Rotation of rigid bodies

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References

University Physics with Modern Physics

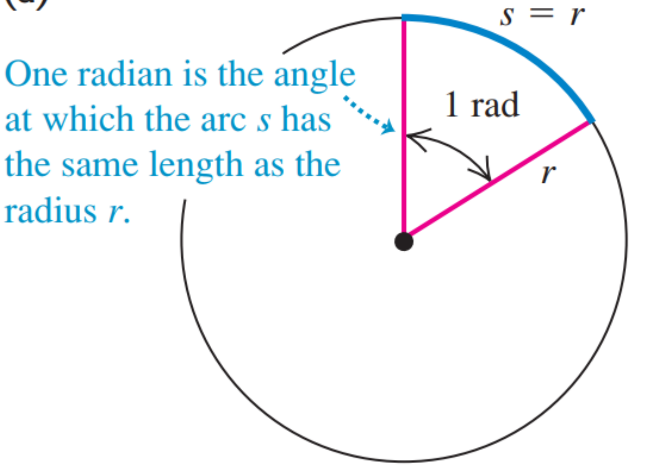
– Hugh D. Young, Roger A. Freedman

Measuring angles in radians

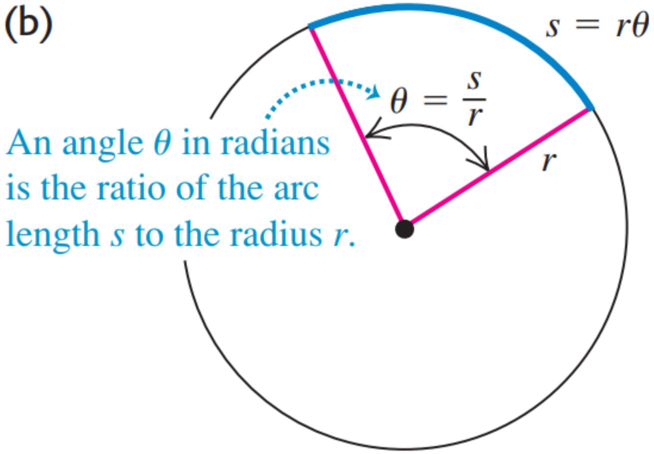
$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta$$

$$1 \text{ rad} = \frac{360°}{2\pi} = 57.3°$$

(a)

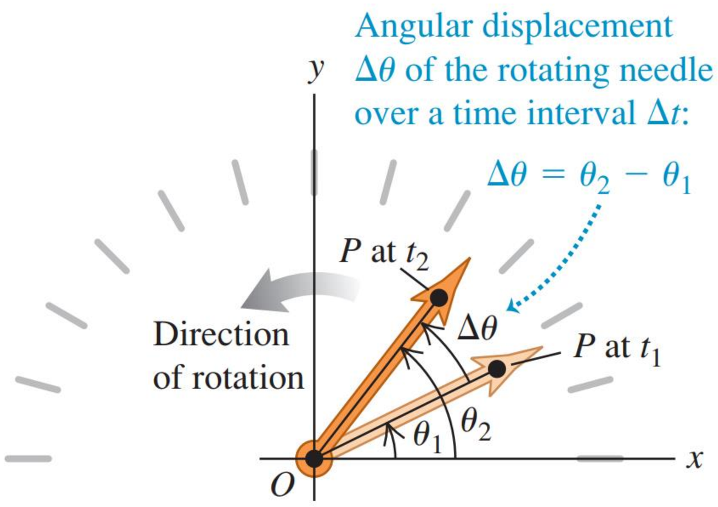


(b)



Angular Velocity

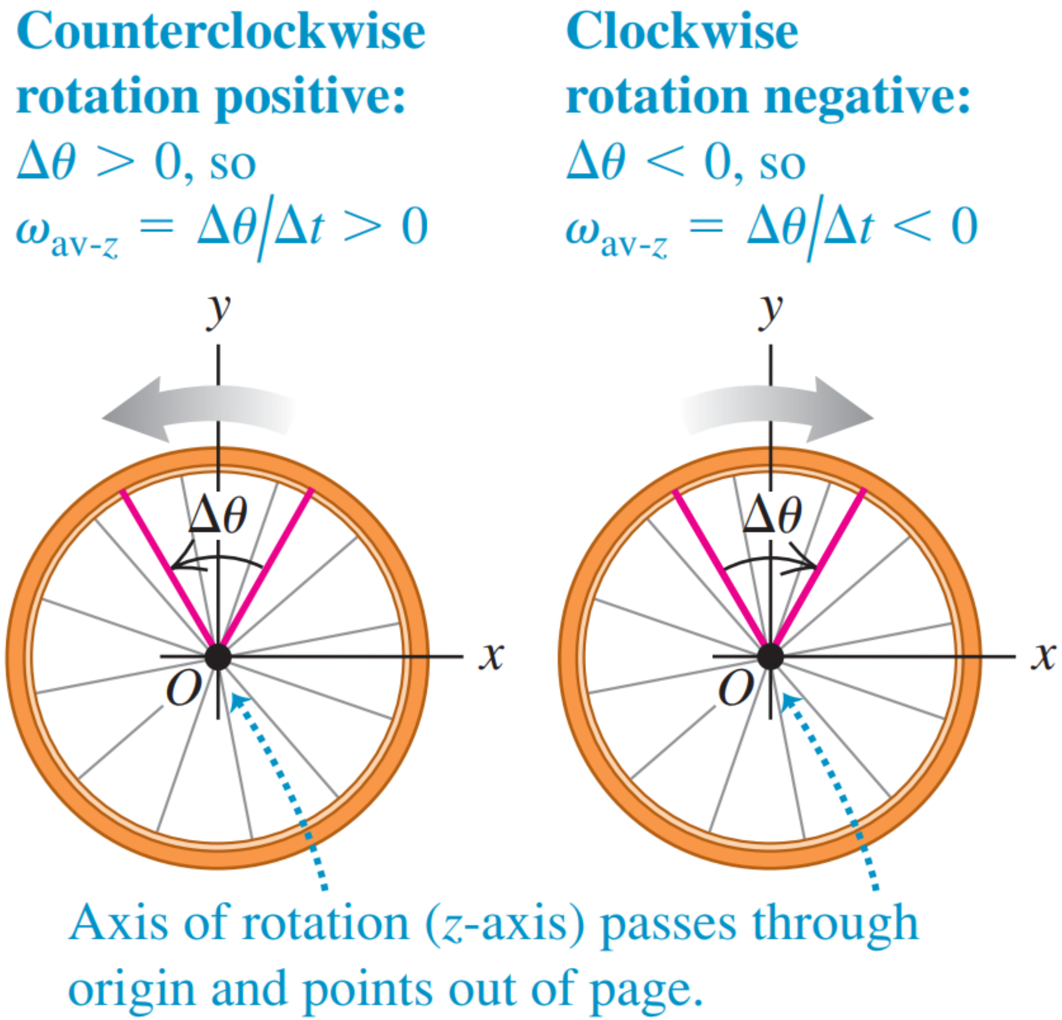
$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$



$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(definition of angular velocity)}$$

Angular Velocity

$$1 \text{ rev/s} = 2\pi \text{ rad/s}$$



Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves over the time interval from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

Calculating angular velocity

EXECUTE: (a) We substitute the values of t into the equation for θ:

$$\theta_1 = (2.0\ \text{rad/s}^3)(2.0\ \text{s})^3 = 16\ \text{rad}$$

$$= (16\ \text{rad})\frac{360°}{2\pi\ \text{rad}} = 920°$$

$$\theta_2 = (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}$$

$$= (250\ \text{rad})\frac{360°}{2\pi\ \text{rad}} = 14{,}000°$$

Calculating angular velocity

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250 \text{ rad} - 16 \text{ rad} = 234 \text{ rad}$.

The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18 \text{ m})(234 \text{ rad}) = 42 \text{ m}$$

We can drop "radians" from the unit for s because θ is a pure, dimensionless number; the distance s is measured in meters, the same as r.

Calculating angular velocity

(c) From Eq. (9.2),

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250 \text{ rad} - 16 \text{ rad}}{5.0 \text{ s} - 2.0 \text{ s}} = 78 \text{ rad/s}$$

$$= \left(78 \frac{\text{rad}}{\text{s}}\right)\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 740 \text{ rev/min}$$

Calculating angular velocity

(d) From Eq. (9.3),

$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}[(2.0\ \text{rad/s}^3)t^3] = (2.0\ \text{rad/s}^3)(3t^2)$$

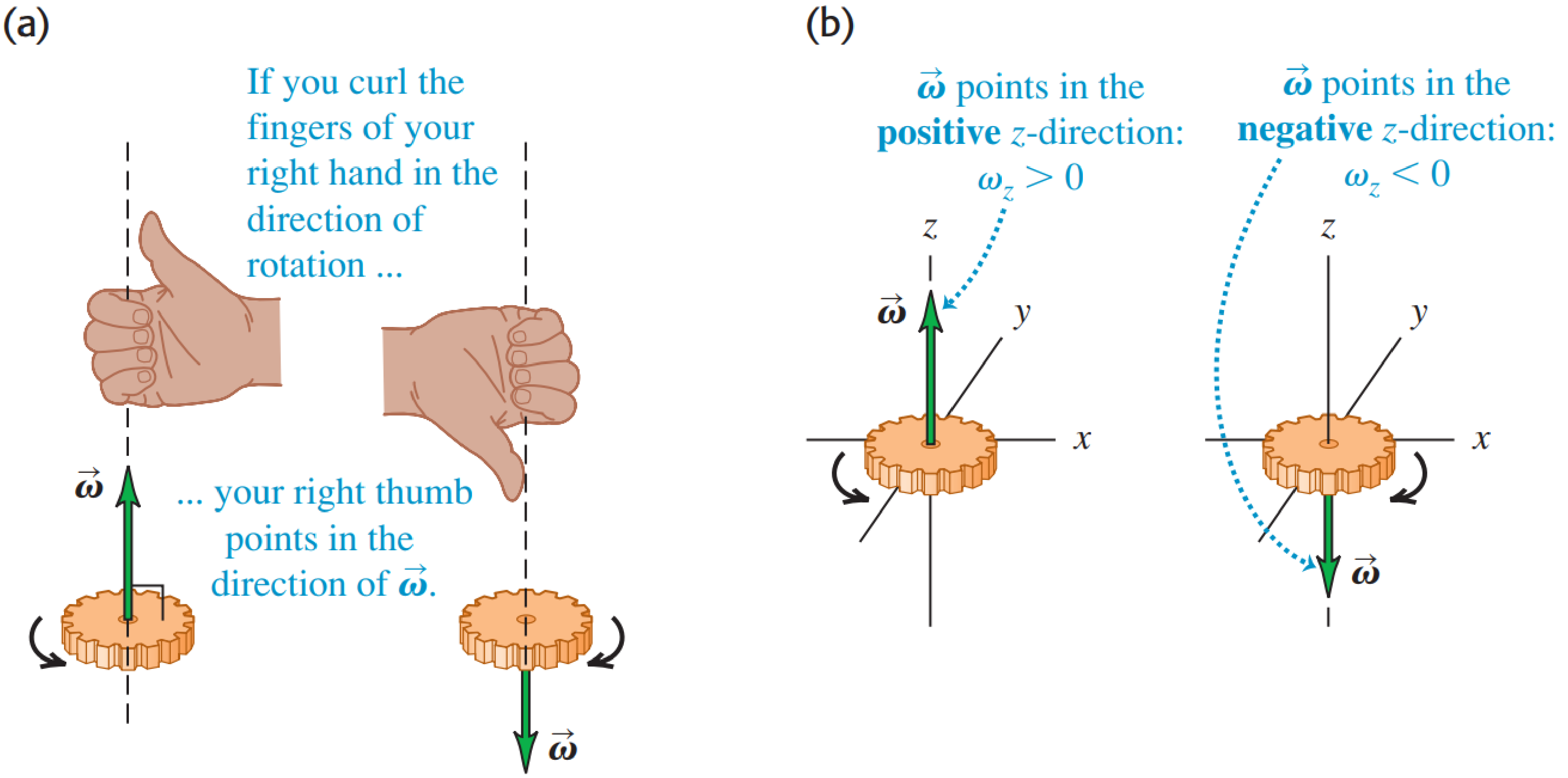
$$= (6.0\ \text{rad/s}^3)t^2$$

At times $t_1 = 2.0$ s and $t_2 = 5.0$ s we have

$$\omega_{1z} = (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}$$

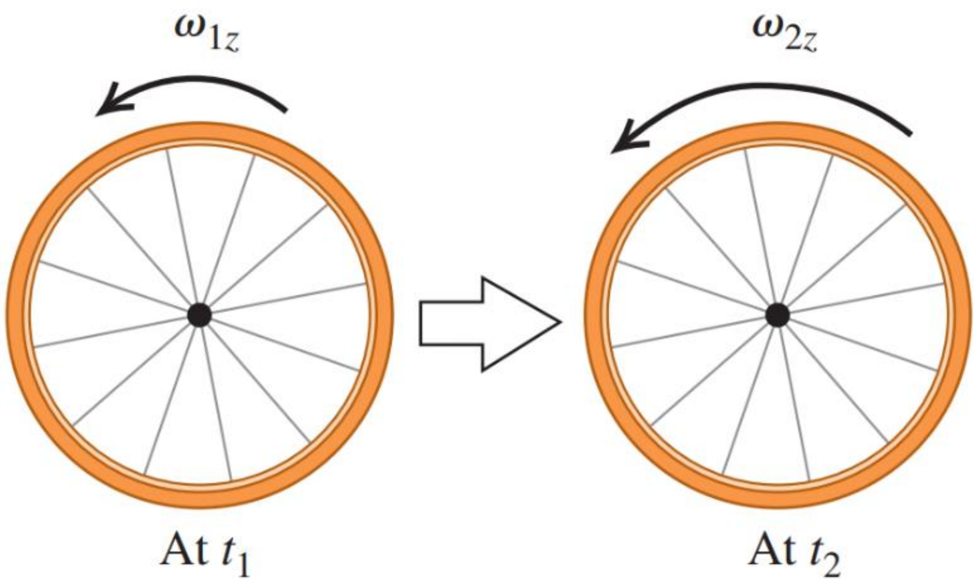
$$\omega_{2z} = (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}$$

Angular Velocity As a Vector



Angular Acceleration

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$



$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \qquad \text{(definition of angular acceleration)}$$

The usual unit of angular acceleration is rad/s^2.

Calculating angular acceleration

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

Calculating angular acceleration

EXECUTE: (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24\ \text{rad/s} \qquad \omega_{2z} = 150\ \text{rad/s}$$

From Eq. (9.4), the average angular acceleration is

$$\alpha_{\text{av-}z} = \frac{150\ \text{rad/s} - 24\ \text{rad/s}}{5.0\ \text{s} - 2.0\ \text{s}} = 42\ \text{rad/s}^2$$

Calculating angular acceleration

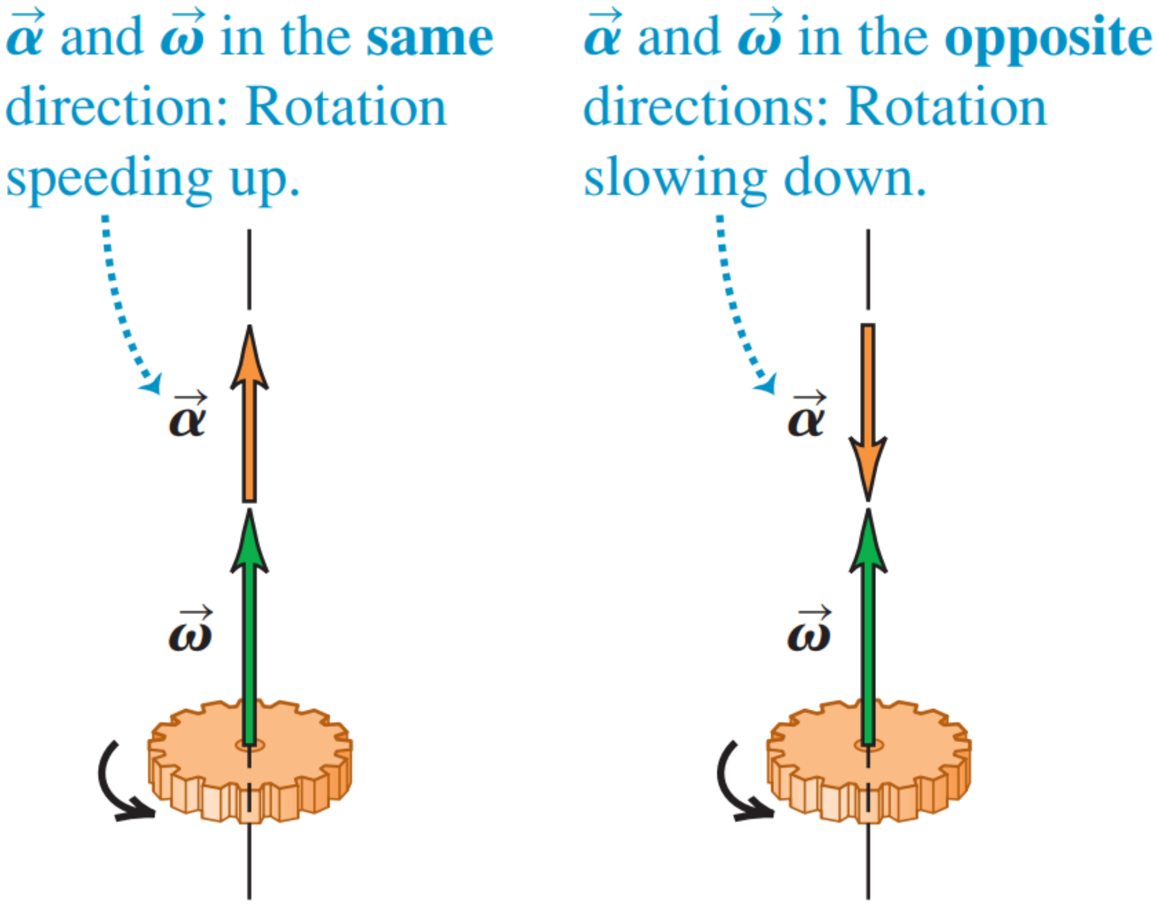
(b) From Eq. (9.5), the value of α_z at any time t is

$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0 \text{ rad/s}^3)(t^2)] = (6.0 \text{ rad/s}^3)(2t)$$
$$= (12 \text{ rad/s}^3)t$$

Hence

$$\alpha_{1z} = (12 \text{ rad/s}^3)(2.0 \text{ s}) = 24 \text{ rad/s}^2$$
$$\alpha_{2z} = (12 \text{ rad/s}^3)(5.0 \text{ s}) = 60 \text{ rad/s}^2$$

Angular Acceleration As a Vector



Comparison of Linear and Angular Motion

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

Linear Acceleration in Rigid-Body Rotation

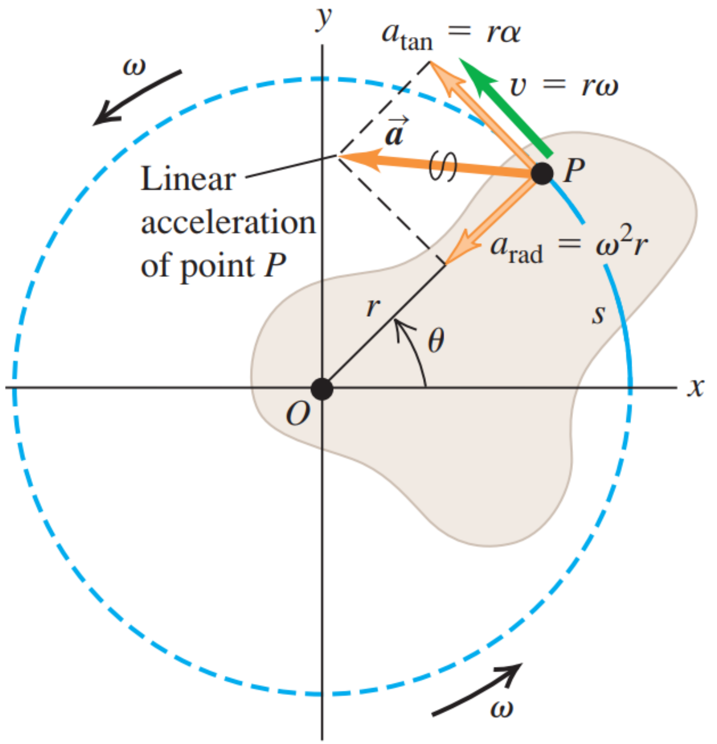
$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$ (tangential acceleration of a point on a rotating body)

$$a_{rad} = \frac{v^2}{r} = \omega^2 r$$ (centripetal acceleration of a point on a rotating body)

CAUTION Use angles in radians in all equations

Radial and tangential acceleration components:

- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



Throwing a discus

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

$$a_{\tan} = r\alpha = (0.800\ \text{m})(50.0\ \text{rad/s}^2) = 40.0\ \text{m/s}^2$$

$$a_{\text{rad}} = \omega^2 r = (10.0\ \text{rad/s})^2(0.800\ \text{m}) = 80.0\ \text{m/s}^2$$

$$a = \sqrt{a_{\tan}{}^2 + a_{\text{rad}}{}^2} = 89.4\ \text{m/s}^2$$

Energy in Rotational Motion

$$K = \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \cdots$$

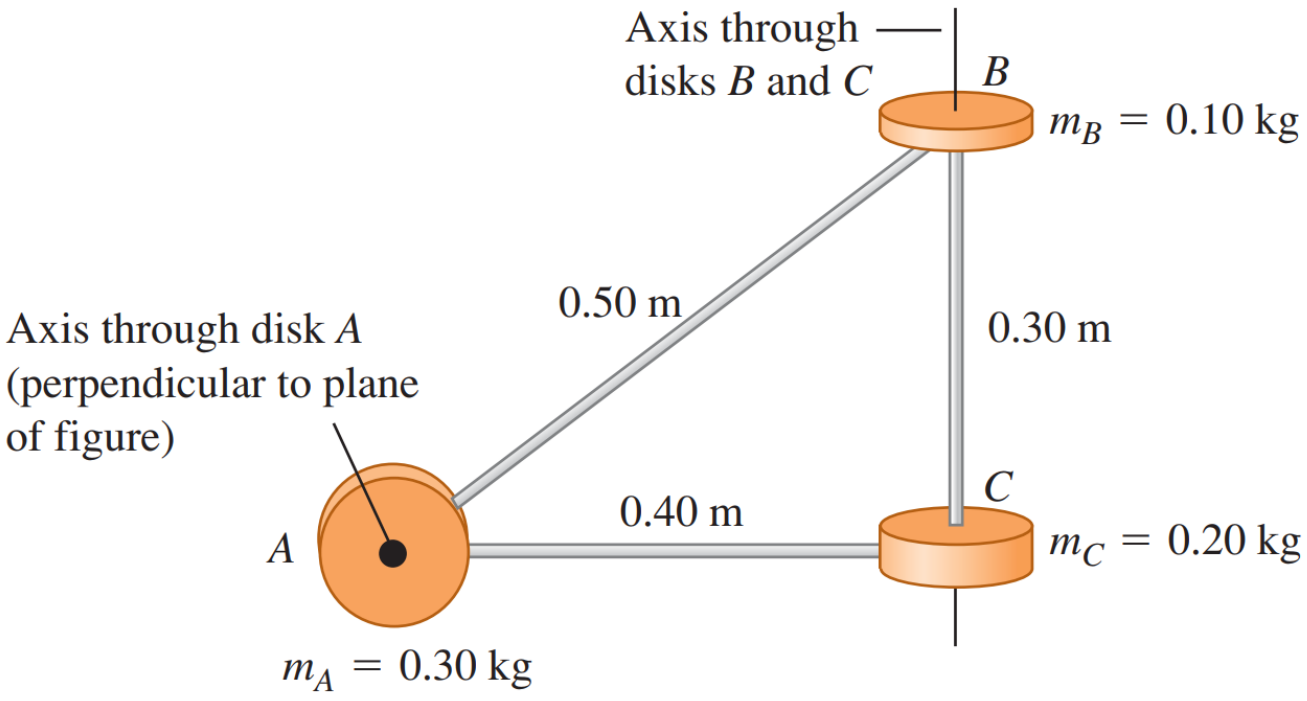
$$= \sum_i \tfrac{1}{2}m_ir_i^2\omega^2 = \tfrac{1}{2}\Big(\sum_i m_ir_i^2\Big)\omega^2$$

$$I = m_1r_1^2 + m_2r_2^2 + \cdots = \sum_i m_ir_i^2$$ (definition of moment of inertia)

$$K = \tfrac{1}{2}I\omega^2$$ (rotational kinetic energy of a rigid body)

Moments of inertia for different rotation axes

A machine part consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about an axis through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about an axis through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about the axis through A with angular speed $\omega = 4.0\ \mathrm{rad/s}$?



CAUTION **Moment of inertia depends on the choice of axis**

Moments of inertia for different rotation axes

EXECUTE: (a) The particle at point A lies *on* the axis through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute, and Eq. (9.16) gives

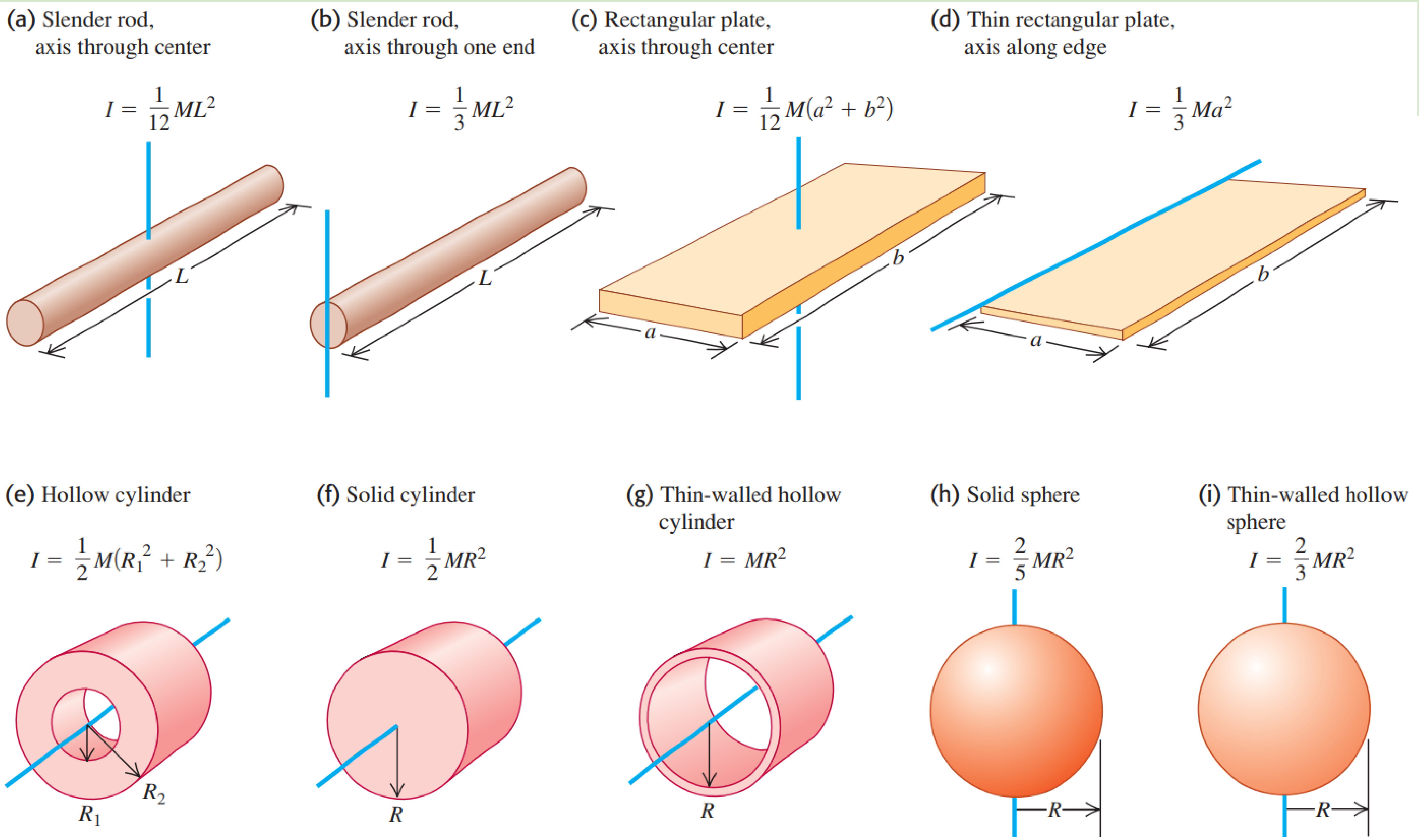
$$I_A = \sum m_i r_i^2 = (0.10\text{ kg})(0.50\text{ m})^2 + (0.20\text{ kg})(0.40\text{ m})^2$$
$$= 0.057\text{ kg}\cdot\text{m}^2$$

(b) The particles at B and C both lie on axis BC, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$I_{BC} = \sum m_i r_i^2 = (0.30\text{ kg})(0.40\text{ m})^2 = 0.048\text{ kg}\cdot\text{m}^2$$

(c) From Eq. (9.17),

$$K_A = \tfrac{1}{2} I_A \omega^2 = \tfrac{1}{2}(0.057\text{ kg}\cdot\text{m}^2)(4.0\text{ rad/s})^2 = 0.46\text{ J}$$



(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$
L
(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$
L
(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
a
b
(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$
a
b
(e) Hollow cylinder
$I = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$
R_1
R_2
(f) Solid cylinder
$I = \frac{1}{2}MR^2$
R
(g) Thin-walled hollow cylinder
$I = MR^2$
R
(h) Solid sphere
$I = \frac{2}{5}MR^2$
R
(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$
R

Readings

University Physics with Modern Physics

– Hugh D. Young, Roger A. Freedman

Chapter 9: Rotation of rigid bodies

9.6 Moment-of-Inertia Calculations