Practise Problems-7

MA 321 (Optimization)

Notations: $S = Fea(P) = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ \text{for } i = 1, 2, .., m\}$,
$D_{x^*} = \{d \in \mathbb{R}^n : g_i(x^* + td) \leq 0,\ \text{for all } i = 1, 2, .., m, \text{and for all } 0 \leq t \leq c_d, c_d > 0\}$,
$F_0 = \{d \in \mathbb{R}^n : \nabla f(x^*)d < 0\}$, $I = \{i \in \{1, \ldots, m\} : g_i(x^*) = 0\}$, $G_0 = \{d \in \mathbb{R}^n : \nabla g_i(x^*)d < 0 \text{ for all } i \in I\}$.

1. Minimize $(x_1 - 2)^2 + (x_2 - 3)^2$
   subject to
   $x_1^2 + x_2^2 \leq 5.$
   $2x_1 + x_2 \leq 4.$
   $-x_1 \leq 0$
   $-x_2 \leq 0.$

   (a) Check whether all the g_i's and f are convex functions.

   (b) Check that $x^* = [1, 2]^T$ is a KKT point (done in class).

   (c) Is x^* a global minimum of f over this feasible region?

   (d) Does there exist an $x^* \in Fea(P)$ such that $G_0 = D_{x^*}$?

   (e) Does there exist an $x^* \in Fea(P)$ such that $G_0 \neq D_{x^*}$?

2. Consider the problem of minimizing
   $f(x_1, x_2) = -x_1x_2 + x_1^2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}$ over R^2.

   (a) Write the first order necessary optimality condition for this problem. Is this condition also sufficient for optimality?

   (b) Is $(0, 0)$ a local minimum point for this problem? If not find a direction d along which the function f will decrease.

3. Consider the following problem:
   Minimize $4x_1^2 - x_2^2 + 8x_1x_2$
   subject to
   $2x_1 + x_1^2 - x_2 \geq 0$
   $x_1 \geq 0,\ x_2 \geq 0.$

   (a) Does $(0, 0)$ satisfy the first and the second order necessary conditions for a local minimum?

   (b) Find all points x^* in the feasible region at which $G_0 = \phi$.

4. Consider the following problem:
   Minimize $-x_1^2 - 4x_1x_2 - x_2^2$
   subject to
   $x_2^2 + x_1^2 = 1.$

   (a) If possible find a FJ point which is not a KKT point.

   (b) If possible find a KKT point which is not an optimal point.

(c) Are the first and the second order necessary conditions (that is for any feasible direction d at x^*, $\nabla f(x^*)d \geq 0$, and if $\nabla f(x^*)d = 0$, then $d^T\nabla^2 f(x^*)d \geq 0$) for a local minimum satisfied at the KKT point/s?

(d) If the objective function is changed to $2x_1^2 - x_1x_2 + x_2^2 - x_2$ and if $S = Fea(P) = \{(x_1, x_2) : x_2^2 + x_1^2 \leq 1\}$ then find all optimum solutions to this problem.

5. For a nonlinear programming problem (P) of the form,

$$\begin{aligned} &\text{Minimize } f(\mathbf{x}) \\ &\text{subject to } g_i(\mathbf{x}) \leq \mathbf{0}, \text{ for } i = 1, \ldots, m, \quad \mathbf{x} \in \mathbb{R}^n, \end{aligned}$$

where all the g_i's and f are continuously differentiable throughout $\mathbb{R}^n$, check the correctness of the following statements with proper justification.

(a) If $x^* \in Fea(P)$ is a KKT point of the above problem then $-\nabla f(x^*)$ lies in the cone generated by $\nabla g_i(x^*), i \in I$, where I gives the indices of the binding constraints (given by g_i's) at x^*.

(b) If $g_2 = -g_3$ then all points $x \in Fea(P)$ are FJ (Fritz John) points.

(c) If $\nabla f(x^*) = 0$ for some $x^* \in Fea(P)$ then x^* is a KKT point.

(d) If x^* is an FJ point and there is a solution to the FJ conditions at x^* with $u_0 = 0$ (u_0 is the coefficient of $\nabla f(x^*)$ in the FJ conditions), then x^* is not a KKT point.

(e) If x^* is a KKT point then it is also an FJ point.

(f) If x^* is an FJ point and $\nabla g_i(x^*)$'s are LD, then $G_0 = \phi$ (or in other words there exists a solution to the FJ conditions with $u_0 = 0$).

(g) If $\nabla g_i(x^*)$'s are LI, then $G_0 \neq \phi$.

(h) If $G_0 = \phi$ then $\nabla g_i(x^*)$'s are LD.

(i) If x^* is not an interior point of the feasible region S of (P) and $F_0 = \phi$ then x^* is a local minimizer.

(j) If all the g_i's and f are convex functions and x^* is such that $F_0 \cap G_0 = \phi$, then x^* is a global minimum of f in S.

(k) If x^* is a local minima of (P) with $F_0 \neq \phi$ but $G_0 = \phi$ then x^* is not a KKT point.

(l) For all $x^* \in S$, $G_0 = D_{x^*}$.

(m) There exists a (P) such that $G_0 \neq D_{x^*}$, for all $x^* \in S$.

(n) There exists a (P) with $Fea(P) \neq \phi$ such that $G_0 = \phi$ for all $x^* \in S$.

(o) There exists a (P) with $Fea(P) \neq \phi$ such that $D_{x^*} = \phi$ for all $x^* \in S$.

6. Give examples of nonconstant functions on $\mathbb{R}^n$ which are both convex and concave and those which are neither convex nor concave.

7. For a linear programming problem (P) of the form, Minimize $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, find the KKT conditions.

8. Consider the linear programming problem.
Minimize $2x_1 - 3x_2$

subject to $x_1 + 2x_2 \leq 3$
$2x_1 + 3x_2 \leq 5$
$x_1 \geq 0, x_2 \geq 0.$

Find the KKT conditions for this problem at a local minimum point of this problem. Solve the KKT conditions for u_i's. Hence find an optimal solution of the dual.