# Lab Session 9

**MA423:** Matrix Computations July-November, 2024 S. Bora

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**Important instructions:**

(i) **Switch to `format long e` for all experiments.**

(ii) **Submit a single livescript program that contains all comments, answers and codes necessary to produce the required outputs. Ensure that the answers are correctly numbered and the file does not include any irrelevant material. The livescript program should be saved as MA423YourrollnumberLab9.mlx**

1. Write a function program $[\texttt{iter}, \texttt{lambda}] = \texttt{Powermethod}(\texttt{A}, \texttt{x}, \texttt{k})$ that performs `k` iterations of the Power Method with $\texttt{A} \in \mathbb{C}^{\texttt{n}\times\texttt{n}}$ and initial vector $\texttt{x} \in \mathbb{C}^{\texttt{n}}$ and returns an $n \times k$ matrix `iter` whose $j$th column is the $j$th iterate $q_j$ and a scalar `lambda` which is the dominant eigenvalue.

2. Run $[\texttt{iter}, \texttt{lambda}] = \texttt{Powermethod}(\texttt{A}, \texttt{x}, \texttt{k})$ with $x = [1\ 1\ 1]^T$ and the following matrices

$$(i)\, A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 9 & 2 \\ 0 & -1 & 2 \end{bmatrix} \qquad (ii)\, A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 9 & 2 \\ -4 & -1 & 2 \end{bmatrix} \qquad (iii)\, A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ -4 & -1 & 2 \end{bmatrix}$$

   In each case check if for large enough values of $j$, $\|\texttt{iter}(:, j+1) - v\|/\|\texttt{iter}(:, j) - v\|$ agrees with the theoretical convergence rate of $|\lambda_2|/|\lambda_1|$ where $\lambda_1$ and $\lambda_2$ are the largest and second largest eigenvalues of $A$ in magnitude and $v$ is an eigenvector corresponding to $\lambda_1$. Try to explain your observations. (Type `format long e` to see more digits. You will have to run $[\texttt{V}, \texttt{D}] = \texttt{eig}(\texttt{A})$ to find $\lambda_1, \lambda_2$ and $v$. Note that you will have to use the same scaling that you perform in your iterations on the column of `V` corresponding to $\lambda_1$ to produce the dominanat eigenvector $v$ to which the iterates are likely to converge.)

3. Repeat the process in (2) for `A` given by (i) and initial vector $\texttt{x} = v_2 + v_3$ where $v_2$ and $v_3$ are eigenvectors corresponding to $\lambda_2$ and $\lambda_3$. Theoretically this implies that $c_1 = 0$, which violates an important necessary condition for the iterations to converge. To check if this happens in practice, run $[\texttt{iter}, \texttt{lambda}] = \texttt{Powermethod}(\texttt{A}, \texttt{x}, \texttt{k})$ for sufficiently large values of `k`. Can you explain your observations?

4. Write a function program $[\texttt{iter}, \texttt{lambda}] = \texttt{Shiftinv}(\texttt{A}, \texttt{x}, \texttt{s}, \texttt{k})$ that *efficiently* performs `k` iterations of Shift and Invert Method using $\texttt{A} \in \mathbb{C}^{\texttt{n}\times\texttt{n}}$, $\texttt{x} \in \mathbb{C}^{\texttt{n}}$ and shift `s` and returns an $n \times k$ matrix `iter` whose $j$th column is the $j$th iterate $q_j$ and a scalar `lambda` is the the eigenvalue of $A$ closest to `s`. The cost per iteration should be $O(n^2)$ flops.

5. Write a function program $[\texttt{L}, \texttt{U}, \texttt{p}] = \texttt{gepphess}(\texttt{A})$ that performe Gaussian Elimination with Partial Pivoting on an $n \times n$ upper Hessenberg matrix $A$ in $O(n^2)$ flops to produce an $LU$ decomposition of $A(p, :)$, $p$ being the permutation vector that records the row interchanges.

6. Write a function program $[\texttt{iter}, \texttt{lambda}] = \texttt{Rayleigh}(\texttt{A}, \texttt{x}, \texttt{k})$ that *efficiently* performs `k` iterations of inverse iterations with Rayleigh quotient shifts using $\texttt{A} \in \mathbb{C}^{\texttt{n}\times\texttt{n}}$ and $\texttt{x} \in \mathbb{C}^{\texttt{n}}$ and

returns an $n \times k$ matrix `iter` whose $j$th column is the $j$th iterate $q_j$ and a scalar `lambda` is the eigenvalue of $A$ to which the Rayleigh quotient shifts converge. If $A$ is not upper Hessenberg, then use the built-in `hess.m` program and your `gepphess.m` program as explained in theory class so that iterations can be performed in $O(n^2)$ flops.

7. Make numerical experiments to compare all three methods for the matrices in Q2 for the same choice of starting vector. You are expected to comment whether the shifting strategies used by you in the Shift and Invert and Rayleigh Quotient Methods accelerate the convergence in comparison to Power Method.