# Elasticity

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## Content

Hooke's law, Breaking stress, Stress-strain diagram, Different types of elasticity, Heat effect on elasticity, Poisson's ratio, Shearing stress and shearing strain, Relations among the elastic constants, Work done in a strain, Work of rupture, Deformation by bending, Bending moment.

## References

Elements of Properties of Matter – D. S. Mathur
Fundamentals of Physics – David Halliday, Jearl Walker, and Robert Resnick

## CHAPTER VIII

# ELASTICITY

**106. Introductory.** All bodies can, more or less, be *deformed* by suitably applied forces. The simplest cases of deformation are those (*i*) in which a wire, fixed at its upper end, is pulled down by a weight at its lower end, *bringing about a change in its length* and (*ii*) in which an equal compression is applied in all directions, so that *there is a change of volume but no change in shape*, or (*iii*) in which a system of forces may be applied to a body such that, although there is no motion of the body as a whole, there is relative displacement of its continuous layers causing a *change in the shape or 'form' of the body with no change in its volume.* In all these cases, the body is said to be *strained* or *deformed.*

When the deforming forces are removed, the body tends to recover its original condition For example, the wire, in the case above, tends to come back to its original length when the force due to the suspended weight is removed from it, or, a compressed volume of air or gas throws back the piston when it is released, in an attempt to recover its original volume. This property of a material body to regain its original condition, on the removal of the deforming forces, is called *elasticity*. Bodies, which can recover completely their original condition, on the removal of the deforming forces, are said to be *perfectly elastic*. On the other hand, bodies, which do not show any tendency to recover their original condition, are said to be *plastic*. There are, however, no *perfectly* elastic or plastic bodies. The nearest approach to a perfectly elastic body is a *quartz fibre* and, to a perfectly plastic body, is *putty*. But even the former yields to large deforming forces and, similarly, the latter recovers from small deformations. Thus, there are *only differences of degree*, and a body is more elastic or plastic when compared to another.

We shall consider here only bodies or substances, which are (*i*) *homogeneous* and (*ii*) *isotropic, i.e.*, which have the *same properties at all points* and *in all directions*. For, these alone have similar elastic properties in every direction, (together with other physical properties like linear expansion, conductivity for heat and electricity, refractive index etc.). Fluids (*i.e.*, liquids and gases), as a rule, belong to this class, but not necessarily all solids, some of which may exhibit different properties at different points and in different directions, *i.e.*, may be *heterogeneous* (or *non-homogeneous*) and *anisotropic* (or *non-isotropic*). Examples of this class of solids are wood, and crystals in general, including those metals, which are crystalline in structure. As a class, however, *metals*,—particularly in the form of *rods* and *wires*,—may be regarded to be more or less wholly isotropic, in so far as their elastic behaviour is concerned.

**107. Stress and Strain.** As a result of the deforming forces applied to a body, *forces of reaction* come into play *internally* in it, due to the relative displacement of its molecules, tending to restore it to its original condition. *The restoring or recovering force per unit area set up inside the body is called stress,* and is measured by the deforming force applied per unit area of the body, being *equal in magnitude* but *opposite in direction* to it, until a permanent change has been brought about in the body, *i.e.*, until its elastic limit has been reached, (see § 108, below). If the force be inclined to the surface, its component, perpendicular to the surface, measured per unit area, is called *normal stress** and the component acting along the surface, per unit area, is called *tangential* or *shearing stress.* Further, the former may be *compressive* or *expansive* (*i.e.*, tensile) according as a decrease or increase in volume is involved. Obviously, being force per unit area, the units and dimensions of stress are the same as those of pressure, *viz.*, $ML^{-1}T^{-2}$, (see page 5).

*The change produced in the dimensions of a body under a system of forces or couples, in equilibrium, is called strain,* and is measured by the *change per unit length* (*linear strain*), per unit volume, (*volume strain*), or the angular deformation, (*shear strain,* or simply, *shear*)† according as the change takes place in *length, volume* or *shape* of the body. Thus, being just a *ratio,* (or *an angle*) it is a *dimensionless* quantity, having no units.

It will be readily seen that *for a perfectly elastic body* (*i*) *the strain is always the same for a given stress* ; (*ii*) *the strain vanishes completely when the deforming force is removed and* (*iii*) *for maintaining the strain, the stress is constant.*

**108. Hooke's Law.** Hooke's law is the fundamental law of elasticity and states that, *provided the strain is small, the stress is proportional to the strain* ; *so that, in such a case, the ratio stress/strain is a constant E, called the* **modulus of elasticity,** (a term first introduced by *Thomas Young*), *or the* **coefficient of elasticity.**

Since stress is just pressure, (or tension per unit area), and strain is just a ratio, the units and dimensions of the modulus of elasticity are the same as those of stress or pressure.

When the stress is continually increased in the case of a solid, a point is reached at which the strain increases more rapidly than is warranted by Hooke's Law. This point is called the **elastic limit,** and if the body happens to be a wire under stretch, it will not regain its original length on being unloaded, *if the elastic limit be passed,* as it acquires what is called a **'permanent set'.** On loading it further, a point is reached when the extension begins to increase still more rapidly and the wire begins to *'flow down'* in spite of the same constant load. This point is called the **'yield point'** ; and, after a large extension, it reaches the **'breaking point'** and the wire snaps. In the case of plastic substances, like lead, there is a long range between the yield point and the breaking point.

*The stress is always normal in the case of a change in the length of the wire, or in the case of a change in the volume of a body, but is tangential in the case of a change in the shape of a body.

†This will be dealt with more fully later in § 109 (3).

Thus, if we were to plot a graph between the load suspended from a wire, fixed to a rigid support at its upper end, and the extension produced thereby, we obtain, in general, a curve of the form shown in Fig 169,—the straight part $OA$ of the curve showing that the extension produced is directly proportional to the load applied, or that Hooke's law is obeyed perfectly up to $A$, and that, therefore, on being unloaded at any point between $O$ and $A$, the wire will come back to its original condition, (represented by $O$). In other words, the wire is perfectly elastic up to $A$, which thus measures the **elastic limit*** of the specimen in question,—the extension here being of the order of $10^{-3}$ of the original length.

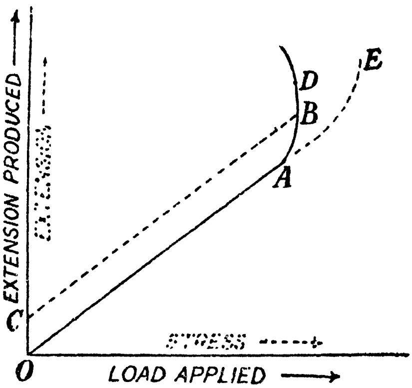

Fig. 169.

On loading the wire beyond the elastic limit, say, up to $B$, the curve takes a bend almost vertically upwards, as shown, and, on being unloaded at any point here, (at $B$, say), it *does not come back to its original condition* but takes the dotted path $BC$, thus acquiring a **'permanent set'** $OC$.

On increasing the load still further, a point $D$ is reached, where the extension is much greater even for a small increase in the load, *i.e.*, Hooke's law is obeyed no longer ; and, beyond $D$, the extension increases continuously, *with no addition to the load*, the wire starting *'flowing down'*, as it were. For, due to its thinning down, the *stress* (or the load per unit area) increases considerably and it cannot support the same load as before ; and, if the wire is to be prevented from *'snapping'*, the load applied to it must be decreased. That is why the curve starts turning towards the extension-axis beyond this point $D$, which thus represents the **yield point** of the wire. And, once the yield point is crossed, the thinning of the wire no longer remains *uniform* or *even*, its cross-section decreasing more rapidly at some points than at others, resulting in its developing small *'necks'* or *'waists'* at the former points, so that the stress is greater there than at the latter points ; and the wire ultimately *'snaps'* at one of these. This point on the curve, at which the snapping or the breaking of the wire actually occurs, is called its **breaking point**,—the corresponding stress and strain there being referred to as the **breaking stress** (or *tensile strength*) and the **breaking strain**, respectively.

Note. The *elastic limit* of a material is also sometimes defined as *the force producing the maximum reversible or recoverable deformation in it*, and may,

*In quite a few cases, Hooke's law is obeyed only up to a point a little below the elastic limit, represented by $A$. The portion of the curve from $O$ to *this point* (below $A$), is then said to indicate the *limit of proportionality*, to distinguish it from the *elastic limit*. The two are thus not always identical, though they are generally regarded to be so, in view of the very small difference between them.

for a given specimen, be determined by loading and unloading it with a number of different loads and measuring its length after *each unloading*, until it acquires a *permanent set*. The latter is then plotted against the load, and from the curve thus obtained, the particular load at which the permanent set *just* starts, can be easily estimated.

Even within the elastic limit, however, few solids come back to their original condition, directly the deforming force is removed. Almost all of them only *'creep'* back to it, (*i.e.*, take some time to do so), though they all do so, ultimately. *This delay in recovering back the original condition, on the cessation of the deforming force, is called* **elastic-after effect.** Glass exhibits this effect to a marked degree, the few *exceptions* to this almost general rule being *quartz*, *phosphor-bronze, silver* and *gold*, which regain their original condition as soon as the deforming force ceases to operate. Hence their use in Cavendish's and Boys' experiments for the determination of $G$, in quadrant electrometers and moving-coil galvanometers etc. etc.

As a natural consequence of the elastic after-effect *the strain in a material*, (*in glass, for example*), *tends to persist or lag behind the stress to which it is subjected*, with the result that during a rapidly changing stress, the strain is greater for the same value of stress, when it is decreasing than when it is increasing, as is clear from the curve in Fig. 170. *This lag between stress and strain is called* **elastic hysterisis**, (the term *'hysterisis'*, meaning *'lagging behind'*). The phenomenon is similar in its implications to the familiar *magnetic hysterisis*, where the magnetic effects tend to persist or lag behind even after the magnetising influence is removed,—the curve referred to above may thus be called the *elastic hysterisis loop*. And, exactly in the same manner the energy, dissipated as *heat*, during a cycle of loading and unloading is given by the *area enclosed by the loop*. There is, however, very little hysterisis in the case of metals or of quartz.

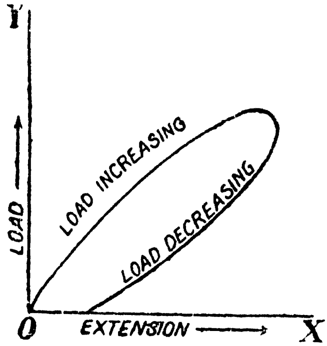

Fig. 170.

Further, it was shown by *Lord Kelvin*, during his investigation of the rate of decay of torsional vibrations of wires, that the vibrations died away much faster in the case of a wire kept vibrating continuously for some time than in that of a fresh wire. The same happens to any elastic body, subjected to an alternating strain. The continuously vibrating wire got *'tired'* or *'fatigued'*, as it were, and found it difficult to continue vibrating. Lord Kelvin fittingly expressed this by the term **'elastic fatigue'**.

A body, thus subjected to repeated strains beyond its elastic limit, has its elastic properties greatly impaired, and may break under a stress, less than its normal breaking stress even within its elastic limit. This phenomenon is, obviously, of great importance in cases like those of the piston and the connecting rods in a locomotive, which, as we know, are subjected to repeated tensions and compressions during each revolution of the crank shaft.

It may be mentioned here that all these elastic properties of a material are linked up with the fine mass of its structure. It is now finally established by careful microscopic examination, that metals are just an aggregation of a large number of fine crystals, in most cases, arranged in a random or a chaotic fashion; *i.e.*, their *cleavage planes* (or the planes along which their constituent atoms can easily slide over each other), being distributed haphazardly, in all possible directions. Now, single crystals, when subjected to deformation, show a remarkable increase in their hardness. Thus, for example, a single crystal of silver, on being stretched to a little more than twice its length, is known to increase to as much as ninety-two times its original strength or stiffness. So that, operations like hammering and rolling, which help this sort of distribution, *i.e.*, which break up the crystal grains into smaller units, result in an increase or extension of their elastic properties ; whereas, operations like annealing (or heating and then cooling gradually) etc., which tend to produce a uniform pattern of orientation of the constituent crystals, by orienting them all in one particular direction and thus forming larger crystal grains, result in a decrease in their elastic properties or an increase in the softness or plasticity of the material.

This is because in the latter case, slipping (or sliding between cleavage planes), starting at a weak spot proceeds all through the crystal and, in the former, the slipping is confined to one crystal grain and stops at its boundary with the adjoining crystal. Indeed, the former may be compared to a small cut, developing into a regular tear all along a fabric and the latter to the tear stopping as it reaches a seam in the fabric. Thus, *'paradoxically', as Sir Lawrence Bragg* puts it, *'in order to be strong, a metal must be weak,'* meaning thereby that metals with smaller grains are stronger than those with larger ones.

A change in the temperature also affects the elastic properties of a material,—a rise in temperature usually decreasing its elasticity and *vice versa*, except in certain rare cases, like that of *invar steel*, whose elasticity remains practically unaffected by any changes in temperature. Thus, for example, lead becomes quite elastic and rings like steel when struck by a wooden mallet, if it be cooled in liquid air. And, again, a carbon filament, which is highly elastic at the ordinary temperature, becomes plastic when heated by the current through it, so much so that it can be easily distorted by a magnet brought near to it.

**109. Three Types of Elasticity.** Corresponding to the three types of strain, we have three types of elasticity, *viz.*,

(*i*) *linear elasticity*, or *elasticity of length, called* **Young's Modulus,** corresponding to linear (or tensile) strain ;

(*ii*) *elasticity of volume* or **Bulk Modulus**, corresponding to volume strain ; and

(*iii*) *elasticity of shape, shear modulus*, or **Modulus of Rigidity,** corresponding to shear strain.

(1) **Young's Modulus.** When the deforming force is applied to the body *only along a particular direction*, the change per unit length in *that* direction is called *longitudinal, linear or elongation strain*, and the force applied per unit area of cross-section is called *longitudinal or linear stress. The ratio of longitudinal stress to linear strain, within the elastic limit, is called Young's Modulus, and is usually denoted by the letter Y.*

Thus, if $F$ be the force applied normally to a cross-sectional area $a$, the *stress* is $F/a$. And, if there be change $l$ produced in the original length $L$, the *strain* is given by $l/L$. So that,

$$\textit{Young's Modulus, } Y = \frac{F/a}{l/L} = \frac{F.L}{a.l}.$$

Now, if $L = 1$, $a = 1$ and $l = 1$, we have $Y = F$.

In other words, if a material of unit length and unit area of cross-section could be pulled so as to increase in length by unity, *i.e.*, to double its length, the force applied would measure the value of Young's Modulus for it.

Since, however, the elastic limit is exceeded when the extension produced is $10^{-3}$ *cm./cm.*, the material will snap before *this much* extension is produced.

In cases, where, elongation produced is not proportional to the force applied, we can still determine Young's Modulus from the ratio $L.dF/a.dL$, where $dF/a$ is the infinitesimal increase in the longitudinal stress and $dL/L$, the corresponding increase in strain.

Or, $$Y = \frac{L}{a} \cdot \frac{dF}{dL}.$$

N.B. The particular case of rubber may, with advantage, be mentioned here, which the beginner finds so confusing, when, in ordinary conversational language, we refer to it as being '*elastic*'. For, he knows well enough that it requires a much smaller force than steel to stretch it, (and that, therefore, its elasticity is much less than that of steel). In fact, the value of Young's Modulus for rubber is about *one-fiftieth* of that of steel. What we mean when we say that it is elastic, therefore, is just that *it has a very large range of elasticity*, for, whereas a crystalline body can be stretched to less than even one per cent of its original length before reaching its elastic limit, rubber can be stretched to about eight times (or 80%) of its original length.

This high extensibility of rubber is due to its molecule containing, on an average, some 4,000 molecules of *isoprene* ($C_5H_8$), whose 20,000 carbon atoms, spreading out in a chain, make it very *long* and *thin*,—about 1/4000 *mm.* in length.

Rubber, in bulk, has thus been rightly compared to an intertwined mass of long, wriggling snakes,—its molecules, like the snakes, tending to uncoil when stretched and getting coiled up again when the stretching force is removed.

(2) **Bulk Modulus.** Here, the force is applied *normally* and *uniformly* to the whole surface of the body ; so that, *while there is a change of volume, there is no change of shape.* Geometrically speaking, therefore, we have here *a change in the scale of the coordinates* of the system or the body. *The force applied per unit area, (or pressure), gives the* **Stress**, *and the change per unit volume, the* **Strain,** *their ratio giving the Bulk Modulus for the body. It is usually denoted by the letter K.*

Thus, if $F$ be the force applied *uniformly and normally* on a surface area $a$, the stress, or pressure, is $F/a$ or $P$ ; and, if $v$ be the change in volume produced in an original volume $V$, the strain is $v/V$. and, therefore,

$$\textit{Bulk Modulus}, K = \frac{F/a}{v/V} = \frac{F.V}{a.v} = \frac{P.V}{v} \qquad [\because F/a = P.$$

If, however, the change in volume be not proportional to the stress or the pressure applied, we consider the infinitesimal change in volume $dV$, for the corresponding change in pressure $dP$ ; so that, we have $$K = dP.V/dV.$$

*The Bulk Modulus is sometimes referred to as* **incompressibility** and hence its reciprocal is called **compressibility** ; so that, *compressibility* of a body is equal to $1/K$, where $K$ is its *Bulk Modulus.* It must thus be quite clear that whereas bulk modulus is stress per unit strain, *compressibility represents strain per unit stress.*

Since fluids (*i.e.*, liquids and gases) can permanently withstand or sustain only a hydrostatic pressure, the only elasticity they possess is Bulk Modulus ($K$), which is, therefore, all that is meant when we refer to their *elasticity*. This, however, is of two types : *isothermal* and *adiabatic*.

For, when a fluid is compressed, there is always some heat produced. If this heat be removed as fast as it is produced, the temperature of the fluid remains constant and the change is said to be *isothermal* ; but if the heat be allowed to remain in the fluid, its temperature naturally rises and the change is then said to be *adiabatic*.

It can be easily shown that the *isothermal elasticity of a gas* (*i.e.*, when its temperature remains constant) *is equal to its pressure P, and its adiabatic elasticity equal to* $\gamma P$, where $\gamma$ is the ratio between $C_p$* and $C_v$* for the gas in question,—its value being 1·41 for air, [see solved Example 1 (*b*) at the end of the Chapter.]

It will thus be readily seen that the Bulk Modulus of a gas (whether isothermal or adiabatic) is *not a constant quantity*, unlike that of a solid or a liquid.

(3) **Modulus of Rigidity**. In this case, while there is a change in the shape of the body, there is no change in its volume. As indicated already, it takes place by the movement of contiguous layers of the body, one over the other, very much in the manner that the cards would do when a pack of them, placed on the table, is pressed with the hand and pushed horizontally. Again, speaking geometrically, we have, in this case, *a change in the inclinations of the coordinate axes* of the system or the body.

Consider a rectangular solid cube, whose lower face $aDCc$, (Fig. 171), is fixed, and to whose upper face a tangential force $F$ is applied in the direction shown. The couple so produced by this force and an equal and opposite force coming into play on the lower fixed face, makes the layers, parallel to the two faces, move over one another, such that the point $A$ shifts to $A'$, $B$ to $B'$, $d$ to $d'$ and $b$ to $b'$, *i.e.*, *the lines joining the two faces turn through an angle* $\theta$†.

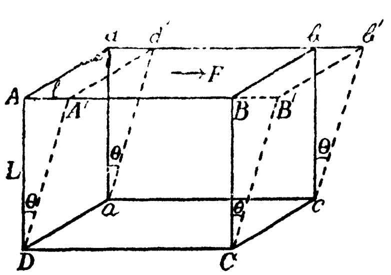


Fig. 171.

*The face ABCD is then said to be* **sheared** through *an angle* $\theta$. *This angle* $\theta$ (*in radians*), *through which a line originally perpendicular to the fixed face is turned, gives the strain or* **the shear strain**, *or the* **angle of shear**, *as it is often called.* As will

*The symbols $C_p$ and $C_v$, stand for the specific heats of a gas at constant pressure and at constant volume respectively,—their ratio $\gamma = C_p/C_v$, being the highest (1·67) for a mono-atomic gas, like helium, goes on decreasing with increasing atomicity of the gas but is always greater than 1.

†As a matter of fact, if this were the only couple acting on the body, it would result in the rotation of the body. This is prevented by another equal and opposite couple, formed by the weight of the body (plus any vertical force applied) and the reaction of the surface on which the body rests.

be readily seen, $\theta = AA'/DA = l/L$, where $l$ is the *displacement* $AA'$ and $L$, the length of the side $AD$ or the *height* of the cube ; or $\theta$ = *relative displacement of plane ABba/distance from the fixed plane aDCc.* So that, if the distance from the fixed plane, *i.e.*, $L = 1$, we have $\theta = l$ = *relative displacement* of plane $ABba$.

Thus, *shear strain (or shear) may also be defined as the relative displacement between two planes unit distance apart.*

And, *stress or* **tangential stress** *is clearly equal to the force F divided by the area of the face ABbd, i.e., equal to F/a. The ratio of the tangential stress to the shear strain gives the* **co-efficient of rigidity** *of the material of the body, denoted by n.*

Thus, *tangential stress* $= F/a$, and *shear strain* $= \theta = l/L$.

And, therefore, *Co-efficient of Rigidity*, or *Modulus of Rigidity* of the material of the cube is given by

$$n = \frac{F/a}{\theta} = \frac{F/a}{l/L} = \frac{F.L}{a \cdot l} \qquad \ldots \qquad \ldots \qquad \ldots (i)$$

*This is a relation exactly similar to the one for Young's Modulus, with the only difference that, here, F is the tangential stress, not a linear one, and l, a displacement at right angles to L, instead of along it.*

Again, if the shearing strain, or shear, be not proportional to the shearing stress applied, we have

$$n = \frac{dF/a}{d\theta},$$

where $d\theta$ is the increase in the angle of shear for an infinitesimal increase $dF/a$ in the shearing stress.

Further, it is clear from relation $(i)$ above, that if $a = 1$, and $\theta = 1$ *radian* (or 57° 18′), we have $n = F$.

*We may thus define modulus of rigidity of a material as the shearing stress per unit shear, i.e., a shear of* 1 *radian, taking Hooke's law to be valid even for such a large strain*.*

**110. Equivalence of a shear to a compression and an extension at right angles to each other.** Consider a cube $ABCD$, (Fig. 172), with the face $DC$ fixed, and let the face $ABCD$ be *sheared* by a force, applied in the direction shown, through an angle $\theta$, into the position $A'B'CD$. Then, clearly, the diagonal $DB$ is *increased in length* to $DB'$, and the diagonal $AC$ is *shortened* to $A'C$.

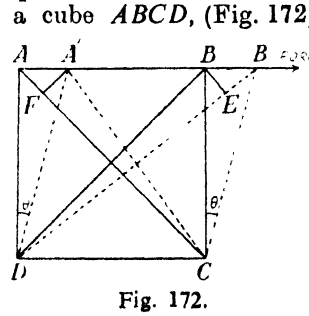

Fig. 172.

The shear is really very small in actual practice, and, therefore, triangles $AFA'$ and $BEB'$ are *isosceles right-angled triangles, (i.e., right-angled 45° triangles).*

*In the case of metals, however, Hooke's law no longer holds even if the shear exceeds 1/200 *radian*, or ·33°.

And, therefore, $EB' = BB'.\cos BB'E = BB'.1/\sqrt{2} = BB'/\sqrt{2}$.
$\because \angle BB'E = 45°$ and $\cos 45° = 1/\sqrt{2}$.

If $AB = l$, then, clearly, $DB = DE = l\sqrt{2}$.

$\therefore$ *extension strain* along diagonal $DB$

$$= \frac{EB'}{DB} = \frac{BB'}{\sqrt{2}}\cdot\frac{1}{l\sqrt{2}} = \frac{BB'}{2l} = \frac{\theta}{2}. \quad [\because BB'/l = \theta$$

Similarly, the *compression strain* along the diagonal $AC$ is given by

$$\frac{AF}{AC} = \frac{AA'.\cos A'AF}{AC} = \frac{AA'.\cos 45°}{l\sqrt{2}}.$$

$$= \frac{AA'}{l\sqrt{2}.\sqrt{2}} = \frac{AA'}{2l} = \frac{\theta}{2}. \qquad [\because AA'/l = \theta.$$

Thus, we see that *a simple shear $\theta$ is equivalent to two equal strains, an extension and a compression, at right angles to each other.*

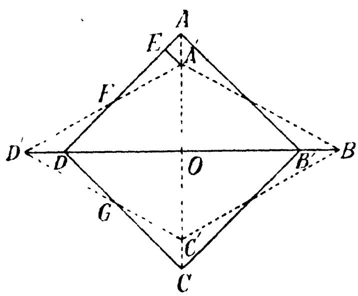

Fig. 173.

**Corollary.** *The converse of the above follows as a corollary, viz., that simultaneous equal compression and extension at right angles to each other are equivalent to a shear,* as will be seen from the following :

Let the cube $ABCD$, of side $l$, be compressed along the diagonal $AC$, so that the new diagonals become $A'C'$ and $B'D'$, (Fig. 173).

Let $\quad AA' = BB' = a$.

And since $\quad OA = AB \cos BAO$
$= AB.\cos 45° = AB/\sqrt{2}$
$= l/\sqrt{2}$,

we have $\qquad OA' = OA - AA' = \left(\frac{l}{\sqrt{2}} - a\right)$,

and $\qquad OB' = OB + BB' = \left(\frac{l}{\sqrt{2}} + a\right)$.

Clearly, $\therefore \qquad (A'B')^2 = (OA')^2 + (OB')^2$

$$= \left(\frac{l}{\sqrt{2}} - a\right)^2 + \left(\frac{l}{\sqrt{2}} + a\right)^2$$

$$= \frac{l^2}{2} - \frac{2al}{\sqrt{2}} + a^2 + \frac{l^2}{2} + \frac{2al}{\sqrt{2}} + a^2 = l^2 + 2a^2.$$

In practice, $2a^2$ is very small as compared with $l^2$, and may, therefore, be neglected.

So that, $\qquad (A'B')^2 = l^2$. Or, $\quad A'B' = l = AB$.

Thus, $A'B'C'D'$ may be rotated through the angle $DGD'$ = angle $AFA'$, so that $D'C'$ coincides with $DC$. Then, it is obvious that $A'D'$ would make an angle $2AFA'$ with $AD$, so that the *angle of shear* is equal to *twice* the angle $AFA'$, *i.e.*, is equal to $2\angle AFA'$.

Or, $\qquad$ *angle of shear* $= 2A'E/EF$, $\qquad$ [$\because$ the angle is small.

where $A'E$ is the perpendicular from $A'$, to $AF$.

Now, $A'E = a/\sqrt{2}$ and $EF = l/2$. $\left[\because A'E = AA' \cos EA'A = a \cos 45° = a/\sqrt{2}\right.$

$$\therefore 2\angle AFA' = \frac{2a}{\sqrt{2}} \div \frac{l}{2} = \frac{2a}{\sqrt{2}} \times \frac{2}{l} = \frac{2a\sqrt{2}}{l}.$$

Denoting this *angle of shear* by $\theta$, we have $\theta = 2a\sqrt{2}/l$.

Now, *compression strain* along the diagonal $AC$ is

$$\frac{AA'}{AO} = \frac{a}{l/\sqrt{2}} = \frac{a\sqrt{2}}{l} = \frac{\theta}{2}.$$

Or, *the compression strain is half the angle of shear, i.e., the angle of shear is twice the angle of compression.*

*Similarly, it can be shown that the extension strain is also half the angle of shear.*

Thus, we see that *simultaneous and equal compression and extension at right angles to each other are equivalent to a shear,* the direction of each strain being at an angle of 45° to the direction of shear.

**111. Shearing stress equivalent to an equal linear tensile stress and an equal compression stress at right angles to each other.** In the case of the cube above, if $F$ were the only force acting on its upper face it would move bodily in the direction of this force. Since, however, the cube is fixed at its lower face $DC$, an equal and opposite force comes into play in the plane of this face, giving rise to a couple $F.l$.*, tending to rotate the cube in the *clockwise* direction, (Fig. 174).

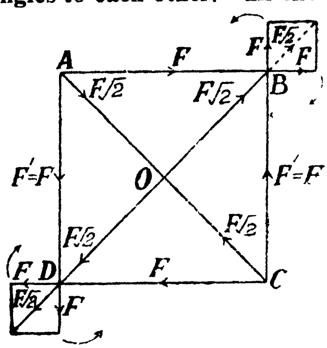

Fig. 174.

Again, since the cube does not rotate, it is obvious that the plane of $DC$ applies an *equal* and *opposite couple* $F'.l$, say, by exerting forces $F'$ and $F'$ along the faces $AD$ and $CB$, tending to rotate it in the *anticlockwise* direction, as shown. Thus, because the cube is in equilibrium under the two couples, we have

$$F.l = F'.l \quad \text{Or,} \quad F = F',$$

*i.e., a tangential force F applied to the face AB results in an equal tangential force acting along all the other faces of the cube in the directions shown.*

Clearly the resultant of the two forces $F$ and $F'$ or $F$ and $F$ along $AB$ and $CB$ respectively is $F\sqrt{2}$ along $OB$, and of those acting along $AD$ and $CD$ is also $F\sqrt{2}$ along $OD$. And, thus, an *outward* pull acts on the diagonal $DB$ of the cube at $B$ and $D$ resulting in its extension, as we have just seen above, (§110). Precisely similarly, an *inward* pull acts on the diagonal $AC$ at $A$ and $C$, thereby bringing about its compression.

*$l$ being the length of each edge of the cube and hence the perpendicular distance between the two forces $F$ and $F$.

Thus, a tangential force F applied to one face of a cube gives rise to a force $F\sqrt{2}$ *outward along one diagonal* (*BD*, in the case shown) and an equal force $F\sqrt{2}$ *inward along the other diagonal* (*AC*) of the cube, resulting in an extension of the former and a compression of the latter.

Now, if the cube be cut up into two halves, by a plane passing through *AC* and perpendicular to the plane of the paper, each face, parallel to the plane, will have an area $l \times l\sqrt{2} = l^2\sqrt{2}$, and, clearly, the outward force $F\sqrt{2}$ along *BD* will be acting perpendicularly to it. So that, we have

$$\text{tensile stress along } BD = F\sqrt{2}/l^2\sqrt{2} = F/l^2.$$

Similarly, if we cut the cube into its two halves by a plane passing through *BD* and perpendicular to the plane of the paper, we shall have an *inward* force $F\sqrt{2}$ along *AC* acting perpendicularly to a face on an area $l^2\sqrt{2}$. So that, we have

$$\text{compression stress along } AC = F\sqrt{2}/l^2\sqrt{2} = F/l^2.$$

Obviously, $F/l^2$ is the shearing stress over the face *AB* of the cube, which produces the shear $\theta$ in it, (see page 281).

Thus, it is clear that *a shearing stress is equivalent to an equal tensile stress and an equal compression stress at right angles to each other.*

**112. Work done per unit volume in a strain.** In order to *deform* a body, work must be done by the applied force. The energy so spent is stored up in the body and is called the *energy strain.* When the applied forces are removed, the stress disappears and *the energy of strain appears as heat.*

Let us consider the work done during the three cases of strain.

(*i*) **Elongation Strain**—(*stretch of a wire*). Let $F$ be the force applied to a wire, fixed at the upper end. Then, clearly, for a small increase in length $dl$ of the wire, the work done will be equal to $F.dl$. And, therefore, during the whole stretch of the wire from 0 to $l$.

$$\text{work done} = \int_0^l F.dl.$$

Now, Young's modulus for the material of the wire, *i.e.*,

$$Y = F.L/a.l.,$$

where $L$ is the original length, $l$, the increase in length, $a$, the cross-sectional area of the wire, and $F$, the force applied.

And $\therefore$ $$F = Y.a.l/L.$$

Therefore, work done during the stretch of the wire from 0 to $l$ is given by

$$W = \int_0^l \frac{Y.a}{L} . l.dl = \frac{Y.a}{L}\int_0^l l.dl.$$

$$= \frac{Y.a}{L} . \frac{l^2}{2} = \frac{1}{2} . \frac{Y.a.l}{L} . l.$$

But $\quad Y.a.l/L = F$, the *force* applied.

Hence $\quad W = \frac{1}{2} F.l = \frac{1}{2} \times$ *stretching force* $\times$ *stretch.*

$\therefore$ work done per *unit* volume $= \frac{1}{2} Fl \times \frac{1}{L.a}$. $\quad \left[\because\right.$ *volume* of the wire $= L \times a$.

$= \frac{1}{2} \cdot \frac{F}{a} \cdot \frac{l}{L} = \frac{1}{2}$ *stress* $\times$ *strain.* $\quad \left[\because F/a = \text{stress}\right.$, and $l/L$ = strain.

**Alternatively,** the same result may also be obtained graphically as follows :

Let a graph $OP$ be plotted between the *streitching force* applied to the wire and the *extension produced* in it, *within the elastic limit*, as shown in Fig. 175.

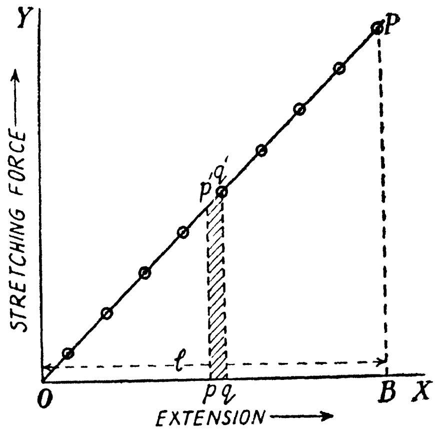

Fig. 175.

Consider a small extension $pq$ of the wire and erect ordinates at $p$ and $q$ to meet the graph in $p'$ and $q'$ respectively, where $pp'$ is very nearly equal to $qq'$, (the extension $pq$ being really small).

Then, clearly, *work done upon the wire or energy stored up in it*
= *stretching force* $pp' \times$ *extension* $pq$.
$= pp' \times pq =$ *area of strip* $pp'q'q$.

So that, imagining the whole extension $OB = l$, of the wire, to be broken up into small bits like $pq$ and erecting ordinates at their extremities, we have
*total work done upon the wire or total energy stored up in it*
= *sum of the areas of all such strips formed*
= *area of the triangle* $OBP = \frac{1}{2} OB \times BP = \frac{1}{2} l \times F$,
where the total extension $OB = l$ and the stretching force corresponding to it is $BP = F$.

Now, if $L$ be the original length of the wire and $a$, its area of cross-section, clearly, *volume of the wire* $= L \times a$.

$\therefore$ *work done*, or *strain energy, per unit volume of the wire*

$$= \tfrac{1}{2} lF/L.a = \frac{1}{2} \cdot \frac{F}{a} \cdot \frac{l}{L} = \frac{1}{2} \text{ stress} \times \text{strain}.$$

(*ii*) **Volume Strain.** Let $p$ be the stress applied. Then, over an area $a$ the force applied is $p.a$, and, therefore, the work done for a small movement $dx$, in the direction of $p$, is equal to $p.a.dx$. Now, $a.dx$ is equal to $dv$, the small change produced in volume. Thus, work done for a change $dv$ is equal to $p.dv$.

And, therefore, total work done for the *whole change in volume*, from 0 to $v$, is given by

$$W = \int_0^v p.dv.$$

Now, $\quad K = p.V/v$ ; so that, $p = K.v/V$, where $V$ is the original volume, and $K$, the *Bulk Modulus*.

$$\text{And } \therefore W = \int_0^v \frac{K.v}{V}\,.\,dv = \frac{K}{V}\int_0^v v.dv = \frac{K}{V}\cdot\frac{1}{2}v^2.$$

$$= \frac{1}{2}\cdot\frac{Kv}{V}.v = \frac{1}{2}p.v = \frac{1}{2}\ \textit{stress} \times \textit{change in volume}.$$

Or, work done per *unit* volume $= \frac{1}{2}\,p.v/V = \frac{1}{2}$ *stress* $\times$ *strain*.

(*iii*) **Shearing Strain.** Consider a cube (Fig. 176), with its lower face $DC$ fixed ; and let $F$ be the *tangential force* applied to its upper face in the plane of $AB$, so that the face $ABCD$ is distorted into the position $A'B'CD$, or *sheared* through an angle $\theta$. Let the distance $AA'$ be equal to $BB' = x$. Then, work done during a small displacement $dx$ is equal to $F.dx$. And, therefore, work done for the *whole of the displacement*, from 0 to $x$, is given by

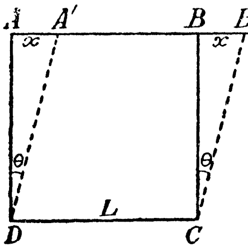


Fig. 176.

$$W = \int_0^x F.dx.$$

Now, $n = F/a\theta$, or $F = n.a.\theta$, and $a = L^2$ ; also $\theta = x/L$, where $L$ is the length of each edge of the cube.

So that, $F = n.L^2.x/L = n.L.x.$

$\therefore$ work done during the *wole stretch* from 0 to $x$, *i.e.*,

$$W = \int_0^x n.L.x.dx = \frac{1}{2}.n.Lx^2.$$

$\therefore$ work done per *unit* volume $= \frac{1}{2}\cdot\frac{n.L.x^2}{L^3}$. $\left[\because \text{ volume of the cube} = L^3\right.$

$$= \frac{1}{2}\cdot\frac{n.x.L}{L^2}\times\frac{x}{L} = \frac{1}{2}\cdot\frac{F}{a}\cdot\frac{x}{L} = \frac{1}{2}\ \textit{stress}\times\textit{strain}.$$

Thus, we see that, in any kind of strain, *work done per unit volume is equal to $\frac{1}{2}$ stress $\times$ strain.*

**113. Deformation of a Cube—Bulk Modulus.** Let $ABDCGHEFA$ be a *unit cube* and let forces $T_x$, $T_y$ and $T_z$ act perpendicularly to the faces $BEHD$ and $AFGC$, $ABDC$ and $EFGH$, and $ABEF$ and $DHGC$ respectively, as shown, (Fig. 177). Then, if $\alpha$ be the increase per *unit* length per *unit* tension *along the direction of the force* and $\beta$, the contraction produced per *unit* length per *unit* tension, *in a direction perpendicular to the force*, the elongation produced in the edges $AB$, $BE$ and $BD$, will, obviously, be $T_x.\alpha$, $T_y.\alpha$ and $T_z.\alpha$, respectively, and the contractions produced *perpendicular to them* will be $T_x.\beta$, $T_y.\beta$, and $T_z.\beta$. The lengths of the edges thus become the following :

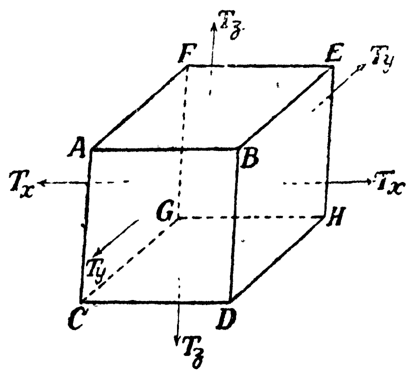


Fig. 177.

$$AB = 1+T_x.\alpha - T_y.\beta - T_z.\beta.$$
$$BE = 1+T_y.\alpha - T_x.\beta - T_z.\beta.$$
$$BD = 1+T_z.\alpha - T_x.\beta - T_y.\beta.$$

Hence the volume of the cube now becomes

$$(1+T_x.\alpha - T_y\,\beta - T_z.\beta)\,(1+T_y.\alpha - T_x.\beta - T_z.\beta)\,(1+T_z.\alpha - T_x\,\beta - T_y.\beta)$$
$$=1+(\alpha-2\beta)\,(T_x+T_y+T_z),$$

neglecting squares and products of $\alpha$ and $\beta$, which are very small compared with the other quantities involved.

If $$T_x = T_y = T_z = T,$$

the volume of the cube becomes $1+(\alpha-2\beta).3T$.

And, therefore, *increase in the volume of the cube*

$$= 1+3T(\alpha-2\beta)-1 = 3T(\alpha-2\beta).$$

If, instead of the tension $T$ outwards, we apply a pressure $P$, compressing the cube, *the reduction in its volume* will similarly be $3P(\alpha-2\beta)$, and, therefore, *volume strain* is equal to $3P(\alpha-2\beta)/1$, or equal to $3P(\alpha-2\beta)$. [$\because$ *original volume* of the cube = 1.

Hence *Bulk Modulus*, $K = \dfrac{stress}{volume\ strain} = \dfrac{P}{3P(\alpha-2\beta)}$.

Or, $$K = \frac{1}{3(\alpha-2\beta)}. \qquad ...(I)$$

And, *Compressibility*, which is the *reciprocal of Bulk Modulus*, is, therefore, equal to $3(\alpha-2\beta)$.

**114. Modulus of Rigidity.** Let the top face $ABHG$, of a cube (Fig. 178), be '*sheared*' by a shearing force $F$, *relative to the bottom face*, such that $A$ takes up the position $A'$ and $B$, the position $B'$, the angle $ADA'$ being equal to the angle $BCB' = \theta$. Then,

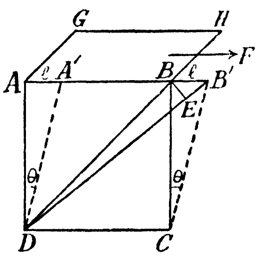


Fig. 178.

$$stress = \frac{F}{area\ of\ the\ face\ ABHG}.$$
$$= \frac{F}{L^2} = T, \text{ say},$$

where $L$ is the length of each edge of the cube.

Let the displacement $AA' = BB' = l$.

Then, *shear strain* $= l/L = \theta$.

And $\therefore$ *coefficient of rigidity*, $n = T/\theta$.

Now, extension of the diagonal $DB$, due to extension along $AB$ is $DB.T.\alpha$, and that due to contraction* along $CA$ is $DB.T.\beta$. Therefore, *total extension in length of the diagonal* $DB$ now becomes

$$DB\times(T.\alpha+T.\beta) = DB\times T(\alpha+\beta).$$
$$= L\sqrt{2}.T(\alpha+\beta). \quad [\because DB = \sqrt{L^2+L^2} = L\sqrt{2}$$

Drop a perpendicular $BE$ from $B$ on to $DB'$.

* See § 117, page 288.

Then, *increase in length of DB* is practically equal to $EB'$.

And, clearly, $EB' = BB'.\cos BB'E.$

$= l\cos 45° = l/\sqrt{2}$ [$\because \angle BB'E = 45°$, very nearly, and $\cos 45° = 1/\sqrt{2}$.

$$\therefore \quad L\sqrt{2}.T(\alpha+\beta) = l/\sqrt{2}.$$

Or, $$\frac{L.T}{l} = \frac{1}{2(\alpha+\beta)}. \quad \text{Or,} \quad \frac{T}{l/L} = \frac{1}{2(\alpha+\beta)}.$$

Or, $$\frac{T}{\theta} = \frac{1}{2(\alpha+\beta)}. \qquad [\because l/L = \theta.$$

And, since $T/\theta = n$, the *coefficient of rigidity* of the material of the cube, we have

$$n = \frac{1}{2(\alpha+\beta)}. \qquad ...(II)$$

**115. Young's Modulus.** If we now imagine a *cube of unit edge*, acted upon by *unit tension* along one edge, the extension produced is $\alpha$. Then, clearly,

$$\textit{stress} = 1, \quad \text{and} \quad \textit{linear strain} = \alpha/1 = \alpha.$$

Therefore, *Young's Modulus*, $Y = 1/\alpha$. ...(*III*)

**116. Relation connecting the Elastic Constants.** We have from relation (*I*), above,

$$\alpha - 2\beta = 1/3K. \qquad ...(i)$$

And, from relation (*II*), $\alpha+\beta = 1/2n.$ .. (*ii*)

$\therefore$ subtracting (*i*) from (*ii*), we have

$$3\beta = \frac{1}{2n} - \frac{1}{3K} = \frac{3K-2n}{6nK},$$

whence, $$\beta = \frac{3K-2n}{18nK}.$$

Again, multiplying (*ii*) by 2 and adding to (*i*), we have

$$3\alpha = \frac{1}{n} + \frac{1}{3K} = \frac{3K+n}{3Kn}. \quad \text{Or,} \quad \alpha = \frac{3K+n}{9Kn}.$$

$$\therefore \quad \frac{1}{Y} = \frac{3K+n}{9Kn}. \quad \text{Or,} \quad Y = \frac{9Kn}{3K+n}. \qquad ..(a)$$

[$\because \alpha = 1/Y$ from (III), above.

Or, $$\frac{9}{Y} = \frac{3K+n}{Kn} = \frac{3K}{Kn} + \frac{n}{Kn},$$

whence, $$\frac{9}{Y} = \frac{3}{n} + \frac{1}{K}. \qquad ...(b)$$

This, then, is the relation connecting the three elastic constants.

**117. Poisson's Ratio.** It is a commonly observed fact that when we stretch a string or a wire, *it becomes longer but thinner, i.e.*, the increase in its length is always accompanied by a decrease in its cross-section (though not sufficient enough to prevent a slight increase in its volume). In other words, *a longitudinal or tangential strain produced in the wire is accompanied by a transverse or a lateral strain in it.* And, of course, what is a true of a wire, is true of all other bodies under strain. Thus, for example, when a cube is subjected to an outward force perpendicular to one pair of its faces, there is elongation produced along this face, but a contraction in a direction perpendicular to it, (as we have seen already in §113).

*The ratio between lateral strain* ($\beta$) *to the tangential strain* ($\alpha$) *is constant* for a body of a given material and is called the Poisson's ratio for that material. It is usually denoted by the letter* $\sigma$.
Thus, *Poisson's ratio = lateral strain/tangential strain* ; or, $\sigma = \beta/\alpha$.

It follows, therefore, that if a body under tension suffers no lateral contraction, the *Poisson's ratio* ($\sigma$) for it is *zero* ; and, because its volume increases, is density decreases.

The relations for $K$ and $n$ above may now be put in terms of *Poisson's ratio, as follows* :

We have, from relation *I*, above,

$$K = \frac{1}{3\alpha\left(1-\frac{2\beta}{\alpha}\right)} = \frac{1}{3\alpha(1-2\sigma)} = \frac{Y}{3(1-2\sigma)}. \left[\because \frac{1}{\alpha} = Y \text{ [see (III) above.}\right.$$

whence, $$Y = 3K(1-2\sigma),\dagger \qquad \ldots \qquad \ldots (iv)$$

Similarly, from relation (II) above, we have

$$n = \frac{1}{2\alpha\left(1+\frac{\beta}{\alpha}\right)} = \frac{1}{2\alpha(1+\sigma)} = \frac{Y}{2(1+\sigma)},$$

whence, $$Y = 2n(1+\sigma)\dagger \qquad .. \qquad .. (v)$$

Now, from relations (*iv*) and (*v*) we have

$$3K(1-2\sigma) = 2n(1+\sigma),$$

whence, $$\sigma = \frac{3K-2n}{6K+2n},$$

which gives the value of *Poisson's ratio* in terms of $K$ and $n$.

Similarly, if we eliminate $\sigma$ from (*iv*) and (*v*), we have

$$Y = \frac{9Kn}{3K+n}, \quad \text{[Same as relation (}a\text{), above.}$$

whence, $$\frac{9}{Y} = \frac{3}{n} + \frac{1}{K}. \quad \text{[Same as relation (}b\text{), above.}$$

**Limiting values of $\sigma$.** We have seen above how

$$3K(1-2\sigma) = 2n(1+\sigma),$$

where $K$ and $n$ are essentially *positive* quantities. Therefore,

(*i*) *if $\sigma$ be a positive quantity*, the right hand expression, and hence also the left hand expression, must be positive, and for this to be so, $2\sigma < 1$, or $\sigma < \frac{1}{2}$ or $\cdot 5$. And,

---

**i.e.*, the lateral strain is proportional to the longitudinal strain. This is, however, so only when the latter is small.

† These relations would not be found to apply in the case of wire specimens of materials for the simple reason that the process of wire-drawing brings about at least a partial alignment of the minute crystals of the substance, which thus no longer remain oriented at random, with the result that the substance loses its isotropic character.

(*ii*) *if* $\sigma$ *be a negative quantity*, the left hand expression, and hence also the right hand expression, must be positive, and this is possible only when $\sigma$ be not less than $-1$.

Thus, the limiting values of $\sigma$ are $-1$ and $\cdot 5$. Or, else, as will be readily seen from relations (*iv*) and (*v*) above, either the bulk modulus or the modulus of rigidity would become *infinite*. Further, a negative value of $\sigma$ would mean that, on being extended, a body should also expand laterally, and one can hardly expect this to happen, ordinarily. At least, we know of no such substance so far. Similarly, a value of $\sigma = 0{\cdot}5$ would mean that the substance *is perfectly incompressible*, and, frankly, we do not know of any such substance either.

In actual practice, the value of $\sigma$ is found to lie between $\cdot 2$ and $\cdot 4$, although Poisson had a theory that the value of $\sigma$ for all elastic bodies should be $\cdot 25$, but this is not borne out by any experimental facts.

**118. Determination of Young's Modulus.** Young's modulus, as we know, is the ratio between *tensil stress* (or tangential force applied per unit area) and *elongation strain* (or extension per unit length). The extension produced is rather small and it is difficult to measure it with any great degree of accuracy. The different methods used are thus merely attempts at measuring this extension accurately. We shall consider here only two methods, *viz.*, one for a wire, and the other for a thick bar.

(*i*) **For a Wire—Searle's Method.** Two wires, $A$ and $B$, of the *same* material, length and area of cross-section, are suspended from a rigid support and carry, at their lower ends, two metal frames, $C$ and $D$, as shown in Fig. 178, one carrying a constant weight $W$ to keep the wire stretched or *taut* and the other, a hanger $H$, to which slotted weights can be slipped on, as and when desired.

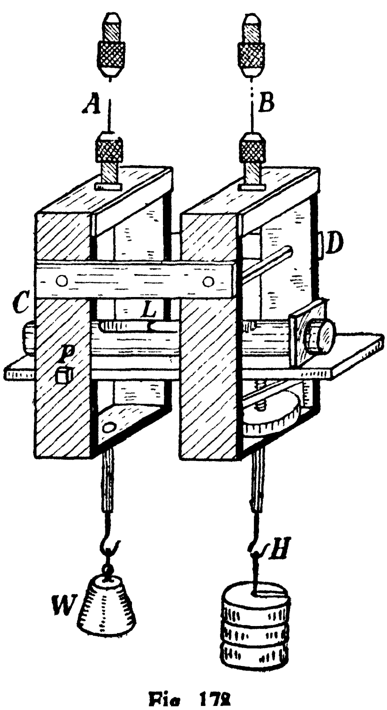

Fig. 178

A spirit-level $L$ rests *horizontally* at a point $P$ in frame $C$, and on the tip of a *micrometer screw* (or spherometer) $S$, working through a nut in frame $D$.

The screw is worked up or down, until the air bubble in the spirit-level is just in the centre. Weights are now slipped on to the hanger, so that the frame $D$ moves down a little due to the extension of wire $B$, and the air bubble shifts towards $P$. The screw is now worked up to restore the bubble back to its central position. The distance through which the screw is moved up is read on the vertical scale, graduated in half-millimetres.

and fixed alongside the disc of the screw. This gives the increase in length of wire $B$. A number of observations are taken by increasing the weight in the hanger by the *same equal steps* and making the adjustment for the level for each additional weight. The mean of all these readings of the screw gives the mean increase in the length of the wire, for the stretching force due to the given weight. Thus, if $l$ *cms.* be the increase in the length of wire $B$, and $L$ *cms.*, its original length, we have

$$\textit{elongation strain} = l/L.$$

And if $W$ *k.gms.* be the weight added each time to the hanger, the stretching force is equal to $W \times 1000$ *gms. wt.* $= W \times 1000 \times 981$ *dynes*, or equal to $F$ *dynes*, say.

So that, if $a$ *sq. cms.* be the area of cross-section ($\pi r^2$) of the wire, we have *tensile stress* $= F/a$.

And, ∴ *Young Modulus* for the material of the wire, *i.e.*,

$$Y = \frac{F}{a} \div \frac{l}{L} = \frac{F \times L}{a \times l}.$$

The other wire $A$ merely acts as a *reference wire*, its length remaining constant throughout, due to the constant weight suspended from it (which need not be known). Any yielding of the support or change in temperature during the experiment affects both the wires equally, and the *relative increase* in the length of $B$ (with respect to $A$) thus remains unaffected by either change.

If a graph be now plotted between the load suspended and the extension produced, it would be found to be a straight line (just like $OA$ in Fig. 169), passing through the origin, showing that the extension produced is directly proportional to the load. Hooke's law also can thus be easily verified.

(*ii*) **For a thick Bar—Ewing's Extensometer Method.** *Ewing's Extensometer* is merely a device to magnify the small extension of the bar under test and consists of two metal arms, $APS$ and $CQD$, (Fig. 179), pivoted at $P$ and $Q$, by means of pointed screws, on the vertical bar $B$ itself, (the *Young's modulus* for the material of which is to be determined), so that they are free to rotate about $P$ and $Q$. The arm $APS$ is bent at right angles, as shown, and carries a *micrometer screw* $S$ at its lower end, and a microscope $M$, fitted with a *micrometer scale*, at the end of an arm, pivoted at its upper end $A$.

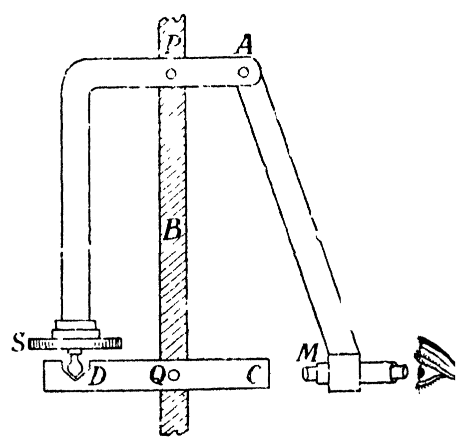


Fig. 179.

The other horizontal arm $CQD$, has a $V$-shaped groove at $D$ for the micrometer screw to rest in, and a fine *horizontal line* marked on the end $C$.

The bar $B$ is fixed at its upper end, the two metal arms are adjusted to be horizontal, by means of the micrometer screw $S$, and the microscope focused on the horizontal line on $C$. The bar is now stretched downwards (by means of a *testing machine*), so that *the horizontal arm CQD gets tilted a little about D as its fulcrum*,—the end $C$, with the fine mark on it, moving down to $C'$, (Fig. 180), and the point $Q$ to $Q'$. The microscope is again focused on the mark and the distance $CC'$ through which it has shifted downwards is measured accurately on the micrometer scale of the eye-piece. Let it be equal to $h$.

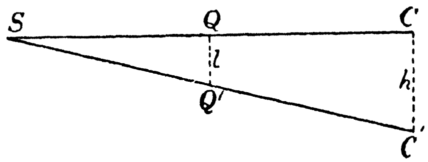


Fig. 180.

Now, obviously, the increase in the length $PQ$ of the rod is $QQ' = l$, say.

Then, clearly, in the two similar triangles $SQQ'$ and $SCC'$,

we have $QQ'/CC' = SQ/SC$. Or, $l/h = SQ/SC$,

whence, $l = SQ.h./SC$.

Thus, knowing $SQ$, $SC$ and $h$, we can determine $l$ to quite a high degree of accuracy.

Then, from the length $PQ$ of the bar, its area of cross-section and the stretching force applied to it, we can easily calculate the value of *Young's Modulus* for its material.

**N.B.** A modification of Ewing's Extensometer, as shown in Fig. 181, is called the *Cambridge Extensometer*, in which there is a vibrating reed $R$ arranged, as shown, the arrangement being such that as the bar $B$ is stretched by the testing machine, *that part of the reed which touches the micrometer screw M, moves downwards through a distance five times the extension of the rod.* Thus, by noting the micrometer screw readings, when the vibrating reed just touches the micrometer screw-point both before and after the rod $B$ has been stretched, we can *directly* obtain the increase $l$ in the length of the rod.

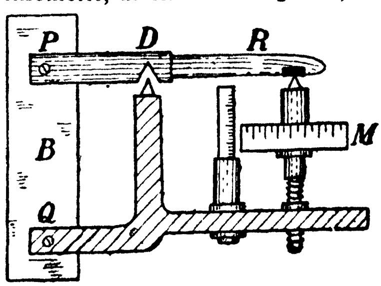


Fig. 181.

**119. Determination of Poisson's Ratio for Rubber.** To determine the value of $\sigma$ for rubber, we take about a metre-long tube $AB$ of it, (Fig. 182), such, for example, as the tube of an ordinary cycle tyre, and suspend it vertically, as shown, with its two ends properly stoppered with rubber bungs and seccotine*. A glass tube $C$, open at both ends, about half a metre long and about 1 *cm*. in diameter, graduated in cubic centimetres, is fitted vertically into it through

* a type of liquid glue.

a suitable hole in the stopper at the upper end $A$, so that a major part of it projects out.

The rubber tube is completely filled with water until the water rises up in the glass tube to a height of about 30 *cms.* from $A$. A suitable weight $W$ is now suspended from the lower end $B$ of the tube. This naturally increases the length as well as the internal volume of the tube. The increase in length is read conveniently on a vertical metre scale $M$, with the help of a pointer $P$, attached to the suspension of $W$, and the increase in volume, from the change in the position of the water column in $C$.

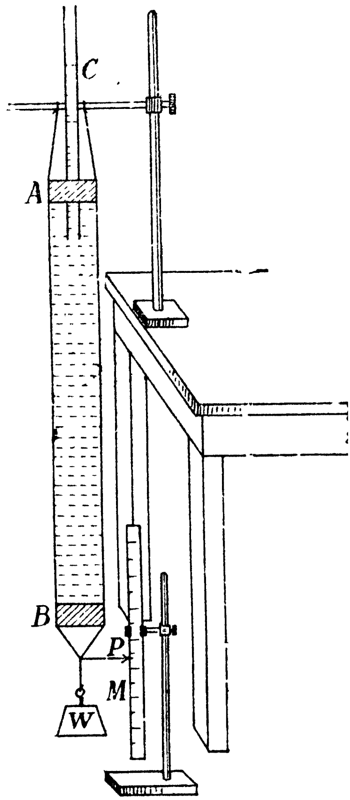

Fig. 182.

Let the original *length*, *diameter* and *volume* of the rubber tube be $L$, $D$ and $V$ respectively.

Then, its *area of cross-section*, $A = \pi(D/2)^2 = \pi D^2/4$, .. .. $(i)$

differentiating which, we have

$$dA = \frac{\pi D}{2} \cdot dD,$$

whence, $\quad dA = 2A.dD/D. \quad \ldots (ii)$

[From $(i)$ above, by eliminating $\pi$.

Now, if corresponding to a small increase $dV$ in the volume of the rubber tube, the increase in its length be $dL$, and the decrease in its area of cross-section be $dA$, we have

$$V+dV = (A-dA)(L+dL). \qquad \left[\because \text{ volume} = \text{area of cross-section} \times \text{length.}\right.$$

$$= AL+A.dL-dA.L-dA.dL$$

Or, $\quad V+dV = V+A\,dL-dA.L, \qquad$ [where $A.L = V$, the original volume of the tube.

neglecting $dA.dL$, as a very small quantity, compared with the other terms in the expression.

So that, $\quad dV = A.dL - dA.L = A.dL - \frac{2AL}{D}dD. \quad$ [Substituting the value of $dA$ from $(i)$, above.

Or, dividing both sides by $dL$, we have

$$\frac{dV}{dL} = A - \frac{2AL}{D} \cdot \frac{dD}{dL}. \quad \text{Or,} \quad \frac{2AL}{D} \cdot \frac{dD}{dL} = A - \frac{dV}{dL}.$$

whence, $$\frac{dD}{dL} = \left(A - \frac{dV}{dL}\right) \bigg/ \frac{2AL}{D} = \frac{AD}{2AL} - \frac{dV}{dL} \cdot \frac{D}{2AL}.$$

$$= \frac{D}{2L} - \frac{dV}{dL} \cdot \frac{D}{2AL}.$$

Or, $$\frac{dD}{dL} = \frac{D}{2L}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right) \qquad ...(iii)$$

Now, *Poisson's ratio*, $\sigma = \frac{lateral\ strain}{longitudinal\ strain} = \frac{dD/D}{dL/L} = \frac{dD}{D} \times \frac{L}{dL}$.

Or, $$\sigma = \frac{L}{D} \cdot \frac{dD}{dL}.$$

Or, substituting the value of $dD/dL$ from relation (*iii*) above, we have

$$\sigma = \frac{L}{D} \cdot \frac{D}{2L}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right) = \frac{1}{2}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right).$$

Thus, knowing the *area of cross-section* ($A$) of the tube, the *change in its volume* ($dV$) and the *change in its length* ($dL$), we can easily calculate the value of $\sigma$ for its material.

N.B.—An identical method may be used for the determination of the value of $\sigma$ for glass, but since the change in its volume is comparatively much too small, we have to use a capillary tube, instead of an ordinary glass tube, to measure it to an adequate degree of accuracy.

**120. Resilience.** By the *resilience* of an elastic body we understand its *capacity for resisting a blow or a mechanical shock, without acquiring a permanent set*,* and we measure it by the *amount of work done in straining the body up to the elastic limit.* Let us consider it for a uniform bar of length $L$ and area of cross-section $a$.

We know that when the bar is subjected to a stretching force $W$, so that it increases in length by $l$, we have

*Young's modulus for the material of the bar*, $Y = \frac{stress}{strain}$

$$= \frac{W/a}{l/L} = \frac{W}{a} \cdot \frac{L}{l} = \frac{F}{\text{strain}},$$

where $F$ denotes the stress $W/a$.

Now, *work done per unit volume in elongation strain* $= \frac{1}{2}$ *stress* $\times$ *strain.*
$\therefore$ *work done in producing extension* $l = \frac{1}{2}$ (*stress* $\times$ *strains*) $\times$ *volume.*

$$= \tfrac{1}{2} F \cdot \frac{F}{Y} \times volume\ (V) = \frac{1}{2} \frac{F^2}{Y} \cdot V = \frac{VF^2}{2Y} \qquad \left[ \because \text{strain} = F/Y. \right.$$

Thus, *work done*, or *resilience of the bar*,

$$= \frac{VF^2}{2Y} = \frac{volume \times (stress)^2}{2 \times Young's\ modulus}.$$

And $\therefore$ *resilience per unit volume of the bar*

$$= \frac{F^2}{2Y} = \frac{(stress)^2}{2 \times Young's\ modulus}.$$

*Height from which the bar can be dropped without acquiring a permanent set.*—Since resilience is a measure of the power to resist a

*The meaning attached to the word '*resilience*' in our common everyday parlance is different, *viz.*, that the body comes back to its normal condition when the applied forces are removed.

So that, $$\frac{(I+I_1)-I}{I} = \frac{t_1^2-t^2}{t^2}.$$ [Subtracting the denominator from the numerator on either side.]

Or, $$\frac{I_1}{I} = \frac{t_1^2-t^2}{t^2}, \text{ whence, } I_1 = \left[\frac{t_1^2-t^2}{t^2}\right]\times I.$$

Thus, knowing $I$, $t$ and $t_1$, we can easily calculate $I_1$, the moment of inertia of the given body.

(*b*) **Comparison of Moments of Inertia.** If, however, it is simply desired to *compare* the moments of inertia of two bodies, we first use one and then the other, *as the disc or rod of the torsional pendulum*, and determine the time-periods $t_1$ and $t_2$ respectively for their torsional vibration about the wire as axis. Then, if $I_1$ and $I_2$, be their respective moments of inertia about this axis and $C$, the torsional couple per unit twist of the wire, we have

$$t_1 = 2\pi\sqrt{I_1/C} \text{ and } t_2 = 2\pi\sqrt{I_2/C}.$$

So that, squaring and dividing one by the other, we have

$$I_1/I_2 = t_1^2/t_2^2,$$

and thus, knowing $t_1$ and $t_2$, the moments of inertia of the two bodies may be easily compared.

Note. In the above cases, the amplitude of vibration need not be small, because it is found that the restoring couple continues to be proportional to the twist $\theta$ in the wire, up to fairly large values of $\theta$. The assumption made, however, that even with different bodies suspended from the wire, resulting in a change in its longitudinal tension, the value of $C$ (or the twisting couple per unit twist of the wire) remains the same is found to be only approximately true.

**129. Bending of Beams—Bending Moment.** We must first be clear about the terms, *beam* and *bending moment.*

**Beam.** A beam is a rod of uniform cross-section, circular or rectangular, whose length is very great compared with its thickness, so that the shearing stresses over any section are small and may be neglected.

**Bending Moment.** When a beam is fixed at one end and loaded at the other, it bends due to the moment of the load, *the plane of bending* being the same as that of the couple applied.* Restoring forces are called into play by this *deformation* of the beam and, *in the equilibrium state, the restoring or resisting couple is equal and opposite to the bending couple, both being in the plane of bending.*

Irrespective of the manner in which the beam is bent by the couple applied, its filaments on the inner or the concave side get shortened or compressed, and those on the outer or the convex side get lengthened or extended, as shown in Fig. 191. Along a section, in between these two portions, there is a layer or surface in which the filaments are neither compressed

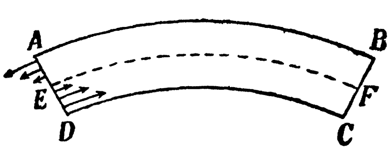

Fig. 191.

*In the case of uniform bending, the longitudinal filaments all get bent into circular arcs in planes parallel to the plane of symmetry, which is then known as the *plane of bending*. And, the straight line, *perpendicular to this plane*, on which lie the centres of curvature of all these bent filaments, is called the *axis of bending*.

nor extended. This surface is called the **neutral surface** and its section ($EF$) by the plane of bending which is perpendicular to it, is called the **neutral axis.**

In the unstrained condition of the beam, the neutral surface becomes a plane surface, and the filament of this unstrained or unstretched layer or surface, lying in the plane of symmetry of the bent beam, is referred to as the **neutral filament**. It passes through the *c.g.* (or the *centroid*) of every transverse section of the beam.

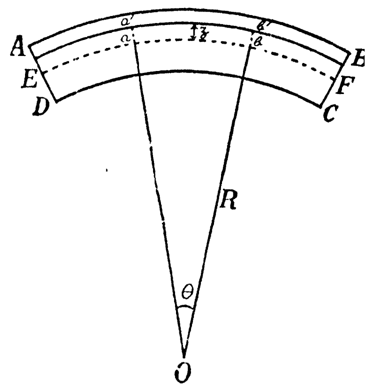

Fig. 192.

The change in length of any filament is proportional to its distance from the *neutral surface.*

Let a small part of the beam be bent, as shown in Fig. 192, in the form of a circular arc, subtending an angle $\theta$ at the centre of curvature $O$. Let $R$ be the radius of curvature of this part of the *neutral axis*, and let $a'b'$ be an element at a distance $z$ from the *neutral axis.*

Then, $a'b' = (R+z).\theta$,

and its original length $ab = R\theta$.

$\therefore$ *increase in length of the filament* $= a'b' - ab$.

$$= (R+z).\theta - R.\theta = z.\theta.$$

And, since the original length of the filament $= R.\theta$, we have

$$strain = z.\theta/R.\theta = z/R,$$

*i.e., the strain is proportional to the distance from the neutral axis.*

Since there are no shearing stresses, nor any change of volume, *the contractions and extensions of the filaments are purely due to forces acting along the length of the filaments.*

If $PQRS$ (Fig. 193), be a section of the beam* *at right angles to its length and the plane of bending*, then, clearly, the forces acting on the filaments are perpendicular to this section, and the line $MN$ lies on the *neutral surface.*

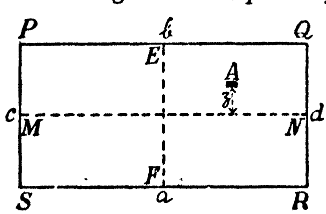

Fig. 193.

Let the *breadth* of the section be $PQ = b$, and its *depth*, $QR = d$.

The forces producing elongations and contractions in filaments act perpendicularly to the upper and the lower halves, $PQNM$ and $MNRS$ respectively, of the rectangular section $PQRS$, their directions being *opposite* to each other.

---

*The section is shown rectangular purely for the sake of convenience.

Consider a small area $\delta a$ about a point $A$, distant $z$ from the ***neutral surface***. The strain produced in a filament passing through this area will be $z/R$, (see above).

Now, $Y = stress/strain$ and $\therefore$ $stress = Y \times strain$.

Therefore, *stress* about the point $A = Y \times z/R$, where $Y$ is the value of *Young's Modulus* for the material of the beam.

And, therefore, *force on the area* $\delta a = \delta a.Y.z/R$

and, *moment of this force about the line* $MN = Y.z \times \delta a \times z/R$.

$$= Y.\delta a.z^2/R.$$

Since the moments of the forces acting on both the upper and the lower halves of the section are in the same direction, the ***total moment of the forces acting on the filaments in the section PQRS*** is given by

$$\Sigma \frac{Y.\delta a.z^2}{R} = \frac{Y}{R} \Sigma \delta a.z^2.$$

Now, $\Sigma \delta a.z^2$ is the *geometrical moment of inertia* $(I_g)$* of the section about $MN$, and, therefore, equal to $ak^2$, where $a$ is the ***whole area*** of the surface $PQRS$ and $k$, its ***radius of gyration*** about $MN$.

Hence, the *moment of the forces about* $MN = \frac{Y}{R} \cdot ak^2 = \frac{YI_g}{R}$.

This, then, balances the couple of moment $M$, say, called the ***bending moment***, acting on the beam due to the load, when the beam is in equilibrium ; for, there is no resultant force acting on the area $PQRS$, and the resultant moment about $EF$, perpendicular to $MN$, is also *zero*. In other words, it is the moment of the stress set up in the beam or the **moment of resistance to bending**, as it is usually called in engineering practice, and is also *of the nature of a couple*, for only a couple can balance a couple. Obviously, it *acts in the plane of* ***bending*** and is *equal to* the *bending moment at the section due to the* ***load***, though, quite frequently, (but, not strictly correctly) it is itself referred to as the ***bending moment***. This forms the very basis of the theory† regarding the bending of beams and is, therefore, a relation of fundamental importance.

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*It is so called because it is proportional to the mechanical moment of inertia of a plane lamina of the same shape as the cross-section. It is denoted, here, by the symbol $I_g$, so that, the student may not confuse it with the ordinary mechanical moment of inertia, denoted by $I$.

†The theory is subject to the limitations mentioned in §131, (page 313), which the student would do well to keep in mind.

Imagine the section as a rectangular plate of ***unit mass per unit area***, (Fig. 194).

Then, area of the strip $AB$, of length $b$ and breadth $dz$, is equal to $b.dz$. And, therefore, its mass $= b.dz.1 = b.dz$.

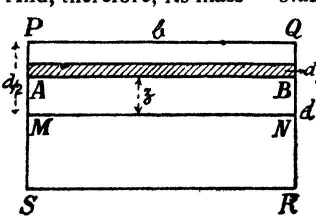


Fig. 194.

Hence, geometric moment of inertia of the strip about $MN = b.dz\, z^2$, and, therefore, moment of inertia of the whole plate or section about $MN$

$$= 2\int_0^{d/2} b.z^2.dz = 2b\left[\frac{z^3}{3}\right]_0^{d/2}$$

$$= 2b\left[\frac{d^3}{24}\right] = \frac{bd^3}{12}.$$

The quantity $Y.I_g = Y.ak^2$ is called the **flexural rigidity** of the beam.

$\therefore$ *bending moment* $= (Y/R) \times$ *geometric moment of inertia of the section.*

$= $ *flexural rigidity*$/R$,

whatever the shape of the cross-section of the beam.

For a *rectangular cross-section*, $a = b \times d$, and $k^2 = d^2/12$.

$$\therefore \qquad I_g = ak^2 = b.d^3/12.$$

Hence, *bending moment for a rectangular cross-section* $= Y.b.d^3/12R$.

For a *circular section*, $a = \pi r^2$ and $k^2 = r^2/4$.

$$\therefore \qquad I_g = ak^2 = \pi r^4/4,$$

*i.e.*, the same as the moment of inertia of a disc about a diameter.

$\therefore$ *bending moment for a circular cross-section* $= Y.\pi r^4/4R$.

**Note.** We have seen above how strain in a beam is proportional to the distance $z$ from its neutral axis, and is equal to $z/R$, where $R$ is the radius of curvature of the portion of the neutral axis under consideration. So that, if $F$ be the stress corresponding to the strain $z/R$, we have

$$Y = \frac{F}{z/R} \cdot \text{ Or, } \frac{F}{z} = \frac{Y}{R}.$$

If, therefore, $F_1$, $F_2$ etc. be the values of stress at distances $z_1$, $z_2$ · from the neutral axis, we have

$$F_1/z_1 = F_2/z_2 = Y/R.$$

And $\therefore$ *bending moment* $M = Y\,I_g/R$.

$$= \frac{F_1}{z_1}.I_g = \frac{F_2}{z_2}.I_g \text{ etc.} = \frac{I_g}{z_1}.F_1 = \frac{I_g}{z_2}.F_2 \text{ etc.}$$

$$= Z_1 F_1 = Z_2 F_2 \text{ etc.},$$

where $Z_1 = I_g/z_1$ and $Z_2 = I_g/z_2$ are called the **moduli of the section** under consideration.

Thus, **modulus of a section** $=$ $\dfrac{\textbf{geometrical moment of inertia}}{\textbf{distance from the neutral axis}}$.

Now, in the case of a flat bar or beam, *of rectangular cross-section*, if the bending be small, there is brought about a change in the shape of the section, such that all lines in it, originally perpendicular to the plane of bending, get bent into arcs, which are all *concentric* and *convex to the axis of bending*. In other words, the layer of the beam, which was originally plane and perpendicular to the plane of bending, and which contained the neutral filament, *now* gets changed into what is called an *anticlastic* surface (Fig. 195), of radius $R$ in the plane of bending (which, here, coincides with the plane of the paper), and, of radius $R'$ in the plane perpendicular to it, the two centres of curvature lying on either side of the beam. This is what is to be expected, because a transverse bending must, of necessity, be associated with a longitudinal bending of the beam, with the curvature of the former opposite to that of the latter. For, the filaments above the neutral axis, which get extended, must obviously suffer a lateral contraction $\sigma$ times as great and, similarly, the filaments below the neutral axis, which get compressed, must suffer a lateral extension.

Thus, by way of illustration, if a rectangular piece of India-rubber be bent longitudinally in the form of an arc, it takes up the form shown in Fig. 196, with its longitudinal fibres bent so as to be concave with respect to a point *below*, and the transverse fibres, so as to be concave with respect to a point *above*, the rubber piece, in the case shown. *It is this bending, which occurs in a plane normal to the longitudinal plane, that gives the rubber piece (or the beam) an anticlastic curvature.*

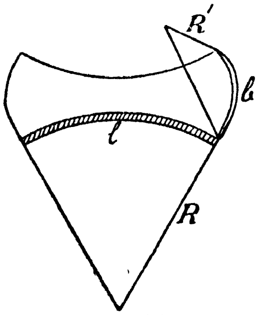

Fig. 195.

And, therefore, as we have seen before, (page 307), the longitudinal and lateral strains in a filament, distant $z$ from the neutral axis will be given by $z/R$ and $z/R'$ respectively. So that, *Poisson's ratio* $\sigma$, for the material of the beam, is given by the expression

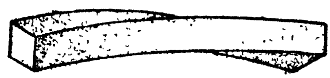
Fig. 196.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{z/R'}{z/R} = \frac{R}{R'}.$$

This, then, gives us a method for the determination of $\sigma$ for the material of a given beam or bar, the two radii being determined directly by attaching suitable pointers to the rod and noting the distances and angles traversed by them, when a known couple is applied to the beam.

**130. The Cantilever.** A *cantilever* is a beam fixed horizontally at one end and loaded at the other.

(*i*) **Cantilever loaded at the free end.** Here, two cases arise, *viz.*, (*a*) when the weight of the beam itself produces no bending, and (*b*) when it does so. Let us consider both the cases.

(*a*) *When the weight of the beam is ineffective.* Let $AB$, (Fig. 197) represent the neutral axis of a cantilever, of length $L$ fixed at the end $A$, and loaded at $B$ with a weight $W$, such that the end $B$ is deflected or depressed into the position $B'$ and the neutral axis takes up the position $AB$, it being assumed that the weight of the beam itself produces no bending.

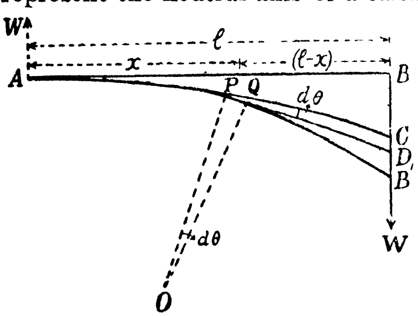

Fig. 197.

Consider a section $P$ of the beam at a distance $x$ from the fixed end $A$.

The moment of the external couple at *this* section, due to the load $W$, or the *bending moment* acting on it

$$= W \times PB' = W(L-x).$$