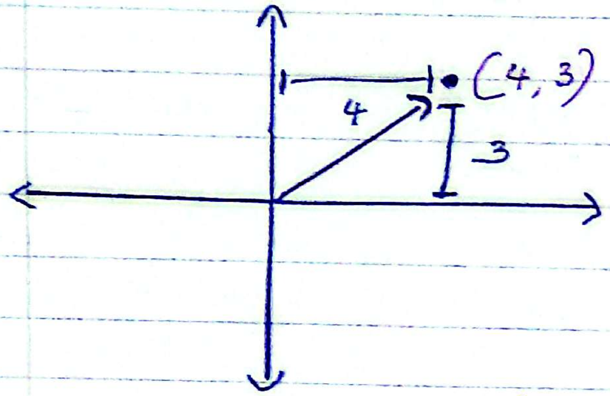
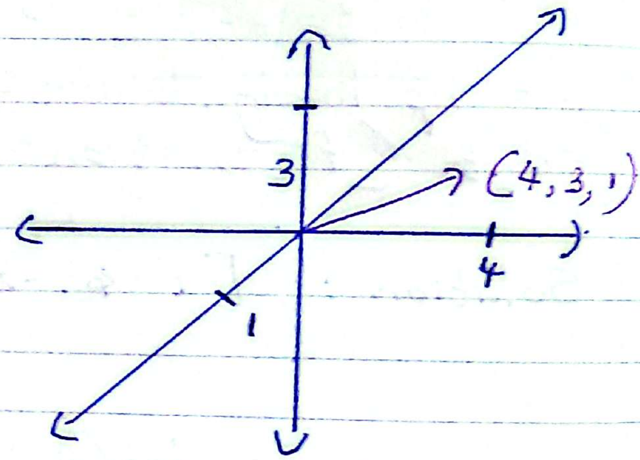
Week 03

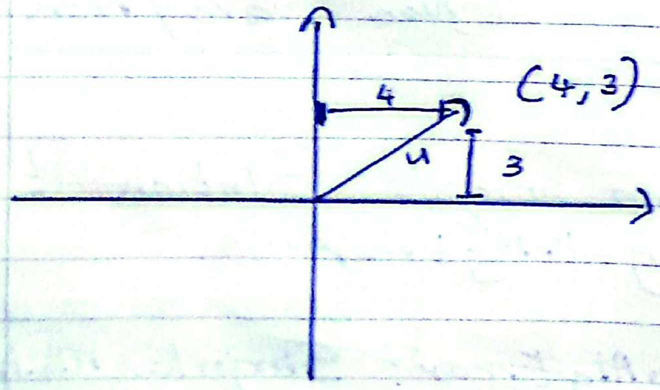
i] Vectors and their properties

Vectors





Norms (size) :



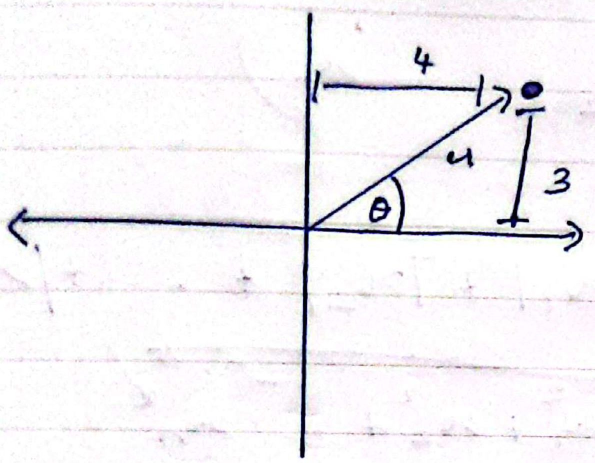
Norm of u

$= \sqrt{4^2 + 3^2}$

$= 25$

$= 5$

Direction of a vector



Direction of u =

$$\tan\theta = \frac{3}{4}$$

$$\theta = \tan^{-1}(3/4)$$

$$\theta = 0.64$$

$$\theta = 36.87^{\circ}$$

Vector notation

*] Row vector

$$x = (x_1, x_2 \ldots x_n)$$

*] Column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Numbered Components

*] $\vec{x}$

*] x

*] $[x_1, x_2 .. x_n]$

* $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

General definition (L_1 and L_2 norms)

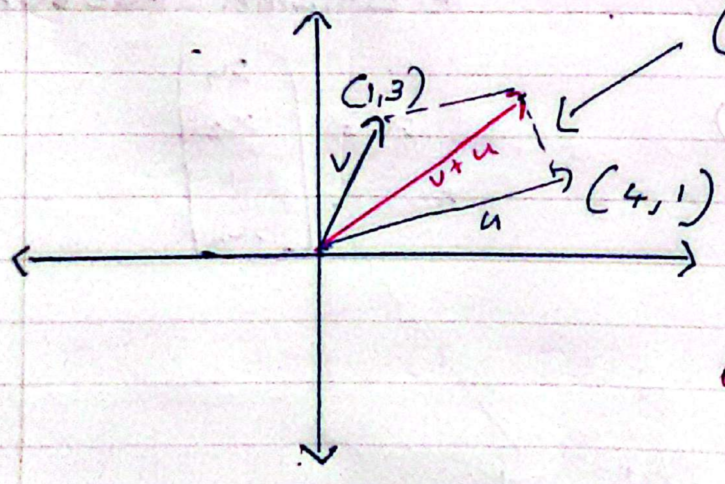
$$x = (x_1, x_2 \ldots x_n)$$

L_1 norm $= \|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$

L_2 norm $= \|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$

2] Vector operations

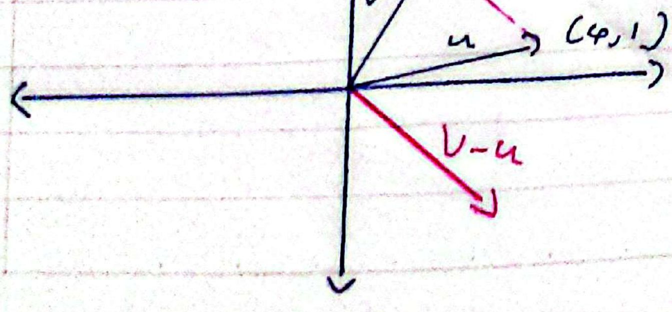
Sum of vectors



(should be a parallelogram)

$$u + v = (4+1, 1+3)$$
$$= (5, 4)$$

Difference of vectors



$$u - v = (4-1, 1-3)$$
$$= (3, -2)$$

General Definition: Sum and difference

(Same number of components)

$x = (x_1, x_2 \dots x_n)$ $y = (y_1, y_2 \dots y_n)$

Sum

$$\boxed{x + y = (x_1 + y_1, x_2 + y_2 \dots x_n + y_n)}$$

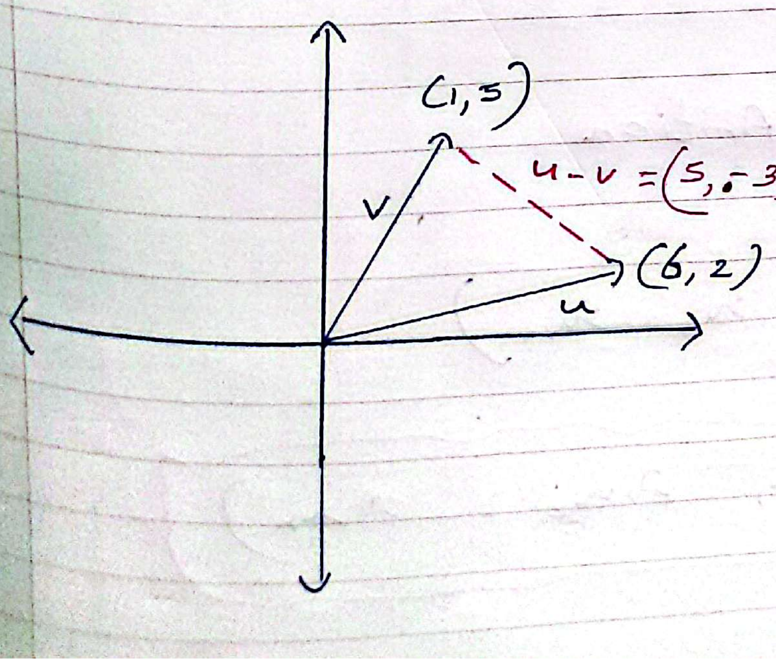
*)(Sum Component by Component)

Difference

$$\boxed{x - y = (x_1 - y_1, x_2 - y_2 \dots x_n - y_n)}$$

*)(Subtract component by component)

Distances



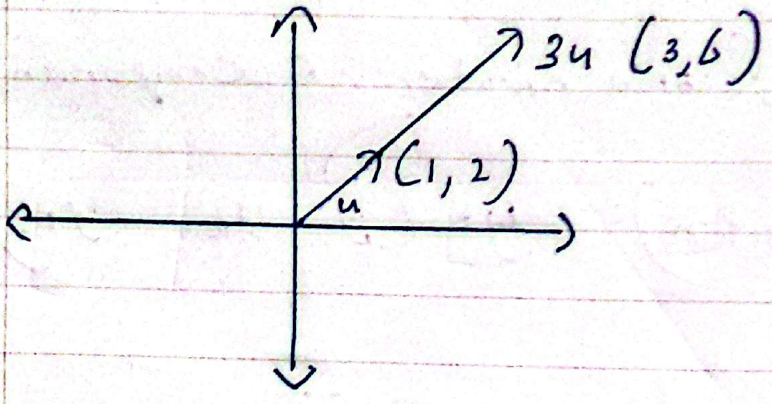
$L_1 = |u - v|_1 =$

$|5| + |-3| = 8$

$L_2 = |u - v|_2 =$

$= \sqrt{5^2 + 3^2} = 5.83$

Multiplying a vector by a scalar

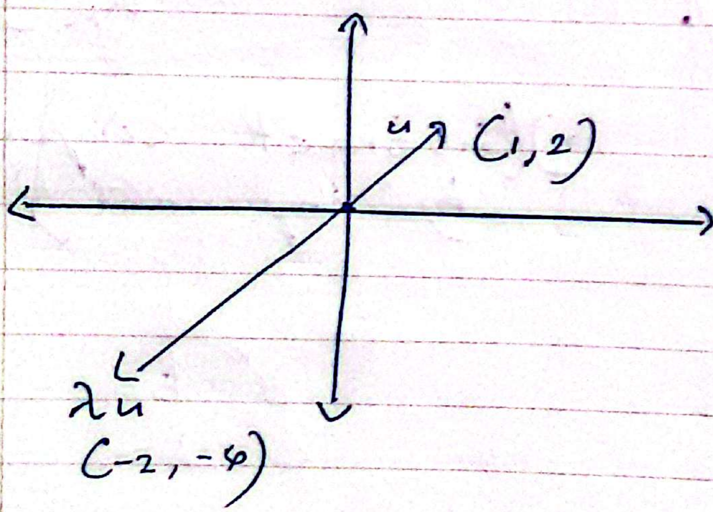


$u = (1, 2)$

$\lambda = 3$

$\lambda u = (3, 6)$

→ If the scalar is negative



$u = (1, 2)$

$\lambda = -2$

$\lambda u = (-2, -4)$

~~Ge~~ General Definition

$$x = (x_1, x_2 \ldots x_n)$$

$$\boxed{\lambda x = (\lambda x_1, \lambda x_2 \ldots \lambda x_n)}$$

3] The dot product

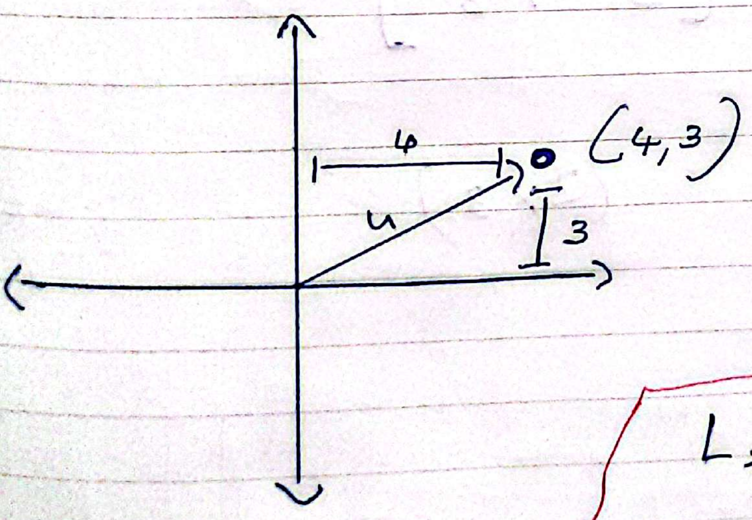
(same number of components)

$x = (x_1, x_2 \dots x_n)$ $\quad$ $y = (y_1, y_2 \dots y_n)$

$$\boxed{x \cdot y = (x_1 \times y_1) + (x_2 + y_2) + \dots (x_n \times y_n)}$$

$\langle x, y \rangle$ is another notation for the dot product

Norm of a vector using dot product



$\sqrt{4^2+3^2} = \sqrt{25} = 5$

$$[4 \ 3]\begin{bmatrix}4\\3\end{bmatrix} = 25$$

$$\boxed{\begin{aligned} L_2 \text{ norm} &= \sqrt{\text{dot product}(u,u)} \\ \lVert u \rVert_2 &= \sqrt{\langle u, u \rangle} \end{aligned}}$$

Date

Vector transpose

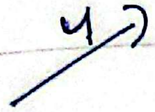
$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Matrix transpose

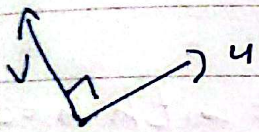
$$\begin{bmatrix} 2 & 5 \\ 4 & 7 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 7 & 3 \end{bmatrix}$$ (Columns → Rows)

(3×2) (2×3)

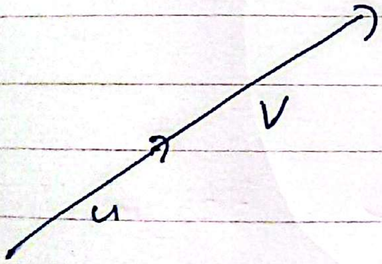
4] Geometric dot product



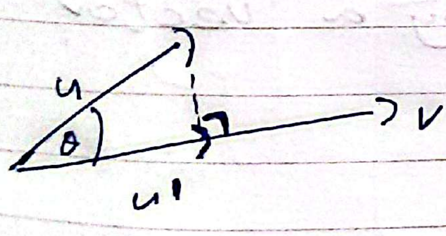
$$\langle u, u \rangle = |u|^2 = |u| . |u|$$



$$\langle u, v \rangle = 0$$

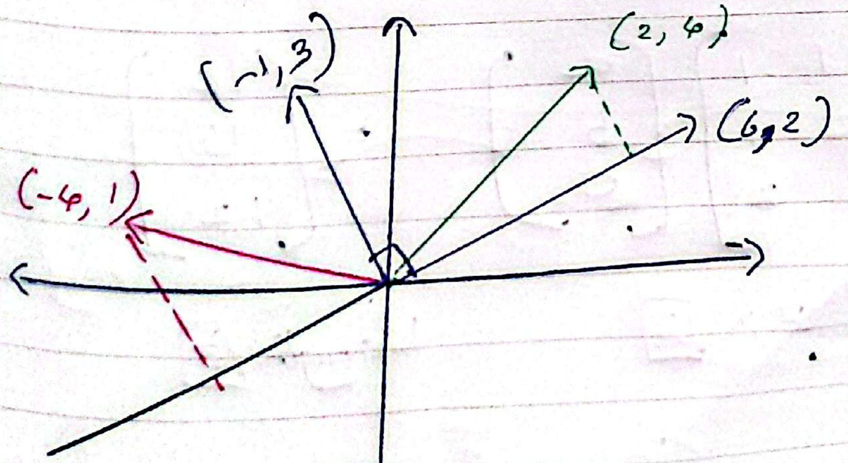


$$\langle u, v \rangle = |u| . |v|$$



$$\langle u, v \rangle = |u'| . |v|$$

$$= |u||v| \cos\theta$$



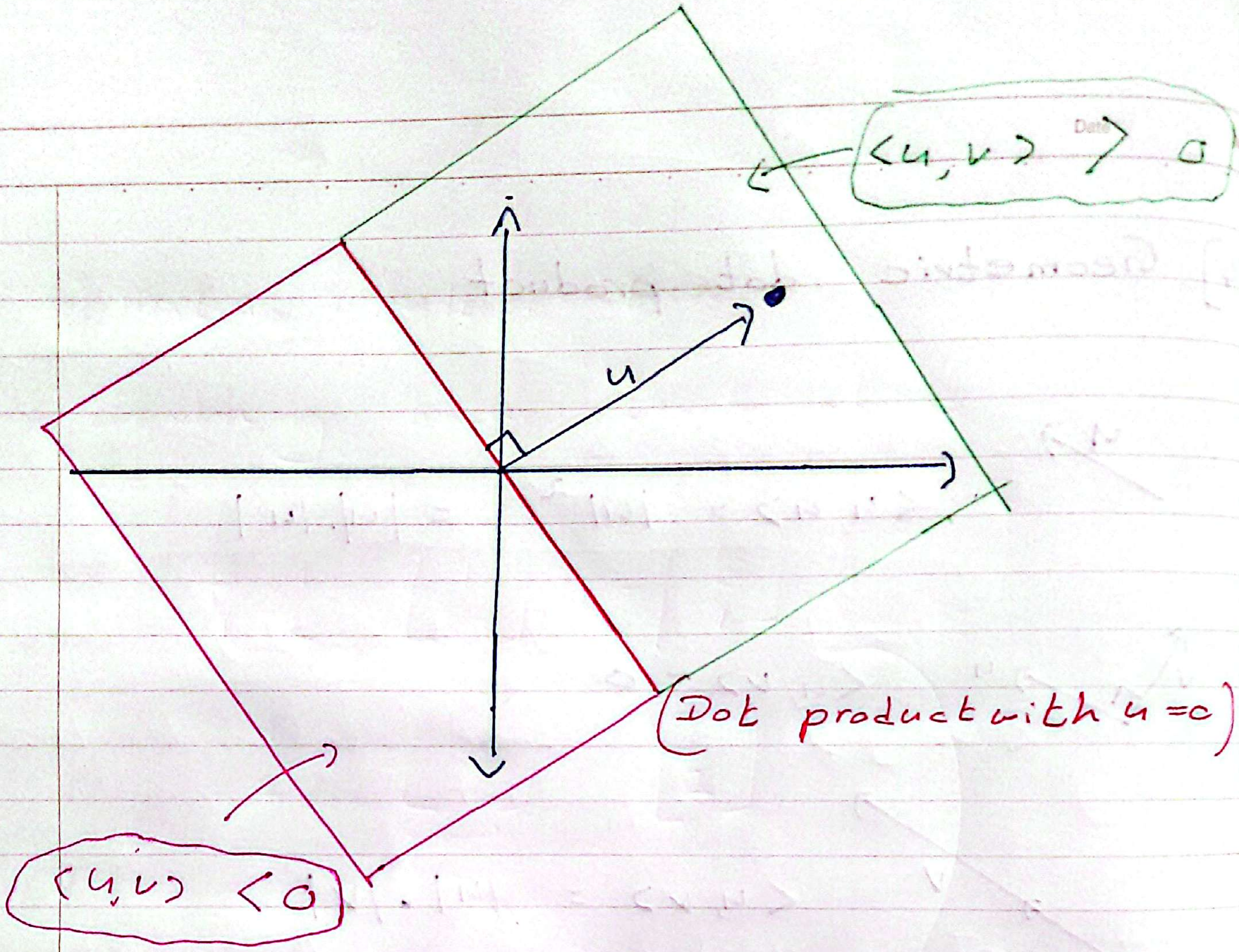
$$\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$$

(positive)

$$\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = -22$$

(Negative

$$\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$



5] Multiplying a matrix by a vector

$a + b + c = 10$
$a + 2b + c = 15$
$a + b + 2c = 12$

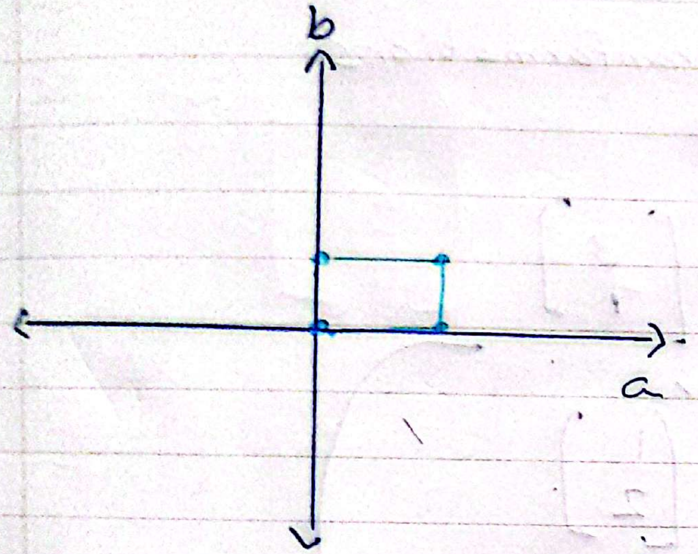
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

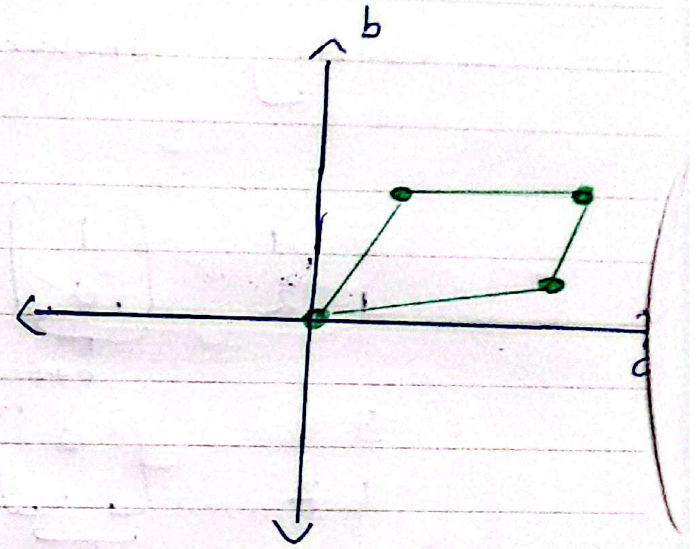
$3 \times \boxed{3}$ Length $\boxed{3}$

Columns = length of vector

(Matrix can be rectangular)

6] Matrices as linear transformations





$$\begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} - \\ - \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}$$

$(0,0) \to (0,0)$
$(1,0) \to (3,-1)$
$(0,1) \to (1,2)$
$(1,1) \to (4,3)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$
$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

7] Linear transformation as matrices

$(0,0) \to (0,0)$
$(1,0) \to (3,-1)$
$(0,1) \to (1,2)$
$(1,1) \to (4,3)$

$$\begin{array}{cccc} a & b & & \\ ? & ? & 0 & \\ ? & ? & 1 & \end{array} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{array}{cccc} a & b & & \\ ? & ? & 1 & \\ ? & ? & 0 & \end{array} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} //$$

8] Matrix multiplication....

Combining linear tranformations

$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{basis vector}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{basis vector}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{matrix} 2 & -1 \\ 0 & 2 \end{matrix} \left.\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} \text{new basis vector} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{matrix} 2 & -1 \\ 0 & 2 \end{matrix} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{new basis vector}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix} \qquad \begin{matrix} 2 & -1 \\ 0 & 2 \end{matrix} \qquad \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

Date No

*) $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$

$$\Downarrow$$

$$\begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -2 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 21 & -6 \\ 1 & -3 & 8 & -4 \end{bmatrix}$$

2×3 $\qquad$ 3×4 $\qquad$ 2×4

*) Columns of first matrix must match rows of second

*) Result takes number of rows from first matrix

*) Result takes number of columns from second matrix

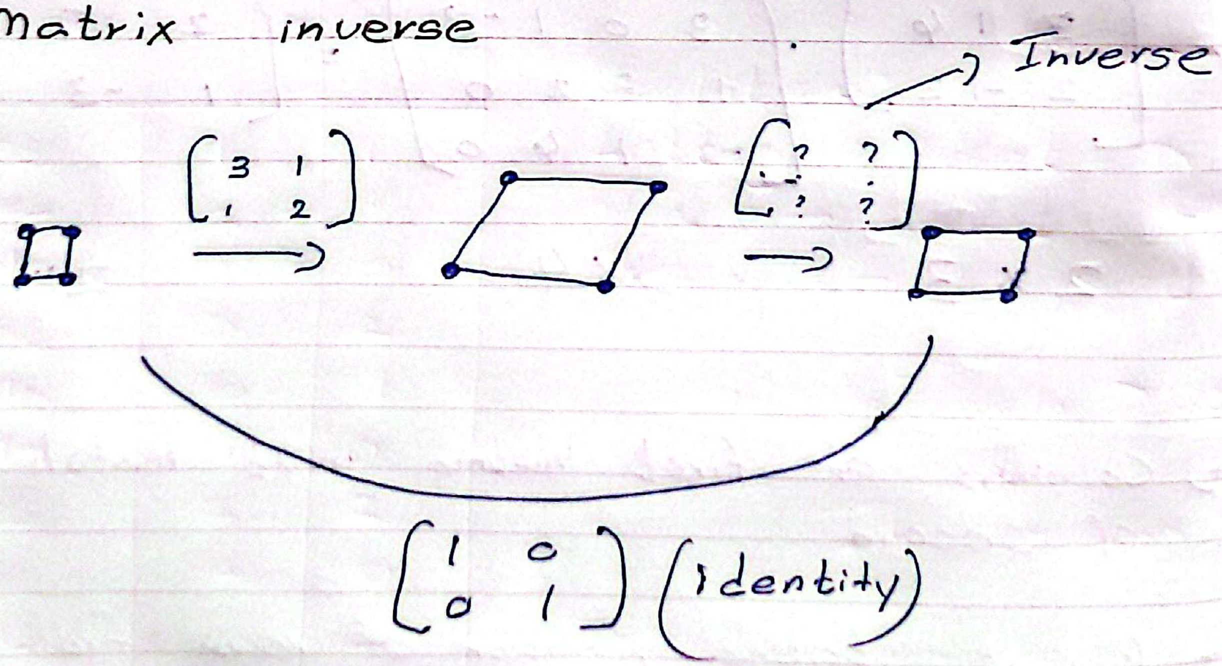
Date

9) Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

1 in diagonal
all others 0
(identity matrix)

10) Matrix inverse



*) $$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = 1 \qquad 3a + 1c = 1$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = 0 \qquad 3b + 1d = 0$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = 0 \qquad 1a + 2c = 0$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = 1 \qquad 1b + 2d = 1$$

$$a = \frac{2}{5} \qquad b = \frac{-1}{5} \qquad c = \frac{-1}{5} \qquad d = \frac{3}{5}$$

Inverse of $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ -1/5 & \frac{3}{5} \end{pmatrix}$

Date

11) Which matrices have an inverse ?

*) Non singular → Inverse ✓

*) Singular → Inverse ✗

12) Neural networks and matrices

Email spam detector

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2

No. of times word appears in the email

Solution :

Lottery → 1 point
Win → 1 point
Threshold → 1.5 point

Scores

if threshold (score) > 1.5
⇓
It is a spam

Ex : "win, win the lottery"
Score = 3
Score > threshold
It is a spam

1	1
2	1
0	0
0	2
0	1
1	0
2	2
2	0
1	2

Model
1
1
1.5

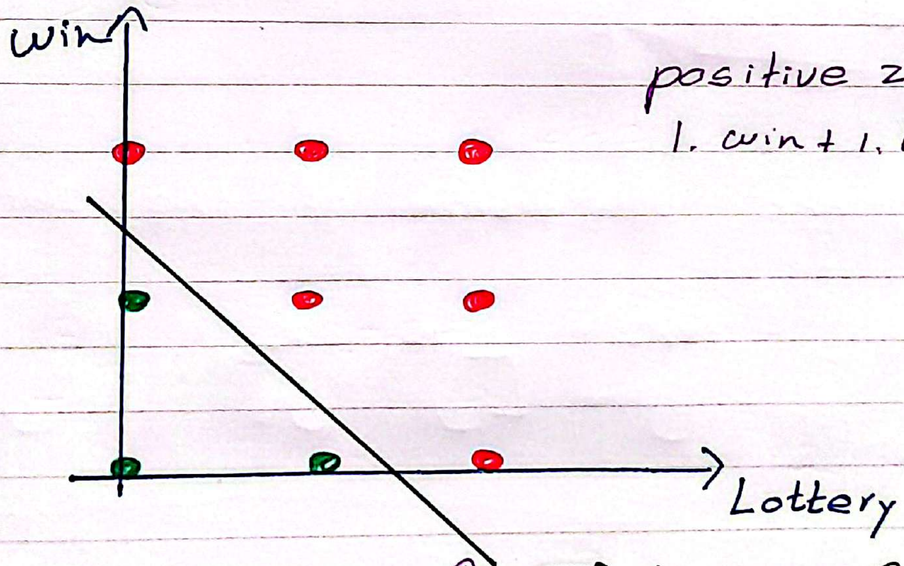
(bias)

=

.prod
2
3
0
2
1
1
4
2
3

check $\longrightarrow$ >1.5?

Spam
Yes
Yes
No
Yes
No
No
Yes
Yes
Yes



positive zone
1. win + 1. Lottery > 1.5

Negative zone
1. win + 1. lottery < 1.5

→ Linear Classifier (NLP)
⇓
Basic neural network (1 layer)

Line
1. win + 1. Lottery = 1.5

$$\boxed{\begin{array}{l} 1.\text{win} + 1.\text{Lottery} > 1.5 \leftarrow (\text{Threshold}) \\ 1.\text{win} + 1.\text{Lottery} - 1.5 > 0 \quad (\text{bias}) \end{array}}$$