Joint Continuous Probability Distributions

X, Y Random Variable
$f_{XY}(x, y)$

Properties
$f_{XY}(x, y) \geq 0$

$$\int_x \int_y f_{XY}(x, y) = 1$$

Marginal PDFs

$$f_X(x) = \int_y f_{XY}(x, y)dy$$

$$f_Y(y) = \int_x f_{XY}(x, y)dx$$

Expected Values

$$E(X) = \int_x x f_X(x)dx$$

$$E(Y) = \int_y y f_Y(y)dy$$

Variance

$$Var(X) = \sigma_x^2 = \int_x x^2 f_X(x)dx - \big(E(X)\big)^2$$

$$Var(Y) = \sigma_y^2 = \int_y y^2 f_Y(y)dy - \big(E(Y)\big)^2$$

Covariance

$$Cov(x, y) = \int_x \int_y xy f_{XY}(x, y)dxdy - E(X)E(Y)$$

Correlation

$$\rho_{XY} = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

Joint Continuous Probability Distributions

X, Y Random Variable where, $0 \leq x \leq 2$ and $0 \leq y \leq 1$

$$f_{XY}(x,y) = \frac{9}{10}xy^2 + \frac{1}{5}$$

Test

$$f_{XY}(x,y) \geq 0$$

Now,

$f_{XY}(1{,}1) = \frac{9}{10}1(1)^2 + \frac{1}{5} = \frac{9}{10} + \frac{1}{5} = 1.1$, So Here, $1.1 > 0$

And, Test

$$\int_x \int_y f_{XY}(x,y) = 1$$

Now,

$$\int_0^2 \int_0^1 f_{XY}(x,y) = \int_0^2 \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) dydx$$
$$= \int_0^2 \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) dydx$$
$$= \int_0^2 \left[\frac{9}{10 \times 3}xy^3 + \frac{1}{5 \times 1}y\right]_0^1 dx$$
$$= \int_0^2 \left(\left(\frac{9}{10 \times 3}x(1)^3 + \frac{1}{5 \times 1}(1)\right) - \left(\frac{9}{10 \times 3}x(0)^3 + \frac{1}{5 \times 1}(0)\right)\right) dx$$
$$= \int_0^2 \left(\left(\frac{9}{10 \times 3}x \times 1 + \frac{1}{5 \times 1} \times 1\right) - (0 + 0)\right) dx$$
$$= \int_0^2 \left(\frac{3}{10}x + \frac{1}{5}\right) dx$$
$$= \left[\frac{3}{10 \times 2}x^2 + \frac{1}{5 \times 1}x\right]_0^2$$
$$= \left(\frac{3}{10 \times 2}(2)^2 + \frac{1}{5 \times 1}(2)\right) - \left(\frac{3}{10 \times 2}(0)^2 + \frac{1}{5 \times 1}(0)\right)$$
$$= \left(\frac{3}{10 \times 2} \times 4 + \frac{1}{5} \times 2\right) - (0 + 0)$$
$$= \left(\frac{3 \times 2}{10} + \frac{2}{5}\right) - 0$$
$$= \frac{6 + 4}{10}$$
$$= \frac{10}{10}$$
$$= 1$$

Marginal PDFs

$$f_{XY}(x,y) = \frac{9}{10}xy^2 + \frac{1}{5}$$

Now,

$$f_X(x) = \int_y f_{XY}(x,y)dy = \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dy$$
$$= \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dy$$
$$= \left[\frac{9}{10 \times 3}xy^3 + \frac{1}{5}y\right]_0^1$$
$$= \left(\frac{9}{10 \times 3}x(1)^3 + \frac{1}{5}(1)\right) - \left(\frac{9}{10 \times 3}x(0)^3 + \frac{1}{5}(0)\right)$$
$$= \frac{3}{10}x + \frac{1}{5}$$

And,

$$f_Y(y) = \int_x f_{XY}(x,y)dx = \int_0^2 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dx$$
$$= \int_0^2 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dx$$
$$= \left[\frac{9}{10 \times 2}x^2y^2 + \frac{1}{5}x\right]_0^2$$
$$= \left(\frac{9}{10 \times 2}(2)^2y^2 + \frac{1}{5}(2)\right) - \left(\frac{9}{10 \times 2}(0)^2y^2 + \frac{1}{5}(0)\right)$$
$$= \frac{9 \times 4}{10 \times 2}y^2 + \frac{2}{5}$$
$$= \frac{9}{5}y^2 + \frac{2}{5}$$

Expected Values

$$f_X(x) = \frac{3}{10}x + \frac{1}{5}$$
$$f_Y(y) = \frac{9}{5}y^2 + \frac{2}{5}$$

Now,

$$E(X) = \int_x x f_X(x)dx = \int_0^2 x\left(\frac{3}{10}x + \frac{1}{5}\right)dx$$
$$= \int_0^2 x\left(\frac{3}{10}x + \frac{1}{5}\right)dx$$
$$= \int_0^2 \left(\frac{3}{10}x^2 + \frac{1}{5}x\right)dx$$
$$= \left[\frac{3}{10 \times 3}x^3 + \frac{1}{5 \times 2}x^2\right]_0^2$$
$$= \left(\frac{3}{10 \times 3}(2)^3 + \frac{1}{5 \times 2}(2)^2\right) - \left(\frac{3}{10 \times 3}(0)^3 + \frac{1}{5 \times 2}(0)^2\right)$$
$$= \frac{3 \times 8}{10 \times 3} + \frac{1 \times 4}{5 \times 2} - 0$$
$$= \frac{4}{5} + \frac{2}{5}$$
$$= \frac{4+2}{5}$$
$$= \frac{6}{5}$$
$$= 1.2$$

And,

$$E(Y) = \int_y y f_Y(y)dy = \int_0^1 y\left(\frac{9}{5}y^2 + \frac{2}{5}\right)dy$$
$$= \int_0^1 y\left(\frac{9}{5}y^2 + \frac{2}{5}\right)dy$$
$$= \int_0^1 \left(\frac{9}{5}y^3 + \frac{2}{5}y\right)dy$$
$$= \left[\frac{9}{5 \times 4}y^4 + \frac{2}{5 \times 2}y^2\right]_0^1$$
$$= \left(\frac{9}{5 \times 4}(1)^4 + \frac{2}{5 \times 2}(1)^2\right) - \left(\frac{9}{5 \times 4}(0)^4 + \frac{2}{5 \times 2}(0)^2\right)$$
$$= \frac{9}{20} + \frac{1}{5} - 0$$
$$= \frac{9+4}{20}$$
$$= \frac{13}{20}$$
$$= 0.65$$

Variance

Now, $f_X(x) = \frac{3}{10}x + \frac{1}{5}, and, E(X) = \frac{6}{5}$

$$\therefore Var(X) = \sigma_x^2 = \int_x x^2 f_X(x)dx - \left(E(X)\right)^2 = \int_0^2 x^2 \left(\frac{3}{10}x + \frac{1}{5}\right) dx - \left(\frac{6}{5}\right)^2$$

$$= \int_0^2 x^2 \left(\frac{3}{10}x + \frac{1}{5}\right) dx - \left(\frac{6}{5}\right)^2$$

$$= \int_0^2 \left(\frac{3}{10}x^3 + \frac{1}{5}x^2\right) dx - \left(\frac{6}{5}\right)^2$$

$$= \left[\frac{3}{10 \times 4}x^4 + \frac{1}{5 \times 3}x^3\right]_0^2 - \left(\frac{6}{5}\right)^2$$

$$= \left(\frac{3}{10 \times 4}(2)^4 + \frac{1}{5 \times 3}(2)^3\right) - \left(\frac{6}{5}\right)^2$$

$$= \left(\frac{3 \times 16}{10 \times 4} + \frac{8}{5 \times 3}\right) - \left(\frac{6}{5}\right)^2$$

$$= \left(\frac{6}{5} + \frac{8}{15}\right) - \frac{36}{25}$$

$$= \left(\frac{18 + 8}{15}\right) - \frac{36}{25}$$

$$= \frac{26}{15} - \frac{36}{25}$$

$$= \frac{26}{15} - \frac{36}{25}$$

$$= 0.2933$$

$$\sigma_x^2 = 0.2933$$

$$\sigma_x = \sqrt{0.2933} = 0.5416$$

And, $f_Y(y) = \frac{9}{5}y^2 + \frac{2}{5}, and, E(Y) = \frac{13}{20}$

$$\therefore Var(Y) = \sigma_y^2 = \int_y y^2 f_Y(y)dy - \left(E(Y)\right)^2 = \int_0^1 y^2 \left(\frac{9}{5}y^2 + \frac{2}{5}\right) dy - \left(\frac{13}{20}\right)^2$$

$$= \int_0^1 y^2 \left(\frac{9}{5}y^2 + \frac{2}{5}\right) dy - \left(\frac{13}{20}\right)^2$$

$$= \int_0^1 \left(\frac{9}{5}y^4 + \frac{2}{5}y^2\right) dy - \left(\frac{13}{20}\right)^2$$

$$= \left[\frac{9}{5 \times 5}y^5 + \frac{2}{5 \times 3}y^3\right]_0^1 - \left(\frac{13}{20}\right)^2$$

$$= \left(\frac{9}{5 \times 5}(1)^5 + \frac{2}{5 \times 3}(1)^3\right) - \left(\frac{13}{20}\right)^2$$

$$= \left(\frac{9}{25} + \frac{2}{15}\right) - \left(\frac{13}{20}\right)^2$$

$$= \frac{9}{25} + \frac{2}{15} - \frac{169}{400}$$

$$= 0.070833$$

$$\sigma_y^2 = 0.070833$$

$$\sigma_y = \sqrt{0.070833} = 0.266$$

Covariance

$$f_{XY}(x,y) = \frac{9}{10}xy^2 + \frac{1}{5}$$
$$E(X) = \frac{6}{5}$$
$$E(Y) = \frac{13}{20}$$

Now,

$$Cov(x,y) = \int_x \int_y xyf_{XY}(x,y)dydx - E(X)E(Y) = \int_0^2 \int_0^1 xy\left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dydx - \frac{6}{5}\times\frac{13}{20}$$
$$= \int_0^2 \int_0^1 xy\left(\frac{9}{10}xy^2 + \frac{1}{5}\right)dydx - \frac{6}{5}\times\frac{13}{20}$$
$$= \int_0^2 \int_0^1 \left(\frac{9}{10}x^2y^3 + \frac{1}{5}xy\right)dydx - \frac{6}{5}\times\frac{13}{20}$$
$$= \int_0^2 \left[\frac{9}{10\times 4}x^2y^4 + \frac{1}{5\times 2}xy^2\right]_0^1 dx - \frac{6}{5}\times\frac{13}{20}$$
$$= \int_0^2 \left(\frac{9}{10\times 4}x^2 + \frac{1}{5\times 2}x\right)dx - \frac{6}{5}\times\frac{13}{20}$$
$$= \left[\frac{9}{10\times 4\times 3}x^3 + \frac{1}{5\times 2\times 2}x^2\right]_0^2 - \frac{6}{5}\times\frac{13}{20}$$
$$= \left(\frac{9\times 8}{10\times 4\times 3} + \frac{1\times 4}{5\times 2\times 2}\right) - \frac{6}{5}\times\frac{13}{20}$$
$$= 0.02$$

Correlation

$$\sigma_x = \sqrt{0.2933} = 0.5416$$
$$\sigma_y = \sqrt{0.070833} = 0.266$$
$$Cov(x,y) = 0.02$$

$$\rho_{XY} = \frac{Cov(x,y)}{\sigma_x\sigma_y} = \frac{0.02}{0.5416\times 0.266} = 0.1388$$