Computational Complexity Theory

Lecture 5: Class co-NP and EXP; Diagonalization

Indian Institute of Science

Recap: Alternate definition of NP

- Definition. An NTM M *accepts* a string $x \in \{0,1\}^*$ iff on input x there ***exists*** a sequence of applications of the transition functions δ_0 and δ_1 (beginning from the start configuration) that makes M reach q_{accept}.

- Definition. A language L is in $NTIME(T(n))$ if there's an NTM M that decides L in $c. T(n)$ time on inputs of length n, where c is a constant.

- Theorem. $NP = \cup_{c>0} NTIME(n^c)$.

Recap: Search versus Decision for NP

- Theorem. Let $L \subseteq \{0,1\}^*$ be NP-complete. Then, the search version of L can be solved in poly-time if and only if the decision version can be solved in poly-time.

- Theorem. *(Bellare-Goldwasser)* If EE $\neq$ NEE then there's a language in NP for which search does not reduce to decision.

- Sometimes, the decision version of a problem can be trivial but the search version is possibly hard. E.g. Computing Nash Equilibrium (see class PPAD).

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Karp reducible implies Cook reducible

Class co-NP

- Definition. For every $L \subseteq \{0,1\}^*$ let $\bar{L} = \{0,1\}^* \setminus L$.

  A language L is in co-NP if $\bar{L}$ is in NP.

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- **Note:** co-NP is <u>***not***</u> complement of NP. Every language in P is in both NP and co-NP.

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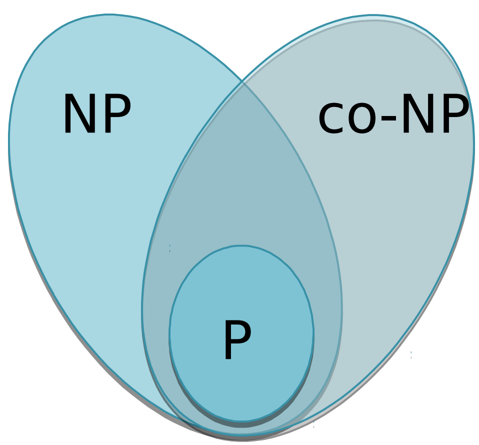

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- Example. $\overline{\text{SAT}}$ = {ϕ : ϕ is *not* satisfiable}.

- Note: $\overline{\text{SAT}}$ is Cook reducible to SAT. But, there's a fundamental difference between the two problems that is captured by the fact that $\overline{\text{SAT}}$ is not known to be Karp reducible to SAT. In other words, there's no known poly-time verification process for $\overline{\text{SAT}}$.

Class co-NP : Alternate definition

- Recall, a language $L \subseteq \{0,1\}^*$ is in NP if there's a *poly-time verifier* M such that

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$\overline{M}$ outputs the *opposite* of M

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$\overline{M}$ is a poly-time TM

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for NP this was $\exists$

co-NP-completeness

- Definition. A language $L' \subseteq \{0,1\}^*$ is *co-NP-complete* if
  - L' is in co-NP
  - Every language L in co-NP is polynomial-time (Karp) reducible to L'.

- Theorem. SAT is co-NP-complete.

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  Proof. Let $\overline{L} \in$ co-NP. Then

  $\Rightarrow$ $\overline{L} \in NP$

  $L \leq_p SAT$

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  $L \le_p \overline{\text{SAT}}$

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- Theorem. Let

$$\text{TAUTOLOGY} = \{\phi : \text{ every assignment satisfies } \phi \}.$$

TAUTOLOGY is co-NP complete.

Proof. Similar (homework)

Class EXP

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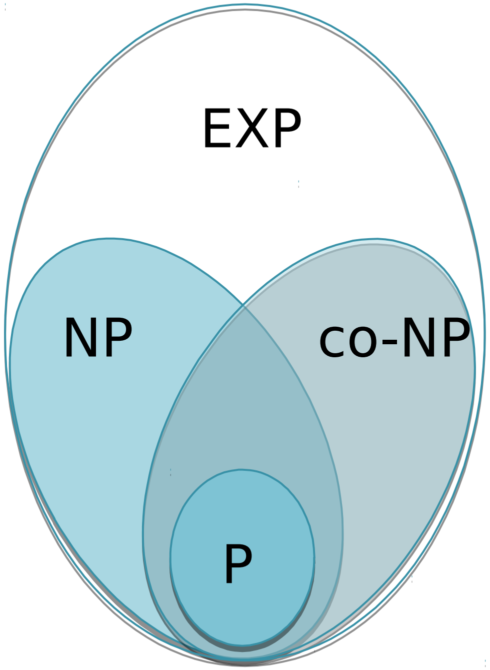

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- Exponential Time Hypothesis. *(Impagliazzo & Paturi)* Any algorithm for 3-SAT takes time $\Omega(2^{\delta.n})$, where δ is a constant and n is the input size.

ETH $\Rightarrow$ P $\neq$ NP

Diagonalization

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     If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

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- These techniques are characterized by two main features:
    1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a *small* overhead.
    2. Every string represents some TM, and every TM can be represented by infinitely many strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be <u>time-constructible</u> functions s.t.,

$$f(n) \,.\, \log f(n) = o(g(n)).$$

e.g. $f(n) = n$, $g(n) = n^2$

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  Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

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1. On input x, compute $|x|^2$.

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D runs in $O(n^2)$ time as n^2 is time-constructible.

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  Claim. There's no TM M with running time O(n) that

  decides L (the language accepted by D).

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- Suppose $M_x(x) = b$
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  - Suppose $M_x(x) = b$
  - D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c.\ |x|.\ \log |x|$ steps. (as $c'.c.\ |x|.\ \log |x| < |x|^2$ for <u>sufficiently large</u> x)

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  - Suppose $M_x(x) = b$
  - D on input x, simulates M_x on x for $|x|^2$ steps. Since M_x stops within $c.|x|$ steps, D's simulation also stops within $c'.c. \, |x|. \log |x|$ steps. And D outputs the opposite of what M_x outputs.

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- Think of a <u>sufficiently large</u> x such that $M = M_x$
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- Hence, $D(x) = 1\text{-}b$

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- Hence, $D(x) = 1\text{-}b$

Contradiction! M does not decide L.

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- Theorem. $P \subsetneq EXP$

  Proof. Similar (homework)

Ladner’s Theorem

- Another application of Diagonalization

NP-intermediate problems

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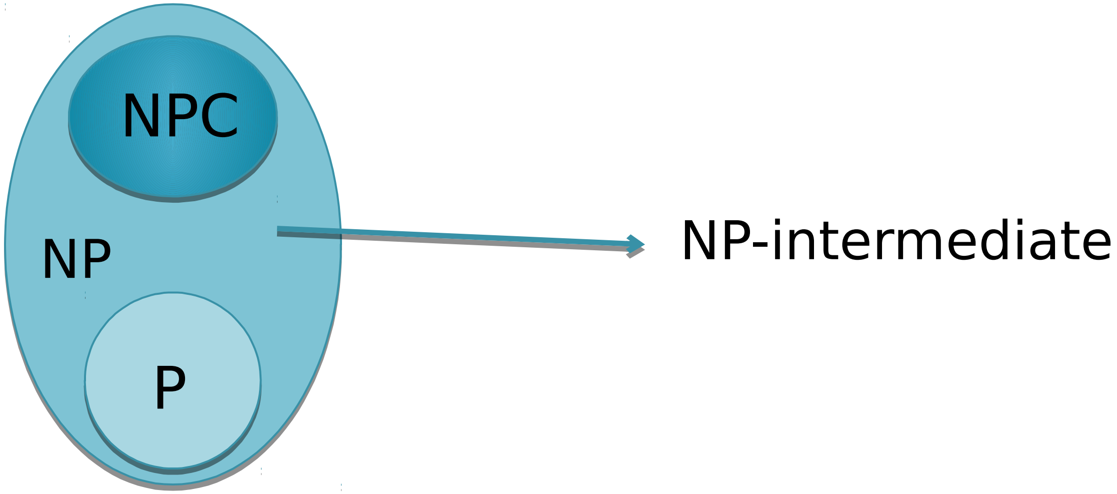

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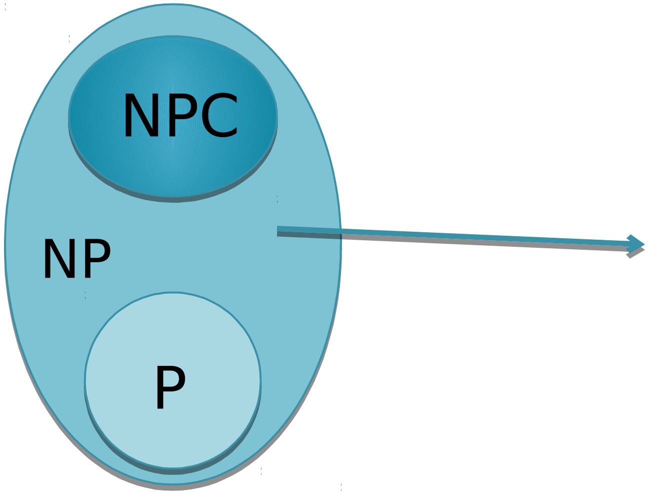

NP-intermediate (integer factoring, graph isomorphism ??)

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… the notion makes sense only if P ≠ NP

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- Theorem. *(Ladner)* If $P \neq NP$ then there is an NP-intermediate language.

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- Theorem. *(Ladner)* If P $\neq$ NP then there is an NP-intermediate language.

  Proof. A delicate argument using diagonalization.

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Let $SAT_H = \{\Psi 0 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

H would be defined in such a way that SAT_H is NP-intermediate

(assuming $P \neq NP$)

Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \rightarrow N$ such that

  1. $H(m)$ is computable from m in $O(m^3)$ time

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Proof: Later (uses diagonalization).

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- This implies a poly-time algorithm for SAT as follows:
  - On input ϕ, find $m = |\phi|$.
  - Compute $H(m)$, and construct the string $\phi\, 0\, 1^{m^{H(m)}}$
  - Check if $\phi\, 0\, 1^{m^{H(m)}}$ belongs to SAT_H
- As $P \neq NP$, it must be that $SAT_H \notin P$.

Ladner’s theorem: Proof

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Ladner's theorem: Proof

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Either m is small (in which case the task reduces to checking if a small Ψ is satisfiable),

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- Compute $H(m)$ and check if $k = m^{H(m)}$.

Either m is small (in which case the task reduces to checking if a small Ψ is satisfiable), or $H(m) > 2c$ (as $H(m)$ tends to infinity with m).

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  - Hence, w.l.o.g. $|f(\phi)| \geq m^{2c}$

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  - On input ϕ, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
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Thus, checking if an n-size formula ϕ is satisfiable reduces to checking if a $\sqrt{n}$-size formula Ψ is satisfiable.

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Do this recursively! Only $O(\log \log n)$ recursive steps required.

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  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

- Hence SAT_H is not NP-complete, as $P \neq NP$.