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UNIT = 5

Power method for finding Eigen value and Eigen vectors of a Matrix

Ques Determine the largest Eigen value and corresponding Eigen vector of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let the initial vector is —

$$X_{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Taking largest value common.

$$= 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

2 is the Eigen value and $X_1 = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$ is the corresponding Eigen Vector.

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I^{st} Iteration

$$A X_{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

2.5 is Eigen value and

$$X_{(2)} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$ is Eigen vector.

II^{nd} Iteration

$$A X_{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix}$$

2.8 is Eigen value and

$$X_{(3)} = \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix}$$ is Eigen vector.

III^{rd} Iteration

Taking numerically largest value common.

$$A X_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.43 \\ 1.86 \end{bmatrix} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$

3.43 is Eigen value and

$$X_4 = \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$ is Eigen vector.

IVth Iteration

$$AX_4 = \begin{bmatrix} +2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$

$$= \begin{bmatrix} 0.74 \\ -3.41 \\ 2.08 \end{bmatrix} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix}$$

Eigen value $= 3.41$

Eigen vector $= \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix}$

Vth Iteration

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \begin{bmatrix} 2.6 \\ -3.41 \\ 2.22 \end{bmatrix}$$

$$= 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix}$$

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Largest Eigen value = 3.41
Vector = $\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix}$

Boundary value problem –

An eqn of the form

$$\frac{d^2y}{dx^2} = \frac{Pdy}{dx} + Qy + R$$

with $y(a) = m$
$y(b) = n$

is known as Boundary value problem.

$$h = \frac{b-a}{n}$$

where, n is the no. of sub-interval

Thus, $x_1 = a+h$
$x_2 = a+2h$
$x_3 = a+3h$

Solution of Boundary value problem

1.) Finite difference Method

If the given Boundary value prblm is

$$\frac{d^2y}{dx^2} = \frac{Pdy}{dx} + Qy + R$$

with boundary values, $y(a) = m$
$y(b) = n$

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Taking $h = \frac{b-a}{n}$ and n is the no. of subintervals then,

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y' = \frac{y_{i+1} - y_{i-1}}{2h}$$

Ques. By finite diff. method solve —

$$\frac{d^2y}{dx^2} = y + x(x-4) \quad \text{—— (1)}, \quad 0 \le x \le 4$$

with $y(0) = y(4) = 0$

Sol^n: $h = \frac{4-0}{4} = 1$ taking $(n=4)$

$x_0 = 0$, $x_1 = x_0 + h = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_{i+1} - 2y_i + y_{i-1}$$

—— in (1)

$$y_{i+1} - 2y_i + y_{i-1} = y_i + x_i(x_i - 4) \quad \text{—— (2)}$$

Taking $i = 1, 2$ & 3 in (2)

$\boxed{i=1}$ $y_2 - 2y_1 + y(0) = y_1 + x_1(x_1 - 4)$

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1.85
2.57
~~1.85~~

$$y_2 - 2y_1 + y_0 = y_1 + x_1(x_1 - 4)$$

$$y_2 - 3y_1 = -3$$

$$-3y_1 + y_2 + 0y_3 = -3 \quad \text{--- (3)}$$

$\boxed{i=2}$

$$y_3 - 2y_2 + y_1 = y_2 + x_2(x_2 - 4)$$

$$y_1 - 3y_2 + y_3 = -4 \quad \text{--- (4)}$$

$\boxed{i=3}$

$$y(4) - 2y_3 + y_2 = y_3 + x_3(x_3 - 4)$$

$$0y_1 + y_2 - 3y_3 = -3 \quad \text{--- (5)}$$

$$y_1 = 1.8571$$

$$y_2 = 2.5714$$

$$y_3 = 1.8571$$

Ques Solve by Method of finite diff.

$\frac{d^2y}{dx^2} = y + x$ with boundary condⁿ ~~$0 \le x \le 2.5$~~ ~~take~~ $h = 0.125$

$y(0) = 2$ (at $x=0$), $y(4) = 2.5$ (at $x=1$) and $h = 0.25$.

Solⁿ: $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \text{in (1)}$$

$$= \frac{y_{i+1} - 2y_i + y_{i-1}}{0.0625} = y_i + x_i$$

$$y_{i+1} - 2y_i + y_{i-1} = 0.0625y_i + 0.0625x_i$$

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$$y_{i+1} - 2.0625y_i + y_{i-1} = 0.0625x_i \quad \text{--- (2)}$$

$i = 1, 2, 3$ in (2)

$\boxed{i=1}$ $y_2 - 2.0625\, y_1 + y(0) = 0.0625 \times 0.25$

$y_2 - 2.0625y_1 + 2 = 0.0625 \times 0.25$

$-2.0625y_1 + y_2 + 0y_3 = -1.9843$

$\boxed{i=2}$ $y_{2+1} - 2.0625y_2 + y_{2-1} = 0.0625 \times 0.5$

$y_3 - 2.0625y_2 + y_1 = 0.03125$

$y_1 - 2.0625y_2 + y_3 = 0.03125$

$\boxed{i=3}$ $y_{(4)} - 2.0625y_3 + y_2 = 0.0625 \times 0.75$

$0y_1 + y_2 - 2.0625y_3 = -2.45312$

On solving

$y_1 = 1.9028$

$y_2 = 1.9402$

$y_3 = 2.1301$

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Eigen value and Eigen function of a differential equation.

Ques Solve the Eigen value problem $\frac{d^2y}{dx^2} + \lambda y = 0$ given that $y(0) = 0$ and $y'(1) = 0$.

Sol^n Let, $\frac{d}{dx} = D$

$$(D^2 + \lambda) y = 0$$

Auxiliary ~~Aucumentary~~ Eq^n : $m^2 + \lambda = 0$

$$m^2 = -\lambda$$

There are following 3 cases —

Case-1 $\lambda = -\mu^2$

Case-2 $\lambda = 0$

Case-3 $\lambda = \mu^2$

Case I : $\lambda = -\mu^2$

a.e., $m^2 = \mu^2$

$$m = \pm \mu$$

$$C.F. = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

at $x = 0$ $y = 0$

$$0 = c_1 + c_2$$

$$\Rightarrow c_2 = -c_1$$

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$y = c_1 [e^{\mu x} - e^{-\mu x}]$

$y' = c_1 [\mu e^{\mu x} + \mu e^{-\mu x}]$

at $x = l$ $y' = 0$

$0 = c_1 \mu [e^{\mu l} + e^{-\mu l}]$

$c_1 \neq 0$, $\mu \neq 0$

$\Rightarrow e^{\mu l} + e^{-\mu l} = 0$

This is immpossible.

Case II : $\lambda = 0$

a.e., $m^2 = 0$

$m = 0, 0$

$y = (c_1 + c_2 x) e^{0x}$

$y = c_1 + c_2 x$

at $x = 0$ $y = 0$

$\boxed{c_1 = 0}$

$y = c_2 x$

$y' = c_2$

at $x = l$ $y' = 0$

$\boxed{c_2 = 0}$

Case III : $\lambda = \mu^2$

a.e. $m^2 = -\mu^2$

$m = \pm i\mu$

$y = c_1 \cos \mu x + c_2 \sin \mu x$

at $x = 0$, $y = 0$

$\boxed{0 = c_1}$

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$y = C_2 \sin \mu x$ —— final Eq^n

$y' = \mu C_2 \cos \mu x$

at $x = l$, $y' = 0$

$0 = \mu C_2 \cos \mu l$

$\Rightarrow$ ~~$\cos \mu l$ is zero~~ $\cos \mu l = 0$

$\cos \mu l$ ~~$\cos$~~ $= \cos (2n-1) \frac{\pi}{2}$

$\Rightarrow \mu = \frac{(2n-1)\pi}{2l}$ ~~substi~~ ← substituting μ in final Eq^n.

Thus, $\lambda = \mu^2$

$$\boxed{\lambda = \frac{(2n-1)^2 \pi^2}{4l^2}}$$

$$y = C_n \sin \frac{(2n-1)\pi}{2l} . x$$

Ques Solve the Eigen value problem $\frac{d^2x}{dt^2} + \lambda x = 0$

given that, at $t = 0$, $x = 0$

at $t = \pi$, $x' = 0$

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Classification of Partial differential eq

~~Ques.~~ ~~Condi~~ Consider the partial differential eq^n of the form.

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

~~Sol~~^{n}. if,
$B^2-4AC<0$, elliptic
$B^2-4AC=0$, Parabolic
$B^2-4AC>0$, Hyperbolic

Ques-(i.) $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x \partial y}+4\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$

Ques-(ii) $(1+x^2)\frac{\partial^2 u}{\partial x^2}+(5+2x^2)\frac{\partial^2 u}{\partial x \partial y}+(4+x^2)\frac{\partial^2 u}{\partial y^2}=0$

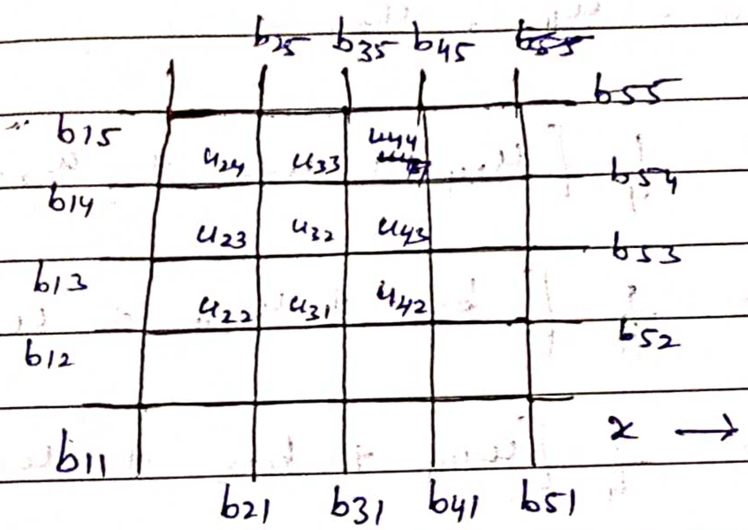
Sol^n:(i) B^2-4AC
$16-4\times1\times4$
$16-16=0$, Parabolic

(ii) B^2-4AC
$(5+2x^2)^2-4\times(1+x^2)\times(4+x^2)$
$25+20x^2+\cancel{4x^4}-16-16x^2-4x^2-\cancel{4x^4}$
$25+\cancel{20x^2}-16-\cancel{20x^2}$
$=9>0$, Hyperbolic

Numerical solⁿ of partial differential eqⁿ —

solⁿ of Elliptic eqⁿ (Laplace eqⁿ)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



first we'll find the interior mesh points by diagonal 5 point formula in the order u_{33}, u_{24}, u_{44}, u_{42}, u_{22}.

$$u_{33} = \frac{1}{4}\left[b_{15} + b_{51} + b_{55} + b_{11}\right]$$

$$u_{24} = \frac{1}{4}\left[b_{15} + u_{33} + b_{35} + b_{13}\right]$$

$$u_{44} = \frac{1}{4}\left[b_{35} + b_{53} + b_{55} + u_{33}\right]$$

$$u_{42} = \frac{1}{4}\left[u_{33} + b_{51} + b_{53} + b_{31}\right]$$

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$$u_{22} = \frac{1}{4}\left[b_{13} + b_{31} + u_{33} + b_{11}\right]$$

Then we'll find remaining interior point by standand five point formula in the order u_{23}, u_{34}, u_{43}, u_{32}

$$u_{23} = \frac{1}{4}\left[u_{24} + u_{33} + u_{22} + b_{13}\right]$$

$$u_{34} = \frac{1}{4}\left[b_{35} + u_{44} + u_{33} + u_{24}\right]$$

$$u_{43} = \frac{1}{4}\left[u_{44} + b_{53} + u_{42} + u_{33}\right]$$

$$u_{32} = \frac{1}{4}\left[u_{33} + u_{42} + b_{31} + u_{22}\right]$$

Iterative formula

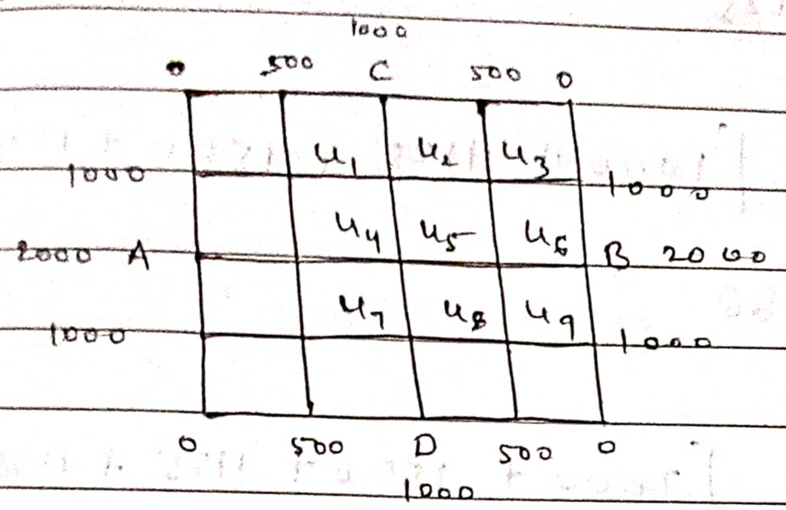
- Jcobi's method

$$u_{ij}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right]$$

- Gauss - Seidal Method

$$u_{ij}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right]$$

Ques Solve the elliptic eq^n — $u_{xx} + u_{yy} = 0$ for the following square mesh.



Sol^n The Boundaries values are symmetrical about the line AB, so we must have

$u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$

Also boundaries values of u are symmetrical about line CD

$u_1 = u_3$, $u_4 = u_6$, $u_7 = u_9$

So, it is sufficient to find only the values u_1, u_2, u_4, u_5

$$u_5 = \frac{1}{4}\left[2000 + 2000 + 1000 + 1000\right] \quad \{\text{St. formula}$$

$$= 1500$$

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$$u_1 = \frac{1}{4}\left[0 + 1500 + 1000 + 2000\right]$$ {diag. formula}

$$= 1125$$

$$u_2 = \frac{1}{4}\left[1000 + 1125 + 1500 + 1125\right]$$ {st. formula}

$$= 1188$$

$$u_4 = \frac{1}{4}\left[2000 + 1500 + 1125 + 1125\right]$$ {st. formula}

$$= 1438$$

Iterative formula (Gauss Seidal)

$$u_1^{(n+1)} = \frac{1}{4}\left[100 + u_2^{(n)} + 500 + u_4^{(n)}\right]$$

$(u_3 = u_1)$ $$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_3^{(n)} + 1000 + u_5^{(n)}\right]$$

$(u_7 = u_1)$ $$u_4^{(n+1)} = \frac{1}{4}\left[2000 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n)}\right]$$

$(u_6 = u_4)$
$(u_8 = u_2)$ $$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_6^{(n)} + u_2^{(n+1)} + u_8^{(n)}\right]$$

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I^{st} Iteration

$(n=0)$

$$u_1^{(1)} = \frac{1}{4}[1000+1188+500+1438] = 1032$$

$$u_2^{(1)} = \frac{1}{4}[1032+1125+1000+1500] = 1164$$

$$u_4^{(1)} = \frac{1}{4}[2000+1500+1032+1125] = 1414$$

$$u_5^{(1)} = \frac{1}{4}[1414+1438+1164+1188] = 1301$$

II Iteration $(n=1)$

$$u_1^{(2)} = \frac{1}{4}[1000+1164+500+1414] = 1020$$

$$u_2^{(2)} = \frac{1}{4}[1020+1032+1000+1301] = 1088$$

$$u_4^{(2)} = \frac{1}{4}[2000+1301+1020+1032] = 1338$$

$$u_5^{(2)} = \frac{1}{4}[1338+1414+1088+1164] = 1251$$

3^{rd} Iteration $(n=2)$

$u_1^{(3)} = 982$

$u_2^{(3)} = 1063$

$u_4^{(3)} = 1313$

$u_5^{(3)} = 1201$

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4^{th} Iteration $(n=3)$

$u_1^{(4)} = 964$

$u_2^{(4)} = 1038$

$u_4^{(4)} = 1288$

$u_5^{(4)} = 1176$

5^{th} Iteration $(n=4)$

$u_1^{(5)} = 957$

$u_2^{(5)} = 1026$

$u_4^{(5)} = 1276$

$u_5^{(5)} = 1157$

6^{th} Iteration $(n=5$

$u_1^{(6)} = 951$

$u_2^{(6)} = 1016$

$u_4^{(6)} = 1266$

$u_5^{(6)} = 1146$

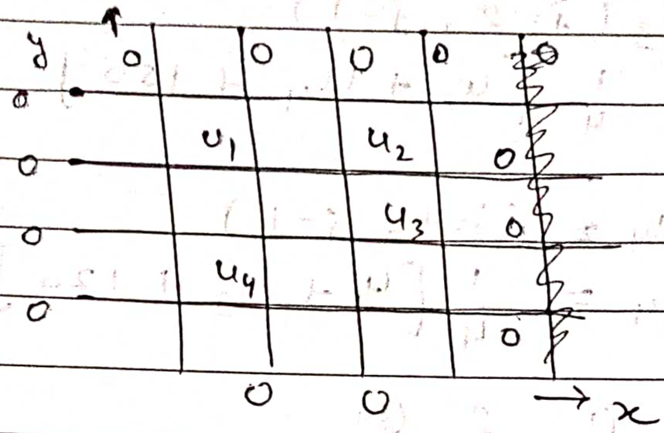
Ques Solution of Poisson's Eq^n of type:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

Sol^n

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jk)$$

Ques Solve the eqn $-\nabla^2 u = -10(x^2+y^2+10)$ over sq. with sides $x=0$, $x=3$, $y=0$, $y=3$ & $u=0$ by the boundary & mesh length $=1$.



Sol^n Given eqn can be written as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = -10(x^2+y^2+10)$$

$$f(x,y) = -10(x^2+y^2+10)$$

The std. five point formula is given by for $h=1$,

$$u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j} = -10(i^2+j^2+10)$$

for u_1 $(i=1,\ j=2)$

$$u_{0,2}+u_{2,2}+u_{1,3}+u_{1,1}-4u_1 = -10(1^2+2^2+10)$$
$$=-150$$

$$0+u_2+0+u_4-4u_1 = -150$$

$$u_1 = \frac{1}{4}\left[u_2+u_4+150\right] \quad \text{— (1)}$$

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for u_2 ($i=2$, $j=2$)

$$u_2 = \frac{1}{4}[u_1 + u_3 + 180] \quad — (2)$$

for u_3 ($i=2$, $j=1$)

$$u_3 = \frac{1}{4}[u_2 + u_4 + 150] \quad — (3)$$

for u_4 ($i=1$, $j=1$)

$$u_4 = \frac{1}{4}[u_1 + u_3 + 120] \quad — (4)$$

from (1) & (3)

$$\boxed{u_1 = u_3}$$

$$u_1 = \frac{1}{4}[u_2 + u_4 + 150]$$

$$u_2 = \frac{1}{2}[u_1 + 90]$$

$$u_4 = \frac{1}{2}[u_1 + 60]$$

I[st] Iteration taking $u_2 = u_4 = 0$

$$u_1 = 37.5$$

$$u_2 = \frac{1}{2}[37.5 + 90] = 64$$

$$u_4 = \frac{1}{2}[37.5 + 60] = 49$$

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II Iteration

$$u_1 = \frac{1}{4}[64 + 49 + 150] = 66$$

$$u_2 = \frac{1}{2}[66 + 90] = 78$$

$$u_4 = \frac{1}{2}[66 + 60] = 63$$

III Iteration

$$u_1 = \frac{1}{4}[78 + 63 + 150] = 73$$

$$u_2 = \frac{1}{2}[73 + 90] = 82$$

$$u_4 = \frac{1}{2}[73 + 60] = 67$$

IV Iteration

$$u_1 = \frac{1}{4}[82 + 67 + 150] = 75$$

$$u_2 = \frac{1}{2}[75 + 90] = 83$$

$$u_4 = \frac{1}{2}[75 + 6] = 68$$

V Iteration

$$u_1 = \frac{1}{4}[83 + 68 + 150] = 75$$

$$u_2 = \frac{1}{2}[75 + 90] = 83$$

$$u_4 = \frac{1}{2}[75 + 60] = 68$$