Dynamics of Circular Motion

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Content

Moment of Inertia, Radius of Gyration, Theorem of perpendicular axes and parallel axes, Moment of inertia for different geometrical shapes and Flywheel.

References

Elements of Properties of Matter – D. S. Mathur
Fundamentals of Physics – David Halliday, Jearl Walker, and Robert Resnick

CHAPTER III

MOMENT OF INERTIA—ENERGY OF ROTATION

26. Moment of Inertia and its Physical Significance—Radius of Gyration. We know that, according to Newton's *first law of motion*, a body must continue in its *state of rest* or of *uniform motion along a straight line*, unless acted upon by an external force. *This inertness or inability of a body to change by itself, its position of rest, or of uniform motion, is called* **inertia***, *and is a fundamental property of matter.* Thus, it is by virtue of its *inertia* that a body, at rest, resists or opposes being put into motion, and a body, in linear or translatory motion, opposes not only being brought to rest but also any change in the magnitude and direction of its motion And, we know, by experience, that the greater the mass of a body, the greater its inertia or opposition to the desired change ; for, the greater is the force required to be applied for the purpose. *The mass of a body is thus taken to be a measure of its 'inertia for translatory motion'*, as it is *this* that opposes the acceleration, (positive or negative), desired to be produced in it by the applied force.

Exactly in the same manner, in the case of *rotational motion* also, we find that a body, free to rotate about an axis, opposes any change desired to be produced in its state of rest or rotation, showing that it possesses '*inertia*' for this type of motion And, obviously, the greater the *couple or torque*, (see §28), required to be applied to a body to change its state of rotation, *i.e.*, to produce in it a desired angular acceleration, the greater its opposition to the desired change, or the greater its '*inertia for rotational motion*'. *It is this 'rotational inertia' of the body which is called its* **moment of inertia**** *about the axis of rotation*,—this name being given to it on the analogy of the moment of the couple, which it opposes.

It will thus be seen that the *moment of inertia* of a body, in the case of rotational motion, plays the same part as, or is the analogue of the *mass* of a body in the case of translatory motion ; and we may, therefore, for purposes of analogy, describe the moment of inertia of a body, in rotational motion as the '*effectiveness of its mass.*' Or, pushing the analogy a little further, we may define *mass* as the '*coefficient of inertia*† *for translatory motion*', and the *moment of inertia*, as the '*coefficient of rotational inertia*'.

Yet, with all this seeming similarity, there is all the difference between the two cases. For, in the case of translatory motion, the

*That is why the comparative slackness or sluggishness of the people of Eastern countries,—a consequence of climatic conditions—is dubbed by the Westerners as the '*Inertia of the East*.'

It is also sometimes referred to as the **'Spin inertia' of the body **about** its axis of rotation.

†The mass of a body being usually referred to as its **inertia coefficient.**

inertia of the body depends wholly upon its *mass* and is, therefore, measured in terms of it alone. In the case of rotational motion, on the other hand, the rotational inertia, or the moment of inertia, of the body, depends not only upon the mass (M) of the body but also upon the '*effective distance*' (K) of its particles from the axis of rotation, and is measured by the expression MK^2, (see next Article).

This '*effective distance*' (K) of the particles of a body from its axis of rotation is called its **radius of gyration** about that axis, and is equal to the *root mean square distance* of the particles from the axis, *i.e.*, equal to the square root of their *mean square distance* (not the square of their mean distance) from it. Or, to give it a clear cut definition, *the radius of gyration of a body, about a given axis of rotation, may be defined as the distance from the axis, at which, if the whole mass of the body were to be concentrated, the moment of inertia of the body about the given axis of rotation would be the same as with its actual distribution of mass.*

Now, it is obvious that a change in the position or inclination of the axis of rotation of a body will bring about a corresponding change in the relative distances of its particles, and hence in their '*effective distance*', from the axis. *i.e.*, in the value of the *radius of gyration* of the body about the axis And, so will the transference of a portion of the matter (or mass) of the body from one part of it to another, or a change in the distribution of the mass about the axis, the total mass of the body remaining the same, in either case.

Thus, whereas the mass of a body remains the *same*, irrespective of the location or inclination of the axis of rotation, *the value of its radius of gyration about the axis depends upon*

(*i*) *the position and direction of the axis of rotation*, and

(*ii* *the distribution of the mass of the body about this axis*; so that, its value for the same body is different for different axes of rotation.

Further, it follows, as a converse of the above, that the radius of gyration of a body about a given axis of rotation gives an indication of the distribution of the mass of the body about it and hence, also, the effect of this distribution of mass on the moment of inertia of the body about that axis.

27. Expression for the Moment of Inertia. Suppose we have a body of mass M, (Fig. 21), and any axis YY'. Imagine the body to be composed of a large number of particles of masses m_1, m_2, m_3 etc., at distances r_1, r_2, r_3...etc from the axis YY'. Then, the moment of inertia of the particle m_1 about YY is $m_1r_1^2$, that of the particle m_2 is $m_2r_2^2$, and so on; and, therefore, the moment of inertia, I, of the whole body, about the axis YY', is equal to the sum of $m_1r_1^2$, $m_2r_2^2$, $m_3r_3^2$......etc.

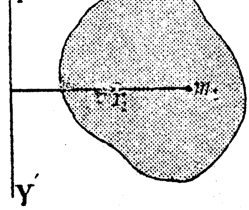

Fig. 21.

Thus, $I = m_1r_1^2+m_2r_2^2+m_3r_3^2+...$

$= \Sigma mr^2$.

Or, $I = MK^2$, [where M is the *mass* and MK^2 the summation ΣMr^2 for the whole body.

K being the *radius of gyration* of the body about the axis YY'.

28. Torque. If we wish to accelerate the rotation of a body, free to rotate about an axis, we have to apply to it a couple. *The moment of the couple, so applied, is called* **torque**, and we say that a torque is applied to the body.

Obviously, the *angular acceleration* of all the particles, *irrespective of their distances* from the axis of rotation, is the *same*, but because their distances are different from the axis, their *linear accelerations are different*, (the linear acceleration of a particle being the product of the angular acceleration and the distance of the particle from the axis of rotation).

If, therefore, $d\omega/dt$ be the angular acceleration of the body, or its particles, we have

linear acceleration of the particle distant r_1 from the axis $= r_1.d\omega/dt$,
,, ,, ,, ,, ,, ,, r_2 ,, ,, ,, $= r_2.d\omega/dt$.
and so on.

Hence, if m be the mass of each particle of the body, the forces on the different particles are $mr_1.d\omega/dt$, $mr_2.d\omega/dt$, etc., and the *moments* o these forces about the axis of rotation will, therefore, be

$$(mr_1.d\omega/dt) \times r_1,\ (mr_2.d\omega/dt) \times r_2 \text{ and so on.}$$

Therefore, total moment for the *whole* body

$$= (mr_1.d\omega/dt) \times r_1 + (mr_2.d\omega/dt) \times r_2 + \ldots\ldots$$
$$= mr_1^2\, d\omega/dt + mr_2^2.d\omega/dt + \ldots\ldots$$
$$= \Sigma mr^2.d\omega/dt,$$
$$= (d\omega/dt).\Sigma mr^2. \qquad [d\omega/dt \text{ being constant.}$$

But $\Sigma mr^2 = I$, the moment of inertia of the body about the axis of rotation. And, therefore,

$$\textit{moment for the whole body} = I.d\omega/dt.$$

This must be equal to the *torque* applied to the body.

So that, $$\textit{torque} = I.d\omega/dt.$$

It will at once be clear that this relation corresponds to the familiar relation, *force* $= m \times a$, in the case of linear motion, where m is the *mass* and a, the *acceleration* of the body.

Thus, in the case of rotatory motion, torque, moment of inertia and angular acceleration are the analogues of force, mass and linear acceleration respectively in the case of linear or translatory motion.

Now, if $d\omega/dt = 1$, clearly, *torque* $= I$.

Or, *the moment of inertia of a body about an axis is equal to the torque, producing unit angular acceleration in it about that axis.*

Incidentally, the expression for torque, obtained above, furnishes us with a method of deducing an expression for the moment of inertia of a particle of mass m, about an axis, distant r from it.

For, if F be the force applied, we have

$$\textit{torque} = F \times r.$$

torque is also $= I\, d\omega/dt$.

And, therefore, $$I = \frac{torque}{d\omega/dt} = \frac{F \times r}{d\omega/dt} \quad \ldots (i)$$

Now, $$F = m \times a,$$

where a is the *linear acceleration* of the particle.

And, since $a = dv/dt$, (where v is the *linear velocity* of the particle), we have $F = m.dv/dt$.

Again, since $v = r\omega$, where ω is the *angular velocity* of the particle, we have

$$F = m.\frac{d(r\omega)}{dt} = m.\left(\frac{d\omega}{dt}.r + \frac{dr}{dt}.\omega\right).$$

Now, the component, $(dr/dt).\omega$, plays no part in the rotation of the body and may, therefore, be ignored ; so that, $F = mr.d\omega/dt$.

Substituting this value of F in relation (i) above, we have

$$I = \frac{mr.(d\omega/dt).r}{d\omega/dt} = mr^2.$$

Thus, the moment of inertia of a particle of mass m, about an axis distant r from it, is equal to mr^2.

29. General Theorems on Moment of Inertia. There are two general theorems of great importance on moment of inertia, which, in some cases, enable us to determine the moment of inertia of a body about an axis, if its moment of inertia about some other axis be known. We shall now proceed to discuss these.

(*a*) **The Principle or Theorem of Perpendicular Axes.**

(*i*) **For a Plane Laminar Body.** According to this theorem, the *moment of inertia of a plane lamina about an axis, perpendicular to the plane of the lamina, is equal to the sum of the moments of inertia of the lamina about two axes at right angles to each other, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it.*

Thus, if I_x and I_y be the moments of inertia of a *plane lamina* about the perpendicular axes, OX and OY, which lie *in the plane of the lamina* and intersect each other at O, (Fig. 22), the moment of inertia about an axis passing through O and *perpendicular to the plane of the lamina*, is given by

$$I = I_x + I_y.$$

For, considering a particle of mass m at P, at distances x and y from OY and OX respectively, and at distance r from O, we have

$$I = \Sigma mr^2,\ I_x = \Sigma my^2 \text{ and } I_y = \Sigma mx^2.$$

So that, $$I_x + I_y = \Sigma my^2 + \Sigma mx^2.$$
$$= \Sigma m(y^2 + x^2).$$
$$= \Sigma mr^2. \quad [\because y^2 + x^2 = r^2.$$

Or, $$I_x + I_y = I.$$

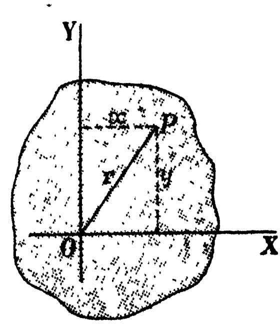

Fig. 22.

(*ii*) **For a Three-Dimensional Body***. Suppose we have a *cubical* or a *three-dimensional* body, shown dotted in Fig. 23, with OX, OY and OZ as its three mutually perpendicular axes, representing its length, breadth and height respectively.

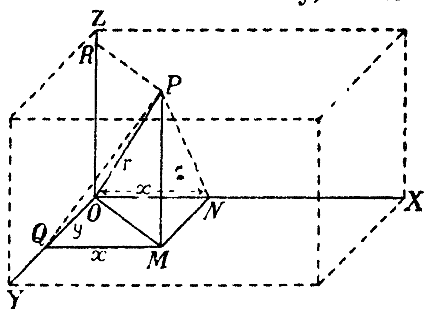

Fig. 23.

Consider a mass m of the body, at a point P, somewhere inside it. Drop a perpendicular PM from P on the xy plane to meet it in M. Join OM and OP, and from M draw MQ parallel to the x-axis and MN, parallel to the y-axis; also, from P draw PR, parallel to OM. Then, clearly, the co-ordinates of the point P are

$$x = ON = QM\,;\ y = OQ = NM \quad \text{and} \quad z = MP = OR.$$

Since the plane xy is perpendicular to the z-axis, any straight line drawn in this plane is also perpendicular to it, and, therefore, OM and PR are both perpendicular to the z-axis, ($\because$ PR is drawn parallel to OM).

Obviously, therefore, $\angle OMP$ is a right angle, because OM is parallel to PR and PM is parallel to OR. Hence, we have

$$OM^2+MP^2 = OP^2.$$

Or, $\quad OM^2+z^2 = r^2,\quad$ where, $OP = r.\quad \ldots(i)$

But $\quad OM^2 = QM^2+OQ^2.$

Or, $\quad OM^2 = x^2+y^2.$

[$\because$ OQM is a right angle, being the angle between the axes x and y.

Therefore, substituting the value of OM^2 in relation (i) above, we have $\quad x^2+y^2+z^2 = r^2.\quad \ldots(ii)$

Join PN and PQ. Then, PN and PQ are the respective normals to the axes of x and y. For, $\angle PMN$ is a right angle, being the angle between the axes y and z, and, therefore,

$$PN^2 = MN^2+PM^2 = y^2+z^2.$$

So that, $\quad x^2+PN^2 = x^2+y^2+z^2 = r^2.\quad$ [From (ii) above

Or, $\quad ON^2+PN^2 = r^2,\quad$ [$\because$ $x = ON$.

from which it is clear that $\angle PNO$ is a right angle, and, therefore, PN is perpendicular to the x-axis.

Similarly, in the right-angled $\triangle PMQ$, we have

$$PQ^2 = MQ^2+PM^2 = x^2+z^2.$$

But $\quad x^2+y^2+z^2 = r^2.$

$\therefore \quad PQ^2+y^2 = r^2.$

Or, $\quad PQ^2+OQ^2 = OP^2.\quad$ [$\because$ $y = OQ$. and $r = OP$.

Or, $\angle PQO$ is a right angle, *i.e.*, PQ is perpendicular to the y-axis.

*Not strictly included in the B.Sc. (Pass or General) course.

Now, moment of inertia of mass m at P, about the z-axis

$$= m \times PR^2 = m.OM^2,$$

because $PR = OM$ is the perpendicular distance of the mass from the axis.

$\therefore$ moment of inertia of the whole body about the z-axis, *i.e.*,

$$I = \Sigma m.OM^2.$$

Or, $$I_z = \Sigma m(x^2+y^2).$$

Similarly, the moment of inertia of the body about the y-axis, *i.e.*,

$$I_y = \Sigma m.PQ^2.$$

Or, $$I_y = \Sigma m.(x^2+z^2).$$ [$\because$ PQ is the $\perp$ distance between the mass and the axis.

And, the moment of inertia of the body about the x-axis, *i.e.*,

$$I_x = \Sigma m.PN^2.$$

Or, $$I_x = \Sigma m(y^2+z^2)$$ [$\because$ PN is the $\perp$ distance between the mass and the axis.

$\therefore$ adding up the moments of inertia of the body *about the three axes*, we have

$$I_x+I_y+I_z = \Sigma m(y^2+z^2)+\Sigma m(x^2+z^2)+\Sigma m(x^2+y^2).$$

$$= 2\Sigma m(x^2+y^2+z^2).$$

Or, $$I_x+I_y+I_z = 2\Sigma mr^2.$$ [$\because$ $x^2+y^2+z^2=r^2$.

Hence the sum of the moments of inertia of a three-dimensional body about its three mutually perpendicular axes, is equal to twice the summation Σmr^2 about the origin.

(8) The Principle or Theorem of Parallel Axes. This theorem (due to **Steiner**) is true both for a plane laminar body as well as a three-dimensional body and states that *the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis, through its centre of mass, plus the product of the mass of the body and the square of the distance between the two axes.*

(*i*) **Case of a Plane Laminar Body**. Let C be the *center of mass* of a body of mass M, (Fig. 24). and I_c, its moment of inertia about an axis through C, *perpendicular to the plane of the paper.*

Now, let it be required to determine the moment of inertia I of the body about a *parallel* axis through O, distant r from C.

Consider any particle P of the body, of mass m, at a distance x from O.

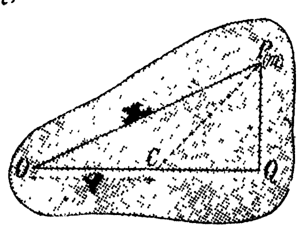

Fig. 24.

Then, the moment of inertia of the body about O is given by

$$I = \Sigma mx^2.$$ [Since $OP^2 = x^2$.

From P drop a perpendicular PQ on to OC produced, and join PC. Then,

$$OP^2 = CP^2+OC^2+2OC.CQ.$$ [By simple geometry.

And $\therefore$ $$m.OP^2 = m.CP^2+m.OC^2+2m.OC.CQ.$$

$\therefore$ $$\Sigma mx^2 = \Sigma m.CP^2+\Sigma mr^2+2r\Sigma m.CQ.$$ [$\because$ $OP = x$ & $OC = r$.

Hence $$I = I_c+Mr^2+2r\Sigma m.CQ.$$ [$\because$ $\Sigma m.CP^2 = I_c$.

Now, since a body always balances about an axis passing through its centre of mass, it is obvious that the *algebraic sum* of the moments of the weights of its individual particles about the centre of mass must be *zero*. Hence, here, $\Sigma mg.CQ$, (the algebraic sum of such moments about C) and, therefore, the expression $\Sigma m.CQ$ is equal to 0, g being constant at a given place. Consequently,

$$2r.\Sigma m.CQ = 0.$$

So that, $$I = I_c + Mr^2.$$

(*ii*) **Case of a Three-Dimensional Body.** Let AB be the axis about which the moment of inertia of a body (shown dotted) is to be determined, (Fig. 25). Draw a parallel axis CD through the centre of mass G of the body, at a distance r from it.

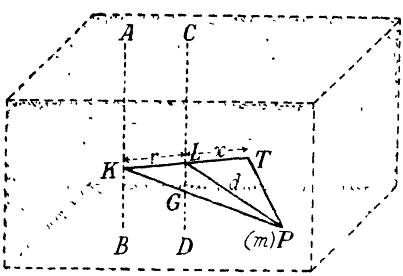

Fig. 25.

Imagine a particle of mass m at any point P, outside the plane of the axes AB and CD and let PK and PL be perpendiculars drawn from P on to AB and CD respectively and PT, the perpendicular dropped from P on to KL produced.

Put $PL = d$, $LK = r$, $LT = x$ and $\angle PLK = \theta$.

Then, if I be the moment of inertia of the body about the axis AB and I_c its moment of inertia about the axis CD (through G), we clearly have

$$I = \Sigma m.PK^2 \text{ and } I_r = \Sigma m.PL^2 = \Sigma m.d^2.$$

Now, from the geometry of the Figure, we have

$$PK^2 = PL^2 + LK^2 - 2PL.LK \cos PLK.$$
$$= d^2 + r^2 - 2d.r \cos \theta.$$

And, in the right-angled $\triangle PTL$, we have

$$\cos PLT = LT/PL,$$

where $\angle PLT = (180° - PLK) = (180° - \ )$. So that,

$$\cos (180° - \theta) = x/d.$$

Or, $$-\cos \theta = x/d,$$

whence, $$d \cos \theta = -x.$$

Substituting this value of $d \cos \theta$ in the expression for PK above, we have

$$PK^2 = d^2 + r^2 + 2rx.$$

And, therefore, $$I = \Sigma m.PK^2 = \Sigma m(d^2 + r^2 + 2rx).$$

Or, $$I = \Sigma md^2 + \Sigma mr^2 + 2r\Sigma mx.$$
$$= I_c + Mr^2 + 2r\Sigma mx,$$

because $\Sigma mr^2 = Mr^2$, where M is the mass of the *whole* body and r, the distance between the two parallel axes and hence a constant. Clearly, $\Sigma mx = 0$, being the total moment about an axis through the centre of mass of the body.

We, therefore, have $$I = I_c + Mr^2,$$

the same result as obtained above in case (*i*) for a plane laminar body.

30. Calculation of the Moment of Inertia of a Body.—Its Units etc. In the case of a *continuous, homogeneous* body of a definite *geometrical shape*, its moment of inertia is calculated by first obtaining an expression for the moment of inertia of an *infinitesimal mass* of it about the given axis—*by multiplying this mass* (m) *by the square of its distance* (r) *from the axis,* (see page 51)—, *and then integrating this expression over the appropriate range, depending upon the shape of the body concerned making full use of the theorems of perpendicular and parallel axes, wherever necessary.*

In case, however, the body is not homogeneous or of a definite geometrical shape, the safest thing to do is to determine its moment of inertia by actual experiment, as explained later, in § 34 and in Chapter VIII.

Now, it will be seen that since the moment of inertia of a body about a given axis is equal to MK^2, where M is its *mass* and K, its *radius of gyration* about that axis, its demensions are 1 *in mass* and 2 *in length,* its dimensional formula being $[ML^2]$. If the mass of the body and its radius of gyration be measured in the $C.G.S.$ units, *i.e.*, its mass in *grams* and radius of gyration in *centimetres*, the moment of inertia of the body is expressed in *gram-centimetre*^{2}, (*i.e.*, in *gm.-cm*^{2}.). And, if the two quantities be measured in the $F.P.S.$ units, *i.e.*, the mass of the body in *pounds* and its radius of gyration in *feet*, the moment of inertia is expressed in *Pound-feet*^{2}, (*i.e.*, in *lb.-ft*^{2}).

And, finally, it must be carefully noted that since the moment of inertia of a body, *about a given axis*, remains unaffected by reversing its direction of rotation about *that* axis, it is just a *scalar quantity.** Thus, the total moment of inertia of a number of bodies, about a given axis, will be equal to the sum of their individual moments of inertia about that axis, in exactly the same manner as the total mass of a number of bodies will be equal to the sum of their individual masses.

Note. The argument is sometimes advanced that since the moment o inertia of a body changes with the direction of the axis of rotation, it is not a scalar quantity ; and, since it is independent of the sense or direction of rotation about that axis, it is not a vector quantity either, and that it is what is called a '*tensor*'.

The author begs to differ. For, the term, '*moment of inertia of a body*' has hardly any meaning unless clear mention is also made of the axis of rotation of that body. And, once the axis of rotation is fixed, the *moment of inertia* of the body, *about that particular axis*, becomes a *scalar quantity*, being independent of the sense of rotation about that axis. Indeed, it would be misleading to call it a *tensor* ; for, the fact is that the *moment of inertia* and the *products of inertia* (see below), at a point, together constitute the components of a *symmetric tensor of the second order*, which simply means that, knowing the system of moments and products of inertia at a point about any three mutually perpendicular axes we can, by means of certain simple transformations, obtain their values for any other set of three mutually perpendicular axes at that very point.

A general tensor, of the second order, in three-dimensional space, has, in general, nine components, say, $C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}$. But, for a symmetric tensor, $C_{12} = C_{21}$, $C_{23} = C_{32}$ and $C_{31} = C_{13}$, so that it has only six distinct components, *viz.*, three moments of inertia and three products of inertia about the three perpendicular axes.

**Scalar quantities* are those which *possess only magnitude*, but *no direction,—e.g., mass, time* etc. On the other hand, *vector quantities* are those which *possess both magnitude as well as direction,—e.g., acceleration, velocity, force,* etc.

us, if x, y, z be the co-ordinates of a particle of mass m, at P, in Fig. 23, we have

(*i*) *moments of inertia* about these three perpendicular axes respectively given by

$$I_x = \Sigma m(y^2+z)^2, \quad I_y = \Sigma m(z^2+x^2), \quad I_z = \Sigma m(x^2+y^2), \text{ and}$$

(*ii*) the *products of inertia* about these axes defined by

$$P_{yz} = \Sigma myz, \quad P_{zx} = \Sigma mzx, \quad P_{xy} = \Sigma mxy,$$

Then, $I_x, I_y, I_z, -P_{yz}, -P_{zx}$ and $-P_{xy}$ are the six components of the symmetric tensor at point P.

It will thus be seen that it is, at best, only a half-truth to say that the moment of inertia of a body about a given axis is a *tensor.*

31. Particular Cases of Moments of Inertia.

1. Moment of Inertia of a Thin Uniform Rod :

(*i*) *about an axis through its centre and perpendicular to its length.*—Let AB, (Fig. 26), be a thin *uniform* rod of length l and mass M, free to rotate about an axis CD through its centre O and perpendicular to its length. Then, its *mass per unit length is* M/l. Consider a small element of length dx of it, at a distance x from O. Its mass is clearly equal to $(M/l).dx$, and its moment of inertia about the axis through $O = (M/l).dx.x^2$.

The moment of inertia I of the *whole* rod about the axis is, therefore, obtained by integrating the above expression between the limits $x = -l/2$ and $x = +l/2$; or between $x=0$ and $x=l/2$ and multiplying the result by 2, to include both halves of the rod.

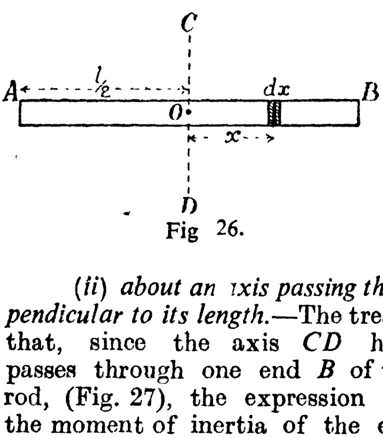

Fig 26.

Thus, $I = 2\int_0^{l/2} \frac{M}{l}.x^2.dx.$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

$$= \frac{2M}{l}\left[\frac{(l/2)^3}{3} - 0\right]$$

Or, $$\mathbf{I} = \frac{2M}{l}.\frac{l^3}{24} = \frac{\mathbf{M}\mathbf{l}^2}{\mathbf{12}}.$$

(*ii*) *about an axis passing through one end of the rod and perpendicular to its length.*—The treatment is the same as above, except that, since the axis CD here passes through one end B of the rod, (Fig. 27), the expression for the moment of inertia of the element dx of the rod is now to be integrated between the limits, $x = 0$, at B and $x = l$, at A.

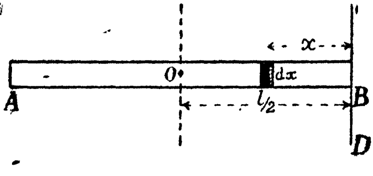

Fig. 27.

Thus, if I be the moment of inertia of the rod about CD, we have

$$I = \int_0^l \frac{M}{l}.x^2.dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$\mathbf{I} = \frac{M}{l}.\frac{l^3}{3} = \frac{\mathbf{M}\mathbf{l}^2}{\mathbf{3}}.$$

Or, we could have arrived at the same result by an application of the *principle of parallel axes,* according to which the moment of inertia of the rod about the axis through B is equal to the *sum* of its moment of inertia about a parallel axis through its centre of mass and the product of its mass and the square of the distance between the two axes.

Thus, $$I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}.$$

2. Moment of Inertia of a Rectangle.

(*i*) *about an axis through its centre and parallel to one of its sides.*—Let $ABCD$ be a rectangle, and let l and b be its length and breadth respectively, (Fig. 28). Let the axis of rotation YY' pass through its centre and be parallel to the side AD or BC.

If M be the mass of the rectangle, (supposed *uniform*), its *mass per unit length* will be M/l.

Consider a small *strip*, of width dx of the rectangle, parallel to the axis. The mass of the strip will obviously be $(m/l).dx$, and, therefore, the moment of inertia of the *strip*, about the axis YY' will be $(M/l).dx.x^2$.

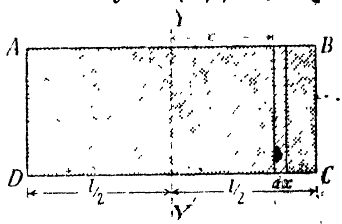

Fig. 28.

The whole rectangle may be supposed to be composed of *such like strips*, parallel to the axis, and therefore, the moment of inertia I of the *whole* rectangle about the axis YY' is obtained by integrating the expression $(M/l).dx.x^2$, for the limits $x=0$ and $x=l/2$ and multiplying the result by 2.

i.e., $$I = 2\int_0^{l/2} \frac{M}{l}.x^2.dx = \frac{2M}{l}\int_0^{l/2} x^2.dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

Or, $$I = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}.$$

It will be seen at once that *if b be small*, the rectangle becomes a *rod*, of length l, whose moment of inertia, about the axis YY' passing through its centre and perpendicular to its length, would be $Ml^2/12$, [as obtained above, §31, case 1, (*i*)].

(*ii*) *about one side.*—We may proceed as above (in case we want an independent proof) except that the expression $(M/l).dx.x^2$ may here be integrated for $x = 0$ and $x = l$. Thus, the moment of inertia of the rectangle about the side AD or BC is given by

$$I = \int_0^l \frac{M}{l}.x^2.dx = \frac{M}{l}\int_0^l x^2.dx.$$

Or, $$I = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{M}{l}\cdot\frac{l^3}{3} = \frac{Ml^2}{3}.$$

Alternatively, proceeding on the basis of the previous article, we may apply *the principle of parallel axes*, according to which the moment of inertia of the rectangle about side AD or BC is given by

$$I = M.I.\ about\ a \parallel axis\ through\ its\ centre + M\left(\frac{l}{2}\right)^2.$$

Or, $$I = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}.$$

(*iii*) *about an axis passing through its centre and perpendicular to its plane.*—This may be obtained by an application of the *principle of perpendicular axes* to case (*i*) above, whence the moment of inertia of the rectangle about an axis through its centre O, perpendicular to its plane, is equal to its moment of inertia about an axis through O, parallel to its breadth b, *plus* its moment of inertia about a perpendicular axis through O, parallel to its length l,

i.e., $$I=\frac{Ml^2}{12}+\frac{Mb^2}{12}=\frac{M(l^2+b^2)}{12}.$$

The above relation is equally valid in the case of *thin* (*i.e.*, *laminar*) or *thick* rectangular plates or bars, no stipulation with regard to its thickness having been made in deducing it. And, after all, a thick rectangular plate or bar may be regarded as just a pile of thin (or laminar) plates or bars, placed one above the other.

The same argument will hold good in all other cases of a similar type, [see cases (*iv*) and (*v*) below].

(*iv*) *about an axis passing through the mid-point of one side and perpendicular to its plane.*—

(*a*) Suppose the axis of rotation passes through the mid-point of AD or BC, (Fig. 28). Then clearly, in accordance with the *principle of parallel axes*, we have

moment of inertia about *this* axis, *i.e.*,

$$I = \textit{moment of inertia about a parallel axis through its centre} + M\times(l/2)^2,$$

where $l/2$ is the distance between the two parallel axes.

i.e., $$I=\frac{M(l^2+b^2)}{12}+M.\left(\frac{l}{2}\right)^2$$
$$=\frac{M(l^2+b^2)}{12}+\frac{Ml^2}{4}=M.\left(\frac{l^2+b^2}{12}+\frac{l^2}{4}\right)$$
$$=M\frac{(l^2+b^2+3l^2)}{12}=\frac{M(4l^2+b^2)}{12}.$$

Or, $$\mathbf{I=M\left(\frac{l^2}{3}+\frac{b^2}{12}\right)}.$$

(*b*) Similarly, if the axis of rotation passes through the mid-point of AB or DC, we have

$$I=M\frac{(l^2+b^2)}{12}+M\left(\frac{b}{2}\right)^2.$$ [∵ the distance between the two ∥ axes is now $b/2$.

Or, $$\mathbf{I=M\left(\frac{l^2}{12}+\frac{b^2}{3}\right)}.$$

3. Moment of inertia of a solid uniform bar of rectangular cross-section, about an axis, perpendicular to its length and passing through its middle point.*

Let *ABCDEFGH* (Fig. 29) be the rectangular uniform bar of length l, breadth b and thickness d, whose moment of inertia about the axis XX', passing through its centre and perpendicular to its length, is desired.

*This is really covered by case 2 above, but is given here for a clearer understanding of the student.

Imagine the whole bar to be made up of a large number of thin rectangular sheets, parallel to the face $CDEH$ and perpendicular to the axis XX', passing through the centre of mass of each sheet. Consider one such sheet, (shown dotted), of mass m, of length and ~~breadth~~ l and d respectively, and centre of mass O, through which the axis XX' is passing perpendicular to its plane.

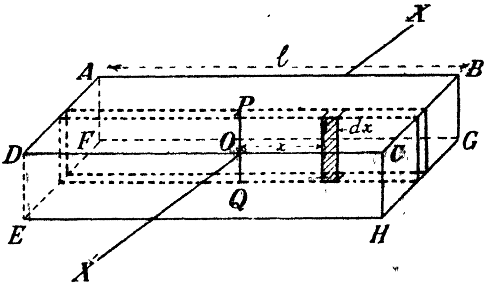

Fig. 29.

Then, the *M.I. of this* sheet about $XX' =$ its *mass* $\times (l^2+d^2)/12$, as can be seen from the following :

Let PQ be an axis through O; *in the plane of this sheet*, and *parallel to its breadth* CH or DE.

Take a thin strip of width dx of *this* strip, *parallel to*, and *at distance* x *from*, the axis PQ. Then, mass of the strip $= (m/l).dx$ and, therefore, its moment of inertia about the axis PQ is

$$= (m/l)\, dx.x^2.$$

$\therefore$ moment of inertia I of the *whole* sheet about PQ is given by

$$2\int_0^{l/2} \frac{m}{l}.x^2.dx = \frac{2m}{l}\int_0^{l/2} x^2.dx$$

$$= \frac{2m}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{2m}{l}\cdot\frac{l^3}{8\times 3}.$$

Or, $$\mathbf{I} = \frac{\mathbf{Ml^2}}{\mathbf{12}}.$$

Similarly, the moment of inertia of the sheet about an axis through O, *in its own plane and perpendicular to PQ, i.e., parallel to its length DC* or *EH will be* $\mathbf{Md^2/12}$.

Therefore, *by the principle of perpendicular axes*, the moment of inertia of the sheet about the axis XX' through O and perpendicular to its plane

$$= \frac{ml^2}{12} + \frac{md^2}{12} = m\left(\frac{l^2+d^2}{12}\right).$$

Hence, moment of inertia of the *whole* bar about the axis XX'

$$\textit{mass of the bar} \times \left(\frac{l^2+d^2}{12}\right).$$

Or, $$\mathbf{I = M\left(\frac{l^2+d^2}{12}\right)}$$ [M being the *mass* of the bar.

4. Moment of Inertia of a Thin Triangular Plate or Lamina about one side.—Let ABC, (Fig. 30), be a thin, triangular plate or lamina, of *surface density* or *mass per unit area*, ρ, whose moment of inertia is to be determined about the side BC.

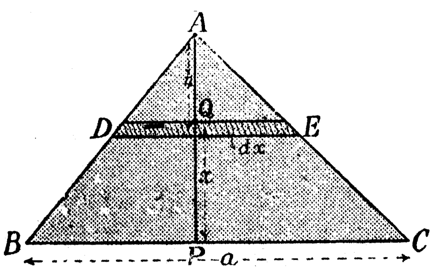

Fig. 30.

Then, if the *altitude* of the plate be $AP = H$, its *area* $= \frac{1}{2}$ *base* $\times$ *altitude*

$= \frac{1}{2}.a.H$, [$\because BC = a$,

and, therefore,

its *mass* $M = \frac{1}{2}.a.H.\rho$. ... (i

Now, let us imagine the triangular plate to be made up of a number of thin *strips*, parallel to BC, and placed side by side ; and, let us consider one such strip DE, of width dx, at a distance x from the base BC. Then, clearly, the *area of this* strip, (which may be considered to be almost rectangular, its width being infinitesimally small) $= DE.dx$. And, therefore,

$$\textit{mass of the strip} = DE.dx.\rho. \qquad \ldots \qquad \ldots \qquad \ldots(ii)$$

Now, in the similar triangles AQD and APB, we have

$$\frac{DQ}{BP} = \frac{AQ}{AP} = \frac{h}{H}, \qquad \text{where, } AQ = h,$$

whence, $$DQ = BP.\frac{h}{H}.$$

Similarly, from the similar triangles AQE and APC, we have

$$\frac{QE}{PC} = \frac{AQ}{AP} = \frac{h}{H}, \text{ whence, } QE = PC.\frac{h}{H}.$$

And, therefore, $DQ+QE = BP.\frac{h}{H}+PC.\frac{h}{H}.$

Or, $$DE = (BP+PC).\frac{h}{H} = BC.\frac{h}{H} = a.\frac{h}{H}.$$

$\therefore$ $$\textit{mass of the strip} = a.\frac{h}{H}.dx.\rho. \qquad \text{[From (ii) above]}$$

Now, clearly, moment of inertia of strip DE about the side BC

$$= \textit{mass of the strip} \times x^2 = a.\frac{h}{H}.dx.\rho.x^2.$$

$$= a.\left(\frac{H-x}{H}\right).dx.\rho.x^2. \qquad [\because h = (H-x).$$

And, therefore, moment of inertia of the *whole* triangular plate about BC is equal to the integral of this expression, between the limits $x = 0$ and $x = H$. So that,

$$\textit{M.I. of the plate about } BC, \textit{ i.e., } I = \int_0^H a.\left(\frac{H-x}{H}\right).\rho.x^2.dx.$$

$$= \frac{a.\rho}{H}\int_0^H (H-x).x^2.dx = \frac{a.\rho}{H}\int_0^H (Hx^2-x^3).dx.$$

$$= \frac{a.\rho}{H}\left[\frac{H.x^3}{3}-\frac{x^4}{4}\right]_0^H = \frac{a.\rho}{H}\left(\frac{H^4}{3}-\frac{H^4}{4}\right)$$

$$= \frac{a.\rho}{H}\left(\frac{4H^4-3H^4}{12}\right) = \frac{a.\rho}{H}.\frac{H^4}{12} = \frac{a.\rho.H^3}{12}.$$

$$= \frac{1}{2}a.H.\rho.\frac{H^2}{6}.$$

But $\frac{1}{2}a.H\rho. = M$, *the mass of the plate.* [See (*i*) above.

$\therefore$ *M. I.* of the triangular plate about side *BC*, *i.e*, $\mathbf{I} = \dfrac{\mathbf{M.H^2}}{\mathbf{6}}.$

5. Moment of Inertia of an Elliptical Disc or Lamina.—*(i) about one of its axes.*—Let $XYX'Y'$ be a thin elliptical plate or lamina, of mass M, and surface density (*i.e.*, mass per unit area), ρ, and let its *major axis* XX' and *minor axis* YY' be equal to $2a$ and $2b$ respectively, (Fig. 31).

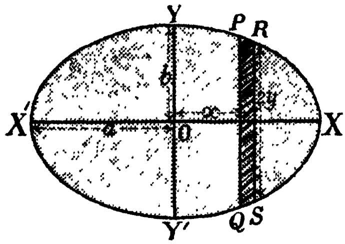

Fig. 31.

Consider a strip $PQSR$ of the plate of width dx, parallel to the minor axis YY' and at a distance x from it. Then, if $2y$ be the length of the strip, its area is clearly equal to $2y.dx$ and, therefore, its *mass* equal to $2y.dx.\rho$.

Obviously, then, $M. I.$ of the strip about the minor axis $YY' = 2y.dx.\rho.x^2$; and, therefore, $M. I.$ of the *whole* elliptical plate about the axis YY' is equal to twice the integral of the above expression, between the limits $x = 0$, and $x = a$. Or, denoting it by I_y, we have

$$I_y = 2\int_0^a 2y.\rho.x^2.dx = 4\rho\int_0^a y.x^2.dx. \quad \dots \quad \dots(\text{I})$$

Now, with the centre of the ellipse as the origin, and with the co-ordinate axes coinciding with its major and minor axes respectively, we have $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$, as the equation to the ellipse ; whence,

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}, \text{ or } y^2 = b^2\left(1-\frac{x^2}{a^2}\right).$$

So that, $\qquad y = b\sqrt{1-x^2/a^2}.$

Substituting this value of y in relation (I) above, we have

$$I_y = 4\rho\int_0^a b\sqrt{(1-x^2/a^2)}.x^2dx.$$

$$= 4\rho b\int_0^a x^2.\sqrt{(1-x^2/a^2)}.dx. \quad \dots \quad \dots \quad \dots(\text{II})$$

Now, putting $x = a \sin\theta$, we have $\dfrac{dx}{d\theta} = a\cos\theta$.

Or, $\qquad dx = a\cos\theta.\,d\theta.$

Substituting these values of x and dx in expression (II) above, we have

$$I_y = 4\rho.b\int_0^{\pi/2} a^2.\sin^2\theta\sqrt{1-\frac{a^2.\sin^2\theta}{a^2}}.a\cos\theta.d\theta.$$

$$= 4\rho.b\int_0^{\pi/2} a^2.\sin^2\theta\sqrt{1-\sin^2\theta}.a\cos\theta.d\theta$$

$$= 4\rho.b\int_0^{\pi/2} a^2.\sin^2\theta.\cos\theta.a\cos\theta.d\theta.$$

$[\because \sqrt{1-\sin^2\theta}=\cos\theta.$

Or, $$I_y = 4\rho b.a^3.\int_0^{\pi/2}\sin^2\theta.\cos^2\theta\,d\theta.$$

$$= 4\rho b.a^3.\frac{1}{4}\int_0^{\pi/2}(2\sin\theta.\cos\theta)^2.d\theta$$

$$= \rho b.a^3. \int_0^{\pi/2} sin^2 2\theta . d\theta.$$

$$= \rho b.a^3. \int_0^{\pi/2} \frac{1-\cos 4\theta}{2} . d\theta. \qquad \left[\because \cos 2\theta = 1 - 2 \sin^2\theta. \right.$$

$$= \rho b.a^3. \frac{1}{2} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$= \rho b.a^3. \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4} \rho b a^3 = \pi . ab\rho . \frac{a^2}{4}.$$

Now, $\pi.a.b.\rho = M$, the *mass of the elliptical plate.*

$\therefore$ $$\mathbf{I_y = M.a^2/4.}$$

Similarly, the moment of inertia (I_x) of the elliptical plate about the major axis XX' is given by the expression,

$$\mathbf{I_x = Mb^2/4.}$$

(*ii*) *about an axis passing through the centre of the plate or lamina and perpendicular to its own plane* —The axis in this case will pass through O, (Fig. 31), and will be perpendicular to the plane of the paper, (or the plane of its two axes, XX' and YY'). Hence, if I be the moment of inertia of the elliptical plate about *this* axis, we have, by the *principle of perpendicular axes,*

$$I = I_x + I_y = I_y + I_x = Ma^2/4 + Mb^2/4.$$

Or, $$\mathbf{I = M. \left(\frac{a^2+b^2}{4}\right).}$$

6. Moment of Inertia of a Hoop or a Circular Ring.—

(*i*) *about an axis through its centre and perpendicular to its plane.*—Let the radius of the hoop or circular ring be R, and its mass, M, (Fig. 32).

Consider a particle of it, of mass m. Then, the moment of inertia of *this* particle about an axis through the centre O of the hoop, and perpendicular to its plane, will obviously be mR^2.

And, therefore, the moment of inertia I of the *entire* hoop about the axis will be ΣmR^2.

Or, $$\mathbf{I = MR^2.} \qquad \left[\because \Sigma m = M \text{ and } R \text{ is the } \textit{same} \text{ for all particles.} \right.$$

(*ii*) *about its diameter.*—Let it be required to determine the moment of inertia of the hoop about the diameter AB, (Fig. 32). Obviously, the moment of inertia of the hoop will be the same about all the diameters. Thus, if I be the moment of inertia of the hoop about the diameter AB, it will also be its moment of inertia about the diameter CD, perpendicular to AB.

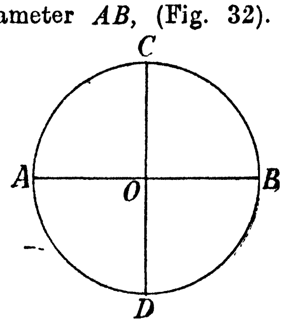

Fig. 32.

Then, by the *principle of perpendicular axes,* its moment of inertia about the axis through the centre O, and *perpendicular to its plane,* is equal to the *sum* of its moments of inertia about the perpendicular axes AB and CD, *in its own plane,* and intersecting at O.

And, therefore, $I+I = MR^2$. Or, $2I = MR^2$. [See case (f).

Or $\mathbf{I = MR^2/2}$.

7. Moment of inertia of a Circular Lamina or Disc.—

(*i*) *about an axis through its centre and perpendicular to its plane.*—Let M be the mass of the disc and R, its radius. Then, since the area of the disc is πR^2, its *mass per unit area* will be $M/\pi R^2$.

Consider a ring of the disc, distant x from O, *i.e.*, of radius x and of width dx, (Fig. 33). Its area is clearly equal to its circumference, multiplied by its width, or equal to $2\pi x \times dx$, and its mass is thus

$$= \frac{M}{\pi R^2} \cdot 2\pi x . dx = \frac{2Mxdx}{R^2}.$$

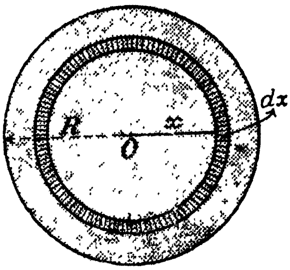

Fig. 33.

Hence, moment of inertia of *this* ring about an axis through O and perpendicular to its plane

$$= \frac{M.2\pi x.dx}{\pi R^2} . x^2 = \frac{2Mx^3dx}{R^2}.$$

Since the whole disc may be supposed to be made up of such like concentric rings of radii ranging from 0 to R, we can get the moment of inertia I of the disc by integrating the above expression for the moment of inertia of the ring, for the limits $x=0$, and $x=R$.

$$\therefore \quad M.I. \text{ of the disc} = \int_0^R \frac{2M}{R^2} . x^3 . dx. = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2} \cdot \frac{R^4}{4}.$$

Or, $\mathbf{I = MR^2/2}$.

(*ii*) *about its diameter.*—Let AB and CD be two perpendicular diameters of a circular disc of radius R and mass M, (Fig. 34). Since the moment of inertia of the disc about one diameter is the same as about any other diameter, the moment of inertia about the diameter AB is equal to the moment of inertia about the diameter CD, perpendicular to AB. Let it be I.

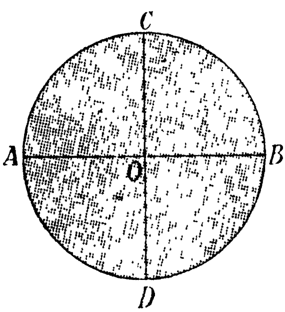

Fig. 34.

Now, we have, by the *principle of perpendicular axes*, $M.I.$ of the disc about AB+its $M.I.$ about CD

= its $M.I.$ about an axis through O and perpendicular to its plane.

Or, $I+I = \frac{MR^2}{2}$ or, $2I = \frac{MR^2}{2}$.

Or, $$\mathbf{I = \frac{MR^2}{4}}.$$

(*ii*) *about a tangent to the disc in its own plane.*—Let AB be the tangent to the circular disc of radius R and mass M, about which its moment of inertia is to be determined, (Fig. 35). Let CD be a diameter of the disc, parallel to the tangent AB. The moment of inertia of the disc about this diameter is, clearly, equal to $MR^2/4$.

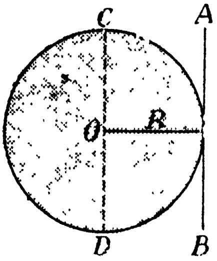

Fig. 35.

So that, by the *principle of parallel axes*, we have

$M.I.$ of the disc about $AB = M.I.$ of the disc about $CD + MR^2$.

Or, $$\mathbf{I} = \frac{MR^2}{4} + MR^2 = \frac{5}{4}\mathbf{MR^2}.$$

(*iv*) *about a tangent to the disc and perpendicular to its plane* —This tangent will obviously be parallel to the axis through the centre of the disc and perpendicular to its plane, the distance between the two being equal to the radius of the disc. Hence, *by the principle of parallel axes*, we have

M.I. about the tangent = *M.I. about the perpendicular axis* $+MR^2$.

Or, $$\mathbf{I} = \frac{MR^2}{2} + MR^2 = \frac{3}{2}\mathbf{MR^2}.$$

8. Moment of Inertia of an Annular Ring or Disc.—

(*i*) *about an axis passing through its centre and perpendicular to its plane.*—An annular disc or ring is just an ordinary disc from which a smaller co-axial disc is removed, so that there is a concentric circular hole in it. Let R and r be the outer and inner radii of the disc, (Fig. 36), and M, its mass. Then, clearly,

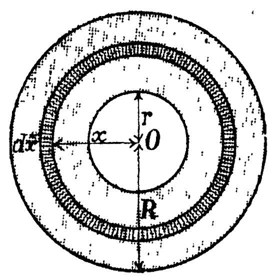

Fig. 36.

face-area of the annular disc = face-area of disc of radius R − face-area of disc of radius r,

$$= \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

And ∴ *mass per unit area of the disc*

$$= M/\pi(R^2 - r^2).$$

Imagine the disc to be made up of a number of thin circular rings, and consider one such ring of radius x and of width dx.

Then, *face-area of this ring* $= 2\pi x.dx$,

and its *mass* $= \dfrac{M}{\pi(R^2 - r^2)}.2\pi x.dx = \dfrac{2Mx}{(R^2 - r^2)}.dx.$

And, therefore, its moment of inertia about an axis through O and perpendicular to its plane $= \dfrac{2Mx}{(R^2 - r^2)}.dx.x^2 = \dfrac{2Mx^3}{(R^2 - r^2)}.dx.$

The moment of inertia of the *whole* annular disc may, therefore, be obtained by integrating the above expression for the limits $x = r$ and $x = R$. Or, moment of inertia of the disc about the axis through O and perpendicular to its plane

$$= \int_r^R \frac{2Mx^3}{(R^2 - r^2)}.dx = \frac{2M}{(R^2 - r^2)}\int_r^R x^3.dx.$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)}\left[\frac{(R^4-r^4)}{4}\right]$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{(R^2+r^2).(R^3-r^2)}{4}\right]$$

Or, $$\mathbf{I = M\left(\frac{R^2+r^2}{2}\right).}$$

It follows at once from the above that if $r = 0$, *i.e.*, if there is no hole in the disc, or that it is just a *plane*, (and *not an annular*) disc, its moment of inertia is $MR^2/2$. [Case 7 (*i*), above.

Again, if $r = R$, *i.e.*, we have a hoop or a circular ring, of radius R, and its $M.I. = \frac{M(R^2+R^2)}{2} = \frac{M.2R^2}{2} = MR^2$. [Case 6 (*i*), above.

(*ii*) *about its diameter.*—Obviously, due to its *symmetrical* shape, the moment of inertia of the annular disc about one diameter will be the same as that about any other diameter. Let it be I. Then, the sum of its moments of inertia about two perpendicular diameters will, by the *principle of perpendicular axes*, be equal to its moment of inertia about an axis through O and perpendicular to its plane, *i.e.*, equal to $M(R^2+r^2)/2$.

Or, $$I+I = \frac{M(R^2+r^2)}{2}, \text{ i.e., } 2I = \frac{M(R^2+r^2)}{2},$$

whence, $$= \mathbf{\frac{M(R^2+r^2)}{4}}.$$

Now, if $r = 0$, *i.e.*, if the disc be a *plane* one, we have

$M.I.$ of the disc about a diameter $= MR^2/4$. [Case 7 (*ii*), above.

Or, if $r = R$, we have a *hoop* or *circular ring* of radius R, and its moment of inertia about its diameter

$$= \frac{M(R^2+R^2)}{4} = \frac{M.2R^2}{4} = \frac{MR^2}{2}.$$ [Case 6 (*ii*) above.

(*iii*) *about a tangent, in its own plane.*—The tangent being parallel to the diameter of the ring or disc, and at a distance R from it, we have, applying the *principle of parallel axes*,

$M.I.$ about the tangent $= M.I.$ about the diameter $+MR^2$.

Or, $$\mathbf{I} = M\frac{(R^2+r^2)}{4} + MR^2 = \mathbf{M\frac{(5R^2+r^2)}{4}}.$$

(*iv*) *about a tangent, perpendicular to its own plane.*—The tangent, in this case, is parallel to the axis through the centre of the ring or disc and perpendicular to its plane, the distance between the two being equal to R. Hence, by the *principle of parallel axes*, we have $M.I.$ about the tangent $= M.I.$ about the perpendicular axis$+ MR^2$.

Or $$\mathbf{I} = M\frac{(R^2+r^2)}{2} + MR^2 = \mathbf{M\frac{(3R^2+r^2)}{2}}.$$

9. Moment of inertia of a Solid Cylinder.—

(*i*) *about its own axis, or its axis of cylindrical symmetry.*—A cylinder is just a thick circular disc, or *a number of thin circular* *discs, piled one upon the other*, and, therefore, its moment of inertia about its axis is the same as that of a circular disc or lamina about an axis through its centre and perpendicular to its plane, *i.e.*, equal to $\mathbf{MR^2/2}$, where M is its mass and R, its radius. [Case 7 (*i*) above.

(*ii*) *about an axis passing through its centre and perpendicular to its own axis of cylindrical symmetry.*—Let M be the mass of the cylinder, R its radius and l, its length, (Fig. 37). Then, obviously, if it be homogeneous, its *mass per unit length* will be M/l. Let YY' be the axis, passing through its centre O and perpendicular to its own axis XX', about which the moment of inertia is to be determined.

Imagine the cylinder to be made up of a number of thin discs and consider one such disc at a distance x from O, and of thickness dx.

Obviously, *the mass of the disc* is $(M/l).dx$ and its *radius*, equal to R; so that, its *moment of inertia about its diameter is equal to mass of the disc* $\times$ $(radius)^2/4$.

$$= \frac{M}{l} \cdot dx \cdot \frac{R^2}{4}.$$

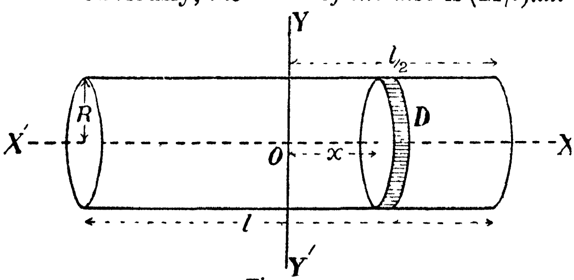

Fig. 37.

And, its moment of inertia about the axis YY', by the *principle of parallel axes*, $= \frac{M}{l} \cdot dx \cdot \frac{R^2}{4} + \frac{M}{l} \cdot dx.x^2$.

Therefore, the moment of inertia of the *whole* cylinder about the axis YY' may be obtained by integrating this expression for the limits $x = 0$ and $x = l/2$ and multiplying the result by 2. Thus,

M.I. of the cylinder about the axis YY'

$$= 2\int_0^{l/2}\left(\frac{M}{l} \cdot \frac{R^2}{4}.dx + \frac{M}{l}.x^2.dx\right) = \frac{2M}{l}\int_0^{l/2}\left(\frac{R^2}{4}.dx + x^2.dx\right)$$

$$= \frac{2M}{l}\left[\frac{R^2x}{4} + \frac{x^3}{3}\right]_0^{l/2} = \frac{2M}{l}\left[\frac{R^2}{4} \cdot \frac{l}{2} + \frac{l^3}{8 \times 3}\right]$$

Or, $$\mathbf{I} = \frac{2M}{l}\left(\frac{R^2 l}{8} + \frac{l^3}{24}\right) = \mathbf{M}\left(\frac{\mathbf{R^2}}{\mathbf{4}} + \frac{\mathbf{l^2}}{\mathbf{12}}\right).$$

(*iii*) *about a diameter of one of its faces.*—It is an easy deduction from the above ; for, by the *principle of parallel axes*, we have

M.I. of the cylinder about the diameter of one face

$$= M\left(\frac{R^2}{4} + \frac{l^2}{12}\right) + M\left(\frac{l}{2}\right)^2.$$

$$= \frac{MR^2}{4} + \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{MR^2}{4} + \frac{Ml^2}{3}.$$

Or, $$\mathbf{I} = \mathbf{M}\left(\frac{\mathbf{R^2}}{\mathbf{4}} + \frac{\mathbf{l^2}}{\mathbf{3}}\right).$$

10. Moment of Inertia of a Solid Cone.—

(*i*) *about its vertical axis.* Let mass of the cone be M, its vertical height, h and radius of its base, R, (Fig. 38).

Then, its *volume* $= \frac{1}{3}\pi R^2 h$.

And, if ρ be the *density* of its material, its *mass* $M = \frac{1}{3}\pi R^2 h\rho$, whence, $\rho = \frac{3M}{\pi R^2 h}$.

Imagine the cone to be made up of a number of discs, parallel to the base, and placed one above the other. Consider one such disc at a distance x from the vertex, and of thickness dx.

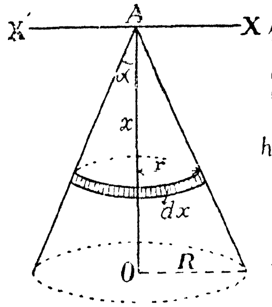

Fig. 38.

If r be the radius of *this* disc, we have

$$r = x\tan\alpha,$$

[where α is the *semi-vertical* angle of the cone.

And, *volume* of the disc $= \pi r^2.dx$.

$\therefore$ its *mass* $= \pi r^2.dx.\rho$.

Now, moment of inertia of the disc about the axis AO, passing through its centre and perpendicular to its plane, *i.e.*, about the vertical axis of the cone, is clearly equal to $\frac{its\ mass \times (its\ radius)^2}{2}$.

$$= \pi r^2.dx.\rho.r^2/2 = \frac{\pi.\rho.r^4}{2}.dx = \frac{\pi.\rho.x^4.\tan^4\alpha}{2}.dx.$$

And, therefore, the moment of inertia of the *whole* cone about its vertical axis AO will be the integral of this expression, for the limits $x = 0$ and $x = h$.

i.e., *M.I.* of the cone about its vertical axis is given by

$$I = \int_0^h \frac{\pi\rho x^4.\tan^4\alpha}{2}.dx = \frac{\pi\rho.\tan^4\alpha}{2}\int_0^h x^4.dx.$$

$$= \frac{\pi\rho.\tan^4\alpha}{2}\left[\frac{x^5}{5}\right]_0^h = \frac{\pi\rho.\tan^4\alpha}{2}.\frac{h^5}{5}.$$

$$= \frac{\pi\rho.R^4}{2.h^4}.\frac{h^5}{5}. \qquad [\because \tan\alpha = R/h.$$

Or, substituting the value of ρ from above, we have

$$\mathbf{I} = \frac{\pi.3M.R^4}{\pi R^2.h.2h^4}.\frac{h^5}{5} = \frac{\mathbf{3MR^2}}{\mathbf{10}}.$$

(*ii*) *about an axis through its vertex and parallel to its base.*— Again, considering the disc at a distance x from the vertex of the cone, we have

$$M.I. \text{ of the } disc \text{ about its diameter} = \pi r^2.dx.\rho.\frac{r^2}{4}.$$

And, therefore, by the *principle of parallel axes,* its moment of inertia about a parallel axis XX', passing through the vertex of the cone

$$= \pi r^2.dx.\rho.\frac{r^2}{4} + \pi r^2.dx.\rho.x^2.$$

Therefore, the moment of inertia of the *whole* cone about the axis passing through the vertex and parallel to the base, *i.e.*, about XX', is obtained by integrating this expression for the limits, $x = 0$ and $x = h$. Thus,

M.I. of the cone about XX'

$$= \int_0^h \left[\frac{\pi(x \tan \alpha)^4.\rho}{4} . dx + \pi(x \tan \alpha)^2.x^2.\rho.dx.\right]$$

$$= \frac{\pi\rho \tan^4 \alpha}{4} \int_0^h x^4.dx + \pi\rho \tan^2 \alpha \int_0^h x^4.dx$$

$$= \frac{\pi\rho}{4} . \frac{R^4}{h^4} \left[\frac{x^5}{5}\right]_0^h + \pi\rho . \frac{R^2}{h^2} \left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{\pi\rho}{4} . \frac{R^4}{h^4} . \frac{h^5}{5} + \pi\rho . \frac{R^2}{h^2} . \frac{h^5}{5} .$$

Or, substituting the value of ρ, we have

$$M.I. \text{ of the cone about } XX' = \frac{\pi.3M}{\pi R^2 h.4} . \frac{R^4}{h^4} . \frac{h^5}{5} + \frac{\pi.3M}{\pi.R^2 h} . \frac{R^2}{h^2} . \frac{h^5}{5} .$$

Or, $$\mathbf{I} = \frac{\mathbf{3MR^2}}{\mathbf{20}} + \frac{\mathbf{3Mh^2}}{\mathbf{5}} .$$

4. Moment of Inertia of a Hollow Cylinder.

(*i*) *about its own axis.*—A hollow cylinder may be considered to consist of a large number of annular discs or rings of the given internal and external radii, placed one above the other, the axis of the cylinder passing through their centre and being perpendicular to their planes, (Fig. 36).

The moment of inertia of the hollow cylinder about its own axis is, therefore, the same as that of an annular disc of the given external and internal radii about an axis through its centre and perpendicular to its plane, *i.e.*, equal to $\mathbf{M(R^2+r^2)/2}$, where M is the mass of the cylinder, R and r, its external and internal radii respectively.

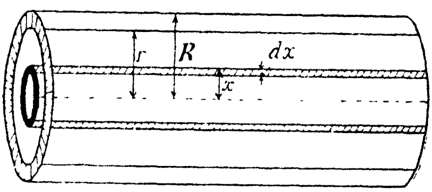

Fig. 39.

Alternative Proof.—Let M be the mass of the cylinder ; R and r, its external and internal radii and l, its length, (Fig. 39).

Then, *face-area* of the cylinder $= \pi(R^2 - r^2)$.

And, *volume* of the cylinder $= \pi(R^2 - r^2)l$;

so that, *its mass per unit volume* $= \dfrac{M}{\pi(R^2 - r^2)l}$.

Now, imagine the cylinder to be made up of a large number of thin co-axial cylinders, and consider one such cylinder of radius x and thickness dx.

Then, its *face area* $= 2\pi x.dx$, its *volume* $= 2\pi x.dx.l$,

and its *mass* $= \dfrac{M}{\pi(R^2-r^2)l} \times 2\pi x.dx.l = \dfrac{2Mx\,dx}{(R^2-r^2)}$.

Since all its particles are equidistant from the axis, its moment of inertia about the axis $= \dfrac{2Mx.dx}{(R^2-r^2)}.x^2 = \dfrac{2M\,x^3.dx}{(R^2-r^2)}$.

And, therefore, the moment of inertia of the *whole* cylinder may be obtained by integrating the above expression for the limits, $x = r$ and $x = R$.

Or, *M.I.* of the cylinder about its axis, *i.e.*,

$$I = \int_r^R \frac{2M.x^3}{(R^2-r^2)}.dx = \frac{2M}{(R^2-r^2)}\int_r^R x^3.dx.$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)}\left[\frac{(R^4-r^4)}{4}\right]$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{(R^2+r^2)(R^2-r^2)}{4}\right].$$

Or, $$\mathbf{I = M\left[\frac{R^2+r^2}{2}\right]}.$$

(*ii*) *about an axis passing through its centre and perpendicular to its own axis.*—As before, let M be the mass of the cylinder, l, its length, and R and r, its external and internal radii respectively; and let YOY' be the axis through its centre O, and perpendicular to its own axis XX', (Fig. 40).

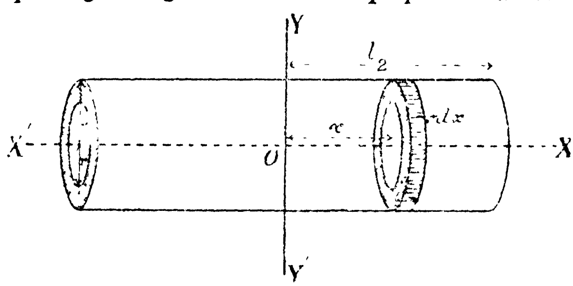

Fig 40.

Then, *face-area* of the cylinder $= \pi(R^2-r^2)$,

and its *volume* $= \pi(R^2-r^2)l$.

$\therefore$ its *mass per unit volume* $= M/\pi(R^2-r^2)l$.

Imagine the cylinder to be made up of a large number of annular discs of external and internal radii R and r, placed one by the side of the other, and consider one such disc at a distance x from the axis YY', and of thickness dx. Then, clearly,

surface area of the disc $= \pi(R^2-r^2)$,

its *volume* $= \pi(R^2-r^2).dx$, and $\therefore$ its *mass* $= M.dx/l$.

Now, moment of inertia of an annular disc of external and internal radii, R and r, about its diameter, is equal to *its mass* $\times$ $(R^2+r^2)/4$. [Case 8 (*ii*), above.

$\therefore$ *M.I.* of the disc about its diameter $= \dfrac{M}{l}.dx \times \dfrac{(R^2+r^2)}{4}$.

And, therefore, its moment of inertia about the parallel axis YY' is, by the *principle of parallel axes*, given by

$$\frac{M}{l}.dx\times\frac{(R^2+r^2)}{4}+\frac{M}{l}.dx.x.$$

And, clearly, therefore, moment of inertia I, of the *whole* cylinder, about the axis YY', is twice the integral of this expression, for the limits, $x = 0$ and $x = l/2$.

i.e., $$I = 2\int_0^{l/2}\left[\frac{M(R^2+r^2)}{4l}.dx+\frac{M}{l}.x^2.dx.\right]$$

$$= \frac{2M}{l}\int_0^{l/2}\left[\frac{(R^2+r^2)}{4}.dx+x^2.dx\right] = \frac{2M}{l}\left[\frac{(R^2+r^2)x}{4}+\frac{x^3}{3}\right]_0^{l/2}$$

Or, $$\mathbf{I} = \frac{2M}{l}\left[\frac{(R^2+r^2)l}{4\times2}+\frac{l^3}{8\times3}\right] = \mathbf{M}\left[\frac{(\mathbf{R}^2+\mathbf{r}^2)}{4}+\frac{\mathbf{l}^2}{12}\right].$$

It follows, therefore, that if $r = 0$, *i.e.*, if the cylinder be a *solid one*, we have

M.I. of the cylinder, (solid), about an axis through its centre and perpendicular to its own axis$=\mathbf{M(R^2/4 + l^2/12)}$. [Case 9, (*ii*) above.

12. Moment of Inertia of a Spherical Shell.

(*i*) *about its diameter.*—

First Method.—Let $ABCD$ be the section of a spherical shell through its centre O and let the mass of the shell be M, and its radius R, (Fig. 41).

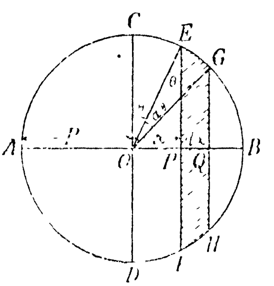

Fig. 41.

Then, area of the shell is equal to $4\pi R^2$, and

$\therefore$ *mass per unit area* of the shell $= M/4\pi R^2$.

Let it be required to determine its moment of inertia about the diameter AB.

Consider a *thin slice* of the shell, lying between two planes EF and GH, perpendicular to the diameter AB, and at distances x and $(x+dx)$ respectively from its centre O.

This *slice* is obviously a *ring* of radius PE, and width EG, (*not* PQ, which is equal to dx).

$\therefore$ area of *this* ring $=$ its circumference $\times$ width.

$$= 2\pi.PE\times EG,$$

and, hence its *mass* $=$ its *area* $\times M/4\pi R^2$.

$$= 2\pi.PE\times EG\times M/4\pi R^2. \quad \dots (i)$$

Join OE and OG, and let $\angle COE = \theta$ and $\angle EOG = d\theta$.

Then, $PE = OE.\cos OEP = R\cos\theta$

[$\because$ $\angle OEP = \angle COE = \theta$.

Similarly, $OP = R\sin\theta$

Or, $x = R\sin\theta$, [$\because$ $OE = R$ and $OP = x$.

Now, differentiating x with respect to θ, we have

$$dx/d\theta = R\cos\theta.$$

Or, $$dx = R\cos\theta.d\theta = PE.d\theta.$$ [$\because R\cos\theta = PE$.

And, $$EG = OE.d\theta = R.d\theta.$$ [$\because$ *arc = radius × angle subtended by the arc.*

$\therefore$ *mass of the ring* $= 2\pi.PE.R.d\theta.M/4\pi R^2.$ [from (*i*).

$$= \frac{M.dx}{2R}.$$ [$\because PE.d\theta = dx$.

Hence, moment of inertia of *the ring* about AB, (an axis passing through its centre and perpendicular to its plane), is equal to its mass × (its radius)^2, *i.e.*, $= \frac{M.dx}{2R}.PE^2$, where $PE^2 = (R^2 - x^2)$.

$\therefore$ moment of inertia of *the ring* about $AB = \frac{M.dx}{2R}.(R^2 - x^2)$.

And, therefore, the moment of inertia I, of the *whole* spherical shell about AB = twice the integral of $\frac{M.dx}{2R}.(R^2 - x^2)$, between the limits, $x = 0$ and $x = R$.

Or, $$I = 2\int_0^R \frac{M}{2R}(R^2 - x^2).dx = \frac{2M}{2R}\int_0^R (R^2 - x^2).dx.$$

$$= \frac{M}{R}\left[R^2.x - \frac{x^3}{3}\right]_0^R = \frac{M}{R}\left[R^3 - \frac{R^3}{3}\right]$$

i.e., $$\mathbf{I} = \frac{M}{R}\cdot\frac{2}{3}R^3 = \frac{2}{3}\,\mathbf{MR^2}.$$

Second Method.—Let M be the mass of the shell and R, its radius.

Consider a particle, of mass m, anywhere on the shell. Then, since the thickness of the shell is negligible, the distance of the particle from the centre of the shell is the same as the radius of the shell, *i e.*, R.

Obviously, therefore, the summation I_0, for all the particles of the shell, about its centre O, is given by the relation,

$$I_0 = \Sigma mR^2. \quad \text{Or,} \quad I_0 = MR^2$$ [All particles being at distance R from O and $\Sigma m = M$.

Now, if I be the moment of inertia of the shell about one diameter, it will be the same about any other diameter also, from the sheer symmetry of the shell. Hence, in accordance with the *principle of perpendicular axes for a three dimensional body*, [§29 (*a*), (*ii*), page 52] the sum of the moments of inertia of the shell about its three mutually perpendicular diameters must be equal to twice the summation I_0, for all its particles, about their point of intersection, *i.e.*, the centre of the shell O; so that,

$$I + I + I = 2I_0, \quad \text{Or,} \quad 3I = 2MR^2,$$ [$\because I_0 = MR^2$.

whence, $$\mathbf{I} = \frac{2}{3}\mathbf{MR^2}.$$

(*ii*) *about a tangent.*—Obviously, a tangent, drawn to the shell at any point, must be parallel to one of its diameters, and at a distance from it equal to R, the radius of the shell. Hence, applying the *principle of parallel axes,* we have

M.I. of the shell about a tangent

= its *M.I.* about a diameter $+ MR^2$.

Or, $$I = \tfrac{2}{3}MR^2 + MR^2 = \tfrac{5}{3}\mathbf{MR^2}.$$

13. Moment of Inertia of a Solid Sphere.

(i) about its diameter.—Let Fig. 42 represent a section of the sphere through its centre O.

Let *mass* of the sphere be M, and its radius, R.

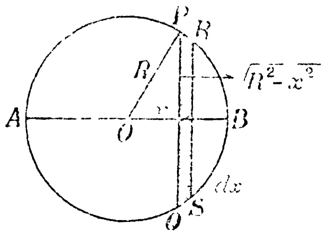

Fig. 42.

Then, clearly, its *volume* $= 4\pi R^3/3$.

And $\therefore$ its *mass per unit volume*

$$= M/\tfrac{4}{3}\pi R^3 = 3M/4\pi R^3.$$

Consider a thin circular *slice* of the sphere at a distance x from the centre O, and of thickness dx.

This slice is obviously a *disc* of radius $\sqrt{R^2-x^2}$, and of thickness dx.

$\therefore$ *surface area* of the slice $= \pi\sqrt{(R^2-x^2)}^2 = \pi(R^2-x^2)$, and its *volume* $=$ area$\times$thickness $= \pi(R^2-x^2).dx$.

And, $\therefore$ its *mass* $=$ its *volume*$\times$*mass per unit volume* of the sphere

$$= \pi(R^2-x^2).dx \times \frac{3M}{4\pi R^3} = \frac{3M.(R^2-x^2)}{4R^3}.dx.$$

Now, the moment of inertia of *this* disc about AB, (an axis passing through its centre and perpendicular to its plane)

$$= \text{its } mass \times (radius)^2/2.$$

$\therefore$ moment of inertia of the disc AB

$$= \frac{3M.(R^2-x^2)}{4R^3}.dx.\frac{(R^2-x^2)}{2} = \frac{3M.(R^2-x^2)^2}{8R^3}.dx. \quad \ldots (1)$$

$\therefore$ moment of inertia I, of the *sphere* about the diameter AB is equal to twice the integral of expression (1) between the limits $x = 0$ and $x = R$.

$$i.e., I = 2\int_0^R \frac{3M(R^2-x^2)^2}{8R^3}.dx = \frac{2\times 3M}{8R^3}\int_0^R (R^2-x^2)^2.dx.$$

$$= \frac{3M}{4R^3}\int_0^R (R^4-2R^2.x^2+x^4).dx = \frac{3M}{4R^3}\Big[R^4x-2R^2.\frac{x^3}{3}+\frac{x^5}{5}\Big]_0^R$$

$$= \frac{3M}{4R^3}\Big(R^5-\frac{2}{3}R^5+\frac{1}{5}R^5\Big) = \frac{3M}{4R^3}\Big[\frac{15R^5-10R^5+3R^5}{15}\Big]$$

Or, $$\mathbf{I} = \frac{3M}{4R^3}\times\frac{8R^5}{15} = \frac{2}{5}\,\mathbf{MR^2}.$$

Now, the moment of inertia of the sphere about one diameter is the *same* as about another diameter, so that we have the moment of inertia of a solid sphere about any diameter given by $\mathbf{I} = \frac{2}{5}\mathbf{MR^2}$.

Alternative Method.—Let M be the mass of the sphere and ρ, the density of its material.

Imagining the whole sphere to be made up of a number of thin, concentric spherical shells, one inside the other, and considering one such shell of *radius* x and *thickness* dx, we have

surface area of the shell $= 4\pi x^2$, and $\therefore$ *volume* of the shell $= 4\pi x^2.dx$ and its *mass* $= 4\pi x^2.dx.\rho$.

$\therefore$ moment of inertia of the *shell* about a diameter

$= \frac{2}{3}\times$(its *mass*)$\times$(its *radius*)^2 $= \frac{2}{3}.4\pi x^2.dx.\rho\times x^2 = \frac{8}{3}\pi\rho x^4\,dx$, [case 11 (*l*).

And, therefore, the moment of inertia *I*, of the *whole* sphere, about its diameter, is obtained by integrating the above expression between the limits. $x = 0$ and $x = R$.

i e.,
$$I = \int_0^R \frac{8}{3}.\pi\rho x^4.dx = \frac{8}{3}\pi\rho\int_0^R x^4.dx.$$
$$= \frac{8}{3}\pi\rho\left[\frac{x^5}{5}\right]_0^R = \frac{8}{3}\pi.\rho.\frac{R^5}{5} = \frac{8}{15}\pi\rho.R^5$$
$$= \frac{4}{3}.\pi R^3.\rho.\frac{2}{5}R^2.$$

t $4\pi R^3/3$ = the *volume* of the sphere ; and, therefore, $4\pi R^3\rho/3 = M$, its *mass*.

$\therefore$ *M.I.* of the sphere about its diameter, *i.e.*, **I = 2.MR²/5**.

(*ii*) *about a tangent.*—A tangent, drawn to the sphere at any point, will obviously be parallel to one of its diameters and at a distance from it equal to *R*, the radius of the sphere.

Therefore, in accordance with the *principle of parallel axes*, we have *M.I.* of the sphere about a tangent

$=$ its *M.I.* about a diameter $+MR^2$.

Or, $\mathbf{I} = 2\,MR^2/5 + MR^2 = \mathbf{7MR^2/5}$.

14. Moment of Inertia of a Hollow Sphere or a Thick Shell.

(*i*) *about its diameter* —A hollow sphere is just a solid sphere from the inside of which a smaller concentric solid sphere has been removed. And so, the moment of inertia of the hollow sphere is equal to the moment of inertia of the bigger solid sphere *minus* the moment of inertia of the smaller solid sphere removed from it, (both about the *same* diameter). If *R* be the radius of the bigger sphere and *r*, that of the smaller sphere, *i.e.*, if *R* and *r* be the external and internal radii of the hollow sphere, and ρ, the density of its material, we have

volume of the bigger sphere $= \frac{4}{3}\pi R^3$, and $\therefore$ its mass $= \frac{4}{3}\pi R^3\rho$.

and, „ „ „ smaller „ $= \frac{4}{3}\pi r^3$ and „ „ „ $= \frac{4}{3}\pi r^3\rho$.

$\therefore$ volume of hollow sphere $= \frac{4}{3}(R^3-r^3)$ and its *mass* $= \frac{4}{3}\pi(R^3-r^3)\rho$.

And $\therefore$ *M.I.* of the bigger sphere about its diameter

$$= \tfrac{2}{5}.(\tfrac{4}{3}\pi R^3\rho).R^2.$$

and *M.I.* of the smaller sphere *about the same diameter*

$$= \tfrac{2}{5}.(\tfrac{4}{3}\pi r^3.\rho).r^2.$$

$\therefore$ *M.I.* of the hollow sphere about that diameter

$$= \tfrac{2}{5}.(\tfrac{4}{3}\pi R^3.\rho).R^2 - \tfrac{2}{5}(\tfrac{4}{3}\pi r^3.\rho).r^2$$
$$= \tfrac{2}{5}.\tfrac{4}{3}\pi\rho.(R^5-r^5) \qquad \dots (1)$$

Now, mass of the hollow sphere, $M = \frac{4}{3}.\pi(R^3-r^3).\rho$.

Or, $3M = 4\pi(R^3-r^3).\rho$. And $\therefore \rho = 3M/4\pi(R^3-r^3)$.

Substituting this value of ρ in relation (1) above, we have moment of inertia I of the hollow sphere about its diameter

$$= \frac{2}{5} \cdot \frac{4}{3}\pi \cdot \frac{3M}{4\pi(R^3-r^3)} \cdot (R^5-r^5).$$

Or, $$I = \frac{2}{5} . M . \frac{(R^5-r^5)}{(R^3-r^3)} .$$

Alternative Method. As in the case of the solid sphere, so also here, we can imagine the sphere to be made up of a number of thin, concentric spherical shells, and considering one such spherical shell of radius x and thickness dx, we have, as before,

$$\text{mass of the shell} = 4\pi x^2.dx.\rho.$$

$$\therefore \textit{M.I.} \text{ of the shell about a diameter} = \tfrac{2}{3}.4\pi x^2.dx.\rho.x^2.$$
$$= \tfrac{8}{3}.\pi.\rho.x^4.dx.$$

[ρ being the density of the material of the sphere.]

Hence, the moment of inertia of the *whole* sphere about its diameter is the integral of the above expression, between the limits, $x = r$ and $x = R$.

Or, *M.I.* of the sphere about a diameter *i.e.*, $I = \int_r^R \frac{8}{3}\pi\, \rho.x^4.dx.$

$$= \frac{8}{3}\pi\,\rho \int_r^R x^4.\,dx = \frac{8}{3}.\pi\,\rho \left[\frac{x^5}{5}\right]_r^R$$

$$= \frac{8}{3}\pi.\rho\left[\frac{R^5}{5} - \frac{r^5}{5}\right] = \frac{8}{15}.\pi.\rho\,[R^5-r^5]$$

$$= \frac{4}{3}\pi\,(R^3-r^3).\rho.\frac{2}{5} . \frac{(R^5-r^5)}{(R^3-r^3)}$$

But $\frac{4}{3}\pi(R^3-r^3).\rho = M$, the mass of the sphere. [See case (*i*) above.]

$\therefore$ $$I = \frac{2}{5}.M(R^5-r^5)/(R^3-r^3).$$

(*ii*) *about a tangent*. Again, as in the case of a solid sphere, the tangent to the sphere, at any point, will be parallel to one of its diameters, and at a distance equal to its *external radius R* from it. Hence, by the *principle of parallel axes*, we have

M.I. of the sphere about a tangent

$$= \text{its } \textit{M.I.} \text{ about a diameter} + MR^2.$$

Or, $$I = \left[\frac{2}{5}M(R^5-r^5)/(R^3-r^3)\right] + MR^2.$$

15. Moment of Inertia of a Flywheel and Axle. A flywheel is just a *large heavy wheel*, with a long, cylindrical *axle*, passing through its centre. *Its centre of gravity lies on its axis of rotation*, so that, when properly mounted over *ball-bearings* (to minimise friction), it may continue to be at rest in any desired position.

Let M be the mass of the flywheel, and m, that of the axle; and let R and r be their respective radii.

Then, for our present purpose, we may regard the flywheel to be a disc, or a small cylinder, from which a smaller, *concentric* disc or cylinder, equal in radius to that of the axle, has been cut off. In other words, we may take it to be an *annular ring*, (or hollow cylinder) with an *outer radius equal to R*, and an *inner radius equal to r*, whose moment of inertia is to be determined about an axis passing through its centre and perpendicular to its plane.

The face area of this wheel or annular disc is clearly equal to the area of the whole disc of radius R *minus* the area of the disc of radius r.

i.e., face area of the wheel $=\pi R^2-\pi r^2=\pi(R^2-r^2)$.

And, if its *mass* be M, clearly,

mass per unit area of the wheel $=M/\pi(R^2-r^2)$.

Now, consider a thin circular ring at a distance x from the centre, and of width dx.

Then, *face area of the ring=its circumference × its width* $=2\pi x.dx$.

And, therefore, its mass $= 2\pi x.dx.M/\pi(R^2-r^2)$.

Now, since the moment of inertia of a ring about an axis through its centre and perpendicular to its plane is equal to its *mass* × (*radius*)², we have

$$M.\ I.\ of\ the\ wheel\ about\ its\ axis = \int_r^R \frac{M}{\pi(R^2-r^2)}.2\pi x.dx.x^2.$$

$$= \frac{2\pi M}{\pi(R^2-r^2)}\int_r^R x^3.dx = \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{R^4}{4}-\frac{r^4}{4}\right] = \frac{2M}{(R^2-r^2)}\left[\frac{R^4-r^4}{4}\right]$$

Or, $$M.I.\ of\ the\ wheel\ about\ its\ axis = \frac{M}{2}(R^2+r^2) = M.\frac{(R^2+r^2)}{2}.$$

The axle, again, is just a disc, (or solid cylinder), and its moment of inertia about its axis is, therefore, just the same as that of a disc or a cylinder about its axis, *i.e.*, = its mass × (radius)²/2.

So that, *M.I. of the axle* $= m.r^2/2$.

Hence, *M.I. of the wheel and axle = M.I. of the wheel + M.I. of the axle.*

Or, $$I = [M(R^2+r^2)/2]+mr^2/2.$$

32. Table of Moments of Inertia. The values of *moments of inertia* for the cases discussed above, together with some other important ones are given in the *Table* below for ready reference of the student, the mass of the body being taken to be M, in all cases.

BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
1. Thin uniform rod, of length l.	(*i*) Through its centre and perpendicular to its length.	$Ml^2/12$
	(*ii*) Through one end and perpendicular to its length.	$Ml^2/3$
2. Thin and rectangular sheet or lamina, of sides l and b.	(*i*) Through its centre and parallel to one side	$Mb^2/12$ or $Ml^2/12$
	(*ii*) About one side.	$Mb^2/3$ or $Ml^2/3$
	(*iii*) Through its centre and perpendicular to its plane.	$M(l^2+b^2)/12$
	(*iv*) Through the mid-point of one side (l or b) and perpendicular to its plane.	$M(b^2/3+l^2/12)$ or $M(l^2/3+$

BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
3. Thick uniform rectangular bar, of length l and thickness d.	Through its mid-point and perpendicular to its length.	$M\left(\frac{l^2+d^2}{12}\right)$
4. Thin triangular plate or lamina, of altitude H.	About one side.	$MH^2/6$
5. Elliptical disc or lamina, of major and minor axes $2a$ and $2b$.	(*i*) About one of the axes, (major or minor).	$Mb^2/4$ or $Ma^2/4$
	(*ii*) Through its centre and perpendicular to its plane.	$M(a^2+b^2)/4$
6. Hoop or circular ring, of radius R.	(*i*) Through its centre and perpendicular to its plane.	MR^2
	(*ii*) About a diameter.	$MR^2/2$
	(*iii*) About a tangent in its own plane.	$3MR^2/2$
	(*iv*) About a tangent, perpendicular to its plane.	$2MR^2$
7. Circular lamina or disc, of radius R.	(*i*) Through its centre and perpendicular to its plane.	$MR^2/2$
	(*ii*) About a diameter.	$MR^2/4$
	(*iii*) About a tangent, in its own plane.	$5MR^2/4$
	(*iv*) About a tangent perpendicular to its plane.	$3MR^2/2$
8. Annular ring or disc of outer and inner radii R and r.	(*i*) Through its centre and perpendicular to its plane.	$M(R^2+r^2)/2$
	(*ii*) About a diameter.	$M(R^2+r^2)4$
	(*iii*) About a tangent, in its own plane.	$M(5R^2+r^2)/4$
	(*iv*) About a tangent perpendicular to its plane.	$M(3R^2+r^2)/2$
9. Solid cylinder of length l and radius R.	(*i*) About its axis of cylindrical symmetry.	$MR^2/2$
	(*ii*) Through its centre and perpendicular to its axis of cylindrical symmetry.	$M\left(\frac{l^2}{12}+\frac{R^2}{4}\right)$
	(*iii*) About a diameter of one face.	$M\left(\frac{l^2}{3}+\frac{R^2}{4}\right)$
10. Solid cone, of altitude h and base radius R.	(*i*) About its vertical axis.	$3MR^2/10$
	(*ii*) Through its vertex and parallel to its base.	$\frac{3MR^2}{20}+\frac{3Mh^2}{5}$
11. Hollow cylinder, of length l and external and internal radii R and r.	(*i*) About its own axis, (*i.e.*, about its axis of cylindrical symmetry).	$M(R^2+r^2)/2$
	(*ii*) Through its centre and perpendicular to its own axis.	$M\left[\frac{R^2+r^2}{4}+\frac{l^2}{12}\right]$

BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
12. Spherical shell, of radius R.	(*i*) About a diameter. (*ii*) About a tangent.	$2MR^2/3$ $5MR^2/3$
13. Solid sphere, of radius R.	(*i*) About a dimeter. (*ii*) About a tangent.	$2MR^2/5$ $7MR^2/5$
14. Thick shell or hollow sphere, of external and internal radii R and r.	(*i*) About a diameter. (*ii*) About a tangent	$\frac{2}{5}M\left(\frac{R^5-r^5}{R^3-r^3}\right)$ $\frac{2}{5}M\left(\frac{R^5-r^5}{R^3-r^3}\right)+MR^2$
15. Flywheel, with radii of wheel and axle, R and r—(mass of axle, m).	About its own axis.	$\frac{M(R^2+r^2)}{2}+\frac{mr^2}{2}$
16. Spheroid of revolution, of equatorial radius R.	About polar axis.	$2MR^2\ 5$
17. Ellipsoid, of axes $2a$, $2b$ and $2c$.	About one axis, ($2a$)	$\frac{M(b^2+c^2)}{5}$
18. Rectangular parallelopiped of edges l, b and d.	Through its centre and perpendicular to one face, (say, face $l-b$)	$\frac{M(l^2+b^2)}{12}$
19. Rectangular prism, of dimensions $2l$, $2b$ and $2d$.	About axis $2l$.	$\frac{M(b^2+d^2)}{3}$
20. Very thin hollow cylinder, of length l and mean radius R	Through its centre and perpendicular to its own axis	$M\left(\frac{l^2}{12}+\frac{R^2}{2}\right)$

33. Routh's Rule. This rule states that *the moment of inertia of a body about any one of the three perpendicular axes of symmetry passing through its centre of mass is given by*

(*i*) *the product of its mass and one-third of the sum of the squares of the other two semi-axes,* **in the case of a rectangular lamina or parallelopiped ;**

(*ii*) *the products of its mass and one-fourth of the sum of the squares of the other two semi-axes,* **in the case of a circular or an elliptical lamina ;**

(*iii*) *the product of its mass and one-fifth of the sum of the squares of the other two semi-axes,* **in the case of a sphere or a spheroid.**

Quite a few of the cases, dealt with in the proceeding pages, may be easily deduced by an application of this rule. Thus, for example,

(*i*) *moment of inertia of a uniform rectangular lamina* (of mass M, length l and breadth b), about an axis passing through its centre O and perpendicular to its plane

$$= M\left(\frac{(l^2/4)+(b^2/4)}{3}\right)=M\left(\frac{l^2+b^2}{12}\right);$$

for, here, the two *semi-axes* of the lamina are clearly, $l/2$ and $b/2$ respectively, (Fig. 43).

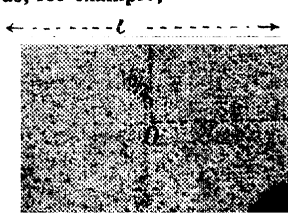

Fig. 43.

(*ii*) *Moment of inertia of a uniform circular lamina or disc*, (of mass M and radius R), about an axis, passing through its entre and perpendicular to its plane is equal to

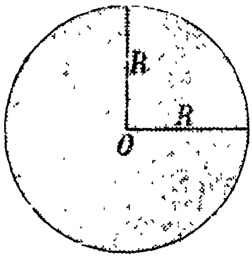

Fig. 44.

$$M.(R^2+R^3)/4 = MR^2/2,$$

because here the two semi-apes of the lamina or disc are obviously R and R, (Fig. 44).

And, again, *moment of inertia of a uniform elliptical lamina*, (of mass M, and with $2a$ and $2b$, as its major and minor axes respectively), about a perpendicular axis passing through its centre, is equal to

$$M(a^2+b^2)/4,$$

because (a) and (b) are the two semi-axes of the lamina, (Fig. 45).

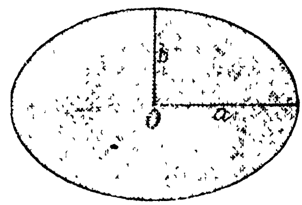

Fig. 45.

(*iii*) *moment of inertia of a solid sphere*, (of mass M and radius R) about its diameter is equal to

$$M.(R^2+R^2)/5 = 2MR^2/5,$$

because here the two semi-axes of the sphere are R and R.

34. Practical methods for the Determination of Moments of Inertia. The principle underlying the experimental determination of the moment of inertia I of a body, about a given axis, is to apply a known couple C to it and to measure the angular acceleration $d\omega/dt$ produced in it. Then, from the relation,

$$C = I.d\omega/dt, \qquad \text{we have} \qquad I = \frac{C}{d\omega/dt},$$

whence, I may be easily calculated.

(*i*) **Moment of Inertia of a Flywheel.**

First Method. The flywheel, whose moment of inertia is to be determined, is mounted on ball-bearings (to minimise friction), and its axle is arranged to be in the horizontal position at a convenient height from the ground, (Fig. 46).

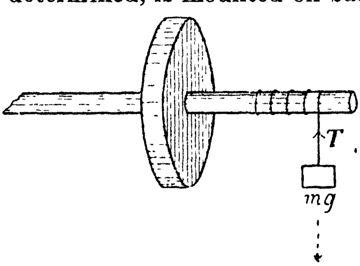

Fig. 46.

A small loop at the end of a small piece of fine cord is then slipped on to a tiny peg on the axle and *the entire length of the cord wound evenly round the latter*, with a suitable mass m suspended from its free end, and properly held in position.

As the mass is released and allowed to fall under the action of its own weight, the cord starts unwinding itself round the axle, thereby setting the wheel in rotation. The length of the cord is so adjusted that the moment the mass reaches the ground, the entire length of it gets *just* unwound from the axle and slips off the peg.

Obviously, the rotation of the wheel, (with the descent of the mass), is due to a *couple* $T.r.$ where T is the tension in the cord and r, the radius of the axle*.

If, therefore, I be the moment of inertia of the flywheel about its axis of rotation and $d\omega/dt$, the angular acceleration produced in it, we have $\quad I.d\omega/dt = T.r.$

The downward force due to the *weight* of the mass, when it has no acceleration, is mg ; but when it has a vertical acceleration a, the force due to it is equal to $m.a.$, and this must clearly be equal to $mg-T$.

Or, $\quad m.a = mg-T, \quad$ whence, $\quad T = m.(g-a)$,

$\therefore \quad I.d\omega/dt = m.(g-a)r.$

But $\quad d\omega/dt = a/r, \quad$ And $\therefore \; I.a/r = m(g-a)r. \; [\because \; a = r.d\omega/dt]$

Or, $$I = mr^2.\frac{(g-a)}{a} = mr^2\left(\frac{g}{a}-1\right) \qquad ...(1)$$

The time-interval between the release of the mass and the slipping of the cord from the axle is carefully noted. Let it be t, and let the distance through which the mass falls down during this interval be S. Then, since the mass starts from rest, we have

$$S = \tfrac{1}{2}\,at^2, \quad \text{whence,} \quad a = 2S/t^2.$$

So that, substituting this value of a in relation (1) above, we have

$$I = mr^2.\left(\frac{g}{2S/t^2}-1\right) = mr^2.\left(\frac{gt^2}{2S}-1\right),$$

whence I, the moment of inertia of the flywheel about its axis of rotation, can be easily calculated.

Second Method.—Proceeding as above, *the loss of potential energy af the falling mass is equated against the gain in kinetic energy of the wheel, the K. E. of the mass itself and the work done against friction.* Thus, when the mass falls through distance S, the potential energy lost by it is equal to $mg.S$. And, if ω be the angnlar velocity of the wheel at the time, the $K.E.$ gained by it is $\frac{1}{2}\,I\omega^2$, the $K.E.$ acquired by the mass being $\frac{1}{2}\,mv^2$, where v is its velocity on descending through distance S.

$$\therefore \; mg.S = \tfrac{1}{2}\,I\omega^2+\tfrac{1}{2}mv^2+\textit{the work done against friction.} \qquad ..(2)$$

To determine the work done against friction, we note the number of turns made by the wheel before coming to rest, *after the mass has been detached from the axle.* Then, obviously, the kinetic energy $\frac{1}{2}\,I\omega^2$, of the wheel, is used up in overcoming the frictional forces at the bearings. If the couple due to friction be C and the number of turns made by the wheel before coming to rest be n, work done by *this* couple is equal to $2\pi n\times C$, ($\because$ *work done = couple × angle,* and the angle, described by the wheel in *one* rotation is equal to 2π). So that, $\quad \frac{1}{2}\,I\omega^2 = 2\pi nC. \quad$ Or, $\quad C = I\omega^2/4\pi n.$

The couple due to friction being thus determined, we can easily calculate the work done against friction during the descent of the

*If the cord be appreciably thick, half of its thickness, added to the radius of the axle, gives the effective value of r.

mass through distance S. For, clearly, the number of turns made by the wheel during the fall of the mass through this distance is $S/2\pi r$; and, therefore, the total angle turned through by it is equal to $2\pi . S/2\pi r = S/r$.

Hence, work done against friction is equal to $C.S/r = SI\omega^2/4\pi nr$.

$\therefore$ our *energy equation* (2) now becomes

$$mg\ S = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2 + \frac{SI\omega^2}{4\pi nr} = \tfrac{1}{2}\ I\omega^2\left(1+\frac{S}{2\pi nr}\right) + \tfrac{1}{2}mv^2.$$

$$= \tfrac{1}{2}\ I.\frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right) + \tfrac{1}{2}mv^2. \qquad [\because\ \omega^2 = v^2/ar^2.$$

Or, $$I.\frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right) + mv^2 = 2mg.\ S.$$

Or, $$I.\ \frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right) = 2mg.S - mv^2. \qquad ..(3)$$

Now, if t be the time taken by the mass to fall through the distance S, its *average velocity* $= S/t$; and since average velocity $=$ (*initial velocity*$+$*final velocity*)$/2$, we have

$$\textit{final velocity},\ v = 2S/t. \qquad \text{Or,} \qquad v^2 = 4S^2/t^2.$$

[$\because$ the initial velocity is *zero*, the mass starting from rest.]

Substituting this value of v^2 in expression (3) above, we have

$$\frac{I}{r^2}\cdot\frac{4S^2}{t^2}\left(1+\frac{S}{2\pi nr}\right) = 2mg.S - m.\frac{4S^2}{t^2}.$$

$$= 2mS\left(g - \frac{2S}{t^2}\right) = 2mS\left(\frac{gt^2-2S}{t^2}\right),$$

whence, $$I = 2mS\left(\frac{gt^2-2S}{t^2}\right)\cdot\frac{r^2.t^2}{4S^2(1+S/2\pi nr)}.$$

Or, $$I = \frac{mr^2(gt^2-2S)}{2S(1+S/2\pi nr)}. \qquad ..(4)$$

Alternative Calculation.—Let the number of rotations made by the wheel, before the cord and the mass slip off from the axle, (*i.e.*, after the mass has fallen through a distance S), be N.* Then, *taking the frictional force to be uniform*, and the work done against it *per rotation* of the wheel to be w, we have

work done against friction during N rotations of the wheel $= N.w$.

Thus, our energy equation (2) becomes

$$mg.S = \tfrac{1}{2}\ I\omega^2 + \tfrac{1}{2}mv^2 + N.w. \qquad ..(5)$$

Now, after the detachment of the mass from the axle, the wheel comes to rest after n rotations, and, therefore, work done against friction during these n rotations of the wheel $= n.w$ and this must obviously be equal to $\tfrac{1}{2}\ I\omega^2$, the $K.E$ of the wheel *at the instant that the mass gets detached from it*. Thus,

$$n\,w = \tfrac{1}{2}\ I\omega^2, \qquad \text{whence}, \qquad w = \tfrac{1}{2}\ I\omega^2/n.$$

Substituting this value of w in equation (5) above, we have

$$mg.S = \tfrac{1}{2}\ I\omega^2 + \tfrac{1}{2}\ mv^2 + \tfrac{1}{2}NI\omega^2/n = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\ I\omega^2(1+N/n).$$

Or. $$2mg.S = mv^2 + I\omega^2(1+N/n).$$

*This is obviously equal to the number of turns of the cord on the axle at the very start.

Or, $$2mg.S - mv^2 = I\omega^2(1+N/n),$$

whence, $$I = \frac{2mg.S - mv^2}{\omega^2(1+N/n)} = \frac{2mg.S - mr^2\omega^2}{\omega^2(1+N/n)}. \qquad [\text{Since } v = r\omega.$$

Or, by dividing both the numerator and the denominator of this expression by ω^2, we have

$$I = \frac{(2mg\,S/\omega^2) - mr^2}{1+N/n} = \frac{m[(2g.S/\omega^2) - r^2]}{1+N/n}. \qquad ..(6)$$

Now, the angular velocity of the wheel at the instant that the mass gets detached from it is ω, and becomes *zero* , when the wheel comes to rest, after time t', say. Hence, if the fractional force *uniformly* retards the rotation of the wheel, its *average* angular velocity, during this interval of time t', may be taken to be equal to $(\omega+0)/2$, *i.e.*, equal to $\omega/2$. And, since the wheel makes n rotations before coming to rest, it describes an angle equal to $2\pi n$ in time t',

$$\therefore \qquad \omega/2 = 2\pi n/t', \qquad \text{whence,} \qquad \omega = 4\pi n/t'.$$

So that, substituting this value of ω in relation (6) above, we have

$$I = \frac{m[(2g\,S\times t'^2/16\pi^2n^2) - r^2]}{1+N/n} = \frac{m[(g\,S\times t'^2/8\pi^2n^2) - r^2]}{(n+N)/n}.$$

$$= m\times\left(\frac{n}{n+N}\right)\left(\frac{g.S.t'^2}{8\pi^2n^2} - r^2\right)$$

Or, $$I = \frac{m}{n+N}\left(\frac{g\,S.t'^2}{8\pi^2 n} - nr^2\right), \qquad ..(7)$$

whence I, the moment of inertia of the flywheel, about its axis of rotation, can be easily calculated.

Accurate value of ω.—In the above treatment, the angular velocity ω of the wheel has been obtained on the supposition that the frictional force remains constant during the time t' that the value of ω falls to zero, after the detachment of the mass from the axle. Obviously, this is by no means a valid assumption, because, as we know, the frictional force decreases with increase of velocity ; so that, the value of I, the moment of inertia of the wheel, deduced on the basis of the above calculations, cannot possibly be quite accurate.

If we aim at accuracy, therefore, we must adopt a *sensitive* method for determining the value of ω, and the one method, which at once suggests itself, is to make use of a tuning fork, as explained below :

A tuning fork, of a known frequency n, is arranged horizontally, (Fig. 47), with a slightly bent metallic style, attached to one of its prongs, such that, when desired, it can be made to lightly press against, or taken off, a strip of smoked paper, wrapped round the rim of the wheel.

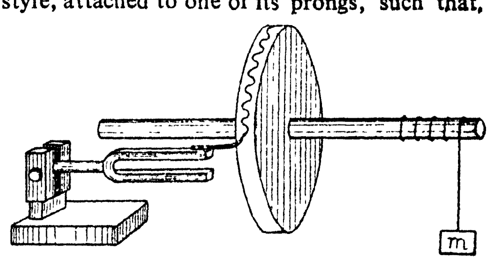

Fig. 47.

Now, with the style kept off the paper-strip, the mass m is allowed to fall down, thus setting the wheel in rotation, and just a second or so before the mass is due to get detached from the axle, the tuning fork is set into vibration (by smartly drawing a bow across it), and the style pressed lightly on to the strip, taking care to take it off soon after the detachment of the mass. A long wavy curve is thus traced out by the style on the smoked strip. The *mean* wavelength λ of this wave is then determined by dividing the tota distance occupied by the wavy curve by the total number of waves constituting it.

Since one wave is traced out by the style during one vibration of the prong of the fork, we have linear distance covered by the wheel during one vibration of the fork $= \lambda$. So that, distance covered by the wheel during n vibrations of the fork $= n\lambda$.

Again, since n vibrations are made by the fork in *one second*, it follows that *distance covered by the wheel in* 1 *second, i.e., the linear velocity* $v = n\lambda$.

But $v = R\omega$, where R is the radius of the wheel and ω, its angular velocity ; so that, we have $R\omega = n\lambda$; whence, $\omega = n\lambda/R$.

Thus, knowing n, λ and R, we can easily calculate the value of ω for the wheel.

This value of ω, substituted in relation (6) above, then gives a much more accurate value of I, the moment of inertia of the flywheel about its axis of rotation.

Note.—The student may, as an interesting exercise, show that expression (4) above can also be reduced to the same form as expression (7). This may be easily done by remembering (*i*) that when the wheel makes *one* full turn, the mass descends through a distance $2\pi r$, the circumference of the axle, and, therefore, when the mass descends through a distance S, the number of rotations made by the wheel is equal to $S/2\pi r$; so that, $S/2\pi r = N$; and further (*ii*) that $t = 2S/v = 2S/r\omega$, where $\omega = 4\pi n/t'$, (see page 81).

(*ii*) **Moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane.**—

(*a*) *Disc suspended by two parallel threads.*—The disc, with a metal axle, is supported on two cords, wound uniformly on the axle on either side, (Fig. 48). On releasing the disc, it begins to fall down until the whole cord is unwound from the axle, say through a distance S.

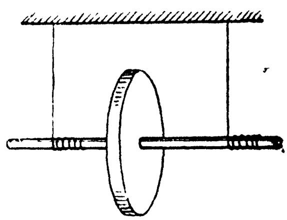

Fig. 48.

Then clearly, $P.E.$ lost by the disc $= mg.S$, where m is the mass of the disc and the axle. This energy will obviously be gained by the disc in the form of kinetic energy of rotation and translation.

If ω be the angular velocity acquired by it after falling through this distance S, its $K.E.$ of rotation will clearly be $\frac{1}{2}I\omega^2$, where I is its moment of inertia about an axis passing through its centre and parallel to the axle, (*i.e.*, perpendicular to its plane) ; and its kinetic energy of translation will be $\frac{1}{2}mv^2$.

$$\therefore \quad mg.S = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2, \text{ (v being its final \textit{linear velocity}),}$$

$$\therefore \quad = (\tfrac{1}{2}I.v^2/r^2) + \tfrac{1}{2}mv^2.$$

[$\because \omega^2 = v^2/r^2$, where r = radius of the disc.

Or, $$\tfrac{1}{2}I\, v^2/r^2 = mg.S - \tfrac{1}{2}mv^2,$$

whence, $$I = (mgS - \tfrac{1}{2}mv^2).2r^2/v^2.$$

Now, *average velocity* $= S/t$, where t is the time taken by the disc in falling through distance S ; and, therefore, velocity v of the disc $= 2S/t$, and $\therefore v^2 = 4S^2/t^2$. So that,

$$I = \left(mg.S - \tfrac{1}{2}m.\frac{4S^2}{t^2}\right) \cdot \frac{2r^2.t^2}{4S^2} = \left(\frac{mg.S.t^2 - 2mS^2}{t^2}\right) \cdot \frac{r^2t^2}{2S^2}.$$

$$= \frac{mgSt^2}{t^2} \times \frac{r^2.t^2}{2S^2} - \frac{2mS^2}{t^2} \times \frac{r^2.t^2}{2S^2} = \frac{mgr^2.t^2}{2S} - mr^2.$$

Or, $$I = mr^2\left(\frac{gt^2}{2S} - 1\right).$$

(*b*) *Disc mounted on axle, rolling on inclined rails.*—Here, the disc, of *mass M* and *moment of inertia I*, is allowed to roll down along inclined rails, as shown in Fig. 49. Let it acquire a *linear velocity v* and an *angular velocity* ω, when it descends a *vertical distance h*, as it rolls down a distance *S along* the rails.

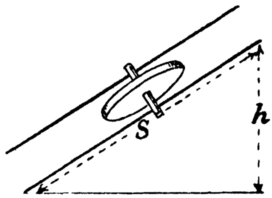

Fig. 49.

Then, clearly, *loss of P.E. of the disc = K.E. of translation gained by disc + K E. of rotation gained by disc.*

Or, $$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2.$$

So that, $$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I.\frac{v^2}{r^2}.$$

[where r = radius of the axle
And $\therefore\ \omega = v^2/r^2$.

Or, $$I.\frac{v^2}{2r^2} = M(gh - \tfrac{1}{2}v^2), \text{ whence, } I = \frac{2Mr^2}{v^2}.(gh - \tfrac{1}{2}v^2)$$

Or, $$I = \frac{Mr^2}{v^2}(2gh - v^2).$$

Or, substituting the value of $v = 2S/t$, (see page 82), where t is the time taken by the disc to cover the distance S, we have

$$I = \frac{Mr^2.t^2}{4S^2}.\left(2gh - \frac{4S^2}{t^2}\right) = \frac{Mr^2t^2.gh}{2S^2} - Mr^2,$$

whence the value of I, the moment of inertia of the disc can be easily calculated.

Note :—For other methods for the determination of moment of inertia, see under Torsional Pendulum, (Chapter VIII)

35. **Angular Moment and Angular Impulse.**—In the case of linear motion, the *momentum* of a body, as we know, is the product of its mass and velocity. On the same analogy, we have, in the case of rotational motion, *the product of the moment of inertia and the angular velocity as the angular momentum of a rotating body.*

Thus, *angular momentum* $= I.\omega$,

where I is the moment of inertia and ω, *the angular velocity of the body about the axis of rotation.*

For, suppose we have a body, rotating about an axis with a velocity ω. Then, all its particles will have the *same* angular velocity ω, but their linear velocities will depend upon their respective distances from the axis of rotation, being equal to the product of the angular velocity and the distance from the axis. Thus, the linear velocity of a particle, distant r_1 from the axis, will be $r_1\omega$; of that distant r_2 from the axis will be $r_2\omega$ and so on.

And, therefore, if m be the mass of each particle, we have, *linear momentum* of the particle, distant r_1 from the axis, equal to $m.r_1\omega$ and, therefore, the *moment of its momentum* about the axis would be $m.r_1.\omega \times r = m.r_1^2.\omega$. Similarly, the *moment of momentum* of the particle, distant r_2 from the axis, would be $m.r_2^2.\omega$ and so on.