# UNIT- 2

12- March [illegible]

## Interpolation -

If we have values of variable $x$ : $x_0, x_1, x_2 \, --- \, x_n$
and corresponding values of $y$ : $y_0, y_1, y_2 \, --- \, y_n$
if we want to find ~~they~~ $y$ for any value of $x \in (x_0, x_n)$, it is called interpolation.

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## Finite differences -

### 1. forward difference operator '$\Delta$'

It is denoted by '$\Delta$' & forward diff. of a function $f(x)$ is defined as $\boxed{\Delta f(x) = f(x+h) - f(x)}$

- Second difference -

$$\begin{aligned}
\Delta^2 f(x) &= \Delta[\Delta f(x)] - \Delta[f(x+h) - f(x)] \\
&= \Delta f(x+h) - \Delta f(x) \\
&= f(x+2h) - f(x+h) - f(x+h) + f(x) \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}$$

### 2. Backward difference operator '$\nabla$'

It is denoted by '$\nabla$' & backward diff. of a function is defined as, $\boxed{\nabla f(x) = f(x) - f(x-h)}$

### 3. Central Difference operator '$\delta$'

It is denoted by '$\delta$' & central diff. of a func$^n$ $f(x)$ is defined as,

$$\boxed{\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}$$

### 4. Average operator '$\mu$'

$$\boxed{\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right]}$$

5. <u>Shifting operator 'E'</u>

$$\boxed{E^n f(x) = f(x+nh)}$$

## Forward Difference Table -

**Ques:** form the forward difference table for

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 1 | 5 | 6 | 8 | -6 | 6 | -15 |
| 3 | 11 | 14 | 2 | 0 | -9 | |
| 5 | 25 | 16 | 2 | -9 | | |
| 7 | 41 | 18 | -7 | | | |
| 9 | 59 | 11 | | | | |
| 11 | 70 | | | | | |

$$\Delta^2 f(1) = \Delta[\Delta f(1)]$$

$\Delta^5(5) = -15$

$\Delta^3(11) = 0$

## Backward difference Table -

**Ques:** form the backward difference table for

| $x$ | $y$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ | $\nabla^5 y$ |
|---|---|---|---|---|---|---|
| 1 | 5 | | | | | |
| 3 | 11 | 6 | | | | |
| 5 | 25 | 14 | 8 | | | |
| 7 | 41 | 16 | 2 | -6 | | |
| 9 | 59 | 18 | 2 | 0 | 6 | |
| 11 | 70 | 11 | -7 | -9 | -9 | -15 |

1B.d  2B.d  3B.d  4B.d  5B.d

$$\nabla f(x) = f(x) - f(x-h)$$

$\nabla^2(70) = -9$

$\nabla^2(59) = 2$

**Ques:** find forward diff. of $\tan x$ & $e^x$

(i) $\Delta \tan x$

(ii) $\Delta e^x$

Sol: (i) $\Delta \tan x$

$= \tan(x+h) - \tan x$

$= \tan\left(\dfrac{(x+h) - x}{1 + x(x+h)}\right)$

$\Delta \tan x = \tan\left(\dfrac{h}{1+x^2+xh}\right)$

(ii) $\Delta e^x$

$= e^{(x+h)} - e^x$

$\Delta e^x = e^x(e^h - 1)$

## Newton's forward interpolation formula - [NFI]

14-Mar-015

$$\boxed{f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0}$$

where, $u = \dfrac{x-a}{h}$

a: I term of 'x' ; $y_0$: I term of 'y'

## Newton's Backward interpolation formula - [NBI]

$$\boxed{f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots}$$

where, $u = \dfrac{x-b}{h}$

b: last term of 'x'

$y_n$: last term of 'y'

**Ques:** Use NFI form. & find the value of f(1.2)

| x | y (f(x)) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 1 | 3.49 | | | |
| | | 1.33 | | |
| 1.4 | 4.82 | | −0.19 | |
| | | 1.14 | | −0.41 |
| 1.8 | 5.96 | | −0.6 | |
| | | 0.54 | | |
| 2.2 | 6.5 | | | |

$x = 1.2$, $a = 1$, $h = 0.4$

$u = \dfrac{x-a}{h} = \dfrac{1.2-1}{0.4} = 0.5$

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 3.49 + 0.5(1.33) + \frac{(0.5)(-0.5)(-0.19)}{2} + \frac{(0.5)(-0.5)(-1.5)}{6} \times (-0.41)$$

$$= \cancel{4.195}\ 4.15$$

**Ans:** from the following table find the value of y

(i) $x = 1.85$

(ii) $x = 2.25$

given that

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ | $\Delta^6 y$ |
|---|---|---|---|---|---|---|---|
| 1.7 | 5.474 | | | | | | |
| | | 0.576 | | | | | |
| 1.8 | 6.050 | | 0.06 | | | | |
| | | 0.636 | | 0.007 | | | |
| 1.9 | 6.686 | | 0.067 | | 0.000 | | |
| | | 0.703 | | 0.007 | | 0.001 | |
| 2.0 | 7.389 | | 0.074 | | 0.001 | | -0.002 |
| | | 0.777 | | 0.008 | | -0.001 | |
| 2.1 | 8.166 | | 0.082 | | 0.000 | | |
| | | 0.859 | | 0.008 | | | |
| 2.2 | 9.025 | | 0.09 | | | | |
| | | 0.949 | | | | | |
| 2.3 | 9.974 | | | | | | |

taking $a = 1.8$, $h = 0.1$, $x = 1.85$

$$u = \frac{x-a}{h} = \frac{1.85 - 1.8}{0.1} = 0.5$$

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0$$

$$= 6.368 + 0.008375 + 0.0004375 + (-0.0000390625) + (-0.0000027734375)$$

$$= 6.376746094 \ / \ 6.359$$

(ii) $x = 2.25$

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)(u+4)(u+5)}{6!}\nabla^6 y_n$$

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ | $\Delta^6 y$ |
|---|---|---|---|---|---|---|---|
| 1.7 | 5.474 | 0.576 | | | | | |
| 1.8 | 6.050 | | | | | | |
| 1.9 | 6.686 | | | | | | |
| 2.0 | 7.389 | | | | | | |
| 2.1 | 8.166 | | | | | | |
| 2.2 | 9.025 | | | | | | |
| 2.3 | 9.974 | | | | | | |

$$= 9.974 + (-0.5)(0.949) + \frac{(-0.5)(0.5)(0.09)}{2}$$

$$+ \frac{(-0.5)(0.5)(1.5)(0.008)}{6}$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(0)}{24} + \frac{(-0.5)(0.5)(1.5)(2.5)(3.5)(-0.001)}{120}$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(3.5)(4.5)(-0.002)}{720}$$

$$= 9.459441016$$

$$= 9.46$$

**Ques:** from the following table estimate the no. of student who obtained marks b/w 40 & 45

| Marks | No. of students | Marks less than (x) | No. of Std (y) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|---|
| 30-40 | 31 | 40 | 31 | | | | |
| | | | | 42 | | | |
| 40-50 | 42 | 50 | 73 | | 9 | | |
| | | | | 51 | | -25 | |
| | | | | | -16 | | 37 |
| 50-60 | 51 | 60 | 124 | | | 12 | |
| | | | | 35 | | | |
| | | | | | -4 | | |
| 60-70 | 35 | 70 | 159 | | | | |
| | | | | 31 | | | |
| 70-80 | 31 | 80 | 190 | | | | |

The given distribution can be arranged ↑

taking, $x = 45$, $a = 40$, $h = 10$

$$u = \frac{x-a}{h} = \frac{45-40}{10} = 0.5$$

Newton's formula —

$$f(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$f(45) = 31 + 0.5(42) + \frac{(0.5)(-0.5)9}{2} + \frac{(0.5)(-0.5)(-1.5)(-25)}{6}$$
$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)(37)}{24}$$

$$= 47.87 \simeq 48$$

Now, No. of students b/w 40 & 45 $= 48 - 31$
$= 17$

**Ques:** find the cubic polynomial which takes the following values, hence/otherwise evaluate $f(4)$.
Give that

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| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 0 | $\boxed{1}$ | | | |
| | | $\boxed{+1}$ | | |
| 1 | 2 | | $\boxed{-2}$ | |
| | | $-1$ | | $\boxed{12}$ |
| 2 | 1 | | 10 | |
| | | 9 | | |
| 3 | 10 | | | |

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$a = 0, \quad h = 1, \quad \Rightarrow u = \frac{x-a}{h} = \frac{x-0}{1}$$

$$\boxed{u = x}$$

$$= 1 + x(1) + \frac{x(x-1)\overset{-1}{\cancel{(-2)}}}{\cancel{2}} + \frac{x(x-1)(x-2)\overset{2}{\cancel{(12)}}}{\cancel{6}}$$

$$= 1 + x - x^2 + x + 2(x^3 - 3x^2 + 2x)$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

taking $x = 4$,

$$f(4) = 128 - 112 + 24 + 1$$

$$f(4) = 41$$

## Central Difference Interpolation formula -

We apply Newton's forward or Backward formula for finding values of y which are nearest to begining term or end term.

whe

$\Rightarrow$ when we want to find value of y which is nearest to middle term we use central difference interpolation formula.

## Gauss's Forward interpolation formula -

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ (p+1)\,p(p-1)(p-2)\,\Delta y_{-2} + \text{---}$$

# Gauss's Backward interpolation formula -

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots$$

16-March-023

**Ques:** Use Gauss's Forward formula to evaluate $y_{30}$. Given that -
$y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3434$,
$y_{37} = 15.5154$

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 21 | $y_{-2}$ 18.4708 | | | | |
| | | −0.6564 | | | |
| 25 | $y_{-1}$ 17.8144 | | −0.051 | | |
| | | −0.7074 | | −0.0054 | |
| 29 | $y_0$ $\boxed{17.1070}$ | | $\boxed{-0.0564}$ | | $\boxed{-0.0022}$ |
| | | $\boxed{-0.7638}$ | | $\boxed{-0.0076}$ | |
| 33 | $y_1$ 16.3432 | | −0.064 | | |
| | | −0.8278 | | | |
| 37 | $y_2$ 15.5154. | | | | |

$x = 30$, $x_0 = 29$, $h = 4$

$$p = \frac{x - x_0}{h} = \frac{30-29}{4} = 0.25$$

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)\Delta^4 y_{-2}}{4!}$$

$$y = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)(-0.0564)}{2}$$
$$+ \frac{(1.25)(0.25)(-0.75)(-0.0076)}{6} + \frac{(1.25)(0.25)(-0.75)(-1.75)(-0.0022)}{24}$$

$= 16.9213375 + 0.000296875 - 0.00003759765625$

$= 16.92159678$

$= 16.92159$

Ques: Interpolate by means of Gauss's backward formula, the popula^n of a town for the year 1974, given that

| Year | Popul^n [in thns] | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 1939 | $y_{-3}$ 12 | | | | | |
| | | 3 | | | | |
| 1949 | $y_{-2}$ 15 | | 2 | | | |
| | | 5 | | 0 | | |
| 1959 | $y_{-1}$ 20 | | 2 | | 3 | $\boxed{-10}$ |
| | | $\boxed{7}$ | | $\boxed{3}$ | | |
| 1969 | $y_0$ $\boxed{27}$ | | $\boxed{5}$ | | $\boxed{-7}$ | |
| | | 12 | | -4 | | |
| 1979 | $y_1$ 39 | | 1 | | | |
| | | 13 | | | | |
| 1989 | $y_2$ 52 | | | | | |

~~Gauss's~~ ba'  $x = 1974$, $x_0 = 1969$, $h = 10$, $p = \dfrac{x - x_0}{h} = 0.5$

Gauss's backward formula -

$$y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$= 27 + (0.5)7 + \frac{(1.5)(0.5)(-0.5)\times 3}{6} + \frac{(2.5)(1.5)(0.5)(-0.5)(-7)}{24}$$

$$+ \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)(-10)}{120} = \begin{array}{l} 32.1875 + 0.27343 \\ + (-0.1171875) \end{array}$$

$$= 32.3437 \text{ thousands} \qquad = 32.34375$$

$$\simeq \underline{32344}$$

$\therefore$ The popula^n of a town for the year 1974 is 32344

Ques: Use Gauss's backward form. estimate the no. of person earning wages b/w Rs. 60 & 70 from the following data.

| | | | | |
|---|---|---|---|---|
| below 40 | 40 - 60 | 60 - 80 | 80 - 100 | 100 - 120 |
| 250 | 120 | 100 | 70 | 50. |

After convers^n of table form,

| Wages below $x$ | | no. of persons $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| 40 | $y_{-1}$ | 250 | | | | |
| | | | 120 | | | |
| $x_0$ 60 | $y_0$ | 370 | | -20 | | |
| | | | 100 | | -10 | |
| 80 | $y_1$ | 470 | | -30 | | 20 |
| | | | 70 | | 10 | |
| 100 | $y_2$ | 540 | | -20 | | |
| | | | 50 | | | |
| 120 | $y_3$ | 590 | | | | |

$h = 20$ , $x_0 = 60$ , $x = 70$ , $p = \frac{x - x_0}{h} = 0.5$

$$y = y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$

$$= 370 + (0.5)(120) + \frac{(1.5)(0.5)(-20)}{2!}$$

$= \cancel{370}$ $422.5$

$\Rightarrow$ $422.5 - x_{\text{below } 60}$.

$= 422.5 - 370 = 52.5$ [after rounding off]

~~483 ye~~ = ~~52 persons~~ 52,500 [in thousands]

The no. of persons earning wages b/w Rs 60-70 are ~~52 persons~~ 52,500 persons

## Choice of interpolation formula

20-Mar-023

1.) To find the tabulated value near the begining of the ~~value~~ table use Newton's forward interpolation formula [NFI formula]

2.) To find the value near the end of the table use Netwon's backword interpolation formula [NBI formula]

3.) To find a value neara the centre of the table then use the following formulae according to —

3a. Whenever mentioned use Gauss's forw. / backw. formula.

3b. If value of P lies b/w $-1/4$ & $1/4$ use Stirling's formula. $[-0.25, 0.25]$

3c. If value of P lies b/w $1/4$ & $3/4$ use Bassel's or Everett's formulae. $[0.25, 0.75]$

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## Stirling's formula -

21-Mar-023

$$\boxed{y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2 \Delta^2 y_{-1}}{2!} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots}$$

**Ques:** Use Stirling's formula to compute $y_{12.2}$ from the following table.

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 10 $y_{-2}$ | 0.23967 | | | | |
| | | 0.04093 | | | |
| 11 $y_{-1}$ | 0.28060 | | $-0.00365$ | | |
| | | $\boxed{0.03728}$ | | $\boxed{0.00058}$ | |
| 12 $y_0$ | $\boxed{0.31788}$ | | $\boxed{-0.00307}$ | | $\boxed{-0.00013}$ |
| | | $\boxed{0.03421}$ | | $\boxed{0.00045}$ | |
| 13 $y_1$ | 0.35209 | | $-0.00262$ | | |
| | | 0.03159 | | | |
| 14 $y_2$ | 0.38368 | | | | |

Taking $x_0 = 12$, $x = 12.2$, $h = 1$ ; $p = \frac{x - x_0}{1} = 0.2$

By Stirling formula,

$$y = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2 \Delta^2 y_{-1}}{2!} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2}$$

$$= (0.31788) + \left[0.2\left(\frac{0.03421 + 0.03728}{2}\right)\right] + \left[\frac{(0.2)^2 \times (-0.00307)}{2}\right]$$

$$+ \left[\frac{0.2\,(0.2^2-1)}{6}\left(\frac{0.00045 + 0.00058}{2}\right)\right] +$$

$$\left[\frac{0.2^2\,(0.2^2-1)}{24} \times (-0.00013)\right]$$

$$= 0.31788 + (0.007149) + (-0.0000614) + (-0.00001648)$$

$$+ (0.000000208)$$

$$= 0.3249$$

**Ques:** Use Stirling formula compute $y_{25}$ given that -

$y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$

where $y$ represent the no. of persons at age $x$

| $x$ | | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|---|
| 20 | $y_{-1}$ | 512 | | | |
| | | | -73 | | |
| 30 | $y_0$ | 439 | | -20 | |
| | | | -93 | | 10 |
| 40 | $y_1$ | 346 | | -10 | |
| | | | -103 | | |
| 50 | $y_2$ | 243 | | | |

taking $x_0 = 30$, $x = 35$, $h = 10$

$$p = \frac{x - x_0}{h} = 0.5$$

by <u>Stirling's formula</u>,

$$y = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left(\frac{p^2}{2!}\Delta^2 y_{-1}\right) + \left(\frac{p(p^2-1)}{3!}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \ldots$$

$$= 439 + 0.5\left(\frac{-93 + (-71)}{2}\right) + \left(\frac{0.5^2}{2}(-70)\right)\left(\frac{10 + 0}{2}\right)$$

$$\approx 394.77$$

$$= 395 \text{ persons}$$

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## <u>Bessel's formula</u> $\left[p \in [\tfrac{1}{4}, \tfrac{3}{4}]\right]$

$$y = y_0 + p\Delta y_0 + \left[\frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)\right] + \left[\frac{(p-\frac{1}{2})\,p\,(p-1)}{2}\Delta^3 y_{-1}\right]$$

$$+ \left[\frac{(p+1)\,p\,(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right)\right] + \ldots$$

<u>Ques</u>: Apply Bessel's form to obtain $y_{25}$ given that -
$y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$

| $x$ | | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|---|
| 20 | $y_{-1}$ | 2854 | | | |
| | | | 308 | | |
| $\boxed{24}$ | $y_0$ | $\boxed{3162}$ | | $\boxed{74}$ | |
| | | | $\boxed{382}$ | | $\boxed{-8}$ |
| 28 | $y_1$ | 3544 | | $\boxed{66}$ | |
| | | | 448 | | |
| 32 | $y_2$ | 3992 | | | |

taking, $x = 25$, & $x_0 = 24$, $h = 4$

$$p = \frac{x - x_0}{h} = 0.25.$$

• 3250.93

$$y = 3162 + (0.25 \times 382) + \left[\frac{0.25(0.25-1)}{2}\left(\frac{66+74}{2}\right)\right]$$

$$+ \left[\frac{(0.25-\frac{1}{2})\,0.25\,(0.25-1)}{6}(-8)\right]$$

$$= 3250.875$$

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27-March-023

## Everettes formula :

$$y_p = q y_0 + \frac{q_0(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$

$$+ p y_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots$$

where, $q = 1-p$

**Ques:** Apply Everette's formula to find the value of log (337.5) given that -

| 'x' | | 'y' log x | Δy | Δ²y | Δ³y | Δ⁴y | Δ⁵y |
|---|---|---|---|---|---|---|---|
| 310 | $y_{-2}$ | 2.49136 | | | | | |
| | | | 0.01379 | | | | |
| 320 | $y_{-1}$ | 2.50515 | | -0.00043 | | | |
| | | | 0.01336 | | 0.00004 | | |
| $x_0$ 330 | $y_0$ | $y_0$ 2.51851 | | -0.00039 | | -0.00003 | |
| | | | 0.01297 | | 0.00001 | | 0.00004 |
| 340 | $y_1$ | 2.53148 | | -0.00038 | | 0.00001 | |
| | | | 0.01259 | | 0.00002 | | |
| 350 | $y_2$ | 2.54407 | | -0.00036 | | | |
| | | | 0.01223 | | | | |
| 360 | $y_3$ | 2.55630 | | | | | |

by Everette's formula,

$$y_p = q y_0 + \frac{q_0(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots + p y_1$$

$$+ \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots$$

where $q = 1-p$

0.629642734

$- 0.0000002307128906$

$0.00002078125$

$0.00$

$= \boxed{0.629663378}$

$1.89861$

taking, $x_0 = 330$, $h = 10$

$x = 337.5$

$$p = \frac{x - x_0}{h} = 0.75$$

$$q = 1 - p = 0.25$$

$$= \overline{(0.25 \times 2.51851)} + \overline{\left[\frac{0.25 \times (0.25^2 - 1^2)}{6} \times (-0.00039)\right]}$$

$$+ \overline{\left[\frac{0.25(0.25^2 - 1^2)}{5!} \times (-0.00003)\right]} + \overline{\left[0.75 \times 2.53148\right]}$$

$$+ \overline{\left[\frac{0.75\,(0.75^2 - 1^2)}{6} \times (-0.00038)\right]} + \overline{\left[\frac{0.75(0.75^2 - 1^2)(0.75^2 - 4)}{5!} \times 0.00001\right]}$$

$$= 1.89861 + 0.629642734 - 0.0000002307128906 + 0.00002078125 + 0.0000000939941406$$

$$= \underline{2.5282}$$

---

## Proof of identities:

1.) $\Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$= (E - 1) f(x)$

$\Rightarrow \Delta = E - 1$

2.) $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= [1 - E^{-1}]\, f(x)$

$\Rightarrow \nabla = 1 - E^{-1}$

3.) $\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$= (E^{1/2} - E^{-1/2})\, f(x)$

$\delta = E^{-1/2} - E^{-1/2}$

4.) $\mu f(x) = \frac{1}{2}\left[f(x + \frac{h}{2}) + f(x - \frac{h}{2})\right]$

$\Rightarrow \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

1) Ques: Prove that: $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

$\Rightarrow$ LHS $= (E^{1/2} + E^{-1/2})(1+\Delta)^{1/2}$

$\because \Delta = E - 1$

$= (E^{1/2} + E^{-1/2})(1 + E - 1)^{1/2}$

$= (E^{1/2} + E^{-1/2})\, E^{1/2}$

$= E + 1$

$= 1 + \Delta + 1$

$= 2 + \Delta$ = R.H.S

---

$\boxed{\text{Note}}$ • formula will be freq. used $\Rightarrow (a+b)^2 - 4ab = (a-b)^2$

---

2) Prove that -

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$$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$$

R.H.S $\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

$$= \frac{1}{2}\delta^2 + \frac{\delta}{2}\sqrt{4 + \delta^2}$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right]$$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right]$$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})\left[(E^{1/2} - E^{-1/2}) + \sqrt{(E^{1/2} + E^{-1/2})^2}\right]$$

$\because (a-b)^2 + 4ab = (a+b)^2$

3) Prove that -

$$\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$

R.H.S $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$= \frac{E-1}{1-\frac{1}{E}} - \frac{1-\frac{1}{E}}{E-1}$$

$$= \frac{(E-1)}{\frac{(E-1)}{E}} - \frac{\left(\frac{E-1}{E}\right)}{(E-1)}$$

$$= E - \frac{1}{E} = (E-1) + \left(1-\frac{1}{E}\right)$$

$$= \Delta + \nabla \quad = L.H.S$$

4) Proove that -

$$\Delta^3 y_2 = \nabla^3 y_5$$

L.H.S.

$$\Delta^3 y_2$$

$$= (E-1)^3 y_2$$

$$= \underbrace{[E^3 - 3E^2 + 3E(-1)^2 - 1]\, y_2}_{\downarrow} = E^3 y_2 - 3E^2 y_2 + 3E y_2 - y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad \text{— (1)}$$

R.H.S

$$\nabla^3 y_5$$

$$= (1-E^{-1})^3 \cdot y_5$$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})\, y_5$$

$$= y_5 - 3E^{-1} y_5 + 3E^{-2} y_5 \cancel{+} - E^{-3} y_5$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \quad \text{— (2)}$$

from (1) & (2),

LHS = RHS.

5) Proove that -

$$1 + \frac{\delta^2}{2} = \sqrt{1 + \delta^2 \mu^2}$$

LHS $= 1 + \frac{\delta^2}{2}$ RHS

$$= \frac{2+\delta^2}{2}$$

$$= \frac{2 + (E^{1/2} - E^{-1/2})^2}{2} = 1 + \frac{(E^{1/2} - E^{-1/2})^2}{2}$$

$$\frac{2+(E^{1/2}-E^{-1/2})}{2} = \frac{\cancel{2}+E+E^{-1}-\cancel{2}}{2}$$

$$= \frac{E+E^{-1}}{2}$$

RHS $= \sqrt{1+\delta^2\mu^2}$

$= \sqrt{1+(E^{1/2}-E^{-1/2})^2 \frac{1}{2}(E^{1/2}+E^{-1/2})}$ / $\sqrt{1^2+(\delta\mu)^2}$

$= \sqrt{1+\frac{1}{2}[E}$ / $\sqrt{1+(E^{1/2}-E^{-1/2})^4 \frac{1}{4}}$

$$= \sqrt{\frac{4+(E^{1/2}+E^{-1/2})^4}{4}}$$

$$= \frac{1}{2}\sqrt{\cancel{4}+\cancel{E^{1/2}}/+/\cancel{E^{?}}\ \cancel{E^{1/2}}\quad E^2+1+\cancel{2E}+1+E^{-2} + \cancel{2E^{-1}} - \cancel{2E} - \cancel{2E^{-1}} - \cancel{4}}$$

$$\frac{1}{2}\sqrt{E^2+1+1+E^{-2}}$$

$$\frac{1}{2}\sqrt{E^2+E^{-2}+2}$$

$$= \frac{1}{2}\sqrt{(E+E^{-1})^2}$$

$$= \frac{E+E^{-1}}{2}$$

---

## Interpolation with unequal intervals:

$\Rightarrow$ Lagrange's interpolation formula

If $x_0, x_1, x_2 ---- x_n$ are values of $x$ & $f(x_0), f(x_1), f(x_2) ---- f(x_n)$ are corresponding val. of $y$ then $f(x)=$

$$f(x) = \frac{(x-x_1)(x-x_2)----(x-x_n)}{(x_0-x_1)(x_0-x_2)----(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)----(x-x_n)}{(x_1-x_0)(x_1-x_2)----(x_1-x_n)} f(x_1)$$

$$+ ------ + \frac{(x-x_0)(x-x_1)-(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)---(x_n-x_{n-1})} f(x_n)$$

Divit

Ques Use lagrange's interpolation formula to find the value of y when $x = 10$ given that

| | | | | |
|---|---|---|---|---|
| $x$ : | 5 | 6 | 9 | 11 |
| $y$ : | 12 | 13 | 14 | 16 |

Ans: $f(x_0)$ $f(x_1)$ $f(x_2)$ $f(x_3)$, by lagrange's formula -

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

Taking $x = 10$

$$f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(+12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)} 13 + \frac{(5)(4)(-1)}{4 \times 3 \times (-2)} \times 14$$

$$+ \frac{5 \times 4 \times 1}{(6)(5)(2)} 16$$

$$= 2 + \left(\frac{-13}{3}\right) + \frac{35}{3} + \frac{16}{3} = \frac{44}{3}$$

Ques Use lagrange's interpolation formula and find the value of $f(9)$

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 5 | 7 | 11 | 13 | 17 |
| $y$ : | 150 | 392 | 1452 | 2366 | 5202 |

Sol $f(x_0)$ $f(x_1)$ $f(x_2)$ $f(x_3)$ $f(x_4)$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

taking $x = 9$

$$f(9) = \frac{(2)(-2)(-4)(-8)}{(-2)(-6)(-8)(-12)} \times 150 + \frac{(4)(-2)(-4)(-8)}{(2)(-4)(-6)(-10)} \times 392$$

$$+ \frac{(4)(2)(-4)(-2)}{(6)(4)(-2)(6)} \times 1452 + \frac{(4)(2)(-2)(-8)}{(8)(6)(2)(-4)} \times 2366$$

$$+ \frac{(4)(2)(-2)(-4)}{(12)(10)(6)(4)} \times 5202$$

$$= -\frac{50}{3} + \left(\frac{-392}{15}\right) - \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5}$$

$$= -\frac{50}{3} + \cancel{34536}\ \frac{3136}{15} - \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5}$$

$$= \frac{-250 + 3136 - 19360 - 11830 + 1734}{15} = -\frac{\cancel{800}}{15}$$

$$= -\frac{12150}{15} = -810$$

$$-\frac{50}{3} + \frac{3136}{15} + \frac{11616}{9} - \frac{2366}{3} + \frac{5202}{45}$$

$$= 810$$

24
32
42

6-Apr-023

## Divided Difference :

If $x_0, x_1, x_2, x_3 ---- x_n$ are values of variable $x$ and $y_0, y_1, y_2 ---- y_n$ are corresponding values of $y$ then divided difference is denoted by '$\Delta$' & it is defined as

$$\boxed{\Delta y_0 = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}}$$

$y_0$ - first term of $y$ always

$$\Delta y_1 = [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

Similarly,

$$\boxed{\Delta^2 y_0 = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \quad \text{or} \quad \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}}$$

## Newton's divided Difference formula :

$$\boxed{y = y_0 + (x - x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + ----}$$

**Ques:** Use Newton's Divided Diff. formula & find the value of $f(9)$ Given that

| $x$ | $f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | |
|---|---|---|---|---|---|
| 5 | 150 | $\frac{392-150}{7-5} = 121$ | $\frac{265-121}{11-5} = 24$ | $\frac{32-24}{13-5} = 1$ | |
| 7 | 392 | $\frac{1452-392}{11-7} = 265$ | $\frac{457-265}{13-7} = 32$ | $\frac{42-32}{17-7} = 1$ | 0 |
| 11 | 1452 | $\frac{2366-1452}{13-11} = 457$ | $\frac{709-457}{17-11} = 42$ | | |
| 13 | 2366 | $\frac{5202-2366}{17-13} = 709$ | | | |
| 17 | 5202 | | | | |

By NDD,

$$y = y_0 + [(x-x_0)\Delta y_0] + [(x-x_0)(x-x_1)\Delta^2 y_0] + [(x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0]$$

$$\Rightarrow y = 150 + (x-5)\,121 + (x-5)(x-7)\,24 + (x-5)(x-7)(x-11)\,1$$

taking $x = 9$

$$\Rightarrow f(9) = 150 + 4\times121 + 4\times2\times24 + 4\times2\times(-2)$$
$$= 810$$

**Ques:** Determine $f(x)$ as a polynomial in $x$ for the following data

**Sol:**

| $x$ | $f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| $-4$ | $\boxed{1245}$ | | | | |
| | | $\frac{33-1245}{-1-(-4)} = \boxed{-404}$ | | | |
| $-1$ | $33$ | | $\frac{-28-(-404)}{0-(-4)} = \boxed{94}$ | | |
| | | $\frac{5-33}{0-(-1)} = -28$ | | $\frac{10-94}{2-(-4)} = \boxed{-14}$ | |
| $0$ | $5$ | | $\frac{2-(-28)}{2-(-1)} = 10$ | | $\frac{13-(-14)}{5-(-4)} = \boxed{3}$ |
| | | $\frac{9-5}{2-0} = 2$ | | $\frac{88-10}{5+1} = 13$ | |
| $2$ | $9$ | | $\frac{442-2}{5-0} = 88$ | | |
| | | $\frac{1335-9}{5-2} = 442$ | | | |
| $5$ | $1335$ | | | | |

By NDD,

$$y = y_0 + [(x-x_0)\Delta y_0] + [(x-x_0)(x-x_1)\Delta^2 y_0] + [(x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0]$$
$$+ [(x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0]$$

$$\Rightarrow 1245 + [(x+4)(-404)] + [(x+4)(x+1)\,94] + [(x+4)(x+1)\,x\,(-14)]$$
$$+ [(x+4)(x+1)(x)(x-2)\times3]$$

$$\Rightarrow 1245 - (x+4)\,404 + (x^2+5x+4)\,94 - (x^3+5x^2+4x)\,14$$
$$+ (x^4+3x^3-6x^2-8x)\times3$$

$\Rightarrow f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$

---

10-Apr-023

## Hermite's Interpolation formula-

Note: In this method values of $x$, values of $y$ & values of $y'$ are given

$$L(x_i) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

$$P(x) = \sum [1-2(x-x_i)L'(x_i)][L(x_i)]^2 y(x_i) + \sum (x-x_i)[L(x_i)]^2 y'(x_i)$$

Ques: Determine the Hermite's interpolation for the following data-

Given that -

| | $x$ | $y(x)$ | $y'(x)$ |
|---|---|---|---|
| $x_0$ | 0 | 0 | 0 |
| $x_1$ | 1 | 1 | 0 |
| $x_2$ | 2 | 0 | 0 |

$$L(x_0) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(-1)(-2)} = \frac{1}{2}[x^2-3x+2]$$

$$L(x_1) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-2)}{(1)(-1)} = -[x^2-2x]$$

$$L(x_2) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{(2)(1)} = \frac{1}{2}[x^2-x]$$

$$P(x) = \Big[\Big([1-2(x-x_0)L'(x_0)][L(x_0)]^2 y(x_0)\Big) + \Big([1-2(x-x_1)L'(x_1)][L(x_1)]^2 y(x_1)\Big) + \Big([1-2(x-x_2)L'(x_2)][L(x_2)]^2 y(x_2)\Big)\Big] +$$
$$(x-x_0)[L(x_0)]^2 y'(x_0) + (x-x_1)[L(x_1)]^2 y'(x_1) + (x-x_2)[L(x_2)]^2 y'(x_2)$$

$$l'(x_0) = \frac{1}{2}[2x-3] \text{ , at } (x = x_0 = 0) \text{ , } = -\frac{3}{2}$$

$$l'(x_1) = -[2x-2] \text{, at } (x = x_1 = 1) \text{ , } = 0$$

$$l'(x_2) = \frac{1}{2}[2x-1] \text{, at } (x = x_2 = 2) \text{ , } = \frac{3}{2}$$

$$P(x) = 0 + [1-2(x-x_1) \times 0] [x^2-2x]^2 \times 1 + 0 + 0 + 0 + 0$$

$$= x^4 - 4x^3 + 4x^2$$

**Ques 8** Apply Hermite's formula to interpolate $\sin(1.05)$, Given that -

| $x$ | $y(x)$ $\sin x$ | $y'(x)$ $\cos x$ |
|---|---|---|
| 1.00 | 0.84147 | 0.54030 |
| 1.10 | 0.89121 | 0.45360 |

$$L(x_0) = \frac{(x-x_1)}{(x_1-x_0)} = \frac{(x-1.1)}{-(0.1)} = -10x+11$$

$$L'(x_0) = -10$$

$$L(x_1) = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(x-1)}{0.1} = 10x-10$$

$$L'(x_1) = 10$$

$$P(x) = [1-2(x-x_0) L'(x_0)][L(x_0)]^2 y(x_0) + [1-2(x-x_1) L'(x_1)] [L(x_1)]^2 y(x_1) + (x-x_0)[L(x_0)]^2 y'(x_0) + (x-x_1) \cdot [L(x_1)]^2 y'(x_1)$$

$$= [1-2(x-1)(-10)](-10x+11)^2 \times 0.84147 + [1-2(x-1.1)(10)][10x-10]^2 \times 0.89121 + (x-1)(-10x+11)^2 \times 0.54030 + (x-1.1)(10x-10)^2 \times 0.45360$$

taking $x = 1.05$

$$P(1.05) = 0.5 \times 0.84147 + 0.5 \times 0.89121 + 0.05 \times 0.25 \times 0.54030$$
$$+ (-0.05)(0.25) \times 0.45360$$
$$= \underline{\underline{0.86742}}$$