Special Theory of Relativity

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Frame of Reference, Einstein’s Special Theory of Relativity, Postulates, Galilean Transformation, Lorentz Transformation, Relativity of time and length, Relativistic Mass and Momentum, Mass less Particles, Mass-Energy Relation

References

Concepts of Modern Physics (6th Ed) – Arthur Beiser

University Physics with Modern Physics (13th Ed) – Young, Freedman

What is a frame of reference?

A set of criteria or stated values in relation to which measurements or judgements can be made.

A system of geometric axes in relation to which measurements of size, position, or motion can be made.

Inertial frame of reference

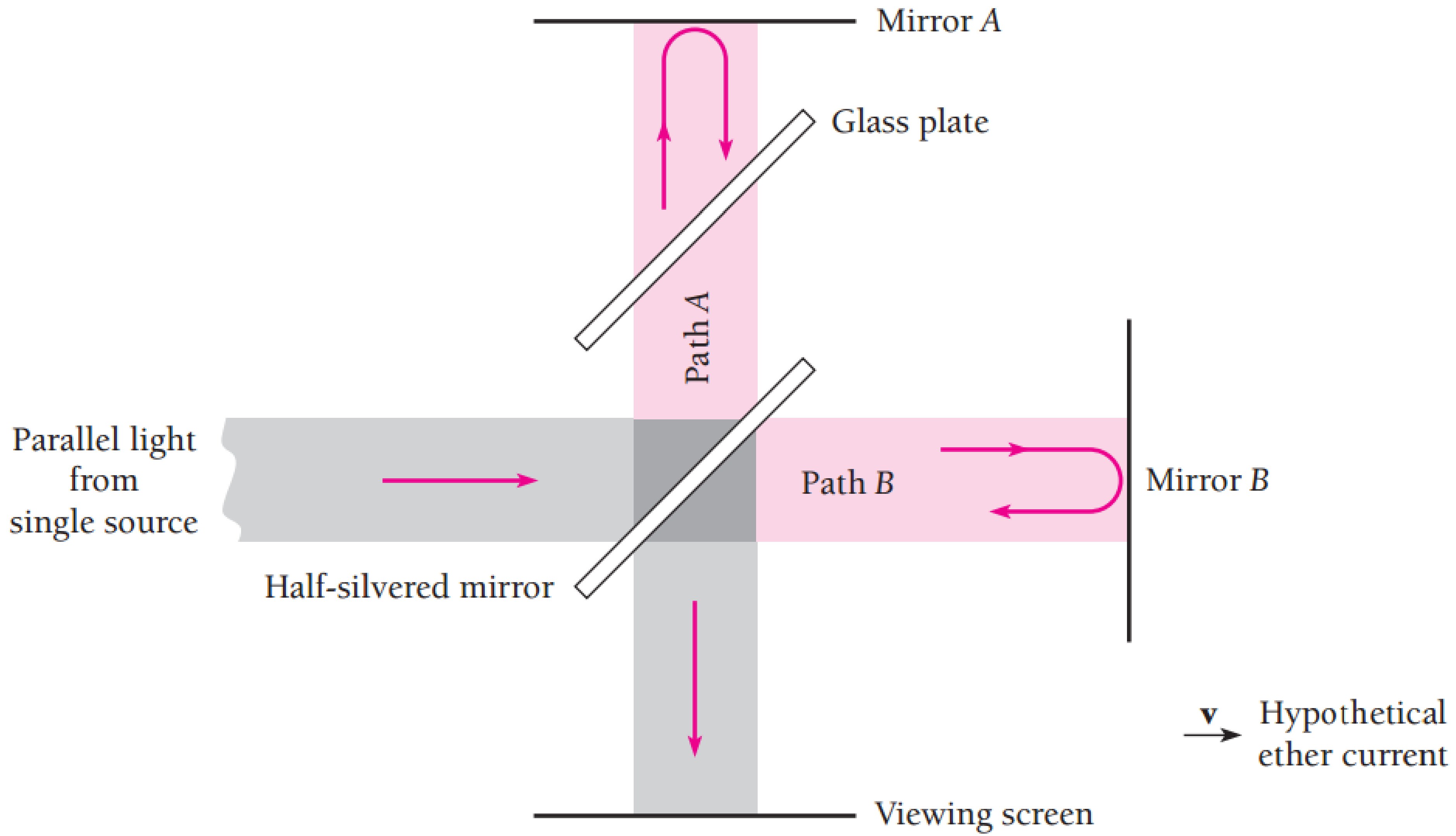
A frame of reference in which Newton’s first law is valid is called an inertial frame of reference.

Postulates of special theory of relativity

First Postulate:
The laws of physics are the same in every inertial frame of reference.

Second Postulate:
The speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source.

The Michelson-Morley experiment



Conclusion

It is impossible for an inertial observer to travel at c, the speed of light in vacuum.

The Galilean Coordinate Transformation

$$x = x' + ut$$

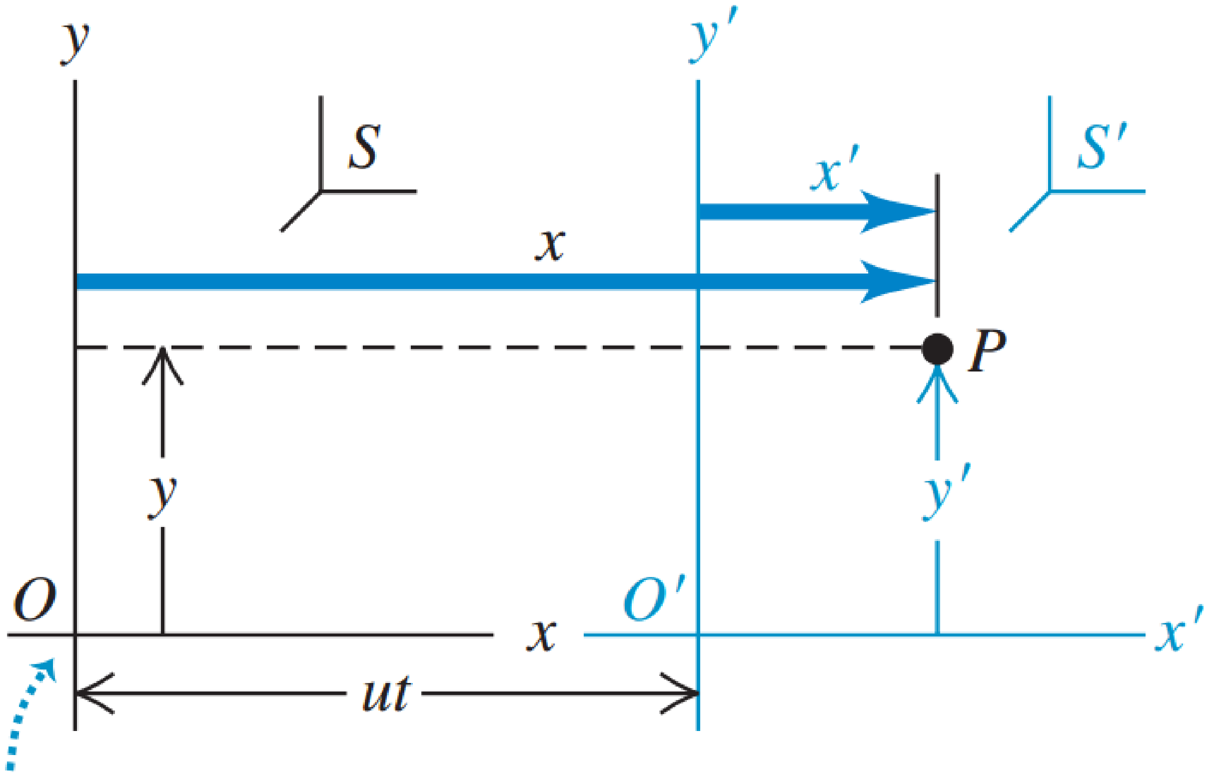
$$y = y'$$

$$z = z'$$

$$\boxed{v_x = v'_x + u}$$

(Galilean velocity transformation)

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.



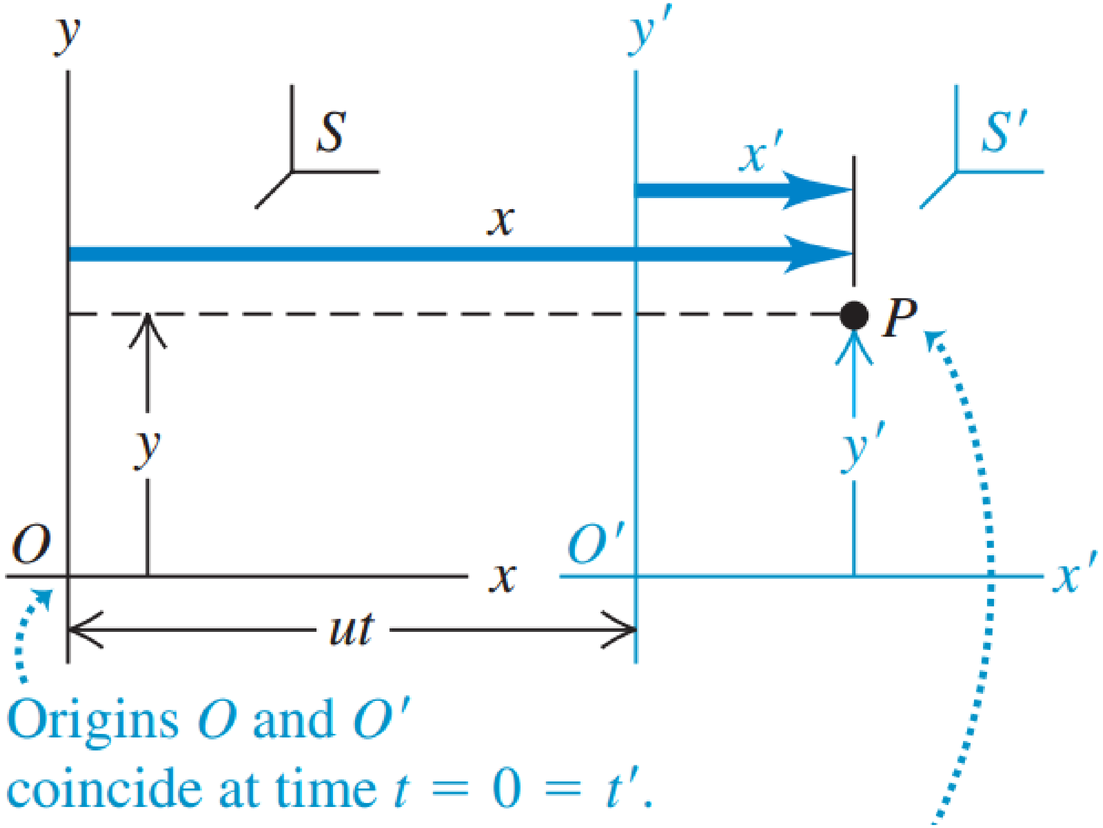
Origins O and O' coincide at time $t = 0 = t'$.

Question

How good is Galilean transformation?

The Lorentz Transformation

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.



$$\boxed{x' = \gamma(x - ut)}$$

The Lorentz coordinate transformation relates the spacetime coordinates of an event as measured in the two frames: (x, y, z, t) in frame S and (x', y', z', t') in frame S'.

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2)$$

(Lorentz coordinate transformation)

Space and time have become intertwined; we can no longer say that length and time have absolute meanings independent of the frame of reference.

Inverse Lorentz Transformation

$$
\begin{aligned}
x &= \frac{x' + ut'}{\sqrt{1 - u^2/c^2}} \\
y &= y' \\
z &= z' \\
t &= \frac{t' + ux'/c^2}{\sqrt{1 - u^2/c^2}}
\end{aligned}
$$

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} \qquad \text{(Lorentz velocity transformation)}$$

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \qquad \text{(Lorentz velocity transformation)}$$

$$v_x = c$$

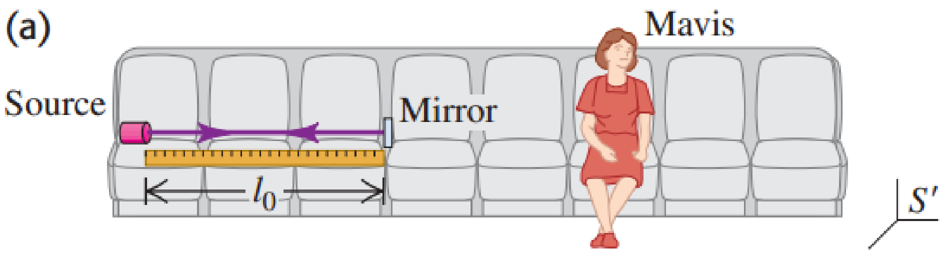
$$\boxed{v'_x = \frac{c - u}{1 - uc/c^2} = \frac{c(1 - u/c)}{1 - u/c} = c}$$

Relativity of Length

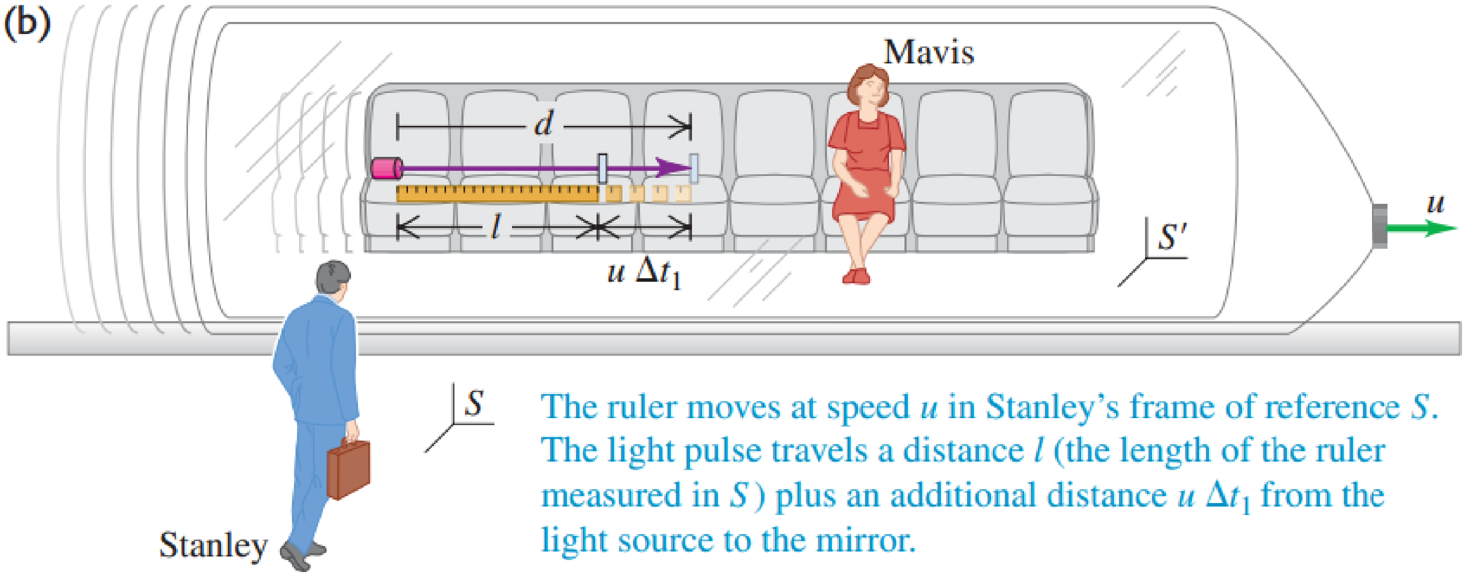
$$\boxed{l_0 = x'_2 - x'_1}$$

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma}$$

(length contraction)

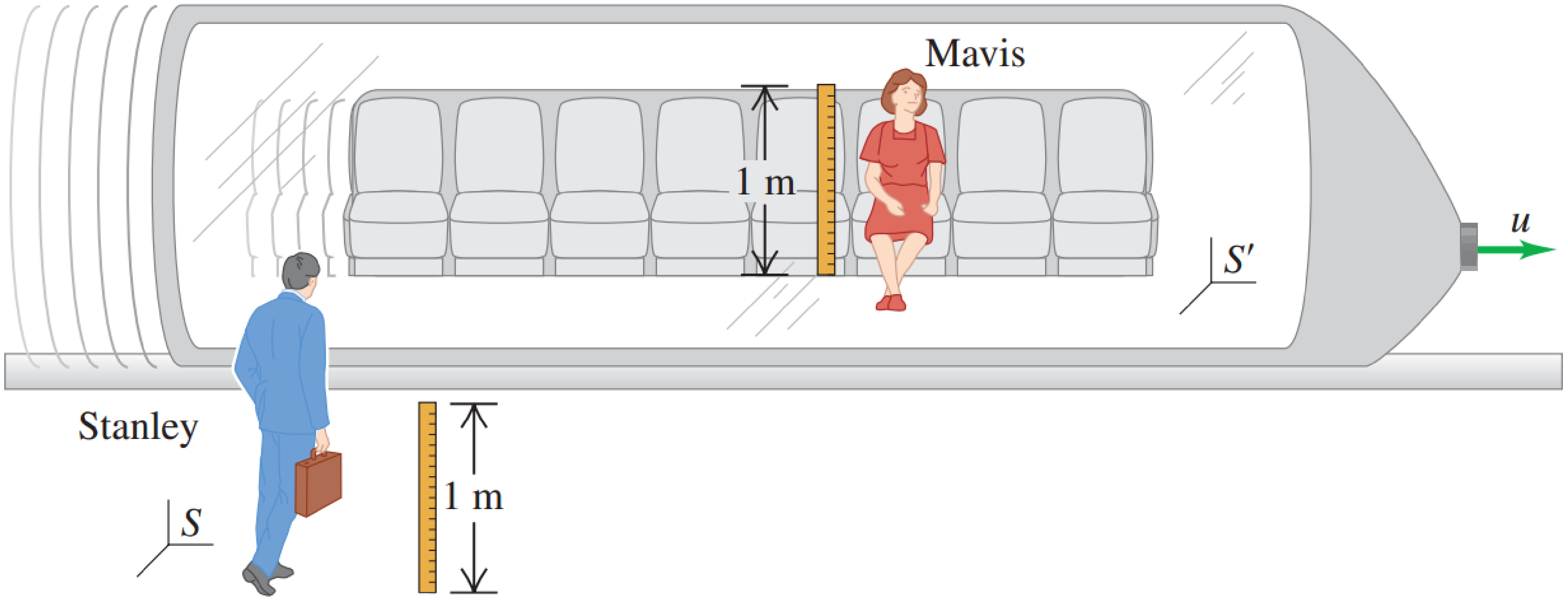


The ruler is stationary in Mavis's frame of reference S'. The light pulse travels a distance l_0 from the light source to the mirror.



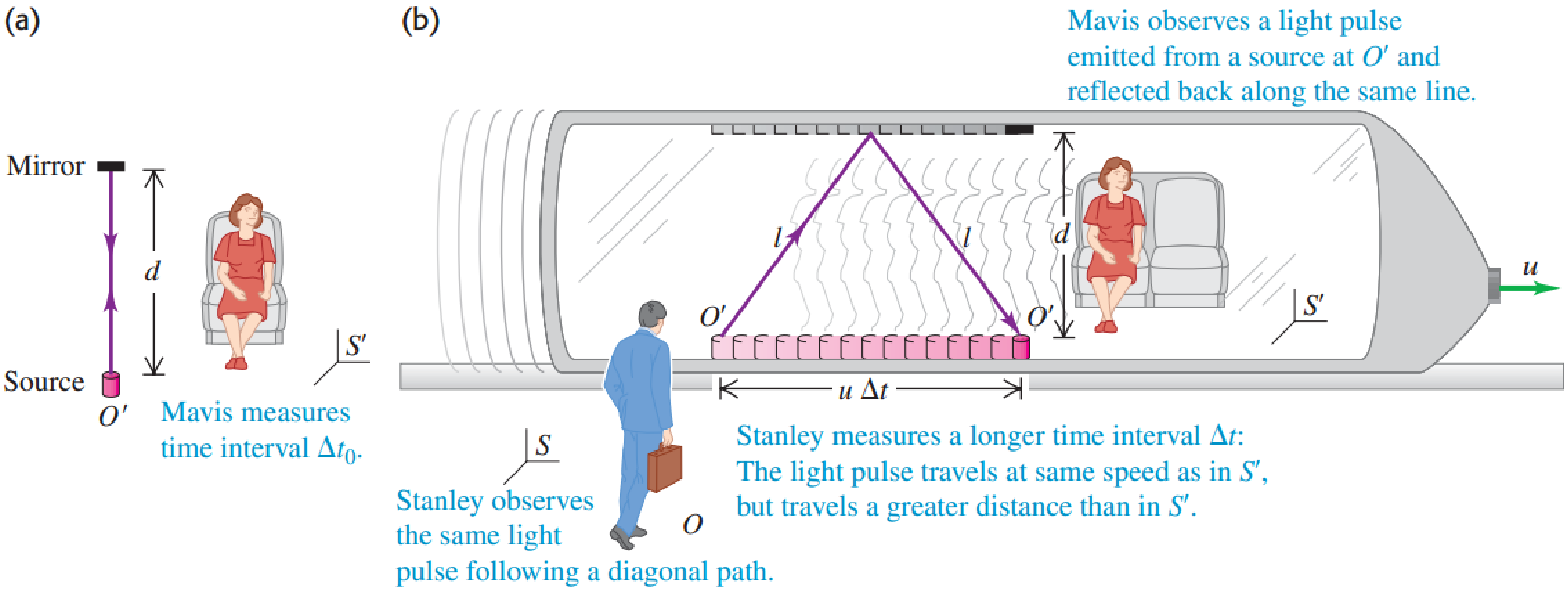
The ruler moves at speed u in Stanley's frame of reference S. The light pulse travels a distance l (the length of the ruler measured in S) plus an additional distance $u\,\Delta t_1$ from the light source to the mirror.

Lengths Perpendicular to the Relative Motion



There is no length contraction perpendicular to the direction of relative motion of the coordinate systems.

Relativity of Time



$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \qquad \text{(time dilation)}$$

Question

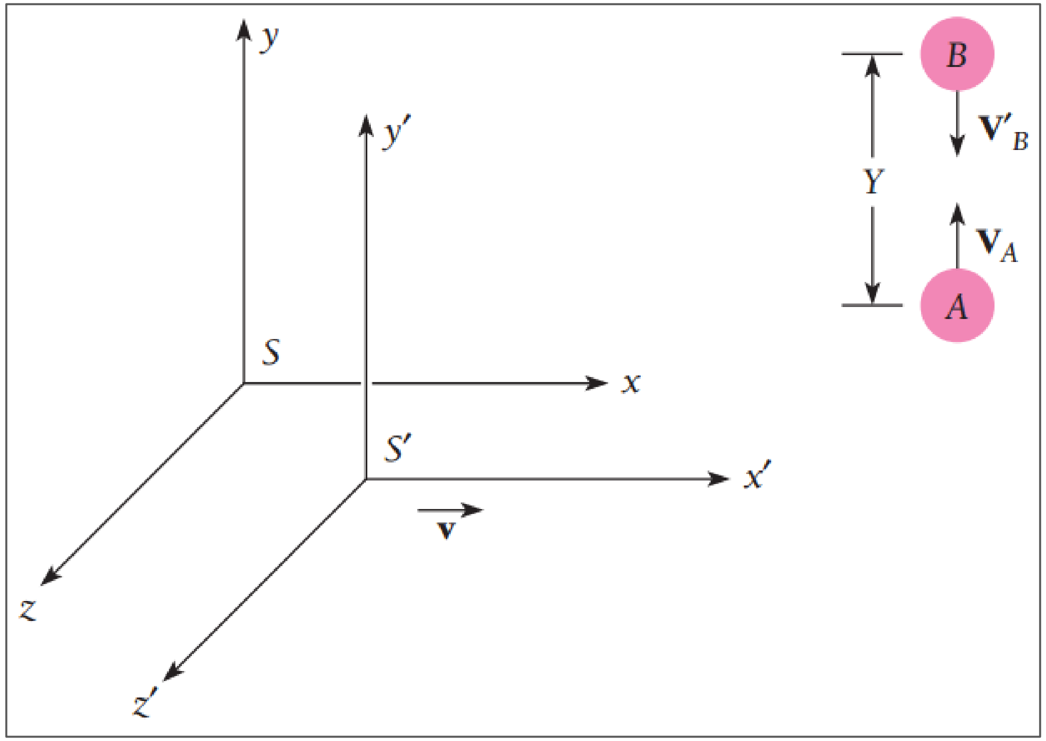
An airplane flies from San Francisco to New York (about 4800 km, or 4.80×10^6 m) at a steady speed of 300 m/s (about 670 mi/h). How much time does the trip take, as measured by an observer on the ground? By an observer in the plane?

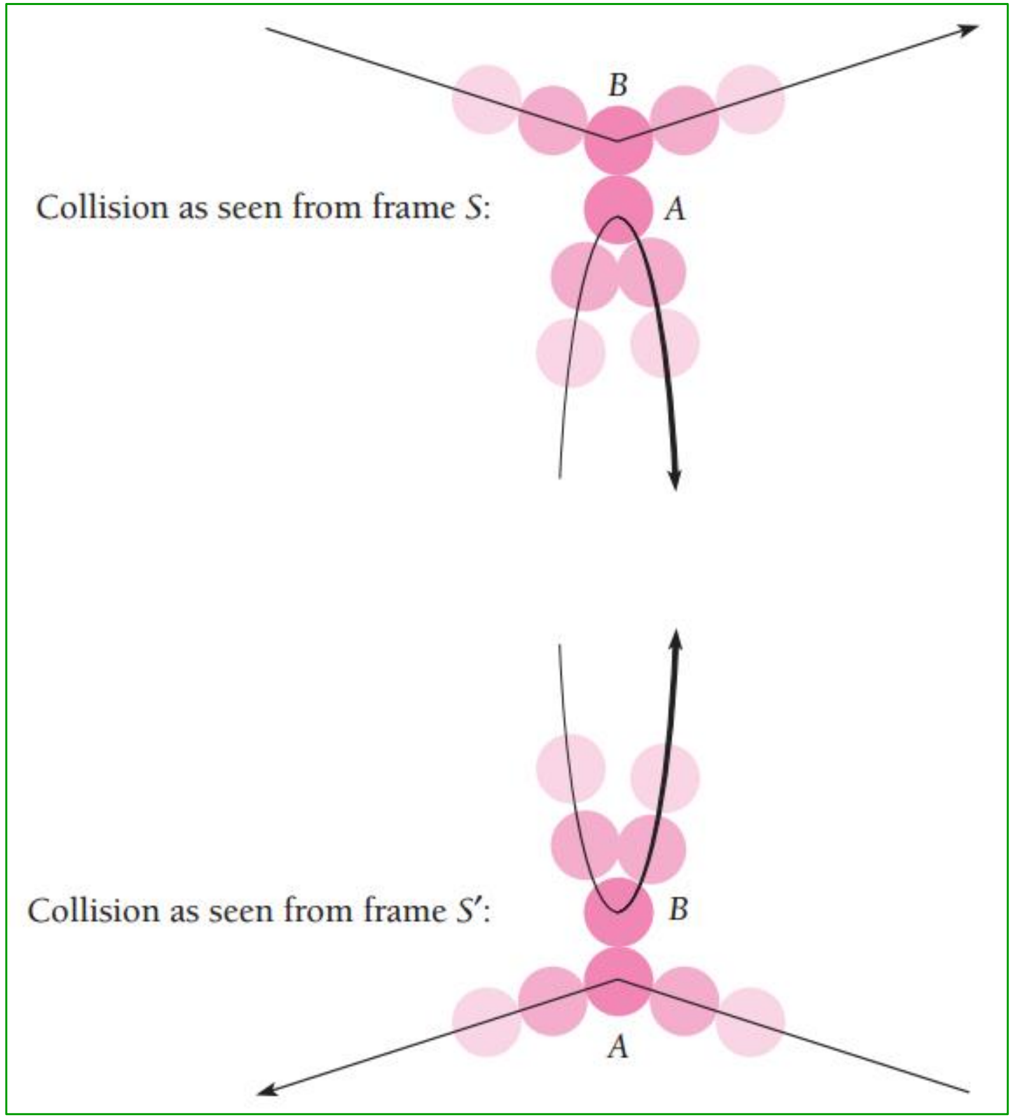
Relativity of simultaneity

- Whether or not two events at different x-axis locations are simultaneous depends on the state of motion of the observer.

- The time interval between two events may be different in different frames of reference.

Relativistic Momentum





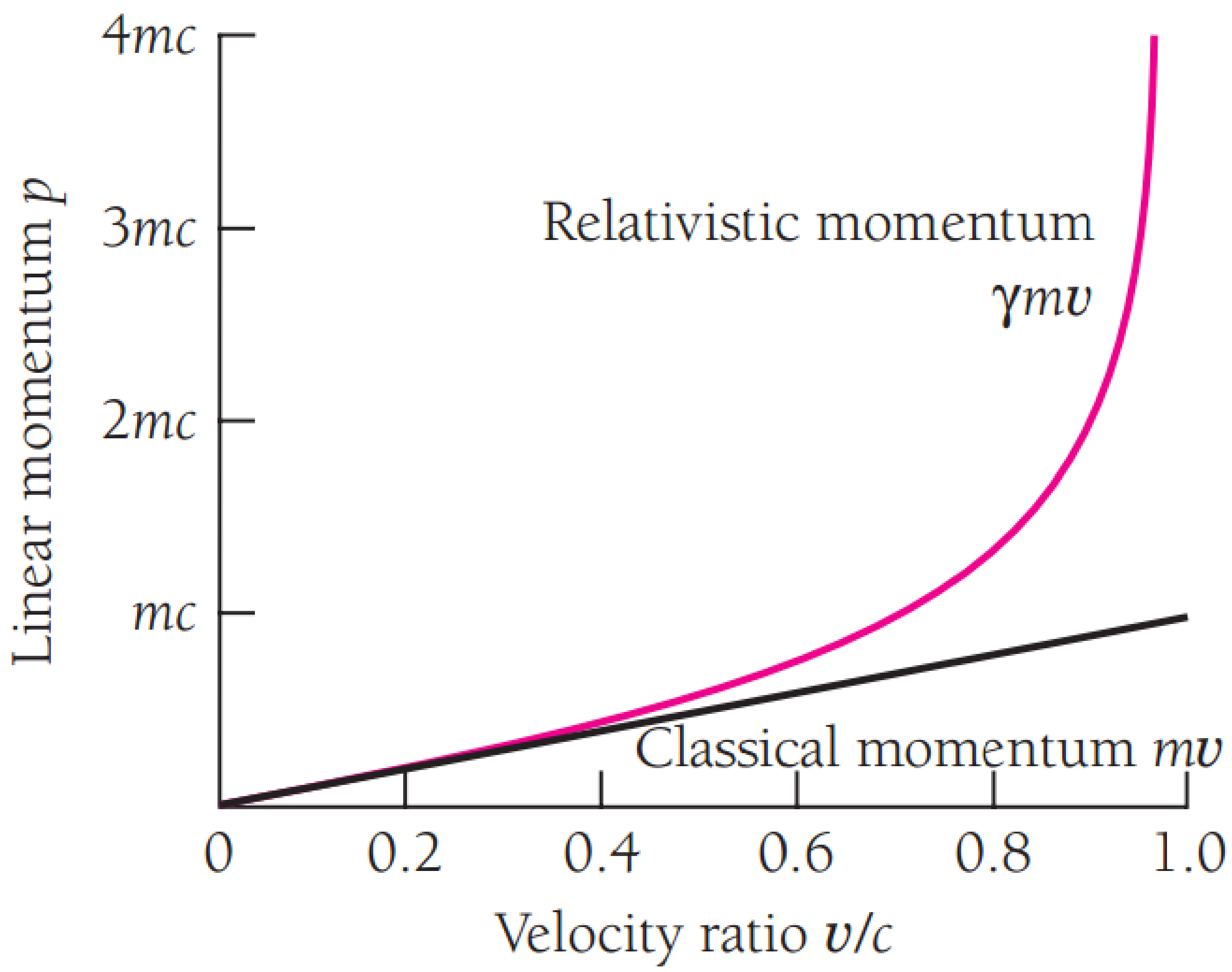
Relativistic Momentum

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

$$m_{\text{rel}} = \frac{m}{\sqrt{1 - v^2/c^2}}$$

m is the rest mass

m_{rel} is the relativistic mass


4mc
3mc
2mc
mc
Linear momentum p
Relativistic momentum
γmv
Classical momentum mv
0
0.2
0.4
0.6
0.8
1.0
Velocity ratio v/c

Relativistic Second Law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\boldsymbol{\gamma} m \mathbf{v})$$

Relativistic Second Law

$$F = \frac{d}{dt}(\boldsymbol{\gamma} m \boldsymbol{v}) = m\frac{d}{dt}\left(\frac{\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2/c^2}}\right)$$

$$= m\left[\frac{1}{\sqrt{1 - \boldsymbol{v}^2/c^2}} + \frac{\boldsymbol{v}^2/c^2}{(1 - \boldsymbol{v}^2/c^2)^{3/2}}\right]\frac{d\boldsymbol{v}}{dt}$$

$$= \frac{ma}{(1 - \boldsymbol{v}^2/c^2)^{3/2}}$$

$$\boxed{a = d\boldsymbol{v}/dt}$$

Relativistic Work and Energy

Consider an object which is initially at rest, starts to move due to a force *F* acting on it. If no other forces act on the object all the work done on it becomes kinetic energy, *K*:

$$
\begin{aligned}
K &= W = \text{Work done} \\
&= \int_0^s F \, ds .
\end{aligned}
$$

Relativistic Work and Energy

$$\begin{aligned} F &= \frac{d}{dt}(\gamma m v) \\ &= \frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \end{aligned}$$

$$\begin{aligned} K &= \int_0^s \frac{d(\gamma m v)}{dt}\, ds \\ &= \int_0^{mv} v\, d(\gamma m v) \\ &= \int_0^v v\, d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \end{aligned}$$

Relativistic Work and Energy

Integrating by parts $\left(\int x\,dy = xy - \int y\,dx\right)$

$$K = \frac{mv^2}{\sqrt{1-v^2/c^2}} - m\int_0^v \frac{v\,dv}{\sqrt{1-v^2/c^2}}$$

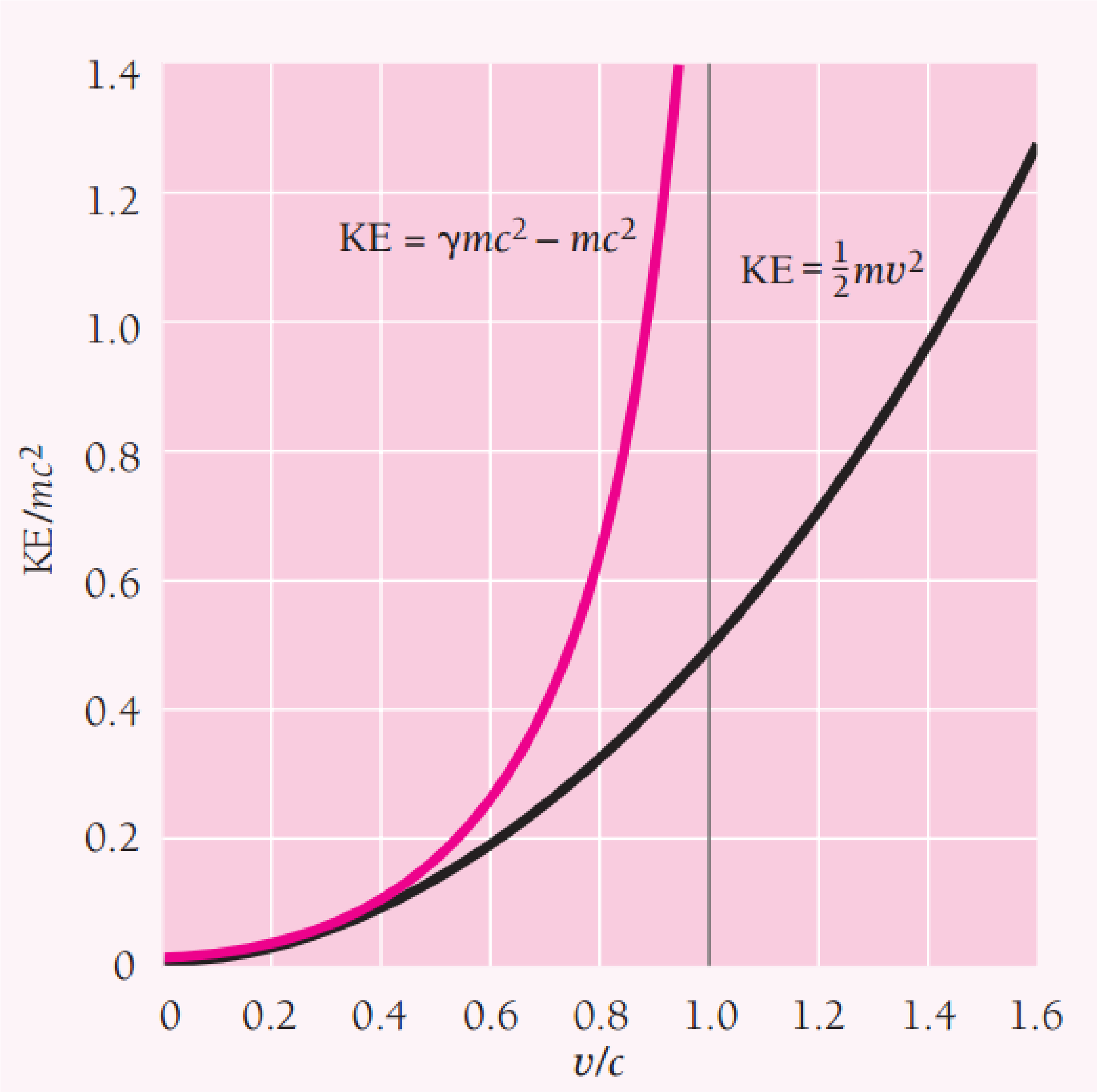
Relativistic Work and Energy

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2$$ (relativistic kinetic energy)

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2$$ (total energy of a particle)

Rest Energy $E = mc^2$

Relativistic Energy



Relativistic Momentum

$$E^2 = (mc^2)^2 + (pc)^2$$ (total energy, rest energy, and momentum)

$$E = pc$$ (zero rest mass)

Massless particle:
e.g. *photon*

$$\epsilon_n = nh\nu \qquad n = 0, 1, 2, \ldots$$

Planck's constant $h = 6.626 \times 10^{-34}\ \text{J} \cdot \text{s}$

Electronvolts $1\ \text{eV} = (1.602 \times 10^{-19}\ \text{C})(1.000\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$

Examples

01. A stationary body explodes into two fragments each of mass 1.0 kg that move apart at speeds of $0.6c$ relative to the original body. Find the mass of the original body.

02. An electron and a photon both have momentum of 2.0 MeV/c. Find the total energy of each.

Examples

03. How much work must be done (classic and relativistic) to increase the speed of an electron from 1.2×10^{8} ms^{-1} to 2.4×10^{8} ms^{-1}?

04. Two protons are initially moving with equal speed in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion of mass 2.40×10^{-28} kg. If all three particles are at rest after the collision, find the initial speed of the protons. Energy is conserved in the collision.