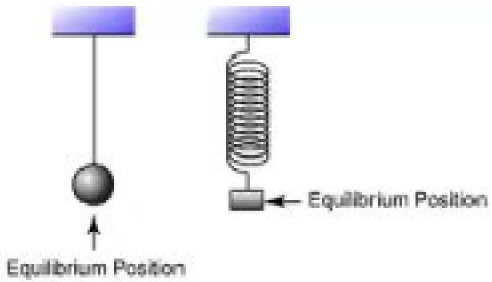
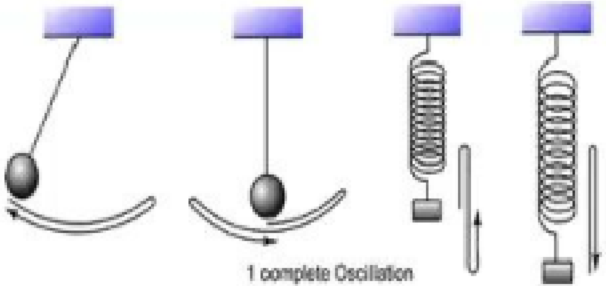
Oscillations

Oscillations

Oscillation refers to any Periodic Motion moving at a distance about the equilibrium position and repeats itself over and over for a period of time. Example The Oscillation up and down of a Spring. The Oscillation side by side of a spring, the Oscillation swinging side by side of a pendulum.





Periodic Motion:

If the motion of a particle is such that it passes through a certain point from the same direction after a definite interval of time, then the motion is called periodic motion. The path of the motion may be circular, elliptical or straight line. Example: the motion of the electric fan, motion of the hands a clock, the motion of the wheel of a cycle.

Simple Harmonic Oscillation:

The type of vibratory motion of a body such that the restoring force or the acceleration acting on the body is directly proportional to the displacement from the mean position and always directed towards the mean position is called the simple harmonic oscillation or motion.

Thus, in case of simple harmonic oscillation. The relationship between acceleration a and displacement x is

$$a \propto x$$

$$\text{or, } a = -kx \ (k = \text{constant})$$

Characteristics of Simple Harmonic Oscillation:

A simple harmonic oscillation has the following characteristics:

1. Its motion is periodic.
2. At particular time interval the motion becomes opposite.
3. Its motion is along a straight line.
4. Its acceleration is proportional to the displacement.
5. Acceleration is opposite to displacement.
6. Acceleration points toward the mean position of the object.

Differential Equation of simple harmonic oscillation:

From the definition of simple harmonic oscillation, acceleration is proportional to displacement but opposite direction. If F be the force acting on a particle executing simple harmonic motion and x its displacement from its mean or equilibrium position

$$F = -kx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

According to Newton's law of motion,

$$F = ma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where, m is the mass of the particle and a its acceleration. From equation (1) & (2) we get,

$$-kx = ma$$

$$\text{Or, } -kx = m\frac{d^2x}{dt^2} \quad \left[a = \frac{d^2x}{dt^2}\right]$$

$$\text{Or, } m\frac{d^2x}{dt^2} + kx = 0$$

$$\text{Or, } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\text{Or, } \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \left[\omega^2 = \frac{k}{m}\right]$$

$$\therefore \frac{d^2x}{dt^2} + \omega^2 x = 0;$$

This is called the differential equation of motion of a body executing simple harmonic motion.

Solution of the differential equation:

From the differential equation of SHM we get,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\text{Or, } \frac{d^2x}{dt^2} = -\omega^2 x \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To obtain a general solution of differential equation of simple harmonic motion, let us multiply both sides of eqⁿ (1) by $2\frac{dx}{dt}$ then we get,

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -\omega^2 x.\,2\frac{dx}{dt}$$

Integrating with respect to time, we have,

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + c \ldots\ldots\ldots\ldots\ldots(2)$$

where, c is a constant of integration. At maximum displacement (or amplitude)

$\frac{dx}{dt} = 0$, when $x = a$,

$$0 = -\,\omega^2 a^2 + c$$

$$\therefore c = \omega^2 a^2$$

Substituting these values in eq^n (2)

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 a^2$$

$$\text{or, } \left(\frac{dx}{dt}\right)^2 = \omega^2 (a^2 - x^2)$$

$$\text{or, } \frac{dx}{dt} = \omega\sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) can be rearranged as,

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt$$

Integrating with respect to time, we have

$$Sin^{-1}\frac{x}{a} = \omega t + \delta$$

$$\text{Or, } \frac{x}{a} = Sin(\omega t + \delta)$$

ii) velocity

We know, $v = \frac{dx}{dt}$

$$= \omega a\, Cos(\omega t + \delta)$$

$$= \omega a \sqrt{1 - Sin^2(\omega t + \delta)}$$

$$= \omega a \sqrt{1 - \frac{x^2}{a^2}}$$

$$= \omega \sqrt{a^2 - x^2}$$

When, $x = 0$, the particle crosses the middle equilibrium position then,

$$v = \omega \sqrt{a^2 - 0} = \omega a$$

and this is the maximum value of the velocity,

$$\therefore v_{max} = \omega a$$

when, $x = a$, the particle reaches at one end,

$$v = \omega \sqrt{a^2 - a^2} = o$$

$$\therefore v_{min} = 0$$

$$v = \omega\sqrt{a^2 - a^2} = o$$

$$\therefore v_{min} = 0$$

(2) *Acceleration:*

We know, $a = \frac{dv}{dt}$

$$= -\omega^2 a\, Sin(\omega t + \delta)$$

$$= -\omega^2 x$$

when, $x = 0$, then $a_{min} = 0$

when, $x = a$, then $a_{max} = -\omega^2 a$.

[Negative sign indicates that acceleration is opposite to displacement]

(3) *Time Period:* Time period is denoted as ***T***.

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

(4) ***Angular Frequency:*** The angular distance travelled by a particle executing simple harmonic motion is called angular frequency. It is denoted by,

$$\omega = \frac{2\pi}{T}$$

$$= 2\pi n$$

$$= 2\pi \times \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}}$$

ω is expressed in rad/sec.

Energy in Simple Harmonic Oscillation:

Suppose, a particle executing simple harmonic oscillation has amplitude a, angular frequency ω, and phase constant δ. If the displacement of the particle in time t is x, then from the equation of simple harmonic oscillation, we know,

$$x = a\,Sin(\omega t + \delta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Potential Energy:

We know that the force acting on a particle executing simple harmonic oscillation towards its equilibrium position is F =— kx. Now to displace the particle from $x = 0$ to $x = x$. The work done by the force would be the potential energy U of the particle at position x.

$$\therefore U = \int_0^x F dx$$

$$\text{Or, } U = \int_0^x kx dx = k\left[\frac{x^2}{2}\right]_0^x$$

$$\therefore U = \frac{1}{2}kx^2$$

Since $x = a\,Sin(\omega t + \delta)$

$$\therefore U = \frac{1}{2}ka^2 Sin^2(\omega t + \delta)\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Kinetic Energy:

At any instant, the kinetic energy of the particle is,

$$k = \frac{1}{2}mv^2$$

But velocity, $v = \frac{dx}{dt} = \omega a\, Cos(\omega t + \delta)$

$$\therefore k = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\omega^2 a^2 Cos^2(\omega t + \delta)$$

$$= \frac{1}{2}m\frac{k}{m}a^2 Cos^2(\omega t + \delta)$$

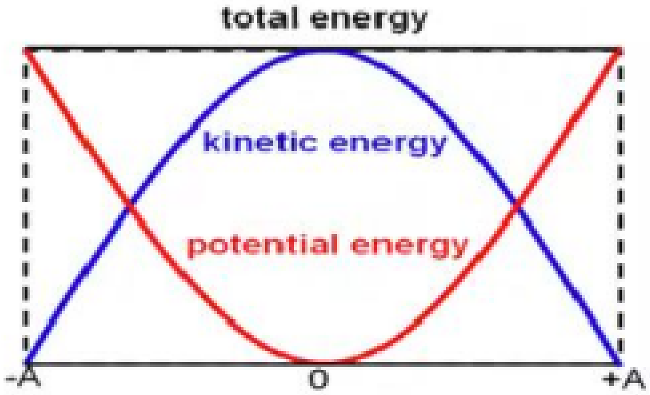
$$\therefore k = \frac{1}{2}ka^2 Cos^2(\omega t + \delta)$$

Total Energy, E:

$$E = K + U$$

$$= \frac{1}{2} k a^2 Sin^2(\omega t + \delta) + \frac{1}{2} k a^2 Cos^2(\omega t + \delta)$$

$$E = \frac{1}{2} k a^2$$



Average Kinetic Energy:

$$K.E_a = \frac{\int_0^T (K.E).dt}{\int_0^T dt}$$

$$= \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 A^2 cos^2(\omega t + \delta).\,dt$$

$$= \frac{m\omega^2 A^2}{2T}\int_0^T \frac{1}{2}\{1 + cos2(\omega t + \delta)\}dt$$

$$= \frac{m\omega^2 A^2}{4T}\left[\int_0^T dt + \int_0^T cos2(\omega t + \delta).\,dt\right]$$

$$= \frac{m\omega^2 A^2}{4T}[t]_0^T + \frac{m\omega^2 A^2}{4T}\left[\frac{\sin 2\,(\omega t+\delta)}{2\omega}\right]_0^T$$

$$= \frac{m\omega^2 A^2}{4} + \frac{m\omega^2 A^2}{8\omega T}[\sin 2(\omega T + \delta) - \sin 2\delta]$$

$$= \frac{m\omega^2 A^2}{4} \qquad [\because \sin 2(\omega T + \delta) = \sin 2\delta]\ldots\ldots.$$

$$K.E_a = \frac{KA^2}{4} \qquad [k = m\omega^2]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$\ldots\ldots\ldots(i)$$

Average Potential Energy:

$$P.E_a = \frac{\int_0^T (P.E).dt}{\int_0^T dt}$$

$$= \frac{kA^2}{2T}\int_0^T sin^2(\omega t + \delta).dt$$

$$= \frac{KA^2}{2T}\int_0^T \frac{1}{2}[1 - \cos\, 2(\omega t + \delta)].dt$$

$$= \frac{KA^2}{4T}\left[\int_0^T dt - \int_0^T \cos 2(\omega t + \delta).dt\right]$$

$$= \frac{KA^2}{4T}\left\{[t]_0^T - \left[\frac{\sin 2(\omega t + \delta)}{2\omega}\right]_0^T\right\}$$

$$= \frac{KA^2}{4}$$

$$\therefore P.E_a = \frac{m\omega^2 A^2}{4}$$

So, in one time period-

Average potential energy = Average Kinetic energy

$$= \frac{KA^2}{4} = \frac{m\omega^2 A^2}{4}$$

Principle of conservation of mechanical energy:

According to this principle, energy is conserved. It can transform one from to another, but total energy remains constant. So, at any instant the total energy of a particle.

$$E = \text{potential energy} + \text{Kinetic energy}$$

$$= \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 (A^2 - x^2)$$

$$= \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 A^2 - \frac{1}{2}m\omega^2 x^2$$

$$= \frac{1}{2}m\omega^2 A^2$$

Again, at the extreme end i.e., at maximum displacement ($x=A$),

its total energy = P.E + K.E

$$= \frac{1}{2} m\omega^2 A^2 + 0 = \frac{1}{2} m\omega^2 A^2$$

$\therefore$ Total energy, E $= \frac{1}{2} m\omega^2 A^2$.

Note: The total energy of the particle does not depend on the its displacement and it remains same through its motion. This is the principle of conservation of mechanical energy.