Numerical solution of generalized Black–Scholes model

A Term Paper Submitted

for the Course

MA473 Computational Finance

by

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ABSTRACT

This term paper is based on survey work that explores a numerical scheme that approximates the option price European call option, governed by the generalized Black–Scholes equation in its degenerate form. The generalized method uses the HODIE scheme in the spacial direction and the two-step backward differentiation formula in the temporal direction. It is learned that the method has second order convergence in space as well as in time. Numerical experiments validate the theoretical results.

Contents

Chapter 1

Introduction

Basic principles underlying the transactions of financial markets are tied to probability and statistics. Only in recent years, spurred by the enormous economical success of financial derivatives, a need for sophisticated computational technology has developed. For example, numerical solution of those partial differential equations that are derived from the Black-Scholes analysis and finite differential methods. Fast and accurate numerical algorithms have become essential tools to price financial derivatives and to manage portfolio risks.

There are different option styles like European, American, Bermudan, etc. These are the financial contracts in which the writer of the option sells the options to the holder and is paid a premium called **option price**. The option which is exercised on the date of expiry of the contract, called the **maturity/expiration date**, is the **European option**.

1.1 The Black–Scholes Equation

Let us assume one risk-free asset and one risky asset constitutes a market which is friction-less and without any arbitrage opportunity.

Definition 1.1.1 (**Black–Scholes equation**)**.** The stock price (S) of the unit risky asset follows the following stochastic differential equation at time τ :

$$dS = (\mu - D)Sd\tau + \sigma SdW \tag{1.1}$$

where μ is the drift, D is the dividend paid, σ is the market volatility and dW is the Wiener process.

Now using Itô's lemma and eliminating the randomness in a complete market, we derive the famous **Black-Scholes equation**.

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = -\frac{\partial C}{\partial \tau}, \quad S > 0, \tau \in (0, T) \tag{1.2}$$

along with the final condition

$$C(S, T) = \max(S - K, 0), \quad S \in [0, \infty) \tag{1.3}$$

where S is the asset price, τ is the current time, r is the risk free interest rate, T is the maturity date of contract and K is the strike price.

Remark 1.1.2. The Black-Scholes equation, in which σ, r and D are constants, can be easily reduced to standard heat equation and further solved to obtain the closed form solution.

$$C(S, \tau) = S\exp(-D(T - \tau))N(d_1) - K\exp(-r(T - \tau))N(d_2) \tag{1.4}$$

where

$$d_1 = \frac{\ln S - \ln K + \left(r - D + \frac{1}{2}\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}}$$
$$d_2 = d_1 - \sigma\sqrt{T-\tau}$$

and the cumulative standard normal distribution function $N(y)$ is defined as

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{1}{2}x^2\right) dx$$

1.2 Problem Formulation

When these parameters σ, r and D are dependent on stock price S and time variable τ, such simplification of the problem is not possible. Also the closed form solution of such generalized problem is not possible.

In this paper we discuss a numerical scheme which efficiently tackles the degeneracy concern of the Black–Scholes equation without performing logarithmic transform of the equation, without truncating the domain so as to exclude the point of degeneracy and without using nonuniform mesh near the point of degeneracy is presented. We implement simultaneous discretization in space and time using High-Order Difference approximation with Identity Expansion (HODIE) scheme explained in chapter 3 in space direction and two-step backward differentiation for temporal discretization.

1.3 Previous Methods

The finite difference methods are the most common numerical approach to approximate the option price as the solution of Black–Scholes model, which avoid probability theory and stochastic methods. The Cubic B-spline method was used to solve generalized Black–Scholes equation after transforming it to non-degenerate uniformly parabolic partial differential equation using logarithmic transform.

Wang developed a first order method after transforming the Dirichlet boundary condition of generalized Black–Scholes equation into homogeneous boundary condition and then writing the partial differential equation in its self-adjoint form.

Recently a penalized nonlinear Black–Scholes equation (with variable parameters) for evaluating European and American options was rewritten in the self-adjoint form and then numerically approximated.

1.4 Organization of the term paper

In this term paper, we have considered Black–Scholes equation governing simple option pricing for European call with variable parameters.
The term report is organized as follows:

In **Chapter 2**, the generalized Black–Scholes partial differential equation with its boundary and terminal conditions is introduced. Some modifications are made so that the problem is in a form which can be dealt numerically.

In **Chapter 3**, the HIODE scheme is explained and explored. We see that applying two-step backward differentiation formula in temporal discretization to achieve second order convergence in time direction and the classical High-Order Difference approximation with Identity Expansion (HODIE) scheme where three nodal auxiliary points are used for identity expansion to achieve third convergence in spatial direction.

Chapter 4 gives the experimental results with two examples, one with constant parameters and one which are dependent on the option price and time in accordance with theoretical claims of presented scheme.

Finally **Chapter 5** concludes the report and discusses some additional merits and concerns of HODIE scheme.

Chapter 2

Generalized Black–Scholes model

Generalized Black–Scholes is a degenerate parabolic partial differential equation. The existence and uniqueness of the classical solution the Black–Scholes final value problem can be ensured after the logarithmic transform of the space variable along with its conversion from final value problem to initial value problem

Definition 2.0.1 (**Generalized Black-Scholes PDE**)**.** The generalized Black-Scholes model for evaluating European call option price $C(S, \tau)$ is

$$-\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2(S,\tau)S^2\frac{\partial^2 C}{\partial S^2}+(r(S,\tau)-D(S,\tau))S\frac{\partial C}{\partial S}-r(S,\tau)C, \quad S > 0, \tau \in (0,T)$$

along with the final condition

$$C(S,T) = \max(S-K,0), \quad S \in [0,\infty)$$

and the boundary conditions

$$C(0,\tau) = 0$$
$$C(S,\tau) \to S, \quad as \quad S \to \infty$$

where S, the asset price is working as the space variable, τ is the time variable, $\sigma(S,\tau)$ is the market volatility, $r(S,\tau)$ is the interest rate and $D(S,\tau)$ is the dividend yield of the asset. σ, r and D are assumed to be continuous and bounded functions of space and time variables within the contract period.

Remark 2.0.2. The initial conditions of the above problems are not smooth, we need to ensure the convergence of the solution obtained from finite difference scheme.

Hence we replace a small ϵ-neighborhood of the point of singularity by a ninth degree polynomial so that the **payoff is a fourth order smooth function**.

Definition 2.0.3 (**Smooth Payoff Function**)**.** For the European option case, define a function $\psi(x)$ as

$$\psi(x) = \begin{cases} x & for x \geq \epsilon \\ c_0 + c_1 x + c_2 x^2 + \ldots + c_9 x^9 & for -\epsilon < x < \epsilon \\ 0 & for x \leq -\epsilon \end{cases}$$

where $\epsilon > 0$ is a small constant and $c_i, i = 0, 1, \ldots, 9$ are the constant coefficients to be determined. Applying the following ten conditions on the function $\psi(x)$:

$$\psi(-\epsilon) = \psi'(-\epsilon) = \psi''(-\epsilon) = \psi'''(-\epsilon) = \psi^{(4)}(-\epsilon) = 0$$
$$\psi(\epsilon) = \epsilon, \psi'(\epsilon) = 1, \psi''(\epsilon) = \psi'''(\epsilon) = \psi^{(4)}(\epsilon) = 0$$

we can uniquely determine the unknown coefficients $c_i, i = 0, 1, \ldots, 9$ viz.:

$$c_0 = \tfrac{35}{256}\epsilon, \quad c_1 = \tfrac{1}{2}, \quad c_2 = \tfrac{35}{64\epsilon}, \quad c_4 = -\tfrac{35}{128\epsilon^3}$$
$$c_6 = \tfrac{7}{64\epsilon^5}, \quad c_8 = -\tfrac{5}{256\epsilon^7}, \quad c_3 = c_5 = c_7 = c_9 = 0$$

The function $\phi(S) = \psi(S - K)$ smooths the payoff where asset price is at-the-money as shown in Fig. 2.1.

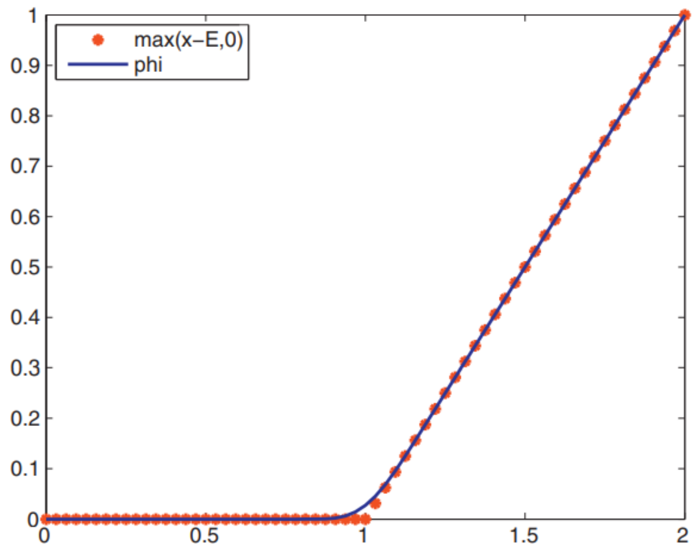

Figure 2.1: Smoothing the payoff of European option

The error estimate due to this smoothing of the final condition in European option case is given by

$$|C(S,\tau) - \hat{C}(S,\tau)| \leq \kappa \|\phi(S) - \max(S-K,0)\|_{L^\infty}, \quad (S,\tau) \in [0,\infty) \times [0,T]$$

where κ is a positive constant independent of ϕ. and our smooth payoff function is

$$\hat{C}(S,T) = \phi(S), \quad S \in [0,\infty)$$

Finally, by changing the final condition to $\hat{C}(S,T)=\phi(S)$ and applying the transformation $\tau=T-t$, where t is new time variable. The above Final boundary value problem is changed into Initial boundary value problem as given below:

$$Lu(S,t)\equiv\frac{\partial u}{\partial t}+\frac{1}{2}\hat{\sigma}^2(S,t)S^2\frac{\partial^2 u}{\partial^2 S}+(\hat{r}(S,t)-\hat{D}(S,t))\frac{\partial u}{\partial S}-\hat{r}(S,t)u=f(S,t)$$

where $S\in(0,S_{\max})$, $\quad t\in(0,T)$ with the initial condition

$$u(S,0)=\phi(S),\quad S\in[0,S_{\max}]$$

and the boundary conditions

$u(0,t)=\hat{g}_1(t)\quad, t\in[0,T]\quad and\quad u(S_{\max},t)=\hat{g}_2(t),\quad t\in[0,T]$

where

$$\hat{g}_1(t)=0\quad and$$

$$\hat{g}_2(t)=S_{\max}\exp\left(-\int_0^t\hat{D}\left(S_{\max},q\right)dq\right)-K\exp\left(-\int_0^t\hat{r}\left(S_{\max},q\right)dq\right)\quad t\in[0,T]$$

Chapter 3

High-Order Difference approximation with Identity Expansion

We discretize the domain Ω uniformly in time and space direction. Let (S_m, t_n), m = 0, 1, ...M, n = 0, 1, ...N be the grid points in the discretized domain Ω_h^k, where M is the total number of intervals in space direction, N is the total number of intervals in time direction, h = $\frac{S_{m+1}+S_m}{2}$, m = 0, 1, ...M - 1 and k = $\frac{t_{n+1}+t_n}{2}$, n = 0, 1, ...N - 1. The discrete approximation of option price u(S,t) at the point (S_m , t_n) is written as U(S_m , t_n). The fully discrete scheme on this mesh is given by

$$\beta_{m,1}^n \left(\delta_t U_m^n\right)+\beta_{m,2}^n \left(\delta_t U_{m+1}^n\right)+\left[\alpha_{m,-}^n U_{m-1}^n + \alpha_{m,c}^n U_m^n + \alpha_{m,+}^n U_{m+1}^n\right] = \beta_{m,1}^n f_m^n+\beta_{m,2}^n f_{m+1}^n,$$

m=1,2, $\ldots$, M-1, n=1,2, $\ldots$, N, where

$$\delta_t U_m^n = \left(U_m^n - U_m^{n-1}\right)/k, \quad n = 1$$
$$\delta_t U_m^n = \left(\tfrac{3}{2}U_m^n - 2U_m^{n-1} + \tfrac{1}{2}U_m^{n-2}\right)/k, \quad n = 2, 3, \ldots, N$$
$$U_m^0 = \phi_m, \quad m = 0, 1, \ldots, M$$
$$U_0^n = \hat{\mathrm{g}}_1^n, \quad n = 0, 1, \ldots, N$$
$$U_M^n = \hat{\mathrm{g}}_1^n, \quad n = 0, 1, \ldots, N$$

Space discretization is done using the classical HODIE scheme with three adjacent mesh points in the space direction (S_{m-1}, t_n) , (S_m, t_n) and (S_{m+1}, t_n) and two auxiliary points (points between first and last stencil point) (S_m, t_n) and (S_{m+1}, t_n) , m = 1 , 2 , . . . , M 1 at each time level t_n , n = 1 , 2 , . . . , N.

The HODIE coefficients $\alpha_{m,-}^n$, $\alpha_{m,c}^n$, $\alpha_{m,+}^n$ are the three coefficients of approximate solution U at the three stencil points at n^{th} time level. The HODIE coefficients $\beta_{m,1}^n$, $\beta_{m,2}^n$, are the two coefficients for identity expansion at the two auxiliary points at n^{th} time level. The HODIE coefficients α's and β's, are computed "locally" by making the above equation exact on P_3 , the space of polynomials of degree less than or equal to 3. We would thus get,

$$\alpha_{m,-}^n = \frac{\beta_{m,1}^n\left(-2a_{2,m}^n + ha_{1,m}^n\right) + \beta_{m,2}^n\left(-2a_{2,m+1}^n - ha_{1,m+1}^n\right)}{2h^2}$$
$$\alpha_{m,+}^n = \frac{\beta_{m,1}^n\left(-2a_{2,m}^n - ha_{1,m}^n\right) + \beta_{m,2}^n\left(-2a_{2,m+1}^n - 3ha_{1,m+1}^n - 2h^2 a_{0,m+1}^n\right)}{2h^2}$$
$$\alpha_{m,c}^n = \frac{\beta_{m,1}^n\left(4a_{2,m}^n - 2h^2 a_{0,m}^n\right) + \beta_{m,2}^n\left(4a_{2,m+1}^n + 4ha_{1,m+1}^n\right)}{2h^2}$$
$$\beta_{m,1}^n = \frac{6ha_{2,m+1}^n + 2h^2 a_{1,m+1}^n}{6ha_{2,m+1}^n + 2h^2 a_{1,m+1}^n + h^2 a_{1,m}^n} \quad and \quad \beta_{m,2}^n = \frac{h^2 a_{1,m}^n}{6ha_{2,m+1}^n + 2h^2 a_{1,m+1}^n + h^2 a_{1,m}^n}$$

With these HODIE coefficients, the fully discrete mesh equation takes the form

$$L_h^k U_m^n \equiv \alpha_{m,-}^n U_{m-1}^n + \left(\alpha_{m,c}^n + \tfrac{1}{k}\beta_{m,1}^n\right) U_m^n + \left(\alpha_{m,+}^n + \tfrac{1}{k}\beta_{m,2}^n\right) U_{m+1}^n =$$
$$\beta_{m,1}^n \left(f_m^n + \tfrac{1}{k}U_m^{n-1}\right) + \beta_{m,2}^n \left(f_{m+1}^n + \tfrac{1}{k}U_{m+1}^{n-1}\right) = F_m^n(say), \quad m = 1, 2, \ldots, M-1, n = 1 \quad and,$$

$$L_h^k U_m^n \equiv \alpha_{m,-}^n U_{m-1}^n + \left(\alpha_{m,c}^n + \tfrac{3}{2k}\beta_{m,1}^n\right) U_m^n + \left(\alpha_{m,+}^n + \tfrac{3}{2k}\beta_{m,2}^n\right) U_{m+1}^n =$$
$$\beta_{m,1}^n \left(f_m^n + \tfrac{2}{k}U_m^{n-1} - \tfrac{1}{2k}U_m^{n-2}\right) + \beta_{m,2}^n \left(f_{m+1}^n + \tfrac{2}{k}U_{m+1}^{n-1} - \tfrac{1}{2k}U_{m+1}^{n-2}\right) =$$
$$F_m^n(say), \quad m = 1, 2, \ldots, M-1, n = 2, 3, \ldots, N$$

amsmath

Chapter 4

Numerical Experiments

In the paper, the author has taken the example of two European options following Black Scholes model and used the above given numerical scheme to find out the solution to the two options. These two options are form a special case of the Black Scholes model, for which the closed form solution is available, and hence, we can calculate the maximum absolute error $\hat{E}^{M,N}_{\max}$ and the Root Mean Square error $\hat{E}^{M,N}_{rms}$ and the corresponding order of convergence $\hat{p}^{M,N}_{\max}$ and $\hat{p}^{M,N}_{rms}$ from the original values, and the ones obtained from the numerical scheme. The expressions are given as,

$$\hat{E}^{M,N}_{\max} = \max_{0 \le m \le M} \left| u^{m,n}\left(S_m, t_N\right) - U^{m,n}\left(S_m, t_N\right) \right|$$

$$\hat{E}^{M,N}_{rms} = \sqrt{\frac{\sum_{m=0}^{M} \left[\left(u^{m,n}(S_m,t_N) - U^{m,n}(S_m,t_N)\right)^2 \right]}{M+1}}$$

$$\hat{p}^{M,N}_{\max} = \log_2 \left(\frac{\hat{E}^{M,N}_{\max}}{\hat{E}^{2M,2N}_{\max}} \right)$$

$$\hat{p}^{M,N}_{rms} = \log_2 \left(\frac{\hat{E}^{M,N}_{rms}}{\hat{E}^{2M,2N}_{rms}} \right)$$

4.1 Example 1

Consider the Black-Scholes equation for European call option price with $\hat{\sigma}(S,t) = 0.4, \hat{r}(S,t) = 0.04,\ \hat{D}(S,t) = 0.02, T = 1$ and $K = 1$. Take

$S_{\max} = 8$ and $\epsilon = 10^{-6}$. The analytical solution and the numerical solution obtained using the scheme are depicted in figure 4.1, 4.2, 4.3, 4.4, 4.5.

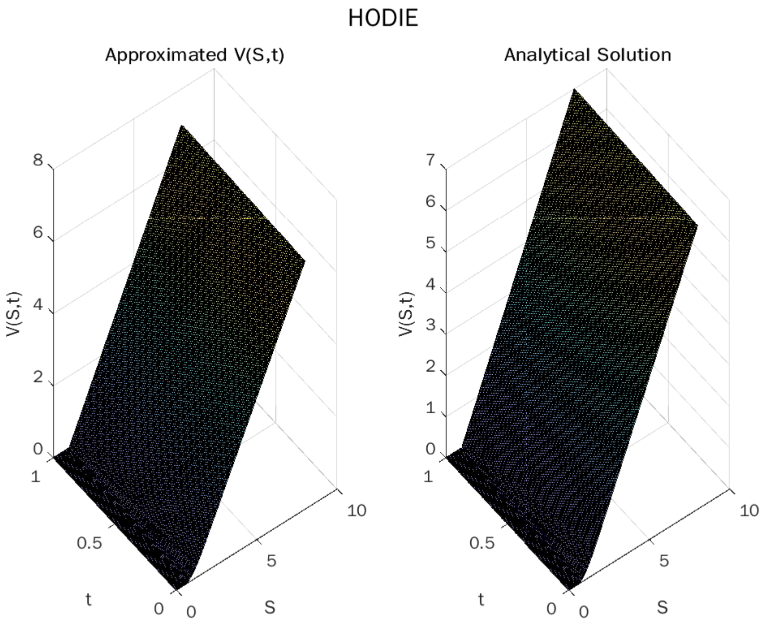

Figure 4.1: analytical and the numerical solution

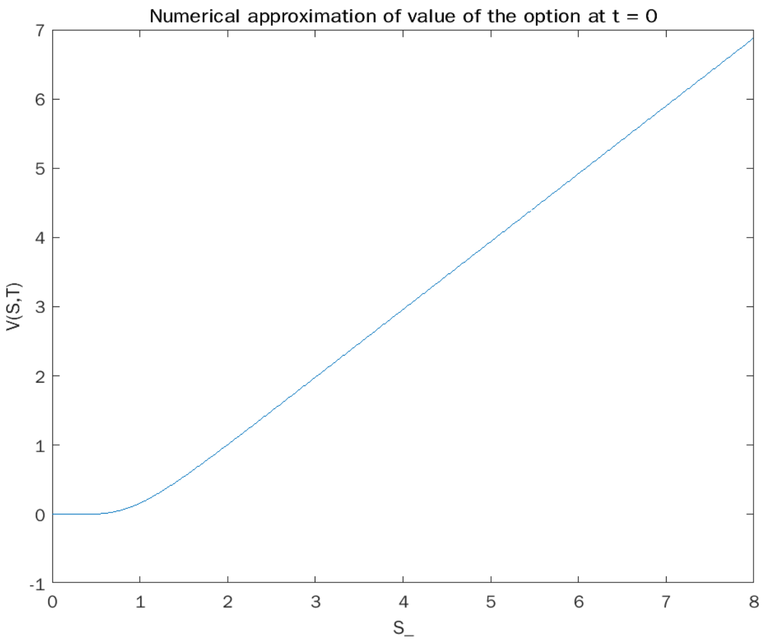

Figure 4.2: value at t = 0 obtained form the numerical solution

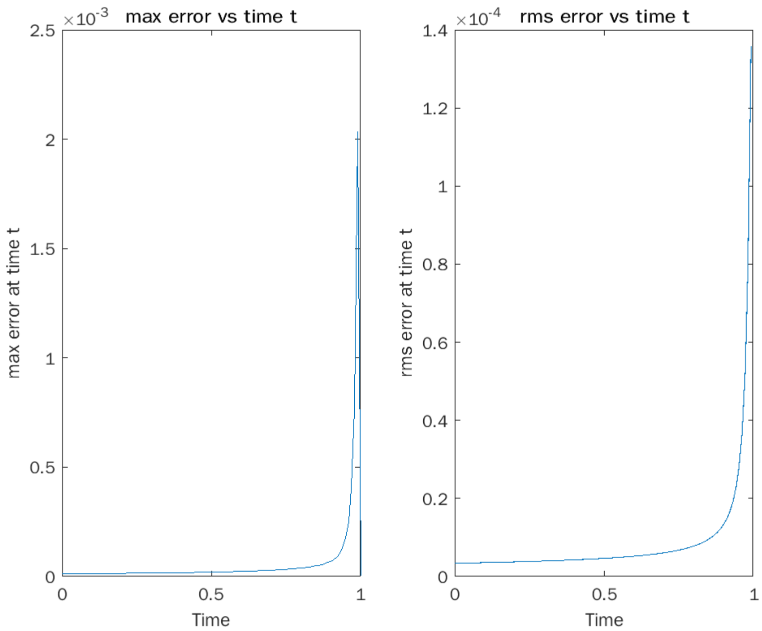

Figure 4.3: plot of the absolute and RMS errors vs time

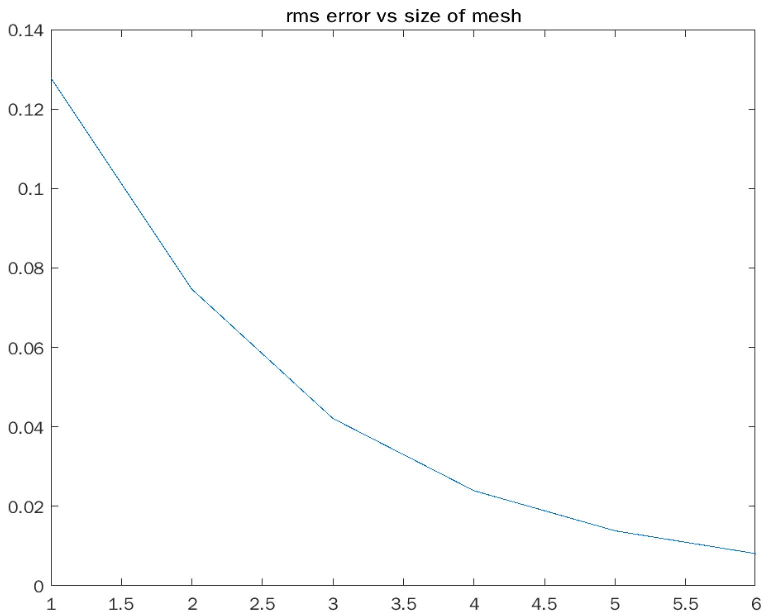

Figure 4.4: plot of the maximum absolute error vs the mesh size

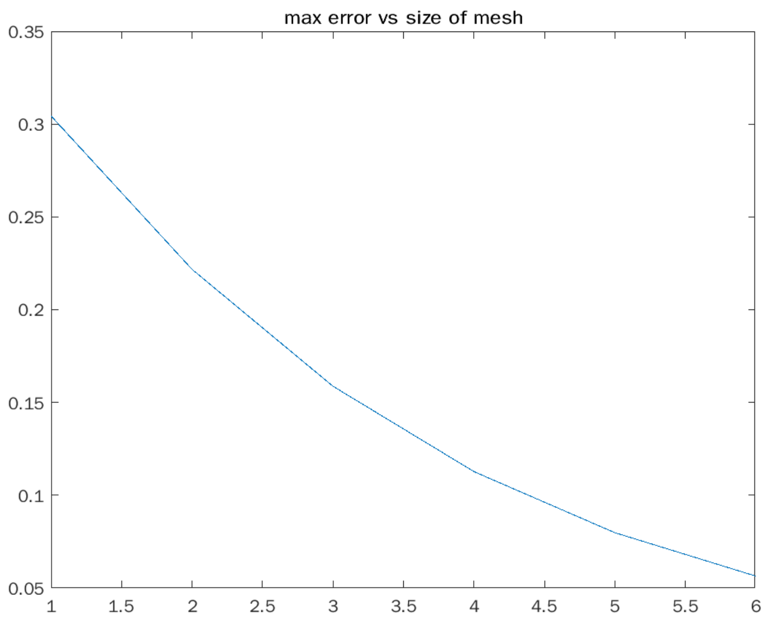

Figure 4.5: plot of the RMS error vs the mesh size

The errors and the corresponding orders of convergence are displayed below.

N	10x2	10x2^2	10x2^3	10x2^4	10x2^5	10x2^6
M	2^5	2^6	2^7	2^8	2^9	2^{10}
$E_{rms}^{M,N}$	0.1278	0.0747	0.0421	0.0239	0.0138	0.0081
$p_{rms}^{M,N}$		0.7759	0.8260	0.8147	0.7901	0.7753
$E_{max}^{M,N}$	0.3044	0.2218	0.1587	0.1127	0.0798	0.0564
$p_{max}^{M,N}$		0.4569	0.4822	0.4936	0.4984	0.5003

4.2 Example 2

Consider the Black-Scholes equation for European call option price with $\hat{\sigma}(S,t) = 0.4, \hat{r}(S,t) = 0.02$, $\hat{D}(S,t) = 0.04, T = 1$ and $K = 1$. Take $S_{\max} = 8$ and $\epsilon = 10^{-6}$. The analytical solution and the numerical solution obtained using the scheme are depicted in figure 4.6, 4.7, 4.8, 4.9, 4.10.

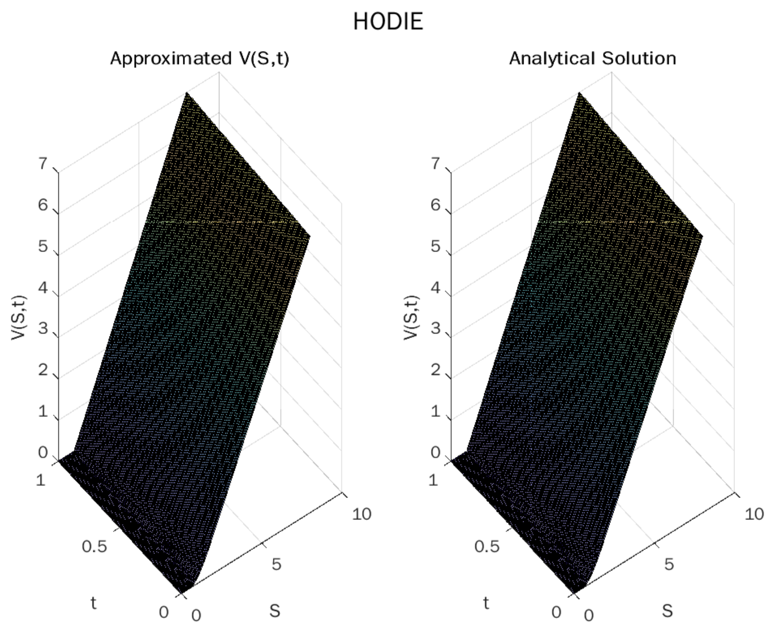

Figure 4.6: analytical and the numerical solution

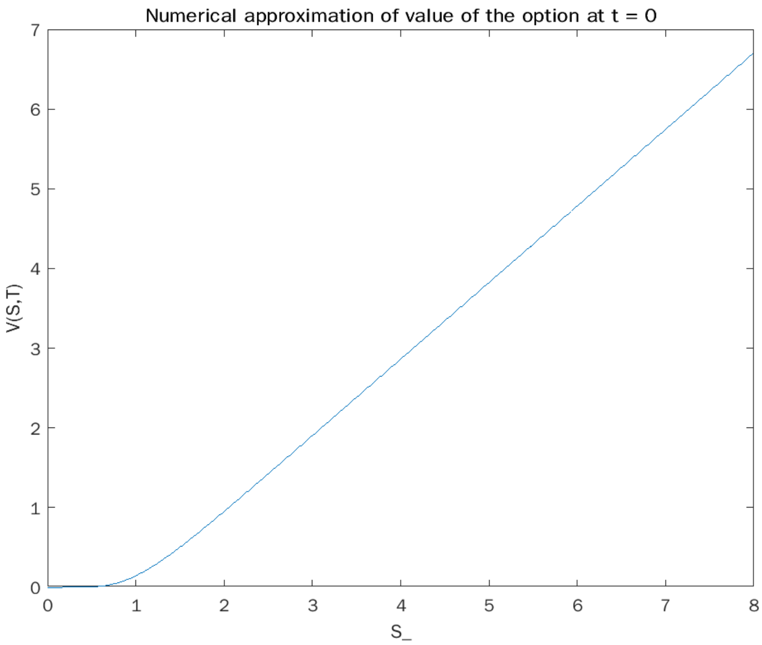

Figure 4.7: value at t = 0 obtained form the numerical solution

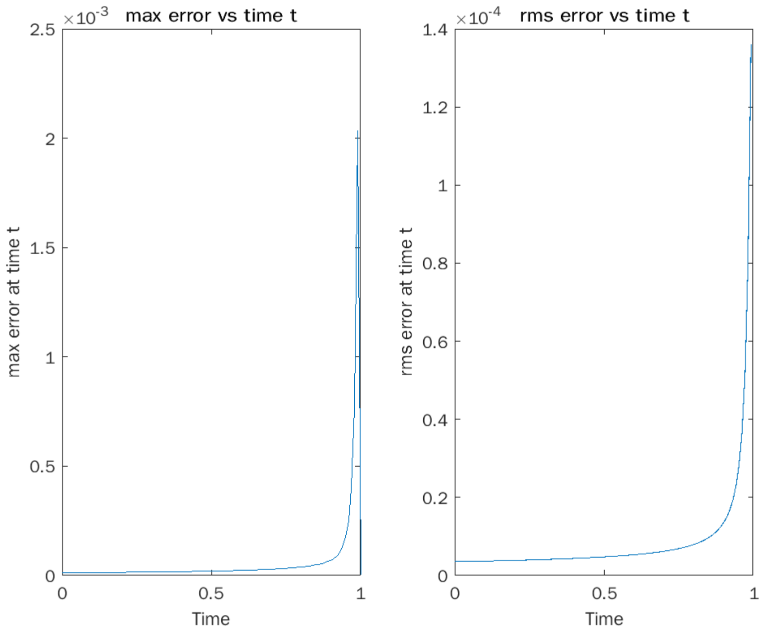

Figure 4.8: plot of the absolute and RMS errors vs time

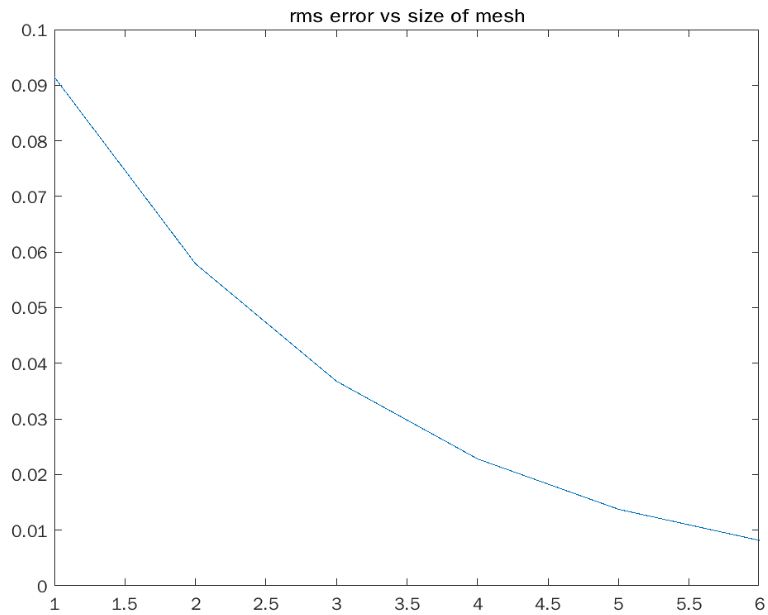

Figure 4.9: plot of the maximum absolute error vs the mesh size

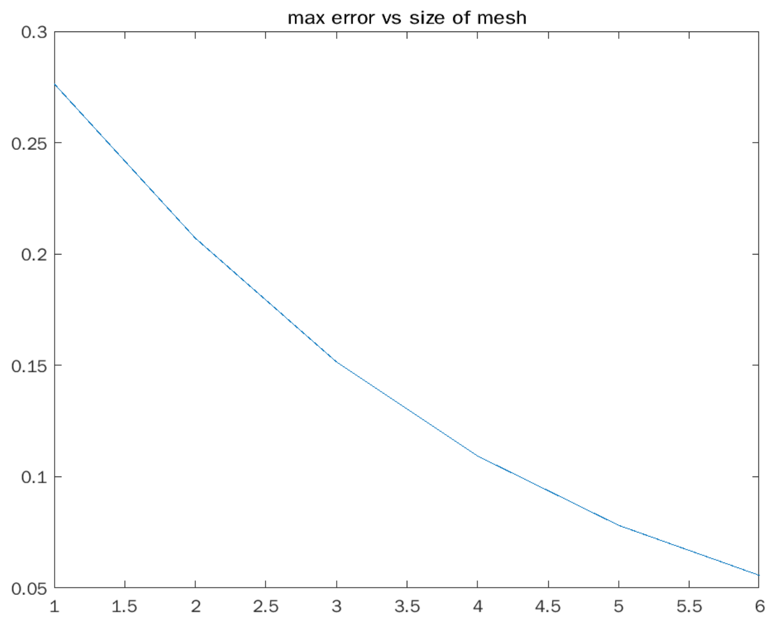

Figure 4.10: plot of the RMS error vs the mesh size

The errors and the corresponding orders of convergence are displayed below.

N	10x2	10x2^2	10x2^3	10x2^4	10x2^5	10x2^6
M	2^5	2^6	2^7	2^8	2^9	2^{10}
$E_{rms}^{M,N}$	0.0915	0.0579	0.0368	0.0228	0.0138	0.0082
$p_{rms}^{M,N}$		0.6591	0.6551	0.6884	0.7305	0.7530
$E_{max}^{M,N}$	0.2765	0.2071	0.1516	0.1093	0.0782	0.0556
$p_{max}^{M,N}$		0.4166	0.4506	0.4716	0.4839	0.4908

Chapter 5

Conclusion

The method mentioned in our Term Paper numerically solves the generalized form of Black–Scholes equation in its almost original form which is degenerate but forward in time, using the HODIE scheme in the space direction and the two-step backward differentiation formula for discretization in the time direction and gives second order convergence in space as well as time direction.

Using these methods and HODIE scheme, We provided numerical examples with two types of solutions where σ, r and D are all constant and one where taken to be functions of space and time variables for proving the theoretical results.

Using HODIE scheme whiche is a high order numerical scheme for pricing European call option governed by generalized Black-Scholes model. one can get second order accuracy both in space and time as proved in the error analysis and is easily observable in the numerical experiments where all the parameters $\hat{\sigma}, \hat{r}$ and $\hat{D}$ are taken as functions of both S and t.

The scheme is constructed in the way that we can simultaneously discretize in space and time directions which leads to simpler convergence analysis. Two step backward differentiation formula is used for temporal discretization and HODIE scheme with three nodal and symmetric auxiliary points is used for spacial discretization. Theoretically the paper have achieved $O\left(k^2 + h^3\right)$ accuracy.