DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
MA 321 (Optimization)
Transportation Example

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1. Consider the following transportation problem (P) with c_{ij}'s, a_i's (40,30,30) and d_j's (30,50,20) as given below:

2	5	1	40
1	4	5	30
1	5	3	30
30	50	20	

Check whether the initial basic feasible solution $\mathbf{x}_0$ with basic cells
$\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$, is optimal for (P) (by taking $v_2 = 0$, where v_2 is the dual variable corresponding to the second demand constraint).
Also find the optimal solution.
Solution: The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by $x_{11} = 30, x_{12} = 10, x_{22} = 30, x_{23} = 0, x_{32} = 10$ as the values of the basic variables (note that it is a degenerate BFS) and the all the other variables (nonbasic variables) $x_{13}, x_{21}, x_{31}, x_{33}$ take the value 0.
To check the optimality of the above BFS we need to calculate the $c_{ij} - u_i - v_j$ values for all the nonbasic cells (for the basic cells $c_{ij} - u_i - v_j$ values are equal to 0) by taking any one of the u_i's or v_j's equal to 0 and solving for the other u_i's and v_j's from the equations $c_{ij} - u_i - v_j = 0$ for the basic cells. If all the $c_{ij} - u_i - v_j$ values are nonnegative then the above table is optimal.
The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ (you can take any one of u_i, v_j values to be equal to 0 whichever one you like, you can check that the $c_{ij} - u_i - v_j$ values will be same as the one given below) for easier solvability and the rest of the u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
(Check that $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$) and hence check that
$c_{13} - u_1 - v_3 = 1 - 5 - 1 = -5, c_{21} - u_2 - v_1 = 1 - 4 - (-3) = 0$, $c_{31} - u_3 - v_1 = 1 - 5 - (-3) = -1$, $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$.

0	0	-5	40
0	0	0	30
-1	0	-3	30
30	50	20	

Since all the $c_{ij} - u_i - v_j$ values are not nonnegative, the above table is not optimal. The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1,3)$, so this will be the entering variable in the basis of the new basic feasible solution.

Consider the unique θ- loop in $\mathcal{B} \cup (1,3)$ which is given by $\{(1,2),(2,2),(2,3),(1,3)\}$. Since $(1,3)$ is the entering variable, so if we give $+\theta$ allocation to cell $(1,3)$ (or value of $x_{13} = +\theta$) then $x_{12} = 10 - \theta$, $x_{22} = 30 + \theta$, $x_{23} = 0 - \theta$ (since the new BFS must satisfy all the supply and the demand constraints so the total amount of allocation in row i must be equal to a_i and the total allocation in column j must be equal to d_j), so the maximum value of θ is equal to 0 since x_{23} has to be nonnegative in the new BFS. So we enter x_{13} in the basis of the new BFS (which is essentially the same BFS but with a different basis) but it takes the value 0 and x_{23} leaves the basis.

So now $\mathcal{B} = \{(1,1),(1,2),(1,3),(2,2),(3,2)\}$ and the values of the basic variables are same as that in the previous BFS.

Now if we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:

$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$

$c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$

$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$

$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$

$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.

Check that $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = 0$ and hence check that $c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0, c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5$, $c_{31} - u_3 - v_1 = 1 - 0 - 2 = -1$, $c_{33} - u_3 - v_3 = 3 - 0 - 1 = 2$.

The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with $\mathcal{B} = \{(1,1),(1,2),(1,3),(2,2),(3,2)\}$.

0	0	0	40
0	0	5	30
-1	0	2	30
30	50	20	

So now the entering variable for the new BFS is x_{31}, since $c_{31} - u_3 - v_1 < 0$ and all other $c_{ij} - u_i - v_j \geq 0$.

Consider the unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(1,1),(1,2)\}$. Since $(3,1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3,1)$ (or value of $x_{31} = +\theta$) then $x_{32} = 10 - \theta$, $x_{11} = 30 - \theta$, $x_{12} = 10 + \theta$, so the maximum value of

θ is equal to 10 since x_{32} has to be nonnegative in the new BFS.
Hence the entering variable for the new BFS is x_{31} and $x_{31} = 10$ and x_{32} is the leaving variable and the new BFS is given by $x_{11} = 20, x_{12} = 20, x_{13} = 0, x_{22} = 30, x_{31} = 10$ (the basic variables) and all the other (nonbasic) variables taking the value 0.
The basic set of cells is given by $\mathcal{B} = \{(1,1),(1,2),(1,3),(2,2),(3,1)\}$.
To check for optimality we need to again calculate the $c_{ij} - u_i - v_j$ values for this BFS. Now if we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
$c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$.
Check that $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1$ and hence check that $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0$, $c_{31} - u_3 - v_1 = 1 - (-1) - 2 = 0$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.
Since all the $c_{ij} - u_i - v_j$ values are nonnegative the above BFS is optimal and the optimal value is given by:
$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{22}x_{22} + c_{31}x_{31} = 2 \times 20 + 5 \times 20 + 1 \times 0 + 4 \times 30 + 1 \times 10 = 270$.