Math 446 Homework 1 Solutions.

3. Given finitely many countable sets $A_1, \dots, A_n$, show that $A_1 \cup \cdots \cup A_n$ and $A_1 \times \cdots \times A_n$ are countable sets.

Proof. Finite unions: We showed in class that a countable union of countable sets is countable. Taking $A_m = \varnothing$ for all $m > n$ therefore shows that the set $A_1 \cup \cdots \cup A_n = \bigcup_{m \in \mathbb{N}} A_m$ is countable.

Finite products: Since each A_i is equinumerous with a subset of $\mathbb{N}$, the set $A_1 \times \cdots \times A_n$ is equinumerous with a subset of $\mathbb{N}^n$. Since subsets of countable sets are countable, it is therefore enough to show that $\mathbb{N}^n \sim \mathbb{N}$. We proceed by induction on n. The case $n = 1$ is trivial. Let $n > 1$ and assume by way of induction that $\mathbb{N}^{n-1}$ is countable, i.e., that $\mathbb{N}^{n-1} \sim \mathbb{N}$. Therefore $\mathbb{N}^n \sim \mathbb{N}^{n-1} \times \mathbb{N} \sim \mathbb{N} \times \mathbb{N}$, and we showed in class that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. Therefore, $\mathbb{N}^n \sim \mathbb{N}$ and the induction is complete. $\square$

4. Show that any infinite set has a countably infinite subset.

Proof. Let A be an infinite set. In particular A is nonempty, so fix any $a_1 \in A$. Assume that $a_1, \dots, a_n \in A$ have been defined with $a_i \neq a_j$ for all $1 \leq i < j \leq n$. Since A is infinite, the set $A \setminus \{a_1, \dots, a_n\}$ is nonempty, so define a_{n+1} to be any element of $A \setminus \{a_1, \dots, a_n\}$. We have inductively defined an infinite sequence $(a_1, a_2, a_3, \dots)$ of elements of A such that for each $n \in \mathbb{N}$ we have $a_{n+1} \neq a_i$ for all $1 \leq i \leq n$. This means that the function $n \mapsto a_n$ is injective, so that the set $\{a_n : n \in \mathbb{N}\}$ is equinumerous with $\mathbb{N}$, i.e., it is a countably infinite subset of A. $\square$

Remark: While it is beyond the scope of this class, I would be remiss if I did not mention that the above argument makes use of a weak form of the Axiom of Choice called the *Axiom of Dependent Choice*, which is stated below. For the statement, recall from class that $A^{<\mathbb{N}}$ denotes the set of all finite strings of elements of A; we also include the emptyset as an element of $A^{<\mathbb{N}}$, viewing it as a string of length zero. Also, if $f : \mathbb{N} \to A$ is a function (i.e., $f \in A^{\mathbb{N}}$) then for each $n \in \mathbb{N} \cup \{0\}$ we let $f \restriction n \in A^{<\mathbb{N}}$ be the string $f \restriction n = (f(1), f(2), \dots, f(n))$ (and $f \restriction 0 = \varnothing$).

Axiom of Dependent Choice: Let A be a set, let P be a subset of $A^{<\mathbb{N}} \times A$, and assume that for every $s \in A^{<\mathbb{N}}$ the set $\{a \in A : (s, a) \in P\}$ is nonempty. Then there exists a function $f : \mathbb{N} \to A$ such that $(f \restriction n, f(n+1)) \in P$ for all $n \in \mathbb{N} \cup \{0\}$.

6. If A is infinite and B is countable, show that A and $A \cup B$ are equinumerous. [Hint: No containment relation between A and B is assumed here.]

Proof. Assume first that A and B are disjoint. By problem 4. we can find a countably infinite subset A_0 of A. Then $A_0 \sim A_0 \cup B$ by problem 1, so fix a bijection $f_0 : A_0 \cup B \to A_0$. Define $f : A \cup B \to A$ by

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in A_0 \cup B \\ x & \text{if } x \in A \setminus (A_0 \cup B) = A \setminus A_0. \end{cases}$$

Then f is surjective since $f(A \cup B) = f(A_0 \cup B) \cup f(A \setminus (A_0 \cup B)) = A_0 \cup A \setminus (A_0 \cup B) = A_0 \cup (A \setminus A_0) = A$ where the second to last equality follows from A and B being disjoint.

Also, f is injective since its restriction to $A_0 \cup B$ is injective and its restriction to $A \setminus (A_0 \cup B)$ is injective, and the sets $f(A_0 \cup B) = A_0$ and $f(A \setminus (A_0 \cup B)) = A \setminus A_0$ are disjoint. Thus f is bijective, so $A \cup B \sim A$.

In general, if A and B are not necessarily disjoint, then A and $B \setminus A$ are disjoint, and $B \setminus A$ is countable since it is a subset of B. Therefore, by the first case, A is equinumerous with $A \cup (B \setminus A) = A \cup B$. □

13. Show that $\mathbb{N}$ contains infinitely many pairwise disjoint infinite subsets.

Proof. We showed in class that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, so fix a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For each $k \in \mathbb{N}$ let $A_k = \{(k, n) : n \in \mathbb{N}\}$. Then $A_0, A_1, \ldots$ are pairwise disjoint infinite subsets of $\mathbb{N} \times \mathbb{N}$, so since f is bijective, $f(A_0), f(A_1), \ldots,$ are pairwise disjoint infinite subsets of $\mathbb{N}$. (Note: moreover, $\mathbb{N}$ is the union of the sets $f(A_0), f(A_1), \ldots$.) □

15. Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable. [Hint: Each one contains a rational!]

Proof. Let $\mathcal{C}$ be a collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$. We showed in class that $\mathbb{Q}$ is countable, so fix a bijection $f : \mathbb{Q} \to \mathbb{N}$. For each $I \in \mathcal{C}$, since I is nonempty and open, the set $\mathbb{Q} \cap I$ is nonempty (by density of $\mathbb{Q}$ in $\mathbb{R}$), and hence $f(\mathbb{Q} \cap I)$ is a nonempty subset of $\mathbb{N}$. Define $F : \mathcal{C} \to \mathbb{N}$ by $F(I) := \min f(\mathbb{Q} \cap I)$. Then F is injective since if $I, J \in \mathcal{C}$ are distinct intervals, then they are disjoint by hypothesis, so $f(\mathbb{Q} \cap I)$ and $f(\mathbb{Q} \cap J)$ are disjoint (since f is a bijection), and hence $\min f(\mathbb{Q} \cap I) \neq \min f(\mathbb{Q} \cap J)$. This shows that $\mathcal{C}$ is countable, since it is equinumerous with a subset of $\mathbb{N}$. □

17. If A is uncountable and B is countable, show that A and $A \setminus B$ are equivalent. In particular, conclude that $A \setminus B$ is uncountable.

Proof. Since B is countable, $A \cap B$ is also countable (being a subset of B). Since A is uncountable and B is countable, the set $A \setminus B$ is uncountable (otherwise A would be countable, being a subset of the countable set $B \cup (A \setminus B)$). Therefore, by problem 6, $A \setminus B$ is equinumerous with $(A \setminus B) \cup (A \cap B) = A$. □

19. Show that the set of all functions $f : A \to \{0, 1\}$ is equivalent to $\mathcal{P}(A)$, the power set of A (i.e., the set of all subsets of A).

Proof. Define $\Phi : \mathcal{P}(A) \to \{0, 1\}^A$ send $B \in \mathcal{P}(A)$ to the function $\Phi(B) : A \to \{0, 1\}$ defined by $\Phi(B)(n) = 1$ iff $n \in B$, and $\Phi(B)(n) = 0$ iff $n \notin B$. To show Φ is a bijection it is enough to show that it has an inverse, i.e., to show that there is a function $\Psi : \{0, 1\}^A \to \mathcal{P}(A)$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity functions on $\mathcal{P}(A)$ and $\{0, 1\}^{\mathbb{N}}$ respectively. Define Ψ by $\Psi(f) := \{n \in \mathbb{N} : f(n) = 1\}$. Then $\Psi(\Phi(B)) = B$ for all $B \in \mathcal{P}(A)$ and $\Phi(\Psi(f)) = f$ for all $f \in \{0, 1\}^A$. Therefore, Φ is bijective, and hence $\mathcal{P}(A) \sim \{0, 1\}^A$. □

Additional problems for honors sections:

5. Prove that a set is infinite if and only if it is equivalent to a proper subset of itself. [Hint: If A is infinite and $x \in A$, show that A is equinumerous to $A \setminus \{x\}$.]

Proof. Let A be a infinite set and we will show it is equivalent to a proper subset of itself. Case 1: A is countably infinite. In this case we can find a bijection $f : \mathbb{N} \to A$. The map $h : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ defined by $h(n) = n + 1$ is a bijection. Therefore, the function $F : A \to A \setminus \{f(1)\}$ defined by $F(a) = f(h(f^{-1}(a)))$ is a bijection between A and a proper subset of A. Case 2: A is uncountably infinite. Fix any $x \in A$. Then A is equinumerous with $A \setminus \{x\}$ by problem 17.

For the other implication we will prove its contrapositive, i.e., assuming that A is finite, we will show that A is not equinumerous with a proper subset of itself. This is clear if $A = \varnothing$, so assume A is a nonempty finite set. It is enough to prove that every injection from A to itself is a surjection. Since A is equinumerous with $\{1, \dots, n\}$ for some n, it is enough to prove that every injection from $\{1, \dots, n\}$ to itself is a surjection. We proceed by induction on $n \geq 1$. The base case $n = 1$ is trivial since there is only one function from $\{1\}$ to itself, and it is bijective. Assume now that the statement is true for n and let f be an injection from $\{1, \dots, n, n + 1\}$ to itself.

Case 1: $n+1 \notin f(\{1, \dots, n\})$. Then the restriction of f to $\{1, \dots, n\}$, is an injection from $\{1, \dots, n\}$ to itself, hence it is surjective by the induction hypothesis. Since f is injective, it must be that $f(n+1) \notin f(\{1, \dots, n\}) = \{1, \dots, n\}$, so $f(n+1) = n+1$. Therefore, f is surjective.

Case 2: $n + 1 \in f(\{1, \dots, n\})$. In this case, fix $p \leq n$ with $f(p) = n + 1$. Define $h : \{1, \dots, n\} \to \{1, \dots, n\}$ by

$$h(i) = \begin{cases} f(i) & \text{if } i \neq p \\ f(n+1) & \text{if } i = p. \end{cases}$$

Then h is injective since it agrees with the injective function f at all values except at p, and for $i \leq n$ with $i \neq p$, since f is injective we have $h(i) = f(i) \neq f(n+1) = h(p)$. By the induction hypothesis then h is surjective. Therefore, $f(\{1, \dots, n, n + 1\}) = h(\{1, \dots, n\}) \cup \{f(p)\} = \{1, \dots, n, n + 1\}$, so f is surjective.

This completes the induction. □

10. Prove that $(0, 1)$ can be put into one-to-one correspondence with the set of all functions $f : \mathbb{N} \to \{0, 1\}$.

    Proof. We went through a proof of this in class (see class notes). □

14. Prove that any infinite set can be written as the countably infinite union of pairwise disjoint infinite subsets.

    Proof. Let A be an infinite set and let B be a countably infinite subset of A. The proof of problem 13 shows that we can find pairwise disjoint, infinite subsets $B_1, B_2, \dots$ of B such that $B = \bigcup_{n=1}^{\infty} B_i$. Let $A_1 = B_1 \cup (A \setminus B)$, and for $n > 1$ let $A_n = B_n$. Then the sets $A_1, A_2, \dots$ are pairwise disjoint, infinite, and their union is all of A. □

20. Prove that $\mathbb{N}$ can contains uncountably many infinite subsets $(N_\alpha)_{\alpha \in \mathbb{R}}$ such that $N_\alpha \cap N_\beta$ is *finite* if $\alpha \neq \beta$.

    Proof. The set $\{0, 1\}^{<\mathbb{N}} = \bigcup_{k=0}^{\infty} \{0, 1\}^k$, of all finite sequences of zeros and ones, is a countable set, being a countable union of finite sets. For each $x = (x_1, x_2, \dots) \in$

$\{0,1\}^{\mathbb{N}}$ let N_x be the subset of $\{0,1\}^{<\mathbb{N}}$ consisting of all finite initial segments of x, i.e., a finite string $s = (s_1, \ldots, s_n)$ belongs to N_x if and only if $s_i = x_i$ for all $1 \leq i \leq n$. Then N_x is infinite, and if $x \neq y$ then $N_x \cap N_y$ is finite since, letting $n_0 := \min\{n : x(n) \neq y(n)\}$, we have $N_x \cap N_y = \{(x_1, \ldots, x_n) : 0 \leq n < n_0\}$.

Fix a bijection $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$. We have $\mathbb{R} \sim \{0,1\}^{\mathbb{N}}$, so fix a bijection $g : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$. Since the family $\{N_x\}_{x \in \{0,1\}^{\mathbb{N}}}$ consists of infinite sets with pairwise finite intersection, the family $\{N_\alpha\}_{\alpha \in \mathbb{R}}$ also does, where $N_\alpha := f(N_{g(\alpha)})$ $\square$