Week 02

1) Solving non-singular system of linear equations

Manipulating equations

1) Multiplying by a constant (Divide)

$$\begin{array}{r} a+b = 10 \\ 7 \times \\ \hline 7a+7b = 70 \end{array}$$

2) Adding two equations (Subtract)

$$\begin{array}{r} a+b = 10 \\ 2a+3b = 22 \\ \hline 3a+4b = 32 \end{array}$$

Example 01:

*) $5a + b = 17 \longrightarrow a + 0.2b = 3.4$

$4a - 3b = 6 \longrightarrow a - 0.75b = 1.5$ } Divide by a coefficient

$$\begin{array}{r} a - 0.75b = 1.5 \\ a + 0.2b = 3.4 \\ \hline 0a - 0.95b = -1.9 \end{array}$$

$\rightarrow b = \frac{-1.9}{0.95}$

$b = 2$ // $\longrightarrow$

$5a + 2 = 17$

$5a = 15$

$a = 3$ //

ProMate

Example 02

*] $5a + b = 17$

$3b = 6 \longrightarrow b = \frac{6}{3}$

$b = 2$

$5a + 2 = 17$

$a = \frac{15}{5} = 3$

2] Solving Singular system of linear ~~Solution~~ equations

Redundant :

$a+b=10 \longrightarrow a+b=10$ — ①

$2a+2b=20 \longrightarrow a+b=10$ — ②

① - ②

$$\begin{array}{r} a+b=10 \\ a+b=10 \\ \hline 0=0 \end{array}$$

$\left.\begin{array}{l} a = x \\ b = 10 - x \end{array}\right\}$ degree of freedom x $\Rightarrow$ Many Solutions

Contradictory :

$a+b=10 \longrightarrow a+b=10$ — ①

$2a+2b=24 \longrightarrow a+b=12$ — ②

① - ②

$0 = 2$

No Solutions

ProMate

3] Solving system of equations with more Variables

Elimanation method

$$a+\cancel{b}+2c=12 \longrightarrow a+b+2c=12$$
$$-2b-7/3c=-11 \longrightarrow b+7/6c=11/12$$
$$-3/2b+c=0 \longrightarrow b-2/3c=0$$

~~Divide last 2 rows by coefficient b~~

$$a+b+2c=12 \longrightarrow a+b+2c=12 \quad -(1)$$
$$3a-3b-c=3 \longrightarrow a-b-\frac{1}{3}c=1 \quad -(2)$$
$$2a-b+6c=24 \longrightarrow a-\frac{b}{2}+3c=12 \quad -(3)$$

(1) - (2)

$$-2b-7/3c=-11 \quad -(3)$$
$$-3/2b+c=0 \quad -(4)$$

$$b+7/6c=11/2 \quad -(3)$$
$$b-2/3c=0 \quad -(4)$$

(4) - (3)

$$-11/6c=-11/2$$
$$c=3$$

$$b+\frac{7}{\cancel{6}_2}\times\cancel{3}^1=\frac{11}{2}$$
$$b=2$$
$$a+2+6=12$$
$$a=4$$

ProMate

4] Matrix row-reduction

$$\begin{aligned} 5a + b &= 17 \\ 4a - 3b &= 6 \end{aligned} \longrightarrow \begin{aligned} a + 0.2b &= 3.4 \\ b &= 2 \end{aligned} \longrightarrow \begin{aligned} a &= 3 \\ b &= 2 \end{aligned}$$

$1a + 0b = 3$

$0a + 1b = 2$

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Upper Diagonal ~~Form~~ Matrix (Row-echelon Form)

Diagonal matrix (Reduced row echelon Form)

Row echelon form

$$\begin{matrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Right to the 1 can be any number

Right to 0 should be 0s

Below the diagonal should be 0

Main diagonal can be 0, 1 or combination of both or numbers with at least one of '1' or '0'

Date

5) Row operations that preserve singularity

Switching rows

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$$

D = 11 D = -11

Multiplying a row by a (Non-zero) Scalar

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \quad 51 \times 10 = 50\ 10 \Rightarrow \begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix}$$

↓ ↓

D = 11 D = 10.11

Adding a row to another row

$$\begin{matrix} 5 & 1 \\ 4 & 3 \end{matrix} \qquad \begin{matrix} 51 \\ 43 \\ \hline 94 \end{matrix} \Rightarrow \begin{matrix} 9 & 4 \\ 4 & 3 \end{matrix}$$

↓

D = 11 D = 11

→) In all these functions singularity will not be changed (preserved)

ProMate

6) Rank of a matrix

*) Ranking used in machine learning heavily on image compressing

*) A image of rank 100 can be reduced to rank like (15-50) without reducing the quality using SVD(Single value decompo-sition)

System of information

System 1	System 2	System 3
*) The dog is black	*) The dog is black	*) The dog
*) The cat is orange	*) The dog is black	*) The dog
↓	↓	↓
Two inform.	One inform.	zero info.
Rank = 2	Rank = 1	Rank = 0

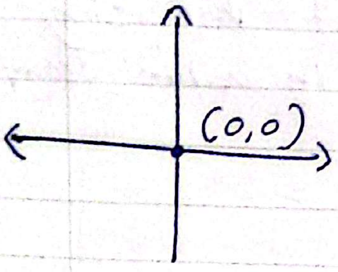
System of equations.

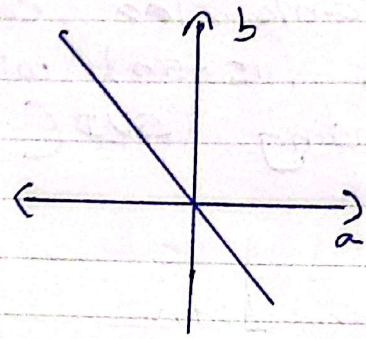
$a+b=0$	$a+b=0$	$0a+0b=0$
$a+2b=0$	$2a+2b=0$	$0a+0b=0$
↓	↓	↓
Rank = 2	Rank = 1	Rank = 0

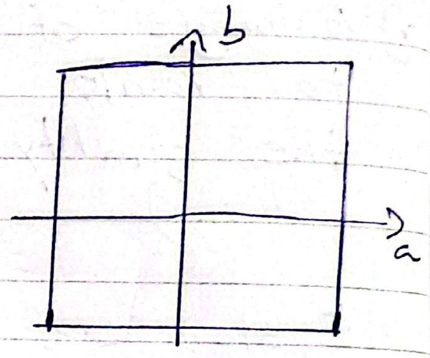
Rank and Solutions

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank = 2 $\qquad$ Rank = 1 $\qquad$ Rank = 0







Dimension of Solution space = 0	Dimension of Solution space = 1	Dimension of solution space = 2

Rank = 2 - Dimension of Solution space

(No. of Rows)

Non-Singular	Singular	Singular
Rank = No. of Rows (Full Rank)	—	—

7] The Rank of a Matrix in general

$a+b+c=0$ ✓	$a+b+c=0$ ✓	$a+b+c=0$ ✓	$\times 0 \quad 0a+0b+0c=0$
$a+2b+c=0$ ✓	$a+b+2c=0$ ✗	$2a+2b+2c=0$ ✗	$\times 0 \quad 0a+0b+0c=0$
$a+b+2c=0$ ✓	$a+b+3c=0$ ✓	$3a+3b+3c=0$ ✗	$\times 0 \quad a+0b+0c=0$
3 info.	2 info	1 info	0 info
Rank 3	Rank 2	Rank 1	Rank 0
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8] Row echelon form

1) $\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix}$

Divide each row by leftmost Coefficient

$\begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix}$

$$\begin{array}{rr} 1 & -0.75 \\ -\;1 & 0.2 \\ \hline 0 & -0.95 \\ \hline \end{array}$$

$\begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ // Rank 2

Divide the second row by the left most non-zero coefficient

*] Target here is to make 0 in the leftmost of 2nd row

[Non - Singular]

e) $\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ ? & ? \end{bmatrix}$

$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$ // Rank 1 (Singular)

3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Rank 0 (Singular)

9] Row echelon form in General

$$\begin{aligned} a + b + 2c &= 12 \\ 3a - 3b - c &= 2 \\ 2a - b + 6c &= 24 \end{aligned} \longrightarrow \begin{aligned} a + b + 2c &= 12 \\ -6b - 7c &= -33 \\ 6c &= 18 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -3 & -1 \\ 2 & -1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -7 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & -5 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank 5   Rank 3

* Zero rows should be at bottom

* Each row has a pivot (leftmost non-zero entry)

* Every pivot is to the right of the pivots on the rows above

* Rank of the matrix is number of pivots

(i) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank 3

Subtract the firstrow from the Second and third ones

(ii)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Subtract the first row from the second and third ones

subtract twice the second row from the third one

Rank 2

(iii)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Subtract twice the first row from the second row

Subtract three times the first row from the third row

Rank 1

10] Reduced row echelon form

Row echelon Form

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{unchanged}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} 0 & 1 \\ & 0.2 \times \\ \hline 0 & 0.2 \end{array} \longrightarrow \begin{array}{cc} 1 & 0.2 \\ 0 & 0.2 - \\ \hline 1 & 0 \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank 5   Rank 3

*] Each pivot is a 1

*] Any number above a pivot is 0

*] Rank of the matrix is the number of pivots

Date No

$$\cancel{\begin{matrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{matrix}}$$

i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Subtract 2 times the second row from the firstone $\longrightarrow$ Add 5 times the the third row to the first one

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ //

Subtract 4 times the third row from the second one

11] The Gaussian Elimination Algorithm

$$
\begin{aligned}
2a - b + c &= 1 \\
2a + 2b + 4c &= -2 \\
4a + b &= -1
\end{aligned}
\longrightarrow
\begin{array}{c}
R_1 \\ R_2 \\ R_3
\end{array}
\begin{bmatrix}
2 & -1 & 1 & 1 \\
2 & 2 & 4 & -2 \\
4 & 1 & 0 & -1
\end{bmatrix}
$$

Augmented matrix

[Should make the pivots to 1 and and all the values below pivots to 0]

*] $R_1 \to \frac{1}{2} R_1$

$$R_1 \to \frac{1}{2}[2 \quad -1 \quad 1 \quad 1] = [1 \quad -1/2 \quad 1/2 \quad 1/2]$$

$$\Downarrow$$

$$
\begin{bmatrix}
1 & -1/2 & 1/2 & 1/2 \\
2 & 2 & 4 & -2 \\
4 & 1 & 0 & -1
\end{bmatrix}
$$

*] $R_2 \to R_2 - 2R_1$

$$
\begin{array}{rrrrr}
 & 2 & 2 & 4 & -2 \\
-2 & 1 & -1/2 & 1/2 & 1/2 \\
\hline
\end{array}
$$

$$R_2 \to [0 \quad 3 \quad 3 \quad -3]$$

*] $R_3 \to R_3 - 4R_1$

$$\begin{array}{ccccc} & 4 & 1 & 0 & -1 \\ -4 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline \end{array}$$

$R_3 \to [\,0 \quad 3 \quad -2 \quad -3\,]$

$\Downarrow$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 3 & -3 \\ 0 & 3 & -2 & -3 \end{bmatrix} \longrightarrow \begin{array}{l} a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2} \\ \;\; + 3b + 3c = -3 \\ \;\; 3b - 2c = -3 \end{array}$$

*] $R_2 \to \frac{1}{3} R_2$

$R_2 \to \frac{1}{3} \quad 0 \quad 3 \quad 3 \; -3 \quad = [\,0 \quad 1 \quad 1 \quad -1\,]$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 3 & -2 & -3 \end{bmatrix}$$

*] $R_3 \to R_3 - 3R_2$

$$\begin{array}{ccccc} & 0 & 3 & -2 & -3 \\ -3 & 0 & 1 & 1 & -1 \\ \hline R_3 \to & 0 & 0 & -5 & 0 \end{array}$$

$$\Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

Date

*) $R_3 \to \frac{-1}{5} R_3$

$R_3 \to -\frac{1}{5} [0 \quad 0 \quad -5 \quad 0] = [0 \quad 0 \quad 1 \quad 0]$

$\Downarrow$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2} \\ b + c = -1 \\ c = 0 \end{array}$$

Back substitution (cancel out numbers above pivot)

*) $R_2 \to R_2 - R_3$

$$\begin{array}{cccc} 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}$$

$R_2 \to [0 \quad 1 \quad 0 \quad -1]$

*) $R_1 \to R_1 - \frac{1}{2} R_3$

$$\begin{array}{cccc} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \\ \hline \end{array}$$

$R_1 \to 1 \quad -1/2 \quad 0 \quad 1/2$

$$\Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow R_1 \rightarrow R_1 + \frac{1}{2} R_2$

$$\begin{array}{rcccc} & 1 & -1/2 & 0 & 1/2 \\ +\frac{1}{2} & 0 & 1 & 0 & -1 \\ \hline R_1 \rightarrow & 1 & 0 & 0 & 0 \end{array}$$

$\Downarrow$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} a = 0 \\ b = 1 \\ c = 0 \end{array}$$

Singular $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 4 & 6 \end{array}\right] \xrightarrow[\text{reduction}]{\text{After row}} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & -7 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If the constant 0
$0a + 0b + 0c = 0$
(Infinite Solutions)

If the constant Non-zero
$0a + 0b + 0c = 0$
(No solution)