AN ASSOCIATION MEASURE OF COORDINATION IN TWO-PLAYER GAMES

A Project Report Submitted

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MA498 Project I

by

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to the

DEPARTMENT OF MATHEMATICS

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CERTIFICATE

This is to certify that the work contained in this project report entitled "An Association Measure of Coordination in Two-Player Games" submitted by Rasesh Srivastava and Rishika Saria (Roll No.: 210123072 and 210123051) to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Bachelor of Technology in Mathematics and Computing has been carried out by him/her under my supervision.

It is also certified that this report is a survey work based on the references in the bibliography.

OR

It is also certified that, along with literature survey, a few new results are established/computational implementations have been carried out/simulation studies have been carried out/empirical analysis has been done by the student under the project.

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November 2024	Project Supervisor

ABSTRACT

This study introduces a statistical association measure for assessing coordination in two-player games. We begin by defining the concept of association in the context of game theory, establishing a novel statistical measure that functions similarly to a correlation coefficient, ranging from -1 to 1. A value of 1 indicates maximum agreement between players, -1 represents minimum agreement. We examine various properties of this measure, including its asymptotic distribution.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Coordination games are a fundamental area of game theory, focusing on situations where players achieve better outcomes by aligning their strategies. These games are widely applicable in fields like economics, social sciences, and computer science, where decision-making and cooperative behavior are critical. Examples of coordination games include companies aligning on production standards, drivers choosing directions at intersections, and nodes in a network synchronizing their activities. When players in these games coordinate effectively, they can reach outcomes that are beneficial to everyone involved. However, when they fail to coordinate, it often results in inefficiency or lost opportunities.

To understand and analyze coordination, we need tools that can measure how well players align their actions over time. Many existing metrics fall short in capturing the complex dependencies in player decisions, especially when coordination changes or evolves with repeated interactions. This project introduces a new statistical association measure specifically designed for two-player games. Unlike basic correlation metrics, this measure aims to reflect the strategic dynamics of player interactions, measuring coordination on a scale from -1 (opposing actions) to 1 (complete alignment), with 0 indicating no association at all.

This measure has potential applications beyond simply describing player behavior. It could also support hypothesis testing, allowing researchers to investigate how changes in game conditions, incentives, or player information affect coordination levels. Such analysis could deepen our understanding of strategic behaviors in game-theoretic models. Applications of this measure range across fields: in economics, it could help evaluate coordination in markets; in behavioral science, it could shed light on teamwork dynamics; and in artificial intelligence, it could improve multi-agent systems where agents must work together toward a common goal.

In this report, we describe the theoretical basis of this association measure, discuss its properties, and provide initial analytical results. Our ongoing work includes exploring its distribution and developing proofs for its theoretical behavior.

1.1 Motivation

The motivation behind this project stems from the critical need for a dedicated tool to assess coordination in strategic settings. Coordination games are used across disciplines to study interactions where alignment influences outcomes, yet current methods for quantifying this alignment are limited. Traditional correlation and association metrics lack the capacity to capture the specific interdependencies and dynamic nature of coordination games, particularly when coordination levels vary or evolve over time.

A specialized association measure would enable researchers to not only observe but also quantify the strength and quality of coordination between players. In summary, this project is motivated by the need for a versatile, rigorous tool to assess coordination in a structured, quantifiable way. By developing an association measure tailored to two-player coordination games, we aim to contribute an essential framework that will advance both descriptive and inferential analyses across a wide range of strategic environments.

1.2 Literature Review

Analyzing coordination in two-player games requires a measure that goes beyond traditional metrics, which may only capture simple associations. This section reviews relevant work on dependence measures, including correlation coefficients, information-theoretic approaches, entropy-based measures, and frameworks for conditional dependence. Each approach offers insights that can help in constructing a more nuanced association measure for coordination games.

- **Dependence and Association Measures**
  The Pearson correlation coefficient is one of the most common measures for capturing linear relationships between two variables. While useful in many situations, Pearson's correlation has limitations in strategic contexts since it only measures linear dependence and does not account for the non-linear dependencies often found in coordination games. This makes it insufficient for capturing the complex interdependencies in such

settings, where player decisions may align or oppose in more intricate ways.

Hoeffding's (1948) work on rank-based statistics addresses some of these limitations by providing a non-parametric approach that captures dependencies through rank orders, making the measure more adaptable to non-linear relationships. This approach is especially useful for coordination games, where the relationships between players' actions may not follow simple patterns. Rank-based methods can be a foundation for developing association measures that detect different levels of alignment in these games, offering more sensitivity than traditional correlation measures.

- **Information-Theoretic Approaches to Association**

  Information theory provides tools for measuring association in ways that go beyond simple linear relationships. Cover and Thomas's Elements of Information Theory (2006) explores measures like mutual information, which captures shared information between variables and accounts for both linear and non-linear dependencies. This broader perspective on interaction dynamics makes mutual information suitable for modeling coordination games, where player alignment may vary widely.

  Building on this, Reshef et al. (2011) introduced the Maximal Information Coefficient (MIC), an association measure capable of detecting complex, non-linear relationships. The MIC is widely used in high-dimensional data analysis because it can adapt to different types of relationships. This adaptability makes it potentially useful for analyzing strategic interactions in coordination games, where player alignments can change over time or in response to game conditions.

- **Statistical Independence and Dependence in Strategic Contexts**

  Pearl's Causality: Models, Reasoning, and Inference (2009) explores conditional independence, which is valuable in strategic settings where actions depend on previous choices. In coordination games, conditional independence frameworks allow researchers to separate direct dependencies from indirect ones, helping to model how players' decisions influence each other. This approach is essential for constructing association measures that capture the complex interdependencies seen in coordination games.

  Dawid's (1980) work on conditional independence further supports this approach by allowing for a separation of independent effects and jointly dependent outcomes. In coordination games, where players' actions often depend on past outcomes, applying these principles can improve an association measure's ability to detect shifts in player strategies over time.

1.3 Research Gap

Although various statistical approaches exist for measuring dependence and association, there are still gaps when it comes to their application in two-player coordination games.

Traditional correlation measures, like Pearson's correlation, are limited to linear relationships and do not account for the dynamic, evolving dependencies that characterize strategic settings. Coordination games often involve non-linear dependencies, where players adapt their strategies based on previous rounds. Basic correlation measures cannot capture these changing patterns, which makes them inadequate for analyzing different levels of alignment such as full, partial, or opposing coordination.

Information-theoretic measures, including mutual information and the Maximal Information Coefficient (MIC), provide greater flexibility by capturing both linear and non-linear relationships. However, these measures are typically designed for high-dimensional data analysis and may lack the specific adaptations needed for game-theoretic contexts. In coordination games, players' strategies often develop sequentially and conditionally on past interactions, creating dependencies that standard information-theoretic approaches do not fully address.

Rank-based methods, such as those developed by Hoeffding, are effective for handling non-linear relationships but are less suited for capturing dynamic interactions that change over multiple game rounds. Since coordination games involve iterative play, these methods may fall short in capturing the adaptive, sequential dependencies that arise as players adjust their strategies.

This research gap highlights the need for a specialized association measure tailored for two-player coordination games, capable of capturing a broader range of coordination dynamics. An ideal measure would:

- Capture both linear and non-linear dependencies,
- Differentiate between levels of alignment, including full coordination, partial coordination, and opposition,
- Be sensitive to sequential and conditional dependencies, reflecting changes in strategy over time,
- Provide a continuous scale to reflect variations in coordination, facilitating statistical inference and hypothesis testing.

Our proposed association measure aims to address these gaps by introducing a method specifically tailored for two-player coordination games. By developing a measure that ranges from -1 (opposition) to 1 (full alignment), our approach will enable a more nuanced understanding of player interactions in strategic settings. Additionally, once fully developed, this measure will allow researchers to conduct statistical inference on coordination dynamics, facilitating hypothesis testing regarding the factors influencing player alignment. This will offer a significant advancement in both the theoretical analysis and practical evaluation of coordination in game-theoretic and multi-agent systems.

Chapter 2

Desired Properties of Score and Key Definitions

2.1 Desired properties of the final score

- A sequence of all 0's should have a score of -1.
- A sequence of all 1's should have a score of 1.
- The distribution of the final score should be symmetric about 0.
- The final score should have an asymptotically normal distribution.
- More weightage is given to more recent agreements, that is, a sequence having an agreement at a later position should have a score higher than that of a similar sequence having an agreement at an earlier position.

2.2 Definitions:

Two players play a coordination game n number of times. Given the binary sequence of length n: $x_0 x_1 ... x_{n-1}$

$$x_i = \begin{cases} 1 & \text{if } i\text{th game was an agreement} \\ 0 & \text{if } i\text{th game was a disagreement} \end{cases}$$

We aim to assign a measure to each sequence that reflects the level of agreement between the 2 players.

Convention: $x_n = 0$.

Factors taken into consideration:

1. *The number of agreements*
2. *The recency of agreements*
3. *The number and location of runs*
4. *Degree of improvement based on learning from outcomes of previous games*

Definition 2.2.1. x-score: Let α be a constant, where $\alpha > 1$. Let $I_1 = \{i : x_i = 1\}$ and $I_0 = \{i : x_i = 0\}$. Calculate the x-score as follows:

$$x\text{-score} = c_x \cdot \left(\sum_{i \in I_1} \alpha^i - \sum_{i \in I_0} \alpha^i \right)$$

where c_x is the normalization constant defined as $c_x = \frac{\alpha - 1}{\alpha^n - 1}$

Definition 2.2.2. run-score: Define:

$$k_{i1} = \max\{k : k \leq i \text{ and } x_k = 0\} \quad \text{and} \quad k_{i0} = \max\{k : k \leq i \text{ and } x_k = 1\}.$$

Let $I_1 = \{i : x_i = 1 \text{ and } x_{i+1} = 0\}$, and $I_0 = \{i : x_i = 0 \text{ and } x_{i+1} = 1\}$.

$$\text{run-score} = c_r \cdot \left(\sum_{i \in I_1} (i - k_{i1}) \cdot \alpha^i - \sum_{i \in I_0} (i - k_{i0}) \cdot \alpha^i \right)$$

where c_r be the normalization constant defined as $c_r = \frac{1}{n \cdot \alpha^{n-1}}$.

Definition 2.2.3. penalty-score: Define k_i as the index of the kth last agreement from the ith index, with $k_i = -\infty$ if there are fewer than k agreements until the ith index. Let:

$$p_i = \begin{cases} 0 & \text{if } x_i = 1, \\ b^{k_i - i} & \text{if } x_i = 0. \end{cases}$$

Calculate the penalty-score as follows:

$$\text{penalty-score} = c_p \cdot \sum_i p_i \cdot \alpha^i,$$

where c_p is the normalization constant defined as $c_p = \left(\max_{x \in \{0,1\}^n} \left(\textit{penalty-score}(x)\right)\right)^{-1}$

Definition 2.2.4. credit-score: Define k_i as the index of the kth last disagreement from the ith index, with $k_i = -\infty$ if there are fewer than k disagreements until the ith index. Let:

$$p_i = \begin{cases} 0 & \text{if } x_i = 0, \\ b^{k_i - i} & \text{if } x_i = 1. \end{cases}$$

Calculate the credit-score as follows:

$$\text{credit-score} = c_c \cdot \sum_i p_i \cdot \alpha^i,$$

where c_c is the normalization constant defined as $c_c = \left(\max_{x\in\{0,1\}^n} (credit\text{-}score(x))\right)^{-1}$

2.2.1 Assumptions:

The maximum penalty score is for the sequence: $\ldots \underbrace{1\ldots1}_{k \text{ times}} 0 \underbrace{1\ldots1}_{k \text{ times}} 0 \ldots \underbrace{1\ldots1}_{k \text{ times}} 0.$

The maximum credit score is for the sequence: $\ldots \underbrace{0\ldots0}_{k \text{ times}} 1 \underbrace{0\ldots0}_{k \text{ times}} 1 \ldots \underbrace{0\ldots0}_{k \text{ times}} 1.$

This gives the value of $c_c = c_p = b^{-k} \cdot \left(\sum_{t=0}^{\lfloor \frac{n}{k+1} \rfloor -1} \alpha^{n-1-t(k+1)}\right)$

2.3 Calculation of final score

Calculate the **final-score** given by:

$$\text{final-score} = w_1 \cdot x\text{-score} + w_2 \cdot \text{run-score} - w_3 \cdot \text{penalty-score} + w_4 \cdot \text{credit-score}$$

where w_1, w_2, w_3, w_4 are > 0.

Normalize the score to lie between -1 and 1

$$\text{normalized-score} = \frac{\text{final-score}}{\text{max-score}}$$

where

$$\text{max-score} = \max_{x\in\{0,1\}^n} (final - score(x))$$

To find the optimal weights SQLSP(Sequential Least Squares Programming method) has been used. It replaces the original problem with a sequence of Quadratic Problems (QP) whose objective are second-order approximations of the Lagrangian and whose constraints are the linearized original constraints. It then uses globalization techniques to guarantee convergence whatever the initial point.

Constraints used while optimizing:

1. *All scores lie between -1 and 1*
2. *All weights lie between 0 and 1*
3. $w_1 + w_2 = 1$ *(this ensures all 0's and all 1's have scores -1 and 1 respectively)*
4. $w_3 = w_4$ *(equal weight to penalty and credit score)*

On setting values of $\alpha = 1.4$, $k = 3$, and $b = 4$, and considering sequences of length 5. The optimal weights come out to be : $w_1 = 0.8696973$, $w_2 = 0.1303027$, $w_3 = 0.14399335$, $w_4 = 0.14399335$
The table 2.1 shows the final score with the above mentioned weights for all sequences of length n = 5.

The final scores obtained are in the range of [-1, 1], whereas the pdf of the normal distribution has the range ($-\infty$, $+\infty$). So, we use Fisher Transformation (described in section 5.3) to transform our final score from the range [-1, 1] to ($-\infty$, $+\infty$). Also, there are 2^n (exponential) number of sequences for each n, where n is the length of the sequence. Therefore, computing the score for large n requires a lot of computing power. Hence, we used the supercomputer Param Kamprupa, to run our code for values of n > 20.

Sequence	Final-Score
00000	-1.0
10000	-0.80553
01000	-0.7216
00100	-0.61833
11000	-0.5399
10100	-0.43662
00010	-0.38663
01100	-0.36014
11100	-0.31197
10010	-0.30246
01010	-0.22598
00001	-0.19341
00110	-0.1227
11010	-0.08027
01001	-0.023
01110	-0.00106
10001	0.00106
10110	0.023
00101	0.08027
11001	0.1227
11110	0.19341
10101	0.22598
01101	0.30246
00011	0.31197
10011	0.36014
11101	0.38663
01011	0.43662
00111	0.5399
11011	0.61833
10111	0.7216
01111	0.80553
11111	1.0

Table 2.1: Final scores with optimal weights for n = 5

Chapter 3

Experimental Results

3.1 Graphs of individual components of the final score

3.1.1 Graphs of x-score

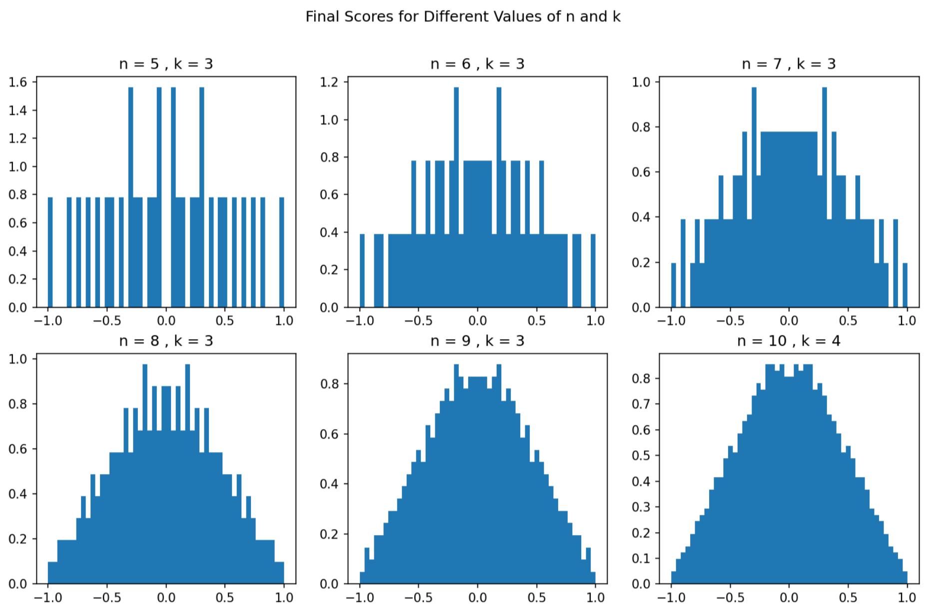

Figure 3.1: x-score with b = 4, α = 1.4 for n = 5 to 10

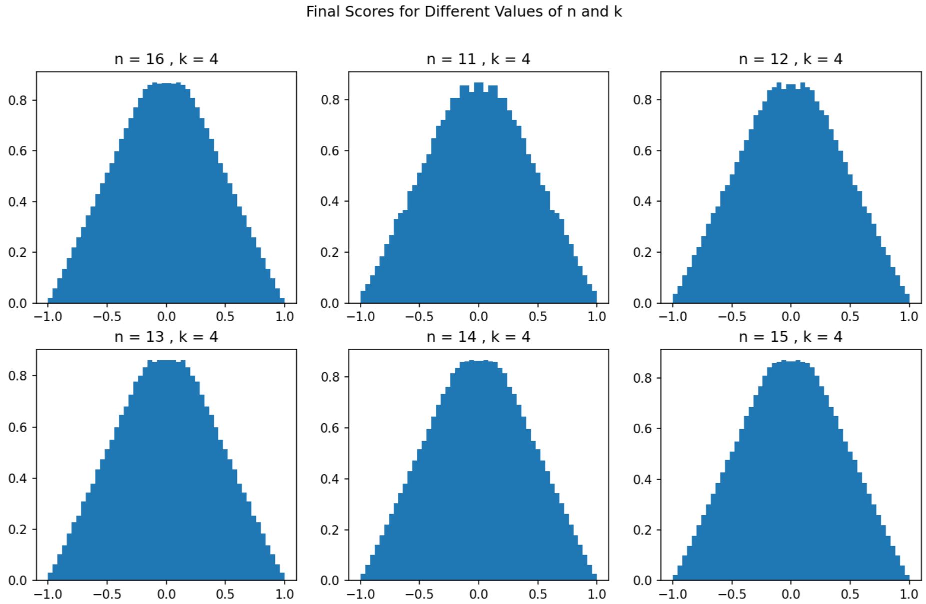

Figure 3.2: x-score with b = 4, α = 1.4 for n = 11 to 16

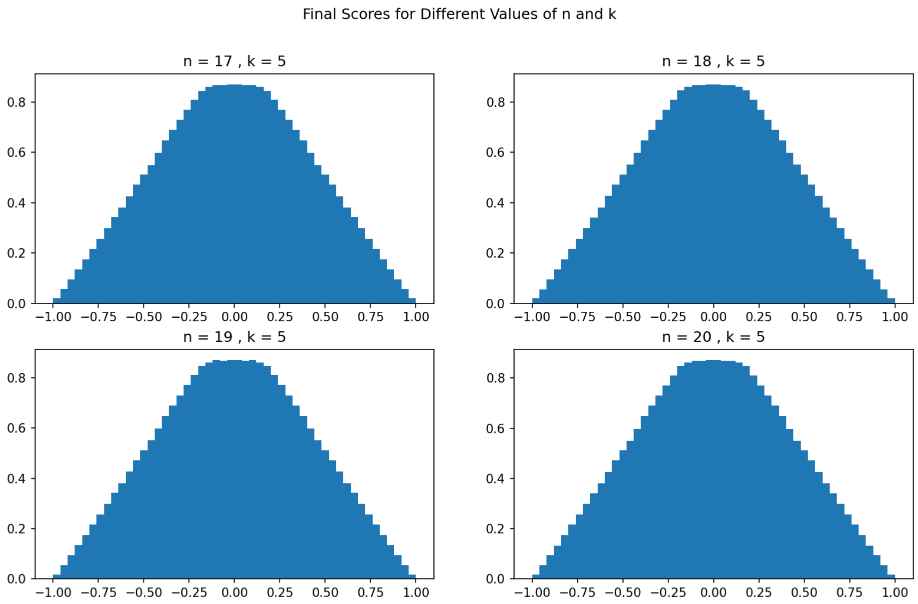

Figure 3.3: x-score with b = 4, α = 1.4 for n = 17 to 20

3.1.2 Graphs of run-score

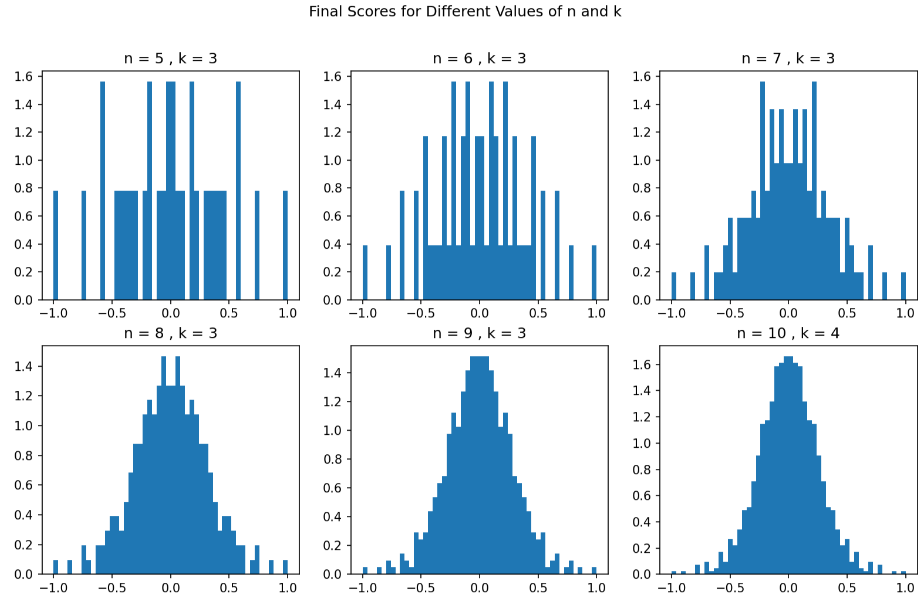

Figure 3.4: run-score with b = 4, α = 1.4 for n = 5 to 10

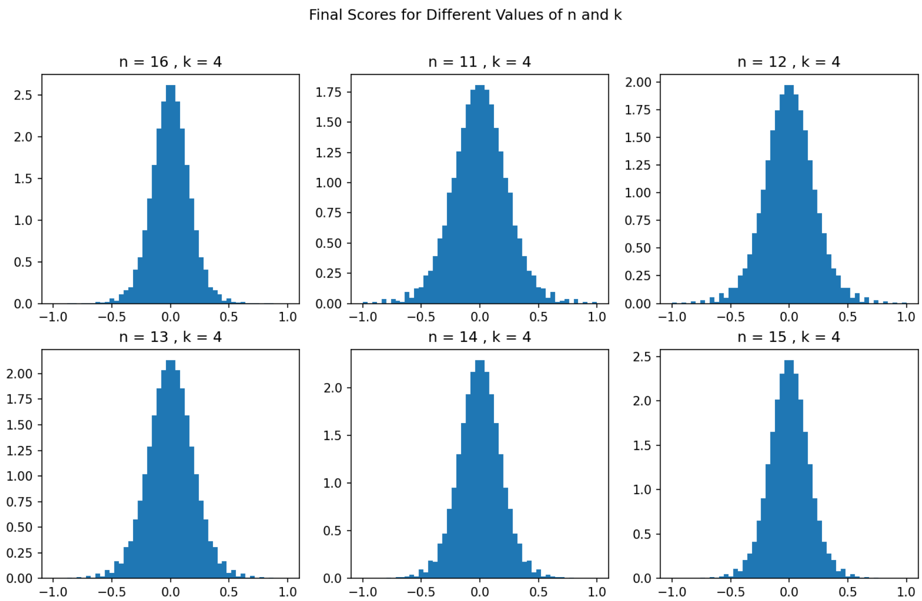

Figure 3.5: run-score with b = 4, α = 1.4 for n = 11 to 16

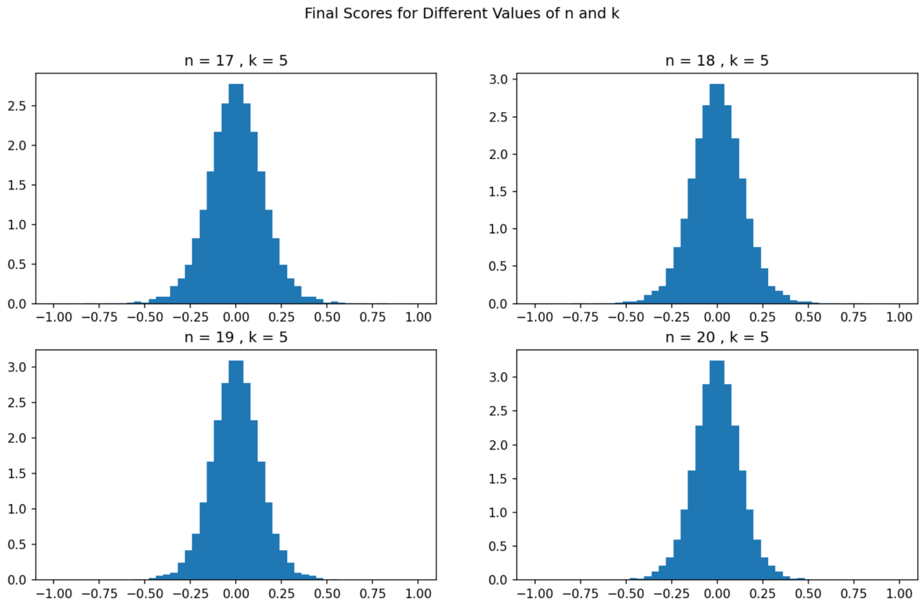

Figure 3.6: run-score with b = 4, α = 1.4 for n = 17 to 20

3.1.3 Graphs of penalty score + credit score

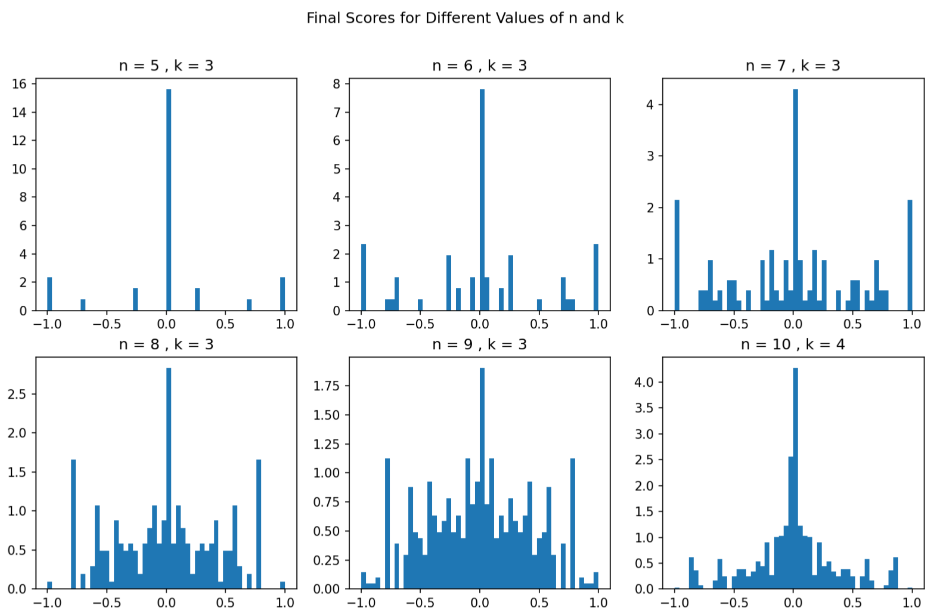

Figure 3.7: penalty score + credit score with b = 4, α = 1.4 for n = 5 to 10

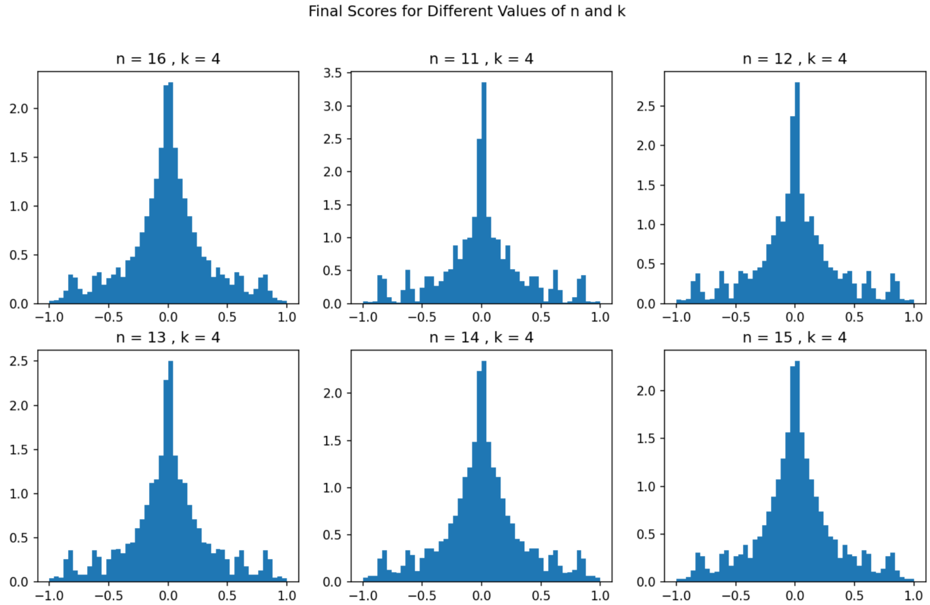

Figure 3.8: penalty score + credit score with b = 4, α = 1.4 for n = 11 to 16

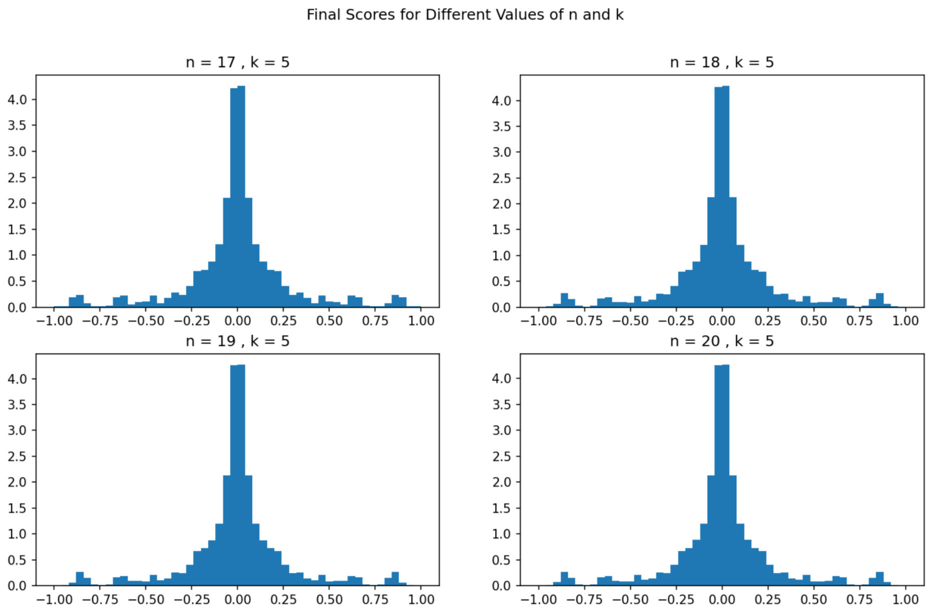

Figure 3.9: penalty score + credit score with b = 4, α = 1.4 for n = 17 to 20

3.2 Graphs of the final score with weights in penalty score and credit score

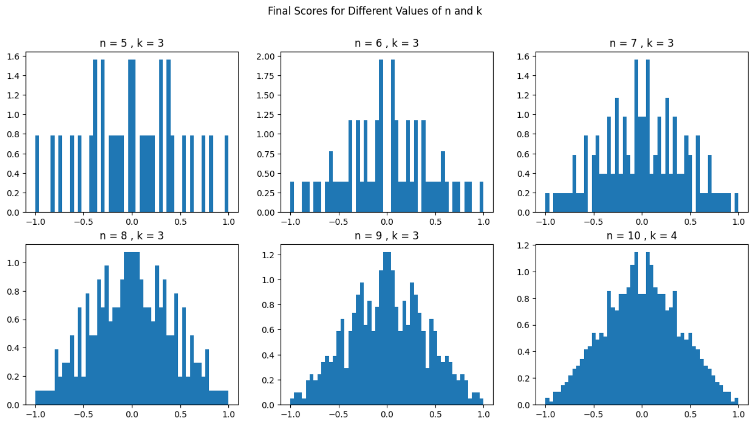

Figure 3.10: Final scores for b = 4, α = 1.4, for n = 5 to 10

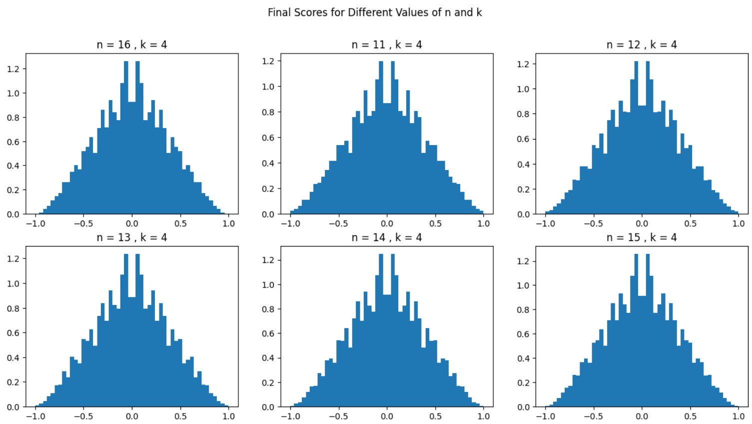

Figure 3.11: Final scores for b = 4, α = 1.4, for n = 11 to 16

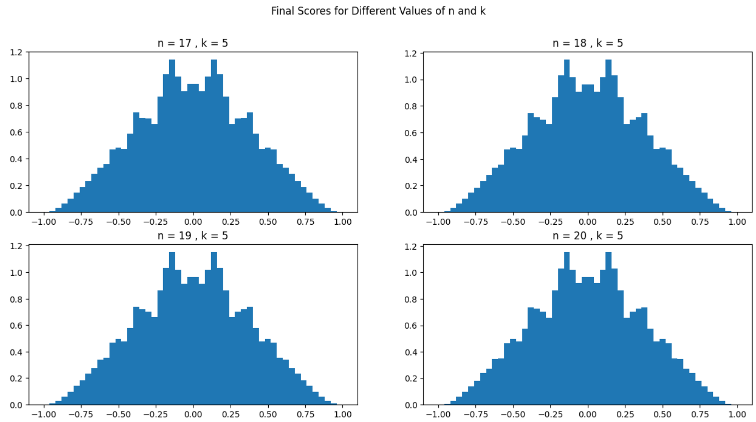

Figure 3.12: Final scores for b = 4, α = 1.4, for n = 17 to 20

3.3 Graphs of the final score without weights in penalty score and credit score

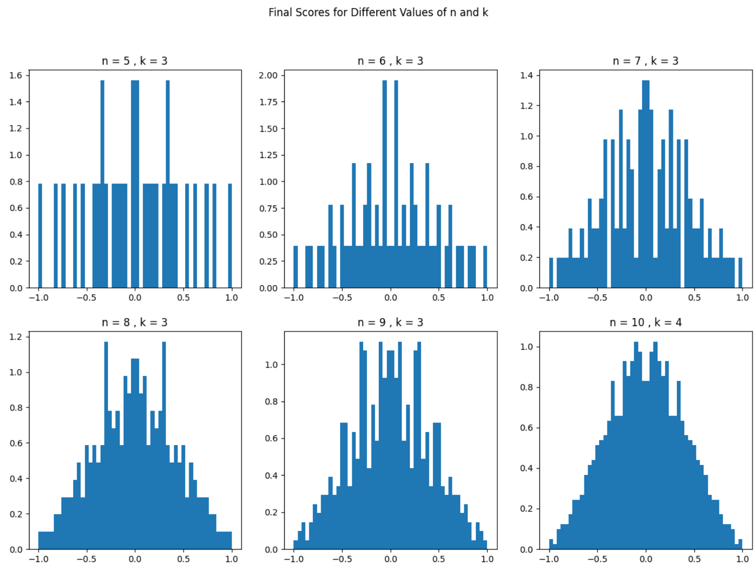

Figure 3.13: Final scores for b = 4, α = 1.4, for n = 5 to 10

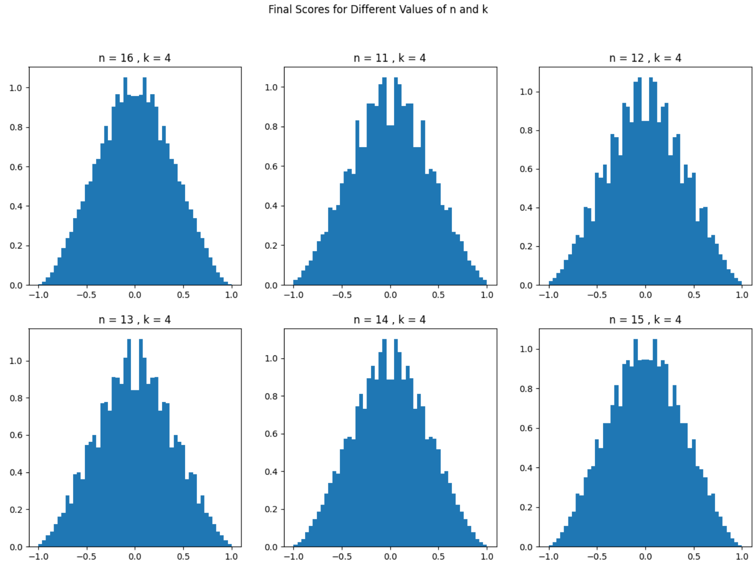

Figure 3.14: Final scores for b = 4, $\alpha = 1.4$, for n = 11 to 16

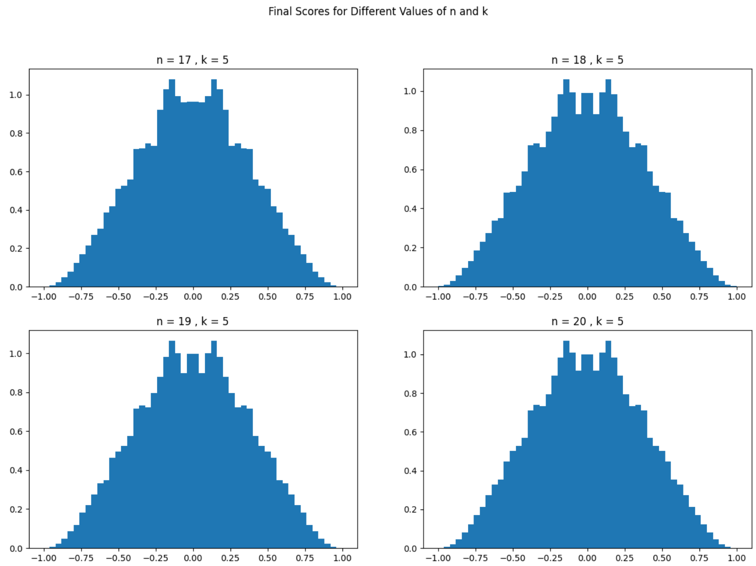

Figure 3.15: Final scores for b = 4, $\alpha = 1.4$, for n = 17 to 20

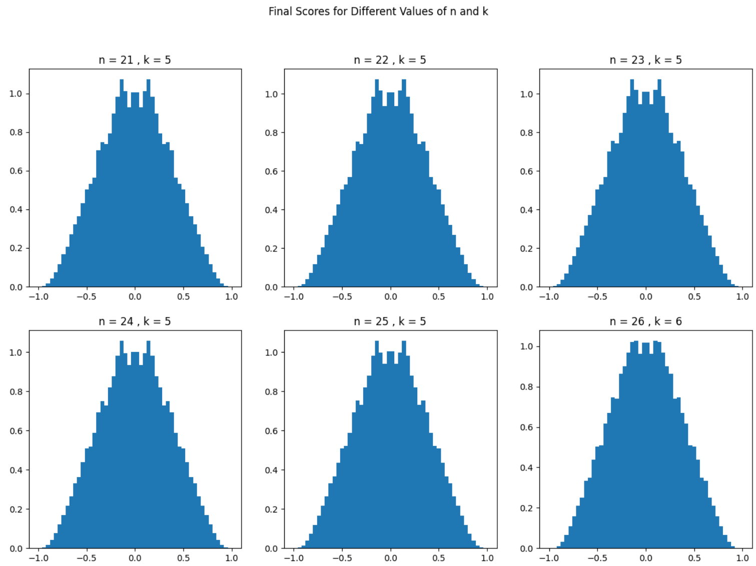

Figure 3.16: Final scores for b = 4, α = 1.4, for n = 21 to 26

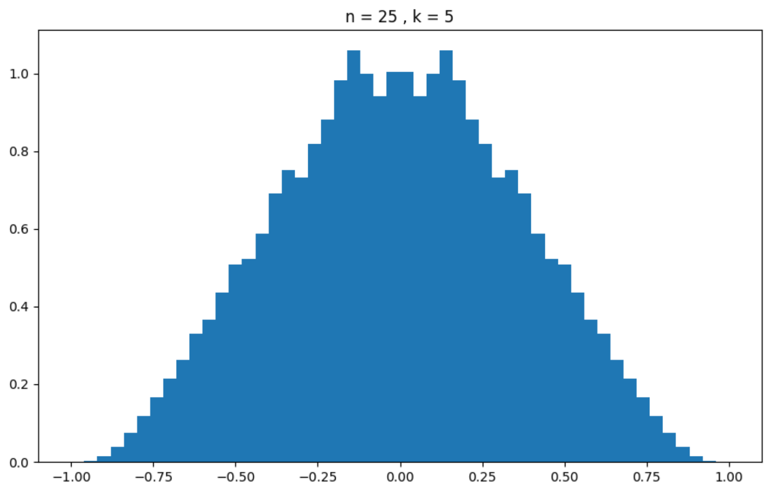

Figure 3.17: Final scores for b = 4, α = 1.4, for n = 25

3.4 Graphs of Fisher Transformation

3.4.1 Fisher Transformation of x-score

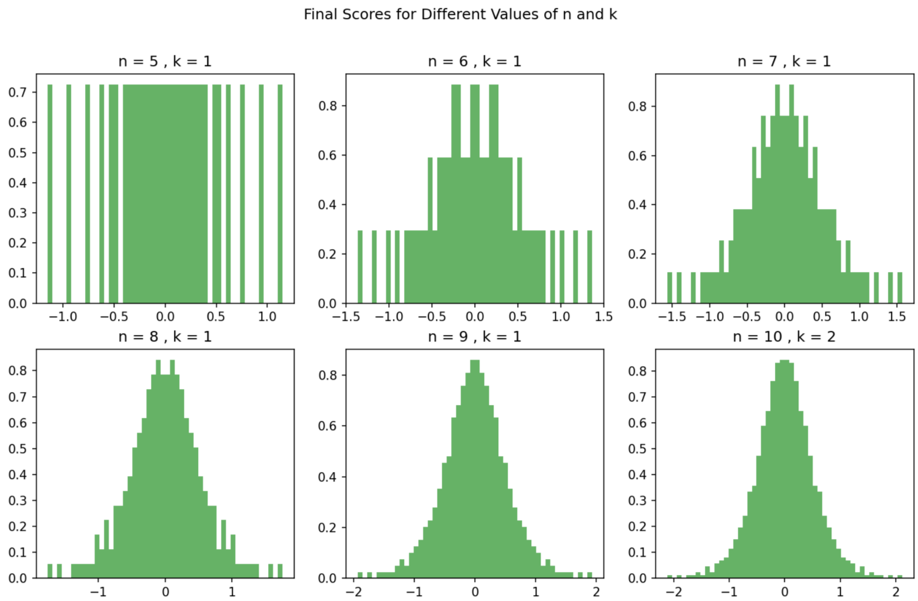

Figure 3.18: Fisher Transformation of x-score for b = 8, $\alpha = 1.4$ for n = 5 to 10

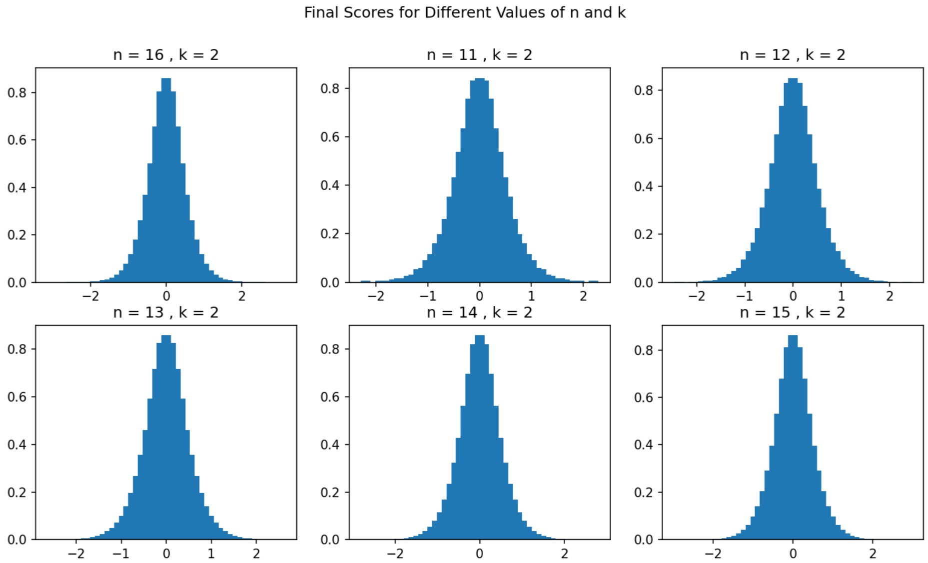

Figure 3.19: Fisher Transformation of x-score for b = 8, α = 1.4 for n = 11 to 16

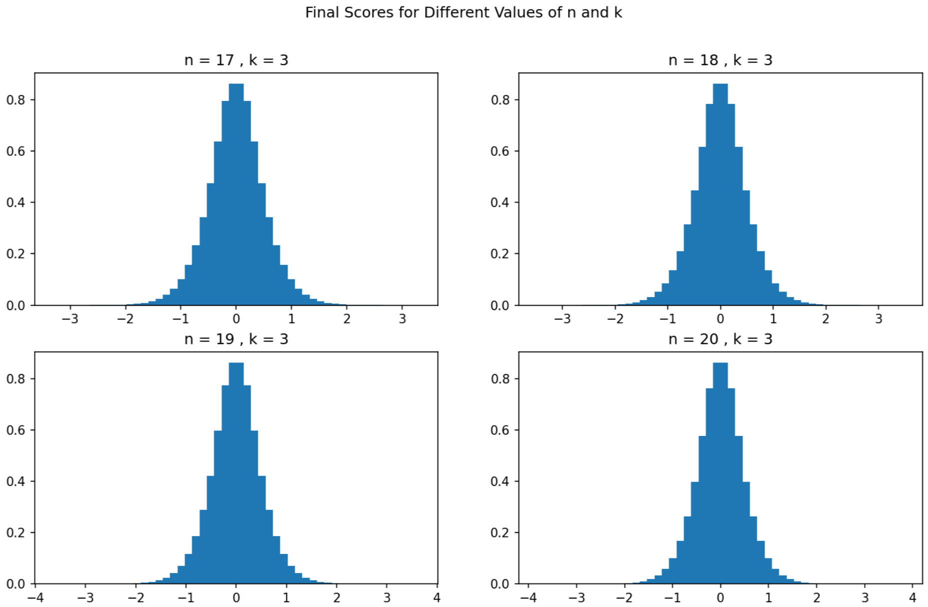

Figure 3.20: Fisher Transformation of x-score for b = 8, α = 1.4 for n = 17 to 20

3.4.2 Fisher Transformation of final score

If we remove weights from penalty score and credit, then fisher transformation of the final score is normal.

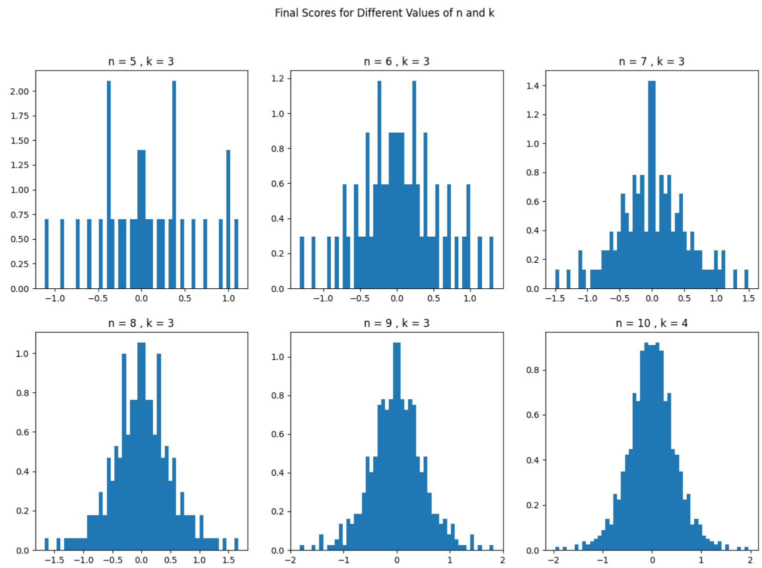

Figure 3.21: Fisher Transformation of final score for b = 8, α = 1.4 and k = $\sqrt{n}$ for n = 5 to 10

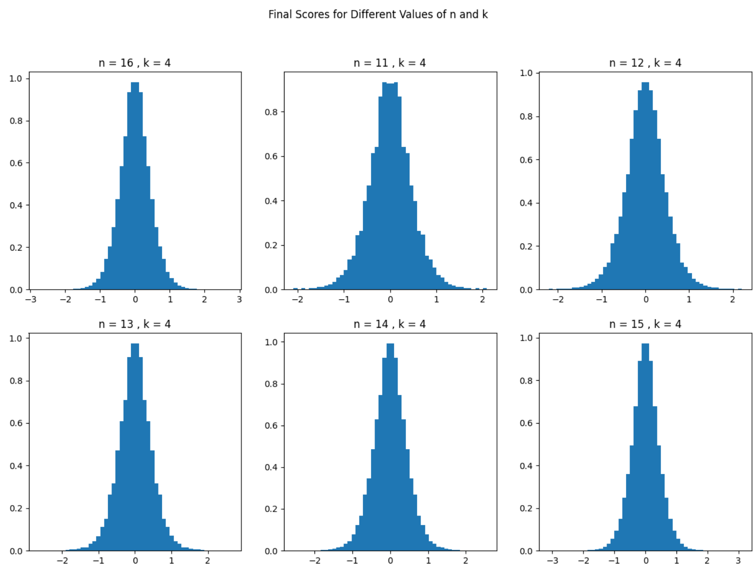

Figure 3.22: Fisher Transformation of final score for b = 8, α = 1.4 and k = $\sqrt{n}$ for n = 11 to 16

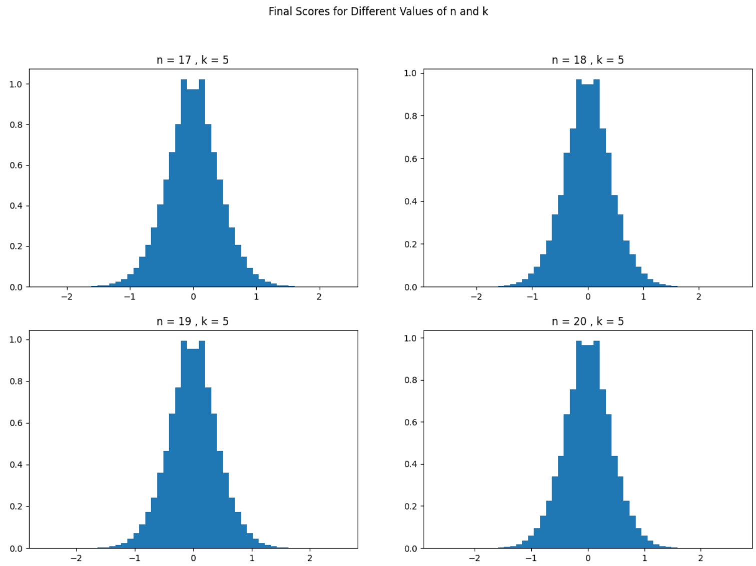

Figure 3.23: Fisher Transformation of final score for b = 8, α = 1.4 and k = $\sqrt{n}$ for n = 17 to 20

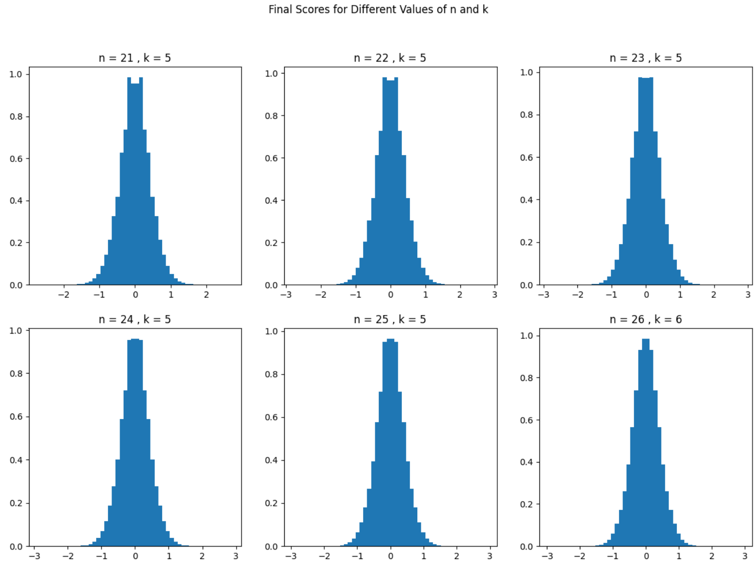

Figure 3.24: Fisher Transformation of final score for b = 8, α = 1.4 and k = $\sqrt{n}$ for n = 21 to 26

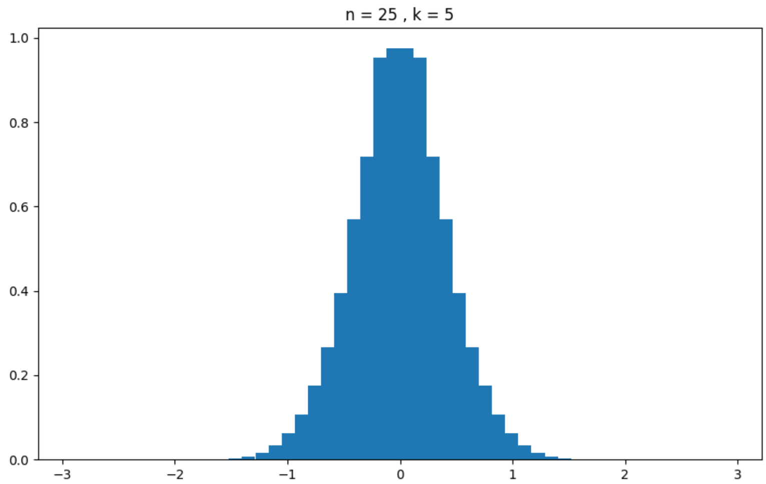

Figure 3.25: Fisher Transformation of final score for b = 8, α = 1.4 and k = $\sqrt{n}$ for n = 25

Chapter 4

Theoretical Proofs

4.1 Finding the distribution of x-score

Theorem 4.1.1.

$$\Rightarrow\ x - score \xrightarrow{d} N(0, \frac{\alpha - 1}{\alpha + 1})$$

Proof. The proof is as follows:

Define a new random variable

$$x_i = \begin{cases} 1 & \text{if } i\text{th game was an agreement} \\ -1 & \text{if } i\text{th game was a disagreement} \end{cases}$$

where i = 0, 1, 2, ..., n-1.

So, we get the x-score as follows:

$$x\text{-score} = c_x \cdot \sum_{i=1}^{n-1} \alpha^i x_i$$

where c_x is the normalization constant defined as $c_x = \frac{\alpha-1}{\alpha^n-1}$ and α is a constant with $\alpha > 1$

The normalization constant remains the same despite changing the random variable used.

Now, we have:

$$E(x_i) = 0$$

$$Var(x_i) = E(x_i^2) - (E(x_i))^2 = 1 - 0 = 1$$

$$E(\alpha^i \cdot x_i) = \alpha^i \cdot E(x_i) = \alpha^i \cdot 0 = 0 \qquad \forall\, i = 0, 1, 2, ..., n-1$$

$$Var(\alpha^i \cdot x_i) = \alpha^{2i} \cdot Var(x_i) = \alpha^{2i} \qquad \forall\, i = 0, 1, 2, ..., n-1$$

$\alpha^i \cdot x_i$ is also a random variable with mean and variance specified above.

Assuming that the x_i's are independent.

The random variables $\alpha^i x_i$'s are independent but not identically distributed. Therefore, we cannot use the Classical Central Limit Theorem (CLT) for them. Hence, we will use Lyapunov CLT described in section 5.2.

For applying Lyapunov CLT, we have the following:

$\{\alpha^0 x_0, \alpha^1 x_1, ..., \alpha^{n-1} x_{n-1}\}$ is a sequence of independent random variables.

$$\mu_i = E(\alpha^i \cdot x_i) = 0 \qquad \forall\, i = 0, 1, 2, ..., n-1$$

$$\sigma_i^2 = Var(\alpha^i \cdot x_i) = \alpha^{2i} \qquad \forall\, i = 0, 1, 2, ..., n-1$$

For $\delta = 1$, the random variables $\alpha^i x_i$ satisfy the Lyapunov's condition.

$$s_{n-1}^2 = \sum_{i=0}^{n-1} \sigma_i^2 = \sum_{i=0}^{n-1} \alpha^{2i} = \frac{\alpha^{2n} - 1}{\alpha^2 - 1}$$

By Lyapunov CLT,

$$\frac{1}{s_{n-1}} \cdot \sum_{i=0}^{n-1} (\alpha^i x_i - \mu_i) \xrightarrow{d} N(0, 1)$$

$$\Rightarrow \sum_{i=0}^{n-1}(\alpha^i x_i - 0) \xrightarrow{d} N(0, s_{n-1}^2)$$

$$\Rightarrow \sum_{i=0}^{n-1}\alpha^i x_i \xrightarrow{d} N(0, s_{n-1}^2)$$

$$\Rightarrow \ c_x \cdot \sum_{i=0}^{n-1}\alpha^i x_i \xrightarrow{d} N(0, c_x^2, s_{n-1}^2)$$

$$\Rightarrow \ x - score \xrightarrow{d} N(0, c_x^2 s_{n-1}^2)$$

We have,

$$c_x^2 s_{n-1}^2 = \frac{(\alpha-1)^2 \cdot (\alpha^{2n}-1)}{(\alpha^n-1)^2 \cdot (\alpha^2-1)} = (\frac{\alpha-1}{\alpha+1}) \cdot (\frac{\alpha^{2n}-1}{(\alpha-1)^2})$$

$$\lim_{n\to\infty}(\frac{\alpha-1}{\alpha+1}) \cdot (\frac{\alpha^{2n}-1}{(\alpha-1)^2}) = \frac{\alpha-1}{\alpha+1}$$

$$\Rightarrow \ x - score \xrightarrow{d} N(0, \frac{\alpha-1}{\alpha+1})$$

For $\alpha = 1.5, \ \frac{\alpha-1}{\alpha+1} = 0.2$

For n = 20, from the code simulation, mean $\approx$ 0 and variance $\approx$ 0.2

From our theoretical derivations, x-score $\sim$ N(0, 0.2) $\square$

4.2 Finding the distribution of run score

Theorem 4.2.1. *For large n, Mean of run-score $\approx$ 0 and Variance of run-score $\approx$ 0*

Proof. The proof is as follows:

E (run-score) = 0 (by the symmetry of 1's and 0's)

Let run-score = X

$$Variance(X) = E(X^2) - (E(X))^2 = (E(X))^2$$

$$(E(X))^2 = E[(\sum_{i \in I_1}(i-k_{i1})\cdot\alpha^i)^2+(\sum_{i \in I_0}(i-k_{i0})\cdot\alpha^i)^2+2(\sum_{i \in I_1}(i-k_{i1})\cdot\alpha^i)(\sum_{i \in I_0}(i-k_{i0})\cdot\alpha^i)]$$

Now, α^{2i} * Expected length of (length of the run of 0's * length of the run of 1's) = 0 since a particular index cannot be the end of a run of 0's as well as 1's.

Therefore,

$2(\sum_{i \in I_1}(i - k_{i1}) \cdot \alpha^i)(\sum_{i \in I_0}(i - k_{i0}) \cdot \alpha^i) \ = \ 0$

By symmetry, $(\sum_{i \in I_1}(i - k_{i1}) \cdot \alpha^i)^2 = (\sum_{i \in I_0}(i - k_{i0}) \cdot \alpha^i)^2$

Hence,

$$(E(X))^2 = 2 * E[(\sum_{i \in I_1}(i - k_{i1}) \cdot \alpha^i)^2]$$

$(\sum_{i \in I_1}(i - k_{i1}) \cdot \alpha^i)^2 = (length\,of\,the\,run\,of\,1's)^2 * \alpha^{2i}$

For runs of 1,

P(length 1) at index = 0 is = $\frac{1}{2}$

P(length 2) at index = 1 is = $(\frac{1}{2})^2$

P(length i + 1) at index = i is = $(\frac{1}{2})^{i+1}$

P(length k) at index i = $(\frac{1}{2^{k+1}})$ (since all values till k + 1 positions from the end are fixed)

$$E[(\sum_{i \in I_1}(i - k_{i1}) \cdot \alpha^i)^2] = \sum_{i=0}^{n-1} \alpha^{2i} \cdot (\frac{1}{2^2} \cdot 1^2 + \frac{1}{2^3} \cdot 2^2 + \cdots + \frac{i^2}{2^{i+1}} + \frac{(i+1)^2}{2^{i+1}})$$

$$\Rightarrow\ E[(\sum_{i\in I_1}(i-k_{i1})\cdot\alpha^i)^2]=\sum_{i=0}^{n-1}\alpha^{2i}\cdot(\frac{(i+1)^2}{2^{i+1}}+\sum_{k=1}^{i}\frac{k^2}{2^{k+1}})$$

Therefore, Variance of run score $= \sigma_i^2 =$

$$2\cdot E[(\sum_{i\in I_1}(i-k_{i1})\cdot\alpha^i)^2]=2\cdot\sum_{i=0}^{n-1}\alpha^{2i}\cdot(\frac{(i+1)^2}{2^{i+1}}+\sum_{k=1}^{i}\frac{k^2}{2^{k+1}})$$

$$\Rightarrow\ \sigma_i^2=\cdot\sum_{i=0}^{n-1}\alpha^{2i}\cdot(\frac{(i+1)^2}{2^{i}}+\sum_{k=1}^{i}\frac{k^2}{2^{k}})$$

$$\Rightarrow\ \sigma_i^2=\alpha^{2i}(6-\frac{(4i+6)}{2^i})$$

$$\Rightarrow\ Var(run-score)=\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\alpha^{2i}(6-\frac{(4i+6)}{2^i})}{c_r^2}$$

$$\Rightarrow\ Var(run-score)=\lim_{n\to\infty}\frac{4(\alpha^{2n})((2n+1)\alpha^2-4n-6)\cdot 2^{-n}-4\alpha^2+24}{(\alpha^2-2)^2\cdot n^2\cdot\alpha^{2n-2}}$$

$$\Rightarrow\ Var(run-score)\to 0\ \ as\ \ n\to\infty$$

□

4.3 Finding a lower bound on the range of α

Theorem 4.3.1. $\alpha \geq \frac{b}{b-1}$

Proof. The proof is as follows:

By our Assumption

The maximum penalty score is for the sequence:

$$\ldots\underbrace{1\ldots1}_{k\text{ times}}0\underbrace{1\ldots1}_{k\text{ times}}0\ldots\underbrace{1\ldots1}_{k\text{ times}}0. \quad (1)$$

It is enough to show that the sequence:

$$\cdots \underbrace{1 \ldots 1}_{k \text{ times}} 00 \tag{2}$$

has a lower penalty score than (1).

Penalty score of sequence (1) $= \alpha^{n-1} \cdot b^{-(k+1)} + \cdots + c$

Penalty score of sequence (2) $= \alpha^{n-2} \cdot b^{-(k+1)} + \alpha^{n-1} \cdot b^{-(k+2)} + \cdots + c$

where c is a constant and n is the length of the sequences being considered.

Penalty score of (1) $\geq$ Penalty score of (2)

$$\Rightarrow \ \alpha^{n-1} \cdot b^{-(k+1)} \geq \alpha^{n-2} \cdot b^{-(k+1)} + \alpha^{i} \cdot b^{-(k+2)}$$

$$\Rightarrow \ \alpha \cdot b \geq b + \alpha$$

$$\Rightarrow \ \alpha \cdot (b-1) \geq b$$

$$\Rightarrow \ \alpha \geq \frac{b}{b-1}$$

□

Chapter 5

Some Theorems/Algorithms used in the project report

5.1 Sequential Least Squares Programming (SLSQP)

Sequential Least Squares Programming (SLSQP) is an iterative optimization algorithm used to solve nonlinear, constrained optimization problems. It belongs to the Sequential Quadratic Programming (SQP) family of methods and is specifically designed to handle cases where both equality and inequality constraints are present.

5.1.1 Problem Formulation

The general form of the optimization problem that SLSQP solves is as follows:

$$\min_{\mathbf{x}} \; f(\mathbf{x})$$

subject to:

$$h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, m$$

$$g_j(\mathbf{x}) \leq 0, \quad j = 1, \ldots, p$$

$$\mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{x}_{\max}$$

where:

- $f(\mathbf{x})$ is the objective function,
- $\mathbf{x}$ is the vector of variables to be optimized,
- $h_i(\mathbf{x})$ are the equality constraints,
- $g_j(\mathbf{x})$ are the inequality constraints, and
- $\mathbf{x}_{\min}$ and $\mathbf{x}_{\max}$ define the lower and upper bounds on the variables in $\mathbf{x}$.

5.1.2 Overview of Sequential Quadratic Programming (SQP)

SLSQP, like other SQP methods, approaches the nonlinear optimization problem by iteratively solving a series of quadratic subproblems. Each iteration of SLSQP involves:

1. **Linearizing the Constraints:** The equality and inequality constraints are approximated linearly around the current point $\mathbf{x}_k$, leading to a linearized form of the constraints.
2. **Quadratic Approximation of the Objective:** The objective function $f(\mathbf{x})$ is approximated by a quadratic function around the current

point. This quadratic approximation is usually obtained from the Taylor series expansion of $f(\mathbf{x})$.

Thus, at each iteration k, SLSQP solves a quadratic programming (QP) subproblem:

$$\min_{\mathbf{d}} \nabla f(\mathbf{x}_k)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}_k \mathbf{d}$$

subject to:

$$h_i(\mathbf{x}_k) + \nabla h_i(\mathbf{x}_k)^T \mathbf{d} = 0, \quad i = 1, \ldots, m$$

$$g_j(\mathbf{x}_k) + \nabla g_j(\mathbf{x}_k)^T \mathbf{d} \leq 0, \quad j = 1, \ldots, p$$

where:

- $\mathbf{d}$ is the search direction from the current point $\mathbf{x}_k$,
- $\nabla f(\mathbf{x}_k)$ is the gradient of the objective function at $\mathbf{x}_k$,
- $\mathbf{H}_k$ is an approximation of the Hessian matrix of the Lagrangian function at $\mathbf{x}_k$,
- $\nabla h_i(\mathbf{x}_k)$ and $\nabla g_j(\mathbf{x}_k)$ are the gradients of the equality and inequality constraints at $\mathbf{x}_k$.

5.1.3 Updating the Solution

After solving the QP subproblem, the algorithm updates the solution:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}$$

where α_k is a step size determined by a line search procedure.

This process is repeated until convergence, where convergence criteria might include changes in the objective function value, constraints, or gradient norms reaching below predefined thresholds.

5.2 Lyapunov CLT

In this variant of the central limit theorem, the random variables X_i have to be independent but not necessarily identically distributed. The theorem also requires that random variables $|X_i|$ have moments of some order $(2+\delta)$ and that the rate of growth of these moments is limited by the Lyapunov condition given below.
Suppose $\{X_1, \ldots, X_n, \ldots\}$ is a sequence of independent random variables, each with finite expected value μ_i and variance σ_i^2. Define

$$s_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

If for some $\delta > 0$, Lyapunov's condition

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}\left[|X_i - \mu_i|^{2+\delta}\right] = 0$$

is satisfied, then a sum of $\frac{X_i - \mu_i}{s_n}$ converges in distribution to a standard normal random variable, as n goes to infinity:

$$\frac{1}{s_n} \sum_{i=1}^{n} (X_i - \mu_i) \xrightarrow{d} \mathcal{N}(0,1).$$

In practice, it is usually easiest to check Lyapunov's condition for $\delta = 1$.

5.3 Fisher Transformation

The Fisher transformation is a statistical technique used to stabilize the variance of the sampling distribution of Pearson's correlation coefficient r. This transformation is useful for inference, such as hypothesis testing and constructing confidence intervals for correlation coefficients.

5.3.1 Formula for Fisher Transformation

The Fisher transformation of a correlation coefficient r is defined as:

$$z = \frac{1}{2} \ln \left(\frac{1+r}{1-r} \right)$$

where:

- r is the correlation coefficient,
- ln denotes the natural logarithm.

This transformation maps r from the range $(-1, 1)$ to the range $(-\infty, +\infty)$.

Chapter 6

Future Plans and Scope

Building on the foundational work completed so far, several important directions exist for future research and development. These directions aim to enhance the theoretical rigour, practical utility, and computational efficiency of the proposed association measure for coordination games. Key areas for future work include the following:

1. Development of a Real-Time Analysis Application Using Shiny

A significant goal is to develop an interactive web application using R's Shiny framework to facilitate real-time analysis of coordination games. This tool would allow users to input parameters, simulate coordination outcomes, and analyze how the proposed measure performs under different game settings. Such an application would provide researchers and practitioners with an intuitive platform to explore the measure's properties, visualize coordination dynamics, and conduct real-time analyses of strategic interactions between players.

2. Exploration of Additional Statistical Properties of the Proposed Measure

To ensure the robustness and reliability of the measure, further research is required to investigate its statistical properties in greater depth. This includes examining aspects such as sensitivity to parameter changes, and consistency under repeated interactions. By thoroughly understanding these statistical characteristics, we can establish a more rigorous theoretical foundation for the measure, strengthening its applicability for empirical research.

3. Proof of Asymptotic Normality for the Final Score

A critical theoretical task is to formally prove that the distribution of the final score is asymptotically normal. Establishing this asymptotic behaviour would validate the measure's suitability for statistical inference and enable hypothesis testing. This will involve applying advanced statistical techniques to rigorously demonstrate the measure's convergence to a normal distribution as sample size increases, which is essential for its broader acceptance and application.

4. Addressing Independence Assumptions in the Lyapunov Central Limit Theorem (CLT) for the X-Score

While attempting to determine the distribution of the X-score using the Lyapunov CLT, we have assumed independence of the X_i values. However, this assumption may not hold in practice, as players' decisions are typically influenced by their previous interactions and miscoordinations. In future work, refining the model to account for these dependencies, possibly through incorporating a dependence structure or a Markov process, will be essential. Adjusting the application of the Lyapunov CLT to account for player learning and sequential dependencies will improve the accuracy of the measure's theoretical framework.

5. Investigation into the Convergence Properties of Penalty and Credit Score Distributions

Another critical area for future exploration is understanding the convergence properties of the distributions of penalty and credit scores. Examining the conditions under which these scores converge will provide insights into the stability and reliability of the proposed measure in long-term or repeated coordination games. This analysis could involve a combination of theoretical proofs and extensive simulations to validate convergence behaviour under various game scenarios.

6. Development of a Quantum Simulation Algorithm for Large-Scale Coordination Games

Conventional computational methods may be inefficient for large-scale coordination games, where n (representing the number of players or game rounds) becomes substantial. As such, developing a quantum simulation algorithm could offer a high-efficiency approach to handling large values of n. By designing and testing quantum algorithms for simulating strategic interactions, we aim to expand the computational feasibility of applying the measure to complex, high-dimensional game environments.

7. Empirical Study on Real-World Applications and Datasets

Finally, to assess the practical relevance of the measure, future research should apply it to real-world datasets across diverse fields such as economics, social behaviour, and multi-agent systems in artificial intelligence. Analyzing empirical data will allow us to evaluate the measure's effectiveness in capturing coordination dynamics, identify potential improvements, and demonstrate its applicability to real-world problems.

These future directions provide a comprehensive roadmap for advancing the proposed measure. By addressing the theoretical assumptions, developing computational tools, and exploring real-world applications, we aim to enhance the measure's rigour, accessibility, and impact in the study of coordination games.

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