Indian Institute of Technology Guwahati
Probability Theory and Random Processes (MA225)
Problem Set 07

1. Let X be a continuous random variable. A real number m is said to be median of X if $F_X(m) = 0.5$. Show that, for all $c \in \mathbb{R}$, $E|X - c| \geq E|X - m|$.

2. Let $\{X_n\}$ be a sequence of random variables with $P(X_n = n) = 1 - \frac{1}{n}$ and $P(X_n = 0) = \frac{1}{n}$. Does X_n converge to some random variable X in distribution? [Note: This example shows that even if a sequence of distribution functions converges, it may not converge to a distribution function.]

3. Let $X_n \to X$ in rth mean, for some $r > 0$. Show that $X_n \to X$ in probability.

4. (a) Show that $|E(X)| \leq E|X|$.
   (b) Show that if $X_n \to X$ in 1st mean, then $E(X_n) \to E(X)$.
   (c) Give an example of a sequence of random variables $\{X_n\}$ such that $E(X_n) \to E(X)$, but $X_n \nrightarrow X$ in 1st mean.

5. Let X_n be a sequence of discrete random variables such that $P(X_n = \frac{k}{2^n}) = \frac{1}{2^n}$ for $k = 1, 2, \ldots, 2^n$. Show that $X_n \to X$ in distribution, where $X \sim U(0, 1)$.

6. Let $\{X_n\}$ be a sequence of $i.i.d.$ random variables with finite variance σ^2. Let $S_n^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$. Show that $\{S_n^2\}$ converges to σ^2 almost surely.

7. Let $\{X_n\}$ be a sequence of identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, where $\sigma > 0$. Also assume that $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. Show that $\overline{X}_n \to \mu$ in probability.

8. Let $\{X_n\}$ be a sequence of $i.i.d.$ random variables with mean 0 and variance 1. Find the limiting distribution of
$$Z_n = \sqrt{n}\frac{X_1X_2 + X_3X_4 + \ldots + X_{2n-1}X_{2n}}{X_1^2 + X_2^2 + \ldots + X_n^2}.$$

9. Let $\{X_n\}$ be a sequence of $i.i.d.$ random variables with mean α and variance σ^2, and let $\{Y_n\}$ be a sequence of $i.i.d.$ random variables with mean $\beta\,(\neq 0)$. Find the limiting distribution of $Z_n = \frac{\sqrt{n}(\overline{X}_n - \alpha)}{\overline{Y}_n}$.

10. Let $\{X_n\}$ be a sequence of $i.i.d.$ random variables with mean μ and finite variance σ^2. Show that $\sqrt{n}\frac{\overline{X}_n - \mu}{S_n} \to Z$ in distribution, where $Z \sim N(0, 1)$.

11. Let X_i and Y_i, $i = 1, 2, \ldots$ are independently and identically distributed $U(0, 1)$ random variables. Let $N_n = \#\left\{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\right\}$. Show that $\frac{4N_n}{n}$ converges to π with probability one.

12. Let X_i, $i = 1, 2, \ldots, 50$, be independent random variables each being uniformly distributed over the interval $(0, 1)$. Find the approximate value of $P\left(\sum_{i=1}^{50} X_i > 30\right)$. You may use the fact that $\Phi(\sqrt{6}) = 0.9928$. Ans: 0.0071.

13. Show that
$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}.$$