Indian Institute of Technology Guwahati
Probability Theory and Stochastic Processes (MA225)
Problem Set 05

1. Let (X, Y) be a continuous random vector with JPDF $f(\cdot, \cdot)$. Show that X and Y are independent if and only if $f(x, y) = g(x)h(y)$ for all $(x, y) \in \mathbb{R}^2$.

2. Let X_1 and X_2 be independent $N(0, 1)$ random variables and let $Y = X_1 + X_2$, $Z = X_1^2 + X_2^2$.

   (a) Show that the joint MFG of (Y, Z) is $M_{Y,Z}(t_1, t_2) = (1 - 2t_2)^{-1} e^{\frac{t_1^2}{1-2t_2}}$ if $t_1 \in \mathbb{R}$ and $t_2 < \frac{1}{2}$.

   (b) Using (a), find $Corr(Y, Z)$.

3. Let (X, Y) be uniform over the interior of the triangle with vertices $(0, 0), (2, 0)$ and $(1, 2)$. Find $P(X \leq 1, Y \leq 1)$.

4. If X_1 and X_2 are independent random variables each having PDF $2xe^{-x^2}(0 < x < \infty)$, then find the PDF of the random variable $\sqrt{X_1^2 + X_2^2}$.

5. Two numbers are independently chosen at random between 0 and 1. What is the probability that their product is less than a constant $k(0 < k < 1)$?

6. A vertical board is ruled with horizontal parallel lines at constant distance b apart. A needle of length $a(< b)$ is thrown at random on the board. Find the probability that it will intersect one of the lines.

7. Let X_1, X_2, X_3 have the joint PDF

$$f(x_1, x_2, x_3) = \begin{cases} 48x_1x_2x_3 & \text{if } 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the marginal distributions of $Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_2}{X_3}$, and $Y_3 = X_3$.

8. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2, and Y_3.

9. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2, and Y_3.

10. Let X and Y be two independent random variables having $Gamma(\alpha_1, \beta)$ and $Gamma(\alpha_2, \beta)$ distributions, respectively. Show that $\frac{X}{X+Y}$ is distributed as $Beta(\alpha_1, \alpha_2)$.

11. Let X_1, X_2, X_3 be i.i.d. with common MGF $M(t) = ((3/4) + (1/4)e^t)^2$, for all $t \in \mathbb{R}$.

    (a) Determine the probabilities $P(X_1 = k)$ for $k \in \mathbb{R}$.

    (b) Find the MGF of $Y = X_1 + X_2 + X_3$, and then determine the probability $P(Y = k)$ for $k \in \mathbb{R}$.

12. Let X be a random variable of continuous type. The integral part, Y, of X has a $P(\lambda)$ distribution and the fractional part, Z, has a $U(0, 1)$ distribution. Find the CDF of X, assuming that Y and Z are independent. Using the CDF find the PDF of X.

13. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U(0, 1)$ random variables. Define $X_{(n)} = \max\{X_1, \ldots, X_n\}$ and $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Find the joint and marginal distributions of $X_{(1)}$ and $X_{(n)}$.

14. Let X_1 and X_2 be i.i.d. $P(\lambda)$ random variables. Find the PMF of $X_{(2)} = \max\{X_1, X_2\}$.