**Notations:**
$\mathbf{x}, \mathbf{d}, \mathbf{b}$, etc, that is characters in boldface represent (column) vectors.
$\tilde{\mathbf{a}}_i$, the $i$th column of $A$.

## Simplex Algorithm

From now on we will consider linear programming problems of the type (P*):

Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $\qquad (*)$
where $rank(A) = m$.
Let us now denote the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ by $Fea(LPP)$.
Note that if we are given a problem of the type:
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
we add some variables and we can convert the system of constraints to
$A_{m\times n}\mathbf{x} + \mathbf{s}_{m\times 1} = [A : I]\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{s} \geq \mathbf{0}$.
So it is now of the form of problem (P*).
If suppose we are given a problem of the type (P**):
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{k\times n}\mathbf{x} = \mathbf{b}_{k\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
where $k > rank(A) = m$.
Let (P**) have atleast one feasible solution and WLOG let $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ be a set of $m$ LI rows of $A$ ( $rank(A) = m$ ).
Let $\mathbf{x}_0 \in \{\mathbf{x} \in \mathbb{R}^n : A_{k\times n}\mathbf{x} = \mathbf{b}_{k\times 1}, \quad \mathbf{x} \geq \mathbf{0}\}$.
Then for all $i = 1, \ldots, k$,
$\mathbf{a}_i^T = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T$ for some real $u_{ji}$'s, $i = 1, \ldots, k$, $j = 1, \ldots, m$, $\qquad (**)$
hence $\mathbf{a}_i^T\mathbf{x}_0 = b_i = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T\mathbf{x}_0 = \sum_{j=1}^m u_{ji}b_j$, which implies $b_i = \sum_{j=1}^m u_{ji}b_j$, for all $i = 1, \ldots, k$. $\qquad (***)$
Hence from (**) and (***) it follows that for any $\mathbf{x}$
$\mathbf{a}_i^T\mathbf{x} = b_i$, for $i = 1, \ldots, m \iff \mathbf{a}_i^T\mathbf{x} = \sum_{j=1}^m u_{ji}\mathbf{a}_j^T\mathbf{x} = \sum_{j=1}^m u_{ji}b_j = b_i$, for all $i = 1, \ldots, k$.
Hence if the system $A_{k\times n}\mathbf{x} = \mathbf{b}_{m\times 1}$, $\mathbf{x} \geq \mathbf{0}$ is consistent( that is it has at least one solution), then we can throw away $k - m$ rows of $A$ and the corresponding components of $\mathbf{b}$ in (P**) to get an equivalent ( having the same set of solutions ) system of equations of the form (*).
Hence an LPP of the form (P**) can again be converted to a problem of the type (P*), by throwing away some of the constraints, such that the feasible region of the changed LPP remains same.
**Note:** Here it is important to point out that if we have a problem (P) of the form:
Max or Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{k\times n}\mathbf{x} \leq \mathbf{b}_{k\times 1}$, $\mathbf{x} \geq \mathbf{0}$,
where $k > rank(A) = m$.
Then if we throw away constraints ( corresponding to LD rows of $A$) from the above set of constraints, then we may get a different feasible region.

An $\mathbf{x} \in Fea(LPP)$ is called a **basic feasible solution** (**BFS**) of the LPP if the columns of the matrix $A$ corresponding to the nonzero components of $\mathbf{x}$ are LI.
An $\mathbf{x}$ satisfying the system $A\mathbf{x} = \mathbf{b}$, and the condition that the columns corresponding to the nonzero components are LI, is called a **basic solution** of the LPP. So a basic solution may not be a nonnegative vector, hence need not be a feasible solution of the LPP.
So a basic feasible solution of a LPP of the form (1), can have at most $m$ strictly positive components (since $rank(A) = m$).

A basic feasible solution is called a **non degenerate** BFS if it has exactly $m$ positive components, otherwise it is said to be a **degenerate** BFS.
If $\mathbf{x}$ is a **non degenerate** BFS, then the columns of $A$ corresponding the nonzero (positive) components of $\mathbf{x}$ form a basis of $\mathbb{R}^m$. WLOG, let the positive components of $\mathbf{x}$ be $x_1, \ldots, x_m$, then $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$,

forms a basis of $\mathbb{R}^m$, where $\tilde{\mathbf{a}}_i$ is the $i$th column of $A$.
The $m \times m$ matrix, $B_{m\times m} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}}_m]$ formed with columns $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$, of $A$ is called the **basis matrix** corresponding to $\mathbf{x}$.
The variables $x_1, \ldots, x_m$ are called the **basic variables**, and $x_{m+1} = x_{m+2} = \ldots = x_n = 0$, are called the **non basic variables** of the BFS $\mathbf{x}$.

If $\mathbf{x}$ is a **degenerate** BFS, then consider the columns of $A$ corresponding to the nonzero components of $\mathbf{x}$. WLOG let them be $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, $k < m$.
Then add $(m-k)$ LI columns of $A$, such that these $(m-k)$ columns of $A$ together with $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_k$, form a set of $m$ LI vectors from the columns of $A$ (you can always do that), hence a basis of $\mathbb{R}^m$.
Let as before the matrix $B_{m\times m}$ formed with these $m$ columns, (WLOG let it be $\tilde{\mathbf{a}}_i, i = 1, \ldots, m$) be called a basis matrix corresponding to $\mathbf{x}$.
The components $x_1, \ldots, x_m$ of $\mathbf{x}$ are called as before, **basic variables** corresponding to $\mathbf{x}$ and the basis matrix $B$. The components $x_{m+1} = \ldots = x_n = 0$ are the **nonbasic variables**.

Note that depending on the choice of the rest of the $(m-k)$ columns which are added, the same $\mathbf{x}$ will correspond to different basis matrices $B$, and hence will have different basic and nonbasic variables.
Suppose if $\mathbf{x}$ is a BFS such that $x_{m+1} = \ldots = x_n = 0$, then since $\mathbf{x}$ is a feasible solution of the LPP, $A\mathbf{x} = \mathbf{b}$, hence we get $\sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$, which implies $B\mathbf{x}_B = \mathbf{b}$ or $\mathbf{x}_B = B^{-1}\mathbf{b}$,
where $\mathbf{x}_B$ are the components of $\mathbf{x}$ corresponding to the basic variables.
Hence a BFS $\mathbf{x}$ is of the form, $\mathbf{x} = \begin{bmatrix} B^{-1}\mathbf{b}_{m\times 1} \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix}$.

Hence in the context of the diet problem, that is if we consider the above problem to be the diet problem ( with $\geq$ inequalities changed to equalities in the original problem), then $\mathbf{x}$ gives the quantities of the various food products $F_j$, $j = 1, \ldots, n$ which are to be included in the diet. The food products $F_1, \ldots, F_m$, which correspond to the basic variables $x_1, x_2, \ldots, x_m$ of $\mathbf{x}$, are the ones which are included in the diet in the quantities $x_j$, the other food products corresponding to the nonbasic variables (they have zero values) are not to be consumed, hence not included in the diet.

**Theorem 1:** Every BFS of the LPP (P*) is an extreme point of $Fea(LPP)$ and also every extreme point of $Fea(LPP)$ is a basic feasible solution of the (LPP), (P*).
The proof is given for your interest but you can leave this proof if you find it difficult.

**Proof: To show every BFS of (P*) is an extreme point of Fea(LPP).**
Let $\mathbf{x}$ be a BFS of the LPP and WLOG let $x_1, \ldots, x_m$ be the basic variables corresponding to $\mathbf{x}$, then $x_{m+1} = \ldots = x_n = 0$ and $A = [B\ N]$ where $B$ is **the** ( or **a**) basis matrix corresponding to $\mathbf{x}$. Hence $\mathbf{x}$ lies on $n$ defining hyperplanes of $Fea(LPP)$ given by $A\mathbf{x} = \mathbf{b}$ and $x_{m+1} = \ldots = x_n = 0$.
We have to show that these hyperplanes are LI.
Let $D$ be the matrix formed by taking the normals of these $n$ hyperplanes as rows, that is
$D = \begin{bmatrix} B_{m\times m} & N \\ \mathbf{0} & I_{n-m} \end{bmatrix}$.
Since $rank(B) = m$ the rows of $B$ are LI.
Let $\alpha_1[\mathbf{b}_1^T : \mathbf{n}_1^T] + \ldots + \alpha_m[\mathbf{b}_m^T : \mathbf{n}_m^T] + \alpha_{m+1}[\mathbf{0} : \mathbf{e}_1^T] + \ldots + \alpha_n[\mathbf{0} : \mathbf{e}_{n-m}^T] = [\mathbf{0}_{1\times m} : \mathbf{0}_{1\times(n-m)}]$ (**),
where $\mathbf{b}_i^T$ is the $i$th row of $B$, $\mathbf{n}_i^T$ is the $i$th row of $N$ and $[\mathbf{b}_i^T : \mathbf{n}_i^T]$ denotes the $i$th row of $A$.
Since the set $\{\mathbf{b}_1^T, \ldots, \mathbf{b}_m^T\}$ is LI (rows of $B$), hence $\alpha_1 = \ldots = \alpha_m = 0$, which implies $\alpha_{m+1} = \ldots = \alpha_n = 0$.
Hence $\mathbf{x}$ is an extreme point fo $Fea(LPP)$.

**To show every extreme point of $Fea(LPP)$ is a BFS of LPP (P*).**
Assume that $\mathbf{x}$ is an extreme point of $Fea(LPP)$ (hence it lies on $n$ LI hyperplanes defining $Fea(LPP)$), to show that it is a basic feasible solution of the LPP of the form (P*).
Note that the $n$ LI hyperplanes on which $\mathbf{x}$ lies can be taken to be $\mathbf{a}_i^T\mathbf{x} = b_i$ for $i = 1, \ldots, m$, where $\mathbf{a}_i^T$ is the $i$ th row of $A$ and $x_{i_1} = x_{i_2} = \ldots = x_{i_{n-m}} = 0$ for some $i_1, \ldots, i_{n-m} \in \{1, 2, \ldots, n\}$.
Let us assume WLOG that $i_1 = m+1, \ldots, i_{n-m} = n$ (otherwise renumber the variables so that this is obtained), then $\mathbf{x}$ satisfies the system of equations

$D\mathbf{x} = \begin{bmatrix} B_{m\times m} & N \\ \mathbf{0} & I_{n-m} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, where $rank(D) = n$.
If we can show that $rank(B) = m$ then we are done.
If not, that is if the rows of $B$ are LD then there exists a $k \in \{1, \ldots, m\}$ such that $\mathbf{b}_k^T$ can be written as a linear combination of the other rows of $B$, that is there exists $\alpha_1, \ldots, \alpha_m$, such that
$\mathbf{b}_k^T = \alpha_1 \mathbf{b}_1^T + \ldots + \alpha_m \mathbf{b}_m^T$ $(\alpha_k = 0)$.
Hence $[\mathbf{b}_k^T : \mathbf{n}_k^T] - \alpha_1[\mathbf{b}_1^T : \mathbf{n}_1^T] - \ldots - \alpha_m[\mathbf{b}_m^T \mathbf{n}_m^T] = [\mathbf{0}_{1\times m} : \mathbf{n}_k^T - \alpha_1\mathbf{n}_1^T - \ldots - \alpha_m\mathbf{n}_m^T]$.
Note that $\mathbf{n}_k^T - \alpha_1\mathbf{n}_1^T - \ldots - \alpha_m\mathbf{n}_m^T$ is a row vector having $(n-m)$ components, hence it can be written as a linear combination of $\mathbf{e}_1^T, \ldots, \mathbf{e}_{n-m}^T$, that is,
$\mathbf{n}_k^T - \alpha_1\mathbf{n}_1^T - \ldots - \alpha_m\mathbf{n}_m^T = \beta_1\mathbf{e}_1^T + \ldots + \beta_{n-m}\mathbf{e}_{n-m}^T$, for some $\beta_1, \ldots, \beta_{n-m}$.
Hence $[\mathbf{b}_k^T : \mathbf{n}_k^T] - \alpha_1[\mathbf{b}_1^T : \mathbf{n}_1^T] - \ldots - \alpha_m[\mathbf{b}_m^T : \mathbf{n}_m^T] - \beta_1[\mathbf{0} : \mathbf{e}_1^T] - \ldots - \beta_{n-m}[\mathbf{0} : \mathbf{e}_{n-m}^T] = [\mathbf{0}_{1\times m} : \mathbf{0}_{1\times(n-m)}]$, which contradicts that $rank(D) = n$, or rows of $D$ are LI.
Hence rows of $B$ are LI, hence $rank(B) = m$ and $\mathbf{x}$ is a BFS of the LPP (P*) with nonbasic variables $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-m}}$.

**Corollary 1:** If $Fea(LPP) \neq \phi$, then the LPP (P*) has at least one basic feasible solution.

Let us assume that (P*) has atleast one feasible solution and let $\mathbf{x}$ be a BFS ( degenerate or non degenerate) of the LPP (P*).
WLOG let $B = [\tilde{\mathbf{a}_1} \ldots \tilde{\mathbf{a}_m}]$ be a (or the) basis matrix corresponding to $\mathbf{x}$.
Then note that for all $k = 1, 2, \ldots, n$,

$$\tilde{\mathbf{a}_k} = \sum_{i=1}^m u_{ik}\tilde{\mathbf{a}_i} = [\tilde{\mathbf{a}_1} \ldots \tilde{\mathbf{a}_m}] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} \qquad (*)$$

for some real $u_{ik}$'s, $i = 1, \ldots, m$, $k = 1, \ldots, n$,
which implies $\begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = B^{-1}\tilde{\mathbf{a}_k}$ for all $k = 1, 2, \ldots, n$.

In the context of the diet problem the above equations (*), mean that in order to obtain the same amount of nutrient as unit amount of $F_k$, $k = 1, \ldots, n$, one needs to consume
$(u_{1k})$ amount of $F_1 +$ $(u_{2k})$ amount of $F_2 + \ldots + (u_{mk})$ amount of $F_m$.
Hence the value of unit amount of $F_k$, if we include only $F_i, i = 1, \ldots, m$ in the diet (which has to be bought from the market) comes out to be
$u_{1k}c_1 + u_{2k}c_2 + \ldots + u_{mk}c_m$, which we denote by $z_k$.

So $z_k = \sum_{i=1}^m u_{ik}c_i = [c_1, \ldots, c_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_k}$.

Note that the cost of the objective function corresponding to this BFS is given by
$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$,
where, $\mathbf{c}_B^T$ are the components of the vector $\mathbf{c}^T$ which correspond to the basic variables.

Also for all $k = 1, \ldots, m$, note that $z_k = c_k$.

Now the simplex table corresponding to the BFS $\mathbf{x}$ will be given by

| | $c_1 - z_1 = 0$ | $c_2 - z_2 = 0$ | $\ldots$ | $c_m - z_m = 0$ | $\ldots$ | $c_s - z_s$ | $\ldots$ | $c_k - z_k$ | $\ldots$ | $c_n - z_n$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $\ldots$ | $B^{-1}\tilde{\mathbf{a}}_m$ | $\ldots$ | $B^{-1}\tilde{\mathbf{a}}_s$ | $\ldots$ | $B^{-1}\tilde{\mathbf{a}}_k$ | $\ldots$ | $B^{-1}\tilde{\mathbf{a}}_n$ | $B^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_1$ | 1 | 0 | $\ldots$ | 0 | $\ldots$ | $u_{1s}$ | $\ldots$ | $u_{1k}$ | $\ldots$ | $u_{1n}$ | $x_1$ |
| $\tilde{\mathbf{a}}_2$ | 0 | 1 | $\ldots$ | 0 | $\ldots$ | $u_{2s}$ | $\ldots$ | $u_{2k}$ | $\ldots$ | $u_{2n}$ | $x_2$ |
| $\vdots$ | 0 | 0 | $\ldots$ | 0 | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\vdots$ |
| $\tilde{\mathbf{a}}_r$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $u_{rs}$ | $\ldots$ | $u_{rk}$ | $\ldots$ | $u_{rn}$ | $x_r$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\vdots$ |
| $\tilde{\mathbf{a}}_m$ | 0 | 0 | $\ldots$ | 1 | $\ldots$ | $u_{ms}$ | $\ldots$ | $u_{mk}$ | $\ldots$ | $u_{mn}$ | $x_m$ |

**Case 1:** $c_k - z_k < 0$ for atleast one $k$, $k = m+1, \ldots, n$.
So if $c_k - z_k < 0$ for some $k = m+1, \ldots, n$, then the market price of unit amount of $F_k$ given by $c_k$ is less than the value of unit amount of $F_k$ (as obtained by consuming only $F_1, \ldots, F_m$) given by $z_k$, hence it might be sensible (profitable) to include this $F_k$ in the diet.
Let $c_s - z_s = min\{c_k - z_k : c_k - z_k < 0, \quad k = m+1, \ldots, n\}$.
**Simplex algorithm** says that, then include $x_s$ in the diet.
If there exists, $s, l$, such that for both $s, l$,
$c_s - z_s = c_l - z_l = min\{c_k - z_k : c_k - z_k < 0, \quad k = m+1, \ldots, n\}$,
then include any **one** of these food products in the diet.
Let $\mathbf{x}'$ be the new solution to be obtained.
Note that (refer to (*)) $u_{is}$ amount of $F_i$, $i = 1, \ldots, m$, is used in the constitution of unit amount of $F_s$, which is now included in the diet in an amount say $x'_s$. Hence the same corresponding amount of the $F_i$'s, $i = 1, \ldots, m$, given by $u_{is}x'_s$ need not be consumed any more.
Hence the components of $\mathbf{x}'$ are given by
$x'_i = x_i - u_{is}x'_s$ for $i = 1, \ldots, m$,
$x'_s \geq 0$ and
$x'_i = 0$ for $i = m+1, \ldots, n$, $i \neq s$.
Then note that if we want $\mathbf{x}'$ to be a feasible solution of the LPP, then we have to choose $x'_s \geq 0$ such that $x'_i \geq 0$ for all $i = 1, \ldots, m$.
Note that if $u_{is} \leq 0$, for some $i = 1, \ldots, m$, then for all $x'_s \geq 0$, $x'_i \geq 0$ for that $i$.
So in order that $x'_i \geq 0$, for all $i = 1, \ldots, m$,
we have to choose $x'_s \geq 0$, such that $x'_s \leq \frac{x_i}{u_{is}}$ for which $u_{is} > 0$ (if at all there is one such $i$).

**Case 1a:** For some (atleast one) $i = 1, \ldots, m$, $u_{is} > 0$ (where $s$ is as defined in Case 1).
Let $\frac{x_r}{u_{rs}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$.
(Note that there could exist $r, t$ such that for both $r, t \in \{m+1, \ldots, n\}$
$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$).

Here $\frac{x_r}{u_{rs}}$ is called the **minimum ratio**.
Then in order that $\mathbf{x}' \geq \mathbf{0}$, $x'_s \leq \frac{x_r}{u_{rs}}$.
Then it can be easily checked that $A\mathbf{x}' = \mathbf{b}$ or $\mathbf{x}' \in Fea(LPP)$, where $\mathbf{x}'$ is as defined in Case 1.
To see this, note that $A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x'_i + \tilde{\mathbf{a}}_s x'_s$, which implies
$A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is} x'_s + \tilde{\mathbf{a}}_s x'_s = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i + x'_s(\tilde{\mathbf{a}}_s - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is}) = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$.

Also $\mathbf{c}^T\mathbf{x}' = \sum_{i=1}^m c_i x'_i + c_s x'_s$, which implies
$\mathbf{c}^T\mathbf{x}' = \sum_{i=1}^m c_i x_i - \sum_{i=1}^m c_i u_{is} x'_s + c_s x'_s = \sum_{i=1}^m c_i x_i + x'_s(c_s - \sum_{i=1}^m c_i u_{is}) = \mathbf{c}^T\mathbf{x} + x'_s(c_s - z_s) \leq \mathbf{c}^T\mathbf{x}$.
Note that $\mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x}$ if $x'_s > 0$, which is possible only if the **minimum ratio** $\frac{x_r}{u_{rs}} > 0$.
In order to reduce cost (or the value of the objective function) as much as possible, $x'_s$ is given its maximum possible value, which is equal to $\frac{x_r}{u_{rs}}$.
Then note that $x'_r = x_r - u_{rs}\frac{x_r}{u_{rs}} = 0$.
The variable $x_s$ is called the **entering variable** for the new basis (choose any one if there are more than one choice for the **entering variable**), and the variable $x_r$ is called a **leaving variable**.
If there exists $r, t$ such that for both $r, t \in \{m+1, \ldots, n\}$
$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$,
then take any **one** of $r, t$ as the **leaving variable**.

So $\mathbf{x}' \in Fea(LPP)$ again has at most $m$ nonzero components, and we can easily check that $\mathbf{x}'$ is a basic feasible solution of the LPP.
To prove this we need to show that the set of columns $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}, \tilde{\mathbf{a}_s}\}$ of $A$ is LI and hence forms a basis of $\mathbb{R}^m$.
If suppose not, then since the collection $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}\}$ is LI, it implies that $\tilde{\mathbf{a}_s}$ can be written as a linear combination of the $(m-1)$ columns, $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}\}$, but this would imply that $u_{rs} = 0$, which is a contradiction.
Hence $\mathbf{x}'$ is an improved (with respect to cost or the value of the objective function) BFS as compared to the BFS $\mathbf{x}$.
Let us denote the new basis matrix corresponding to $\mathbf{x}'$ as $B' = [\tilde{\mathbf{a}_1} \ldots \tilde{\mathbf{a}_{r-1}} \tilde{\mathbf{a}_{r+1}} \ldots \tilde{\mathbf{a}_m} \tilde{\mathbf{a}_s}]$.

The simplex table corresponding to the BFS $\mathbf{x}'$ will be obtained again by expressing each of the vectors $\tilde{\mathbf{a}}_i$, $i = 1, \ldots, n$, and $\mathbf{b}$ in terms of the new set of basis vectors $\{\tilde{\mathbf{a}_1}, \ldots, \tilde{\mathbf{a}_{r-1}}, \tilde{\mathbf{a}_{r+1}}, \ldots, \tilde{\mathbf{a}_m}, \tilde{\mathbf{a}_s}\}$.
Since $\tilde{\mathbf{a}_s} = \sum_{i=1, i\neq r}^{m} u_{is} \tilde{\mathbf{a}}_i + u_{rs} \tilde{\mathbf{a}_r}$,
so $\tilde{\mathbf{a}_r} = \frac{\tilde{\mathbf{a}_s}}{u_{rs}} - \sum_{i=1, i\neq r}^{m} \frac{u_{is}}{u_{rs}} \tilde{\mathbf{a}}_i$. $\qquad (*)$
Also since for any $k = 1, \ldots, n$,
$\tilde{\mathbf{a}_k} = \sum_{i=1}^{m} u_{ik} \tilde{\mathbf{a}}_i = \sum_{i=1, i\neq r}^{m} u_{ik} \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}_r} u_{rk}$. $\qquad (**)$
By substituting the expression for $\tilde{\mathbf{a}_r}$ given in $(*)$ in the equation $(**)$, we get
$\tilde{\mathbf{a}_k} = \sum_{i=1, i\neq r}^{m} u_{ik} \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}_r} u_{rk} = \sum_{i=1, i\neq r}^{m} (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}_s} \frac{u_{rk}}{u_{rs}}$.

Let us denote the new coefficients corresponding to $\tilde{\mathbf{a}}_i$, in the above expression as $u'_{ik}$, for $i = 1, \ldots, r-1, r+1, m, s$, and $k = 1, \ldots, n$.
Hence for all $k = 1, \ldots, n$,
$u'_{ik} = u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}$ for $i = 1, \ldots, r-1, r+1, m$
and $u'_{sk} = \frac{u_{rk}}{u_{rs}}$.

Note that the coefficients corresponding to $\mathbf{b}$, when $\mathbf{b}$ is expressed as a linear combination of the new basis of $\mathbb{R}^m$, ( which was already obtained and the philosophy behind explained ) can be recalculated as follows:
Recall that $\mathbf{b} = \sum_{i=1, i\neq r}^{m} \tilde{\mathbf{a}}_i x_i + \tilde{\mathbf{a}_r} x_r$.
Now again substituting the expression for $\tilde{\mathbf{a}}_i$ given in $(*)$ in the above equation we get
$\mathbf{b} = \sum_{i=1, i\neq r}^{m} \tilde{\mathbf{a}}_i x_i + x_r (\frac{\tilde{\mathbf{a}_s}}{u_{rs}} - \sum_{i=1, i\neq r}^{m} \frac{u_{is}}{u_{rs}} \tilde{\mathbf{a}}_i)$.
On simplifying the above expression we get
$\mathbf{b} = \sum_{i=1, i\neq r}^{m} (x_i - \frac{u_{is}}{u_{rs}} x_r) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}_s} (\frac{x_r}{u_{rs}})$.
Hence Let us denote the new values of $z_k$ now corresponding to $\mathbf{x}'$, by $z'_k$. Then we get
$z'_k = \sum_{i=1, i\neq r}^{m} u'_{ik} c_i + u'_{sk} c_s$
$= \sum_{i=1, i\neq r}^{m} (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}) c_i + \frac{u_{rk}}{u_{rs}} c_s = \sum_{i=1, i\neq r}^{m} u_{ik} c_i - \sum_{i=1, i\neq r}^{m} (\frac{u_{is}}{u_{rs}} u_{rk}) c_i + \frac{u_{rk}}{u_{rs}} c_s$
$= (\sum_{i=1}^{m} u_{ik} c_i - u_{rk} c_r) - \frac{u_{rk}}{u_{rs}} (\sum_{i=1}^{m} u_{is} c_i - u_{rs} c_r) + \frac{u_{rk}}{u_{rs}} c_s$
$= z_k - u_{rk} c_r - \frac{u_{rk}}{u_{rs}} z_s - \frac{u_{rk}}{u_{rs}} (-u_{rs} c_r) + \frac{u_{rk}}{u_{rs}} c_s = z_k + \frac{(c_s - z_s)}{u_{rs}} u_{rk}$.

From this we get the new values of $c_k - z'_k$ as
$c_k - z'_k = (c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$.

Hence the simplex table corresponding to the new BFS $\mathbf{x}'$ is given by

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $c_1 - z_1'$ | $\dots$ | $c_m - z_m'$ | $\dots$ | $c_s - z_s'$ | $\dots$ | $c_k - z_k'$ | $\dots$ | $c_n - z_n'$ | |
| | $0$ | $\dots$ | $0$ | $\dots$ | $c_s - z_s - \frac{(c_s - z_s)}{u_{rs}} u_{rs} = 0$ | $\dots$ | $(c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$ | $\dots$ | ... | |
| | $B'^{-1}\tilde{\mathbf{a}}_1$ | $\dots$ | $B'^{-1}\tilde{\mathbf{a}}_m$ | $\dots$ | $B'^{-1}\tilde{\mathbf{a}}_s$ | $\dots$ | $B'^{-1}\tilde{\mathbf{a}}_k$ | $\dots$ | $B'^{-1}\tilde{\mathbf{a}}_n$ | $B'^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_1$ | $1$ | $\dots$ | $0$ | $\dots$ | $u_{1s} - \frac{u_{1s}}{u_{rs}} u_{rs} = 0$ | $\dots$ | $u_{1k} - \frac{u_{1s}}{u_{rs}} u_{rk}$ | $\dots$ | ... | $x_1 - \frac{u_{1s}}{u_{rs}} x_r$ |
| $\tilde{\mathbf{a}}_2$ | $0$ | $\dots$ | $0$ | $\dots$ | $u_{2s} - \frac{u_{2s}}{u_{rs}} u_{rs} = 0$ | $\dots$ | $u_{2k} - \frac{u_{2s}}{u_{rs}} u_{rk}$ | $\dots$ | ... | $x_2 - \frac{u_{2s}}{u_{rs}} x_r$ |
| $\vdots$ | $0$ | $\dots$ | $0$ | $\dots$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ | $\vdots$ | $\vdots$ |
| $\tilde{\mathbf{a}}_r$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ | $\frac{u_{rs}}{u_{rs}} = 1$ | $\dots$ | $\frac{u_{rk}}{u_{rs}}$ | $\dots$ | $\frac{u_{rn}}{u_{rs}}$ | $\frac{x_r}{u_{rs}}$ |
| $\vdots$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ | $\vdots$ | $\vdots$ |
| $\tilde{\mathbf{a}}_m$ | $0$ | $\dots$ | $1$ | $\dots$ | $u_{ms} - \frac{u_{ms}}{u_{rs}} u_{rs} = 0$ | $\dots$ | $u_{mk} - \frac{u_{ms}}{u_{rs}} u_{rk}$ | $\dots$ | $\dots$ | $x_m - \frac{u_{ms}}{u_{rs}} x_r$ |

The entry $u_{rs}$ of the previous table which is made 1 (by dividing) in this table is called the **pivot element**.

**Case 1b:** For all $i = 1, \dots, m$, $u_{is} \leq 0$ (where $s$ is as defined in **Case 1**).
In that case $\mathbf{x}' \geq \mathbf{0}$, for all $x_s' \geq 0$, hence $\mathbf{x}' \in Fea(LPP)$ for all values of $x_s' \geq 0$.
Also since $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + x_s'(c_s - z_s)$, and $(c_s - z_s) < 0$, so given any real number $M$, by taking $x_s'$ sufficiently large we can make $\mathbf{c}^T\mathbf{x}'$ smaller than $M$, hence in this case the LPP has unbounded solution, and does not have an optimal solution.

Also note that $\mathbf{x}' = \mathbf{x} + x_s' \begin{bmatrix} -u_{1s} \\ \vdots \\ -u_{ms} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{x} + x_s' \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$,

where the 1 in the above vector occurs at the $s$ th position.

Let us call this vector $\begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ as $\mathbf{d}$, then since $\mathbf{x} + x_s'\mathbf{d} \in Fea(LPP)$ for all $x_s' \geq 0$, from definition, $\mathbf{d}$ should be a direction of $Fea(LPP)$.

**Result 1:** The set of all directions of $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, rank(A) = m\}$, is given by
$D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq 0, A_{m\times n}\mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}\}$.
**Proof:** Exercise.

**Result 2:** If for some basis matrix $B$ and a column $\tilde{\mathbf{a}}_s$ of $A$, $B^{-1}\tilde{\mathbf{a}}_s \leq \mathbf{0}$ then

$$\mathbf{d} = \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is an extreme direction of $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where the entry 1 in the above vector is at the $s$ th position.

**Proof:** That it is a direction of $S$, follows from the direction of a set.
It can also be checked that

$$A\mathbf{d} = [\tilde{\mathbf{a}}_1 \ldots \tilde{\mathbf{a}_m}|\tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [B|\tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$= -\tilde{\mathbf{a}_s} + \tilde{\mathbf{a}_s} = \mathbf{0}$.
To check that $\mathbf{d}$ is an extreme direction, let there exist directions $\mathbf{d}_1, \mathbf{d}_2$ of $S$ and $\alpha_1, \alpha_2 > 0$ such that $\mathbf{d} = \alpha_1\mathbf{d}_1 + \alpha_2\mathbf{d}_2$.
Since $\alpha_1, \alpha_2 > 0$ and $\mathbf{d}_1, \mathbf{d}_2 \geq 0$

$$\mathbf{d}_1 = \begin{bmatrix} \mathbf{c}_{m\times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

for some $\mathbf{c}_{m\times 1} \geq \mathbf{0}$ and $u \geq 0$, where $u$ is at the $s$ th position.

But form result 1, $A\mathbf{d}_1 = [B : \tilde{\mathbf{a}_{m+1}} \ldots \tilde{\mathbf{a}_s} \ldots \tilde{\mathbf{a}_n}] \begin{bmatrix} \mathbf{c}_{m\times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$

$\Rightarrow B\mathbf{c} = -u\tilde{\mathbf{a}_s}$.
Hence $\mathbf{c} = -uB^{-1}\tilde{\mathbf{a}_s}$ or $\mathbf{d}_1 = u\mathbf{d}$, and $u > 0$.
Similarly we get $\mathbf{d}_2 = v\mathbf{d}$ for some $v > 0$.
Hence $\mathbf{d}$ is an extreme direction.
**Alternatively** note that this $\mathbf{d}$ lies on $(n-1)$ LI hyperplanes (check this) of the $m+n$ hyperplanes defining $D$.
It lies on $m$ LI hyperplanes given by $A\mathbf{d} = \mathbf{0}$ and also on the $(n-m-1)$ LI hyperplanes given by $d_i = 0$ for $i = m+1, \ldots, n$, $i \neq s$.
It is easy to see that the $(n-1)$ hyperplanes are LI.
(Hint: The normals corresponding to these $(n-1)$ hyperplanes written together as a matrix with $(n-1)$ rows look somewhat like

$$\begin{bmatrix} B & \tilde{\mathbf{a}_{m+1}} & \ldots & \tilde{\mathbf{a}_s} & \ldots & \tilde{\mathbf{a}_n} \\ \mathbf{0} & 1 & \ldots & 0 & \ldots & 0 \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 0 \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}.$$

Note that the column below the vector $\tilde{\mathbf{a}_s}$ in the above matrix is the zero column, since $d_s \neq 0$.)

Since the LPP has unbounded solution, so there should exist at least one extreme direction $\mathbf{d}$ of the LPP such that $\mathbf{c}^T\mathbf{d} < 0$.
Check that if $\mathbf{d}$ is as defined above then

$$\mathbf{c}^T\mathbf{d} = [\mathbf{c}_B^T \mid c_{m+1}\dots c_s\dots c_n]\left[\begin{array}{c} -B^{-1}\tilde{\mathbf{a}_s} \\ \hline \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] = -\mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_s} + c_s = c_s - z_s < 0.$$

**Case 2:(Optimality Condition)** $c_k - z_k \geq 0$ for all $k = 1, \dots, n$.

Then to show that $\mathbf{x}$ is optimal for the LPP.
Among the various ways to check this, one way is produce a feasible solution of the dual of this problem say $\mathbf{y}$ such that $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$. Yet another way (which is actually same as the previous ) is to produce a feasible solution $\mathbf{y}$ of the dual and to show that this $\mathbf{y}$ satisfies the complementary slackness property with this BFS $\mathbf{x}$ of the primal.
Since $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$, and since this has to be equal to $\mathbf{b}^T\mathbf{y} = \mathbf{y}^T\mathbf{b}$ for some feasible solution $\mathbf{y}$ of the dual, so we can start by checking if $\mathbf{y}_0^T = \mathbf{c}_B^T B^{-1}$ gives a feasible solution of the dual.

Recall that the problem (LPP) is given by,

Min $\mathbf{c}^T\mathbf{x}$
subject to
$A_{m\times n}\mathbf{x} = \mathbf{b}$,
$\mathbf{x} \geq \mathbf{0}$.

The dual of the above problem is given by,

Max $\mathbf{b}^T\mathbf{y}$
subject to
$A_{n\times m}^T\mathbf{y} \leq \mathbf{c}$.

**Proof :** In order to see this note that,
if $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$ then
$\mathbf{a}_i^T\mathbf{x} \geq b_i$ for $i = 1, \dots, m$ (*)
and $-\mathbf{a}_i^T\mathbf{x} \geq -b_i$ for $i = 1, \dots, m$ (**),
where $\mathbf{a}_i^T$ is the $i$ th row of the matrix $A$.
The above inequalities can be written as
$\begin{bmatrix} A \\ -A \end{bmatrix}\mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$.

Hence the LPP (primal problem) can be written as:
Min $\mathbf{c}^T\mathbf{x}$
subject to
$\begin{bmatrix} A \\ -A \end{bmatrix}\mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$, $\mathbf{x} \geq \mathbf{0}$.
Since each constraint of the primal corresponds to a variable of the dual, let $u_i$ correspond to the $i$th constraint of the type $(*)$, $i = 1, \dots, n$, and let $w_i$ correspond to the $i$th constraint of the type $(**)$, $i = 1, \dots, n$.

Then the dual of the above LPP problem is given as follows:
Given an LPP
Max $\mathbf{b}^T\mathbf{u} - \mathbf{b}^T\mathbf{w}$
subject to

$[A^T \ \ -A^T]\begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = A^T\mathbf{u} - A^T\mathbf{w} \leq \mathbf{c}$
$\mathbf{u} \geq \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$.

Hence if we let $\mathbf{y} = \mathbf{u} - \mathbf{w}$ then the above dual reduces to
Max $\mathbf{b}^T\mathbf{y}$
subject to
$A^T\mathbf{y} \leq \mathbf{c}$,
where $\mathbf{y}$ is unrestricted in sign.

**Case 2 (continued):**
If $c_k - z_k \geq 0$ for all $k = 1, \ldots, n$, then $z_k \leq c_k$ for all $k = 1, \ldots, n$.
Then note that $z_k = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_k} = \mathbf{y}_0^T\tilde{\mathbf{a}_k} \leq c_k$ for all $k = 1, \ldots, n$, or $\mathbf{y}_0$ satisfies the condition:
$A^T\mathbf{y}_0 \leq \mathbf{c}$, which implies that $\mathbf{y}_0$ is a feasible solution of the dual satisfying $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}_0$, hence $\mathbf{x}$ and $\mathbf{y}_0$ are optimal solutions of the LPP (primal problem) and its dual, respectively.

Also note that if $(A^T\mathbf{y}_0)_k = \mathbf{y}_0^T\tilde{\mathbf{a}_k} = \mathbf{c}_B^T B^{-1}\tilde{\mathbf{a}_k} = z_k < c_k$, the $x_k$ has to be a nonbasic variables, then $x_k = 0$. Hence $\mathbf{y}_0 \in Fea(D)$ satisfies the complementary slackness property with this $\mathbf{x}$ as expected.
(The complementary slackness condition for this LPP and its dual reduces to only one condition:
$x_i = 0$, whenever $(A^T\mathbf{y})_k < c_k$.)

**Remark:** Note that if instead the LPP would have been a maximization problem as given below:
Max $\mathbf{c}^T\mathbf{x}$
subject to
$A_{m\times n}\mathbf{x} = \mathbf{b}$,
$\mathbf{x} \geq \mathbf{0}$, $rank(A) = m$,
then **Case 1** and **Case 2** conditions would change accordingly, as given below:

**Case 1:** $c_k - z_k > 0$ for at least one $k$, $k = m+1, \ldots, n$.
And the condition for the **entering variable** becomes,
$s$ th variable will enter the basis if $c_s - z_s = max\{c_k - z_k : c_k - z_k > 0, \quad k = m+1, \ldots, n\}$.

**Case 2:** (**Optimality condition**) $c_k - z_k \leq 0$ for all $k = 1, \ldots, n$.