Fluid Mechanics and Viscosity

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Content

Fluid, Rate of flow, Different fluid motions, Equation of continuity, Bernoulli's equation, Speed of efflux: Torricelli's theorem, Venturimeter, Viscosity, Newton's law of viscous flow, coefficient of viscosity, Reynold Number, Poiseuille's equation and corrections, Capillary flow method.

References

Elements of Properties of Matter – D. S. Mathur
Fundamentals of Physics – David Halliday, Jearl Walker, and Robert Resnick

CHAPTER XII

FLOW OF LIQUIDS—VISCOSITY

200. Rate of Flow of a Liquid. A liquid, for our present purpose, is taken to be *perfectly mobile* and practically *incompressible* and, therefore, the same amount of it flows across every section of a tube in a given time. *The rate of flow of a liquid is, therefore, defined as the volume of it that flows across any section in unit time.*

If the velocity of flow of a liquid be v, in a direction perpendicular to two sections A and B, (Fig. 256), of area a, and distance l apart, and if t be the time taken by the liquid to flow from A to B, we have

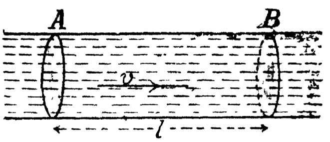

Fig. 256.

$$vt = l.$$

Obviously, the volume of liquid flowing through the section AB, in *this* time, is equal to the cylindrical column $AB = l \times a$, or $= vt \times a$. This, therefore, is the volume of the liquid flowing across the section in time t.

$$\therefore \quad \textit{rate of flow of liquid} = \frac{vt \times a}{t} = v \times a.$$

$$= \textit{velocity of liquid} \times \textit{area of cross-section of the tube.}$$

Sometimes, the rate of flow of a liquid is also expressed in terms of the *mass of the liquid flowing across any section in unit time*; so that, in this case,

rate of flow of liquid = mass of liquid flowing across any section per unit time.

= velocity of liquid × area of cross-section × density of liquid.

$$= v \times a \times \rho.$$

201. Lines and Tubes of Flow. In a simple flow of liquid, *i.e., when it is not turbulent*, but *steady*, the velocity at every point in the liquid remains constant, (in magnitude, as well as direction), the energy needed to drive the liquid being used up in overcoming the *'viscous drag'* between its layers. In other words, each particle follows exactly the *same path* and has the *same velocity* as its predecessor and the liquid is said to have an *orderly* or a **stream-line flow**. In such a case, if we consider a line along which a particle of the liquid moves, the direction of the line at any point is the direction of the velocity of the liquid at that point. Such a line is called a **stream-line.** More correctly, *a stream-line may be defined as a curve the tangent to which at any point gives the direction of flow of the liquid at that point*; for, *it may be straight or curved, according as the lateral pressure on it is the same throughout or different,—in the latter case the pressure being greater on the convex side than on the concave one.*

This holds good, however, only so long as the velocity of the liquid does not exceed a particular limiting value, called its *critical velocity*, beyond which the flow of the liquid loses all its steadiness or orderliness, and becomes *zig zag* or *sinuous*, acquiring what is called a *turbulent motion*. This may be easily seen by introducing a small jet of colouring matter into a tube through which a liquid may be made to flow with a gradually increasing velocity; when, as long as the velocity remains below its critical value, we see only a thin streak of the colouring matter along the axis of the tube. [Fig. 257, (*a*)], representing a stream-line motion, but when the velocity reaches *this* value, the colouring matter takes a *zig zag* path, [Fig. 257, (*b*)], and later, when this value is exceeded, the colouring matter spreads out in all directions, filling the entire tube, showing that the motion is no longer steady or orderly but has become '*turbulent*'. The energy needed to drive the liquid is here dissipated, for the most part, in setting up eddy currents in the liquid.

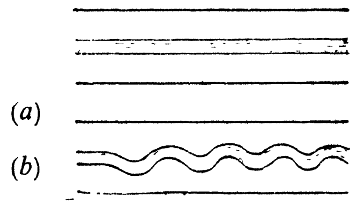

Fig. 257.

Consider two areas, *A* and *B*, at right angles to the direction of flow of the liquid, (Fig. 258), and draw *stream lines* through their boundaries; then, a *tube AB of the liquid* is obtained. This is known as a **tube of flow**.

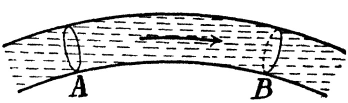

Fig 258.

As explained above, the volume of liquid passing through section *A* is equal to that passing through section *B*. For, the sides of the tube being everywhere in the direction of flow of the liquid, no liquid can cross the sides but must enter or leave through the ends. Since the velocity is constant over a section, *i.e.*, *the motion is steady*, (if the tube be narrow), the volume of the liquid entering section *A* is equal to $a_1.v_1$ *per sec.*, and the volume of the liquid leaving section *B* is equal to $a_2.v_2$ *per sec.*, where a_1, a_2, and v_1, v_2, are the areas of cross-section and velocities at sections *A* and *B* respectively.

$\therefore$ we have $$a_1.v_1.\rho_1 = a_2.v_2.\rho_2,$$
where ρ_1 and ρ_2 are the densities of the liquid at the two sections respectively.

The liquid being incompressible, $\rho_1 = \rho_2$, and so we have
$$a_1.v_1 = a_2.v_2,$$
i.e., the volume of the liquid entering section A is equal to that leaving section B.

202. Energy of the Liquid. Since a liquid has *inertia*, it possesses *kinetic energy*, when in motion. It is also subject to *pressure*, and may also have *potential energy*, due to its position. We have thus three types of energy possessed by a liquid in flow, *viz.*, (*i*) *kinetic energy*, (*ii*) *potential energy, and* (*iii*) *pressure energy.*

(*i*) **Kinetic Energy.** We know that $K.E. = \frac{1}{2}mv^2$, so that the kinetic energy of a mass *m* of a liquid, flowing with a velocity *v*, is given by $\frac{1}{2}mv^2$. If we consider *unit volume* of the liquid, $m = \rho$, the *density* of the liquid, and, therefore, we have

kinetic energy per unit volume of the liquid $= \frac{1}{2}\rho v^2$.

And, if we consider *unit mass* of the liquid, $m = 1$, and, therefore,

kinetic energy per unit mass of the liquid $= \frac{1}{2}v^2$.

(*ii*) **Potential Energy.** We have $P.E. = mgh$; so that, the potential energy of a liquid of mass m at a height h above the earth's surface (*i.e.*, in its gravitational field) is equal to mgh. Again, if we consider *unit volume* of the liquid, $m = \rho$, the *density* of the liquid, and, therefore,

$$P.E. \text{ per unit volume of the liquid} = \rho.g.h,$$

But, if we consider *unit mass* of the liquid, $m = 1$ and we have

$$P.E. \text{ per unit mass of the liquid} = gh.$$

(*iii*) **Pressure Energy.** Consider a tank A, containing a liquid of density ρ, provided with a narrow side tube T, of cross-sectional area a, properly fitted with a piston P that can be smoothly moved in and out, (Fig. 259). Let the *hydrostatic pressure* due to the liquid, at the level of the axis of the side tube, be p, so that the force on the piston is $= p.a$. If, therefore, more liquid is to be introduced into the tank, this much force has to be applied to the piston in moving it inwards. Let the piston be moving slowly inwards through a distance x, so that the velocity of the liquid be very small and *there may be no kinetic energy acquired by it*. Then, clearly, a volume of the liquid $a.x.$, or a mass $a.x.\rho$ of it, is forced into the tank, and an amount of work $p.a.x$ is performed to do so. This work, (or energy), $p.a.x$, required to make the liquid move against pressure p, *without imparting any velocity to it*, thus becomes the energy of the mass $a.x.\rho$ of the liquid in the tank, for it can do the same amount of work in pushing the piston back, when escaping from the tank. It is referred to as the *pressure energy* of the liquid.

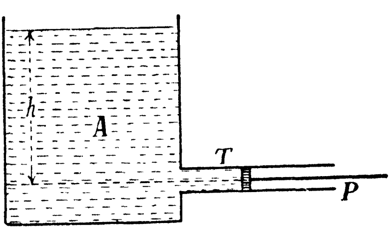

Fig. 259.

Thus pressure energy of a mass $a.x.\rho$ of the liquid is equal to $p.a.x$, and, therefore.

$$\textit{pressure energy per unit mass of the liquid} = \frac{p.a.x}{a.x.\rho} = \frac{p}{\rho} = \frac{pressure}{density}.$$

Now, if we consider *unit volume* of the liquid, we have

pressure energy of volume $a.x$ of the liquid $= p.a.x$,

and $\therefore$ *pressure energy per unit volume of the liquid* $= \dfrac{p.a.x}{a.x} = p$, *the pressure of the liquid.*

The three types of energy possessed by a liquid under flow are mutually convertible, one into the other. For, consider a liquid of density ρ contained in a vessel, and let its depth be h, (Fig. 260). Then, pressure due to the liquid column h at the bottom of the vessel is $p = h\rho.g$. If we take unit mass of the liquid from the bottom B to the surface A, clearly, $h.g.$ units of work has to be done against gravity, and, therefore, the potential energy of the liquid increases by this much amount ; or this much work is done by gravity if unit mass of the liquid comes down through a depth h. Hence, potential energy of unit mass of the liquid is equal to $h.g$. And, since pressure at a depth h, is given by $p = h.\rho.g$, and pressure energy per unit mass of the liquid = pressure/density, we have *pressure energy per unit mass of the liquid* = $h.\rho.g/\rho = h.g$ = *potential energy lost by the liquid in descending through* h.

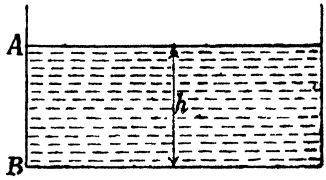

Fig. 260.

Thus, we see that pressure energy and potential energy are convertible, one into the other, and, therefore, their sum for a liquid at rest is constant.

Again, consider the flow of liquid through a tube, (Fig. 261). If the liquid has a constant velocity, there is no resultant force acting upon it. But, if the flow is accelerated, there must be a *pressure gradient* along the tube of flow. Let the change of pressure for a distance dx be dp, *i.e.*, let the pressure gradient be dp/dx, which may be taken to be constant for a short length of the tube.

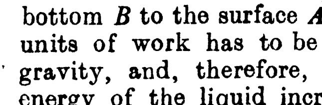

Fig. 261.

If the direction of flow be from A to B, the pressure decreases from A to B. If, therefore, p be the pressure at the cross-section B, that at A will be greater by $\delta x.dp/dx$, if the small distance AB be δx, *i.e.*, the pressure at A will be $p - \delta x.dp/dx$. The resultant force on the slice AB of the liquid will, therefore, be $a.\delta x.dp/dx$, where a is the cross-section of the tube, (*force being equal to pressure* $\times$ *area*).

Let the *velocity gradient* along the tube of flow be dv/dx ; then, if v be the velocity at A, the velocity at B will be $v+\delta x.dv/dx$, because *the velocity increases in the direction A to B*, and, therefore, increase in velocity through the distance δx will be $\delta x.dv/dx$.
If the liquid covers this distance in time δt, we have

$$\delta t = \delta x/v, \qquad \text{whence,} \qquad v = \delta x/\delta t.$$

Or, *in the limit*, $v = dx/dt$.

Now, *acceleration = rate of change of velocity* and, therefore, acceleration at the section $AB = dv/dt$, and mass of liquid in the section $= a.\delta x.\rho$; so that, force on it $= a.\delta x\ \rho\ dv/dt$, (because *force = mass* $\times$ *acceleration*).

But force on this slice of the liquid is also equal to $a.\delta x.dp/dx$.

$$\therefore \qquad -a.\frac{dp}{dx}.\delta x = a.\delta x.\rho.\frac{dv}{dt},$$

the *—ve sign* merely indicating that the pressure and velocity gradients are opposite in sign, *i.e., whereas the pressure decreases, the velocity increases along AB.*

Or, $$-\frac{dp}{dx} = \rho\cdot\frac{dv}{dt} = \rho\cdot\frac{dv}{dx}\cdot\frac{dx}{dt} = \rho.v.\frac{dv}{dx}. \qquad [\because dx/dt = v.$$

And $\therefore$ $-dp = \rho.v.dv$. Or, $-dp/\rho = v.dv$.

Or, $$-\frac{1}{\rho}\int_{p_1}^{p_2} dp = \int_{v_1}^{v_2} v.dv. \quad \left\{\begin{array}{l}\text{where } p_1, p_2 \text{ and } v_1, v_2 \text{ are pressures} \\ \text{and velocities at sections 1 and 2, res-} \\ \text{pectively.}\end{array}\right.$$

Or, $$-\frac{p_2}{\rho} - \left(-\frac{p_1}{\rho}\right) = \frac{v_2^2}{2} - \frac{v_1^2}{2}.$$

Or, $$\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2),$$

i.e., pressure energy and kinetic energy are convertible, one into the other.

Since pressure energy is also convertible into potential energy, it follows that the *three types of energy are mutually convertible into each other.*

203. Bernoulli's Theorem and its Important Applications. *Bernoulli's theorem states that the total energy of a small amount of liquid flowing from one point to another, without any friction, remains constant throughout the displacement.*

We have seen that pressure energy and potential energy of a liquid are convertible, one into the other, and so are its pressure energy and kinetic energy. It follows, therefore, that in any *stream-line flow* of liquid, the loss of energy in one form is equal to the gain of energy in another, or that the sum total of its energy, *viz.*,

potential energy + pressure energy + kinetic energy = a constant.

Or, $$hg + p/\rho + \tfrac{1}{2}v^2 = C, \text{ a constant} \qquad \ldots \quad (i)$$

This relation is known as **Bernoulli's Equation.**

If we divide relation (i) by g, we have

$$h + \frac{p}{\rho g} + \frac{1}{2}\cdot\frac{v^2}{g} = C', \text{ another constant.} \quad \ldots \quad (ii)$$

Now, h is what is called the *gravitational head*, $p/\rho g$, *the pressure head* and $\frac{1}{2}v^2/g$, the *velocity head**. Thus,

gravitational head + pressure head + velocity head = a constant.

We may, therefore, also state Bernoulli's theorem in another way, *viz.*, that *at all points, in the stream-line flow of a liquid, the sum of the gravitational head, the pressure head and the velocity head remains constant throughout.*

It follows at once from relation (ii) that if the flow of the liquid be horizontal, the gravitational head h is a constant ; so that, here,

$$\frac{p}{\rho.g} + \frac{1}{2}\cdot\frac{v^2}{g} = a \text{ constant.}$$

Similarly, from relation (i), we would have $p/\rho + \frac{1}{2}v^2 =$ *a constant*, since the potential or gravitational energy hg would be a constant.

Or, $$p + \tfrac{1}{2}\rho v^2 = a \text{ constant.} \qquad \ldots (iii)$$

*For, the liquid must fall through this much height to attain the velocity v.

Here, p is referred to as the *static pressure* of the liquid and $\frac{1}{2}\rho v^2$ as its *dynamic or velocity pressure* So that, we may express this result by saying that *for a horizontal motion of the liquid, the sum of its static and dynamic pressures remains a constant.*

Thus, if in a liquid, flowing herizontally, the pressure and velocity at one point be p_1 and v_1 and at another, p_2 and v_2 respectively, we have

$$\frac{p_1}{\rho}+\frac{1}{2}v_1^2=\frac{p_2}{\rho}+\frac{1}{2}v_2^2,$$

which shows that pressure and velocity (and, therefore, kinetic energy) can only increase at the expense of one another, *i.e., points of maximum pressure correspond to those of minimum velocity,* and *vice versa**. This principle is made use of in various important practical applications, (see § 204).

204. Important Applications of Bernoulli's Equation

(*i*) **Velocity of Efflux of a Liquid.** Let the surface of the liquid be at a height h above the level of the orifice O in a tank, (Fig. 262). If the tank be sufficiently wide, the velocity at the liquid surface may be taken to be *zero,* the pressure there being, clearly, atmospheric. Since the pressure is also atmospheric at the orifice, where the liquid emerges, it plays no part in the flow of the liquid. If v be the velocity at the level of the orifice, we have, considering a tube of flow beginning at A and ending at O,

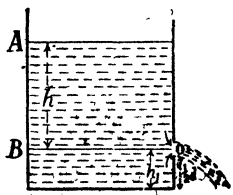

Fig. 262.

total energy at A = pressure energy + potential energy + kinetic energy

$$= 0+hg+0,$$

because *pressure* at $A = 0$, $P.E. = gh$ and $K.E. = 0$. [$\because v = 0$].

And, *total energy at O,* the level of the orifice

$$= 0+0+\tfrac{1}{2}v^2,$$

because pressure at $O = 0$, $P.E. = 0$, and $K.E. = \frac{1}{2}v^2$.

Since total energy remains the *same,* we have

$$\tfrac{1}{2}v^2 = hg. \quad \text{Or,} \quad v^2 = 2gh,$$

whence, $$v = \sqrt{2gh}.$$

This, then, is the *velocity of efflux* of the liquid at the orifice O.

This result was first obtained by *Torricelli* (in the year 1644) and hence is known as **Torricelli's Theorem,** or *the Law of Efflux,* and may be stated as follows :

The velocity of efflux of a liquid through an orifice is equal to that which a body attains in falling freely from the surface of the liquid to the orifice.

For, clearly, if the liquid had fallen freely through this height h, its velocity would be given by the relation, $v^2 = 2gh$, to be equal to $v = \sqrt{2gh}$, the same as obtained above.

This *ideal* velocity is, however, seldom reached, for no liquid is perfectly free from friction (or viscosity).

* This result is also true for compressible fluids and is sometimes referred to as **Hawksbee's law.**

Now, the liquid-jet flows out in the form of a *parabola*, and takes time equal to $\sqrt{2h_1/g}$ to fall through a height h_1 to a plane, in level with the bottom of the vessel, striking the plane at a distance d, called its **range**, such that

$$d = v \times \sqrt{\frac{2h_1}{g}} = \sqrt{2gh \times \frac{2h_1}{g}} = 2\sqrt{h \times h_1}.$$

For a given height $(h+h_1)$, of the liquid column, this range will be a maximum when $h = h_1$. And, obviously, if the jet were directed upwards, it should theoretically rise to the level A of the free surface of the liquid. But, again, due to air-resistance and viscosity, the height attained is actually less than this ideal one.

(*ii*) **Vena Contracta**. The *whole* of the liquid entering the orifice does not move perpendicularly to it, but comes from all directions, as shown in Fig. 263, the stream-lines near the edges being curved. The liquid coming from the sides of the vessel, as it enters the orifice, has still a lateral velocity due to inertia and continues to move inwards towards the centre of cross-section of the jet, until the increasing outward pressure is balanced by the atmospheric pressure at the jet. The liquid jet thus *contracts* at C, a little outside the orifice, to a *neck*, called the *Vena Contracta. It is here that the jet becomes uniform and the velocity becomes the same throughout*, and it is *this* velocity which is given by Torricelli's equation, (see § 204, (*i*), above, page 423).

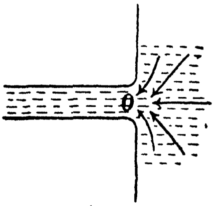

Fig. 263.

Obviously, the area of the jet at the *Vena Contracta* is smaller than the area of the orifice and is found to be about ·62 times the latter. The volume of the liquid passing out through the orifice in unit time is, therefore, equal to $\cdot 62a\sqrt{2gh}$. This ratio between the area of the *Vena Contracta* and the orifice is called the *coefficient of contraction.*

N.B. If *outflow tubes* of suitable shapes be used, the *Vena Contracta* may be almost completely avoided, but the velocity of efflux always suffers a diminution in its value due to a loss in the kinetic energy of the liquid, caused by its internal friction or viscosity, —this diminution being quite independent of the *Vena Contracta.*

(*iii*) **Venturimeter**. It is an arrangement to measure the amount of flow of a liquid in a pipe,—usually water, when it is called a *venturi water-meter.*

The principle underlying it is that *when a liquid flows through a tube of a varying bore or cross-section, the velocity and pressure vary along the tube, the pressure being the least where the velocity is the greatest, and vice versa.*

For, if we have a tube KLM, with a constriction at L, (Fig. 264), the velocity of the liquid will be greater at L, the narrowest part of the tube, than that at K or M. Let the velocity at L be v_l and that at K be v_k. Then, $v_l > v_k$.

Applying *Bernoulli's theorem*, we have

(*potential energy + pressure energy + kinetic energy*) at L

= (*potential energy + pressure energy + kinetic energy*) at K.

Or, $h_l . g + \frac{p_l}{\rho} + \frac{1}{2} v_l^2$

$$= h_k . g + \frac{p_k}{\rho} + \frac{1}{2} v_k^2$$

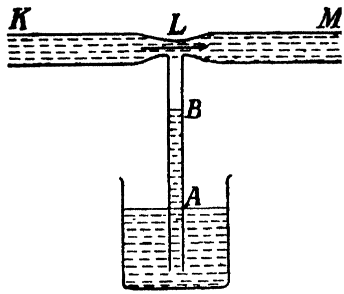

Fig. 264.

where h_l, p_l and v_l are the *height*, *pressure* and *velocity* of the liquid at L and k_k, p_k and v_k, their corresponding values at K, ρ being the *density* of the liquid, supposed constant, because the liquid is taken to be incompressible.

If the tube be horizontal, $h_l = h_k$, so that the above relation becomes

$$\frac{p_l}{\rho} + \frac{1}{2} v_l^2 = \frac{p_k}{\rho} + \frac{1}{2} v_k^2. \quad \text{Or,} \quad \frac{1}{2} v_l^2 - \frac{1}{2} v_k^2 = \frac{p_k}{\rho} - \frac{p_l}{\rho}. \quad ...(i)$$

Since $v_l > v_k$, it is clear that $p_k > p_l$, *i.e.*, the pressure at L is less than at K. This can be shown by attaching a vertical tube, connected to KLM at L and dipping it into a liquid, not miscible with the one in KLM, when the liquid rises up in the vertical tube, as shown at AB, and it will be seen that the *narrower the bore at L, the greater the rise of the liquid in the vertical tube.*

Let us now consider a pipe through which water is flowing, such that it has a cross-section a_1 at K and a_2 at L, (Fig, 265). Then, if v_1 and v_2 be the velocities of water at K and L respectively, we have

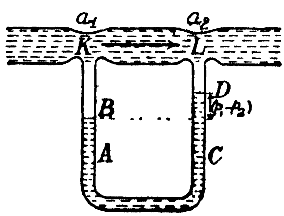

Fig. 265.

$$a_1 v_1 = a_2 v_2, \text{ [see § 201, (page 419).}$$

whence, $\quad v_2 = a_1 v_1 / a_2.$

And, since ρ is 1 for water, relation (*i*) above becomes,

$$\frac{1}{2} v_2^2 - \frac{1}{2} v_1^2 = p_1 - p_2,$$

where p_1 and p_2 are the pressures at K and L, respectively.

Or, substituting the value of v_2, we have

$$\frac{1}{2} \frac{a_1^2 v_1^2}{a_2^2} - \frac{1}{2} v_1^2 = (p_1 - p_2). \quad \text{Or,} \quad \frac{1}{2} v_1^2 \left(\frac{a_1^2}{a_2^2} - 1 \right) = (p_1 - p_2).$$

$$\text{Or,} \quad \frac{1}{2} v_1^2 \left(\frac{a_1^2 - a_2^2}{a_2^2} \right) = (p_1 - p_2). \quad \text{Or,} \quad v_1^2 = \frac{2a_2^2 (p_1 - p_2)}{a_1^2 - a_2^2},$$

$$\text{whence,} \quad v_1 = a_2 \sqrt{\frac{2(p_1 - p_2)}{a_1^2 - a_2^2}}. \quad \text{So that,} \quad a_1 v_1 = a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{a_1^2 - a_2^2}}.$$

Thus, if we know a_1, a_2 and $(p_1 - p_2)$, we can determine $a_1 v_1$, the volume of the liquid flowing across the section K *per second.* The difference of pressure $(p_1 - p_2)$ at K and L is read directly on the vertical tubes AB and CD joined together to form a *manometer,* as shown.

(*iv*) **Pitot Tube.** This arrangement is also used to measure the amount of flow of water through a pipe and is based on the same principle as the venturimeter.

It consists of two vertical tubes, with small apertures at their lower ends, —the plane of the aperture of one tube PQ (Fig. 266), being parallel to the direction of flow of water and the aperture of the other tube RS, facing the flow. The rise of the liquid column in the tube PQ, therefore, measures the pressure at Q. And since the water is stopped in the plane of the aperture S of the tube RS, its velocity *there* becomes *zero*. Therefore, its kinetic energy is reduced from $\frac{1}{2}v^2$ to *zero*, where v is the velocity of flow of water. Its pressure, therefore, increases by an amount $\frac{1}{2}v^2$, and the water consequently rises to a higher level in the tube RS than in PQ. If h be the difference of level in the two tubes, we have

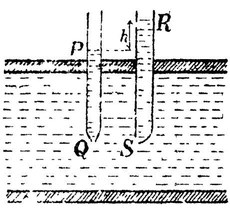

Fig. 266.

$$\tfrac{1}{2}v^2 = hg. \quad \text{Or,} \quad v^2 = 2gh,$$

whence, $$v = \sqrt{2gh}.$$

This multiplied by a, the cross-section of the pipe, where the tubes are placed, gives the volume of water flowing *per second* past that section and the amount of flow of water is thus easily measured.

(v) Other Common Applications of Bernoulli's Theorem.

1. The Steam Injector. It is a simple device to accelerate the ejection of the exhaust steam from the cylinder of a steam engine, and consists of a tube A, (Fig. 267), narrowing down into a nozzle N at its lower end, inside another tube B, having a side-tube C, which is connected to the cylinder of the engine.

A jet of steam is introduced into A, and as it issues out of the nozzle N, its velocity is considerably increased, resulting in a corresponding fall in pressure there, and the steam from the engine-cylinder thus rushes into this region of reduced pressure, whence, it is ejected out through the lower end of B.

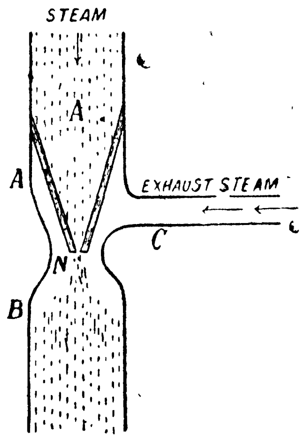

Fig. 267.

2. The Filter Pump. It is also based on the same principle and is used to reduce the pressure in a vessel. Here, a stream of water from a tap, flowing through a tube A, (Fig. 268), issues out in the form of a jet from its narrow orifice O, which results in a great rise in its velocity and proportionate fall in its pressure, which is thus soon reduced to a value, below that of the atmosphere The air from the vessel, connected through a side-tube B to this region of reduced pressure, then rushes into it, and is carried away by the stream of water as it flows down through C.

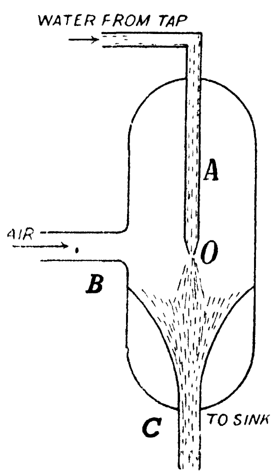

Fig 268

In this way, the pressure in the vessel is ultimately reduced to just a little above the vapour pressure of water, in a comparatively very short time.

If the inlet water tube be a twisted, instead of a straight, one, the exhaustion proceeds more rapidly, due to the rotating water-jet in the tube breaking up more readily and mixing up easily with the incoming air from the vessel.

3. The Atomizer. The *atomizer* or *sprayer*, used for spraying scents etc. is yet another example of a fall in pressure due to an increase in velocity.

Here, (Fig. 269), air is blown through a tube T, (usually by compressing a rubber bulb) fitted on the tube at one end, which, when it rushes out of the aperture O, where the tube narrows down, acquires high velocity. The pressure in the vicinity of O is thus greatly reduced, and since O lies directly above the vertical tube, dipping in the liquid in vessel V, the liquid rises up through it, when, on issuing out of the aperture at the top, it is blown into a fine spray by the air stream from T.

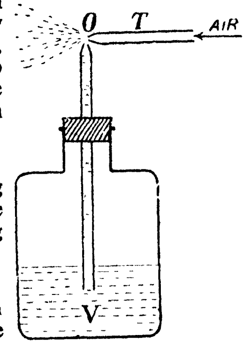

Fig. 269.

4. The Attracted-disc Paradox. The following is a simple and interesting experiment, which the student may well try for amusement at a small gathering at home.

DE is a flat card-board disc (Fig. 270), over which is placed another flat disc BC, fitted with a tube A, the opening of which is in *flush* with BC.

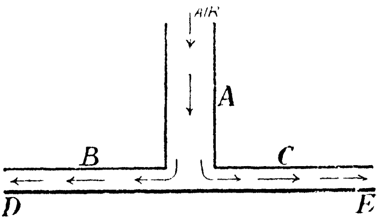

Fig. 270.

On blowing air down through A, on to DE, the latter, instead of being blown away from BC, as one might ordinarily expect, sticks on to it more and more closely, and might even be lifted up a little.

This seeming paradox is, however, easily explained. For, as the air from A rushes through the narrow space in between BC and DE, its velocity increases and consequently the pressure there decreases, so that it soon falls below the atmospheric pressure on DE, which thus pushes it up towards BC.

5. The Bunsen-Burner. This too is a familiar example of a fall of pressure due to increased velocity. For, as the gas issues out with a great velocity from the fine nozzle, down below, the pressure falls in its immediate neighbourhood, and the air is thus sucked in through the hole O, (Fig. 271), and gets mixed up with the gas.

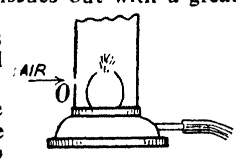

Fig. 271.

6. The Magnus Effect. If a ball, or a sphere be rotated about an axis through it, perpendicular to the plane of the paper, the air surrounding it is also set into motion,—the streamlines taking the form of concentric circles in planes, parallel to the plane of the paper, their direction being the same as that of the rotation of the ball, shown in Fig. 272 (*a*). And, obviously, the rougher the surface of the ball, the thicker the layer of air thus set into motion.

If, however, the ball be given only a linear forward motion, it pushes aside the air in front of it, to make room for itself, and this displaced air then flows along its sides on to its back or the rear end, the form of the streamlines being as shown in Fig. 272 (*b*).

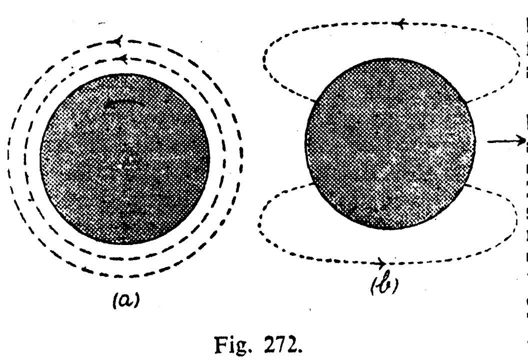

Fig. 272.

And, finally, if the ball be given both, a rotatory and a circular motion simultaneously, it is clear from Figs. 272 (*a*) and (*b*) that the streamlines due to the two motions run in opposite directions on the underside of the ball, but in the same direction on its upper side. Thus, there is a *decrease of velocity* or an *increase of pres-*

sure, on the lower, and an *increase of velocity* or a *decrease of pressure* on its upper, side. The ball, as a result of this difference in the lateral pressure on it, takes a *curved path* which is convex towards the greater pressure side. This is what is called the *Magnus Effect* and is easily observed when a tennis or a golf ball is given a spin.

7. **The Cylindrical Shape of a Bullet.** We have already seen [§ 46, (*ii*), page 98], that for giving directional stability, it is desirable to give the shot or bullet a rapid '*spin*' about an axis, along its direction of motion, and how this object is achieved by '*rifling*' the barrel of a gun or rifle, (*i.e.*, by cutting spiral grooves inside it).

Now, if the bullet were spherical in shape, there will come about, as explained above in case 6, a difference in the lateral pressure on it during its passage through air, on account of its simultaneously possessing a rotatory and a linear motion and the bullet will thus be deflected from its straight path To avoid this, the bullet is made cylindrical in shape, so that the lateral pressure on it remains uniform and it flies undeflected along its path.

8. **Streamline Bodies.** The student has no doubt heard of streamline bodies of automobiles, particularly of racing cars etc. We shall discuss in brief here as to what this *streamlining* of a body really connotes in the language of Science.

As a body moves through air, or through a fluid, in general, it carries a part of it along with itself, pushing the rest on to either side. The steamlines of the fluid, directed towards the body, open out to either side to make way for it, as it were, and meet some distance behind it. This fluid at the rear of the body, enclosed by the streamlines meeting there is thus carried by the body as a sort of a '*tail*'. Some extra work has thus to be done by the body, in carrying this extra burden, resulting in an appreciable decrease in its kinetic energy and velocity. In fact, the body has to encounter a double opposition to its forward motion, *viz.*, (*i*) *an increased pressure in front*, called the **head pressure**, and (*ii*) *a decreased pressure* or the **tail suction** behind, (which exerts a backward pull on it).

Naturally, the surrounding fluid flows into this rear region of decreased pressure or *tail suction*, and is thus thrown up into vortices (*i.e.*, whirls and eddies) there, which results in a further fall in pressure in this region. These vortices are thus responsible for dissipating away a fairly good part of the energy of the body, thus decreasing its velocity or offering resistance to its motion.

If, therefore, the resistance to the forward motion of the body is to be minimised, *it should be given a shape similar to that of the fluid forming its 'tail', so that there is no tailsuction region formed at all at its rear*, and no energy is thus dissipated in the formation of whirls and eddies. The body is thus made with a gradually decreasing cross-section, tapering towards the rear, and having no sharp corners or edges anywhere. The body is then said to have a *streamline shape* and the resistance to its forward motion is considerably decreased. This explains the shape of the bodies of big airliners and of most of the modern cars.

205. Viscosity. When a liquid flows slowly and steadily over a fixed horizontal surface, *i.e*, when its flow is streamline, its layer in contact with the fixed surface is stationary and the velocity of the layers increases with the distance from the fixed surface, *i.e.*, *the greater the distance of a layer from the fixed surface, the greater its velocity.*

Considering any particular layer of the liquid, we have the layer immediately *below* it moving *slower* than it, and the one immediately *above* it moving *faster* than it, so that the former tends to retard its motion and the latter tends to accelerate it. The two layers thus tend to destroy their relative motion, *as though there were a backward dragging force, acting tangentially on the layers.* If, therefore the relative velocity between the two layers is to be maintained, an external force must be applied to overcome this *backward drag.* In the absence of any such outside force, the relative motion between the layers is destroyed and the flow of the liquid ceases. *This property of a liquid by virtue of which it opposes relative motion between its different layers is known as viscosity or internal friction of the liquid.*

206. Coefficient of Viscosity (η). *Newton* showed that the backward dragging, or *viscous*, force, acting tangentially on any liquid layer, is directly proportional to its surface area A, and velocity v, and inversely proportional to its distance x from the stationary layer. Denoting this force by F, therefore, we have

$$F \propto A\ ;\quad F \propto -v\ ;\quad F \propto \frac{1}{x}\ ;$$

the $-ve$ *sign* of v merely indicates that the direction of the force is opposite to that of velocity.

Or, $$F \propto -\frac{A.v}{x}, \qquad i.e., \qquad F = -\eta.\frac{A.v}{x},$$

where η is a constant, depending upon the nature of the liquid, and is called its *coefficient of viscosity.**

Now, v/x may be put as dv/dx, which gives the *rate of change of velocity with distance*, and is called the *velocity gradient* ; so that, we have

$$F = -\eta.A.\frac{dv}{dx}. \qquad \ldots I.$$

This is know as *Newton's law of viscous flow in streamline motion.*

If, $A = 1$ *sq. cm.*, and $dv/dx = 1$, we have $F = \eta$.

Thus, the **coefficient of viscosity** *of a liquid may be defined as the tangential force required per unit area to maintain a unit velocity gradient, i.e., to maintain unit relative velocity between two layers unit distance apart.* And, clearly, if this tangential force be unity, the *coefficient of viscosity* of the liquid is unity, and is called *Poise*, after **Poiseuille,** whose work on viscosity is important.

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* This coefficient n is sometimes referred to as the *dynamic viscosity of the liquid*, with *Poise* as its *C.G.S.* unit (see below). On the other hand, the ratio n/ρ (where ρ is the density of the liquid) is called its *kinematic viscosity* (k), and the corresponding *C.G.S.* unit for it is the *stokes.*

Dimensions of η It is clear from relation *I* above that

$$\eta = \frac{F}{A.dv/dx}.$$

So that, the dimensions of η are those of $\frac{force}{area \times velocity\ gradient}$.

$$= \frac{MLT^{-2}}{[L^2].\left[\frac{L/T}{L}\right]} = \frac{MLT^{-2}}{L^2.T^{-1}}. \quad \left\{\because \frac{dv}{dx} = \left[\frac{L/T}{L}\right]\right.$$

Or, $\eta = ML^{-1}T^{-1}$.

Viscosity in liquids corresponds to solid friction in so far as, like the latter, it also opposes relative motion between two layers. It, however, differs from solid friction in that, *unlike solid friction, it depends upon (i) the surface area of the liquid layer, (ii) its distance from the stationary layer end (iii) its velocity with respect to the stationary layer.*

207. Fugitive Elasticity. The expression for F, above, may be re-arranged and put as

$$\eta = \frac{F/A}{dv/dx}, \qquad \ldots(a)$$

i.e., $\quad$ *coefficient of viscosity* $= \frac{tangential\ stress^*}{velocity\ gradient}$.

This is an expression similar to the one for the coefficient of rigidity, *viz.,*

$$n = \frac{F/A}{\theta} = \frac{F/A}{dy/dx}. \quad \ldots(b) \left[\because \theta = \frac{dy}{dx}.\right.$$

$$= \frac{tangential\ stress^*}{displacement\ gradient}.$$

Maxwell, therefore, considered a liquid to possess a certain amount of rigidity, breaking down continually under a shearing stress. Very fittingly, he imagined viscosity of a liquid to be the limiting case of the rigidity of a solid, when the latter breaks down under the shear applied. A liquid is thus regarded as capable of exerting and sustaining an amount of shearing stress for a short time, after which it breaks down and the shear is formed over again. In other words, a liquid offers a *fugitive resistance* to shearing stress, which is continually breaking down, and it may thus be said to possess a fugitive rigidity.

Now, if the rate at which the shear (θ) breaks down be taken to be proportional to shear, we have

rate of the breakdown of shear $\propto \theta$.

Or, ,, ,, ,, ,, ,, $= \lambda.\theta$, where λ is a constant.

And, clearly, the rate of formation of the shear

$$= \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{dv}{dx}.$$

[v being the velocity in the *same plane.*

*It will be noted from expressions (a) and (b) that whereas in a fluid, the viscous drag is proportional to the velocity gradient, perpendicular to the direction of motion, the shearing stress, in a solid, is proporlional to the displacement gradient, perpendicular to the direction of shear.

Thus, when the motion of the fluid becomes quite steady, the rate of formation of the shear must be the same as that of its breakdown. So that,

$$\lambda.\theta = \frac{dv}{dx}. \quad \text{Or,} \quad \theta \Big/ \frac{dv}{dx} = \frac{1}{\lambda}. \qquad \ldots(c)$$

Now, dividing relation (a) by (b), above, we have $\frac{\eta}{n} = \frac{\theta}{dv/dx}$.

$\therefore$ substituting $\frac{\eta}{n}$ for $\theta \Big/ \frac{dv}{dx}$ in relation (c) above, we have

$$\eta/n = 1/\lambda.$$

This quantity $1/\lambda$ is called the *'time of relaxation of the medium'* and gives the time taken by the shear to disappear, provided no fresh shear is applied.

208. Critical Velocity. It was *Osborne Reynolds* who first showed by direct experiment that the *critical velocity* v_c of a liquid is given by the relation, $v_c = k.\eta/\rho r$, called *Osborne Renyold's formula*, where η is its coefficient of viscosity, ρ, its density and r, the radius of the tube, the constant k being called *Reynold's number*, its value being about 1000 for narrow tubes.

The expression for v_c may, however, be easily deduced by the method of dimensions, as explained below :

Since v_c is found to depend upon (i) η, (ii) ρ, and (iii) r, we have

$$v_c = k.\eta^a \rho^b r^c, \text{ say.} \qquad [\, k \text{ being a constant.}$$

So that, putting the dimensions of the quantities involved, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a[ML^{-3}]^b[L]^c$$

Or, $$[LT^{-1}] = [M^{a+b}L^{-a-3b+c}T^{-a}]$$ [k having no dimensions being a constant.

Since the dimensions on the two sides of the equation must be the same, (by the *principle of homogeneity of dimensions*), we have

$$a+b = 0 \quad \ldots(i) \; ; \quad -a-3b+c = 1 \quad \ldots(ii) \; ; \quad -a = -1 \quad \ldots(iii)$$

So that, adding (i) and (iii), we have $b = -1$.

Substituting this value of (b) in (i), we have $a = 1$,

[or, directly from (iii).

And, substituting the values of a and b in (ii), we have $c = -1$.

Hence $$v_c = k.\eta/\rho r,$$

where k (*Reynold's number*) is, as mentioned above, near about 1000 for narrow tubes. Thus, *for narrow tubes*, $v_c = 1000.\eta/\rho r$.

It must be emphasized again that this relation applies only to narrow tubes. For tubes of wide bores, the value of v_c is very much greater, and may be even a thousand times greater than that given by the above relation.

Now, a mere glance at the expression for v_c, deduced above, will show that

$$(i)\ v_c \propto \eta \; ; \; (ii)\ v_c \propto 1/\rho \; ; \text{ and } (iii)\ v_c \propto 1/r,$$

i.e., the critical velocity of a liquid is (i) directly proportional to its viscosity, (ii) inversely proportional to its density and (iii) inversely proportional to the radius of the tube through which it flows.

It follows, therefore, that *narrow tubes, and liquids of high viscosity, and low density tend to promote orderly motion*, whereas tubes of wide bores, and liquids of low viscosity and high density lead to turbulence.

Again, if we have a perfectly *mobile* or *inviscid* liquid, *i.e.*, a liquid. for which $\eta = 0$, then, obviously, $v_c = 0$; so that, its flow would be turbulent and not orderly, even for the smallest velocity and in the narrowest of tubes.

Thus, we see that *it is the viscosity of a liquid alone, due to which its flow may possibly be orderly and thus approximate to that of a perfect fluid.*

209. Poiseuille's Equation for flow of liquid through a tube. Imagine a cylindrical layer, or shell of liquid, of radius x, flowing through a capillary tube of radius r. Then, the velocity of flow *at all points* on this cylindrical shell will be the *same*. Let it be v. As the velocity of the layers in contact with the walls of the tube is *zero* and goes on increasing towards the axis, it is obvious that the liquid inside the imaginary cylinder is moving faster than that outside it, and the *backward tangential force* due to the outer slower, moving liquid on the inner faster moving liquid is, in accordance with relation *I* above, given by $\eta.2\pi x.l.dv/dx$, where η is the coefficient of viscosity of the liquid, [because, here, *surface area* (A) of the cylindrical shell of radius x is equal to $2\pi x.l$, where l is the length of the capillary tube, and dv/dx is the *velocity gradient* there].

Let the difference of pressure at the two ends of the capillary tube be P Then, the *forward force* on the cylindrical liquid shell, in the direction of flow, is clearly equal to $P\times\pi x^2$, and tends to accelerate the motion of the liquid. If, therefore, the motion of the liquid be steady, we have

$$\eta.2\pi x.l\frac{dv}{dx} = -P.\pi x^2,$$

the *—ve sign* showing that the two forces act in opposite directions.

And $\therefore$ $$dv = \frac{-P.\pi x^2 dx}{\eta.2\pi x.l} = \frac{-P.x.dx}{2\eta l}.$$

Integrating this expression for dv, we have $v = -\dfrac{P}{2\eta l}\displaystyle\int x.dx.$

Or, $$v = \frac{-P}{2\eta l}\cdot\frac{x^2}{2}+C_1 = \frac{-Px^2}{4\eta l}+C_1,$$

where C_1 is a constant of integration.

Now, $v = 0$, when $x = r$, because the layers in contact with the sides of the tube are stationary.

$\therefore$ $$0 = \frac{-Pr^2}{4\eta l}+C_1, \text{ whence, } C_1 = \frac{Pr^2}{4\eta l}.$$

$\therefore$ $$v = \frac{Pr^2}{4\eta l}-\frac{Px^2}{4\eta l} = \frac{P}{4\eta l}(r^2-x^2).$$

This, therefore, is the *velocity of flow of the liquid* at a distance x from the axis of the tube, and a glance at the expression for v will show that the *profile* or the *velocity distribution curve* of the advancing liquid in the tube is a *parabola*, (Fig. 273),—the velocity increasing from 0 at the walls of the tube to a maximum at its centre.

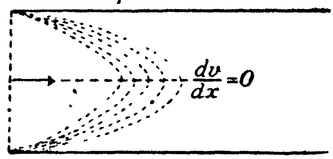

Fig. 273.

Now, imagine another *co-axial* cylindrical shell of the liquid, of radius $(x+dx)$. The cross-sectional area between the two shells is clearly $2\pi x.dx$ and, since v is the velocity of the flow of liquid in-between the two shells, the volume of liquid flowing *per second* through the cross-sectional area is given by $dV = 2\pi x.dx\, v$. If we imagine the whole of the tube to be made up of such like concentric cylindrical shells, the volume V of the liquid flowing through all of them, *i.e.*, through the capillary tube, *in unit time*, will be obtained by integrating the expression for dV between the limits, $x = 0$ and $x = r$.

Or,
$$V = \int_0^r 2\pi x.dx.v = \int_0^r 2\pi x.\frac{P}{4\eta l}.(r^2-x^2)dx.$$
$$= \frac{\pi P}{2\eta l}\int_0^r (xr^2-x^3).dx = \frac{\pi P}{2\eta l}\left[\frac{x^2r^2}{2}-\frac{x^4}{4}\right]_0^r$$
$$= \frac{\pi P}{2\eta l}\left(\frac{r^4}{2}-\frac{r^4}{4}\right) = \frac{\pi P.r^4}{8\eta l},$$

whence,
$$\eta = \frac{\pi Pr^4}{8Vl}. \qquad ..II$$

Thus, if we know P, r, V and l, the coefficient of viscosity of the liquid (η) can be easily determined.

The above relation holds good only when

(*i*) *the flow is steady and streamline, i.e.*, when its average velocity is less than its critical velocity ;

(*ii*) *the pressure is constant over every cross-section, i.e., there is no radial flow* ; and

(*iii*) *the liquid in contact with the sides of the tube is stationary.*

When the velocity of flow is small, and the tube is a narrow one, these assumptions are more or less valid. It is clear, therefore, that for tubes of wide bores, the relation breaks down ; for, in their case, the value of the critical velocity is much smaller ($\because v_c \propto 1/r$) and the flow of the liquid becomes turbulent. Thus, if we were to plot a graph between the pressure difference P between the two ends of the outflow tube, and the rate of flow V of the liquid, (*i.e.*, the volume of the liquid flowing out of it per second), we get a curve, as shown (Fig. 274), where the portion OA of the curve corresponds to the velocities lesthan the critical velocity and the portion AB, to those above its.

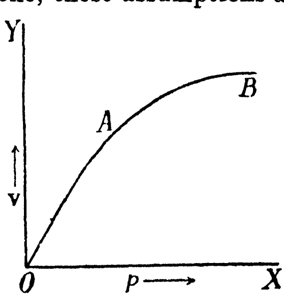

Fig. 274.

It is found that when the velocity of the liquid is below the critical value, the rate of flow V is proportional to P, the pressure difference, (as indicated by the straight part OA of the curve). Thus, within this range of velocity, the rate of flow of a liquid depends chiefly on its viscosity (η), quite in accordance with Poiseuille's formula.

Beyond the critical velocity, however, the pressure difference (P) is almost wholly utilized in combating the turbulence set up in the liquid, and in imparting kinetic energy to it ; so that, its rate of flow is now no longer proportional to P, and hence no longer depends upon its viscosity. In fact, it now depends mainly on the density of the liquid (ρ) and is approximately proportional to $\sqrt{P}$.

The following interesting consequences follow from the above :

(*i*) Since in turbulent motion, the rate of flow of a liquid is quite independent of its viscosity, it obviously follows that *all liquids, irrespective of their different viscosities, would require the same pressure difference to be driven through a tube at velocities higher than their critical velocities*. Thus, for example, a viscous liquid, like treacle, would require the same pressure difference to be driven through a tube, at a velocity greater than its critical velocity, as would be needed to drive water through it at the *same* velocity.

(*ii*) Since the critical velocity of a liquid is inversely proportional to the radius of the tube through which it flows, it is clear that liquids of all viscosities would flow equally readily through tubes of sufficiently wide bores. Thus, in a wide tube, treacle will flow just as freely as water. A typical natural example of this is the free flow of the highly viscous *lava* down the sides of an erupting volcano,—its rate of flow being about the same as we would expect in the case of water.

210. Experimental determination of η for a liquid—Poiseuille's method*. A capillary tube T, of known length l and radius r, is fixed *horizontally* near to the bottom of a vessel A, (Fig. 275), the liquid level in which can be kept constant at any desired height by means of an over-flow arrangement O. A clean and dry beaker, of known weight, is placed below the outer end of tube T to collect the liquid flowing out through it. The liquid is allowed to flow out in a slow trickle and collected in the beaker for a known time, and the beaker is then weighed again. The difference of the two weights gives the *mass* of the liquid flowing out in that time. Then, knowing the density of the liquid its volume can be determined and, dividing it by the time for which the liquid was allowed to flow, its volume V flowing out *per second* is known. Substituting the value of V, so obtained, in relation II,

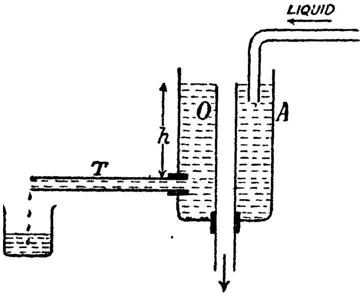

Fig. 275.

*The method is suitable only for comparatively less viscous liquids, like water.

above the coefficient of viscosity (η) of the liquid can be easily calculated.

There are two important sources of error in the above experiment, *viz.*, (*i*) part of the thrust, due to the difference of pressure between the two ends of the flow-tube, imparts kinetic energy to the liquid and the whole of it, therefore, is not used *simply* in overcoming the viscous resistance of the liquid. This may be corrected for by taking the effective value of the pressure difference to be $P - \frac{V^2.\rho}{\pi^2 r^4}$, instead of P; (*ii*) the motion of the liquid, where it enters the flow-tube, is accelerated, with the result that the velocity of flow is not uniform for the first short length of the tube. This is eliminated by taking the effective length of the flow-tube to be $(l+1{\cdot}64r)$, instead of l. Thus, the corrected relation for η becomes

$$\eta = \frac{\pi P r^4}{8V.(l+1{\cdot}64\,r)} - \frac{V.\rho}{8\pi(l+1{\cdot}64\,r)}.$$

A much better apparatus, however, is the following, in which the flow-tube F is a *long* one, and of a uniform circular cross-section, and the difference of pressure for a length AB of it is given directly by means of a manometer M, whose limbs are arranged over two fine holes at A and B, as shown, (Fig. 276), where A and B lie at a distance of at least 10 *cms.* from the two ends of the flow-tube respectively, so that the velocity of the out-flowing liquid becomes uniform near about them. This very much minimises the two sources of error referred to above,—the second one, almost completely. So that, with a slow rate of flow of the liquid and a fairly small size of the holes at A and B, no further corrections are necessary.

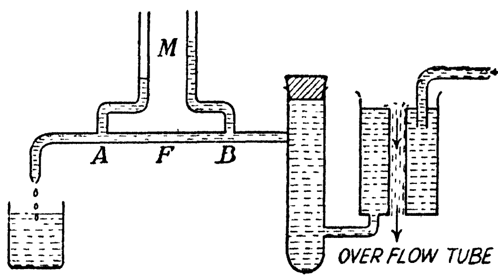

Fig. 276.

Note. In either of these apparatus, it is essential that the outflow tube should have a *perfectly uniform* bore. The uniformity of the bore may be tested in a manner similar to that employed in the construction of a mercury thermometer, *i.e.*, by introducing a small thread of mercury into the tube and measuring its length in the different parts of the tube. In no part should the length vary by more than 5%.

And, since the *4th* power of the radius occurs in the formula for η, it should be determined most accurately. The tube is, therefore, properly dried and filled with mercury, and the length of the mercury thread measured most carefully by means of a vernier microscope, making the necessary correction for the curvature of the ends of the thread. The mercury is then taken out in a clean, dry and weighed watch glass and its mass determined as accurately as possible. Then, if m be its mass, ρ, its density at the then-temperature l', the length of its thread in the tube, and r, the radius of the tube, we clearly have

$$\pi.r^2.l'.\rho = m, \quad \text{whence,} \quad r = \sqrt{\frac{m}{\pi.l'.\rho}}.$$

So that, knowing m, l' and ρ, the accurate value of the radius r of the tube can be calculated.

211. Motion in a Viscous Medium. When a body falls through a viscous medium, its motion is opposed by a force, *frictional in nature*, due to the fact that whereas the layer of the liquid medium in immediate contact with it is carried along with it, that at an *infinite* distance from it is at rest. Energy is being continually absorbed by the medium and is converted into heat. Possibly also, eddy currents and waves are set up in the medium—particularly when the body moving is a fast one, like high speed cars and airplanes or projectiles--and these absorb still more energy. That is why cars etc., are streamlined these days to minimise the absorption of energy in this way. Even if the body be moving so slowly that no eddy currents or waves are set up, energy is still wasted due to the *viscous drag* it has to overcome.

Now, this opposing force, increases with the velocity of the body, until, in the case of small bodies, it becomes just equal to the *motive* or the *driving force*, and the body then attains a constant velocity, called its *terminal velocity*.

Stokes showed that the *retardation* F, due to to the *viscous drag*, for a spherical body of radius r, moving with velocity v, in a medium whose coefficient of viscosity is η, is given by

$$F = 6\pi v r \eta.$$

This relation, known as **Stokes' law**, may be deduced as follows, by the method of dimensions :

For slow moving bodies,

$F \propto$ *velocity* v ; $F \propto$ *radius* r *of the body* ;

$F \propto$ *coefficient of viscosity* η *of the medium* ;

$F \propto$ *density* σ *of the medium.*

Or, $$F = K.v r^a \eta^b \sigma^c,$$

where K is a constant and a, b and c, the dimensional coefficients of r, η and σ respectively.

Now, putting the proper dimensions of the different terms, we have

$$[MLT^{-2}] = [LT^{-1}]\,[L^a][M^bL^{-b}T^{-b}]\,[M^cL^{-3c}]$$
$$= [M^{b+c}L^{1+a-b-3c}T^{-1-b}]$$

whence (*i*) $b+c = 1$, (*ii*) $1+a-b-3c = 1$, and (*iii*) $-1-b = -2$.

Therefore, from relation (*iii*), $b = 1$; and hence from (*i*) we have $c = 0$; so that, from (*ii*), $a = 1$.

$\therefore$ $F = K.v.r.\eta$; and the value of K was found by *Stokes* to be 6π ; so that, $F = 6\pi v r \eta$, as stated above.

If the density of the spherical body be ρ,

its *weight* $=$ *volume* $\times \rho \times g = \frac{4}{3}.\pi r^3 \times \rho \times g$,

and the *upthrust on it due to the displaced medium* $= \frac{4}{3}\pi r^3.\sigma.g$.

$\therefore$ *resultant downward force on the body* $= \frac{4}{3}\pi r^3.\rho.g. - \frac{4}{3}\pi r^2 \sigma.g$
$= \frac{4}{3}\pi r^3.g.(\rho - \sigma)$.

Equating this against the value of F, we have

$6\pi v r \eta = \frac{4}{3}\pi r^3.g(\rho-\sigma)$, whence, $v = \frac{4}{3}\pi r^3.g(\rho-\sigma)/6\pi r \eta$.

Or, $$v = \frac{2}{9}\cdot\frac{r^2.g(\rho-\sigma)}{\eta}. \quad \dots (i)$$

Thus, the *terminal velocity* of a body, (of course, of a small size), falling through a viscous medium, is (*i*) *directly proportional to the square of its radius* (r^2), (*ii*) *directly porportional to the difference in the densities of the body and the medium*, ($\rho-\sigma$), *and* (*iii*) *inversely proportional to the coefficient of viscosity of the medium* (η).

In arriving at the above result, *Stokes* made the following assumptions :

(*a*) That the medium through which the body falls is *infinite in extent.*

(*b*) That the spherical body is *perfectly rigid and smooth.*

(*c*) That there is *no slip between the spherical body and the medium.*

(*d*) That the medium is *homogeneous*, so far as the spherical body is concerned, *i.e.*, the diameter of the spherical body is large compared with the spaces between the molecules of the medium.

(*e*) That there are *no eddy currents or waves set up in the medium due to the motion of the body through it* ;—in other words, that the body is moving very slowly through it, or that the motion of the medium is smooth and not turbulent. *Stokes* found that the relation holds good only when v is smaller than $\eta/\sigma r$*, called the *critical velocity*.

A striking example of a body falling through a viscous medium is that of the tiny rain drops that form what we call *clouds*. These tiny drops of water have a radius as small as ·001 *cm.* and their terminal velocity, as they fall through air, for which η = ·00018, comes to about 1·2 *cms./sec.*, [from relation (*i*) above]. That is why they remain suspended in the air and appear to us to be floating about as clouds.

Bigger rain drops, on the other hand, have a radius about 10 times as great (*i.e.*, = ·01 *cm.*) and their terminal velocity, therefore, comes to about 120 *cms./sec.* ; so that, they *fall* through the air, instead of floating in it, (v being proportional to r^2).

Also, if the density of the medium in which the body falls be greater than that of the body itself, *i.e.*, if $\sigma > \rho$, it is clear that the terminal velocity v will have a *negative* value. In such a case, therefore, the body will have an upward terminal velocity. That is why bubbles of air or gas can rise up through water or any other liquid,—the smaller the bubble, the smaller its velocity.

212. Determination of coefficient of viscosity of a liquid—Stokes' method. The relation for v obtained above, (§ 211), has been used to determine the viscosity of a liquid. The method consists in finding the time of fall of small spheres, such as ball-bearings etc., in the liquid, and then to apply *Stokes'* relation,

$$v = \frac{2}{9} \cdot \frac{r^2.g(\rho-\sigma)}{\eta}, \text{ whence, } \eta = \frac{2}{9} \cdot \frac{r^2 g(\rho-\sigma)}{v}. \quad ...(ii)$$

*It was shown by Arnold, however, that in actual practice v should be less than ·6$\eta/\sigma r$.