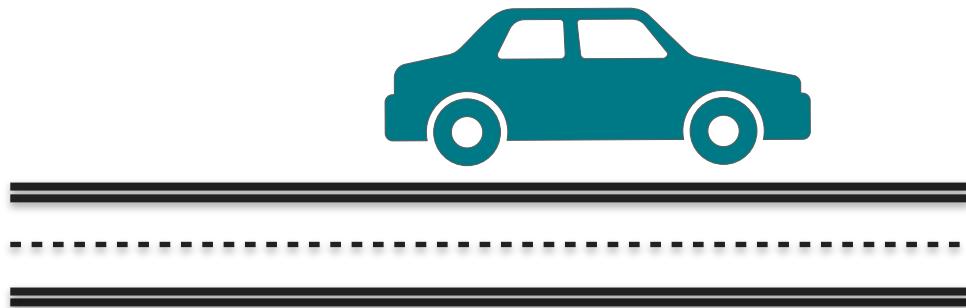


Understanding Second Derivative



x Distance

Understanding Second Derivative

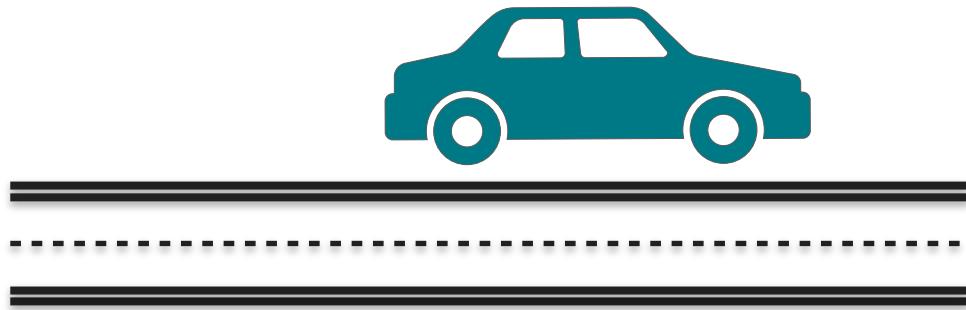


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative



x Distance

v Velocity

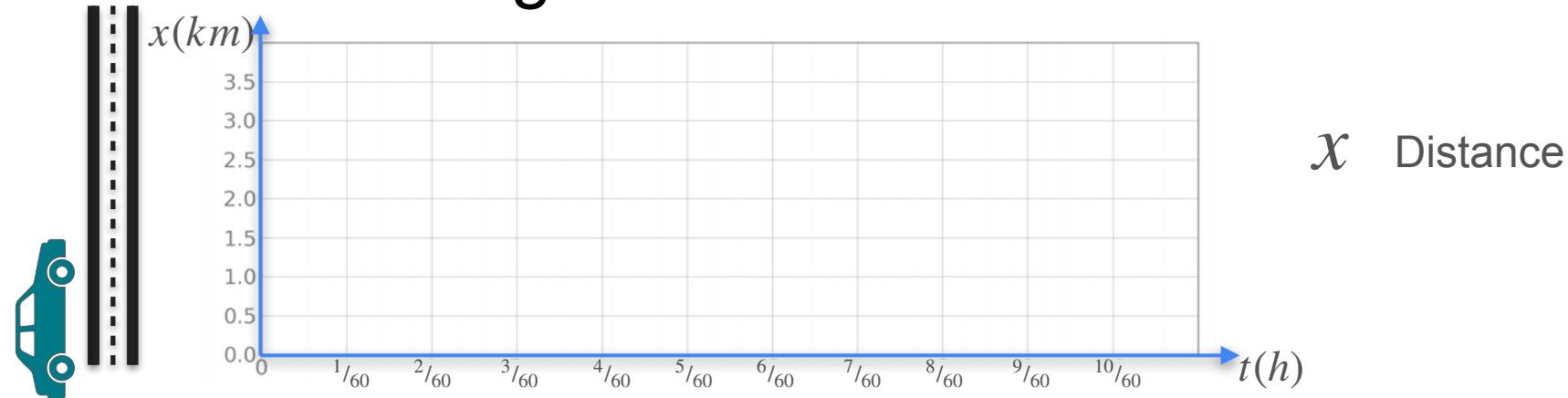
a Acceleration

$$\frac{dx}{dt}$$

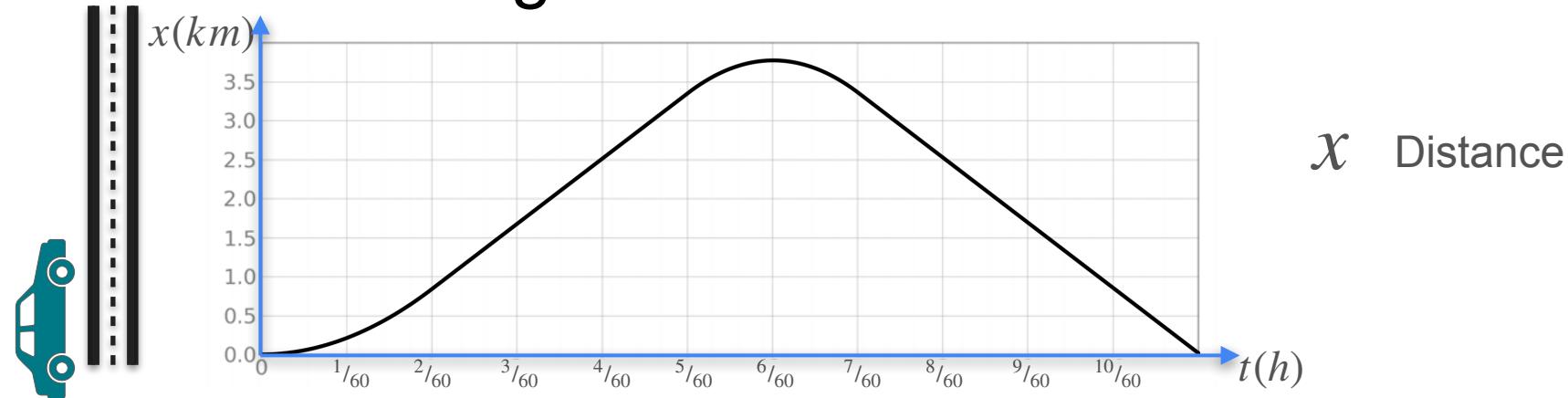
$$\frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Understanding Second Derivative

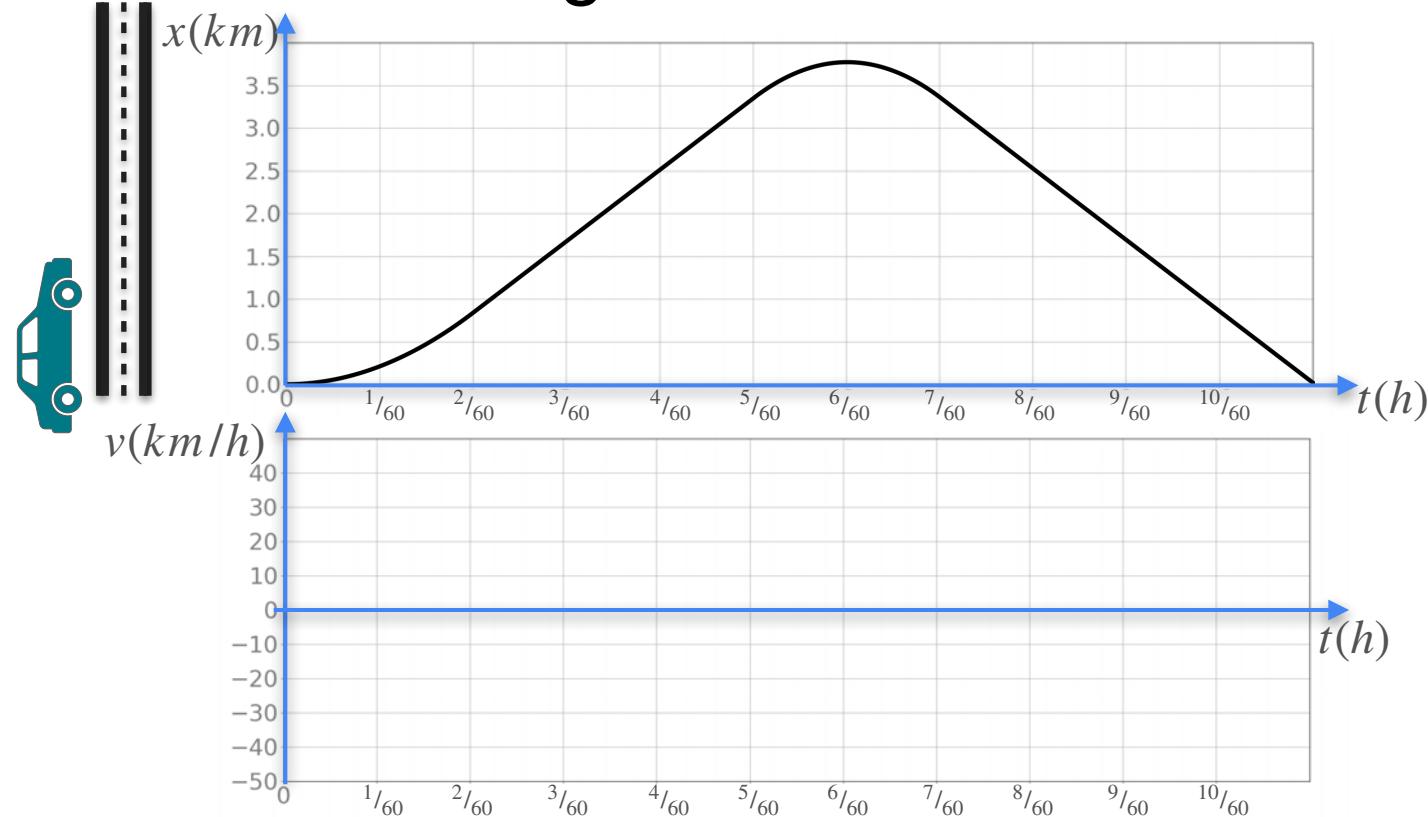
Understanding Second Derivative



Understanding Second Derivative



Understanding Second Derivative

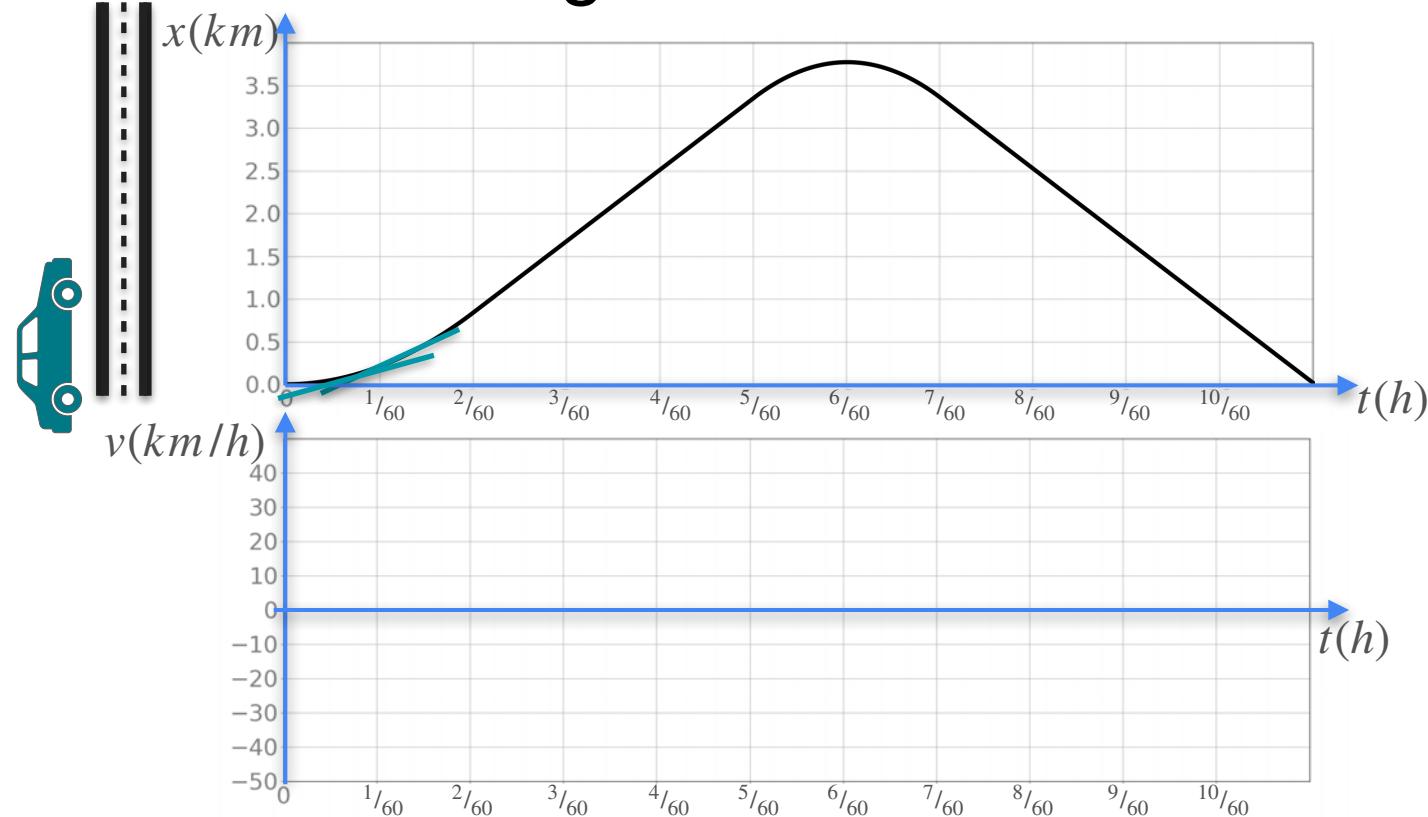


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

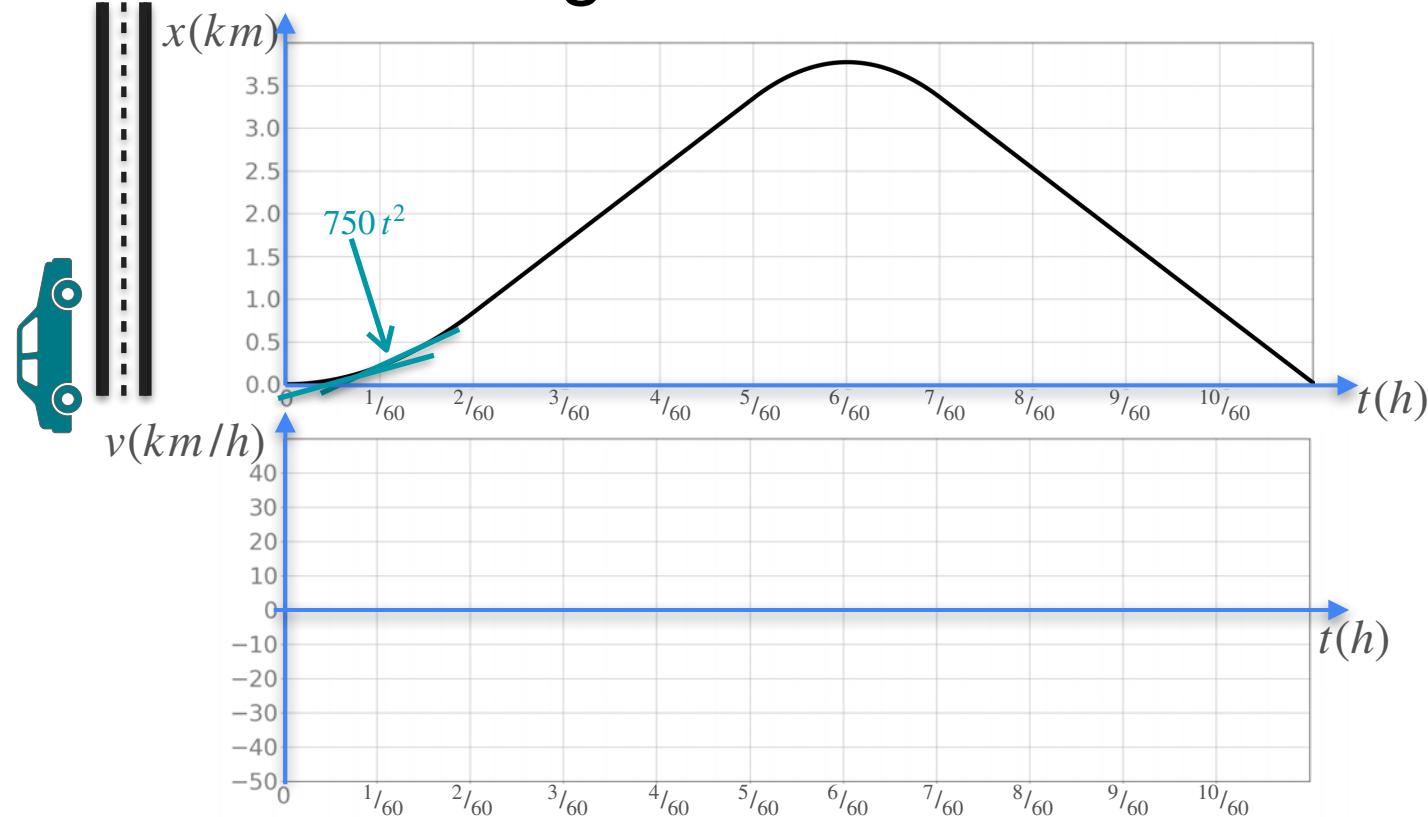


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

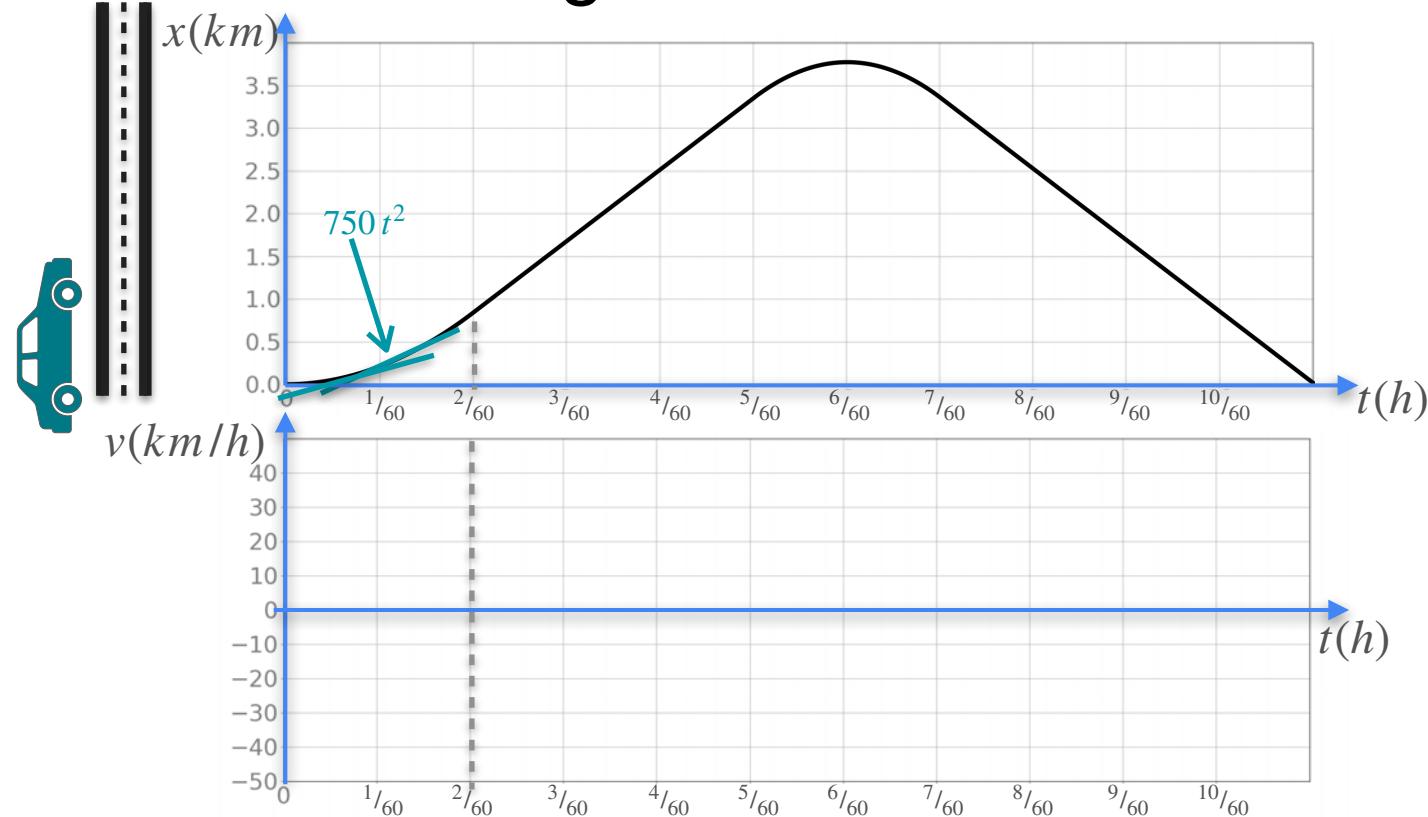


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

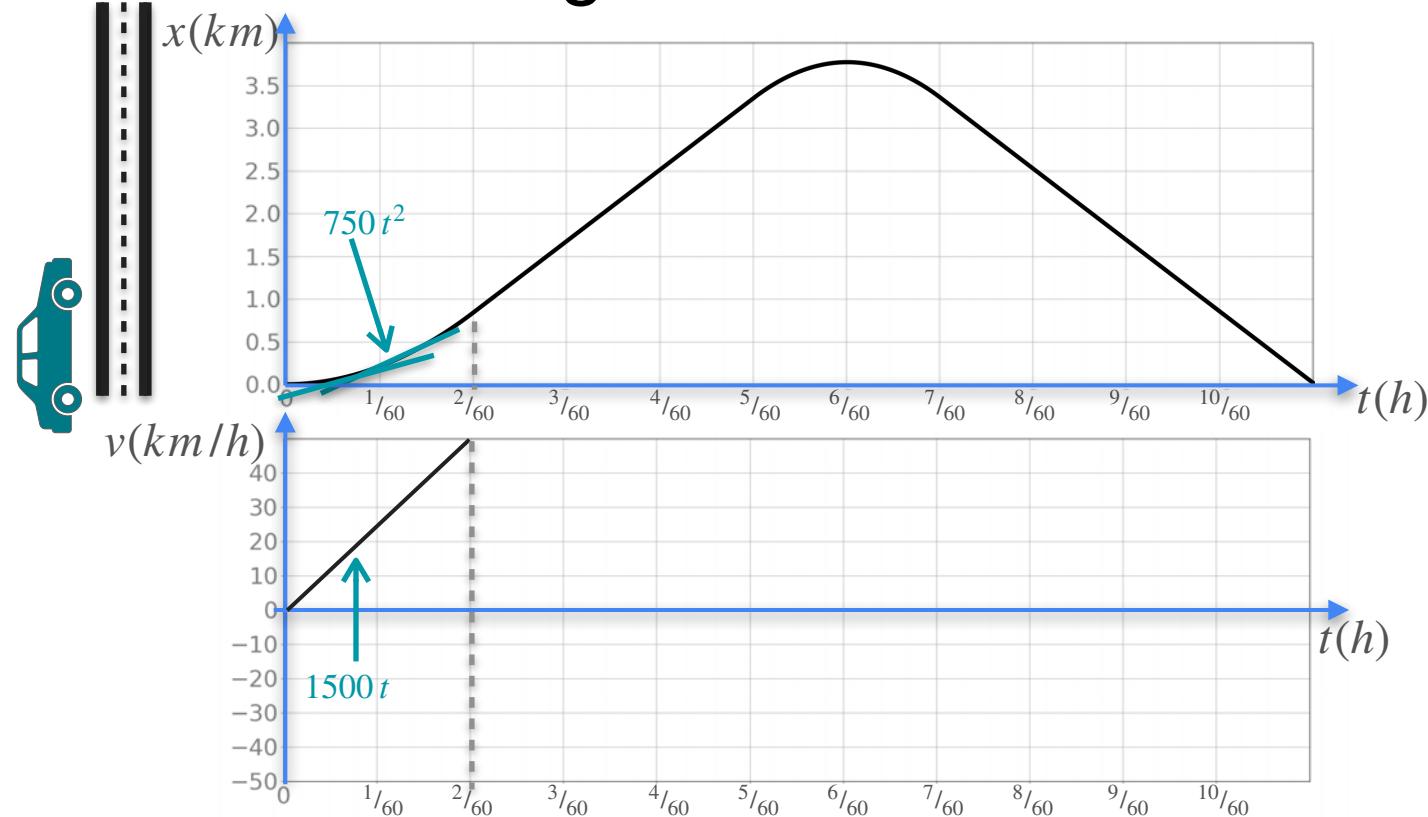


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

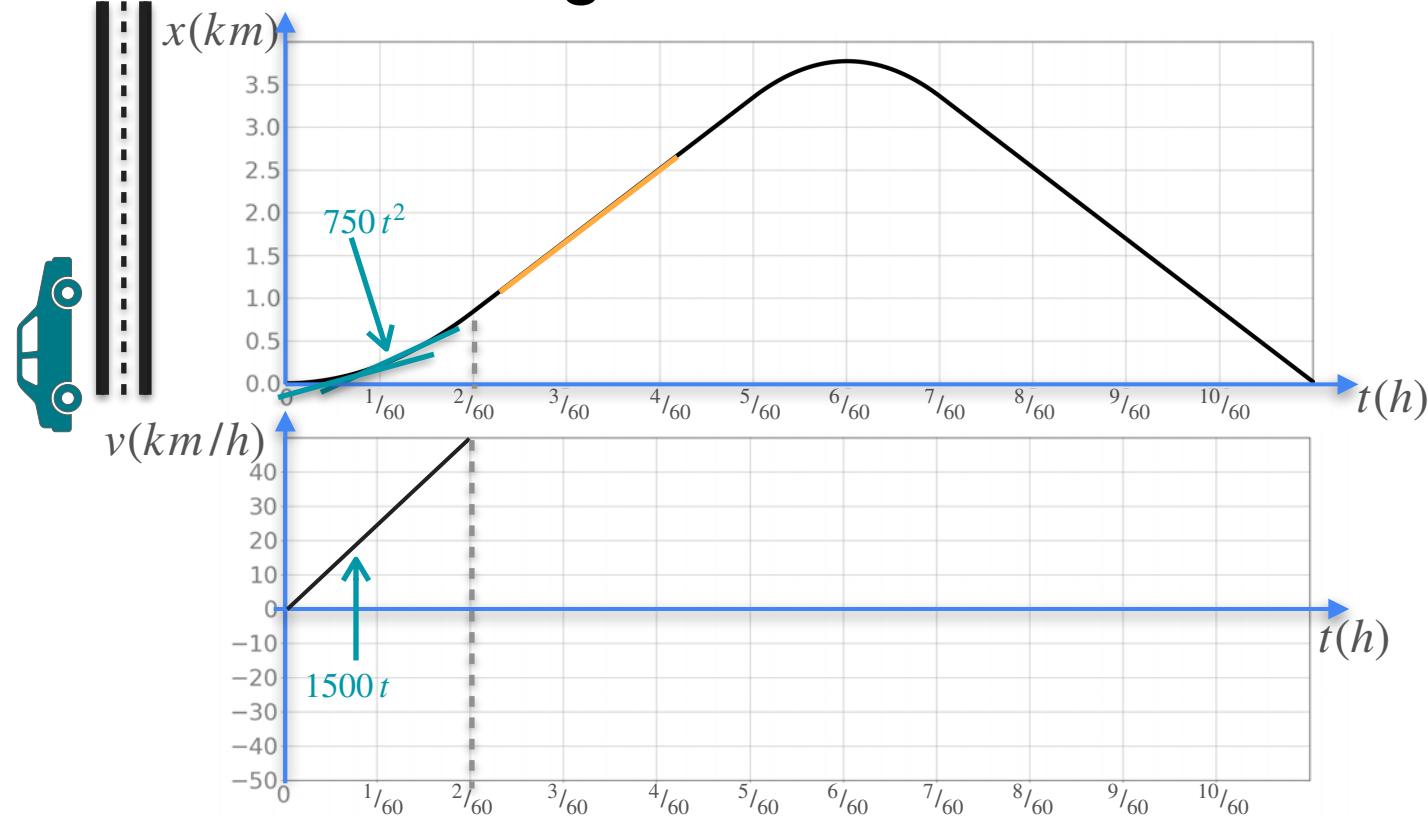


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

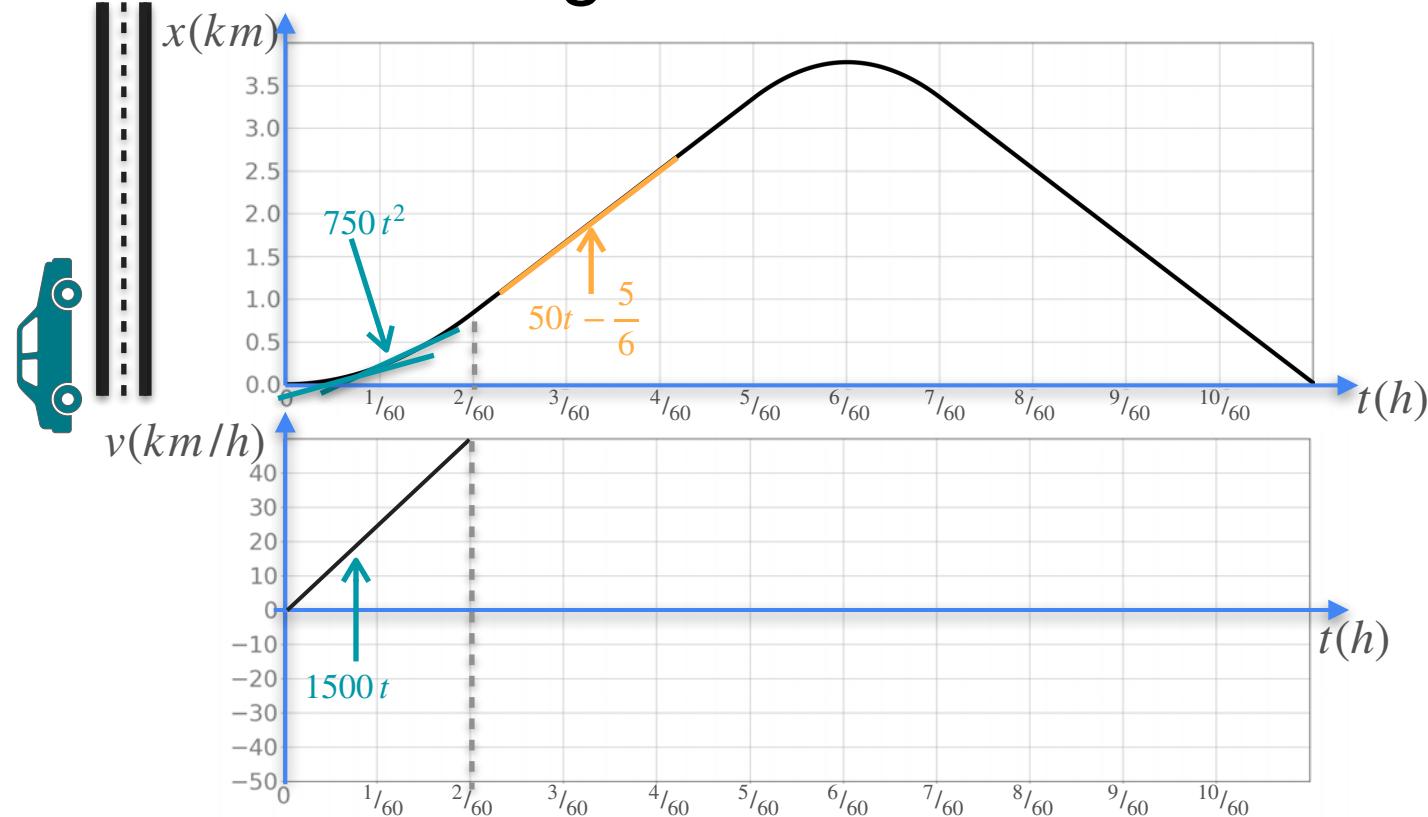


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

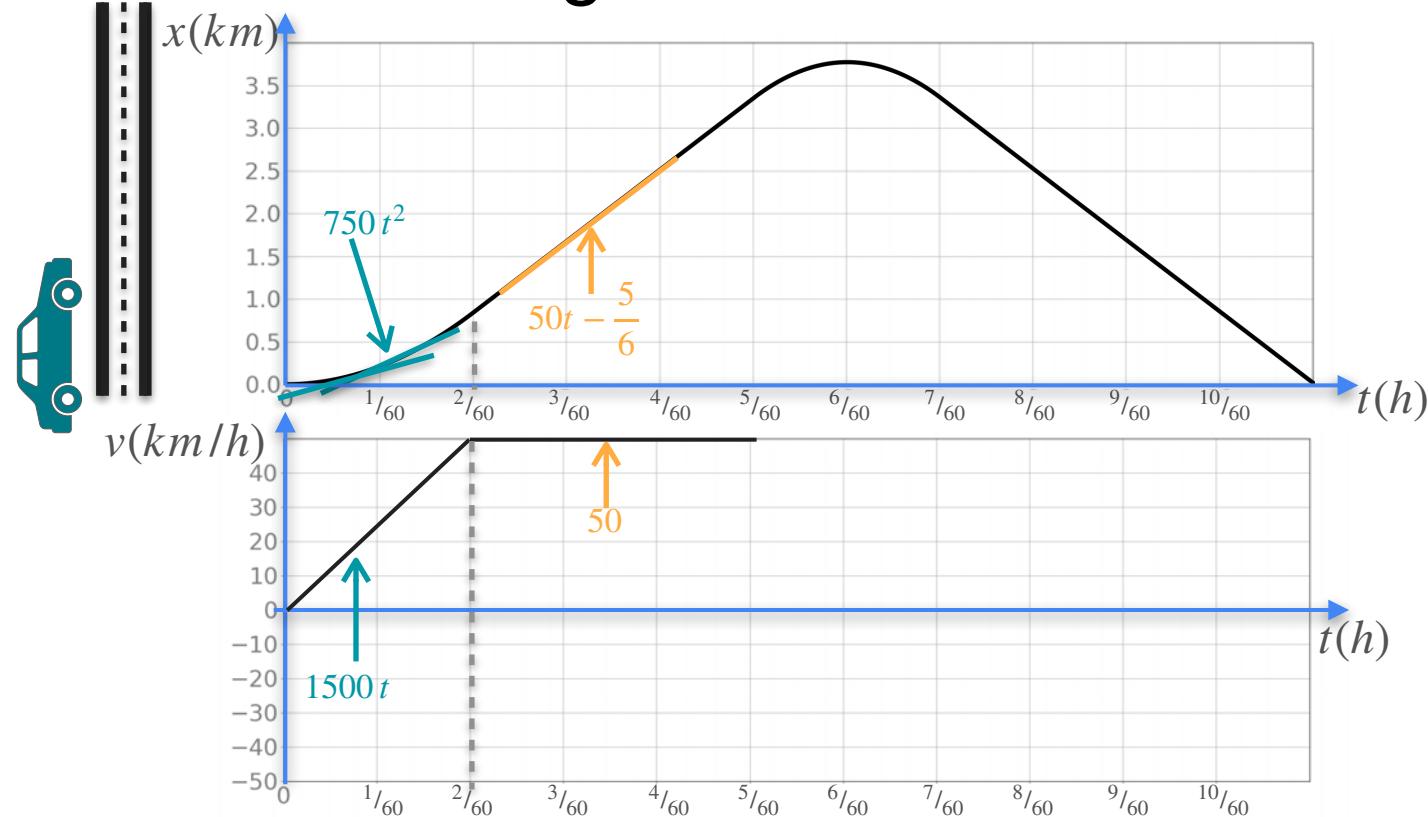


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

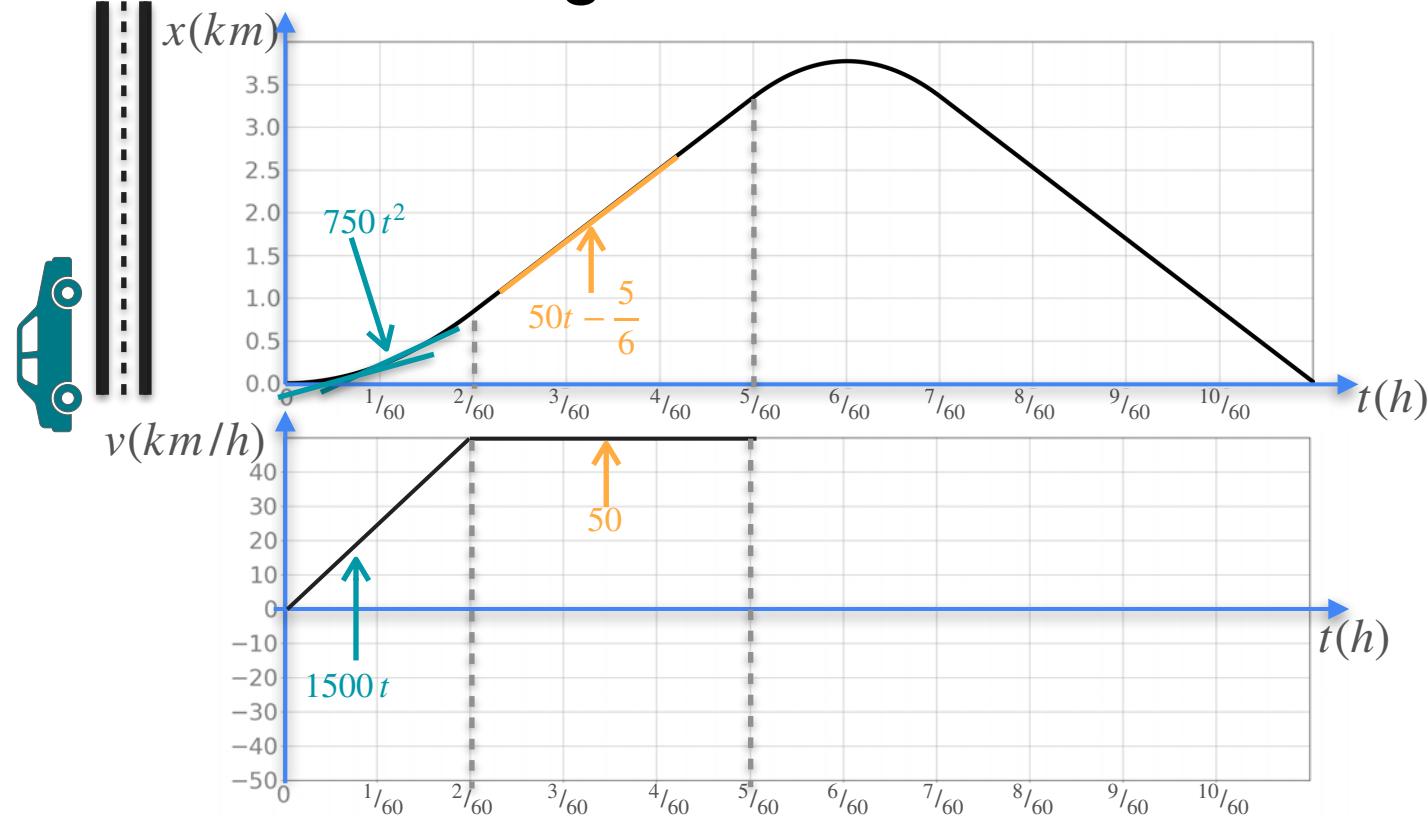


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

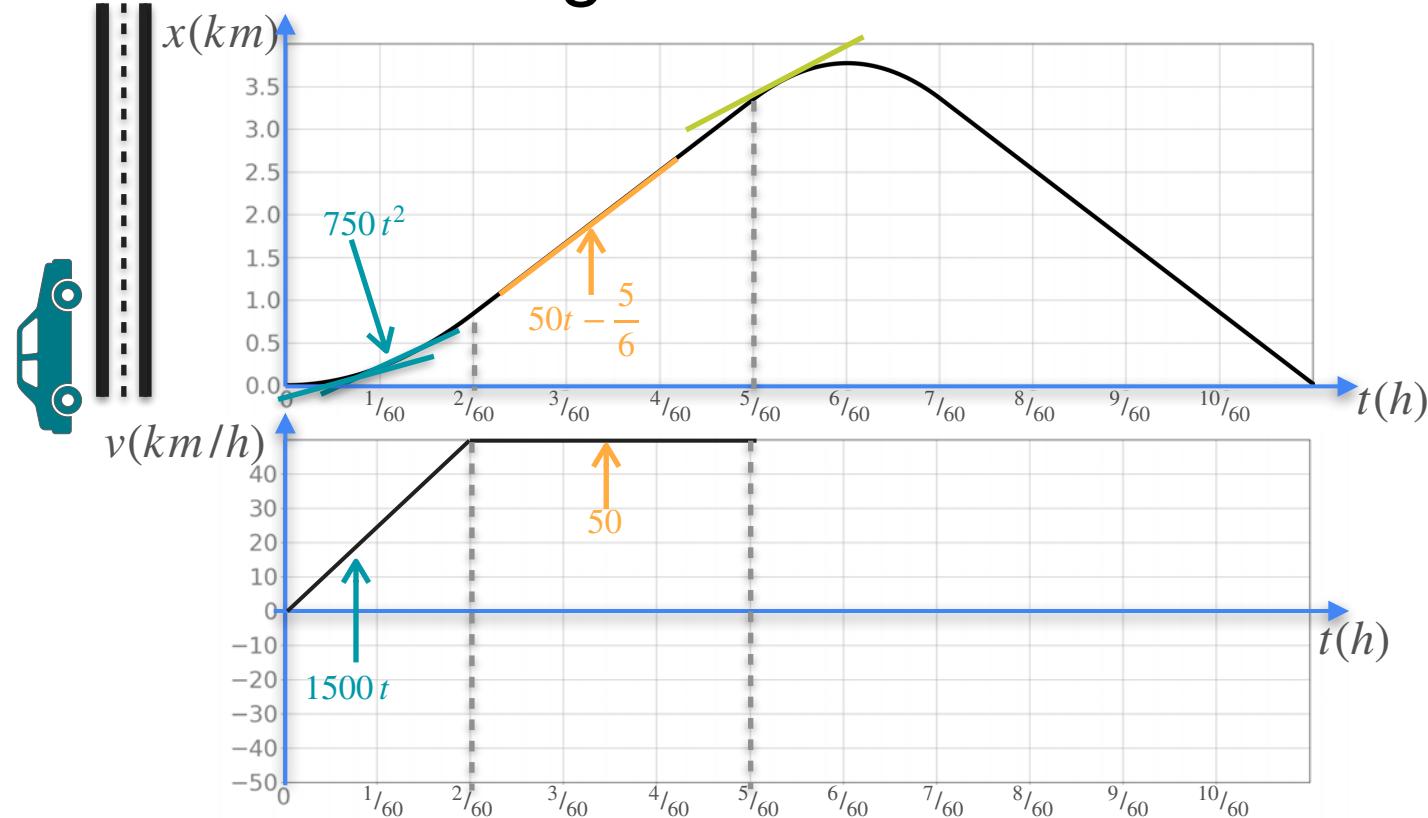


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

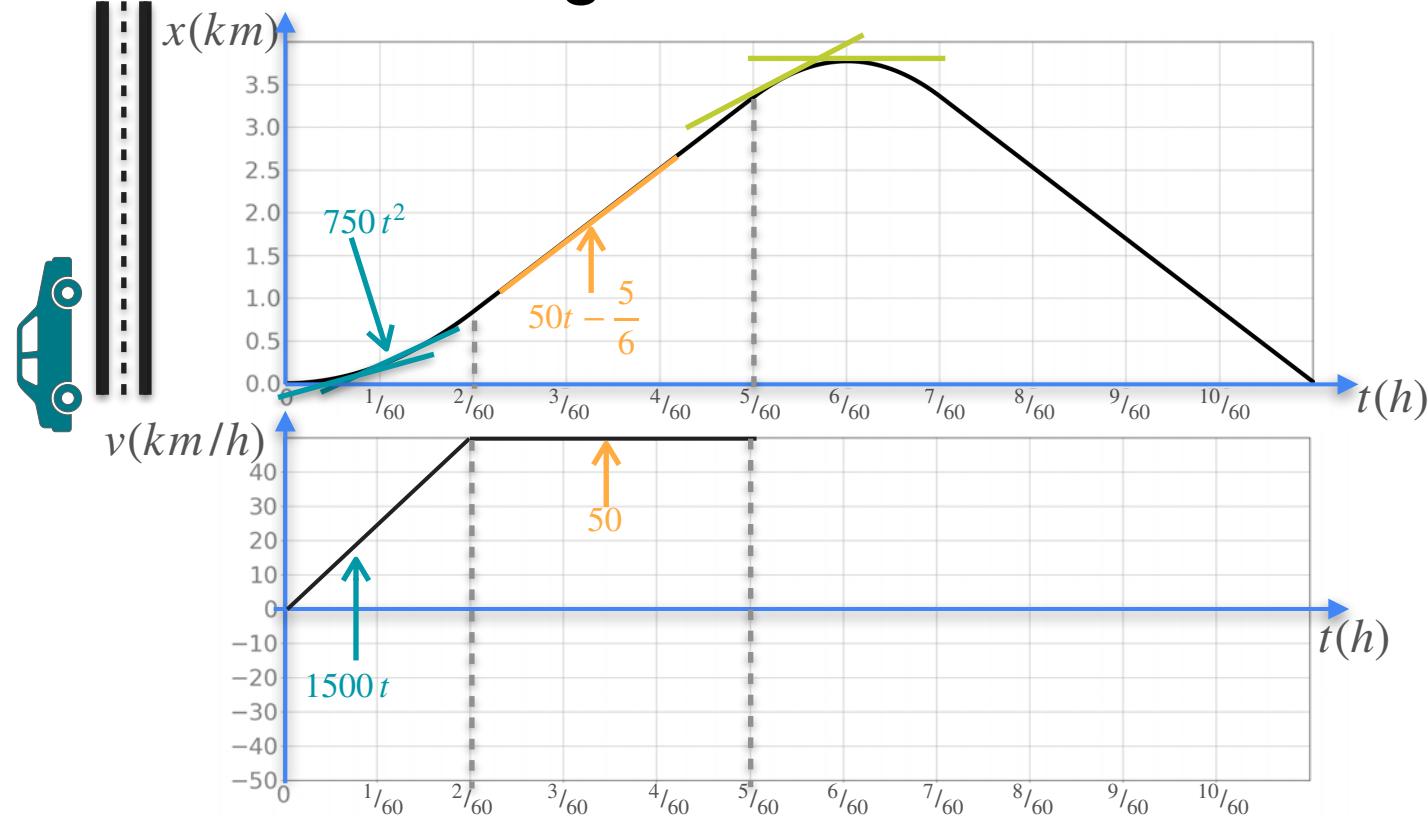


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

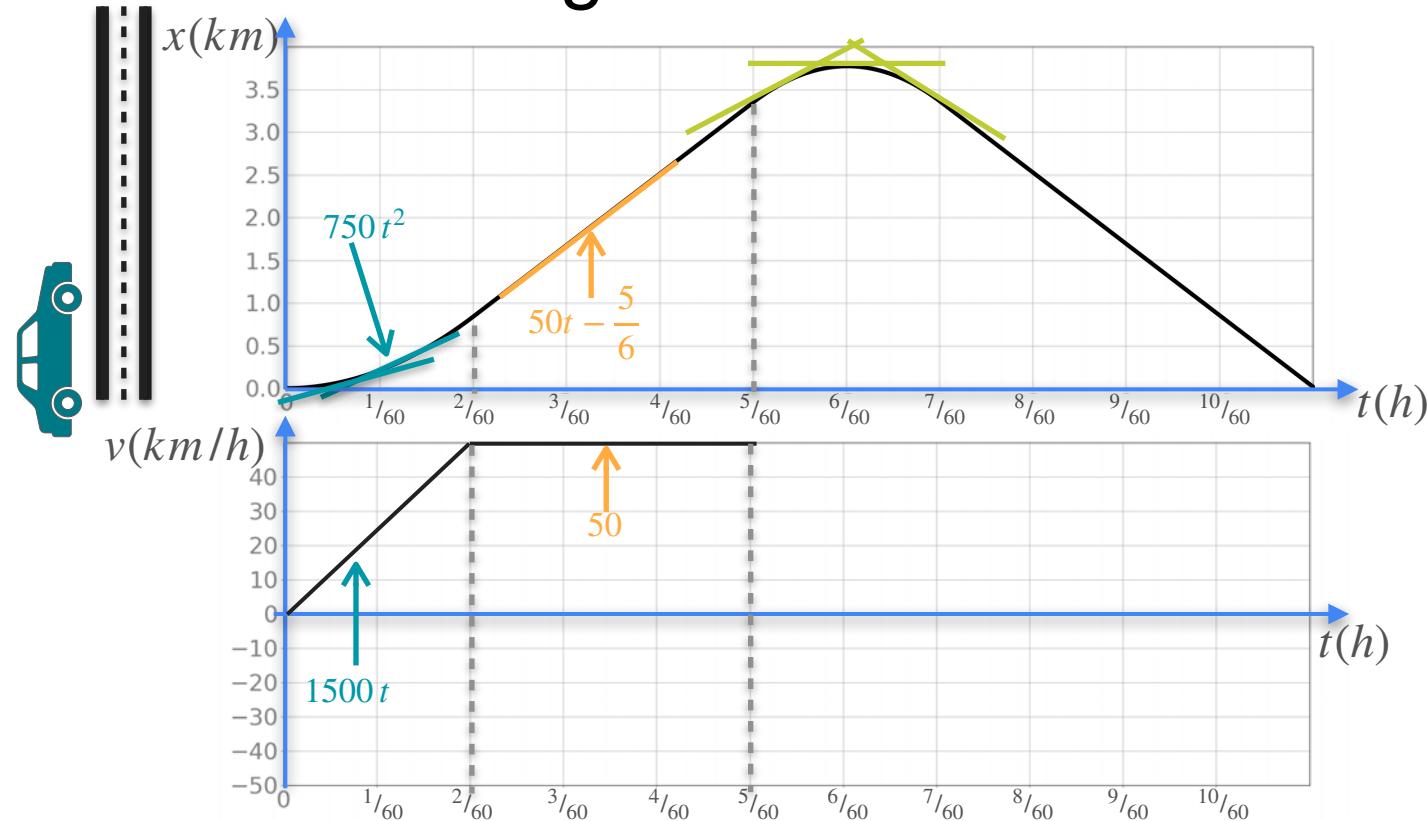


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

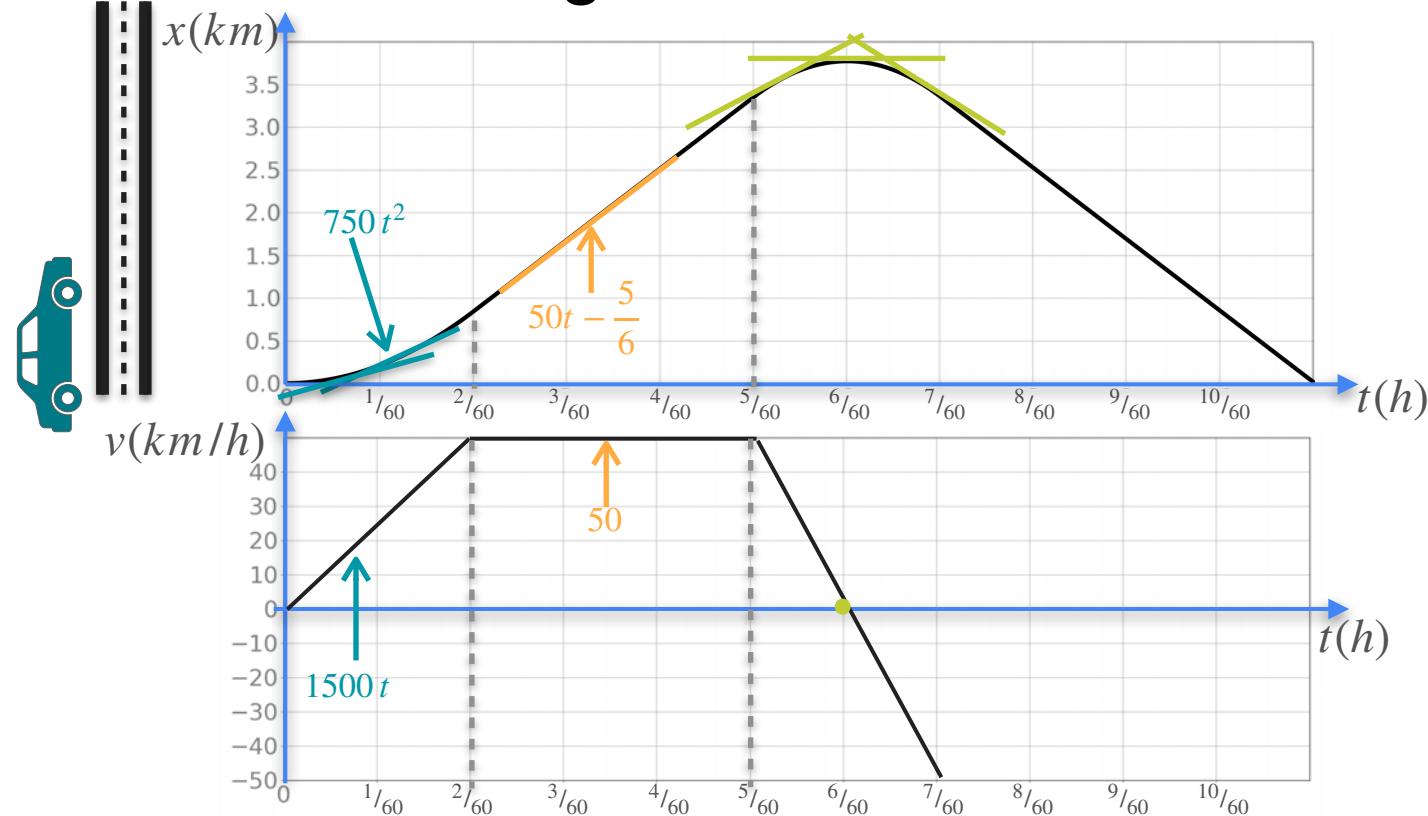


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

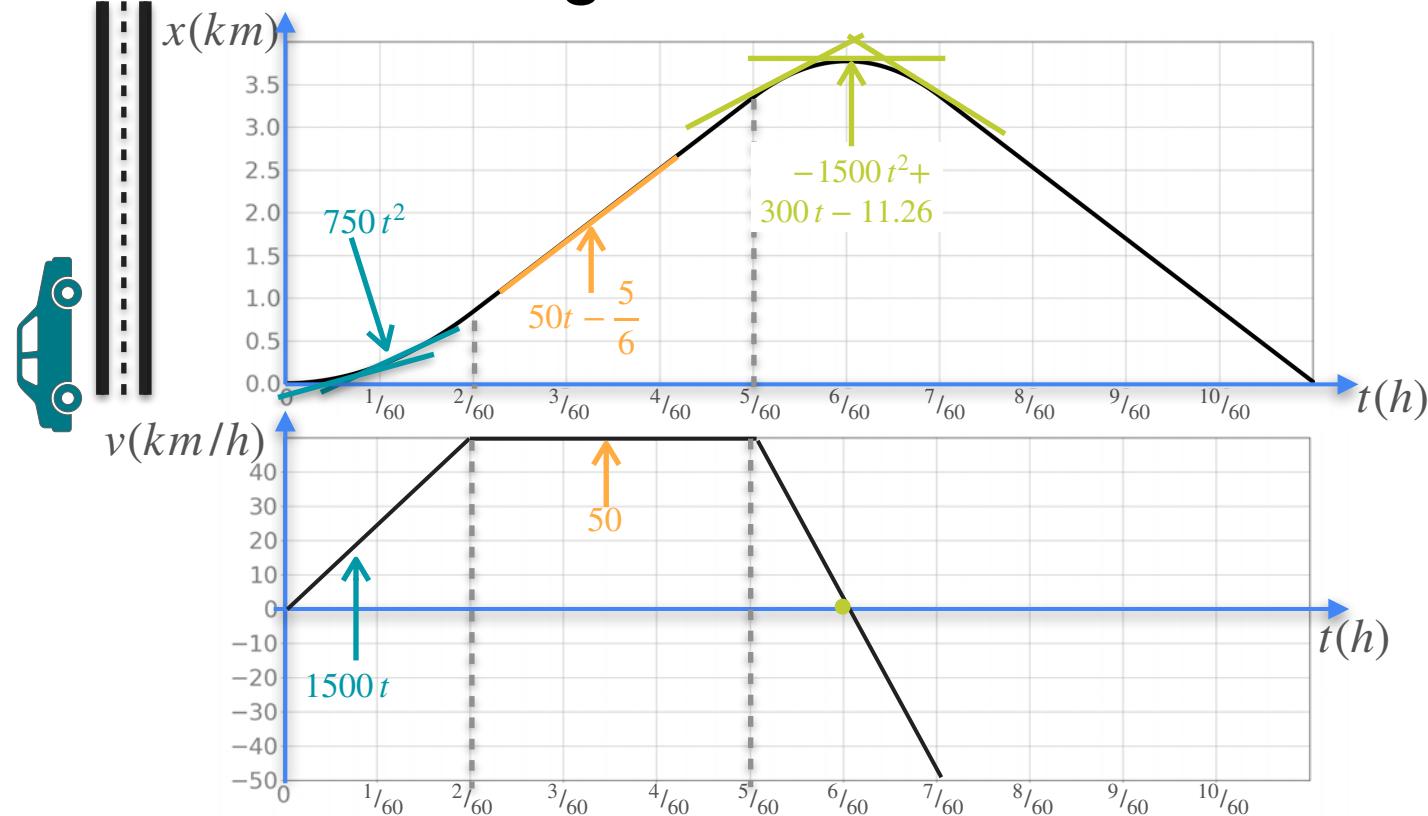


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

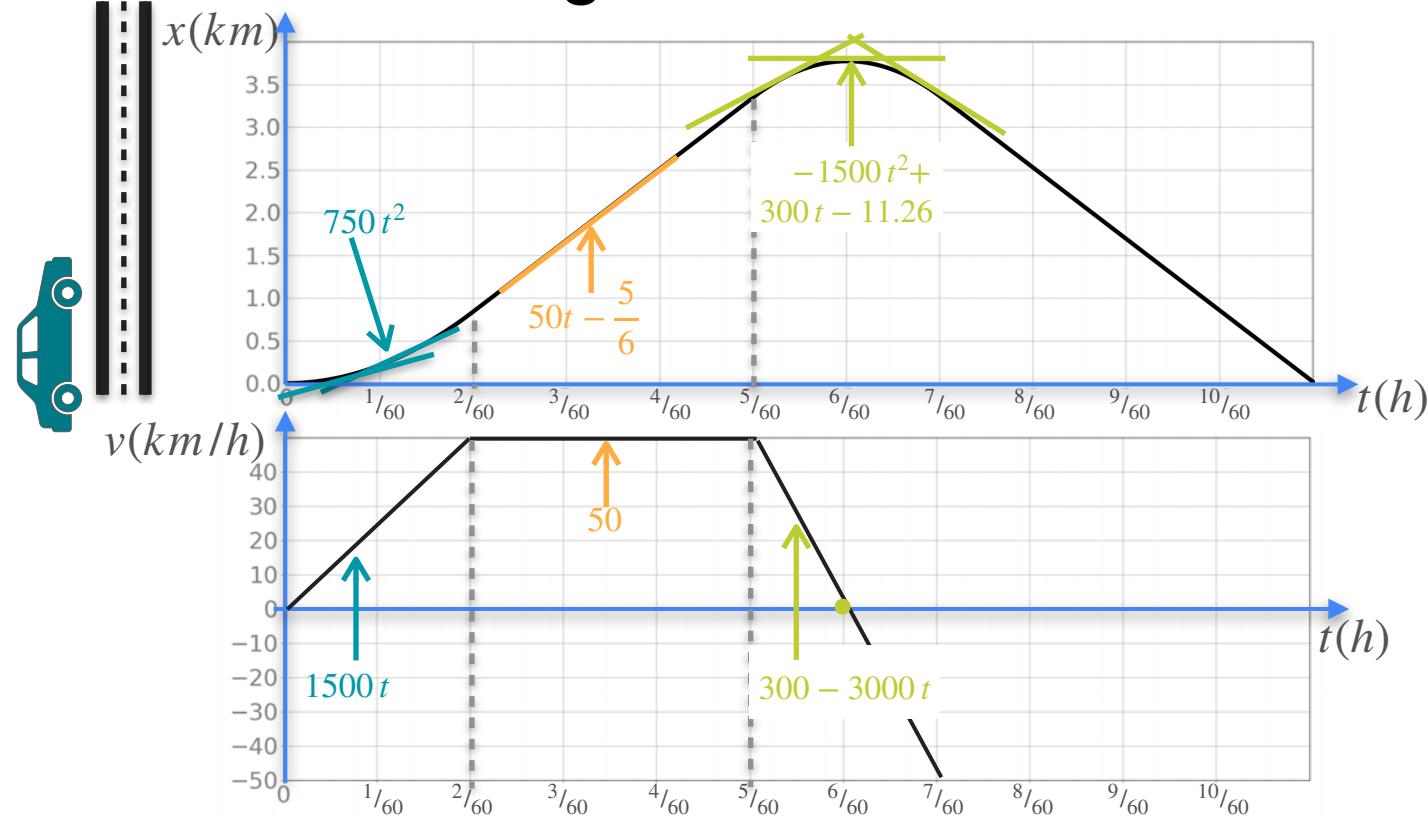


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

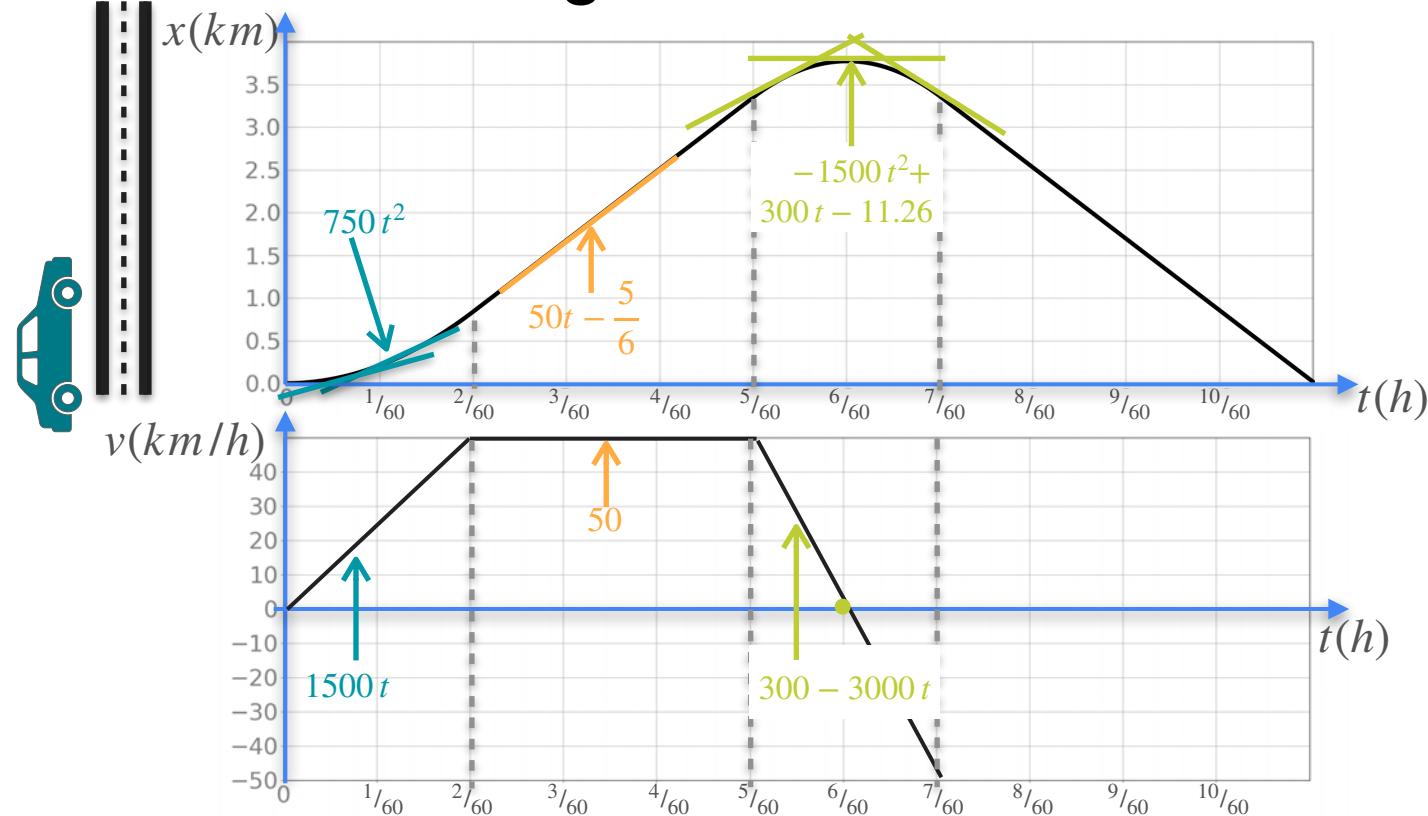


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

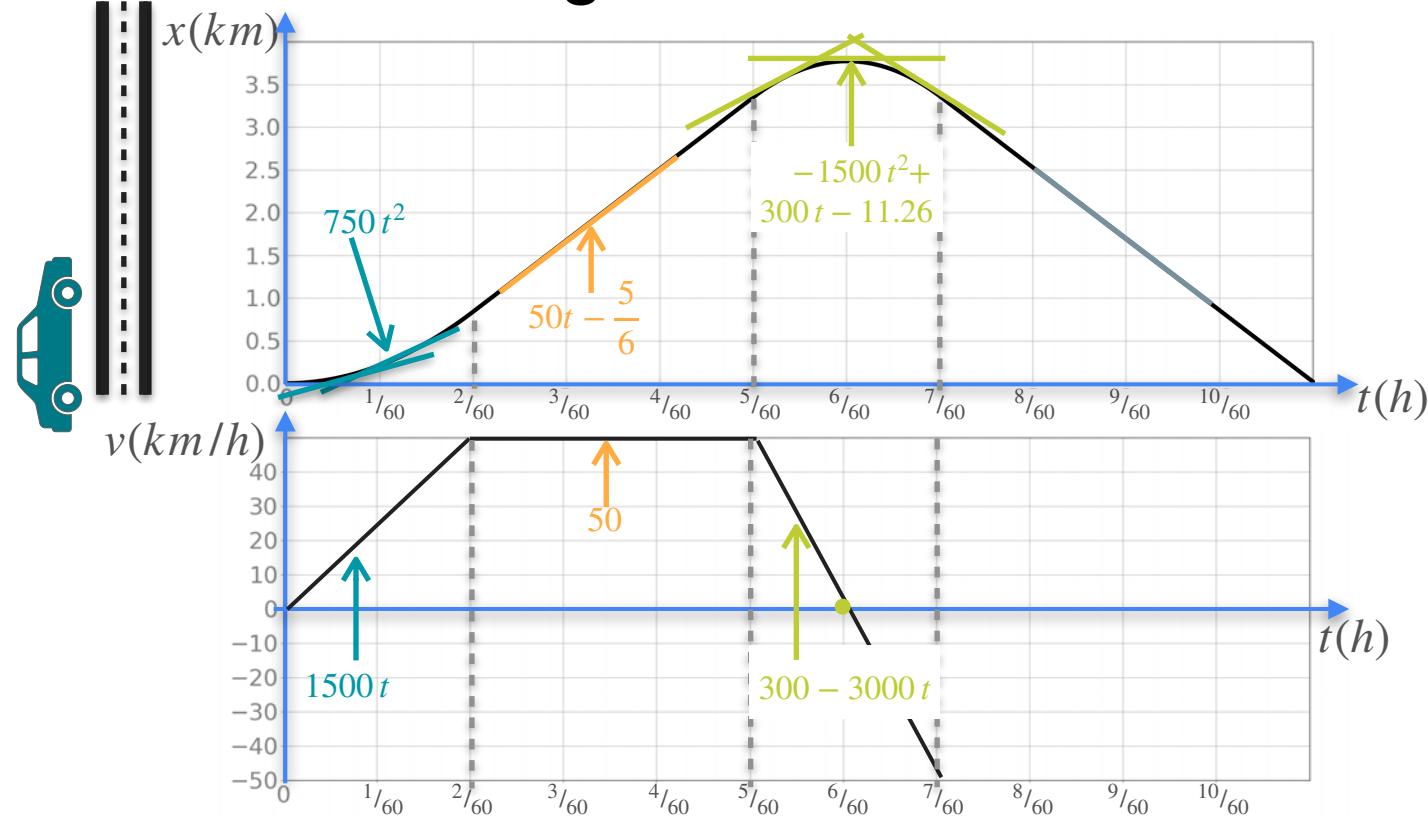


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

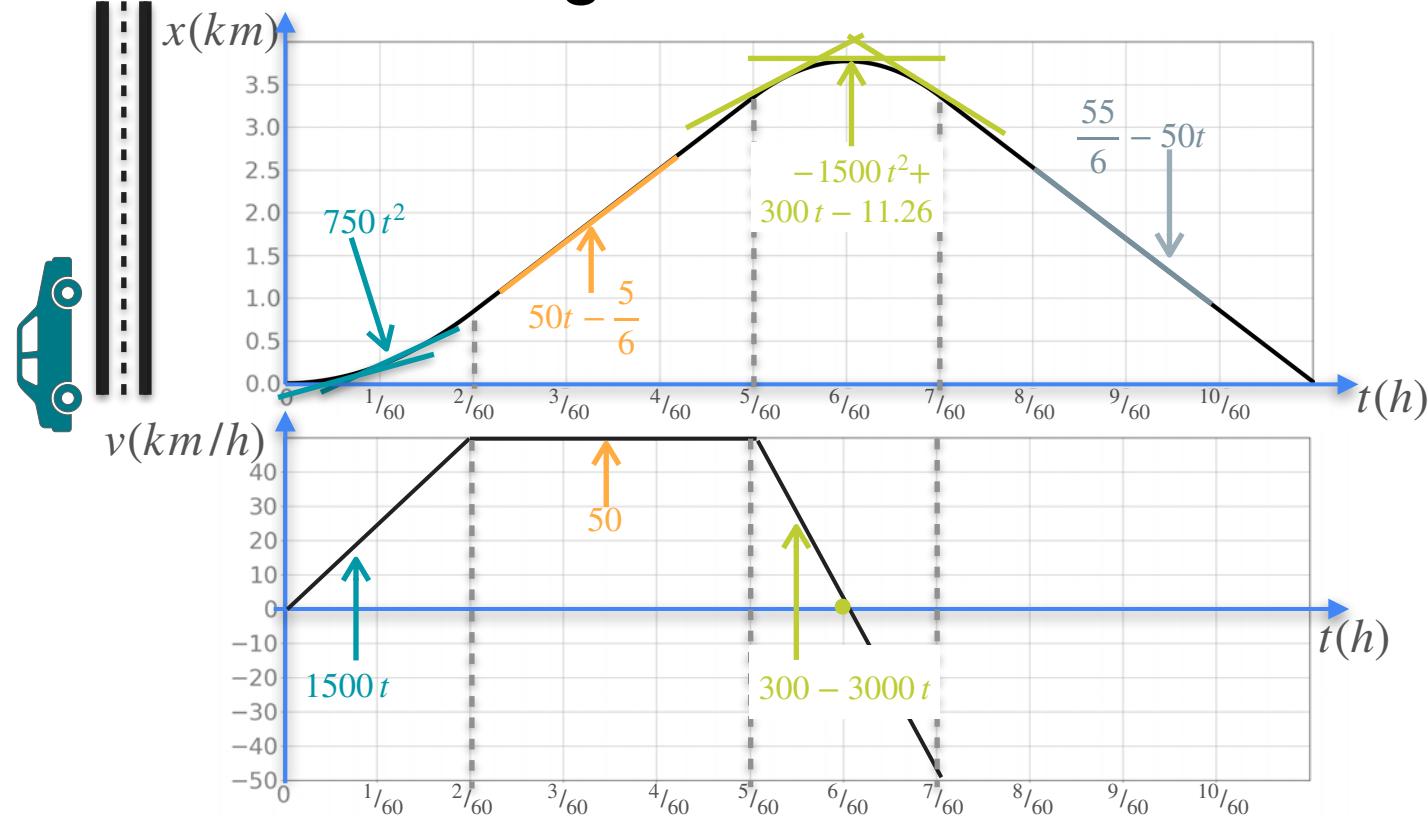


\mathcal{X} Distance

\mathcal{V} Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

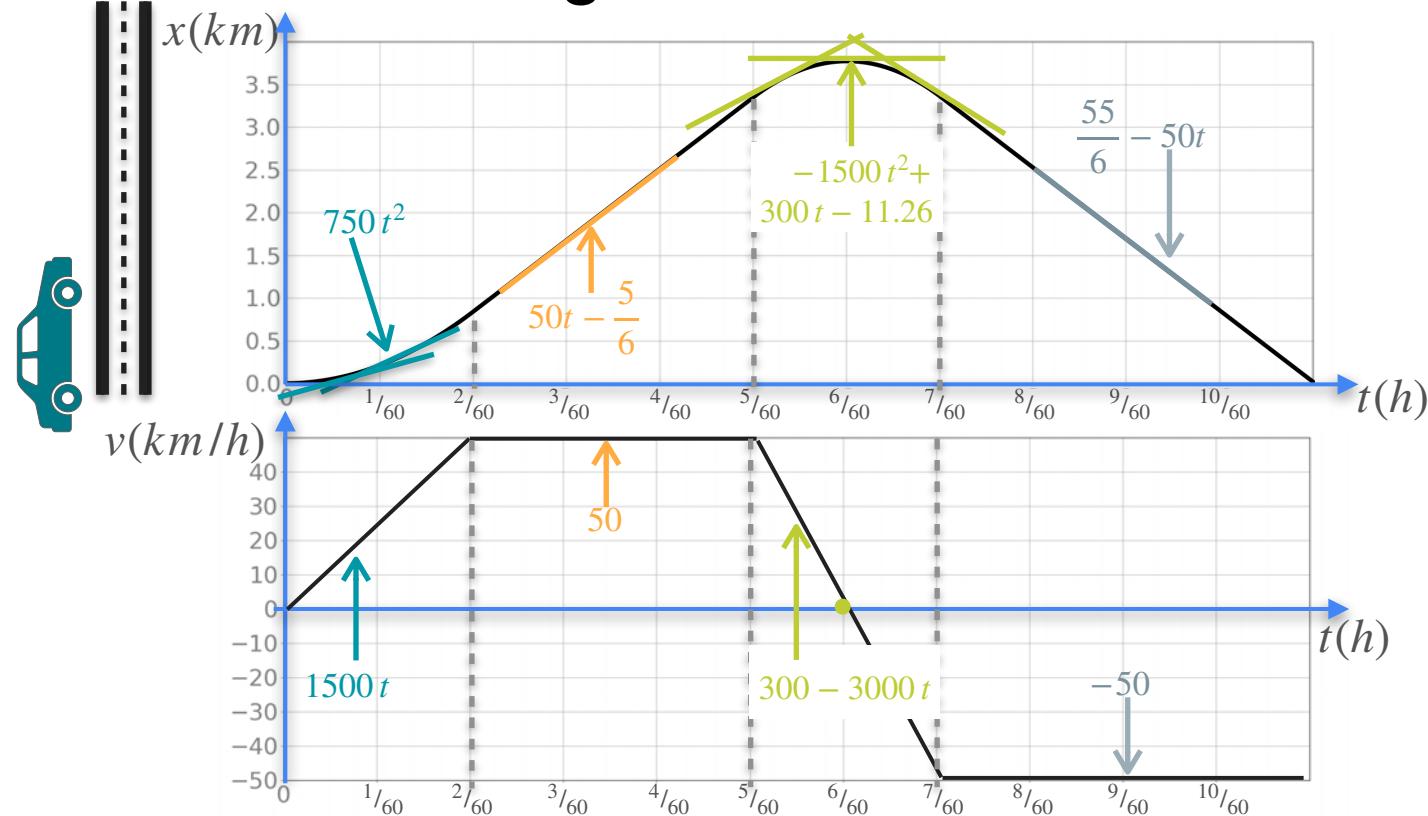


x Distance

v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative

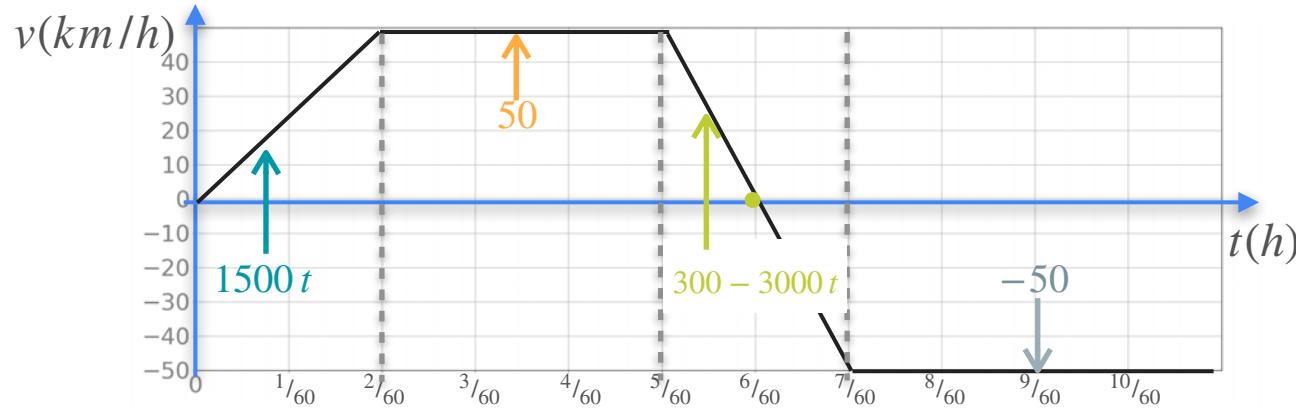


x Distance

v Velocity

$$\frac{dx}{dt}$$

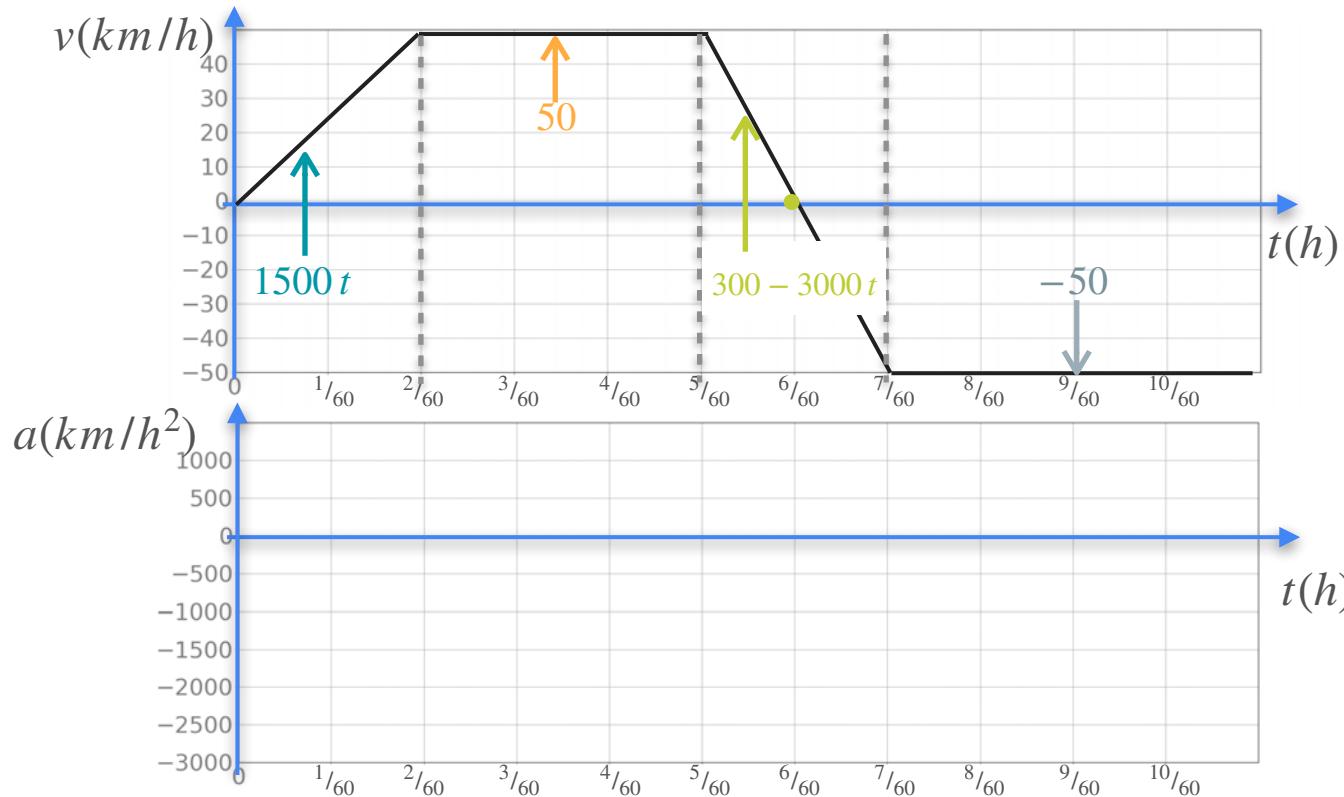
Understanding Second Derivative



v Velocity

$$\frac{dx}{dt}$$

Understanding Second Derivative



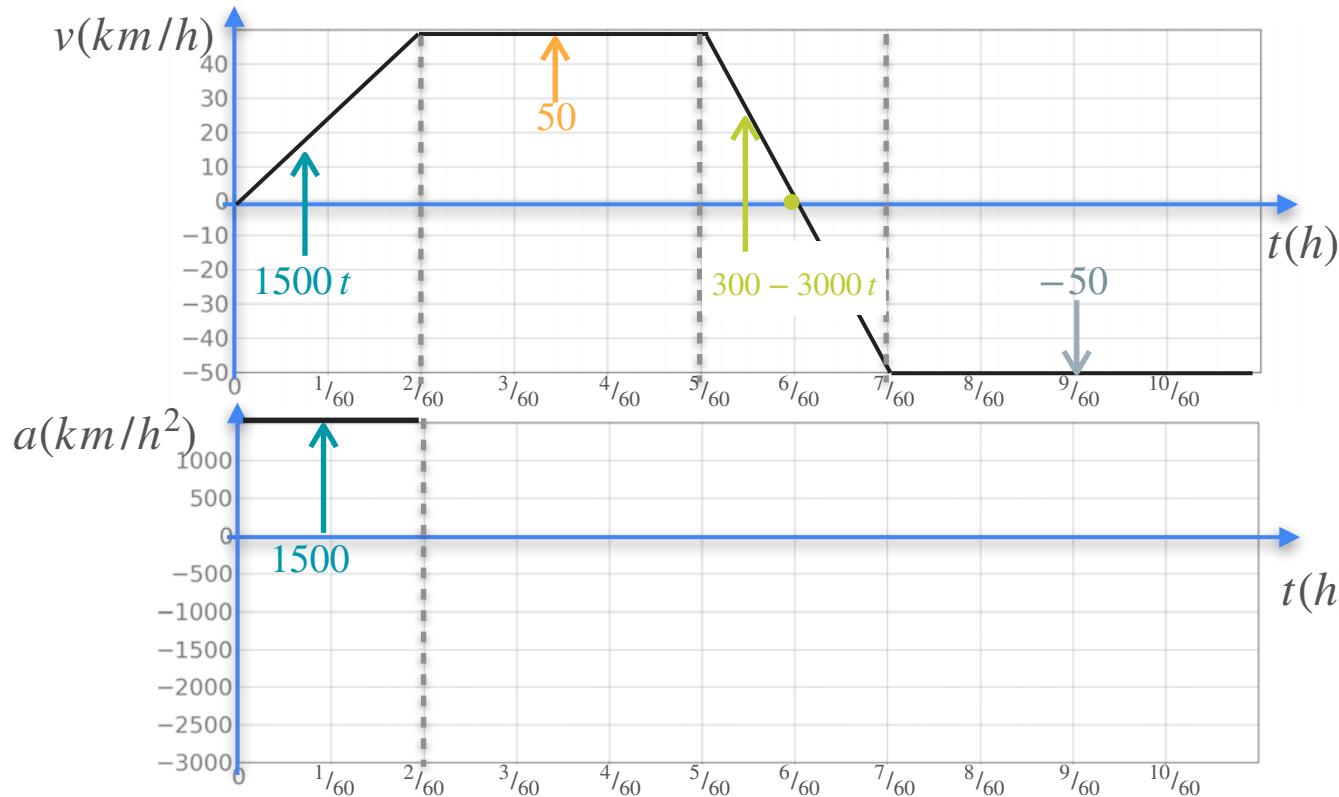
v Velocity

$$\frac{dx}{dt}$$

a Acceleration

$$\frac{dv}{dt}$$

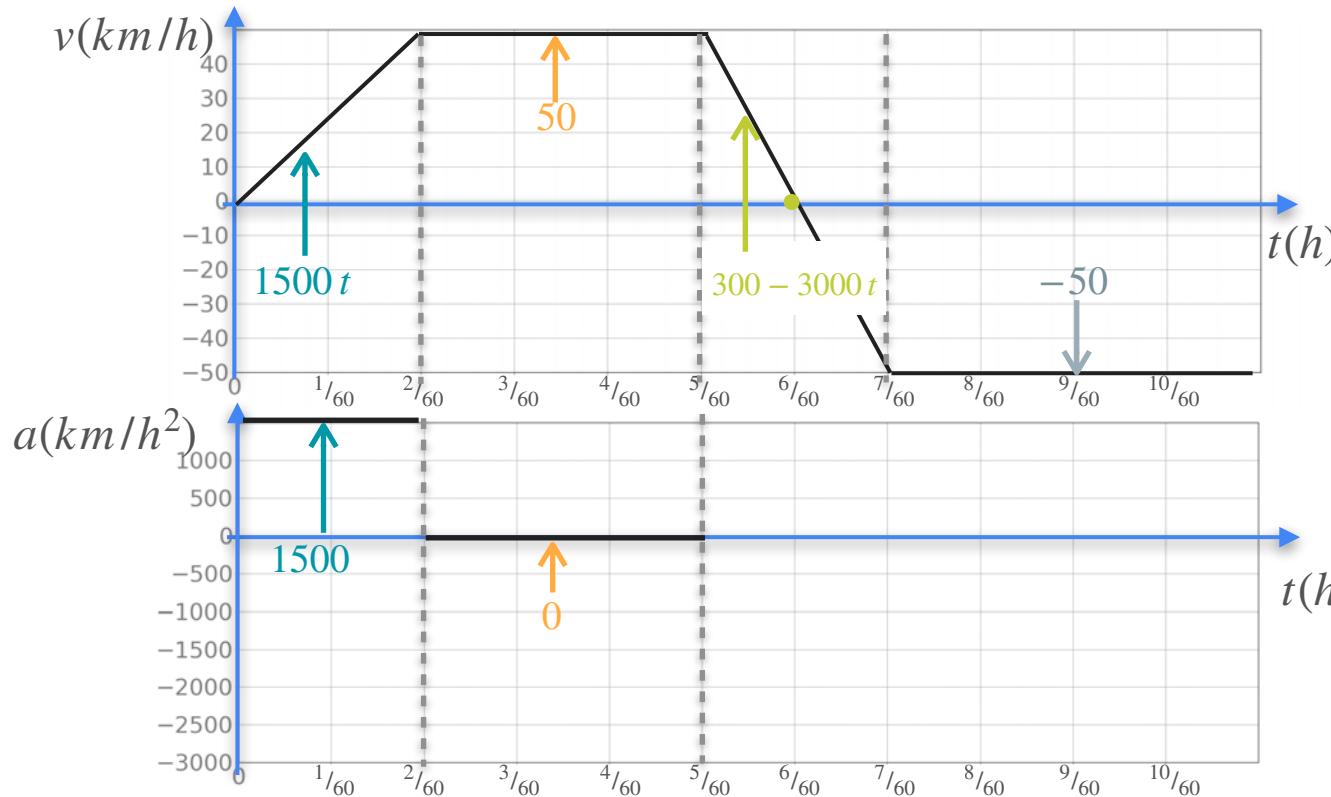
Understanding Second Derivative



v Velocity $\frac{dx}{dt}$

a Acceleration $\frac{dv}{dt}$

Understanding Second Derivative



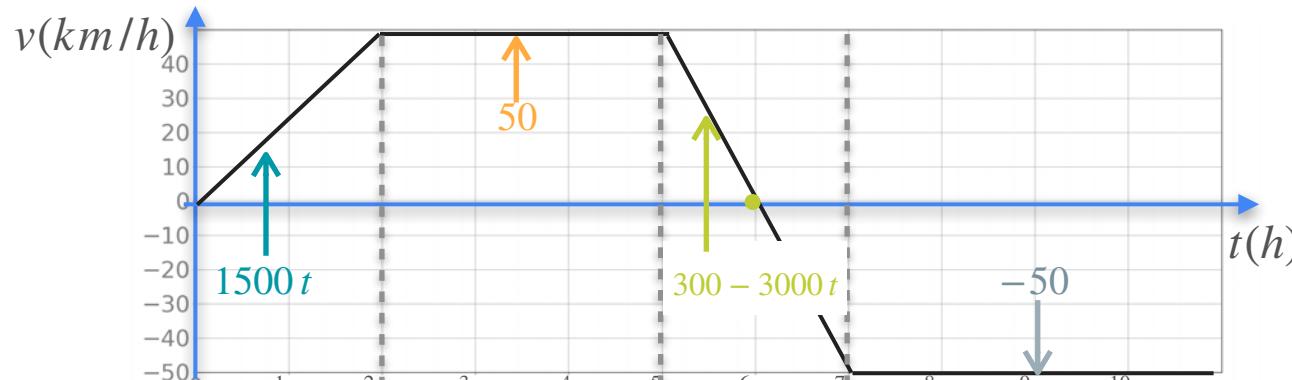
v Velocity

$$\frac{dx}{dt}$$

a Acceleration

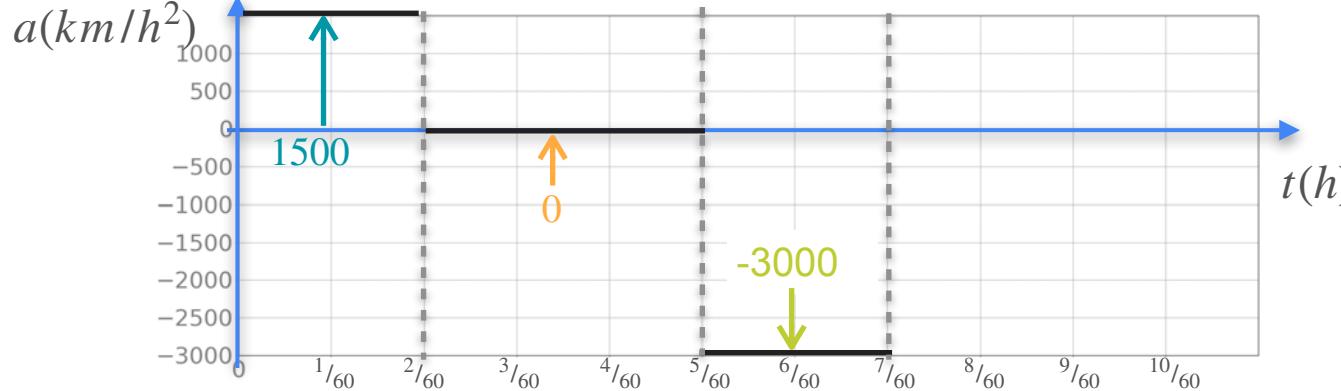
$$\frac{dv}{dt}$$

Understanding Second Derivative



v Velocity

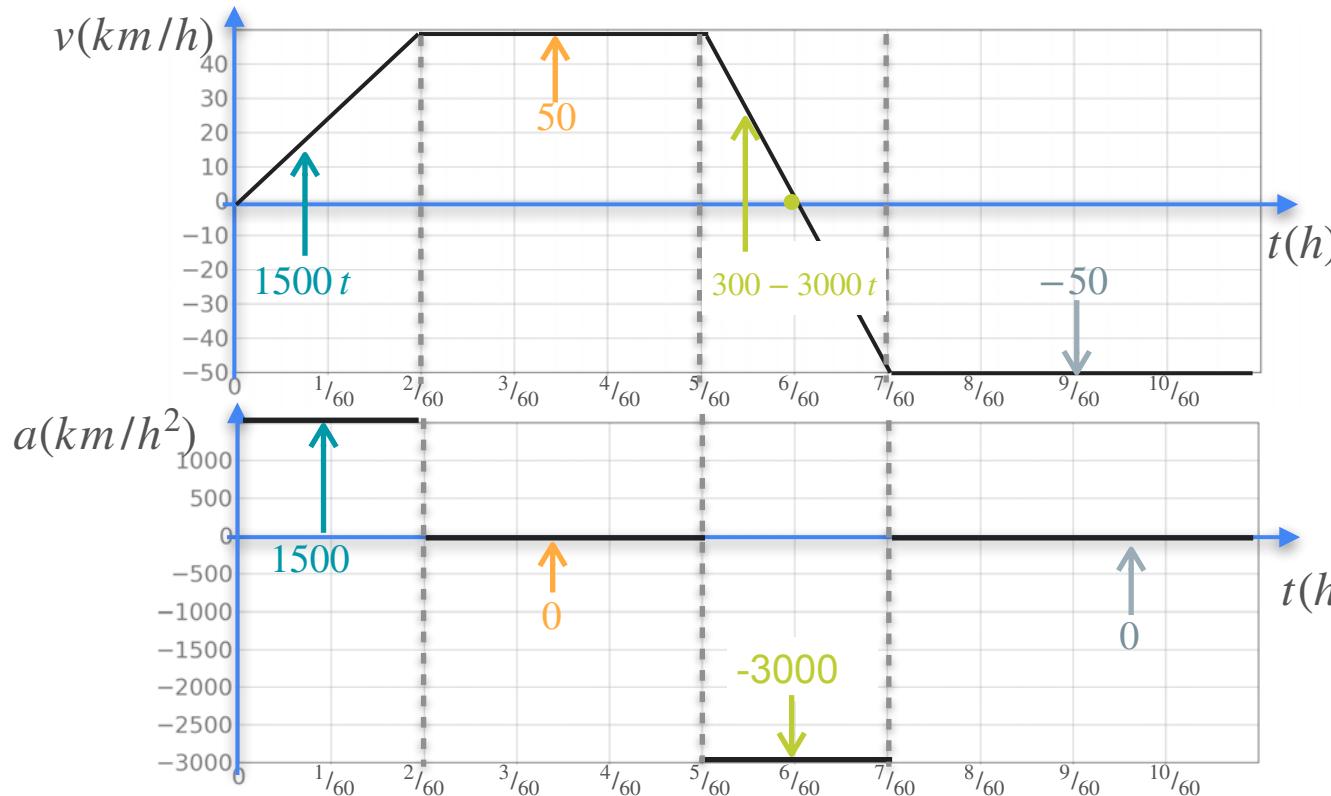
$$\frac{dx}{dt}$$



a Acceleration

$$\frac{dv}{dt}$$

Understanding Second Derivative



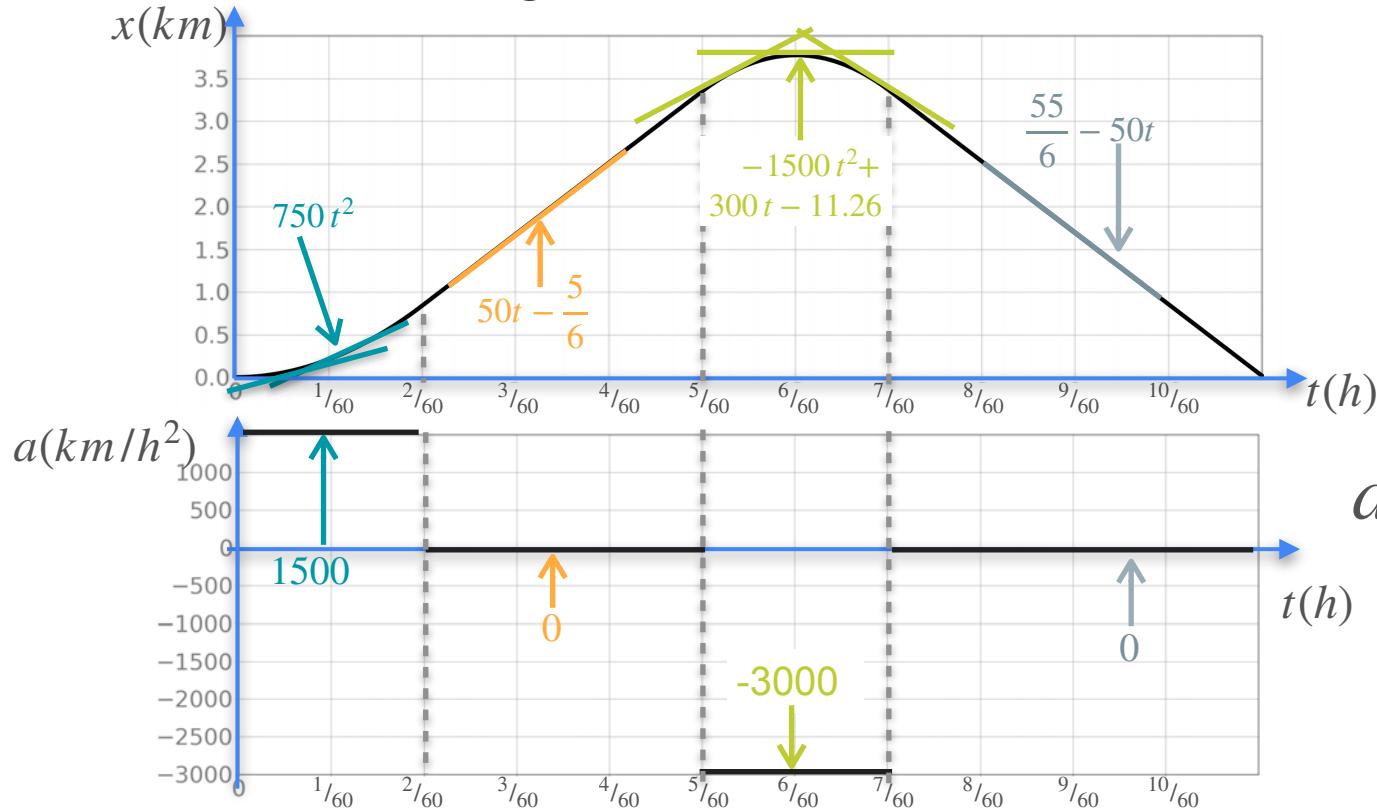
v Velocity

$$\frac{dx}{dt}$$

a Acceleration

$$\frac{dv}{dt}$$

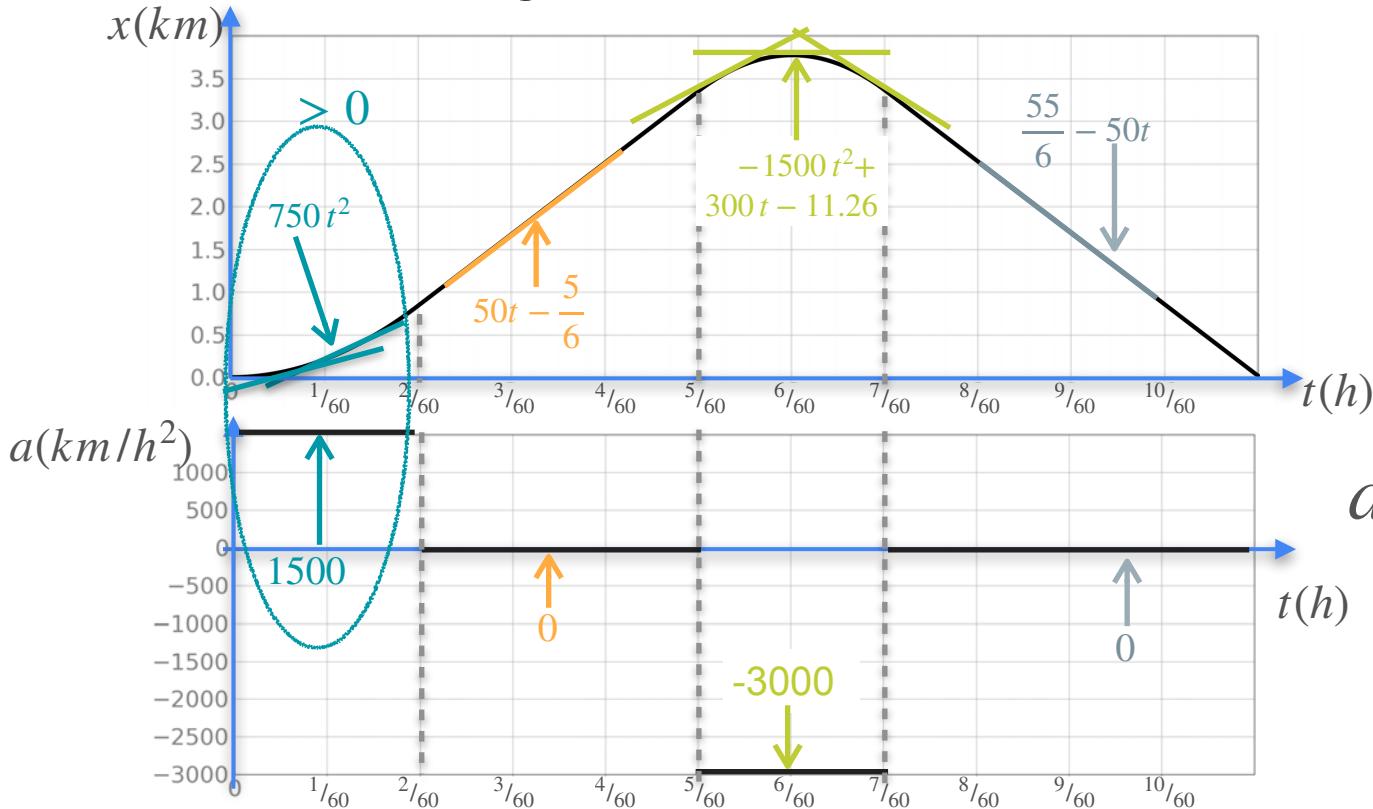
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

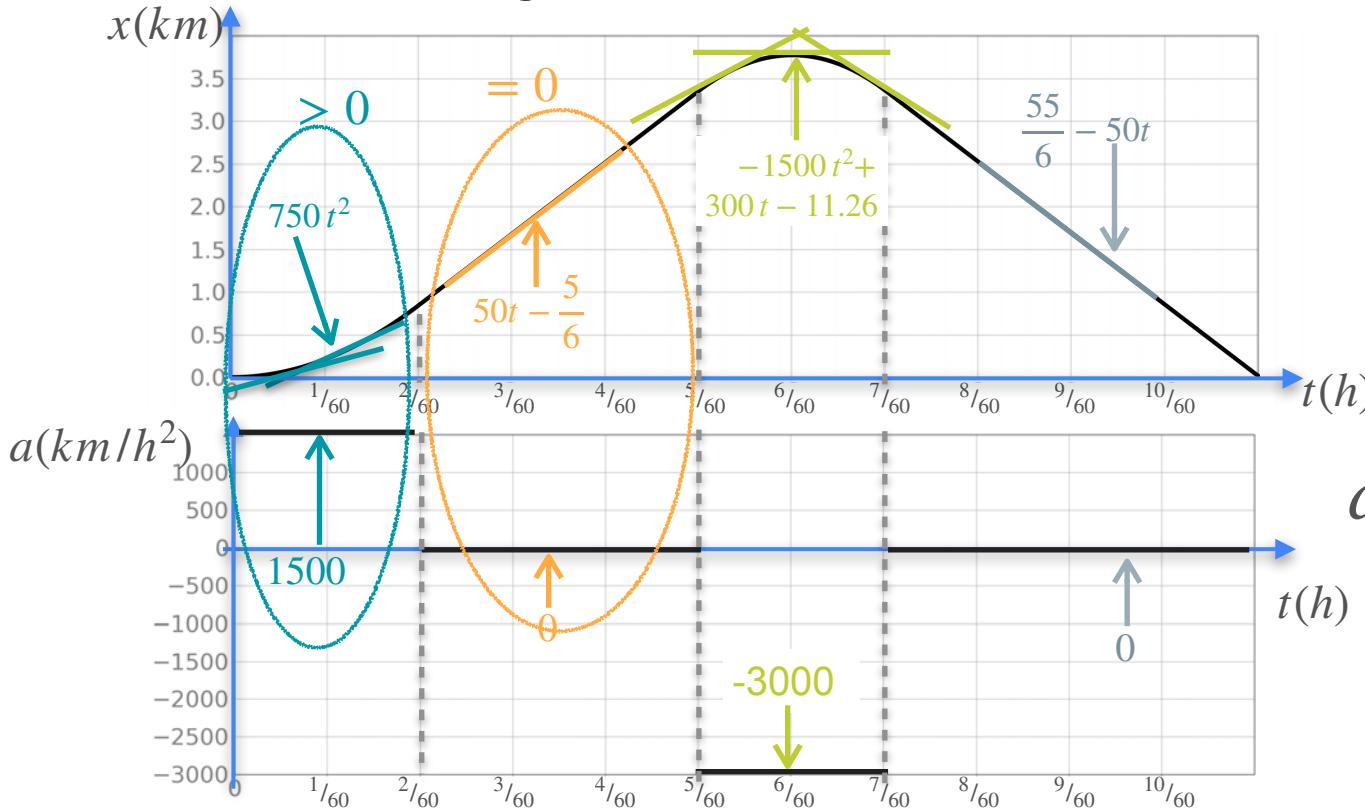
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

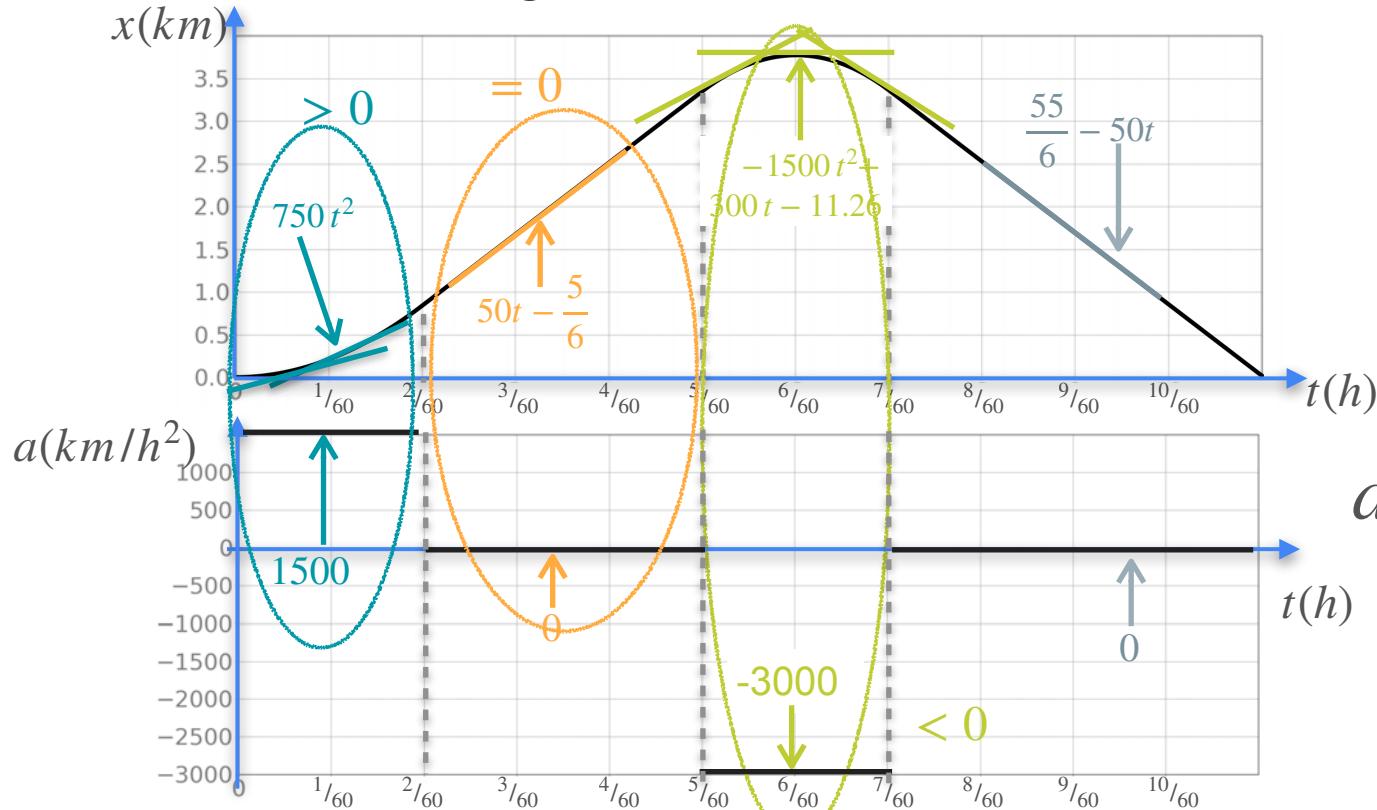
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

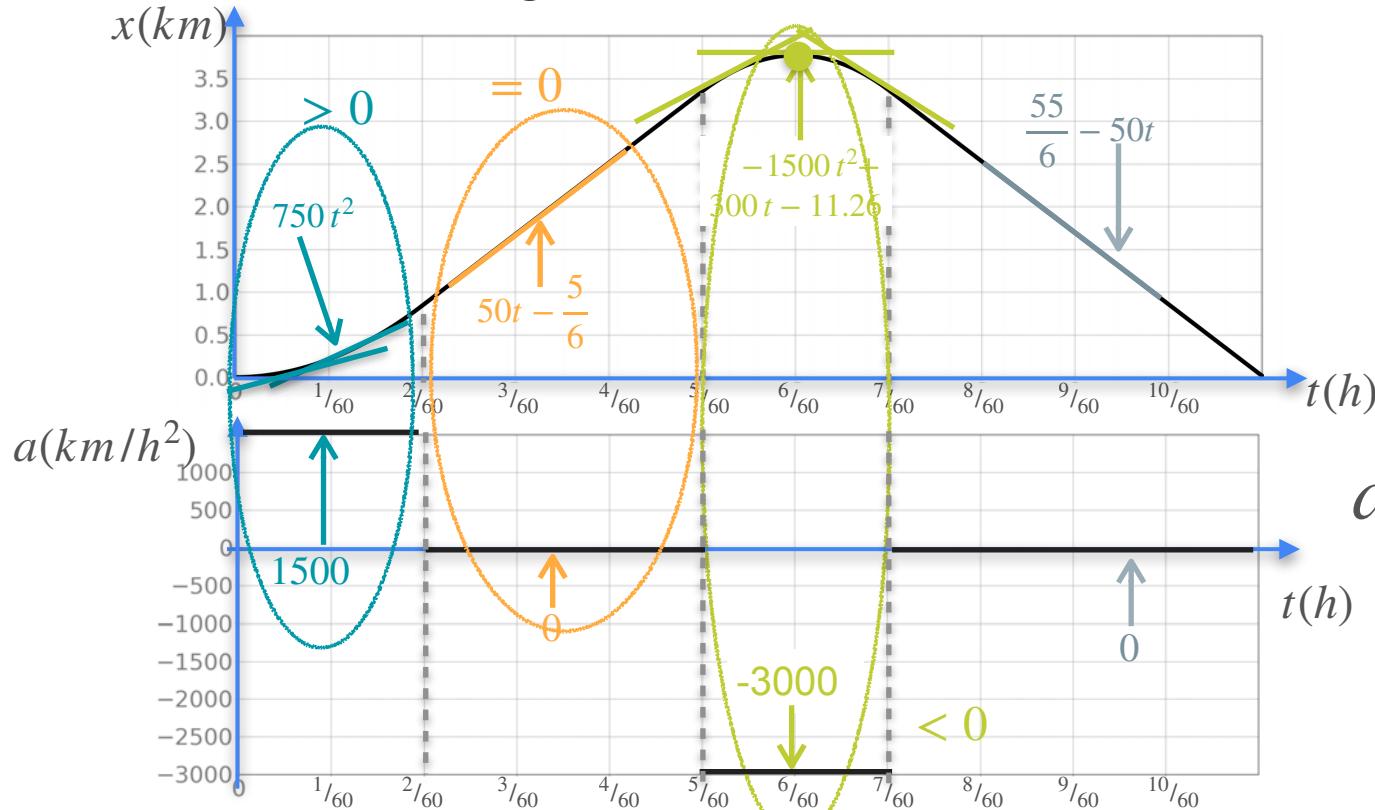
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

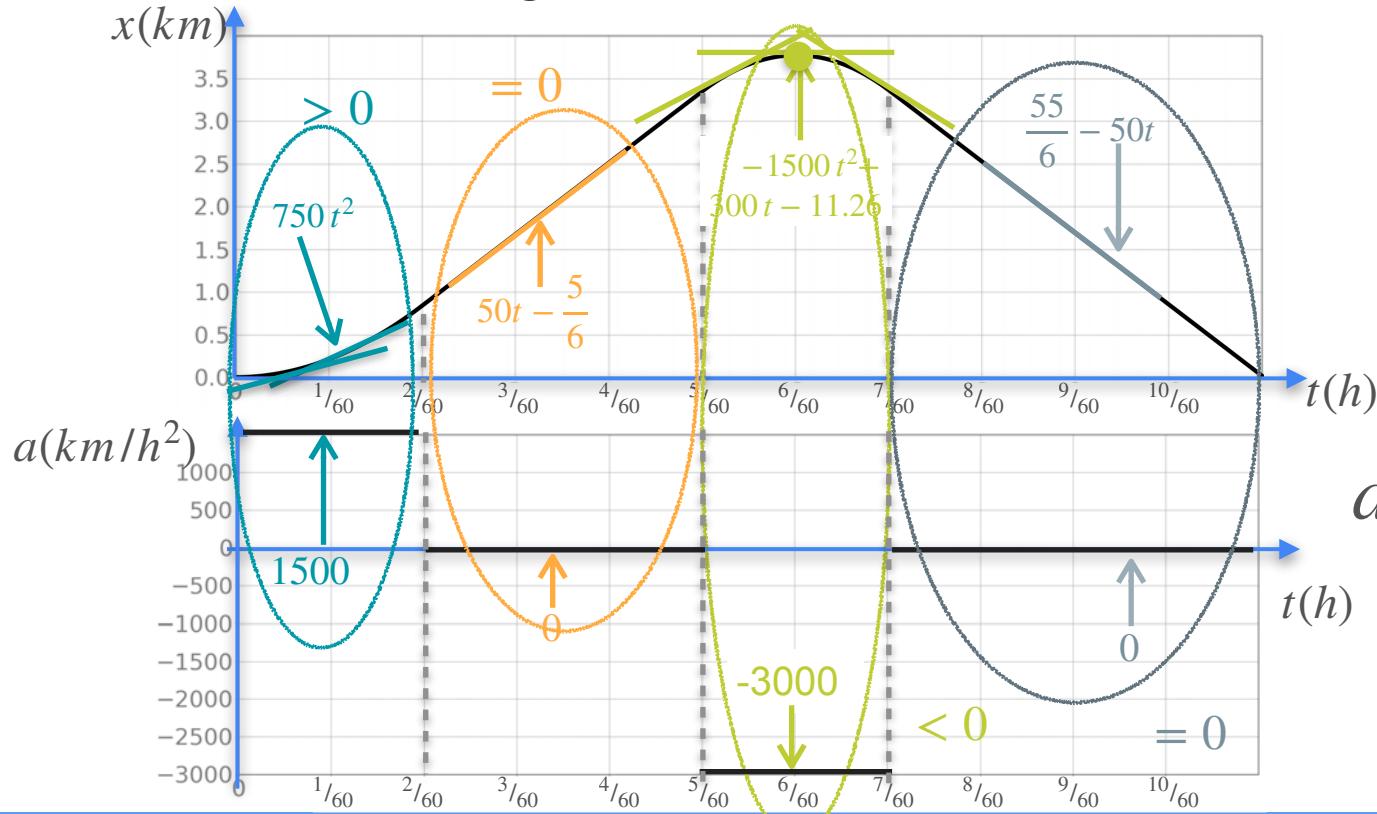
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

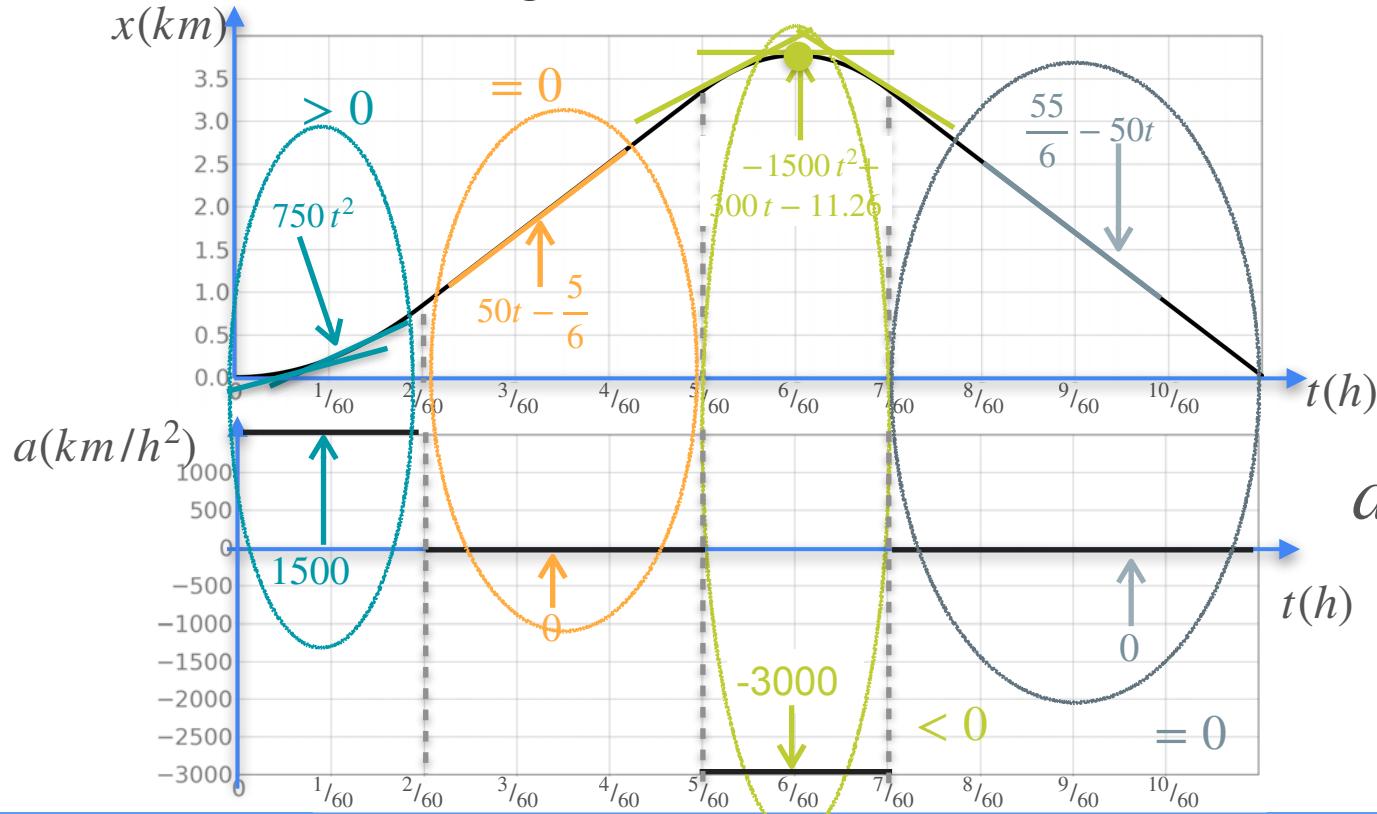
Understanding Second Derivative



x Distance

a Acceleration $\frac{d^2x}{dt^2}$

Understanding Second Derivative



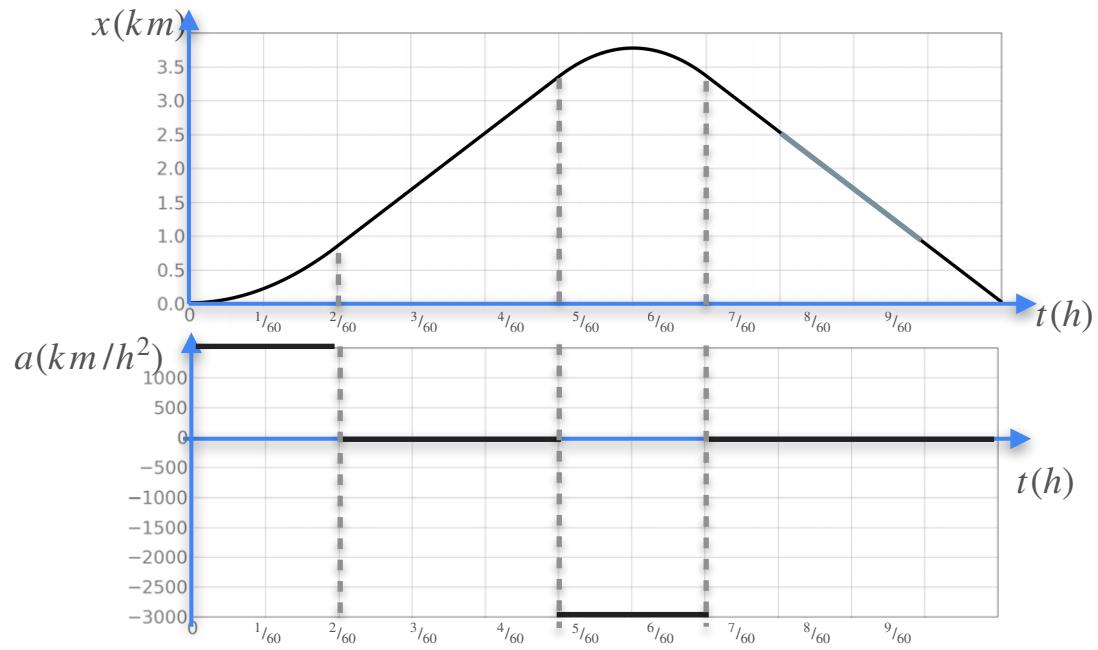
x Distance

Second derivative tells us about the curvature

a Acceleration $\frac{d^2x}{dt^2}$

Curvature

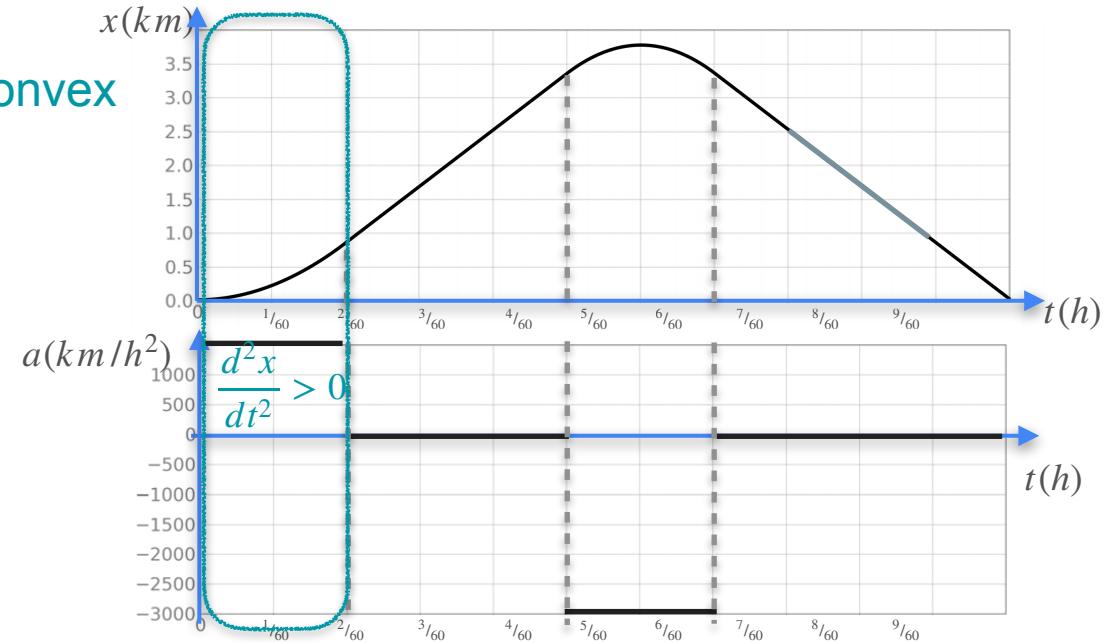
Curvature



Curvature

$$\frac{d^2x}{dt^2} > 0$$

Concave up or convex



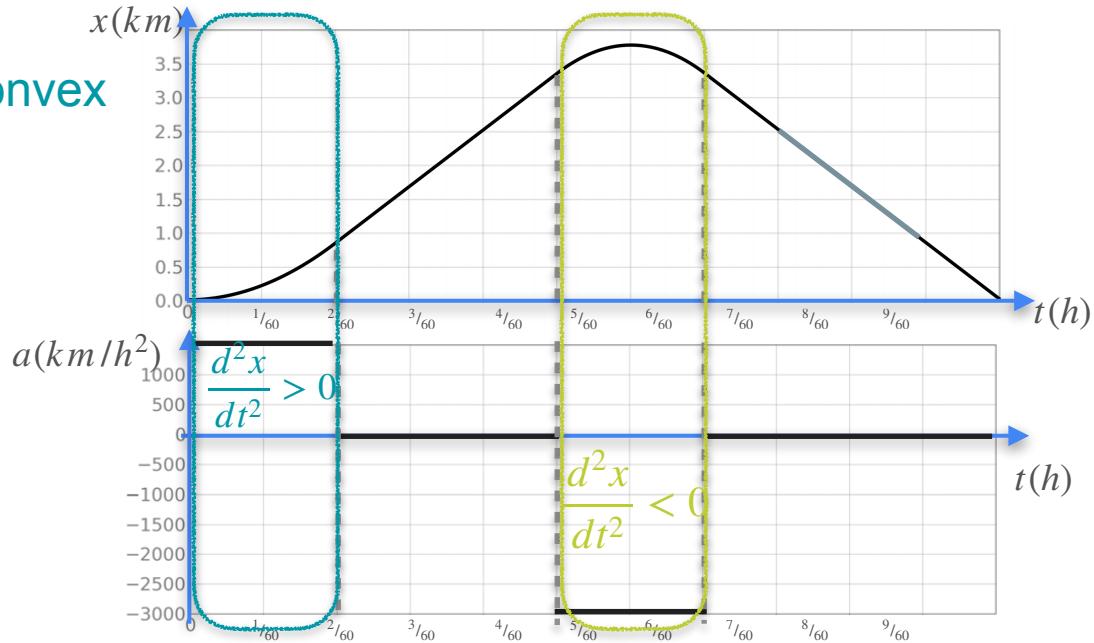
Curvature

$$\frac{d^2x}{dt^2} > 0$$

Concave up or convex

$$\frac{d^2x}{dt^2} < 0$$

Concave down



Curvature

$$\frac{d^2x}{dt^2} > 0$$

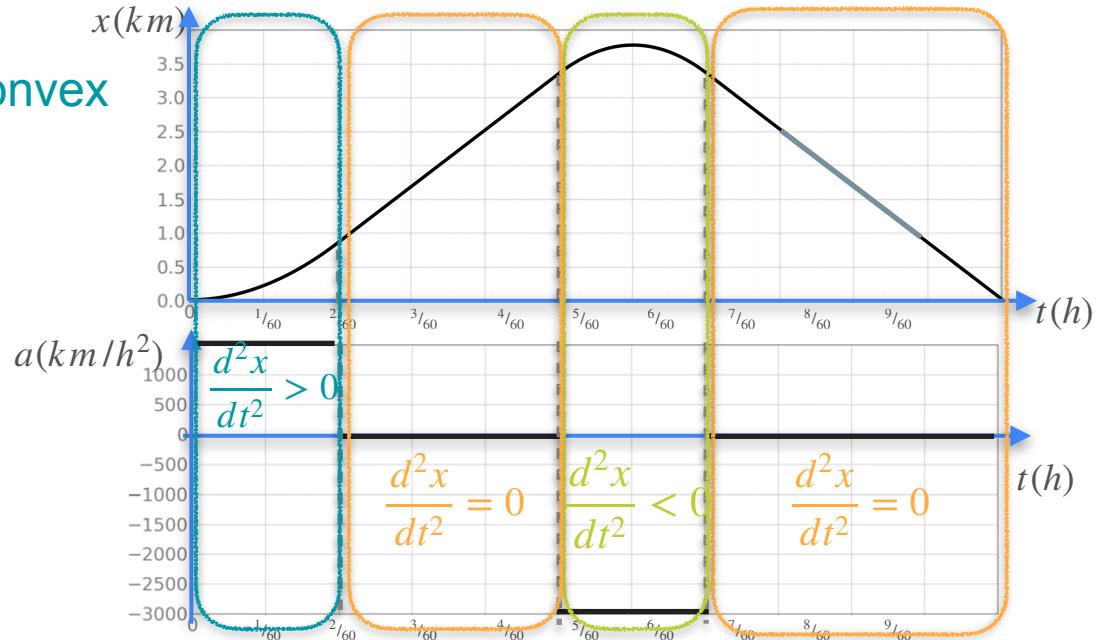
Concave up or convex

$$\frac{d^2x}{dt^2} < 0$$

Concave down

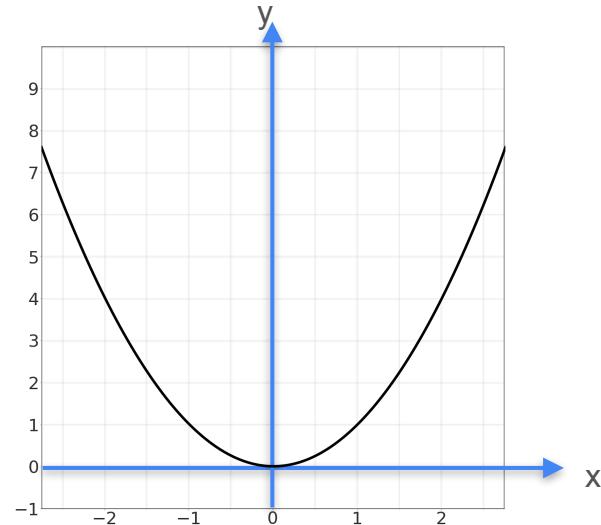
$$\frac{d^2x}{dt^2} = 0$$

Need more information

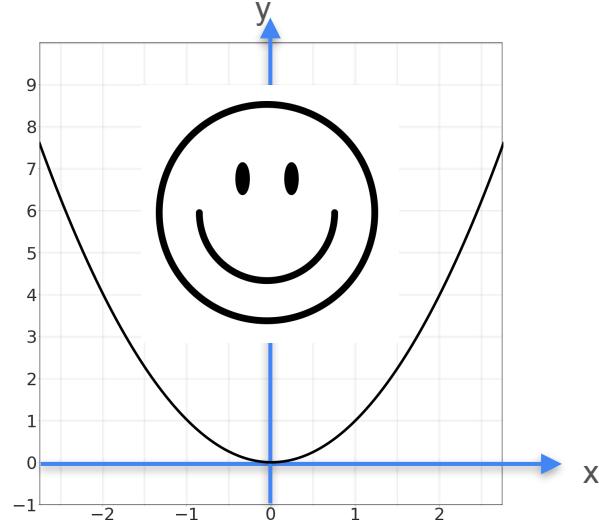


Curvature

Curvature



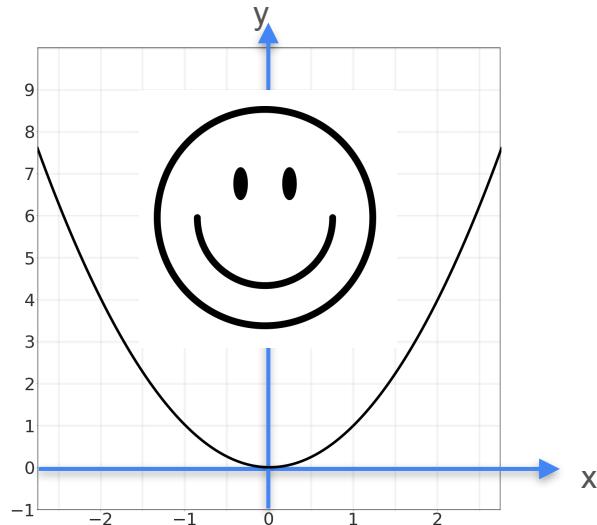
Curvature



Concave up or convex

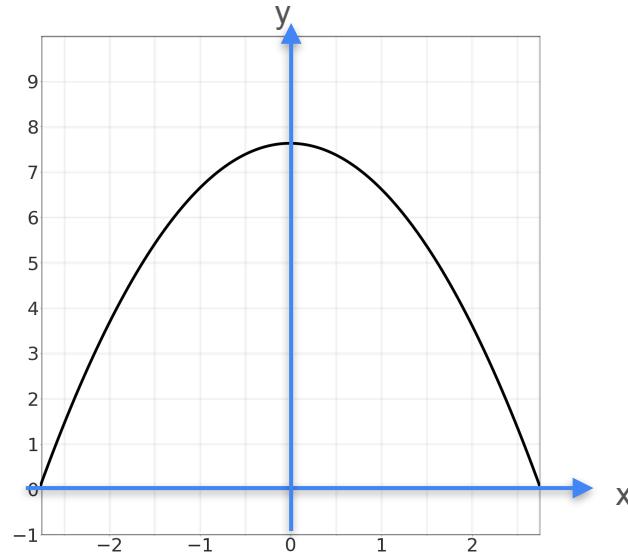
$$f''(0) > 0$$

Curvature

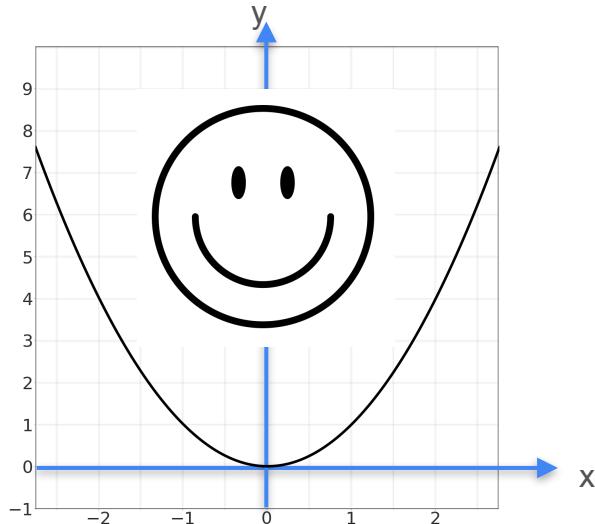


Concave up or convex

$$f''(0) > 0$$

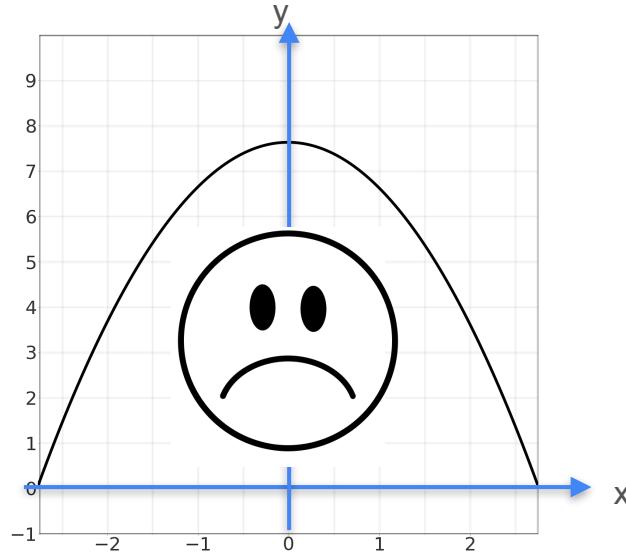


Curvature



Concave up or convex

$$f''(0) > 0$$



Concave down

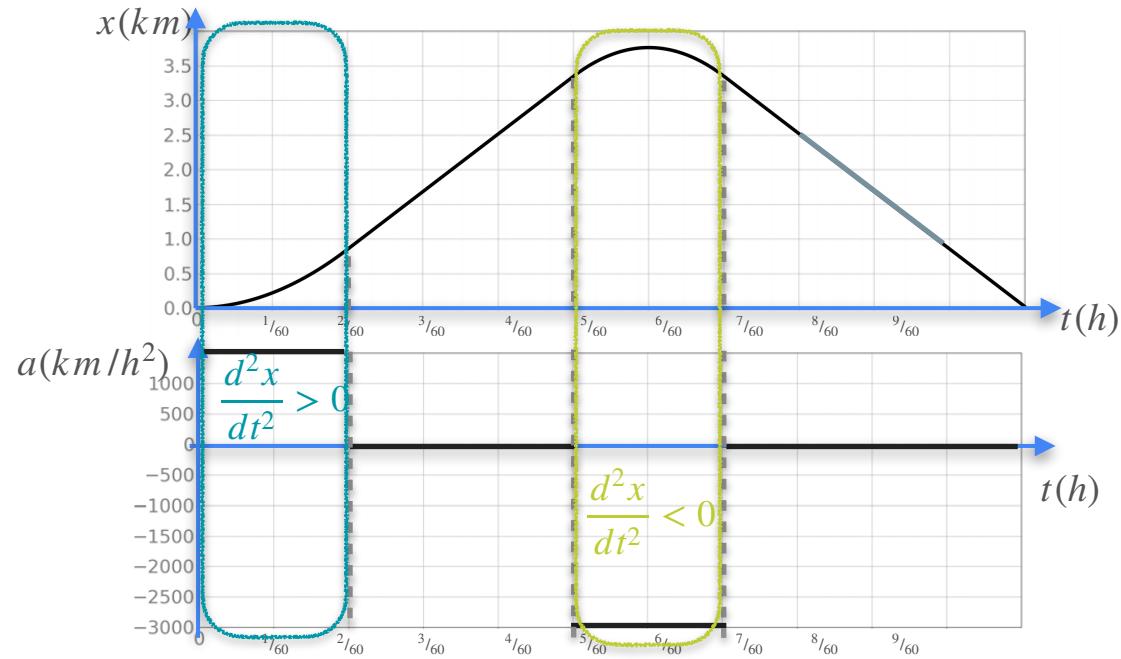
$$f''(0) < 0$$

Second Derivative and Optimization

$$\frac{d^2x}{dt^2} > 0$$

$$\frac{d^2x}{dt^2} < 0$$

$$\frac{d^2x}{dt^2} = 0$$

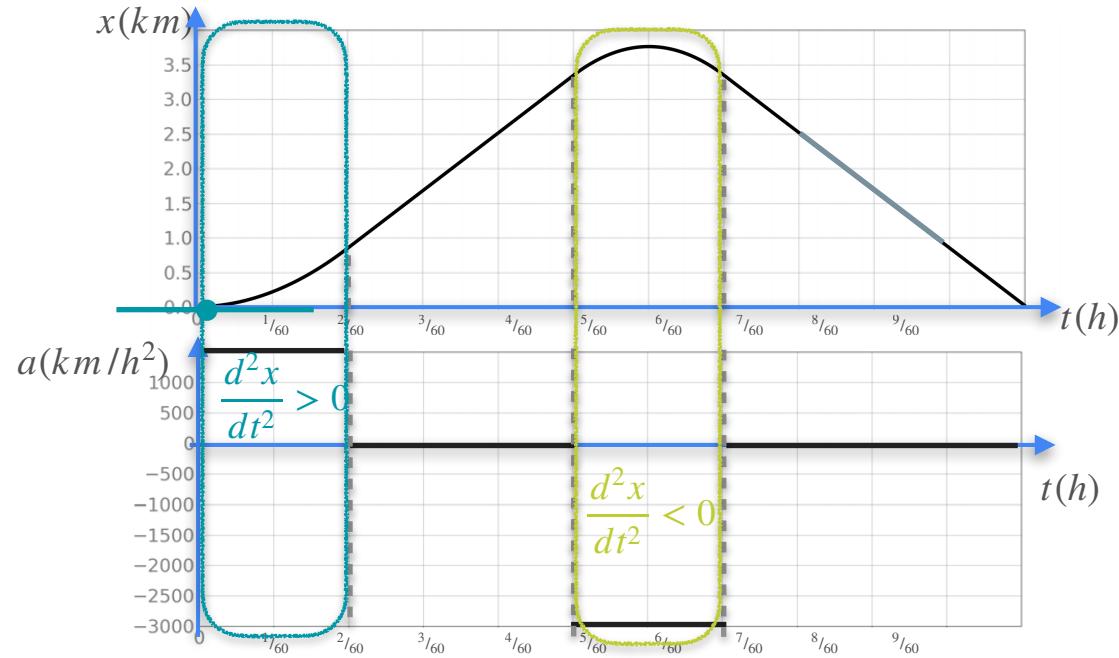


Second Derivative and Optimization

$\frac{d^2x}{dt^2} > 0$ (Local) Minimum

$\frac{d^2x}{dt^2} < 0$

$\frac{d^2x}{dt^2} = 0$

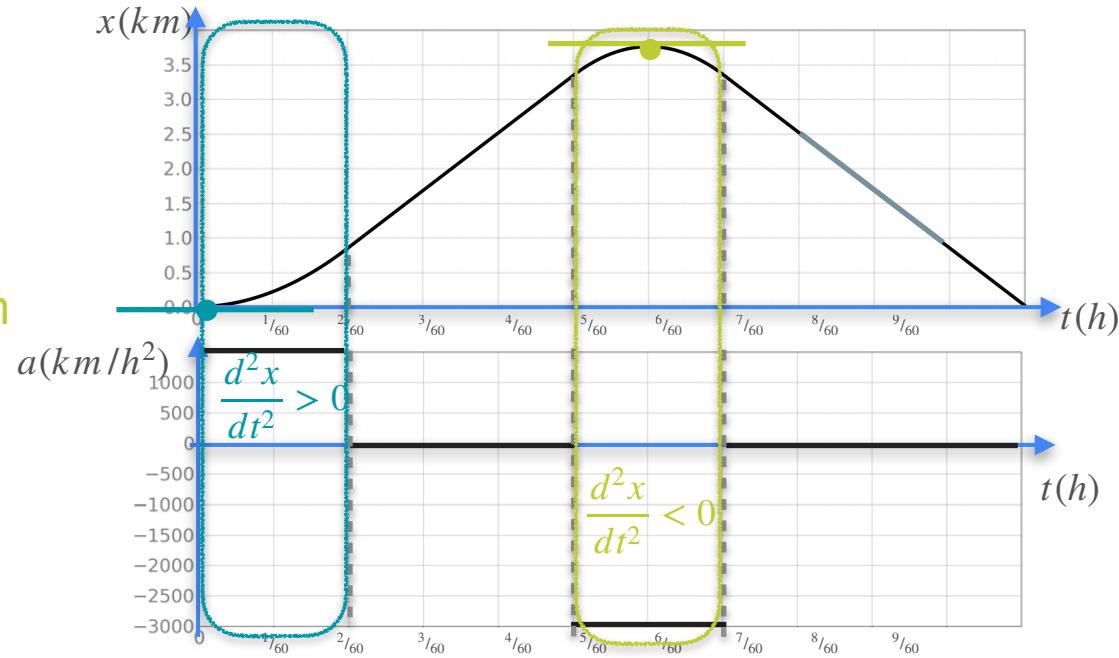


Second Derivative and Optimization

$$\frac{d^2x}{dt^2} > 0 \quad (\text{Local}) \text{ Minimum}$$

$$\frac{d^2x}{dt^2} < 0 \quad (\text{Local}) \text{ maximum}$$

$$\frac{d^2x}{dt^2} = 0$$

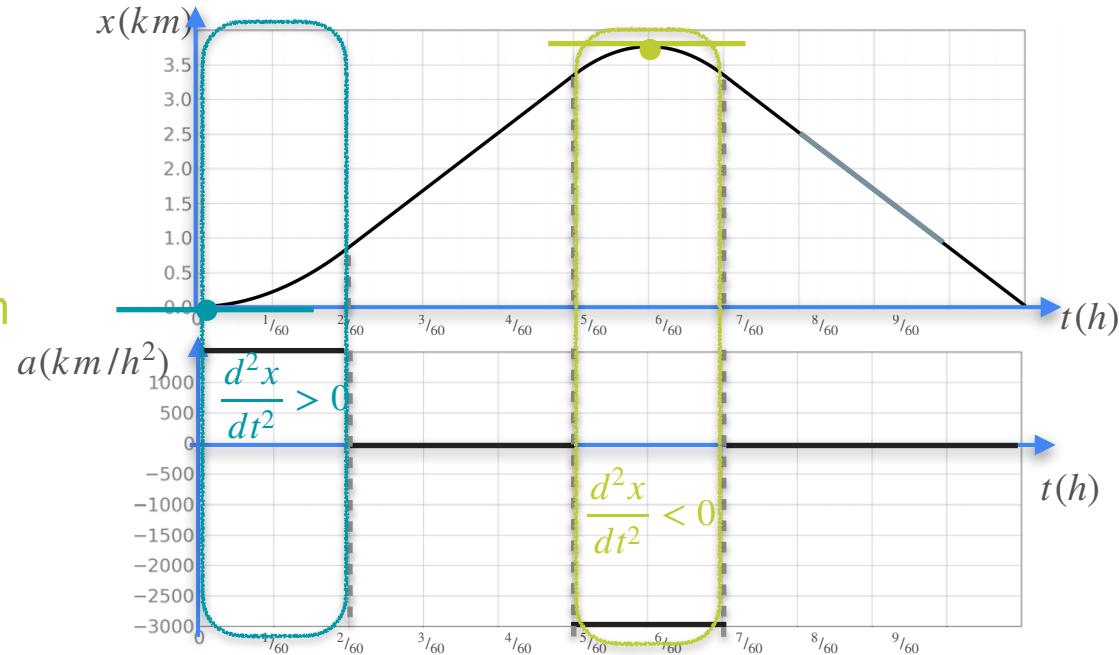


Second Derivative and Optimization

$\frac{d^2x}{dt^2} > 0$ (Local) Minimum

$\frac{d^2x}{dt^2} < 0$ (Local) maximum

$\frac{d^2x}{dt^2} = 0$ Inconclusive



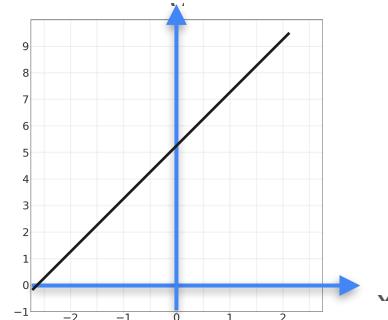
Curvature

First derivative

Second derivative

Curvature

First derivative



Increasing

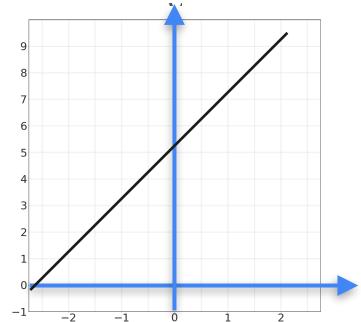
$$f'(0) > 0$$

Second derivative



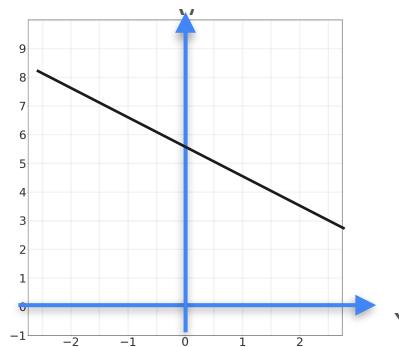
Curvature

First derivative



Increasing

$$f'(0) > 0$$



Decreasing

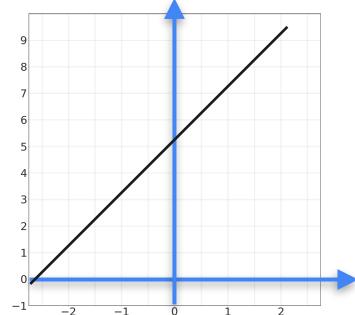
$$f'(0) < 0$$

Second derivative



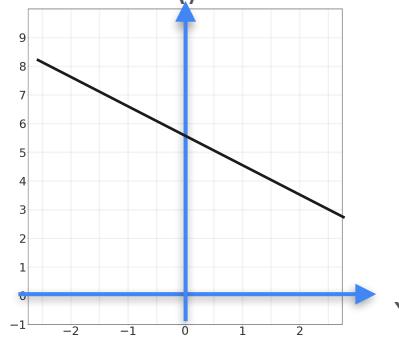
Curvature

First derivative



Increasing

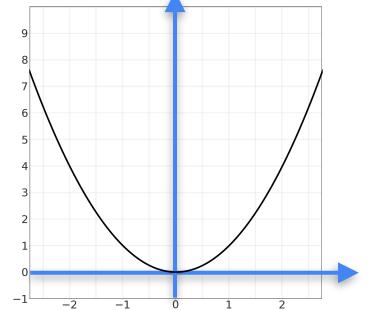
$$f'(0) > 0$$



Decreasing

$$f'(0) < 0$$

Second derivative

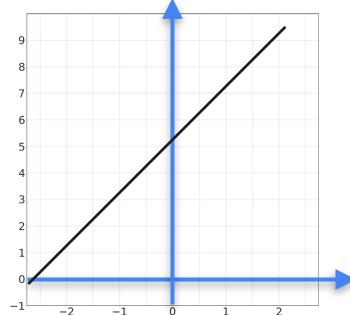


Concave up

$$f''(0) > 0$$

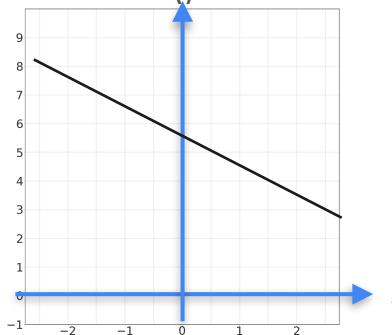
Curvature

First derivative



Increasing

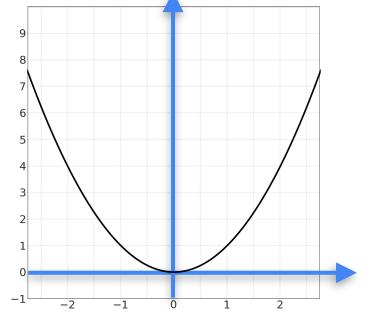
$$f'(0) > 0$$



Decreasing

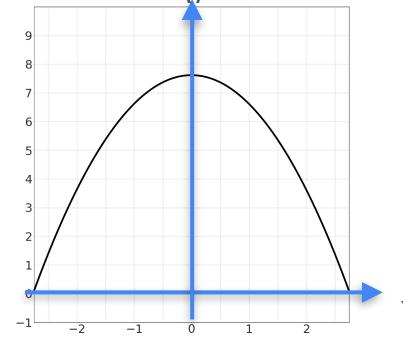
$$f'(0) < 0$$

Second derivative



Concave up

$$f''(0) > 0$$



Concave down

$$f''(0) < 0$$



DeepLearning.AI

Optimization in Neural Networks and Newton's Method

The Hessian

Second Derivative

Second Derivative

1 variable

2 variables

Second Derivative

	1 variable	2 variables
Function	$f(x)$	

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ Rate of change w.r.t x

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ Rate of change w.r.t x $f_y(x, y)$ Rate of change w.r.t y

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ $f_y(x, y)$ $\nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$

Second Derivative

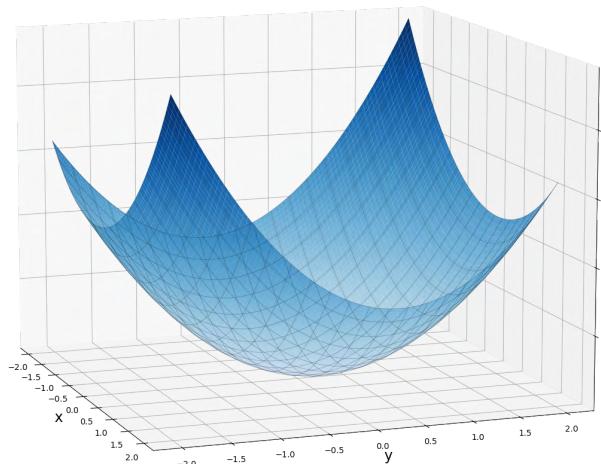
	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ $f_y(x, y)$ $\nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$
Second derivative	$f''(x)$ Rate of change of the rate of change of $f(x)$	

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ Rate of change w.r.t x $f_y(x, y)$ Rate of change w.r.t y $\nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$
Second derivative	$f''(x)$ Rate of change of the rate of change of $f(x)$???

Second Derivative

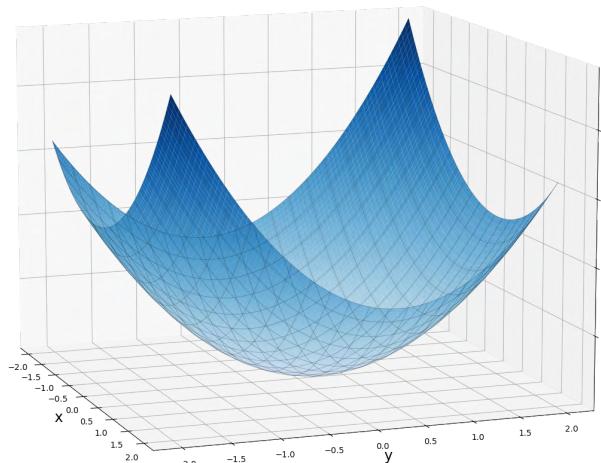
Second Derivative



$$f(x, y) =$$

$$2x^2 + 3y^2 - xy$$

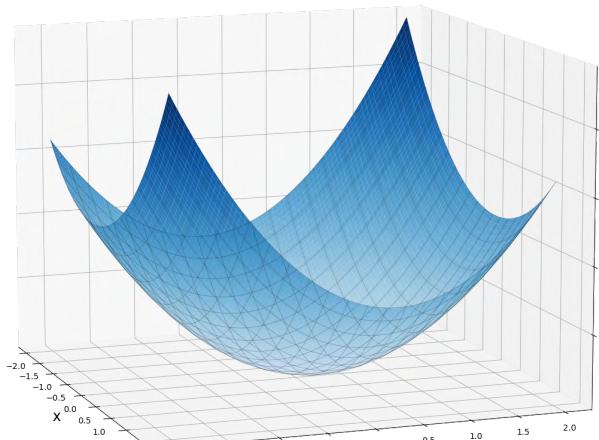
Second Derivative



$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$\begin{matrix} 4x - y \\ x \end{matrix}$$

Second Derivative

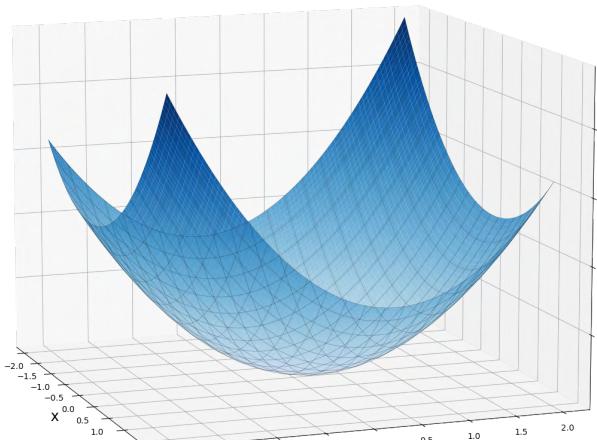


$$f(x, y) = 2x^2 + 3y^2 - xy$$

A diagram illustrating the second derivatives of the function $f(x, y) = 2x^2 + 3y^2 - xy$. A central point is connected by arrows to four surrounding points, representing the second partial derivatives:

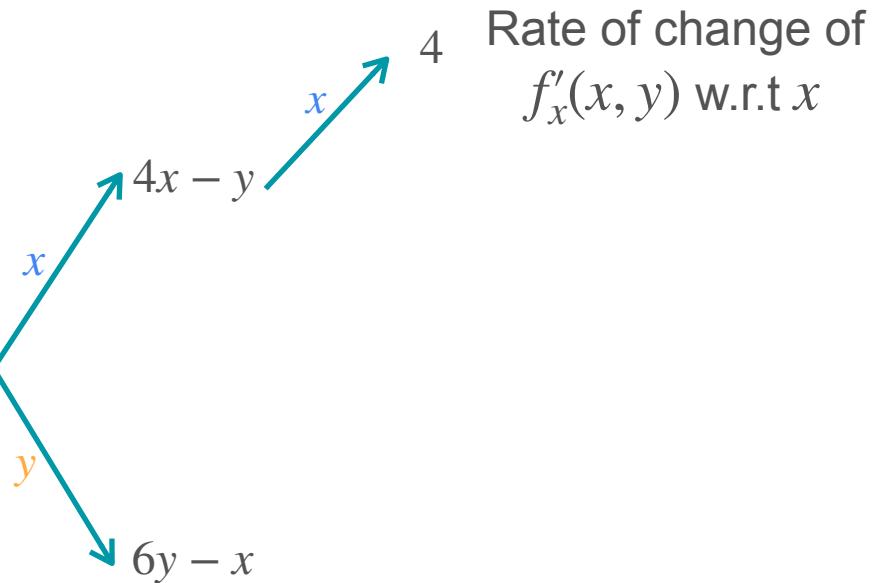
- Upward arrow: $4x - y$
- Rightward arrow: x
- Downward arrow: y
- Leftward arrow: $6y - x$

Second Derivative



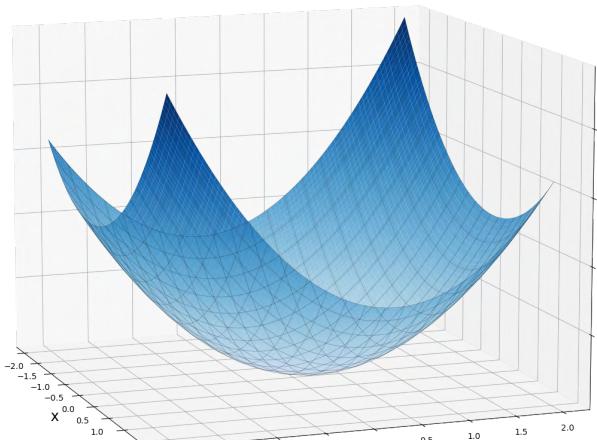
$$f(x, y) =$$

$$2x^2 + 3y^2 - xy$$



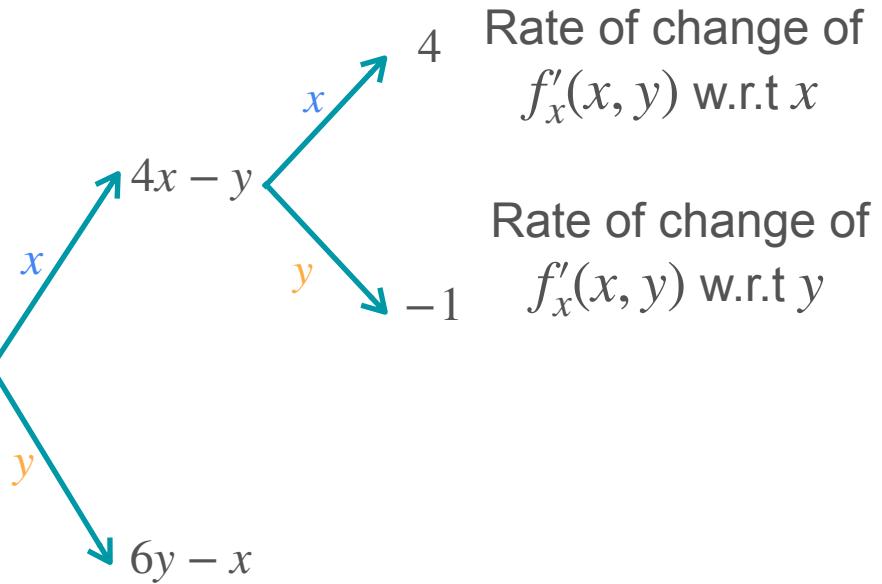
Rate of change of
 $f'_x(x, y)$ w.r.t x

Second Derivative

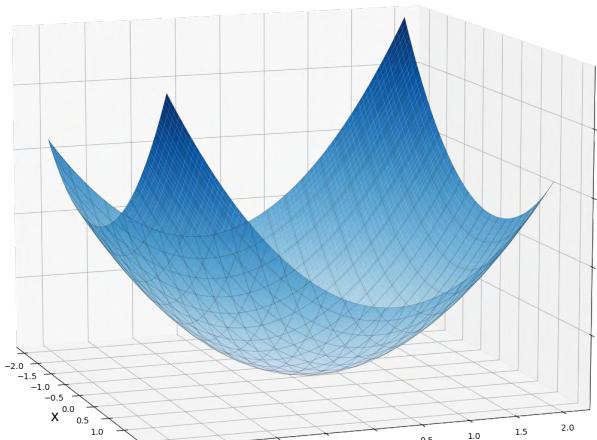


$$f(x, y) =$$

$$2x^2 + 3y^2 - xy$$

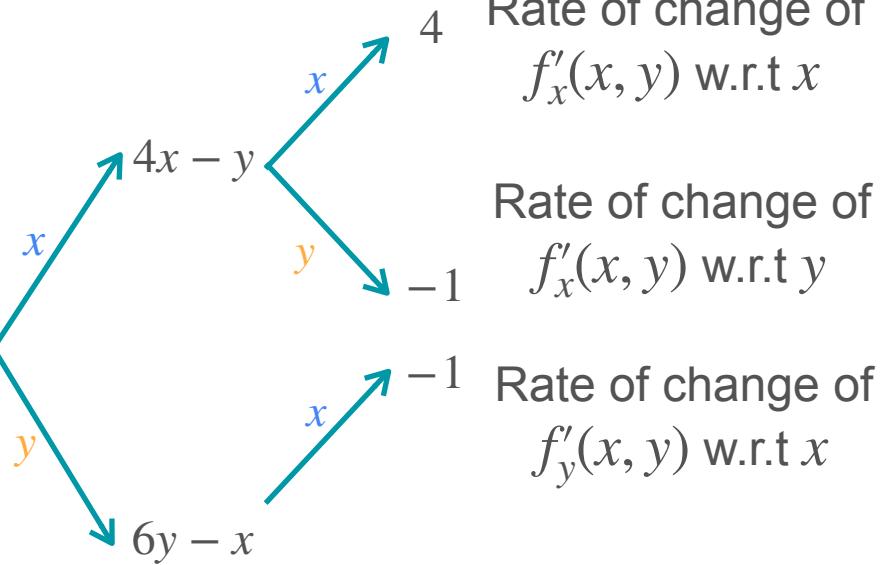


Second Derivative

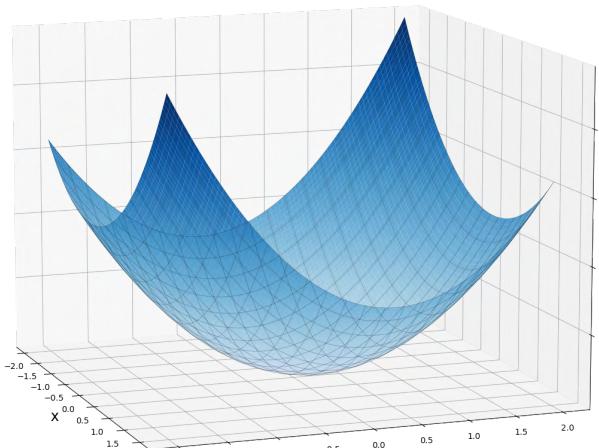


$$f(x, y) =$$

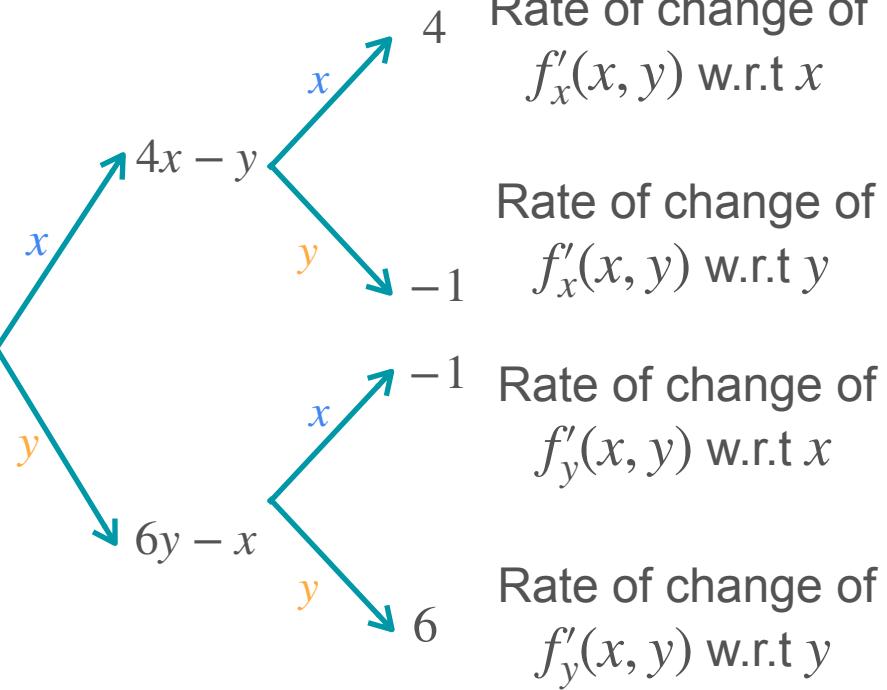
$$2x^2 + 3y^2 - xy$$



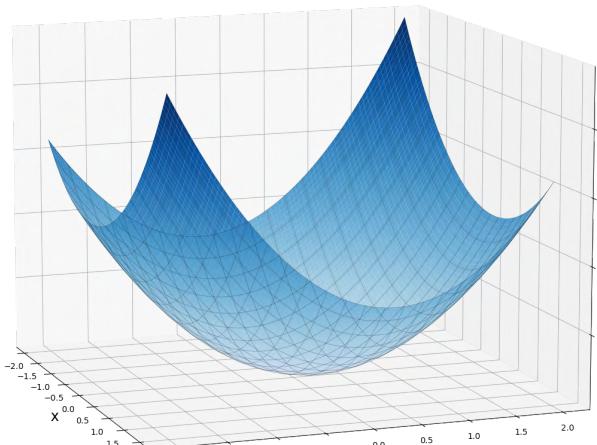
Second Derivative



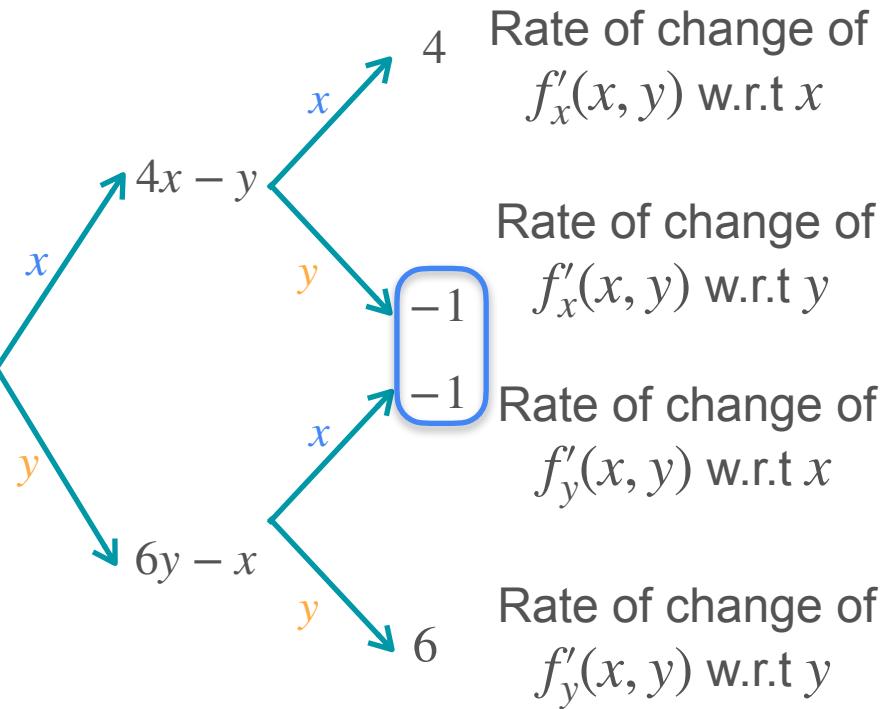
$$f(x, y) =$$
$$2x^2 + 3y^2 - xy$$



Second Derivative



$$f(x, y) =$$
$$2x^2 + 3y^2 - xy$$



What Do These Mean?

Rate of change of
 $f_x(x, y)$ w.r.t x

Rate of change of
 $f_y(x, y)$ w.r.t y

Rate of change of
 $f_x(x, y)$ w.r.t y

Rate of change of
 $f_y(x, y)$ w.r.t x

What Do These Mean?

Rate of change of
 $f_x(x, y)$ w.r.t x

Rate of change of
 $f_y(x, y)$ w.r.t y

Change in the change in the function
w.r.t tiny changes in x and y

Rate of change of
 $f_x(x, y)$ w.r.t y

Rate of change of
 $f_y(x, y)$ w.r.t x

What Do These Mean?

Rate of change of

$f_x(x, y)$ w.r.t x

Rate of change of

$f_y(x, y)$ w.r.t y

Rate of change of

$f_x(x, y)$ w.r.t y

Rate of change of

$f_y(x, y)$ w.r.t x

Change in the change in the function
w.r.t tiny changes in x and y

Same idea as
with one
variable!

1. Change in the slope along one coordinate axis w.r.t tiny changes along an orthogonal coordinate axis

What Do These Mean?

Rate of change of

$f_x(x, y)$ w.r.t x

Rate of change of

$f_y(x, y)$ w.r.t y

Rate of change of

$f_x(x, y)$ w.r.t y

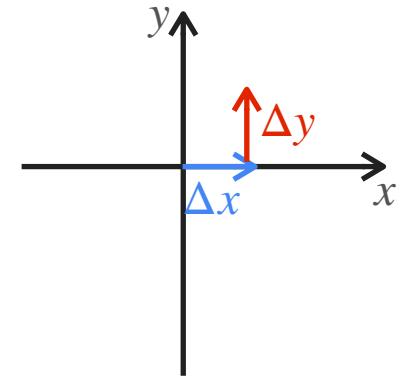
Rate of change of

$f_y(x, y)$ w.r.t x

Change in the change in the function
w.r.t tiny changes in x and y

Same idea as
with one
variable!

1. Change in the slope along one coordinate axis w.r.t tiny changes along an orthogonal coordinate axis



What Do These Mean?

Rate of change of

$f_x(x, y)$ w.r.t x

Rate of change of

$f_y(x, y)$ w.r.t y

Rate of change of

$f_x(x, y)$ w.r.t y

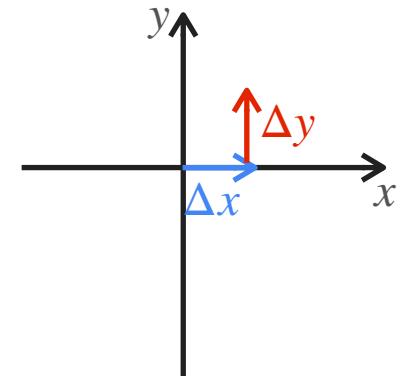
Rate of change of

$f_y(x, y)$ w.r.t x

Change in the change in the function
w.r.t tiny changes in x and y

Same idea as
with one
variable!

1. Change in the slope along one coordinate axis w.r.t tiny changes along an orthogonal coordinate axis
2. They are the same!



What Do These Mean?

Rate of change of

$f_x(x, y)$ w.r.t x

Rate of change of

$f_y(x, y)$ w.r.t y

Rate of change of

$f_x(x, y)$ w.r.t y

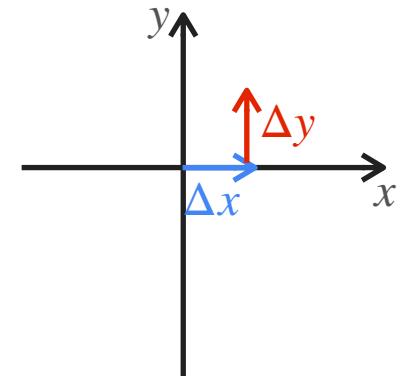
Rate of change of

$f_y(x, y)$ w.r.t x

Change in the change in the function
w.r.t tiny changes in x and y

Same idea as
with one
variable!

1. Change in the slope along one coordinate axis w.r.t tiny changes along an orthogonal coordinate axis
2. They are the same!
(In most cases)



Notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f'_x(x, y)$ w.r.t y

Rate of change of
 $f'_y(x, y)$ w.r.t x

Notation

Leibniz's notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f'_x(x, y)$ w.r.t y

Rate of change of
 $f'_y(x, y)$ w.r.t x

Notation

Leibniz's notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

$$\frac{\partial^2 f}{\partial x^2}$$

Rate of change of
 $f'_y(x, y)$ w.r.t y

$$\frac{\partial^2 f}{\partial y^2}$$

Rate of change of
 $f'_x(x, y)$ w.r.t y

Rate of change of
 $f'_y(x, y)$ w.r.t x

Notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f'_x(x, y)$ w.r.t y

Rate of change of
 $f'_y(x, y)$ w.r.t x

Leibniz's notation

$$\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x}$$

Notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f'_x(x, y)$ w.r.t y

Rate of change of
 $f'_y(x, y)$ w.r.t x

Leibniz's notation

$$\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x}$$

Lagrange's notation

Notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f''_x(x, y)$ w.r.t y

Rate of change of
 $f''_y(x, y)$ w.r.t x

Leibniz's notation

$$\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x}$$

Lagrange's notation

$$f_{xx}(x, y)$$

$$f_{yy}(x, y)$$

Notation

Rate of change of
 $f'_x(x, y)$ w.r.t x

Rate of change of
 $f'_y(x, y)$ w.r.t y

Rate of change of
 $f''_x(x, y)$ w.r.t y

Rate of change of
 $f''_y(x, y)$ w.r.t x

Leibniz's notation

$$\frac{\partial^2 f}{\partial x^2}$$

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$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x}$$

Lagrange's notation

$$f_{xx}(x, y)$$

$$f_{yy}(x, y)$$

$$f_{xy}(x, y)$$

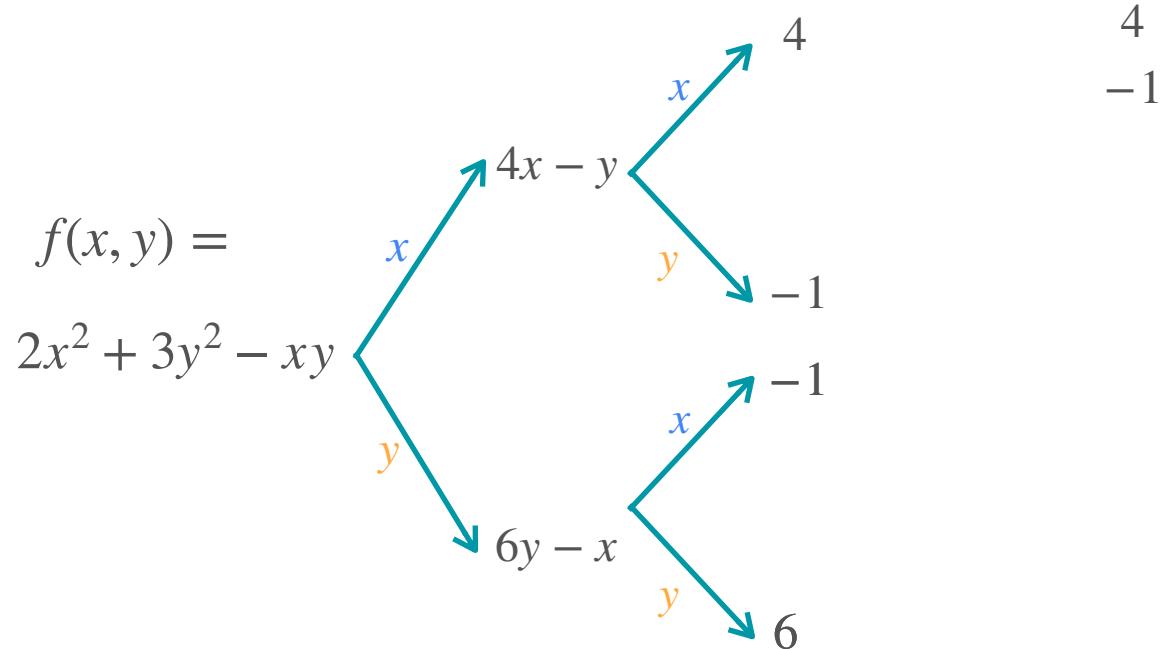
$$f_{yx}(x, y)$$

Hessian Matrix

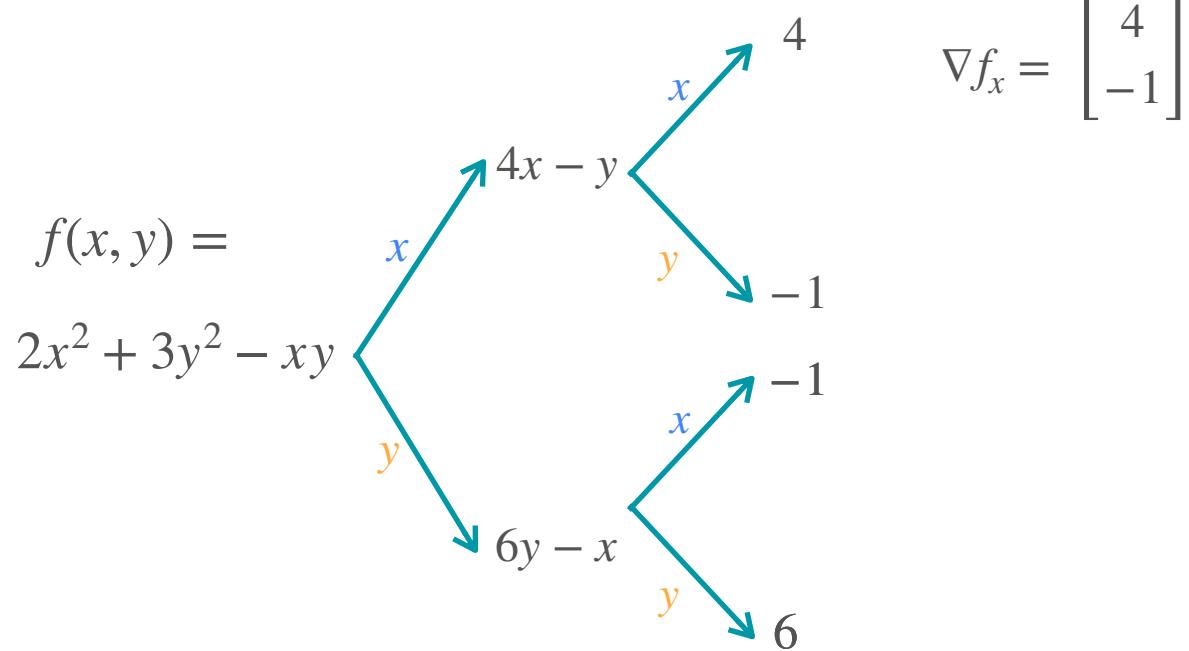
Hessian Matrix

$$f(x, y) = 2x^2 + 3y^2 - xy$$
$$\begin{matrix} & \begin{matrix} 4 & \\ & -1 \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{pmatrix} 4x - y & \\ & 6y - x \end{pmatrix} \\ & \begin{matrix} -1 & \\ & 6 \end{matrix} \end{matrix}$$

Hessian Matrix



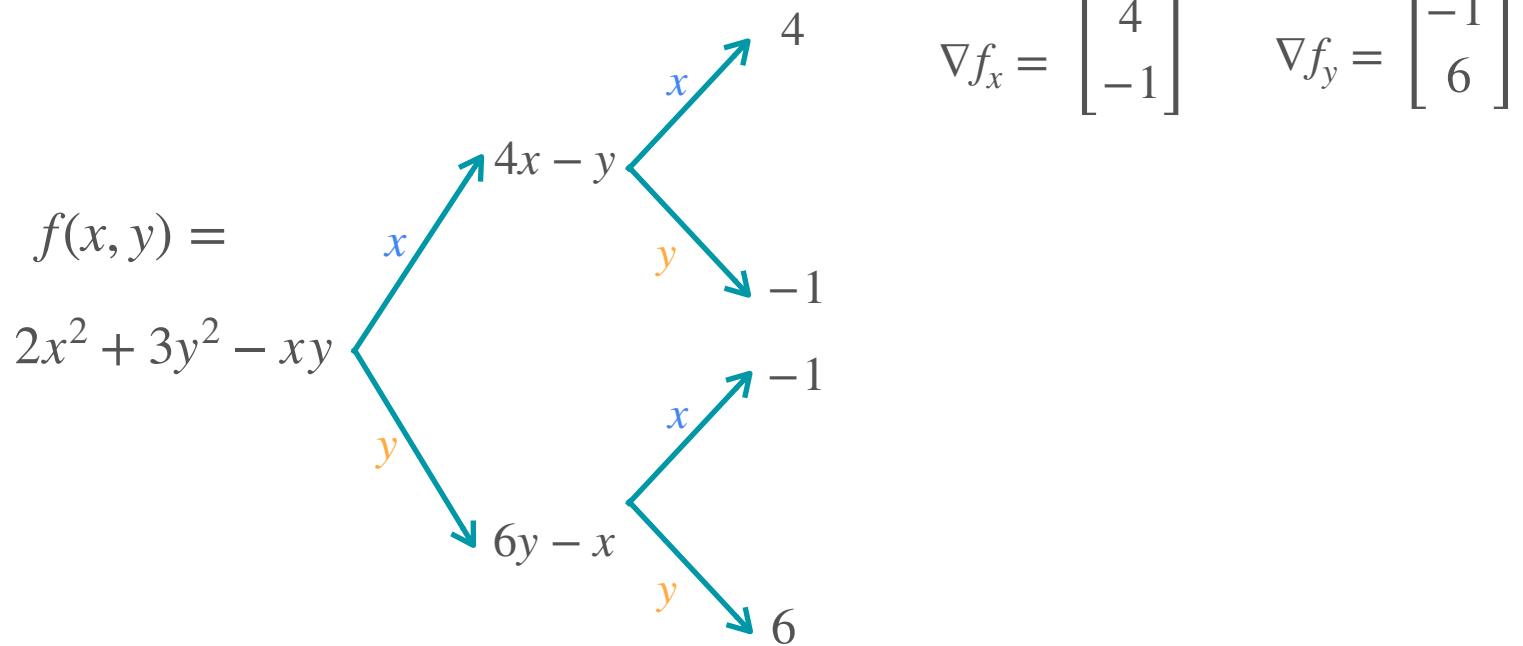
Hessian Matrix



Hessian Matrix

$$f(x, y) = 2x^2 + 3y^2 - xy$$
$$\nabla f_x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

Hessian Matrix



Hessian Matrix

$$f(x, y) = 2x^2 + 3y^2 - xy$$
$$\begin{matrix} & \begin{matrix} x \\ y \end{matrix} & \\ \begin{matrix} x \\ y \end{matrix} & \begin{matrix} 4x - y \\ 6y - x \end{matrix} & \begin{matrix} 4 \\ -1 \\ -1 \\ 6 \end{matrix} \end{matrix}$$

$$\nabla f_x = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \nabla f_y = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\begin{matrix} 4 & -1 \\ -1 & 6 \end{matrix}$$

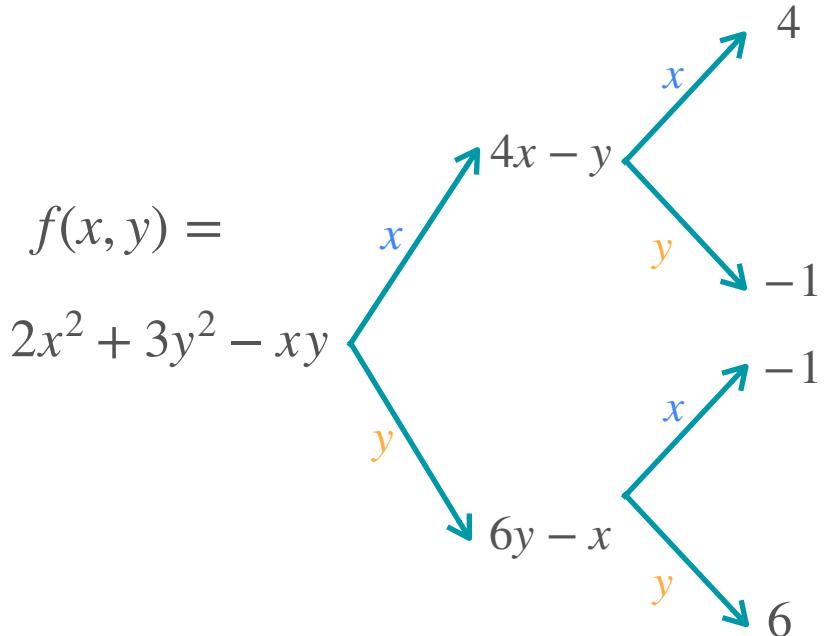
Hessian Matrix

$$f(x, y) = 2x^2 + 3y^2 - xy$$
$$\begin{matrix} & \begin{matrix} 4x - y & 4 \\ 6y - x & 6 \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \end{matrix}$$

$$\nabla f_x = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \nabla f_y = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} \nabla f_x^T \\ \nabla f_y^T \end{bmatrix}$$

Hessian Matrix



$$\nabla f_x = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \nabla f_y = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

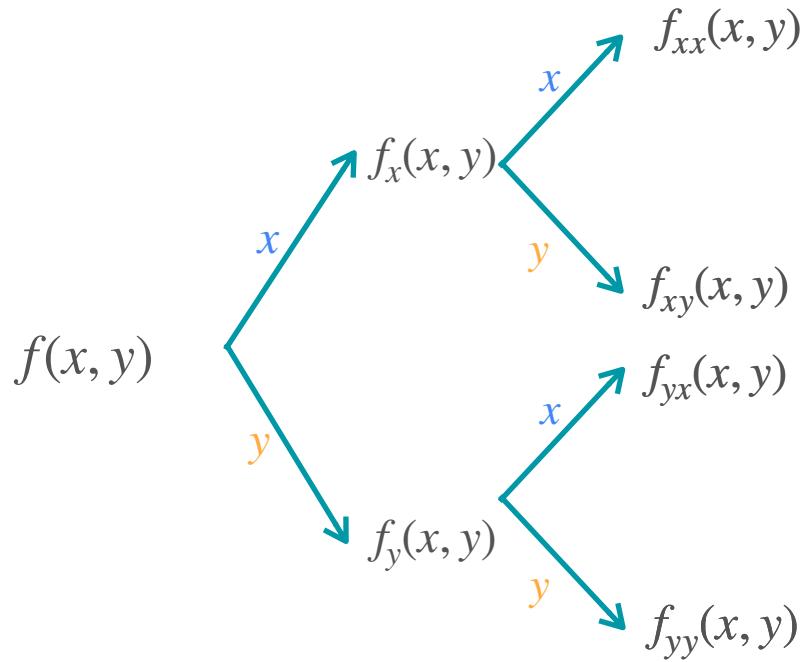
$$H = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} \nabla f_x^T \\ \nabla f_y^T \end{bmatrix}$$

**Hessian
matrix**

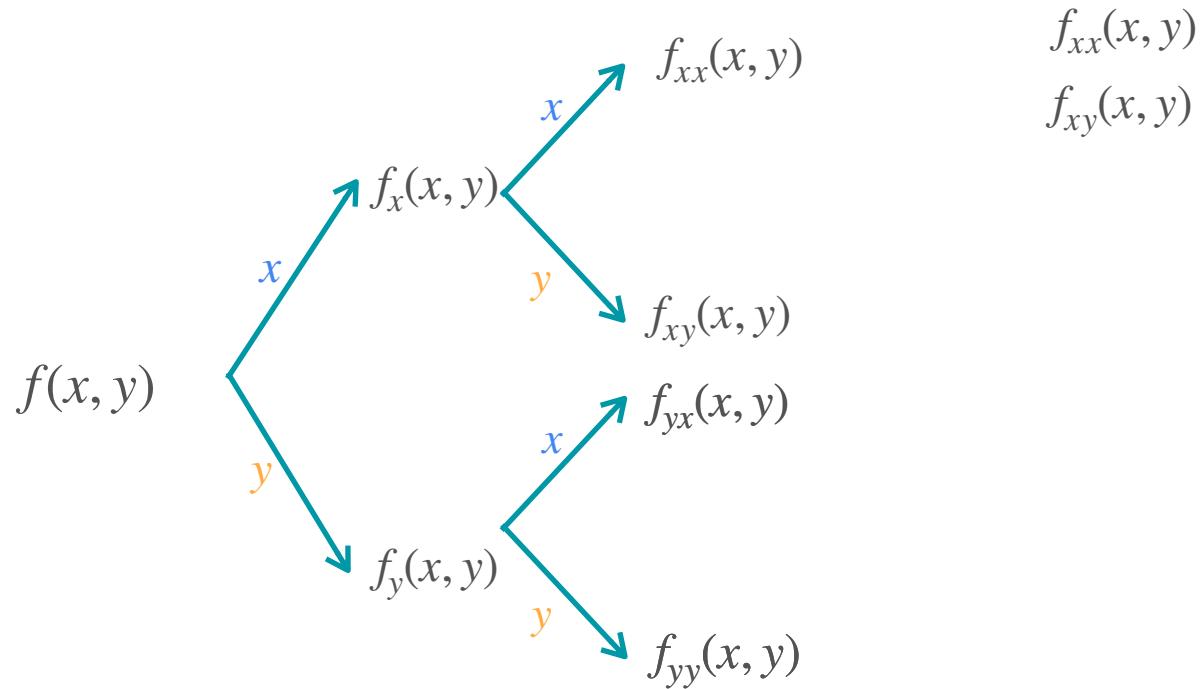
All information
about second
derivatives

Hessian Matrix - General Case

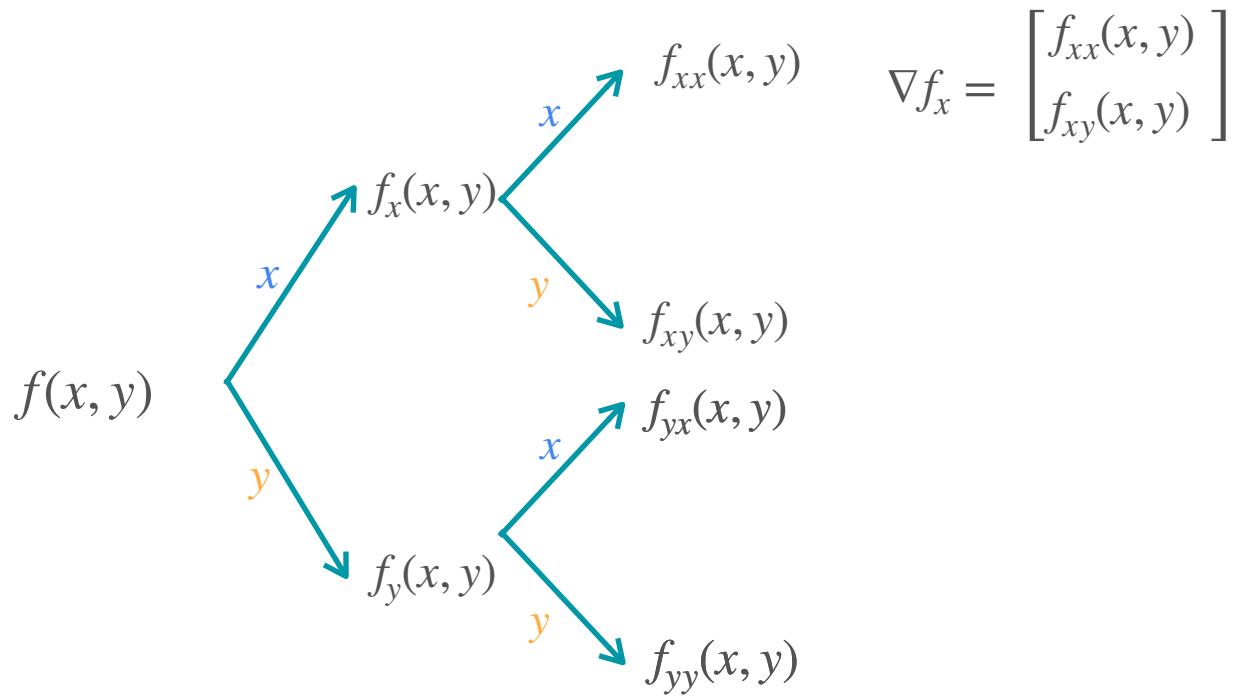
Hessian Matrix - General Case



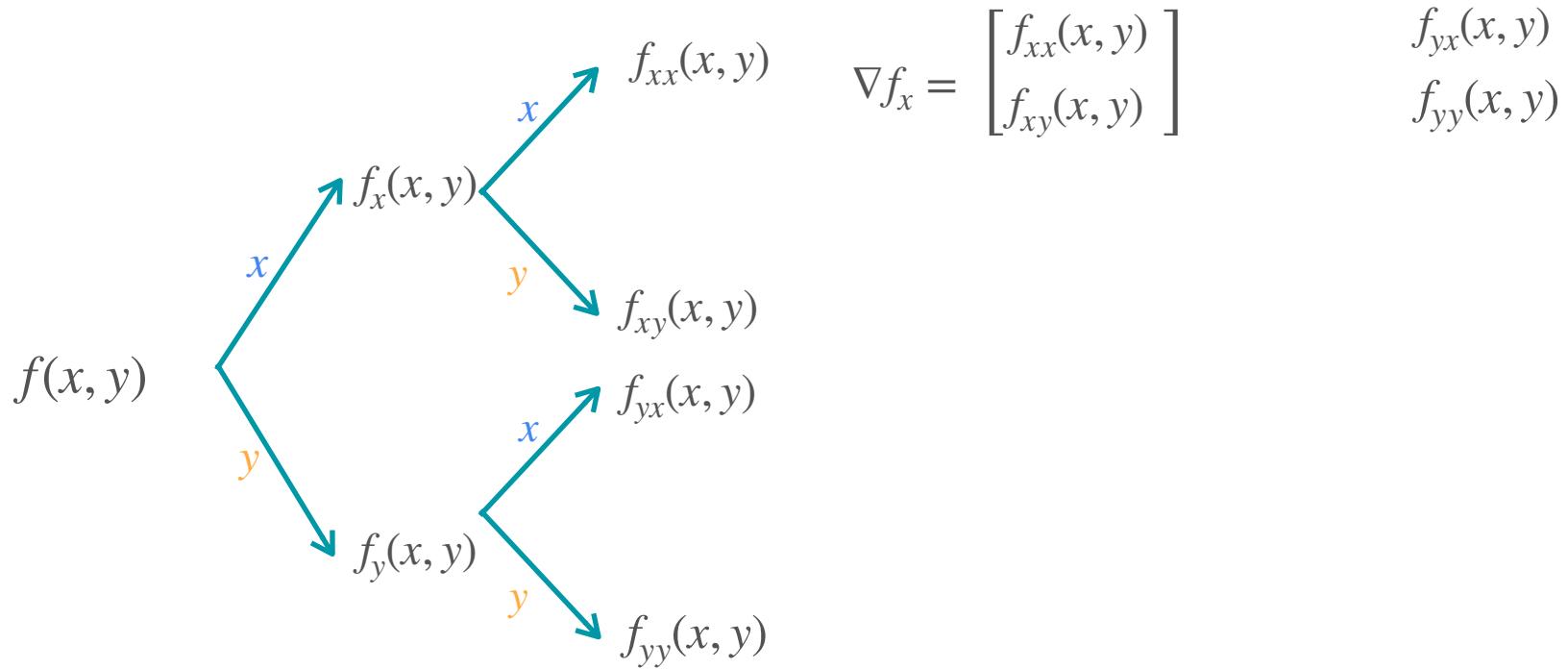
Hessian Matrix - General Case



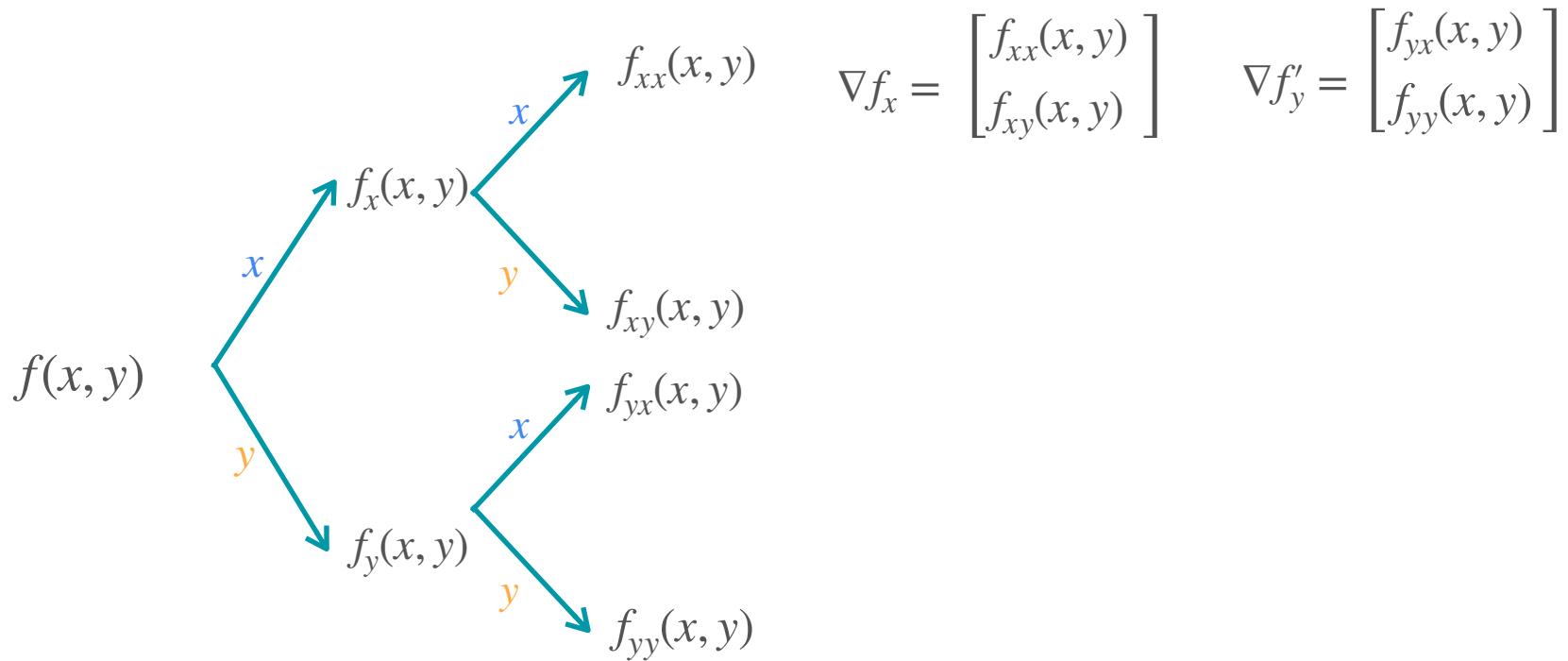
Hessian Matrix - General Case



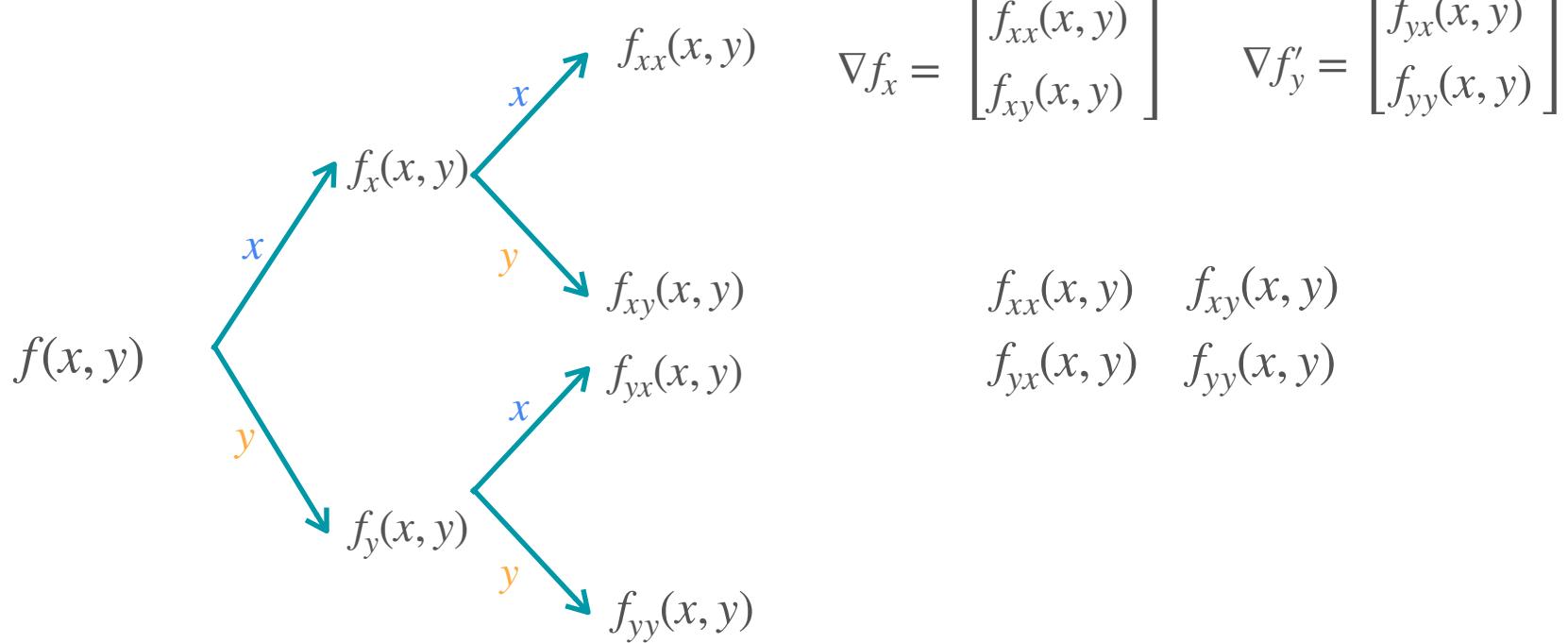
Hessian Matrix - General Case



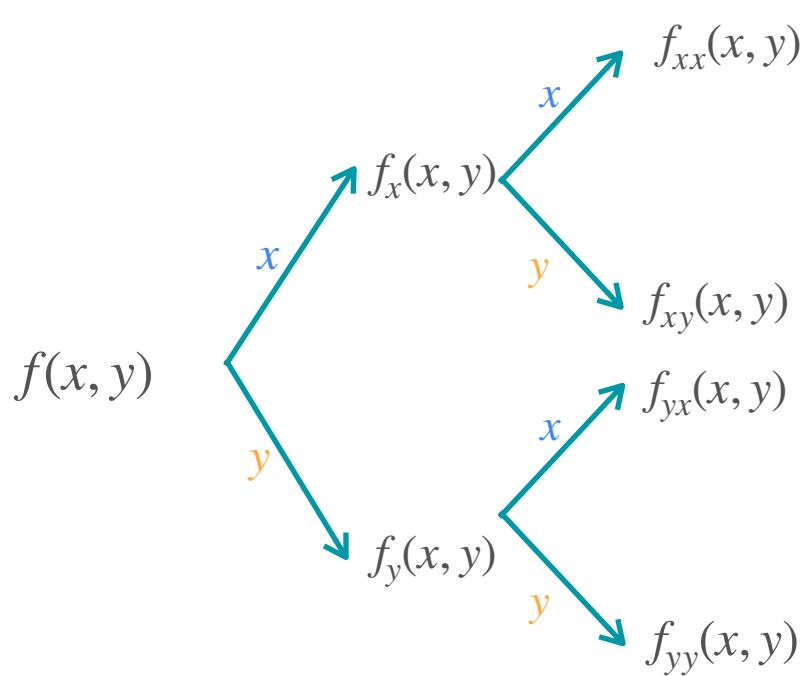
Hessian Matrix - General Case



Hessian Matrix - General Case



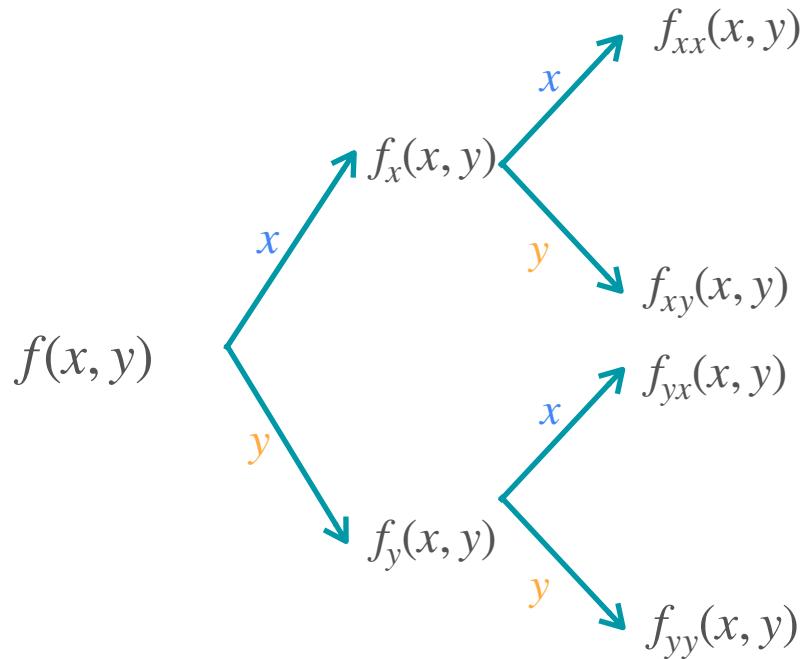
Hessian Matrix - General Case



$$\nabla f_x = \begin{bmatrix} f_{xx}(x, y) \\ f_{xy}(x, y) \end{bmatrix} \quad \nabla f'_y = \begin{bmatrix} f_{yx}(x, y) \\ f_{yy}(x, y) \end{bmatrix}$$

$$\begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \begin{bmatrix} \nabla f_x^T \\ \nabla f_y^T \end{bmatrix}$$

Hessian Matrix - General Case



$$\nabla f_x = \begin{bmatrix} f_{xx}(x, y) \\ f_{xy}(x, y) \end{bmatrix} \quad \nabla f'_y = \begin{bmatrix} f_{yx}(x, y) \\ f_{yy}(x, y) \end{bmatrix}$$

$$H = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \begin{bmatrix} \nabla f_x^T \\ \nabla f_y^T \end{bmatrix}$$

**Hessian
matrix**

All information
about second
derivatives

Second Derivative

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ Rate of change w.r.t x $f_y(x, y)$ Rate of change w.r.t y $\nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$
Second derivative	$f''(x)$ Rate of change of the rate of change of $f(x)$	

Second Derivative

	1 variable	2 variables
Function	$f(x)$	$f(x, y)$
First derivative	$f'(x)$ Rate of change of $f(x)$	$f_x(x, y)$ Rate of change w.r.t x $f_y(x, y)$ Rate of change w.r.t y $\nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$
Second derivative	$f''(x)$ Rate of change of the rate of change of $f(x)$	$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$



DeepLearning.AI

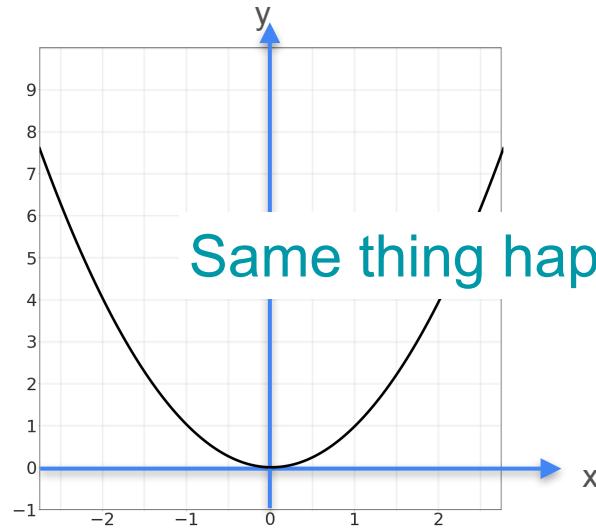
Optimization in Neural Networks and Newton's Method

Hessians and concavity

Remember...

Same thing happens for many variables!

Remember...

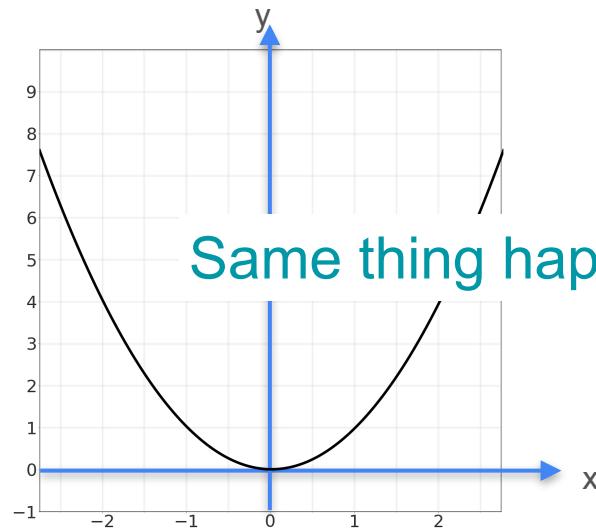


Same thing happens for many variables!

Concave up or convex

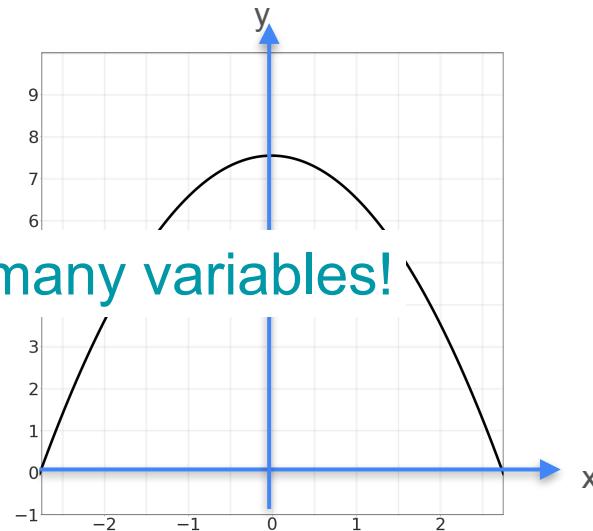
$$f''(0) > 0$$

Remember...



Concave up or convex

$$f''(0) > 0$$

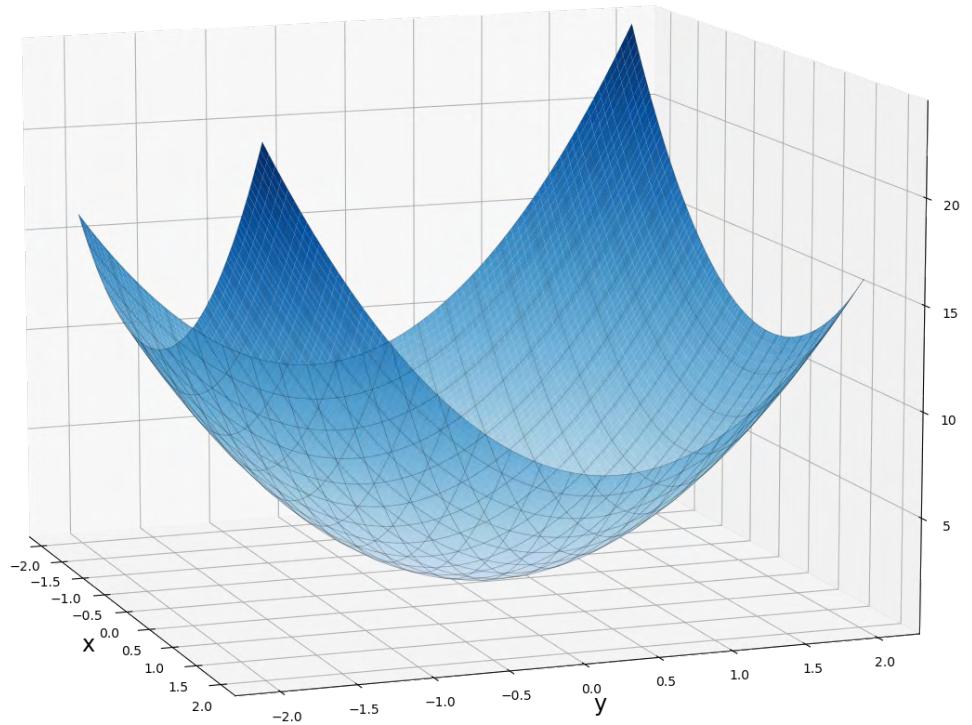


Concave down

$$f''(0) < 0$$

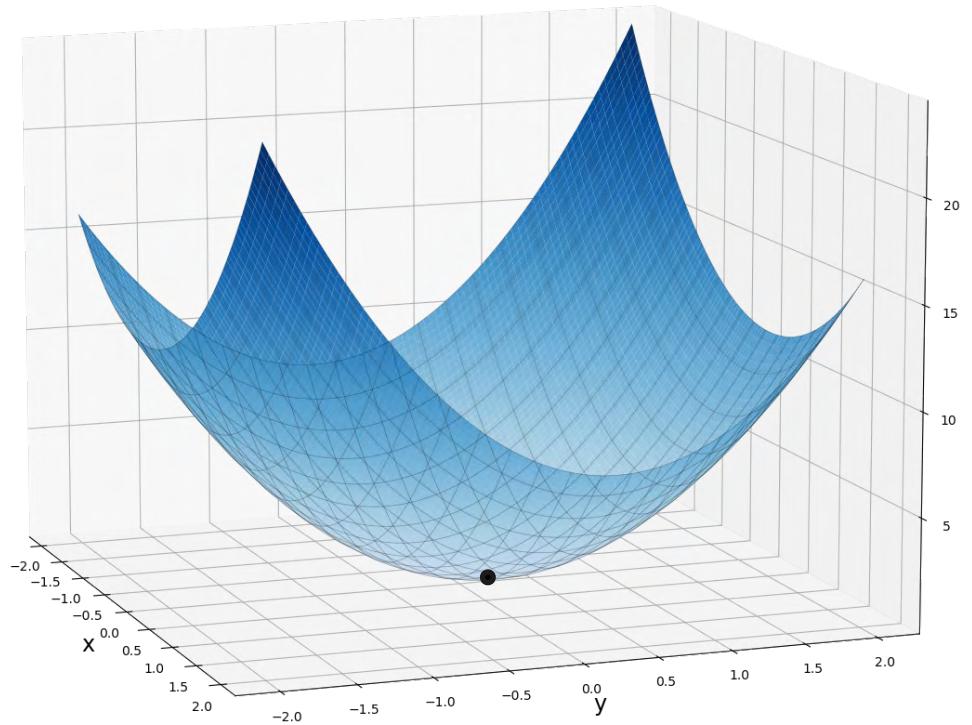
Concave Up

Concave Up



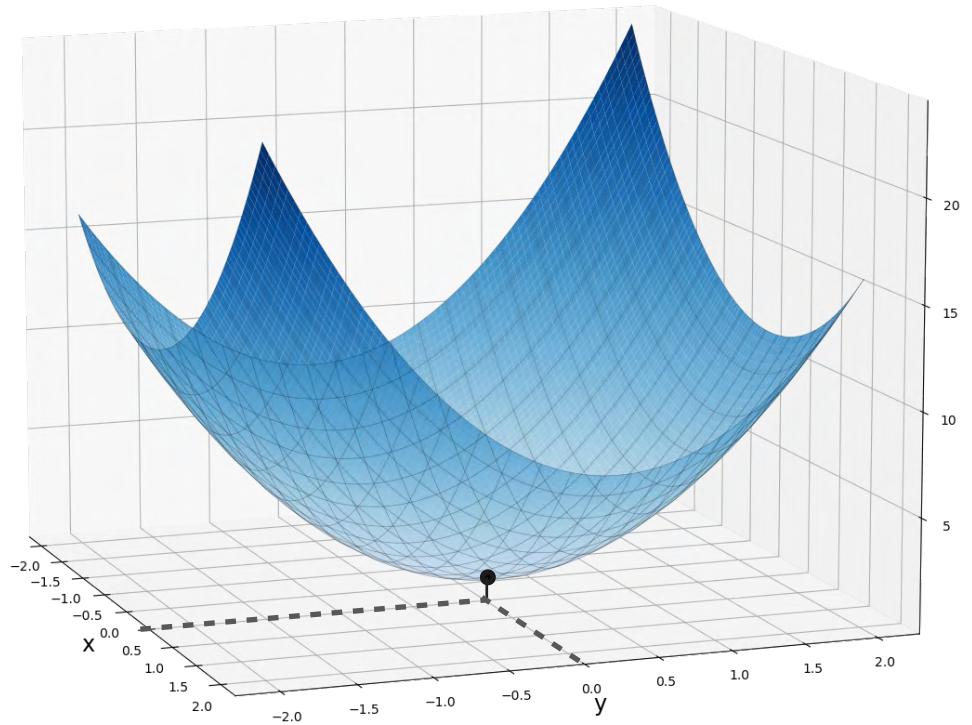
$$f(x, y) = 2x^2 + 3y^2 - xy$$

Concave Up



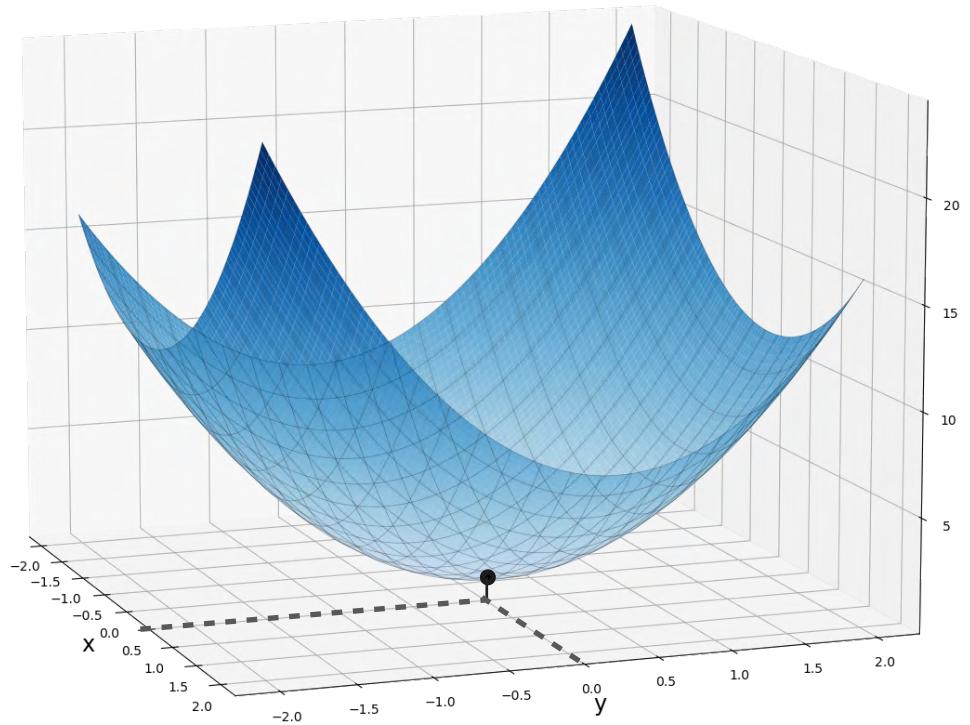
$$f(x, y) = 2x^2 + 3y^2 - xy$$

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

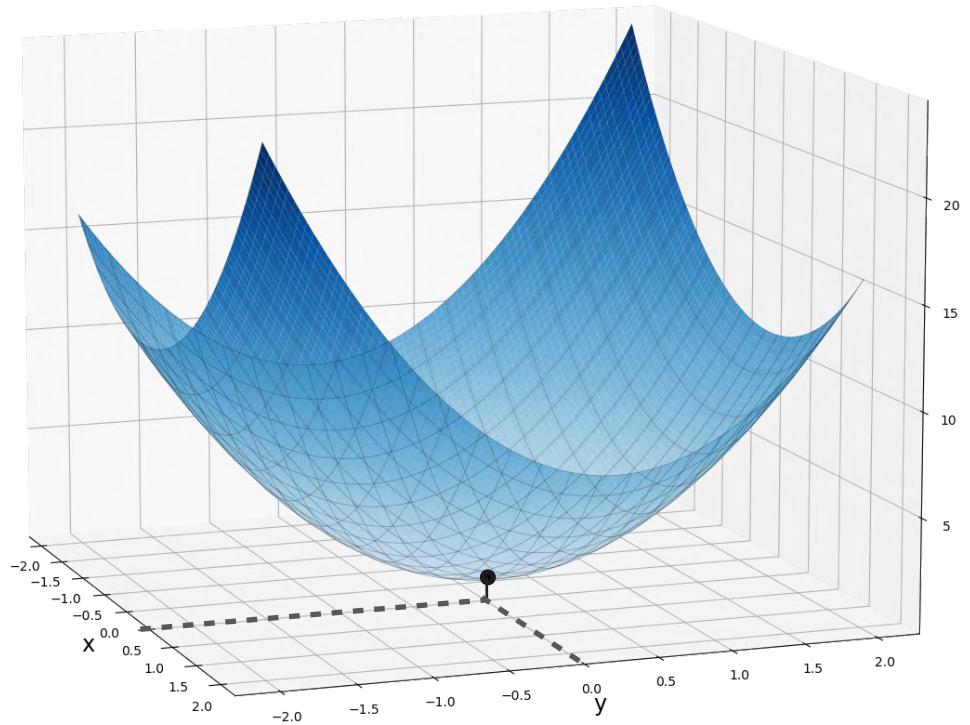
Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

Concave Up

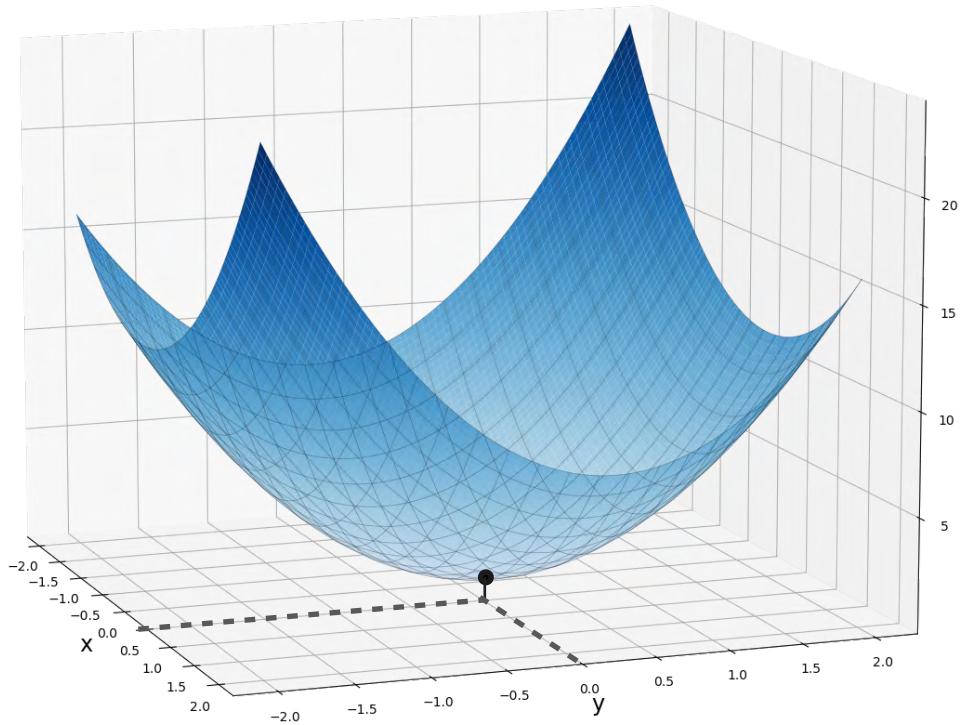


$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) =$$

Concave Up

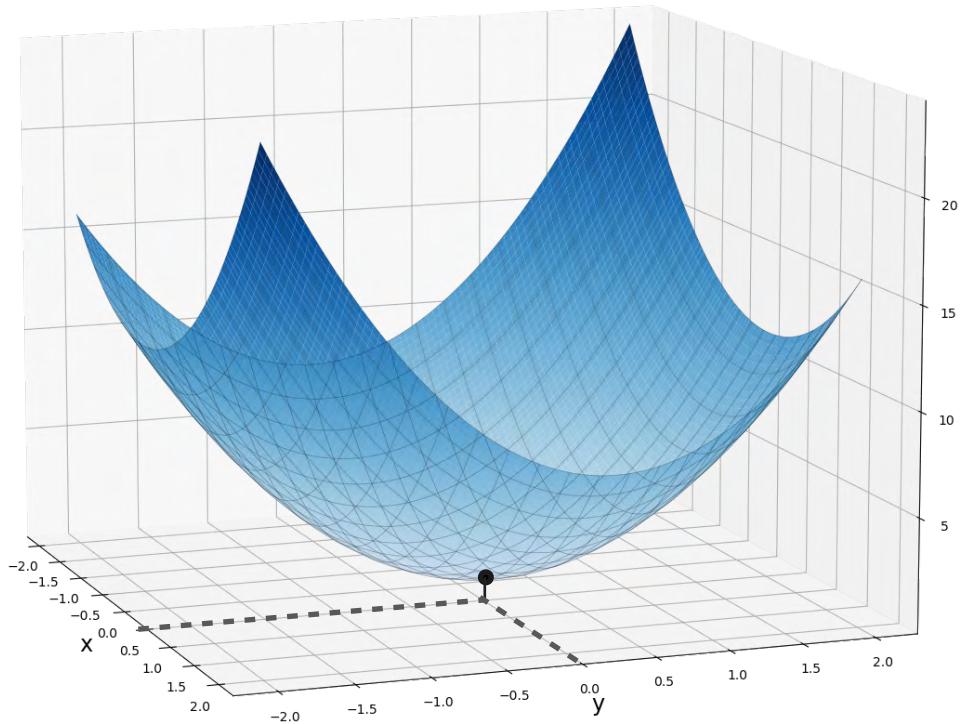


$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$

Concave Up

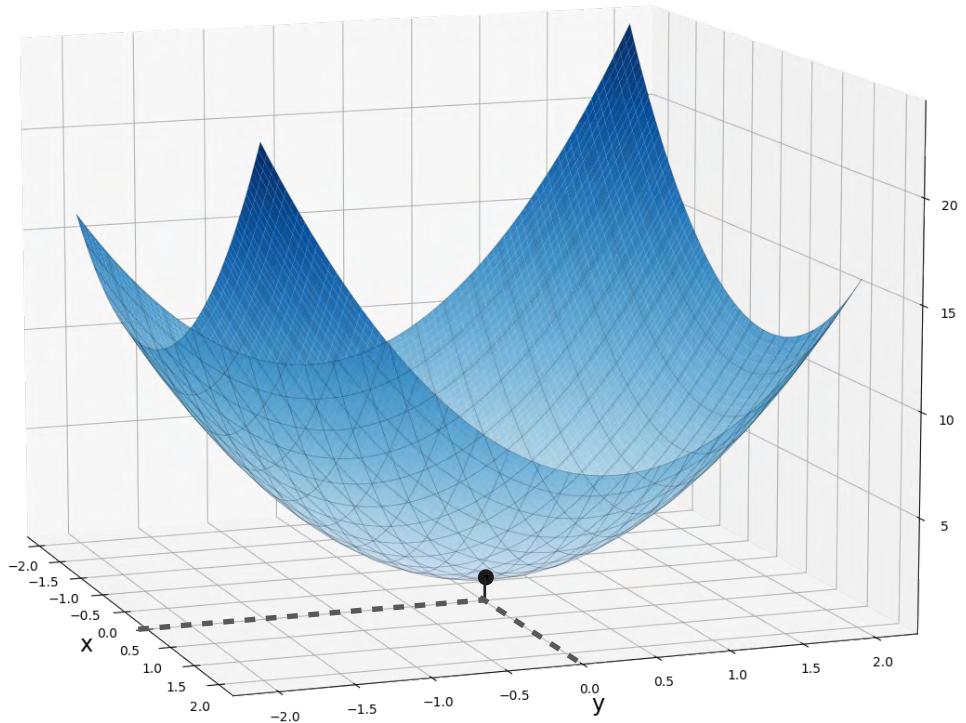


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$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$
$$= (4 - \lambda)(6 - \lambda) - (-1)(-1)$$

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

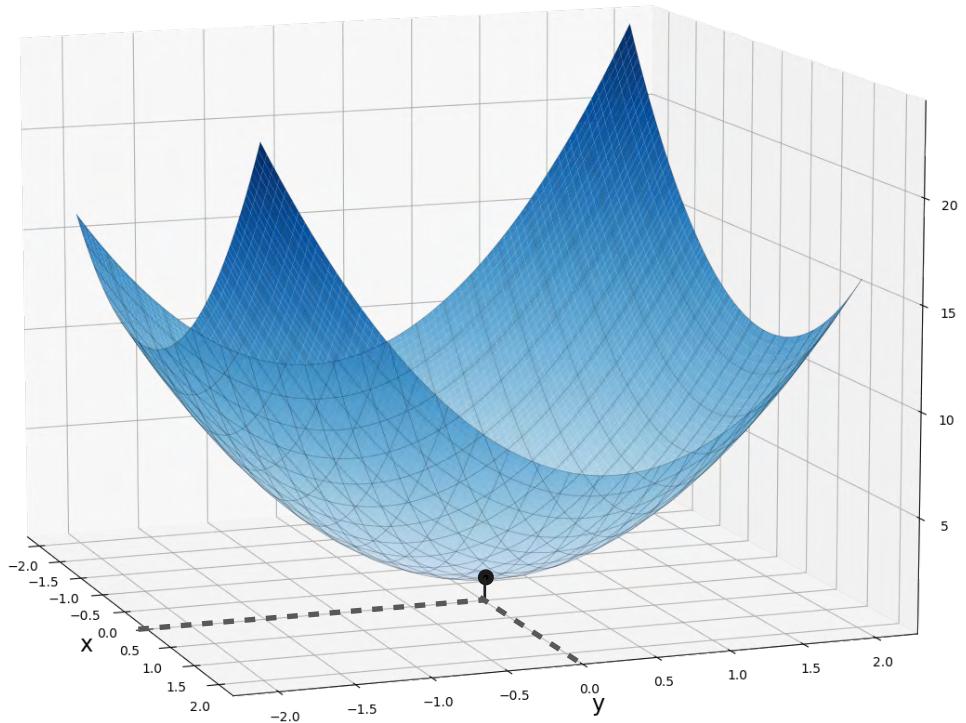
$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$

$$= (4 - \lambda)(6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 - 10\lambda + 23$$

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

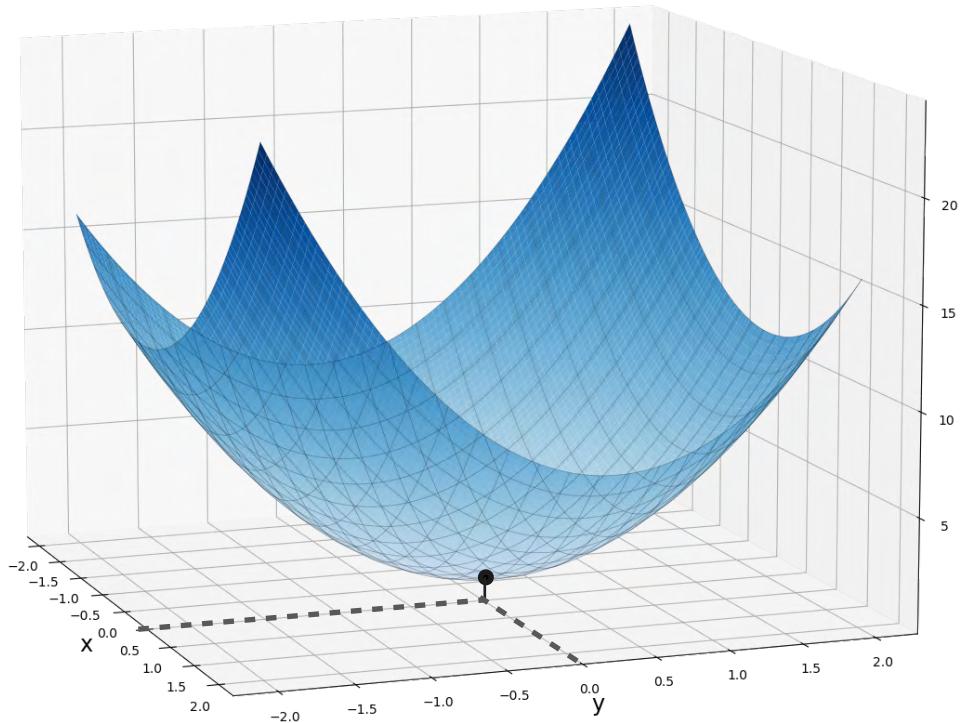
$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$

$$= (4 - \lambda)(6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 - 10\lambda + 23 \rightarrow \lambda_1 = 6.41$$

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

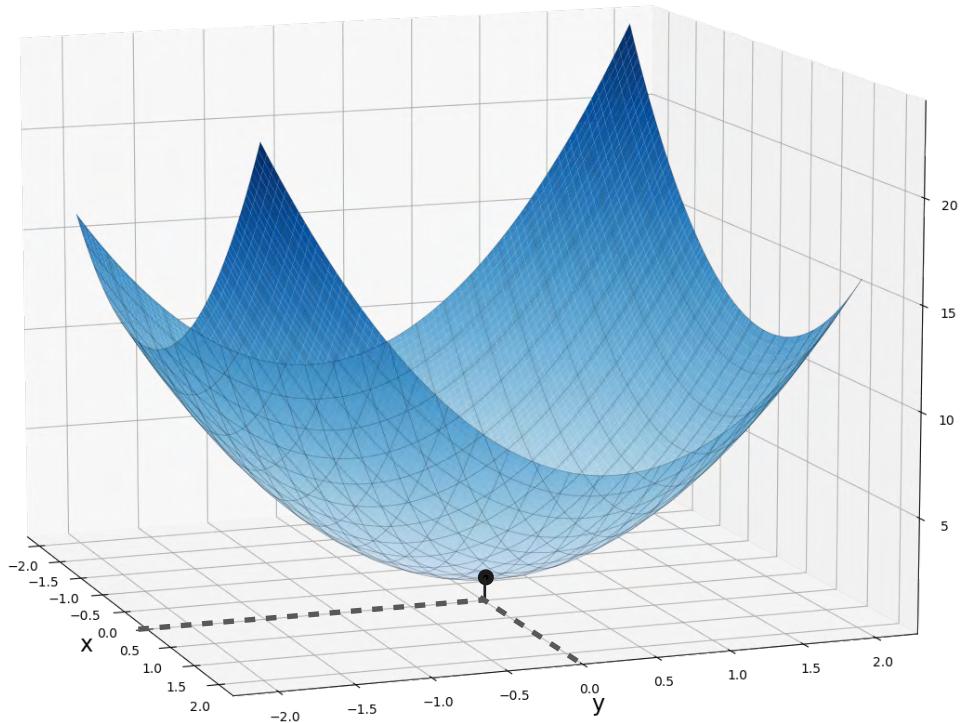
$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$

$$= (4 - \lambda)(6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 - 10\lambda + 23$$

$\lambda_1 = 6.41$
 $\lambda_2 = 3.59$

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) = \det \left(\begin{bmatrix} 4 - \lambda & -1 \\ -1 & 6 - \lambda \end{bmatrix} \right)$$

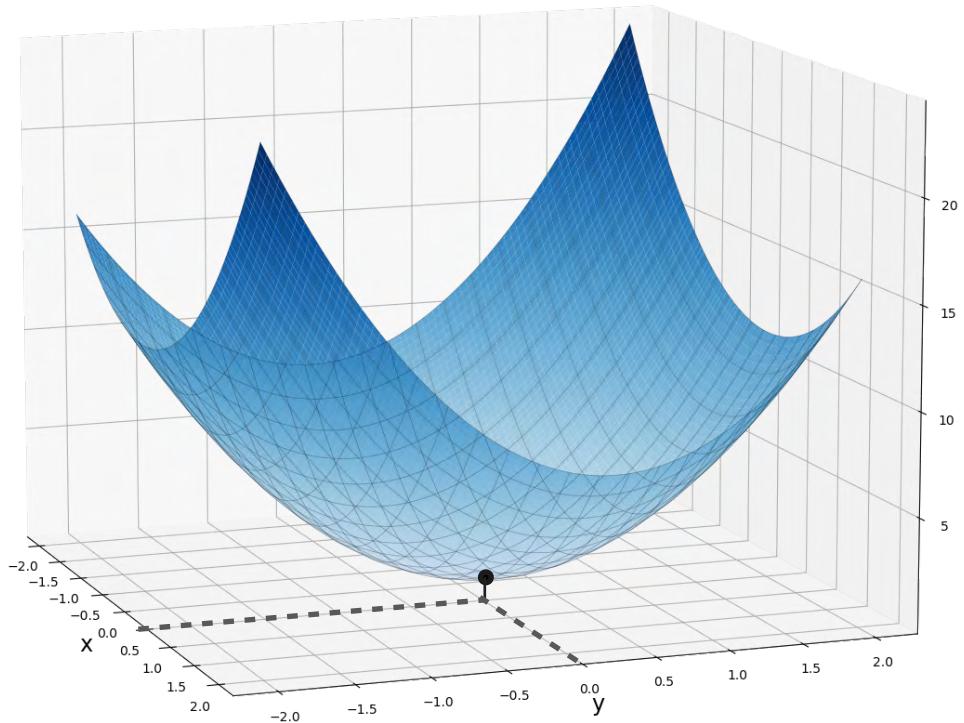
$$= (4 - \lambda)(6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 - 10\lambda + 23$$

$$\lambda_1 = 6.41$$
$$\lambda_2 = 3.59$$

> 0

Concave Up



$$f(x, y) = 2x^2 + 3y^2 - xy$$

$$H(0,0) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

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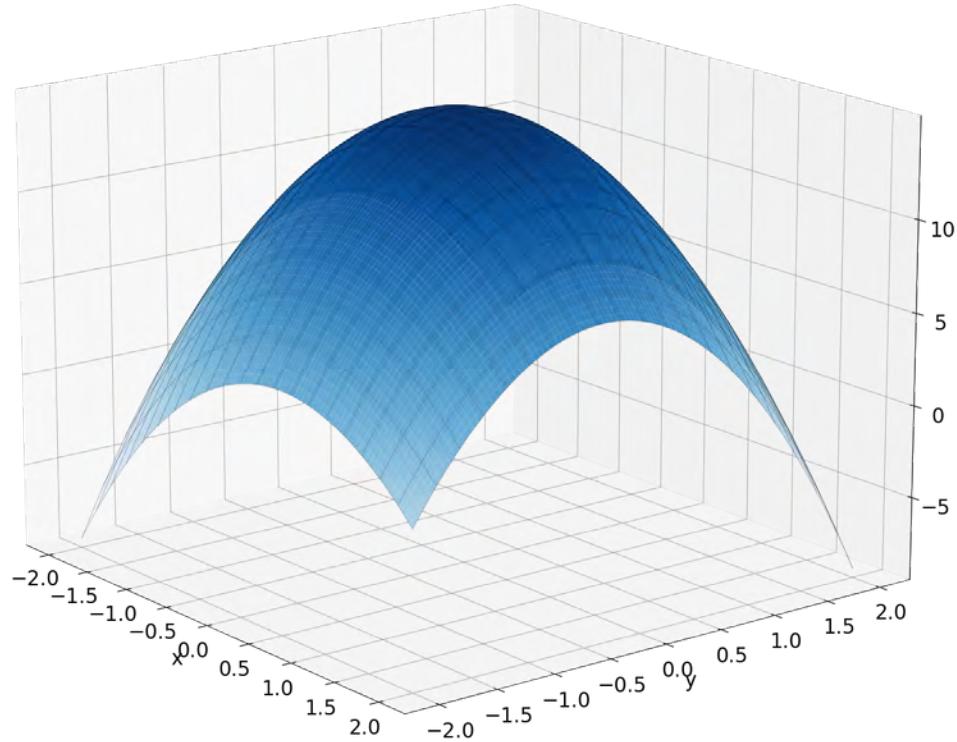
$$\lambda_1 = 6.41$$
$$\lambda_2 = 3.59$$

(0,0) is a minimum!

> 0

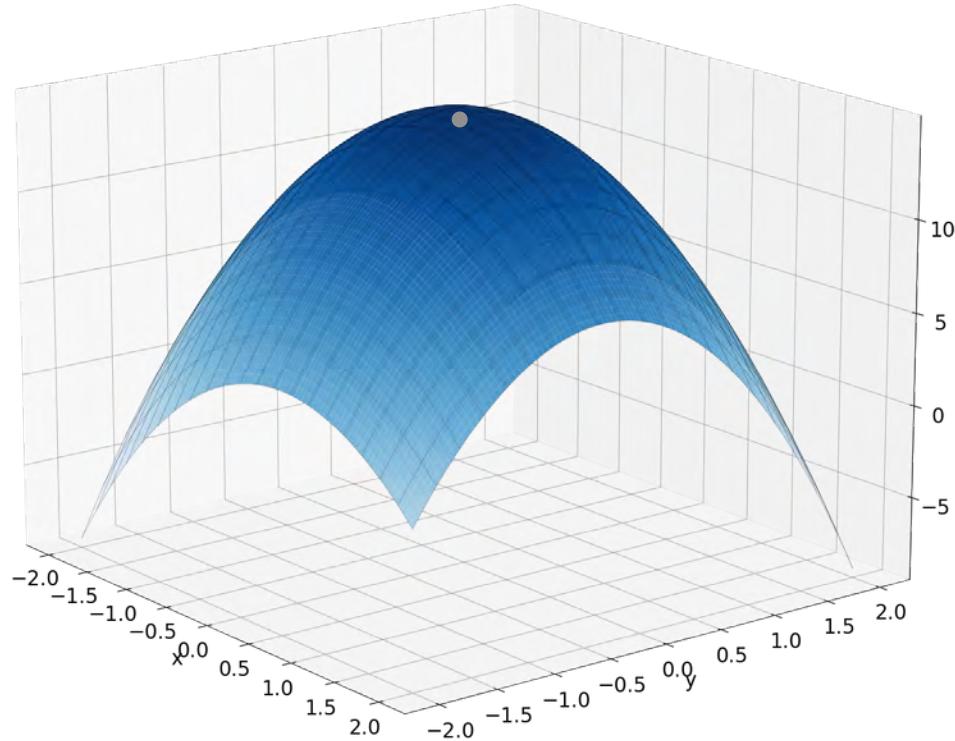
Concave Down

Concave Down



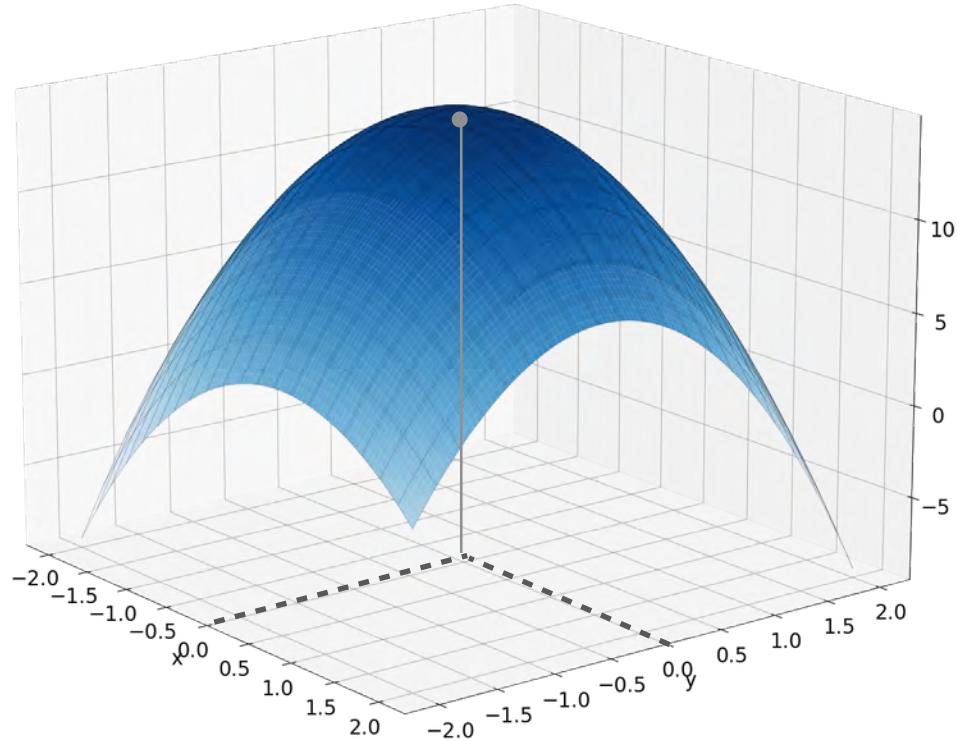
$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

Concave Down



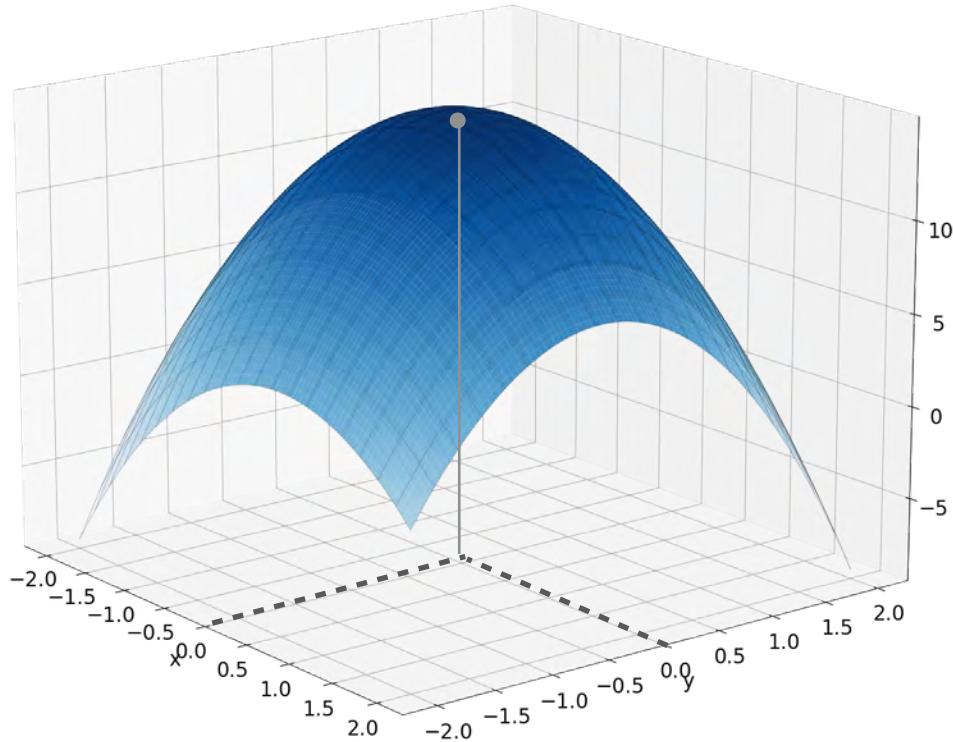
$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

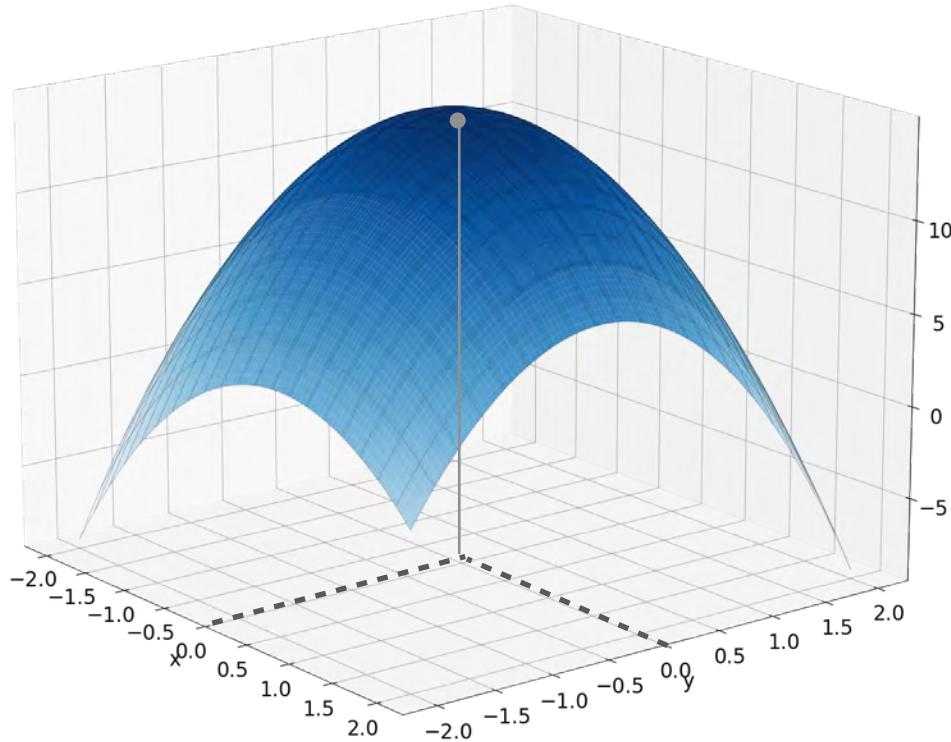
Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

Concave Down

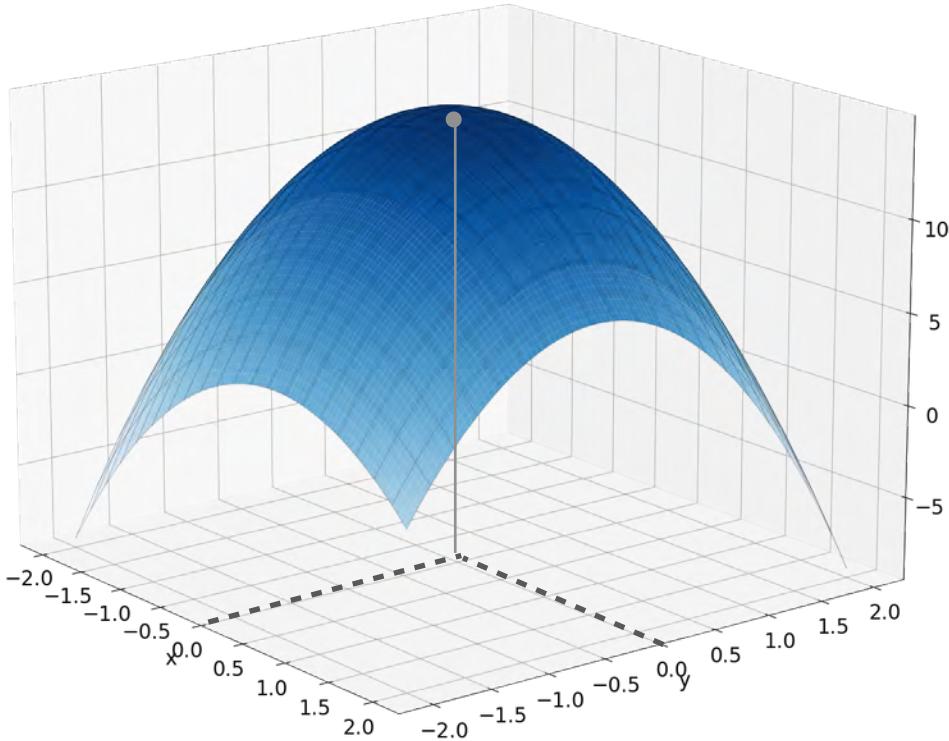


$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

Concave Down



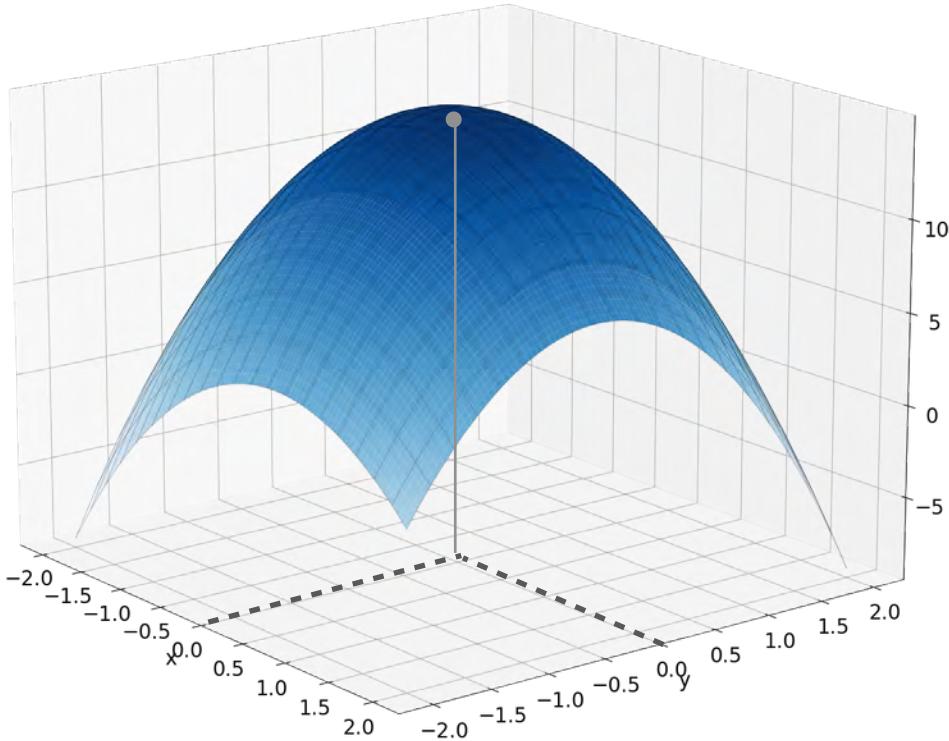
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$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) =$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

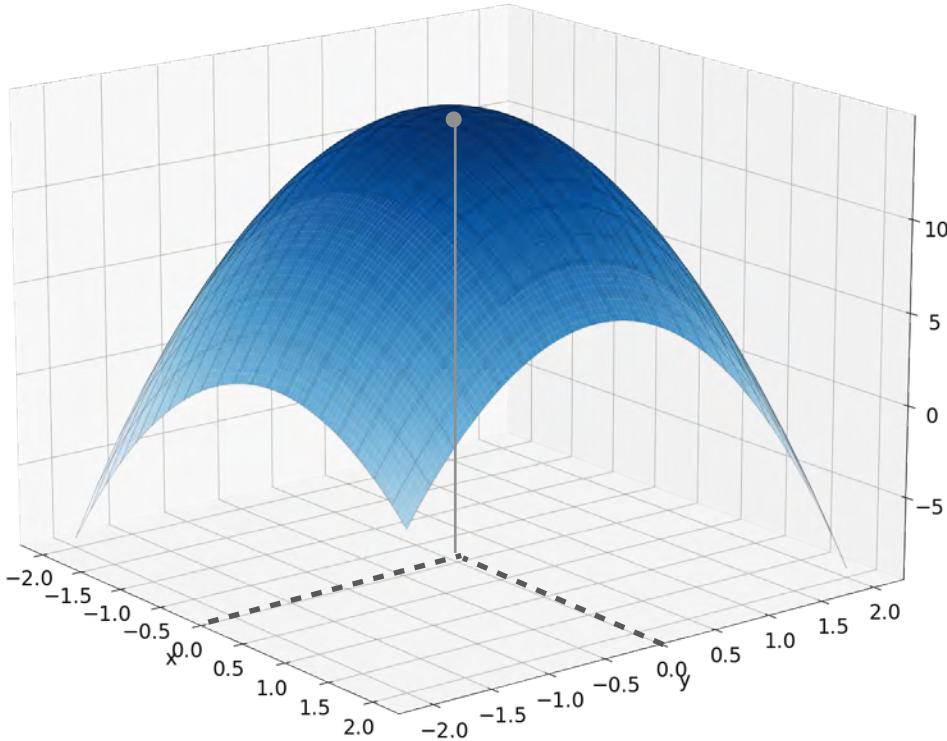
$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

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$$\det(H(0,0) - \lambda I) =$$

$$(-4 - \lambda)(-6 - \lambda) - (-1)(-1)$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

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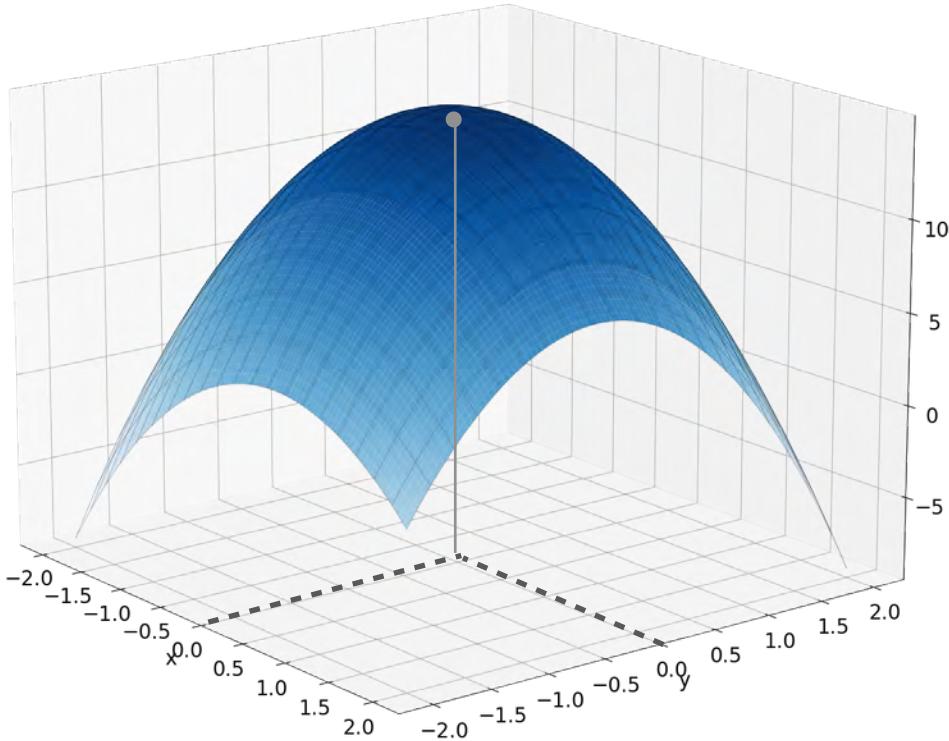
$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

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$$= \lambda^2 + 10\lambda + 23$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

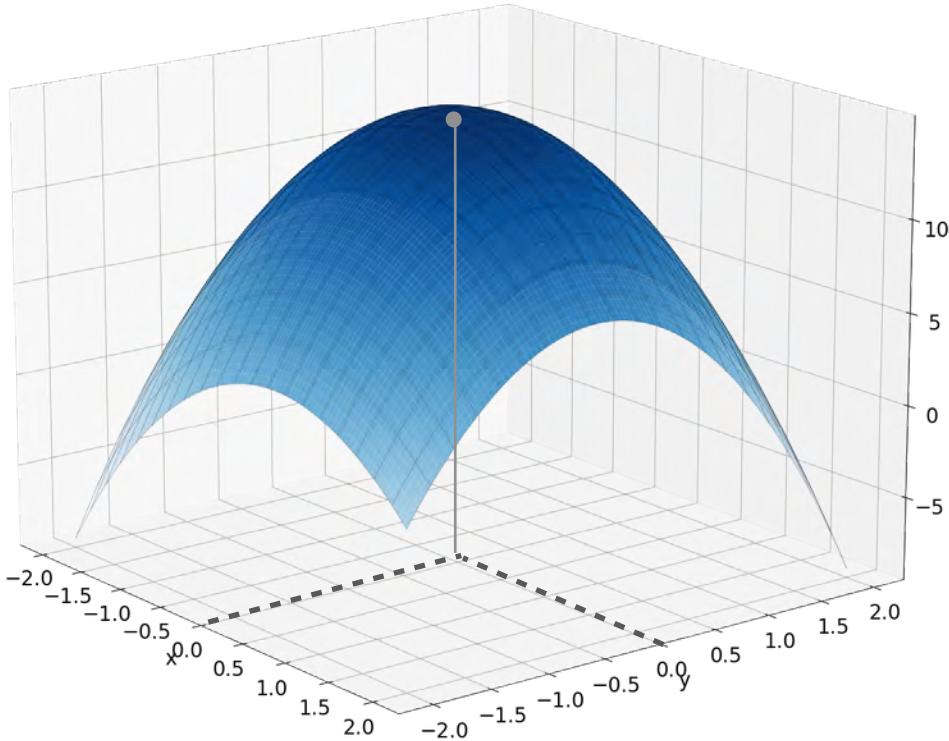
$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) =$$

$$(-4 - \lambda)(-6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 + 10\lambda + 23 \rightarrow \lambda_1 = -3.49$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

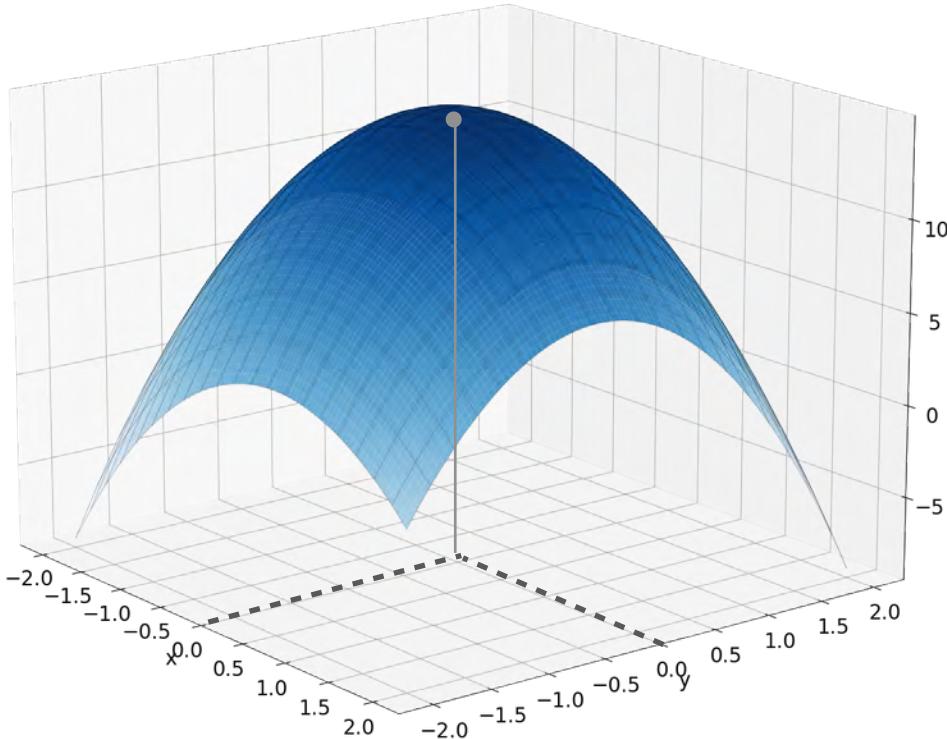
$$\det(H(0,0) - \lambda I) =$$

$$(-4 - \lambda)(-6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 + 10\lambda + 23$$

$$\begin{array}{l} \lambda_1 = -3.49 \\ \lambda_2 = -6.41 \end{array}$$

Concave Down



$$f(x, y) = -2x^2 - 3y^2 - xy + 15$$

$$\nabla f(x, y) = \begin{bmatrix} -4x - y \\ -x - 6y \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} -4 & -1 \\ -1 & -6 \end{bmatrix}$$

$$\det(H(0,0) - \lambda I) =$$

$$(-4 - \lambda)(-6 - \lambda) - (-1)(-1)$$

$$= \lambda^2 + 10\lambda + 23$$

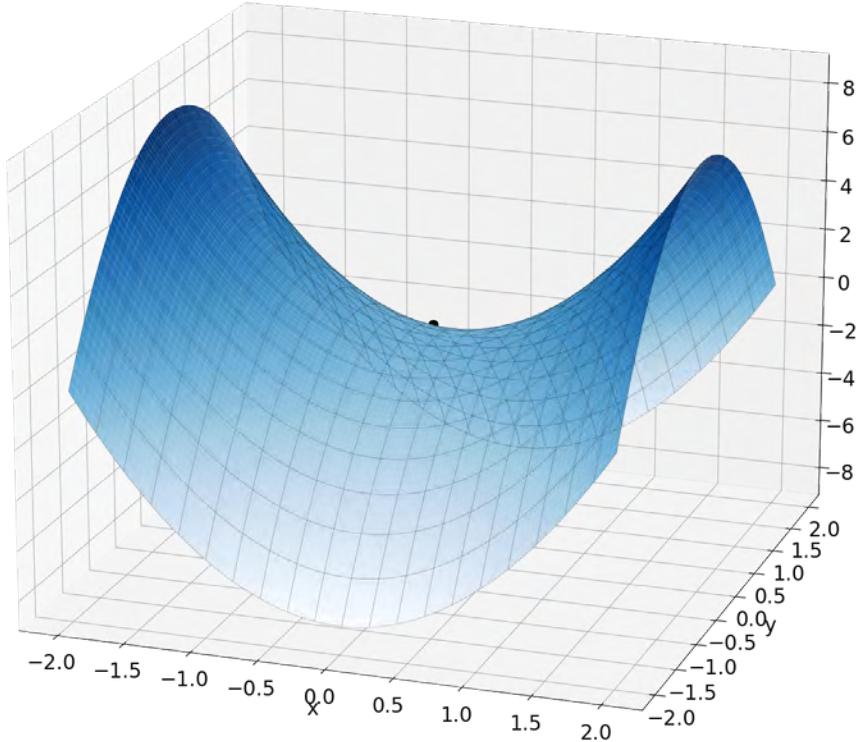
$$\lambda_1 = -3.49$$
$$\lambda_2 = -6.41$$

(0,0) is a maximum!

< 0

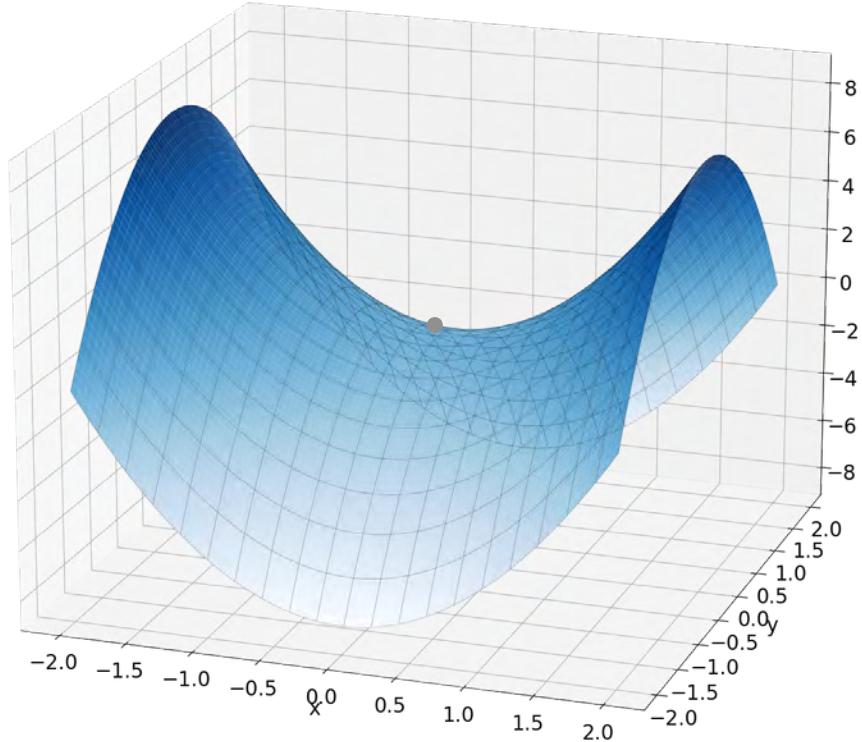
Saddle Point

Saddle Point



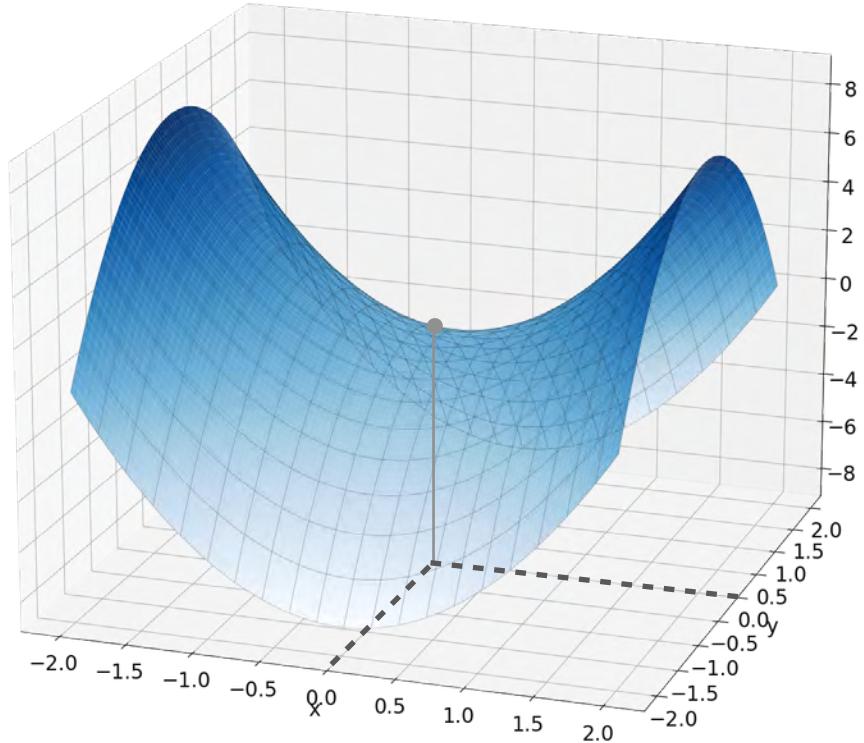
$$f(x, y) = 2x^2 - 2y^2$$

Saddle Point



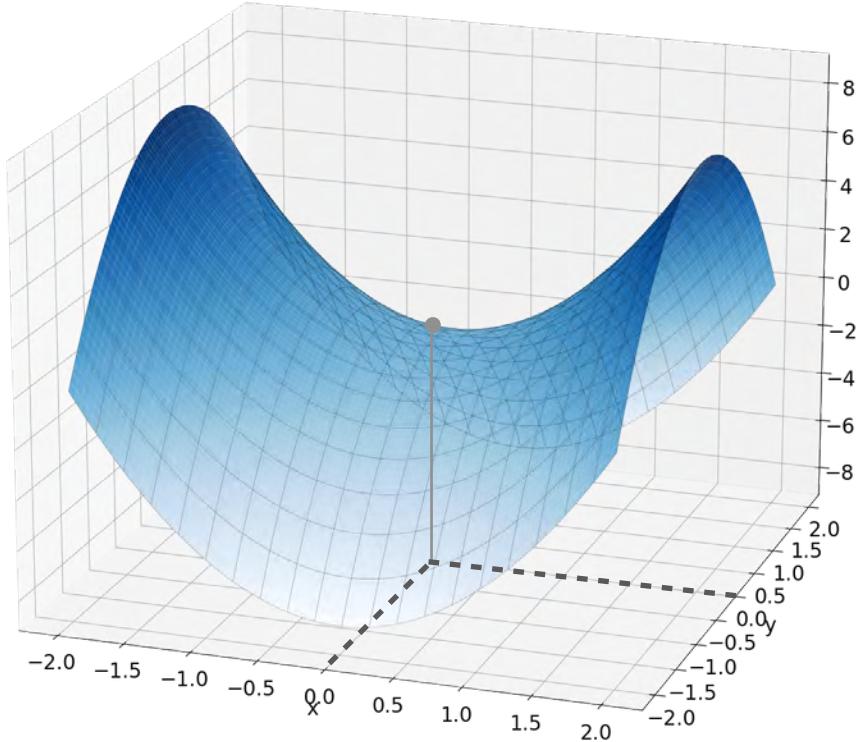
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Saddle Point



$$f(x, y) = 2x^2 - 2y^2$$

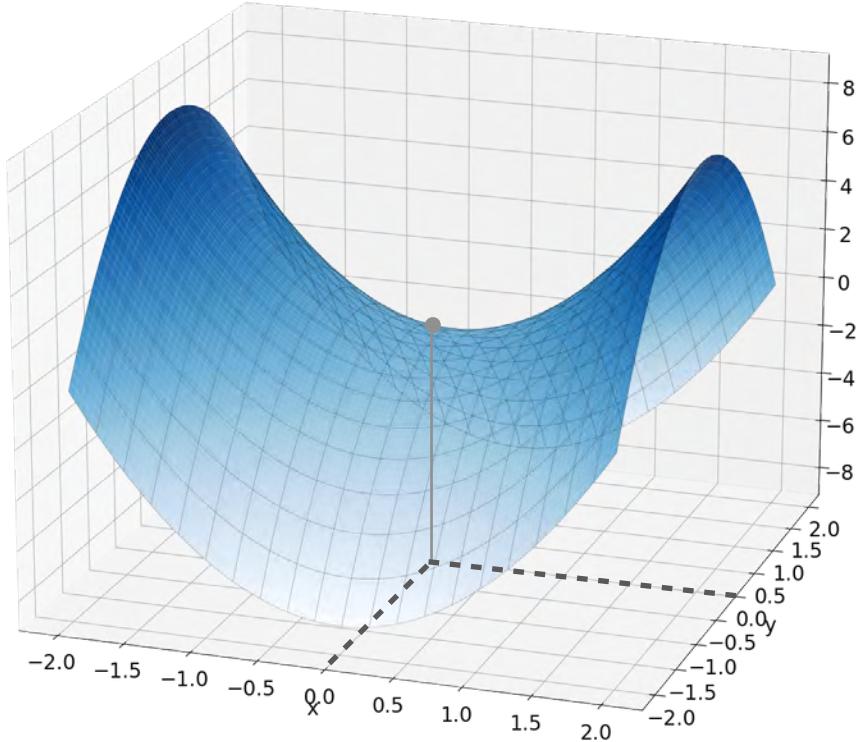
Saddle Point



$$f(x, y) = 2x^2 - 2y^2$$

$$\nabla f(x, y) = \begin{bmatrix} 4x \\ -4y \end{bmatrix}$$

Saddle Point

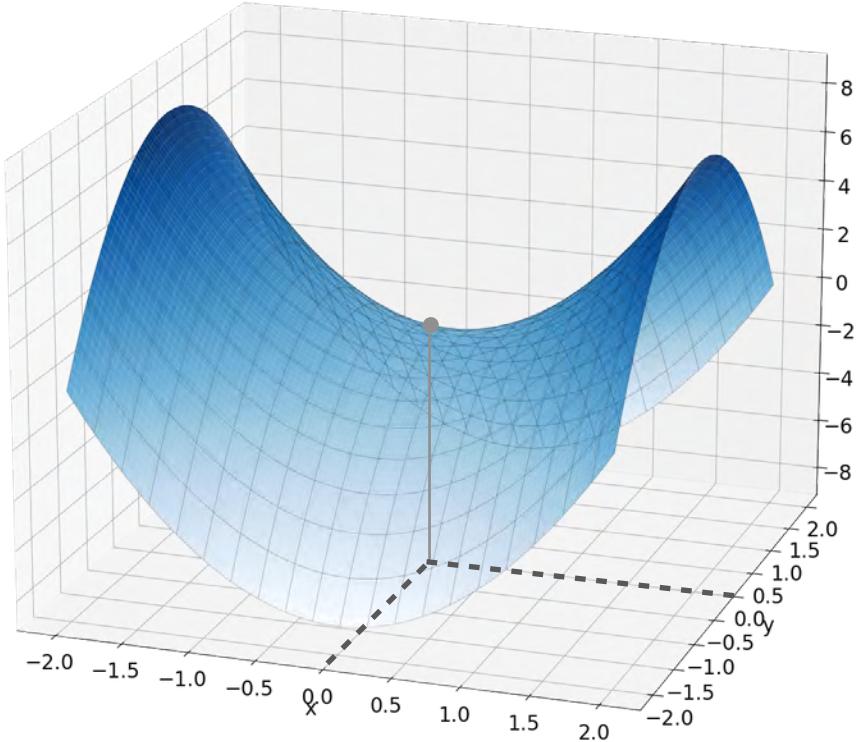


$$f(x, y) = 2x^2 - 2y^2$$

$$\nabla f(x, y) = \begin{bmatrix} 4x \\ -4y \end{bmatrix}$$

$$H(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

Saddle Point



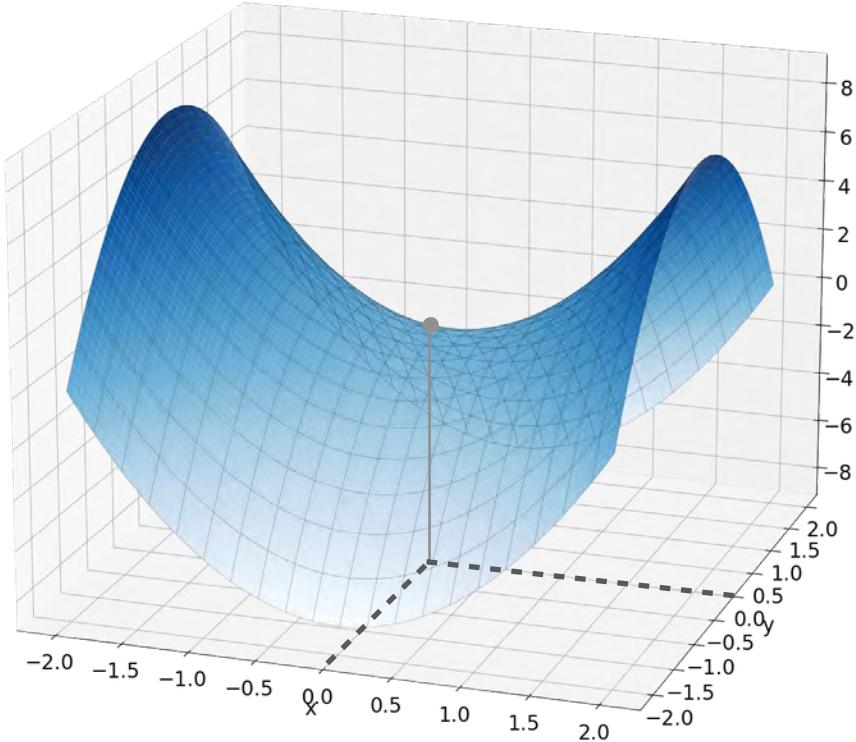
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$$\det(H(0,0) - \lambda I) =$$

Saddle Point



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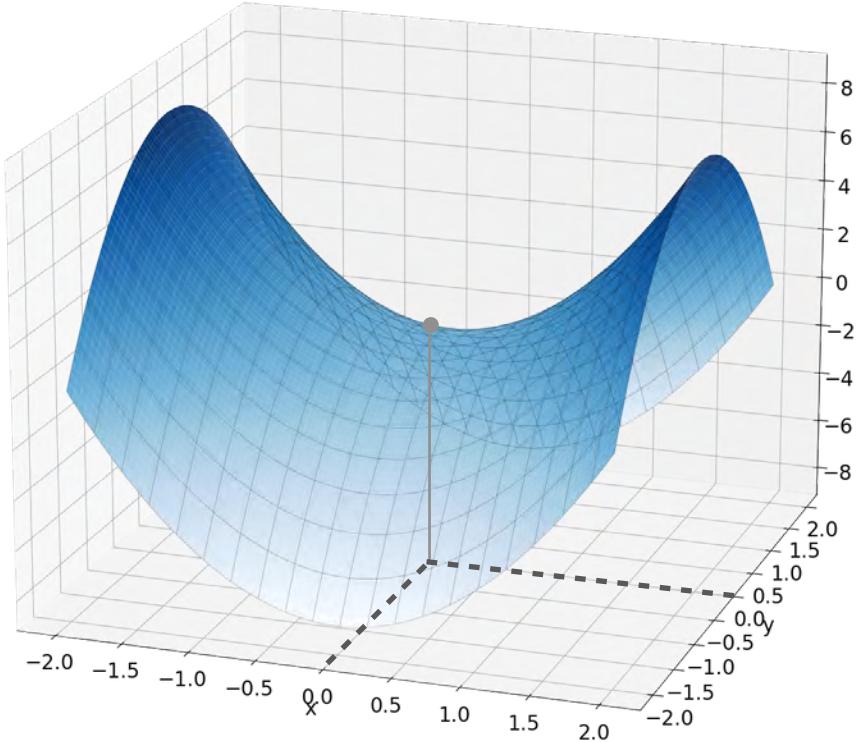
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$$(4 - \lambda)(-4 - \lambda) - 0$$

$$\lambda_1 = -4$$

Saddle Point



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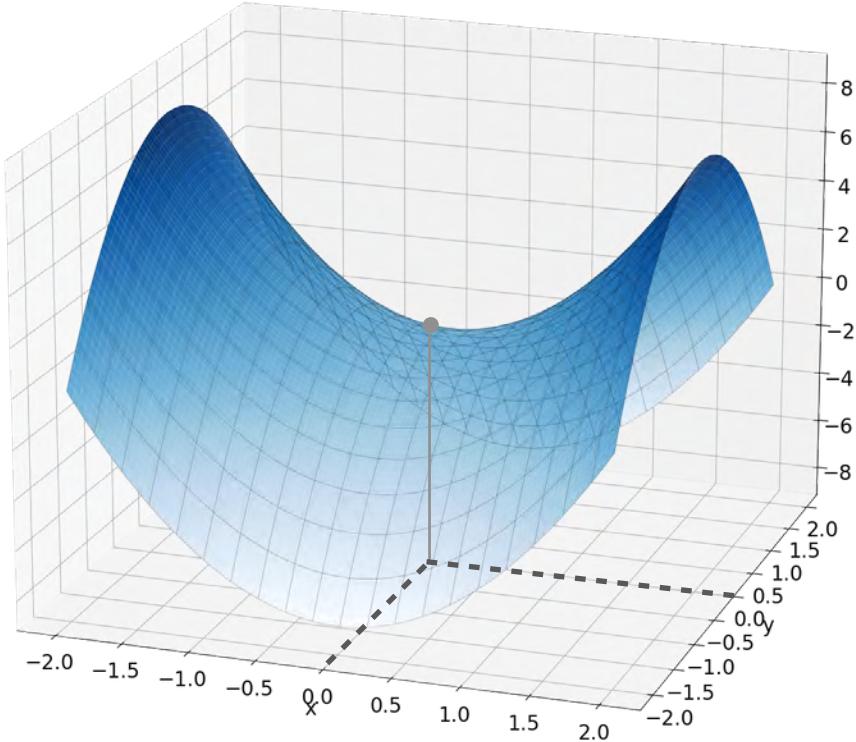
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$$\begin{array}{l} \xrightarrow{\hspace{1cm}} \lambda_1 = -4 \\ \xrightarrow{\hspace{1cm}} \lambda_2 = 4 \end{array}$$

Saddle Point



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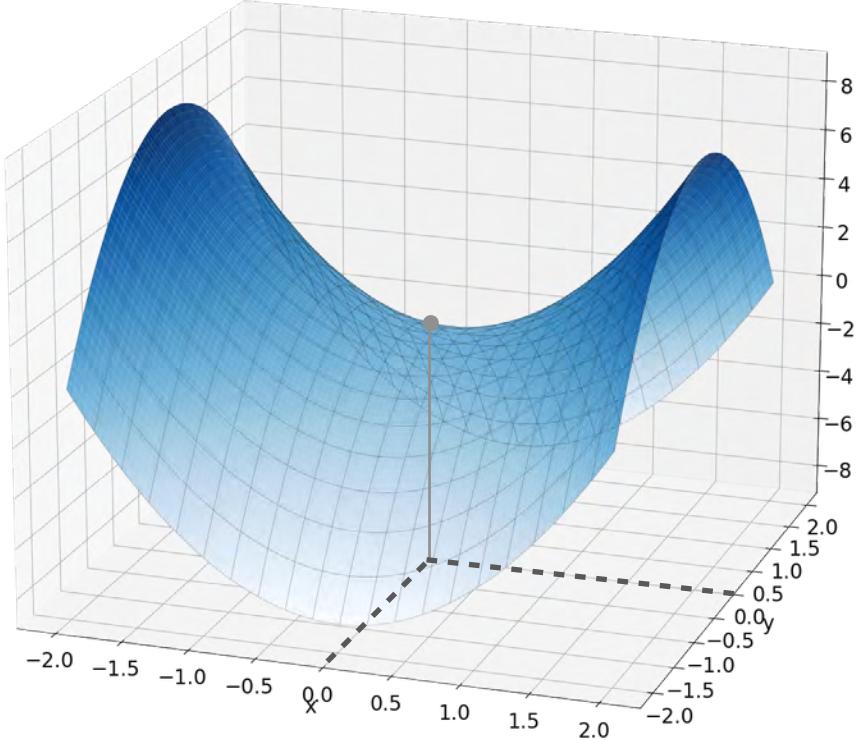
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$$(4 - \lambda)(-4 - \lambda) - 0 < 0$$

$\lambda_1 = -4$
 $\lambda_2 = 4$

(0,0) is saddle point

Saddle Point



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$$> 0$$

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(0,0) is saddle point

Summary

Summary

1 variable
 $f(x)$

2 variables
 $f(x, y)$

More variables
 $f(x_1, x_2, \dots, x_n)$

Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$		

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Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$

Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$
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Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$
(Local) maxima	Sad face $f''(x) < 0$	Down paraboloid $\lambda_1 < 0 \text{ & } \lambda_2 < 0$	

Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$
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Need more information	$f''(x) = 0$		

Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
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Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$
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Need more information	$f''(x) = 0$	Saddle point $\lambda_1 > 0 \text{ & } \lambda_2 < 0$ $\lambda_1 < 0 \text{ & } \lambda_2 > 0$ Or some $\lambda_i = 0$	

Summary

	1 variable $f(x)$	2 variables $f(x, y)$	More variables $f(x_1, x_2, \dots, x_n)$
(Local) minima	Happy face $f''(x) > 0$	Upper paraboloid $\lambda_1 > 0 \text{ & } \lambda_2 > 0$	All $\lambda_i > 0$
(Local) maxima	Sad face $f''(x) < 0$	Down paraboloid $\lambda_1 < 0 \text{ & } \lambda_2 < 0$	All $\lambda_i < 0$
Need more information	$f''(x) = 0$	Saddle point $\lambda_1 > 0 \text{ & } \lambda_2 < 0$ $\lambda_1 < 0 \text{ & } \lambda_2 > 0$ Or some $\lambda_i = 0$	Some $\lambda_i > 0$ and some $\lambda_j < 0$ OR At least one $\lambda_i = 0$



DeepLearning.AI

Optimization in Neural Networks and Newton's Method

**Newton's method for two
variables**

Newton's Method

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

Newton's Method

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$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

2 variables

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix}$$

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} -$$

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - H^{-1}(x_k, y_k)$$

Newton's Method

1 variable

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k - f''(x_k)^{-1} f'(x_k)$$

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - H^{-1}(x_k, y_k) \nabla f(x_k, y_k)$$

Newton's Method

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - H^{-1}(x_k, y_k) \nabla f(x_k, y_k)$$

Newton's Method

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \textcolor{orange}{H^{-1}(x_k, y_k)} \quad \textcolor{teal}{\nabla f(x_k, y_k)}$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \nabla f(x_k, y_k) \quad H^{-1}(x_k, y_k)$$

Newton's Method

2 variables

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Newton's Method

2 variables

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \underbrace{\mathbf{H}^{-1}(x_k, y_k)}_{2 \times 2} \underbrace{\nabla f(x_k, y_k)}_{2 \times 1}$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \nabla f(x_k, y_k) \cancel{- \mathbf{H}^{-1}(x_k, y_k)}$$

Newton's Method

2 variables

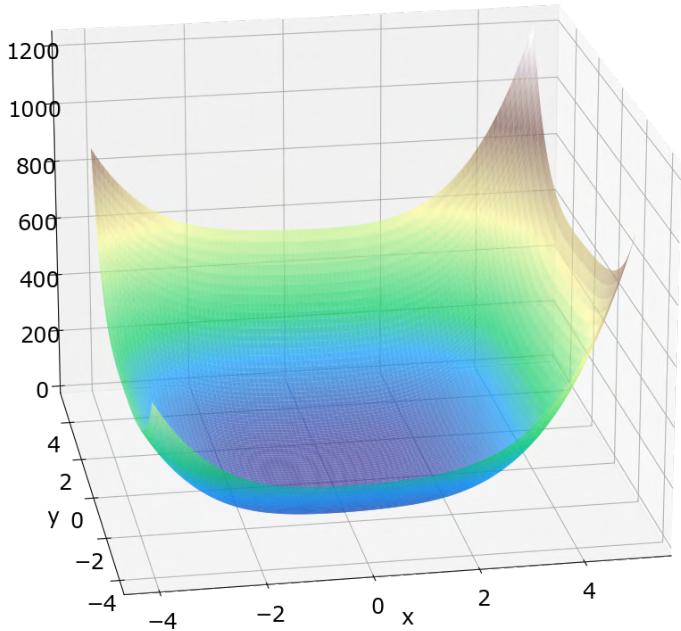
$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \underbrace{\mathbf{H}^{-1}(x_k, y_k)}_{2 \times 2} \underbrace{\nabla f(x_k, y_k)}_{2 \times 1}$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \nabla f(x_k, y_k) \cancel{- H^{-1}(x_k, y_k)}$$

When working with 2 variables the order is crucial!

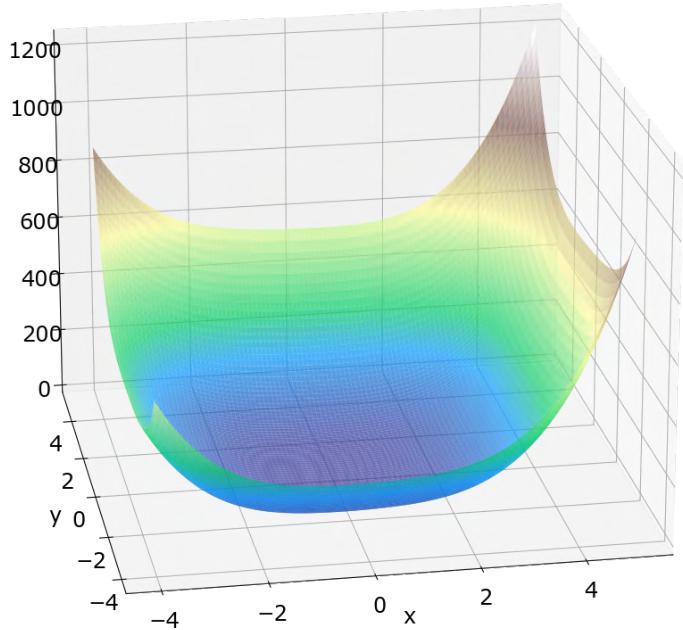
An Example

An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

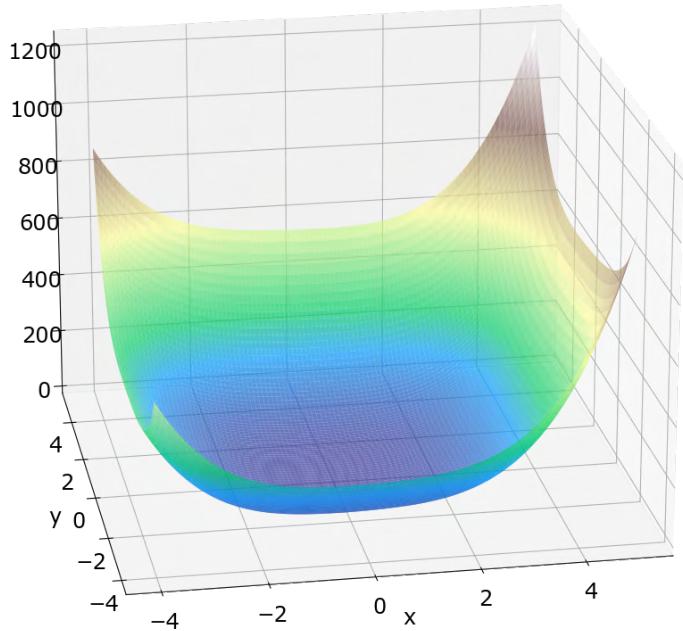
An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

$$f(x, y) \rightarrow 4x^3 + 8x - y - 0.4xy$$

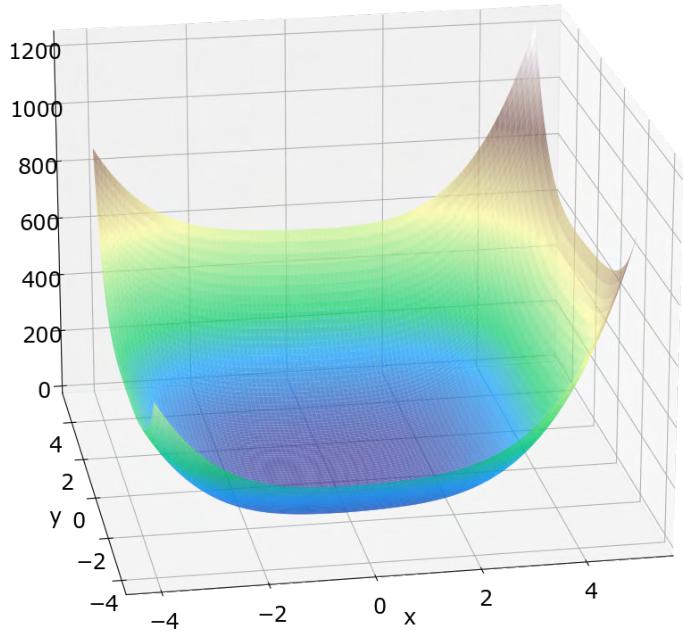
An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

$$\begin{array}{l} \textcolor{blue}{x} \nearrow 4x^3 + 8x - y - 0.4xy \\ f(x, y) \\ \textcolor{orange}{y} \searrow 3.2y^3 + 4y - x - 0.2x^2 \end{array}$$

An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

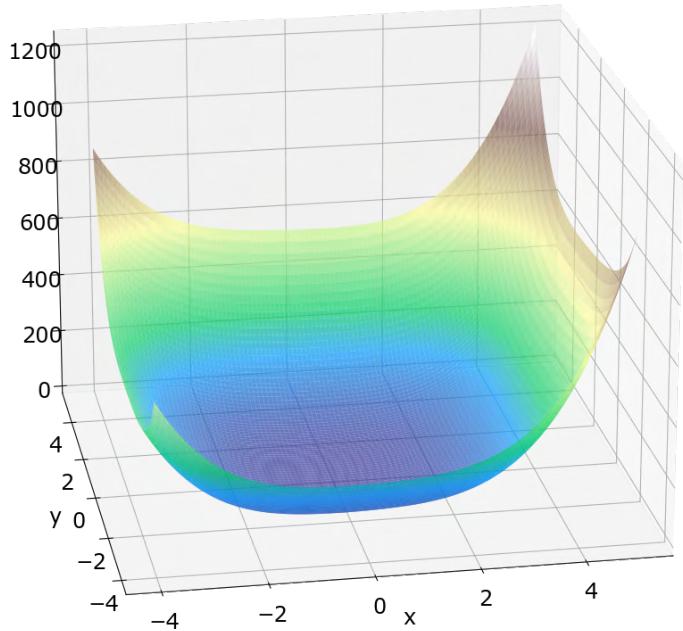
$f(x, y)$

x $12x^2 + 8 - 0.4y$

$4x^3 + 8x - y - 0.4xy$

y $3.2y^3 + 4y - x - 0.2x^2$

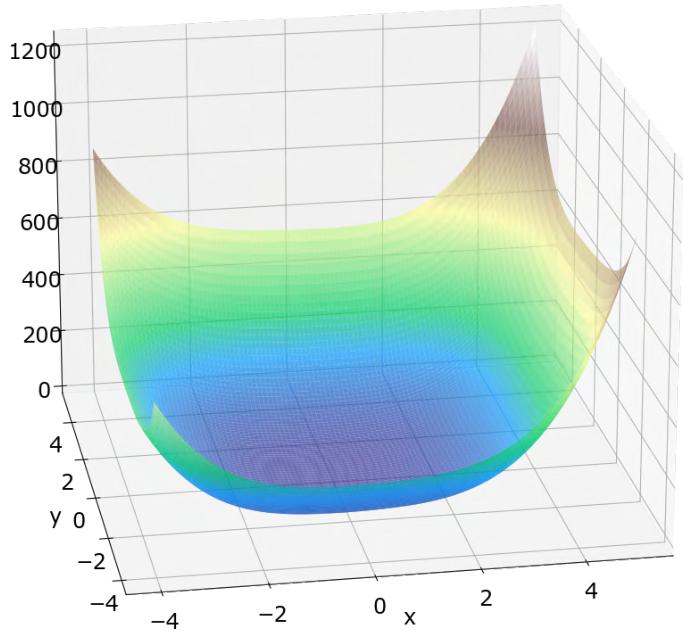
An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

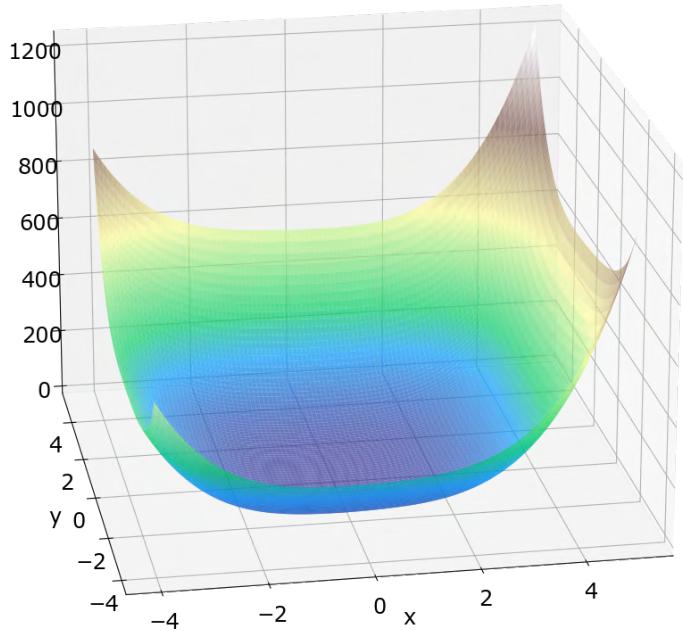
$$\begin{aligned} f(x, y) & \xrightarrow{x} 4x^3 + 8x - y - 0.4xy \\ & \xrightarrow{y} 3.2y^3 + 4y - x - 0.2x^2 \\ & \quad \quad \quad \xrightarrow{x} 12x^2 + 8 - 0.4y \\ & \quad \quad \quad \xrightarrow{y} -1 - 0.4x \end{aligned}$$

An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$
$$\begin{aligned} f(x, y) &\xrightarrow{x} 4x^3 + 8x - y - 0.4xy & \xrightarrow{y} 12x^2 + 8 - 0.4y \\ &\xrightarrow{y} 3.2y^3 + 4y - x - 0.2x^2 & \xrightarrow{x} -1 - 0.4x \end{aligned}$$

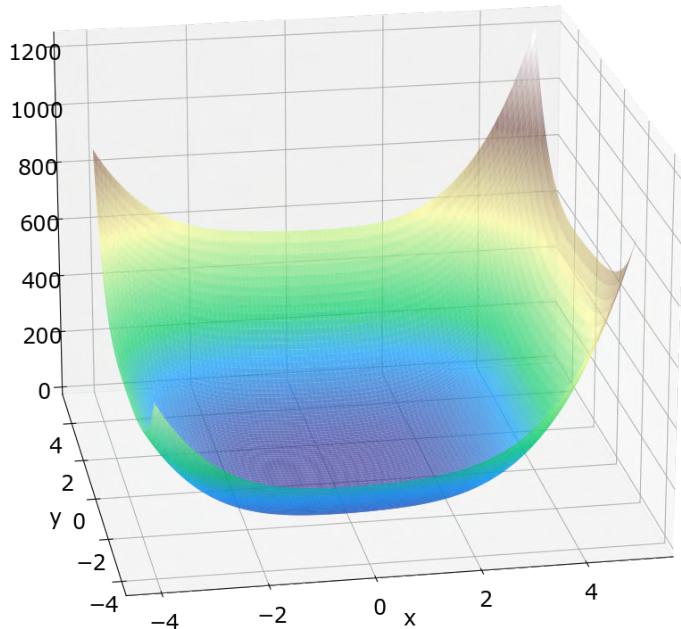
An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

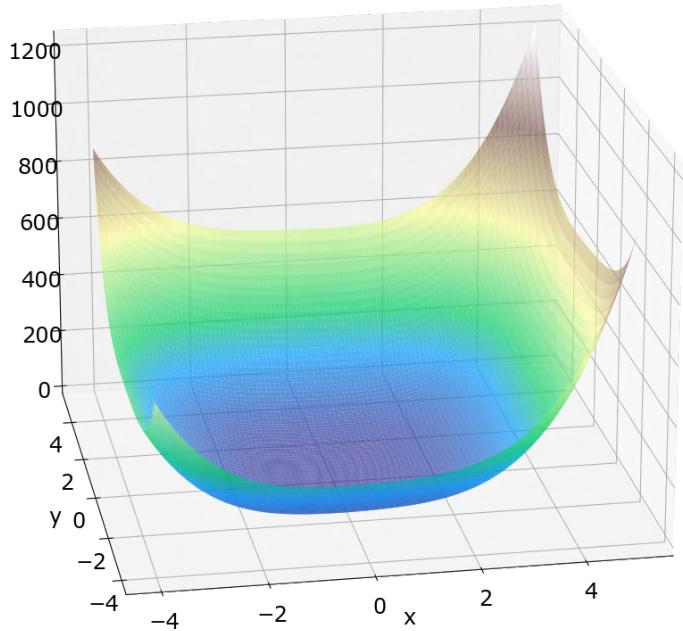
$$\begin{aligned} f(x, y) &\quad \begin{array}{l} \nearrow x \\ \searrow y \end{array} \\ &= 4x^3 + 8x - y - 0.4xy && \begin{array}{l} \nearrow x \\ \searrow y \end{array} & 12x^2 + 8 - 0.4y \\ &\quad \begin{array}{l} \nearrow x \\ \searrow y \end{array} && & -1 - 0.4x \\ &= 3.2y^3 + 4y - x - 0.2x^2 && \begin{array}{l} \nearrow x \\ \searrow y \end{array} & -1 - 0.4x \\ &\quad \begin{array}{l} \nearrow x \\ \searrow y \end{array} && & 9.6y^2 + 4 \end{aligned}$$

An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

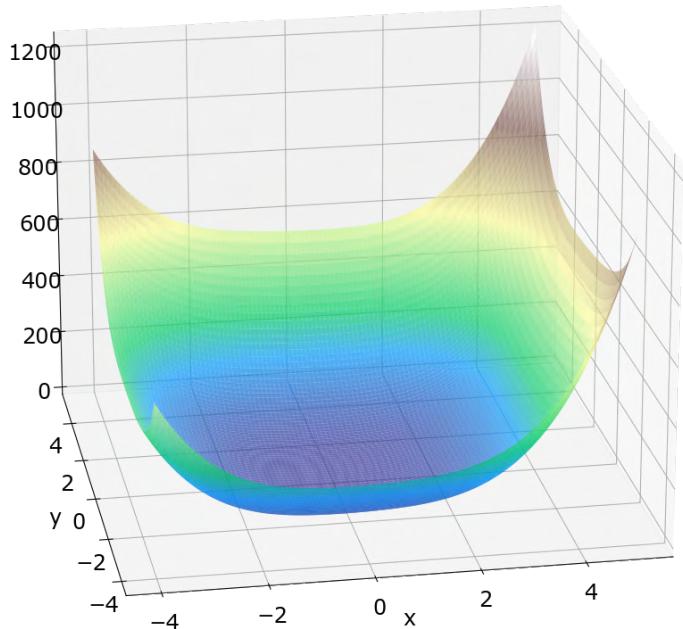
An Example



$$f(x, y) = x^4 + 0.8y^4 + 4x^2 + 2y^2 - xy - 0.2x^2y$$

$$\nabla f(x, y) = \begin{bmatrix} 4x^3 + 8x - y - 0.4xy \\ 3.2y^3 + 4y - x - 0.2x^2 \end{bmatrix}$$

An Example

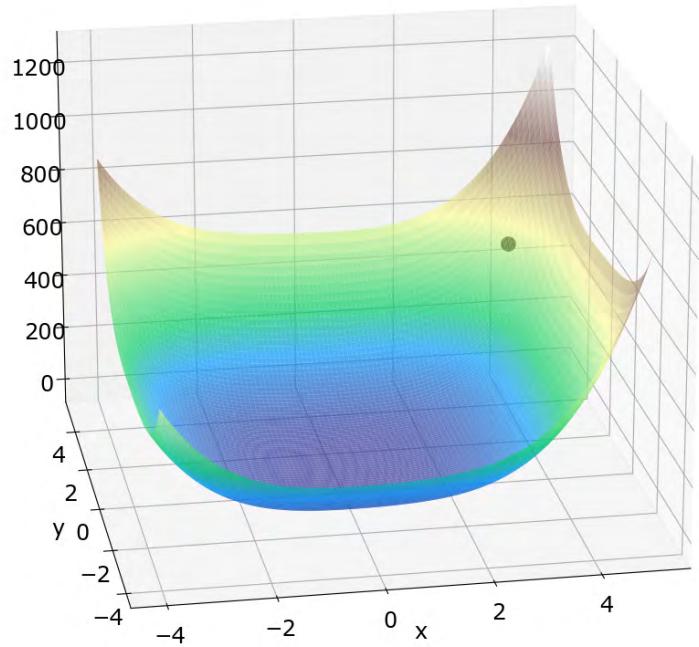


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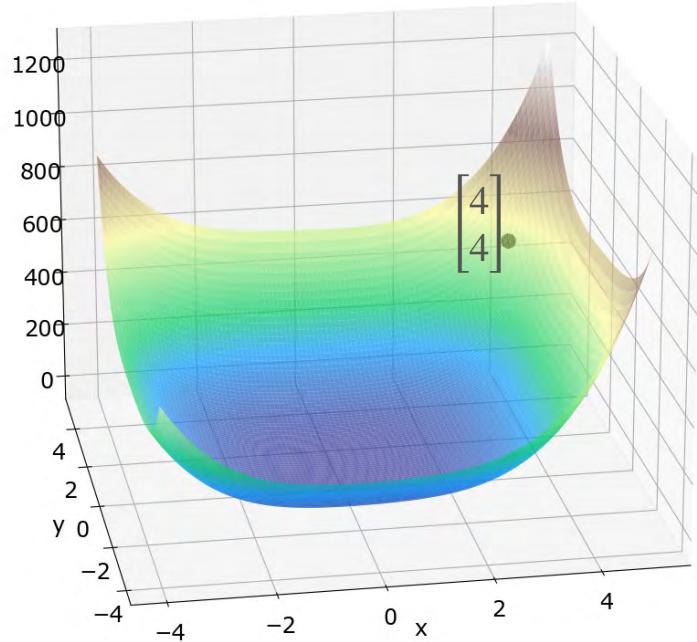
$$\nabla f(x, y) = \begin{bmatrix} 4x^3 + 8x - y - 0.4xy \\ 3.2y^3 + 4y - x - 0.2x^2 \end{bmatrix}$$

$$H(x, y) = \begin{bmatrix} 12x^2 + 8 - 0.4y & -1 - 0.4x \\ -1 - 0.4x & 9.6y^2 + 4 \end{bmatrix}$$

An Example

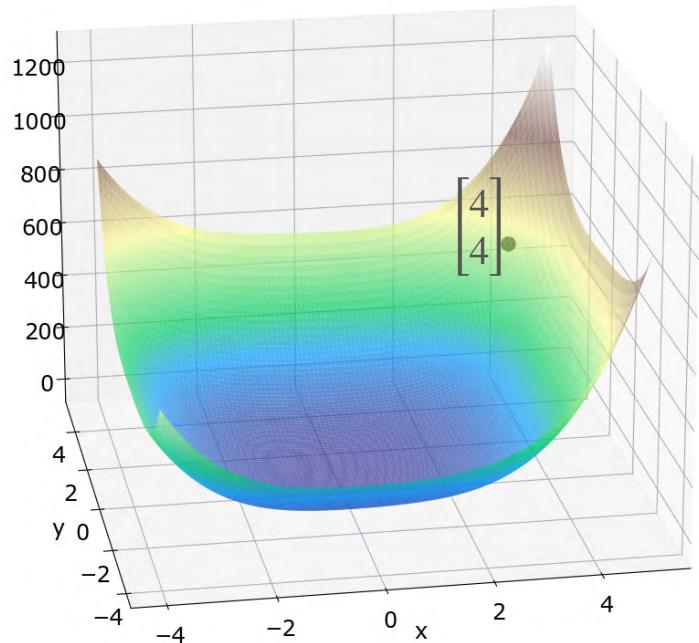


An Example



Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

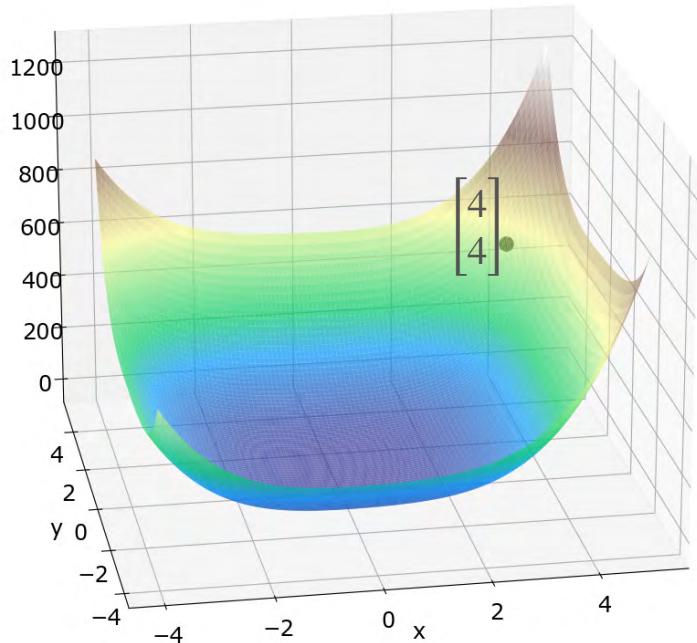
An Example



Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$$\nabla f(4,4) = \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix}$$

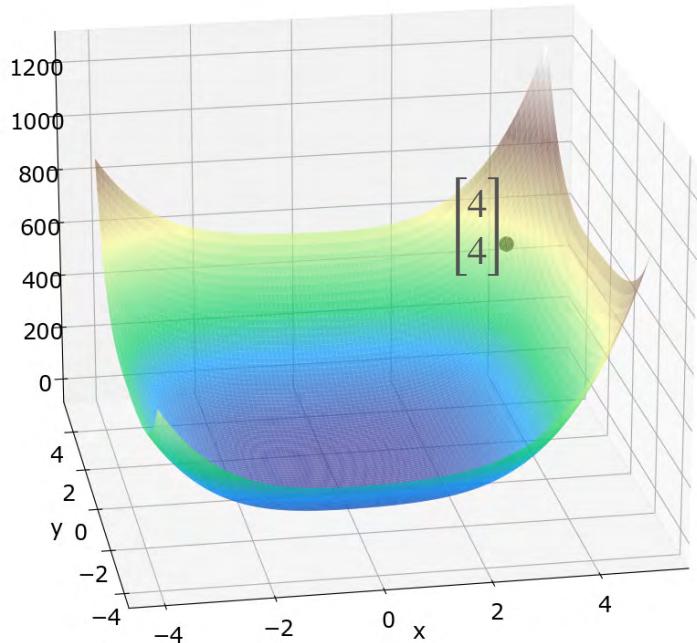
An Example



Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$$\nabla f(4,4) = \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix} \quad H(4,4) = \begin{bmatrix} 198.4 & -2.6 \\ -2.6 & 157.6 \end{bmatrix}$$

An Example

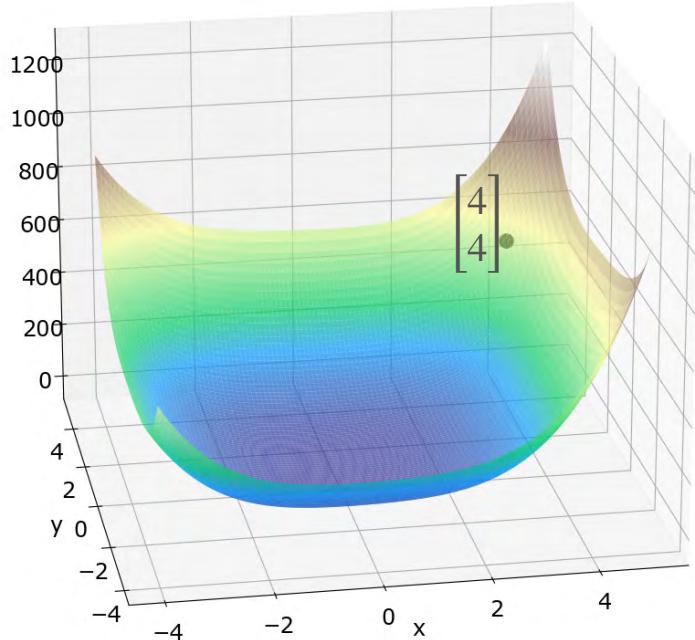


Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$$\nabla f(4,4) = \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix} \quad H(4,4) = \begin{bmatrix} 198.4 & -2.6 \\ -2.6 & 157.6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} -$$

An Example

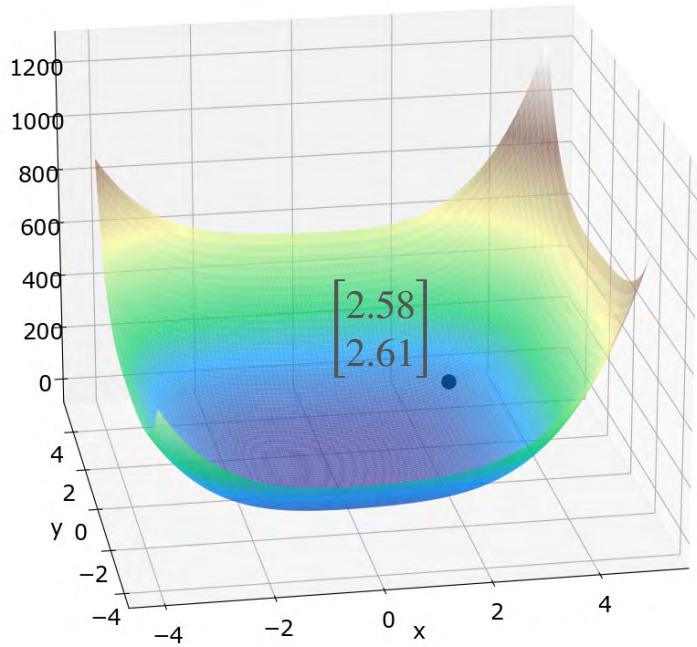


Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$$\nabla f(4,4) = \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix} \quad H(4,4) = \begin{bmatrix} 198.4 & -2.6 \\ -2.6 & 157.6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 198.4 & -2.6 \\ -2.6 & 157.6 \end{bmatrix}^{-1} \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix}$$

An Example



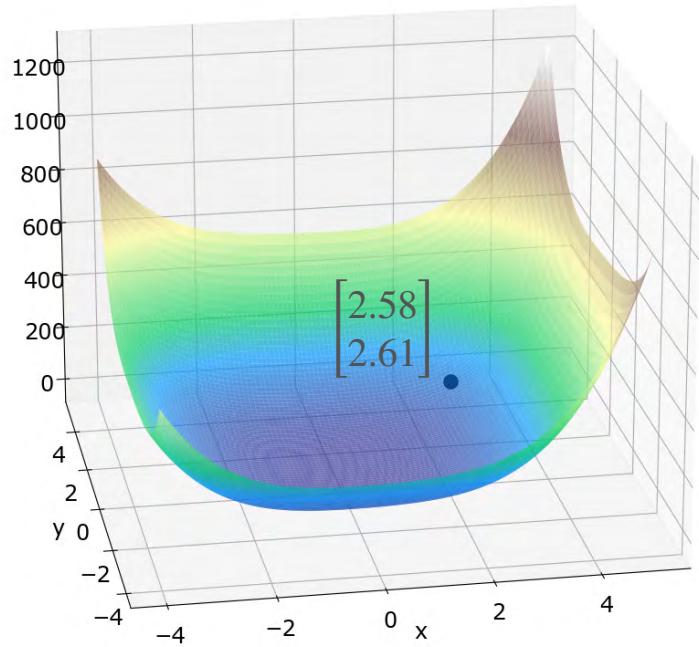
Start at some point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

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$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 198.4 & -2.6 \\ -2.6 & 157.6 \end{bmatrix}^{-1} \begin{bmatrix} 277.6 \\ 213.6 \end{bmatrix}$$

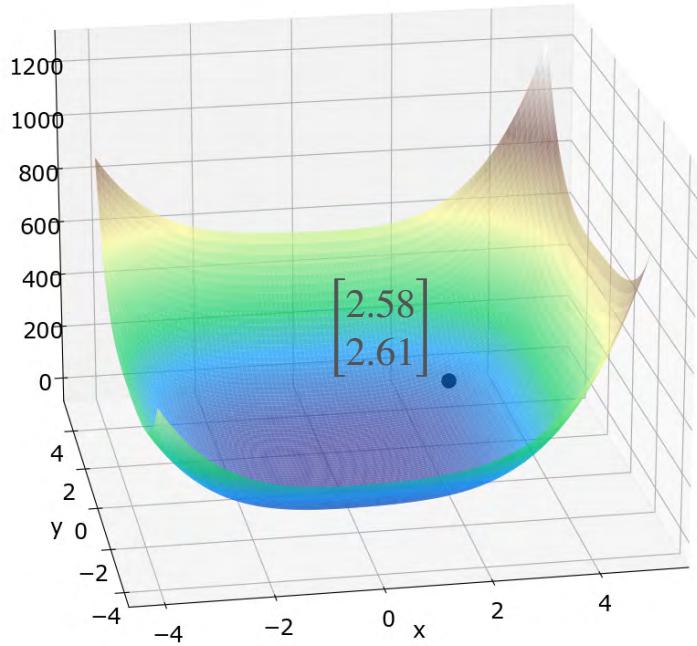
$$= \begin{bmatrix} 2.58 \\ 2.62 \end{bmatrix}$$

An Example



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

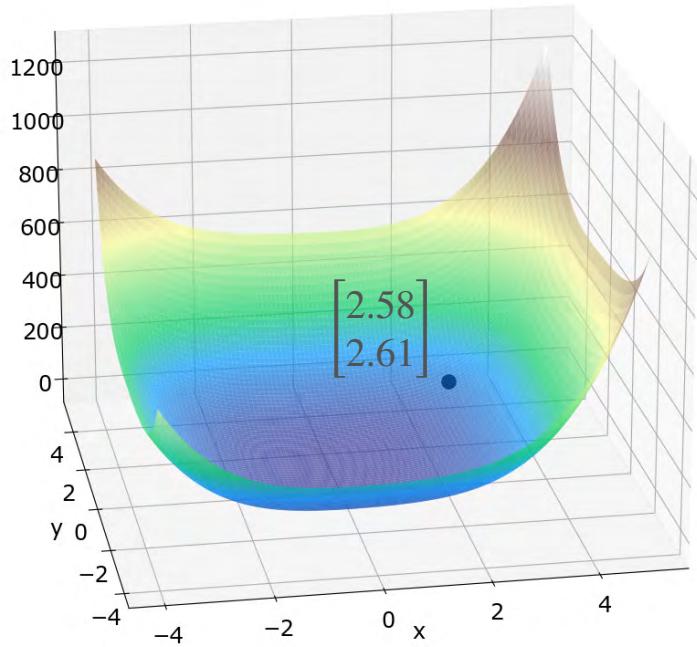
An Example



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

$$\nabla f(2.58, 2.61) = \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix}$$

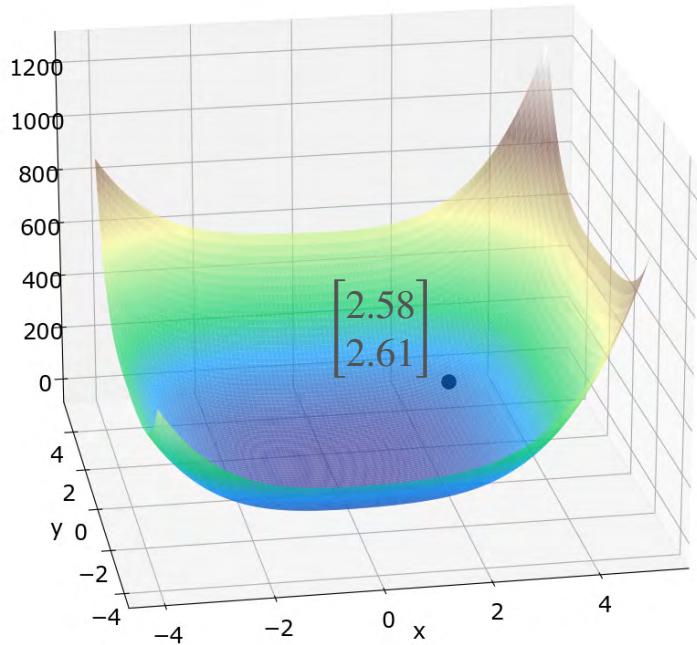
An Example



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

$$\nabla f(2.58, 2.61) = \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix} \quad H(2.58, 2.61) = \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}$$

An Example

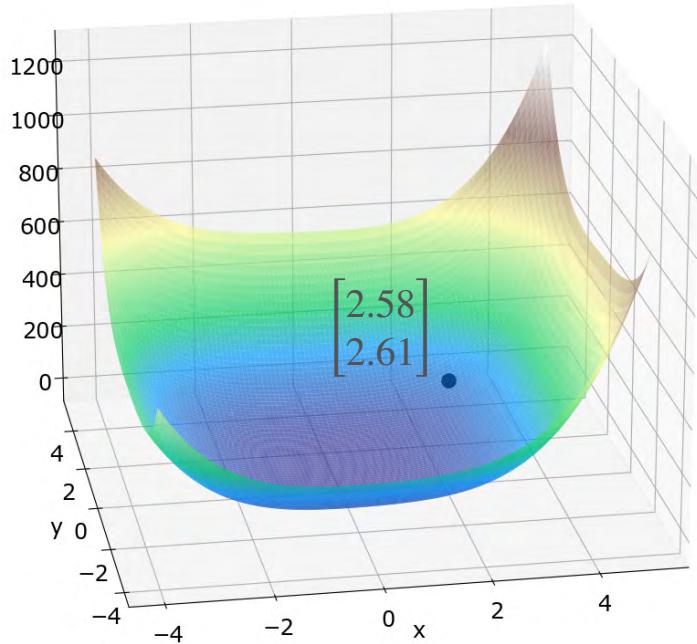


$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

$$\nabla f(2.58, 2.61) = \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix} \quad H(2.58, 2.61) = \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix} -$$

An Example

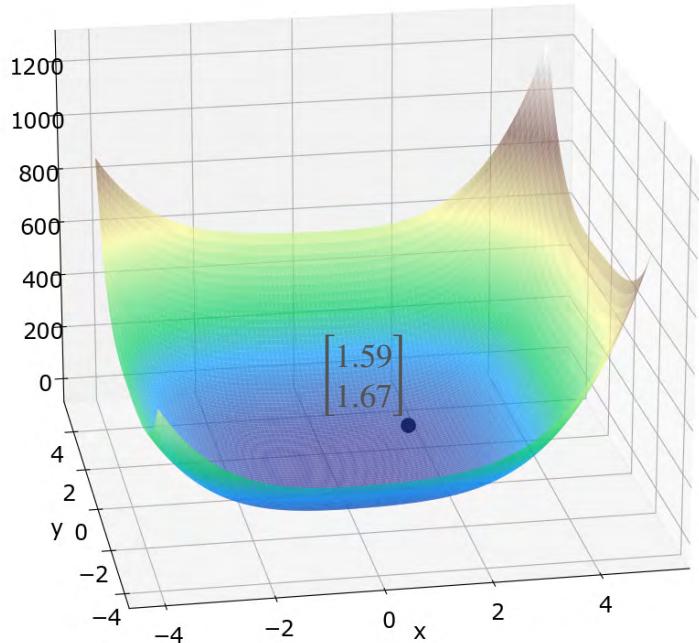


$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

$$\nabla f(2.58, 2.61) = \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix} H(2.58, 2.61) = \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix} - \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}^{-1} \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix}$$

An Example



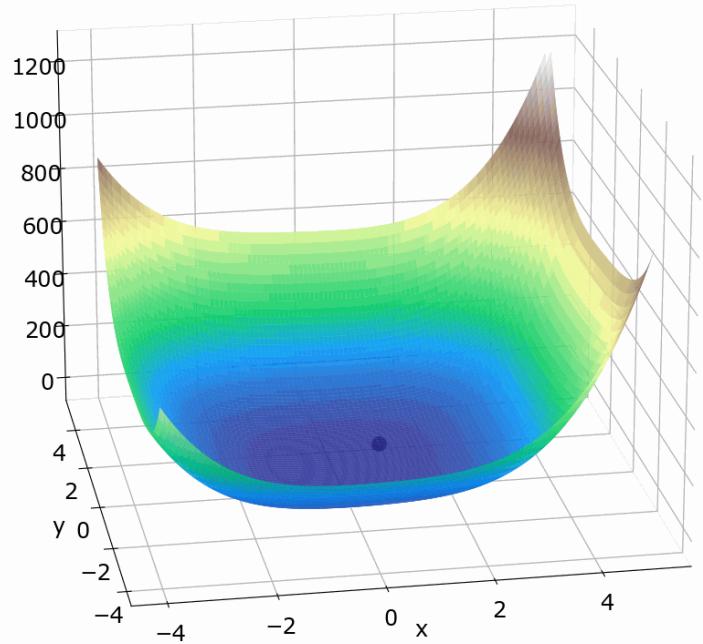
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix}$$

$$\nabla f(2.58, 2.61) = \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix} H(2.58, 2.61) = \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}$$

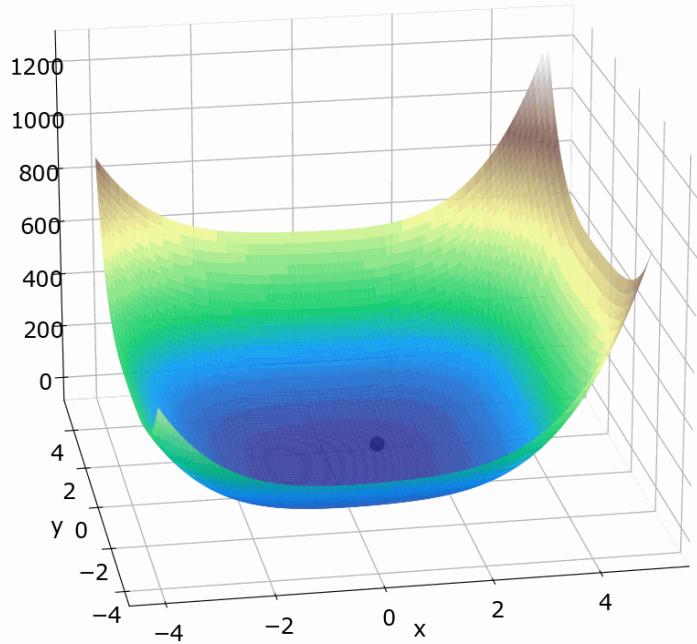
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.58 \\ 2.61 \end{bmatrix} - \begin{bmatrix} 86.83 & -2.032 \\ -2.032 & 69.39 \end{bmatrix}^{-1} \begin{bmatrix} 84.25 \\ 63.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.59 \\ 1.67 \end{bmatrix}$$

An Example

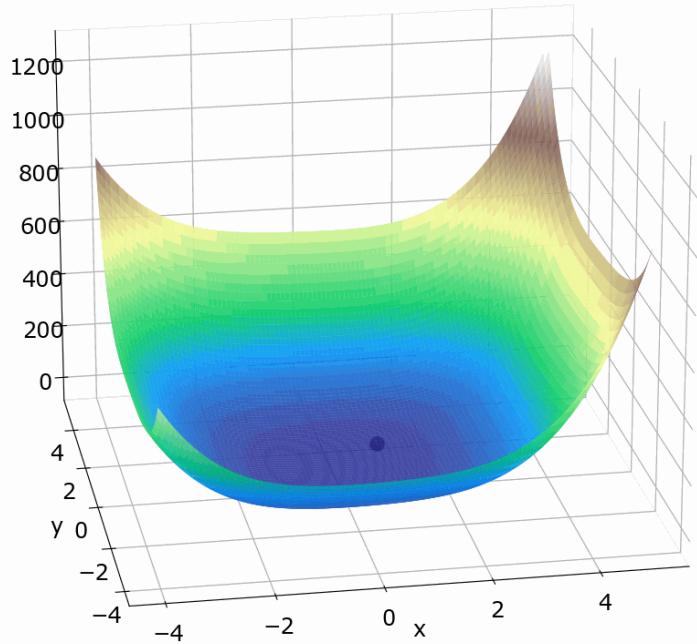


An Example



Repeat until you are close enough to the actual zero!

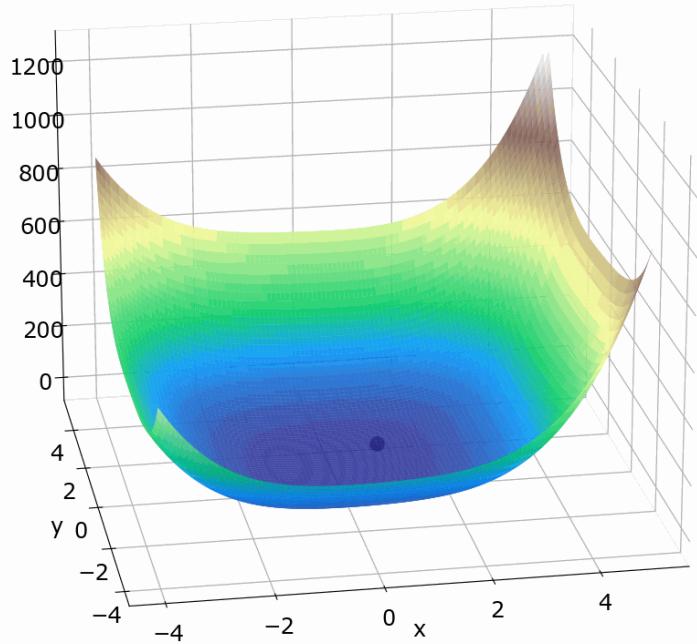
An Example



Repeat until you are close enough to the actual zero!

Needed $k = 8$ steps

An Example

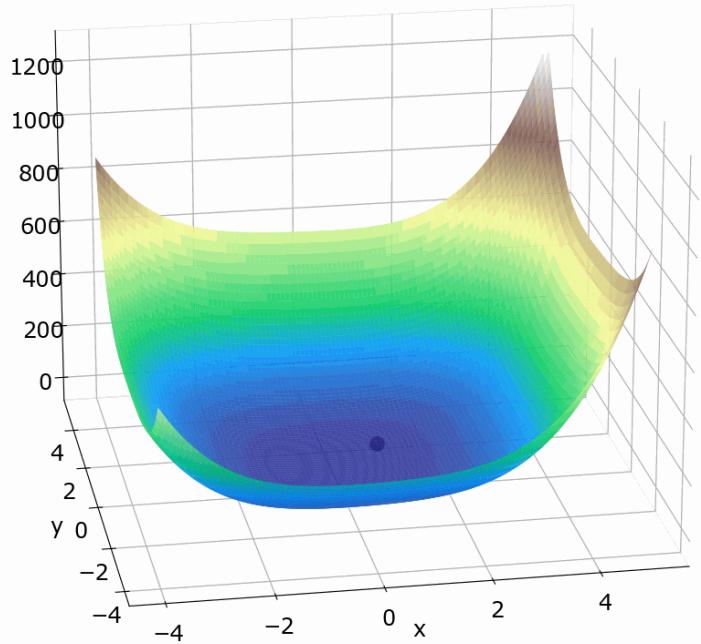


Repeat until you are close enough to the actual zero!

Needed $k = 8$ steps

$$\begin{bmatrix} x_8 \\ y_8 \end{bmatrix} = \begin{bmatrix} 4.15 \cdot 10^{-17} \\ -2.05 \cdot 10^{-17} \end{bmatrix}$$

An Example



Repeat until you are close enough to the actual zero!

Needed $k = 8$ steps

$$\begin{bmatrix} x_8 \\ y_8 \end{bmatrix} = \begin{bmatrix} 4.15 \cdot 10^{-17} \\ -2.05 \cdot 10^{-17} \end{bmatrix}$$

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



DeepLearning.AI

Optimization in Neural Networks and Newton's Method

Conclusion