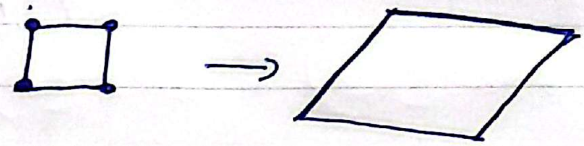
Date No

Week 04

1) Singularity and rank of linear transformations

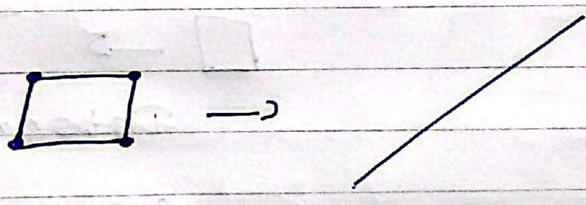
*) $\begin{array}{cc} a & b \\ 3 & 1 \\ 1 & 2 \end{array}$



Dimension (2)
(Rank 2)

Non - Singular (covers every single point of the plane)

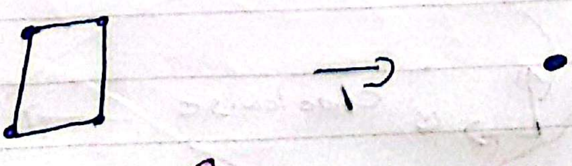
*) $\begin{array}{cc} a & b \\ 1 & 1 \\ 2 & 2 \end{array}$



Dimension (1)
(Rank 1)

Singular (Does not get sent to the right plane (one line segment)

*) $\begin{array}{cc} a & b \\ 0 & 0 \\ 0 & 0 \end{array}$



(Rank 0)

Singular (Does not covers every single point of the plane)

ProMate

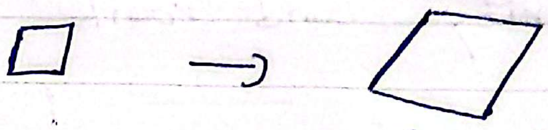
2) Determinant as an area

*) a b

3 1

1 2

Determinant = 5



Area = 5

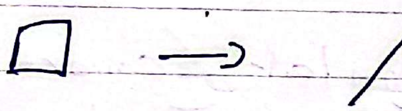
(Non - Singular)

*) a b

1 1

2 2

Determinant = 0



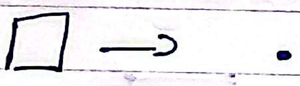
Area = 0

(singular)

*) a b

0 0

0 0



Area = 0

Determinant = 0

(Singular)

Negative Determinant

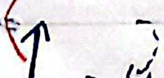
a b

1 3

2 1

Determinant = −5

(Area = −5)



Clockwise

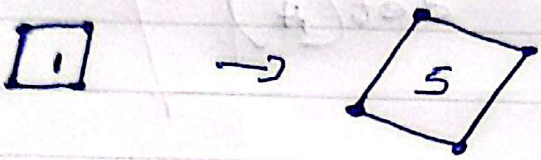
Counter Clockwise (−)

(order opposite)

$$\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix} \qquad \begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}3\\ 1\end{bmatrix}$$

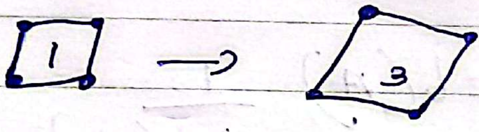
3] Determinant of a product

*) $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ Det = 5



Blows the area by 5

*) $\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ Det = 3



Blows the area by 3

* Combining both blows the area by 3×5

$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -5 & 0 \end{bmatrix} \Rightarrow \boxed{\text{Det } 15}$

$$\boxed{\det(AB) = \det(A)\det(B)}$$

The area of the parallegoram spanned by two vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in R^2 given by the absolute value of the determinant of the matrix formed by placing the vectors in columns.

$\Rightarrow (2,5) / (3,1) \Rightarrow$ Area $\left|\det \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\right| = 13$

ProMate

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4) Determinants of inverses

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\underset{(1)}{\det(I)} = \det(A)\frac{1}{\det(A)}$$

5) Bases in Linear Algebra

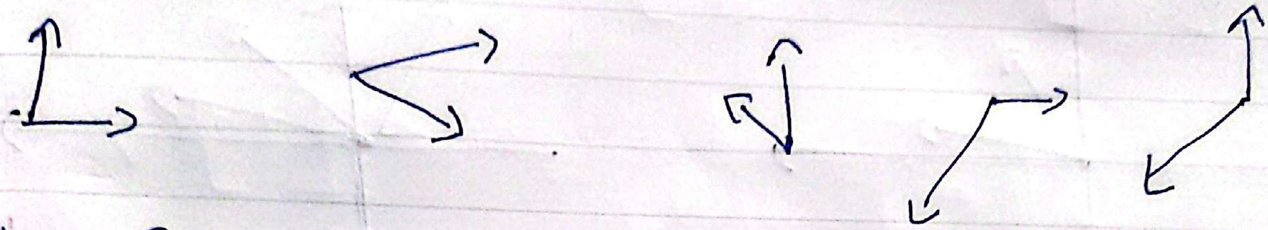
A basis of a vector space V is a set of vectors in V that satisfies two important properties

*) Linear Independence : No vector in the set can be written as a combination of the others

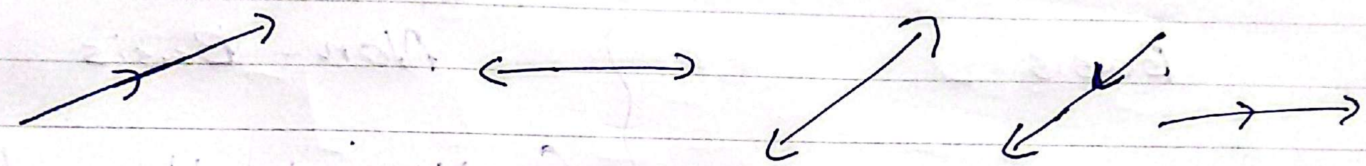
*) Spanning : Any vector in V can be written as a linear Combination of the vectors In the space

ProMate

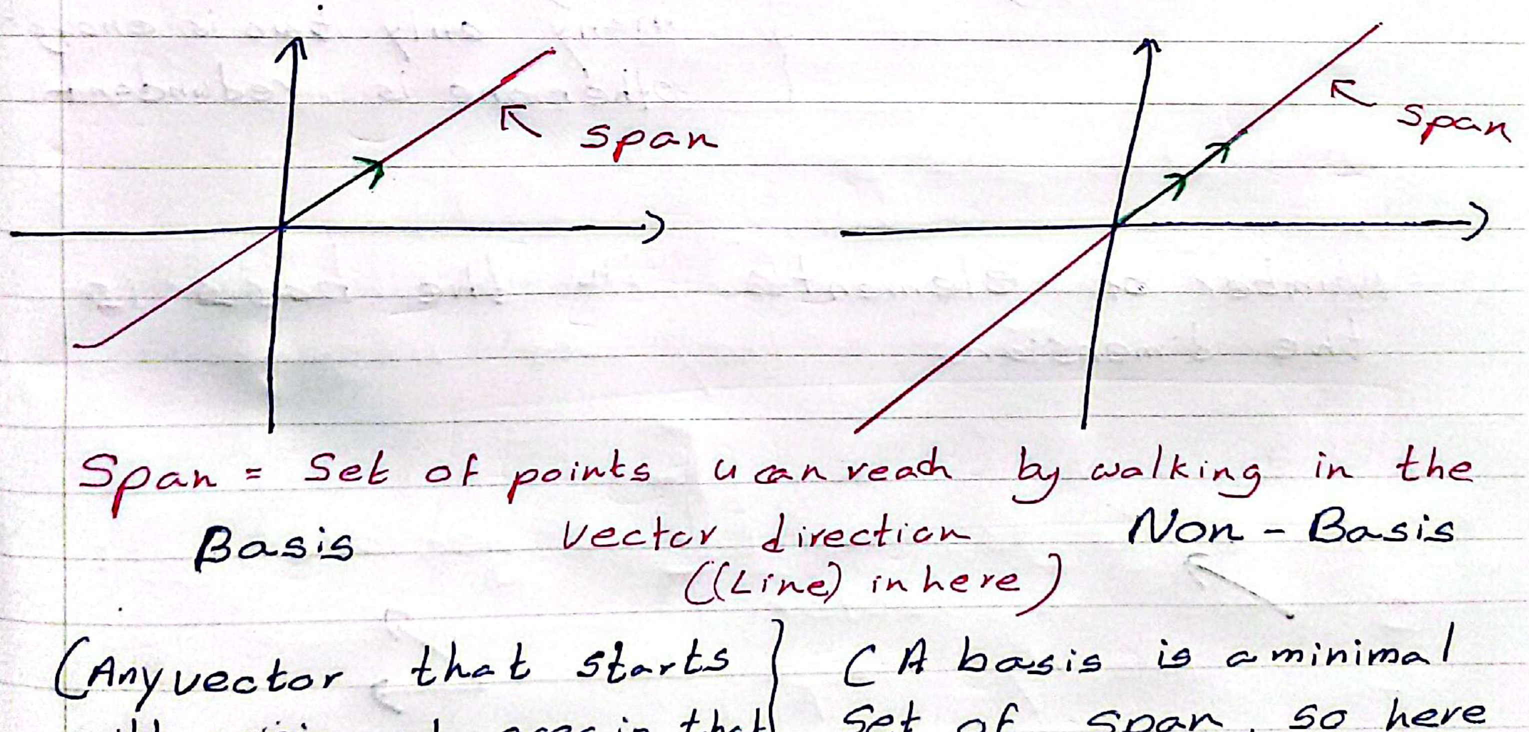
Bases (plural for basis)



Non - Bases



6] Span in Linear Algebra



Span = Set of points u can reach by walking in the vector direction ((Line) in here)

Basis | Non - Basis

(Any vector that starts with origin and goes in that same direction is also a basis of that line)

(A basis is a minimal set of span, so here there are too many, basis need to be a minimal spaning set

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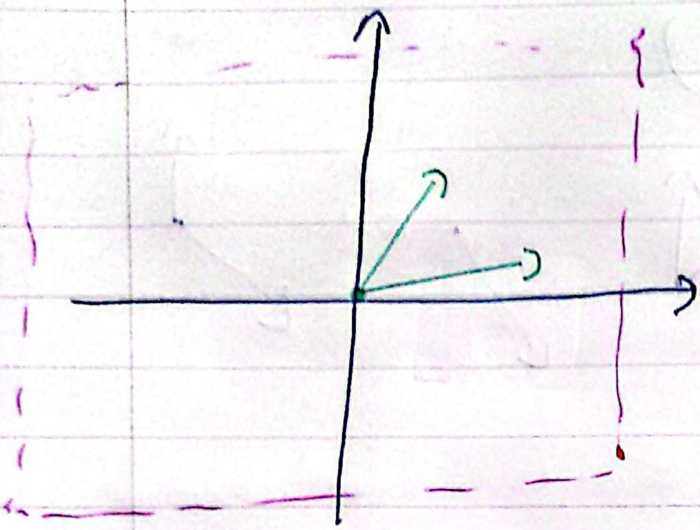
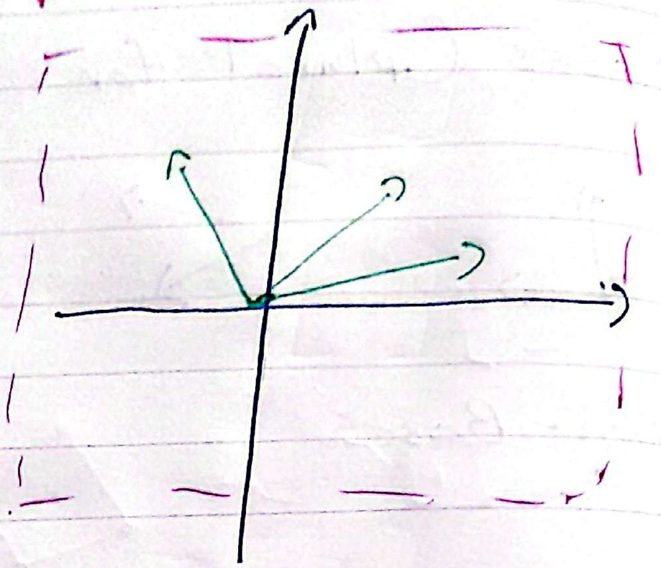
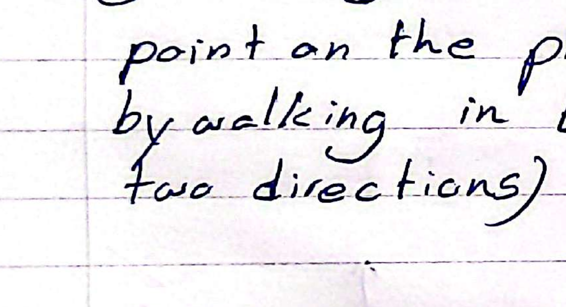
Spans a plane



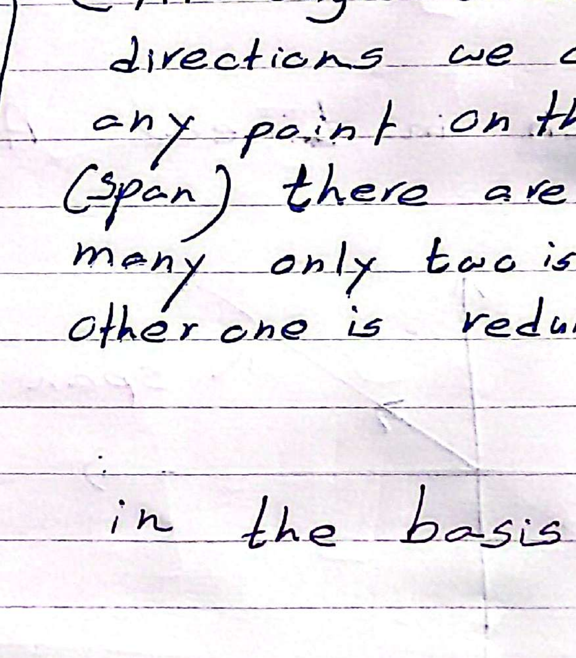


Basis	Non - Basis
(we can get to any point on the plane by walking in these two directions)	(Although these lines directions we can get to any point on the plane (span) there are too many only two is enough other one is redundant

Number of elements in the basis is the dimension.



1 element
Dimension = 1.



2 elements
dimension = 2

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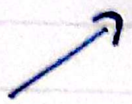
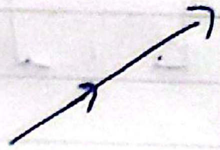


Linearly independent

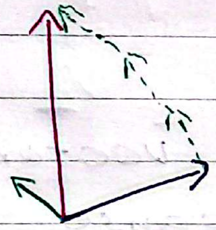
(Basis)

Linearly independent

(Basis)



Linearly dependent
(Non - Basis)
(One vector canbe obtain by using the other)



*) These three are linearly dependent because third line (pink) can be obtained by one blue and 3 green vectors

*) If you have more vectors than the dimension of the space you trying to span always linearly dependent

Date

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Find if these 3 vectors lineary depend-ent or independent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$-\alpha + 2\beta = 5 \qquad \alpha = \frac{11}{3} \qquad \beta = \frac{-2}{3}$$
$$\alpha + \beta = 3$$

*) This set of vectors are linearly dependent because there are solutions for α and β

*) If there are no solutions for the coefficients they are linearly indepen-dant.

A basis is a set of vectors that:

*) Spans a vector

*) is linearly independent

ProMate

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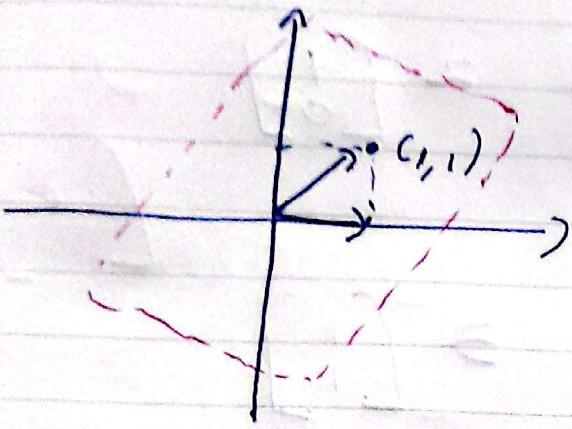
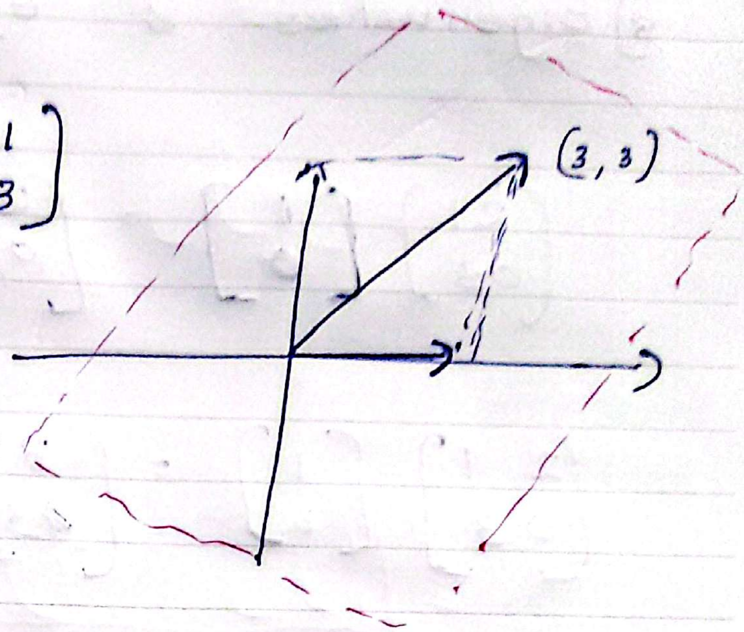
7] Eigenbases



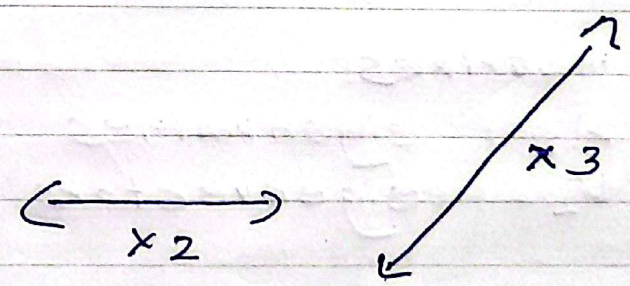
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$



$(1,0) \to 2,0$
$(1,1) \to (3,3)$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Strecteching



*) This is a eigenbasis

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8) Eigenvalues & Eigenvectors

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

matrix (x) scalar (x)

$$\boxed{\begin{aligned} A v_1 &= \lambda_1 v_1 \\ A v_2 &= \lambda_2 v_2 \end{aligned}}$$

A matrixes

λ_1, λ_2 eigenvalues

v_1, v_2 eigenvectors

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenbasis

ex:-

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \Big[-3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big]$$

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

steps

1) The inverse of eigenbasis

2) multiply by vector

$$= -3 \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3 \cdot 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \cdot 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

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*] $Av = \lambda v$ for each eigenvector / eigenvalue

*] Eigenvectors : direction of stretch

*] Eigenvalues : how much strech

*] Eigenbasis : the set of matrix's eigenvectors can be arranged as a matrix with one eigenvector in each column

*] Save work and characterize a transformation

9] Calculating Eigenvectors and EigenValues

EigenValues

$$\boxed{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{For infinity many } \begin{bmatrix} x \\ y \end{bmatrix}}$$

$$\left[\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]\begin{matrix} x \\ y \end{matrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det(A - \lambda I)$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ infintely many solutions

charecteristic polynomial $= (2-\lambda)(3-\lambda) - 1.0 = 0$

$$\boxed{\lambda = 2 \quad \lambda = 3}$$

ProMate

Eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$2x + y = 2x$

$0x + 3y = 2y$

$x = 1$

$y = 0$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ //

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

$2x + y = 3x$

$0x + 3y = 3y$

$x = 1$

$y = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

*) If the matrix is not square then it does not have any eigenvectors or eigenvalues

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10] On the number of Eigenvectors

*] $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Eigen values λ_1, λ_2

if $\lambda_1 \neq \lambda_2 \longrightarrow$ 2 eigen vectors
(2 directions different)

if $\lambda_1 = \lambda_2 \longrightarrow$ 1 eigen vector
(1 direction)

$\searrow$ 2 eigen vectors
(2 different directions)

*] $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ Eigen values $\lambda_1, \lambda_2, \lambda_3$

if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \longrightarrow$ 3 eigen vectors
(3 different directions)

if $\lambda_1 = \lambda_2 \neq \lambda_3 \longrightarrow$ 2 eigen vectors
(2 different directions)

$\searrow$ 3 eigen vectors
(3 different directions)

ProMate

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If $\lambda_1 = \lambda_2 = \lambda_3 \longrightarrow$ 1 eigen vector (1 direction)

$\searrow$ 2 eigenvectors (2 different directions)

$\searrow$ 3 eigenvectors (3 different directions)

11] Dimensionality Reduction and projection

Dimensionality Reduction

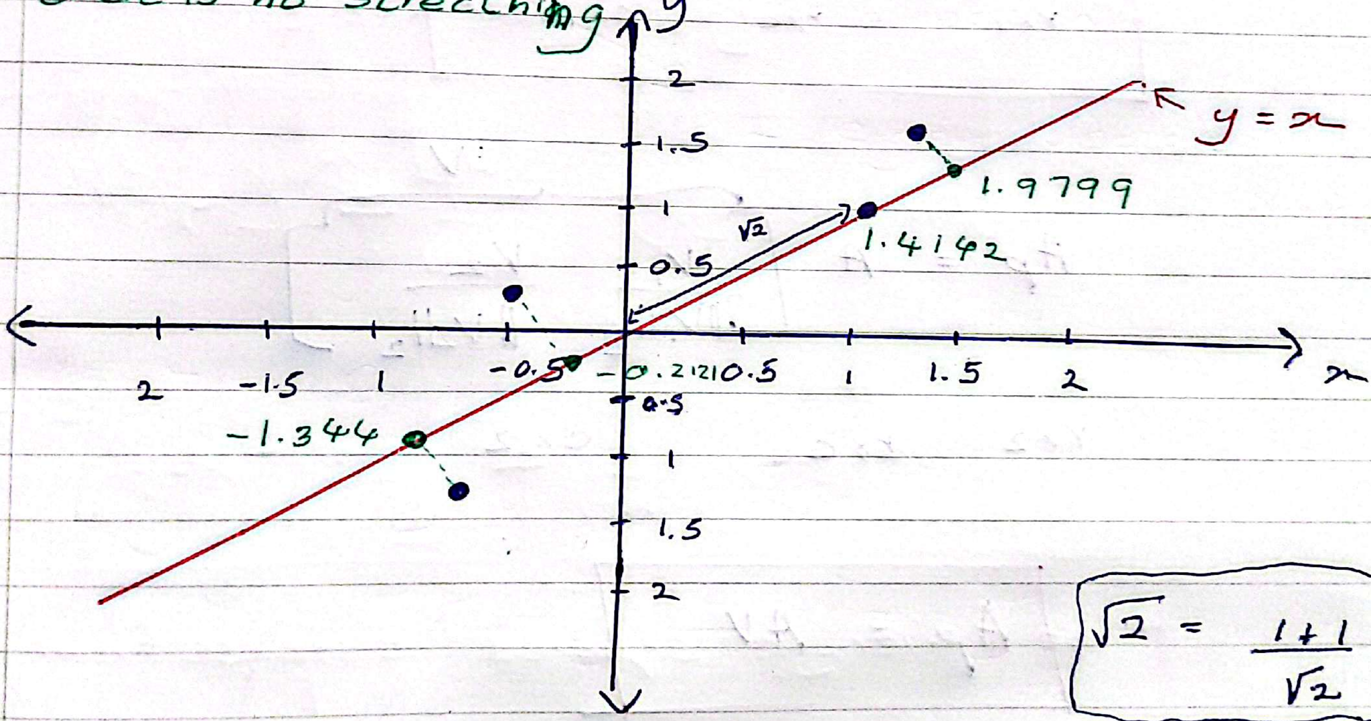
*] Reduce dimensions (columns) of dataset
*] preserve as much information as possible
*] Leads to smaller datasets
*] Easier to visualize

*] Easiest approach is to delete columns (features) But it loses valuable information .

projection

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

Multiplying the vector projects the points along that vector and dividing by the vector's norm ensure there is no streching



$$\sqrt{2} = \frac{1+1}{\sqrt{2}}$$

1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{array}{l} (1+1)/\sqrt{2} \Rightarrow 1.4142 \\ (1.2+1.6)/\sqrt{2} \Rightarrow 1.9799 \\ (-0.5+0.2)/\sqrt{2} \Rightarrow -0.2121 \\ (-1.3+(-0.6))/\sqrt{2} \Rightarrow -1.344 \end{array}$$

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To project a matrix A onto a vector V :

$$\cancel{A_p = \frac{A}{\|V\|_2}}$$

$r \times 1$

$$\boxed{\underset{r\times 1}{A_p} = \underset{r\times c}{A}\ \underset{c\times 1}{\frac{V}{\|V\|_2}}}$$

$$\underset{r\times 2}{A_p} = \underset{r\times c}{A}\ \underset{c\times 2}{\overbrace{\left[\frac{V_1}{\|V_1\|_2} \quad \frac{V_2}{\|V_2\|_2}\right]}^{V}}$$

$$\boxed{A_p = AV}$$

Note :-

$$x = y \longrightarrow \vec{V} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2x = y \longrightarrow \vec{V} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ProMate

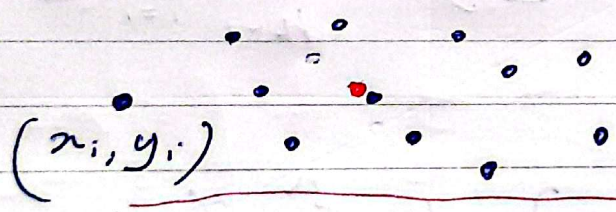
Goal of PCA is to find the projection that preserves the maximum possible spread in your data, even as you reduce the dimensionaility of the dataset

More spread $\longrightarrow$ more information
Less spread $\longrightarrow$ Less information

12] Variance and Covariance

Mean

*] The average of the data



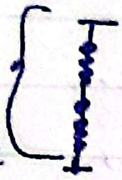
$$\text{Mean}(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

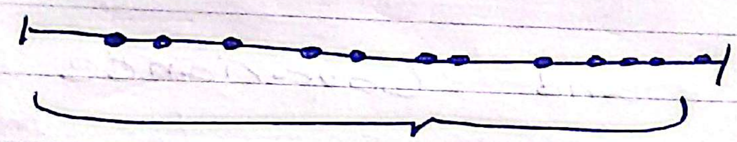
$$\text{Mean}(y) = \frac{1}{n} \sum_{i=1}^{n} y_i$$

Variance

*) How spreadout your data is

Small Variance





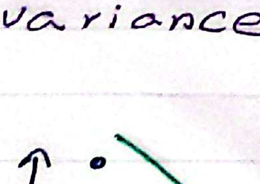
large Variance

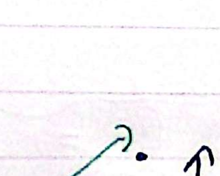
$$\text{Variance}(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \text{Mean}(x))^2$$

$$\text{Var}(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)^2$$

"The average squared distance from mean"

Covariance





*) Same Variance
Solution → Covariance

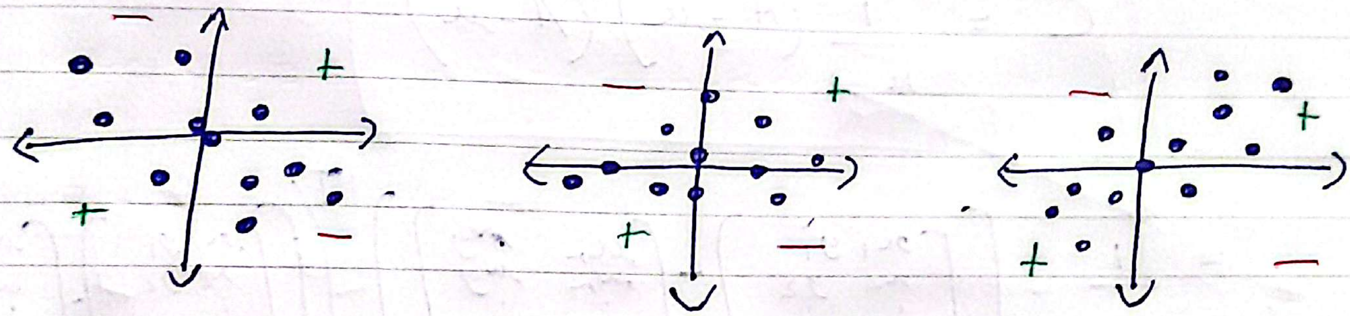
Negative Covariance

Positive Covariance

ProMate

$$\boxed{Cov_{(x,y)} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}$$

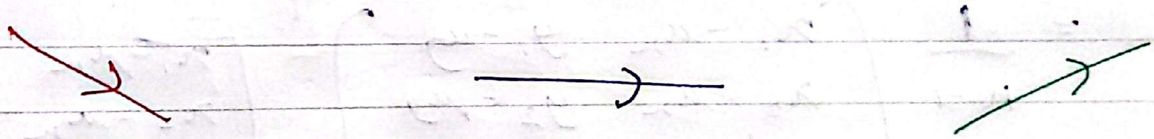
" The direction of the relationship between two variables



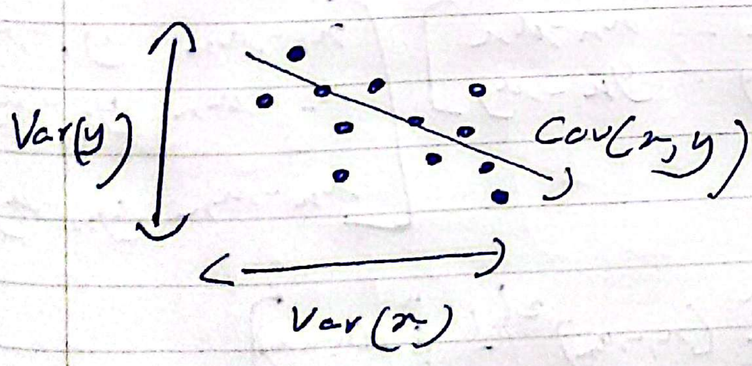
Negative Covariance

Covariance zero or small

Positive Covariance



13) Covariance Matrix



~~C = Cov(x,x)~~

$$\boxed{C = \begin{bmatrix} Var_{(x)} & Cov_{(x,y)} \\ Cov_{(y,x)} & Var_{(y)} \end{bmatrix}}$$

$Cov_{(x,x)} = Var_{(x)}$

Date

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\boxed{C = \frac{1}{n-1}(A-\mu)^T(A-\mu)}$$

$$= \frac{1}{n-1}\left[\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right]^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & \cancel{\mu_x} y_n-y_\mu \end{bmatrix}$$

$$= \frac{1}{n}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \cdots & x_n-\mu_x \\ y_1-\mu_y & y_2-\mu_y & \cdots & y_n-\mu_y \end{bmatrix}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$\cancel{n-1}\ \begin{bmatrix} \underbrace{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)^2}_{var(x)} & \underbrace{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y)}_{cov(x,y)} \\ Cov(x,y) & var(y) \end{bmatrix}$$

ProMate

Example :-

x	y
10	5
12	3
6	9
6	4
5	11
14	2
8	1
3	13

A

$\mu_x = 8$

$\mu_y = 6$

$\longrightarrow$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

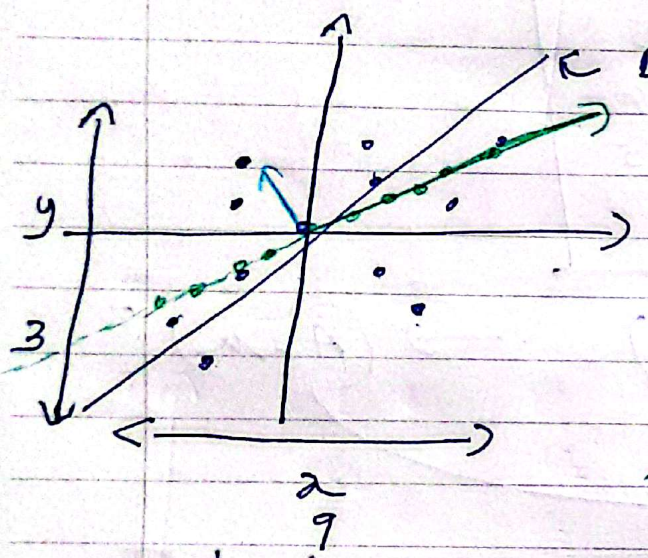
$(A - \mu) \longrightarrow (A - \mu)^T$

$$\frac{1}{8-1} \times \begin{array}{c|cccccccc} x - \mu_x & 2 & 4 & -2 & -2 & -3 & 6 & 0 & 5 \\ y - \mu_y & -1 & -3 & 3 & -2 & 5 & -4 & -5 & 8 \end{array} \times$$

$$(A - \mu) \implies C = \begin{bmatrix} 14 & -11.86 \\ -11.86 & 19.71 \end{bmatrix}$$

14] PCA Overview

Principal Component Analysis



← Best line = most variance

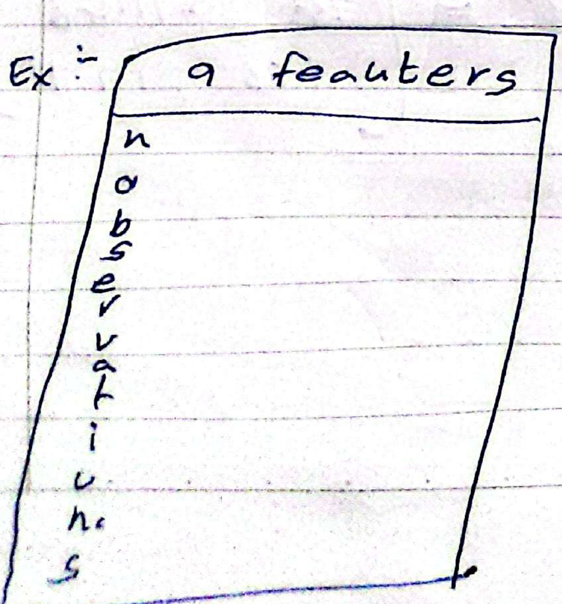
$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{(Eigen vectors)}$$

11 max value   1 - (Eigen Values)

*) Covariance matrix is alway symetric therfore all the eigen vectors are orthoganal

Ex:-



$$C = \begin{bmatrix} & \\ & \end{bmatrix}_{9\times 9}$$

Eigen Values / Vectors

v_1	λ_1
v_2	λ_2
v_2	λ_3
⋮	⋮

If u want to reduce the data set to 2 variables ;

$$\underset{\text{9 features}}{\boxed{\quad}} \times \left[\frac{v_1}{\|v_1\|_2} \quad \frac{v_2}{\|v_2\|_2} \right] \Rightarrow \underset{\text{two featuers}}{\boxed{x \mid y}}$$

15) PCA - mathematical Formualation

You have n observations of 5 variables $(x_1, x_2, x_3\ x_4, x_5)$

Goal : Reduce to 2 variables

1] Create matrix

$$x = \begin{array}{c} \text{5 variables} \\ \left[\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{15} \\ x_{21} & x_{22} & \cdots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n5} \end{array} \right] \end{array} \begin{array}{l} n \\ \text{Observations} \end{array}$$

2] Center Data

$$x - \mu \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \cdots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \cdots & x_{25} - \mu_5 \\ \vdots & & & \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \cdots & x_{n5} - \mu_5 \end{bmatrix}$$

Date

3) Calculate Covariance Matrix.

$$C = \frac{1}{n-1}(X-\mu)^T(X-\mu)$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}_{5\times 5}$$

4) Eigenvectors and Eigenvalues

Big

$\lambda_1 \quad V_1$

$\lambda_2 \quad V_2$

$\lambda_3 \quad V_3$

$\lambda_4 \quad V_4$

$\lambda_5 \quad V_5$

$\vdots$

Small

5) Create projection Matrix

$$V = \begin{bmatrix} \frac{V_1}{\|V_1\|_2} & \frac{V_2}{\|V_2\|_2} \end{bmatrix}$$

6) Project Centered data

$$X_{pca} = (X-\mu)V$$

16] Discrete Dynamical Systems

*) A discrete Dynamical system is like pressing a "next button" repeatedly where each press applies the same rule to go to the next state

*) Depending on the rule, the system might settle down, cycle or become chaotic

*) $\vec{x}_{n+1} = A\vec{x}_n$

where A is a matrix, this is where eigenvalues and eigenvectors come in they describe long-term behaviour

$$\underbrace{\begin{bmatrix} 0.80 & 0.45 & 0.30 \\ 0.15 & 0.35 & 0.40 \\ 0.05 & 0.20 & 0.30 \end{bmatrix}}_{\text{Transition Matrix } (P)} \underbrace{\begin{bmatrix} 0.6665 \\ 0.2223 \\ 0.1112 \end{bmatrix}}_{\text{Equilibrium Vector } (X_\infty)}$$

Markup Matrix ← (P)
(Non-Negative, adduptoi)

*) Long run probabilities