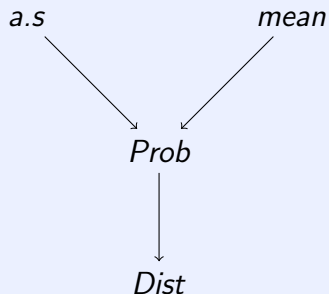


PROBABILITY THEORY AND RANDOM PROCESSES (MA225)

LECTURE SLIDES

Lecture 19 (September 25, 2019)

Relation between Modes of Convergence



Counter Examples

Example 1: Let $\mathcal{S} = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$ and P be the uniform measure. Define $X_n = n1_{[0, \frac{1}{n}]}$. X_n converges to 0 in probability and almost surely but not in r th mean for any $r \geq 1$.

Example 2: Let $X_1 = 1_{[0, 1/2]}$, $X_2 = 1_{[1/2, 1]}$
 $X_3 = 1_{[0, 1/4]}$, $X_4 = 1_{[1/4, 1/2]}$, $X_5 = 1_{[1/2, 3/4]}$, $X_6 = 1_{[3/4, 1]}$...
Then X_n converges in r th mean and in probability but not almost surely.

Example 3: Let X be a $N(0, 1)$ RV defined on some probability space $(\mathcal{S}, \mathcal{F}, P)$. Define $X_n = X$ for all n . Then X_n converges in distribution to $-X$ but not in probability.

Theorem: Suppose $\{X_n\}$ is a sequence of RVs defined on a single probability space and X_n converges in distribution to some constant c , then X_n also converges in probability to c .

Theorem: Let $\{X_n\}$ be a sequence of random variables with moment generating functions $M_n(t)$. Let X be a random variable with moment generating function $M(t)$. If $M_n(t) \rightarrow M(t)$ for all t in an open interval containing zero, then X_n converges to X in distribution.

Example 4: Let $X_n \sim \text{Bin}(n, p_n)$, where $p_n \rightarrow 0$ and $np_n = \lambda(> 0)$. Let $X \sim \text{Poi}(\lambda)$. Then X_n converges to X in distribution.



Theorem: Let $\{X_n\}$ be a sequence of DRV's with PMF $f_n(\cdot)$. Let X be a DRV with PMF $f(\cdot)$. If $f_n(x) \rightarrow f(x)$ for all x , then X_n converges to X in distribution.

Example 5: Prove the claim of the previous example using the above Theorem.

Theorem: Let $\{X_n\}$ be a sequence of CRV's with PDF $f_n(\cdot)$. Let X be a CRV with PDF $f(\cdot)$. If $f_n(x) \rightarrow f(x)$ for all x , then X_n converges to X in distribution.

Example 6: Let $X_n \sim U(0, 1 + 1/n)$ and $X \sim U(0, 1)$. Then X_n converges to X in distribution.