Computer Vision and Image Processing (EC 336)

Lecture 7: Introduction to frequency domain analysis

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Fourier Series and Fourier Transform

- Fourier Series

  Any periodic function can be expressed as the sum of sines and /or cosines of different frequencies, each multiplied by a different coefficients

- Fourier Transform

  Any function that is not periodic can be expressed as the integral of sines and /or cosines multiplied by a weighing function

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Impulse, train of impulses, & shifting property

A *unit impulse* of a continuous variable t located at t=0, denoted $\delta(t)$, defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and is constrained also to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The *sifting property* $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

A *unit impulse* of a discrete variable x located at x=0, denoted $\delta(x)$, defined as

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and is constrained also to satisfy the identity

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

The *sifting property* $\sum_{x=-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

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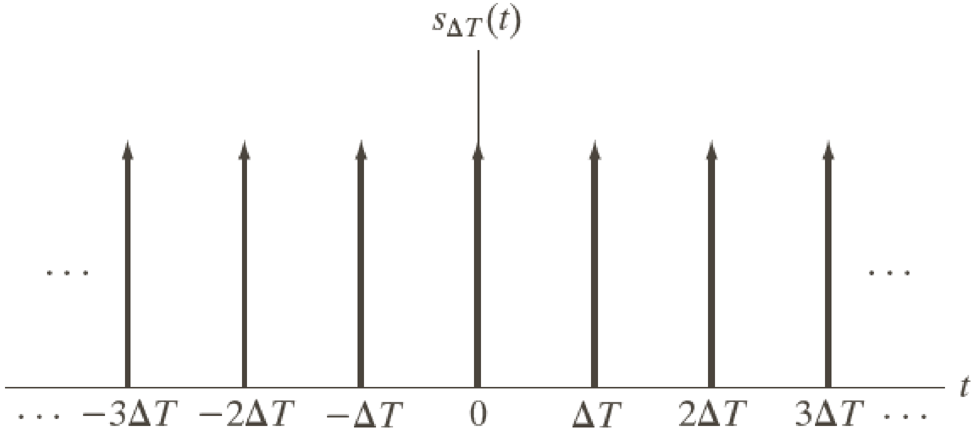
Train of impulses

$$impulse\ train\ s_{\Delta T}(t),$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



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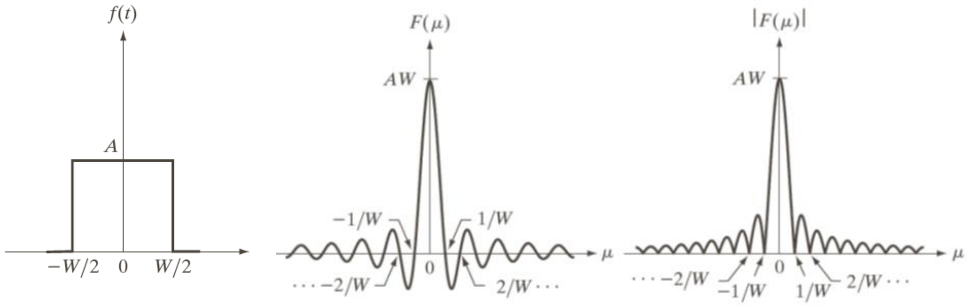
Fourier Transform
Continuous function & single variable

The *Fourier Transform* of a continous function $f(t)$

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

The *Inverse Fourier Transform* of $F(\mu)$

$$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$



The Fourier transform of a unit impulse located at the origin:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$
$$= e^{-j2\pi\mu 0}$$
$$=1$$

The Fourier transform of a unit impulse located at $t = t_0$:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$
$$= e^{-j2\pi\mu t_0}$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$
$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{A}{j2\pi W}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$
$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

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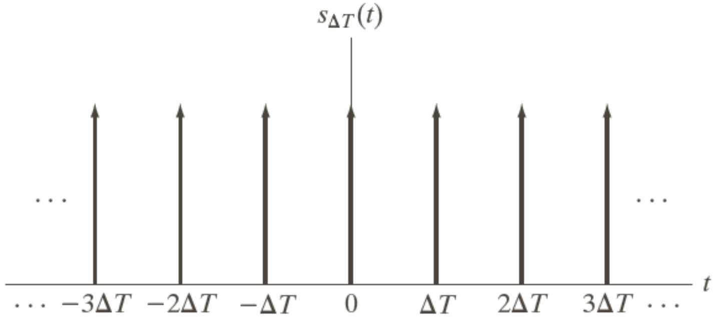
Fourier Transform of Impulse Trains



Impulse train $s_{\Delta T}(t)$, $\quad s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$

The fourier series expansion will be $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Let $S(\mu)$ denote the Fourier transform of the periodic impulse train $S_{\Delta T}(t)$

$$S(\mu) = \Im\{S_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

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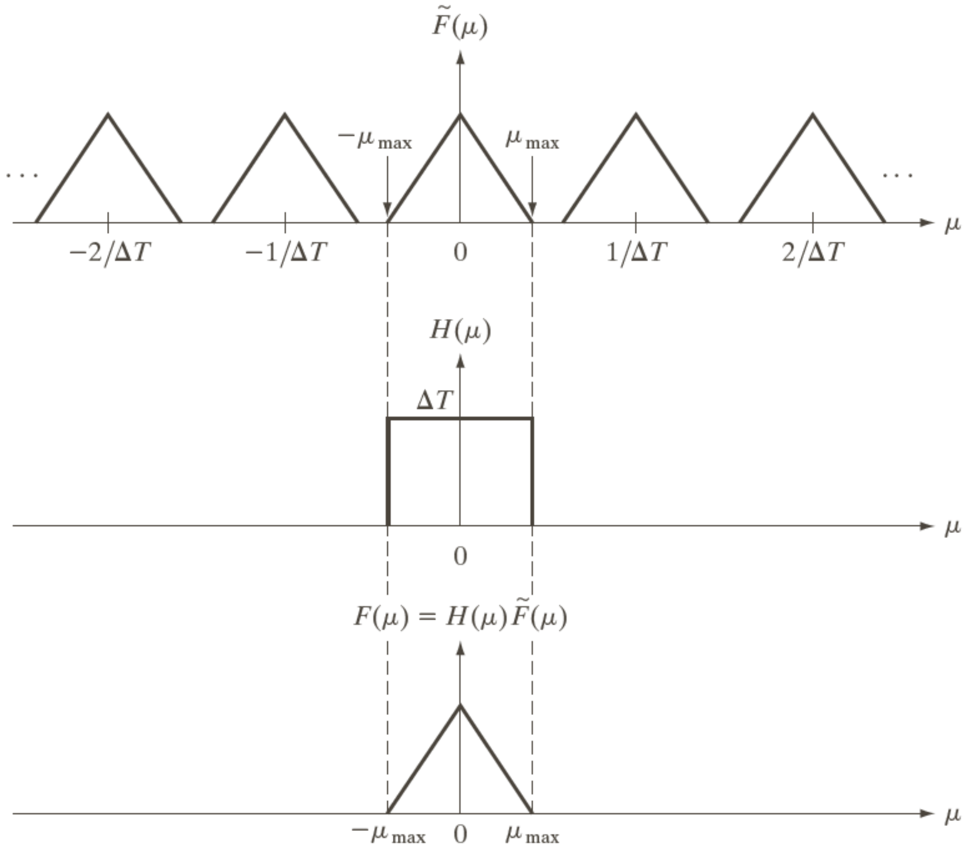
Nyquist–Shannon sampling theorem

- A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

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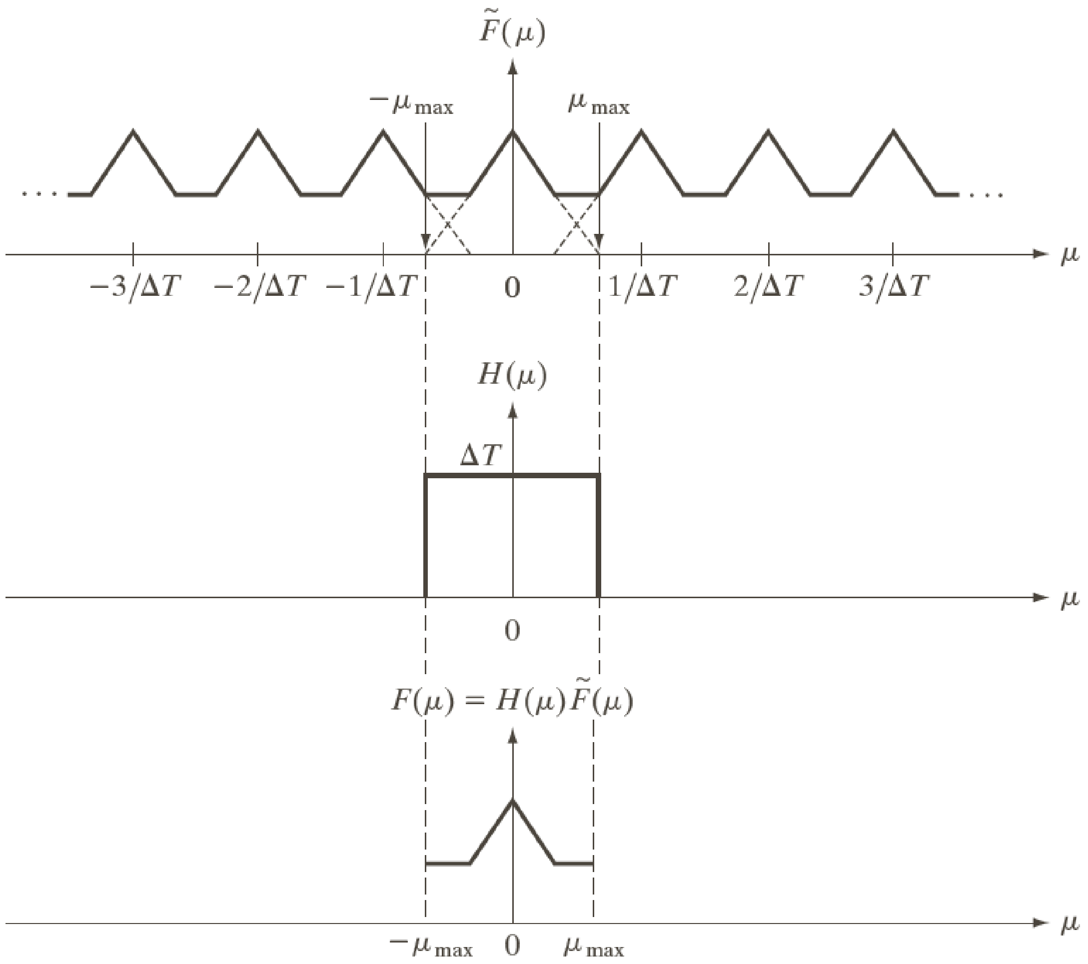
If a band-limited function is sampled at a rate that is less than twice its highest frequency then the inverse transform will yield a corrupted function.
This effect is known as frequency aliasing or simply as **aliasing**.

Aliasing



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

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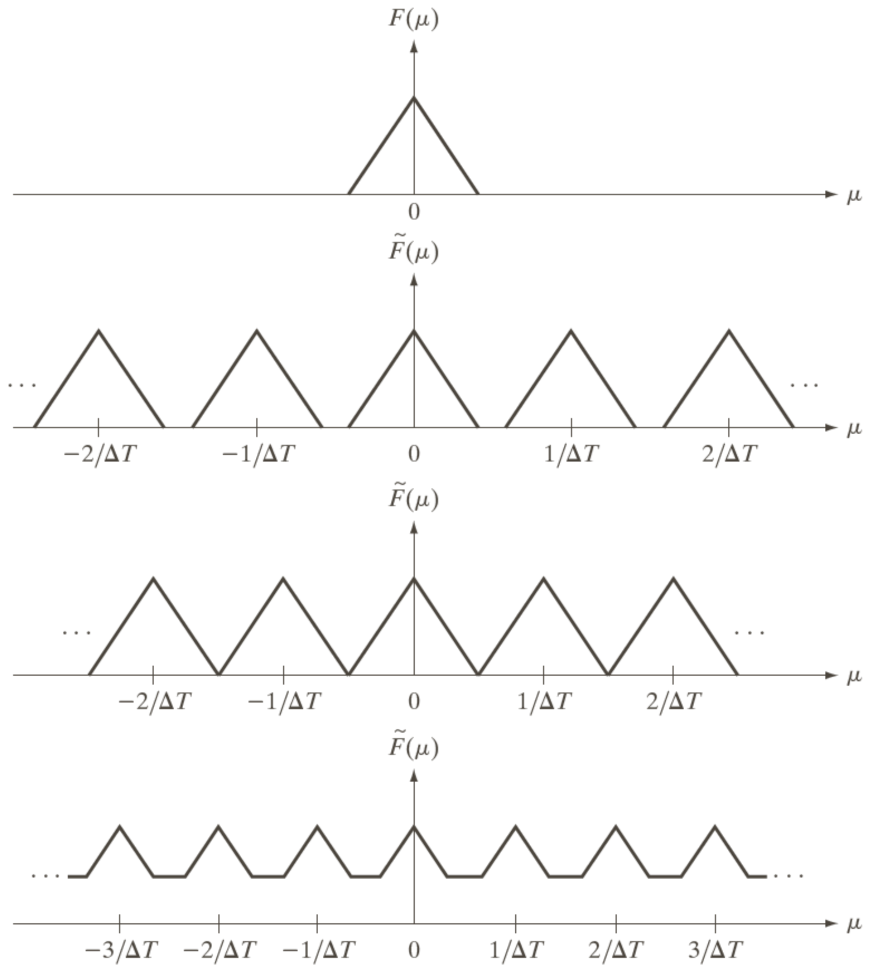
Fourier transform of sampled function



Over-sampling

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

Critically-sampling

$$\frac{1}{\Delta T} = 2\mu_{\max}$$

under-sampling

$$\frac{1}{\Delta T} < 2\mu_{\max}$$

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1D Discrete Fourier transform

$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi\mu x/M}, \qquad \mu = 0, 1, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{\mu=0}^{M-1} F(\mu) e^{j2\pi\mu x/M}, \qquad x = 0, 1, 2, \ldots, M-1$$

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2-D Continuous Fourier Transform

$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

and

$$f(t,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

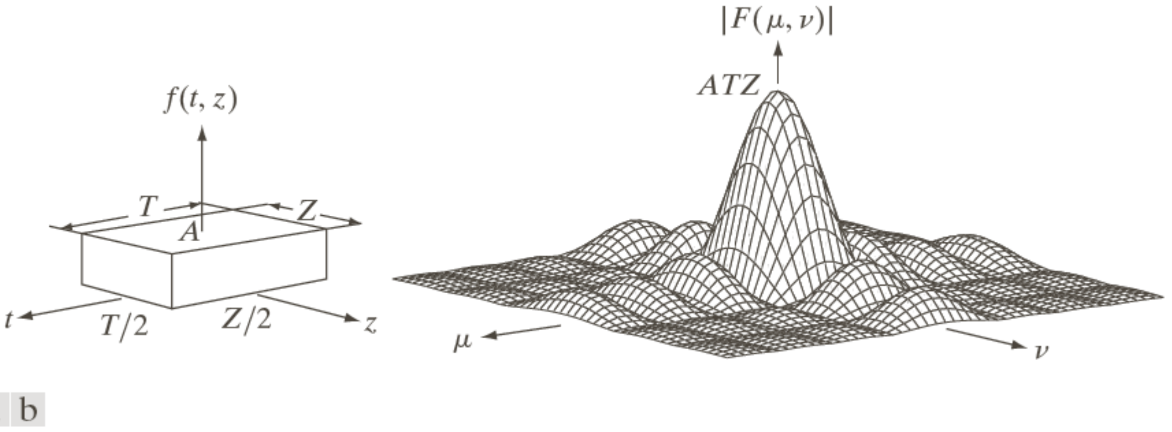


FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

Example:

$$\begin{aligned}F(\mu,\nu)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz\\&=\int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}dtdz\\&=ATZ\left[\frac{\sin(\pi\mu T)}{\pi\mu T}\right]\left[\frac{\sin(\pi\nu T)}{\pi\nu T}\right]\end{aligned}$$

2-D Sampling Theorem

A continuous, band-limited function $f(t,z)$ can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{\max}} \text{ and } \Delta Z < \frac{1}{2\nu_{\max}}$$

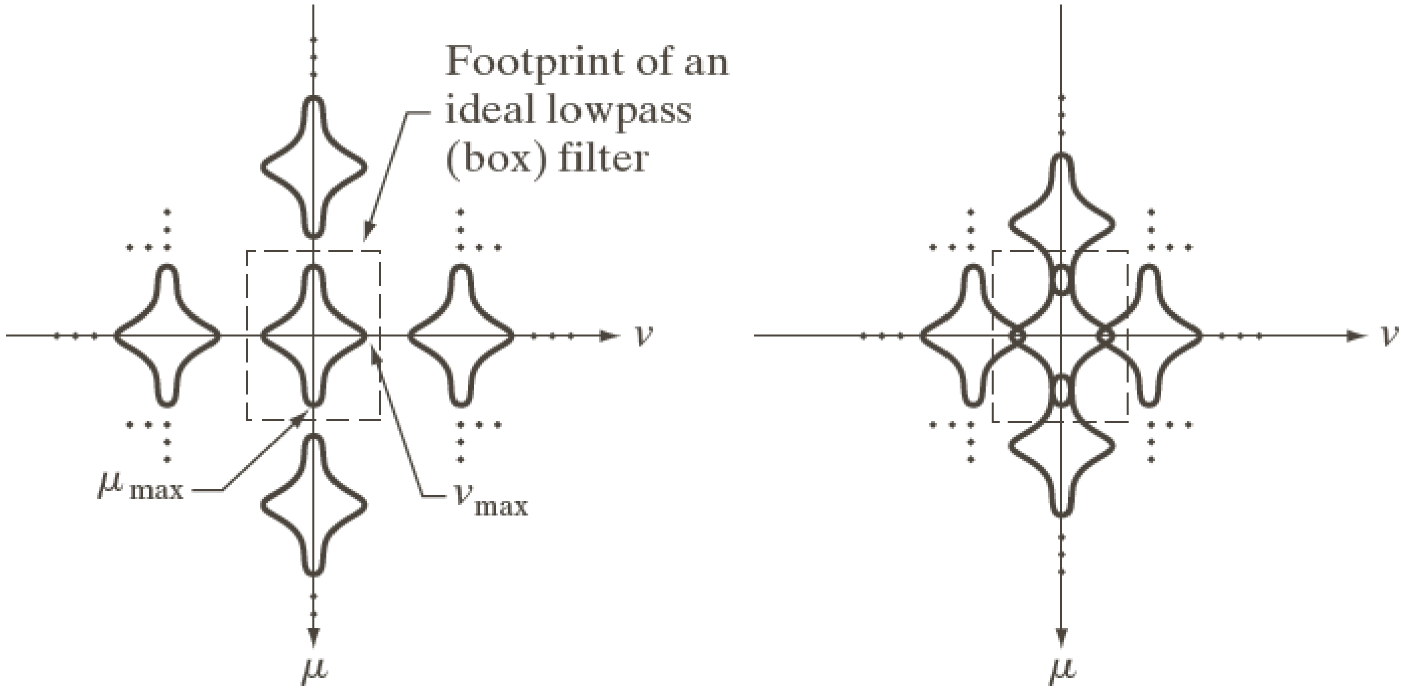
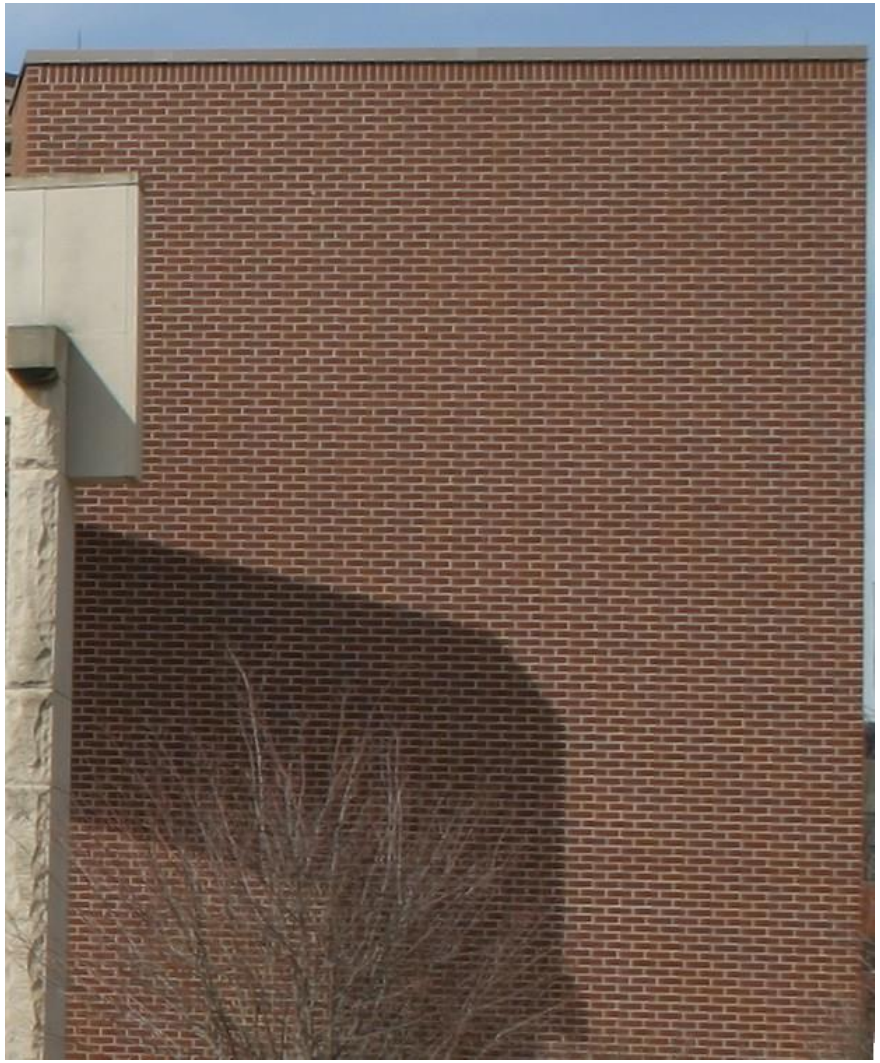


FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

Aliasing in Images



A moiré pattern formed by incorrectly down-sampling the former image

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2-D Discrete Fourier Transform

DFT of a digital image $f(x, y)$ of size $M \times N$ is given by

$$F(\mu,\nu) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)e^{-j2\pi(\mu x/M+\nu y/N)} \quad , \quad \mu = 0,1,2,..., M-1; \nu = 0,1,2,..., N-1;$$

IDFT:

$$f(x, y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(\mu,\nu)e^{j2\pi(\mu x/M+\nu y/N)}$$

Relationships between spatial and frequency intervals

Let ΔT and ΔZ denote the separations between samples, then the seperations between the corresponding discrete, frequency domain variables are given by

$$\Delta\mu = \frac{1}{M\Delta T}$$

and $$\Delta\nu = \frac{1}{N\Delta Z}$$

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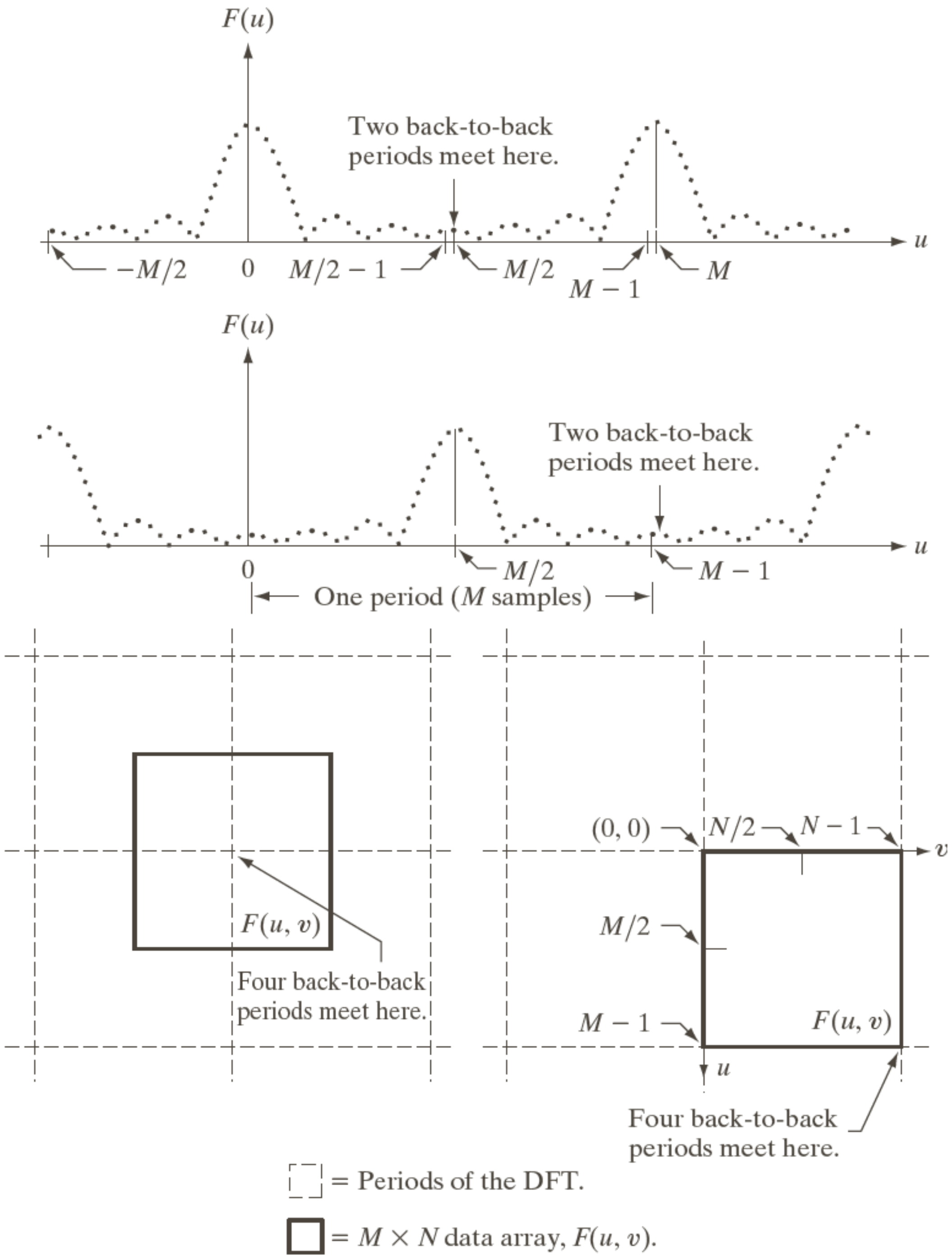
Centering of Fourier transform

$$f(x,y)(-1)^{x+y} \rightarrow F(u - \frac{M}{2}, v - \frac{N}{2})$$



a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).

Centering of Fourier transform

- DFT of *f(x)*

$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi\mu x/M}, \quad \mu = 0, 1, \ldots, M-1$$

- Circular frequency shift property of DFT

$$DFT[f(x) \times e^{-j2\pi\mu_0 x/M}] = F(\mu - \mu_0)$$

proof

$$DFT[f(x) \times e^{-j2\pi\mu_0 x/M}] = \sum_{x=0}^{M-1} [f(x) e^{-j2\pi\mu_0 x/M}] e^{-j2\pi\mu x/M}$$

$$= \sum_{x=0}^{M-1} [f(x) e^{-j2\pi(\mu+\mu_0)x/M} = F(\mu - \mu_0)$$

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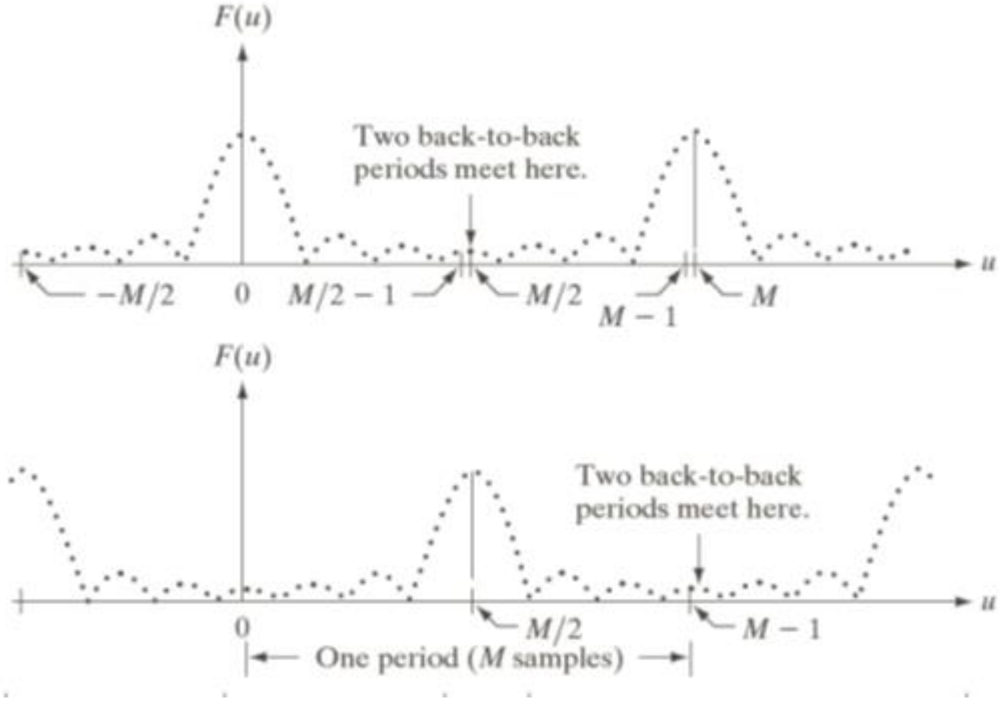
Centering of Fourier transform

$$\text{when } \mu_0 = \frac{M}{2},$$

$$\text{DFT}[\, f(x) \times e^{\frac{-j2\pi(M/2)x}{M}} \,] = F(\mu + \frac{M}{2})$$

$$\Rightarrow \text{DFT}[\, f(x) \times e^{-j\pi x} \,] = F(\mu + \frac{M}{2})$$

$$\Rightarrow \text{DFT}[\, f(x) \times (-1)^x \,] = F(\mu + \frac{M}{2})$$



Similarly for 2D DFT $\quad f(x,y)(-1)^{x+y} \rightarrow F(u - \frac{M}{2}, v - \frac{N}{2})$

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Example:

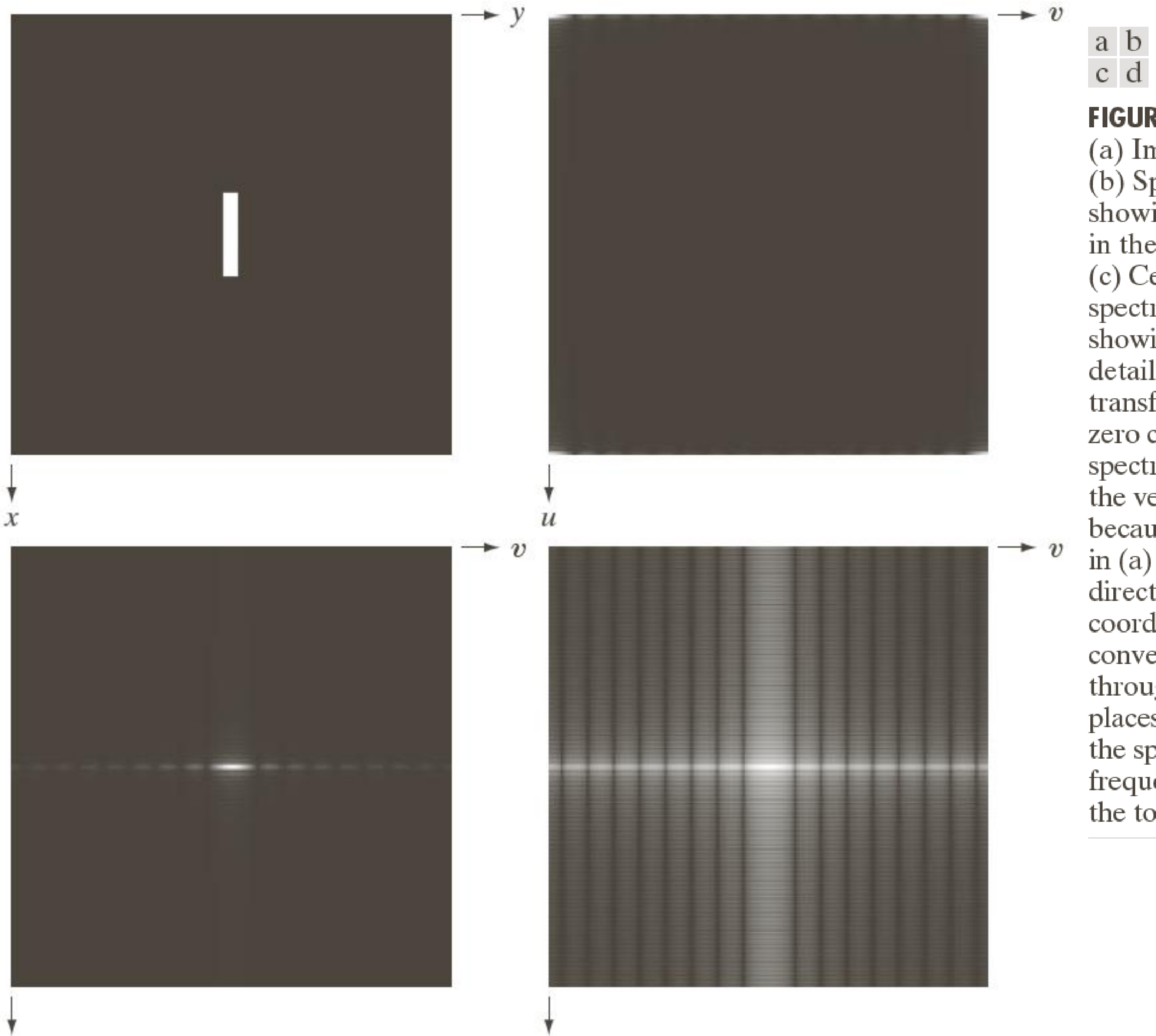

FIGURE 4.24
(a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

Fourier Transform and Convolution

In 1D

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

In 2D

$$f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$$

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Zero padding in convolution

- Let *f(x,y)* and *h(x,y)* be two image arrays of sizes $A \times B$ and $C \times D$ pixels, respectively. Wraparound error in their convolution can be avoided by padding these functions with zeros

$$f_p(x, y) = \begin{cases} f(x, y) & 0 \le x \le A\text{-}1 \; and \; 0 \le y \le B\text{-}1 \\ 0 & A \le x \le P \; or \; B \le y \le Q \end{cases}$$

$$h_p(x, y) = \begin{cases} h(x, y) & 0 \le x \le C\text{-}1 \; and \; 0 \le y \le D\text{-}1 \\ 0 & C \le x \le P \; or \; D \le y \le Q \end{cases}$$

$$\text{Here } P \ge A + C - 1; Q \ge B + D - 1$$

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The Basic Filtering in the Frequency Domain

- Modifying the Fourier transform of an image
- Computing the inverse transform to obtain the processed result

$$g(x, y) = \Im^{-1}\{H(u,v)F(u,v)\}$$

$F(u,v)$ is the DFT of the input image

$H(u,v)$ is a filter function.

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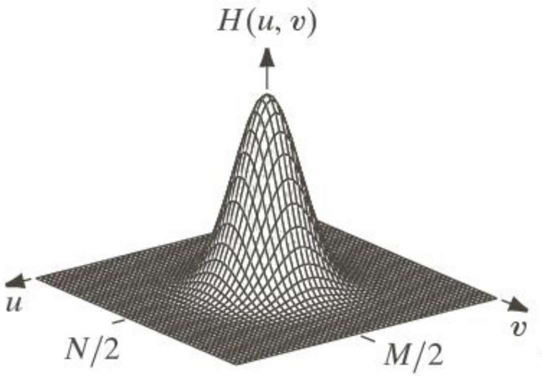
The Basic Filtering in the Frequency Domain



a	b	c
d	e	f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Zero-Phase-Shift Filters

$$g(x,y) = \Im^{-1}\{H(u,v)F(u,v)\}$$

$$F(u,v) = R(u,v) + jI(u,v)$$

$$g(x,y) = \Im^{-1}\left[H(u,v)R(u,v) + jH(u,v)I(u,v)\right]$$

Filters affect the real and imaginary parts equally,

and thus no effect on the phase.

These filters are called **zero-phase-shift** filters

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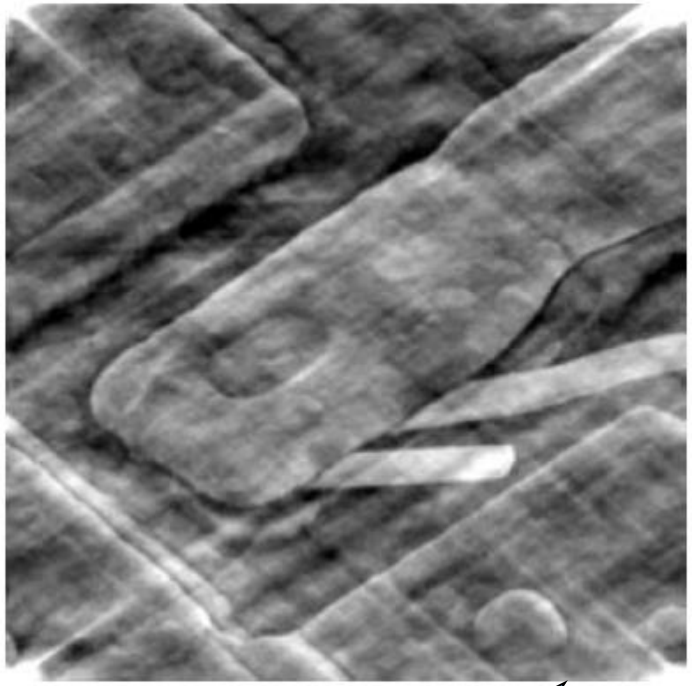
Examples: Nonzero-Phase-Shift Filters



a b

FIGURE 4.35 (a) Image resulting from multiplying by 0.5 the phase angle in Eq. (4.6-15) and then computing the IDFT. (b) The result of multiplying the phase by 0.25. The spectrum was not changed in either of the two cases.

Even s… n the phase ang… ramatic (usuall… effects on the … t

Phase angle is multiplied by 0.5

Phase angle is multiplied by 0.25

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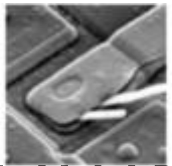
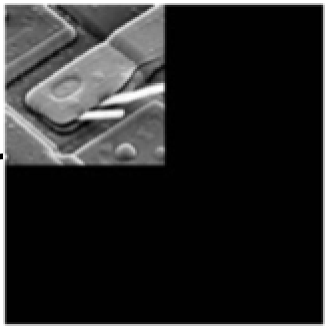
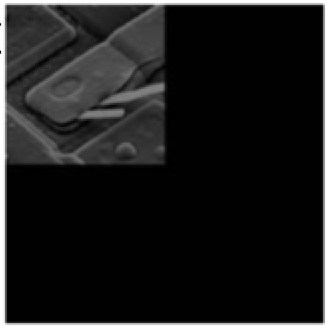
Steps for Filtering in the Frequency Domain





1. Given an input image $f(x,y)$ of size MxN, obtain the padding parameters P and Q.

   Typically, $P = 2M$ and $Q = 2N$.

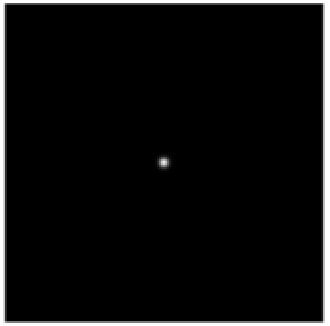


2. Form a padded image, $f_p(x,y)$ of size PxQ by appending the necessary number of zeros to $f(x,y)$

3. Multiply $f_p(x,y)$ by $(-1)^{x+y}$ to center its transform

4. Compute the DFT, $F(u,v)$ of the image from step 3

5. Generate a real, symmetric filter function*, $H(u,v)$, of size PxQ with center at coordinates $(P/2, Q/2)$



*generate from a given spatial filter, we pad the spatial filter, multiply the padded array by $(-1)^{x+y}$, and compute the DFT of the result to obtain a centered H(u,v).

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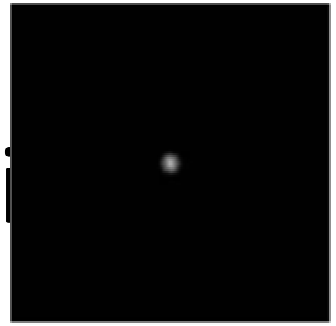
Steps for Filtering in the Frequency Domai

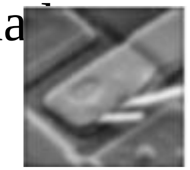


6. Form the product *G(u,v) = H(u,v)F(u,v)* using array multiplication

7. Obtain the processed image

$$g_p(x,y) = \left\{ real\left[\Im^{-1}\left[G(u,v)\right]\right]\right\}(-1)^{x+y}$$

8. Obtain the final processed result, *g(x,y)*, by extracting the *MxN* region from the top, left qua of $g_p(x,y)$



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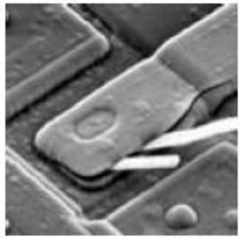

a	b	c
d	e	f
g	h	

FIGURE 4.36
(a) An $M \times N$ image, f.
(b) Padded image, f_p of size $P \times Q$.
(c) Result of multiplying f_p by $(-1)^{x+y}$.
(d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$.
(f) Spectrum of the product HF_p.
(g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p.
(h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

Examples of frequency domain filters

- Image smoothing using frequency domain filters (LOW PASS FILTERS)
- Image sharpening using frequency domain filters (HIGH PASS FILTERS)

- These will be covered in the next lab presentation

It's important.
Do not miss the lab presentation

- **Next theory lecture**: Laplacian in frequency domain,

Unsharp Masking, Highboost Filtering and High-Frequency-Emphasis Fitering,

Homomorphic Filtering

Frequency selective filtering

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Properties of the 2-D DFT

	Spatial Domain[†]		Frequency Domain[†]
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

[†]Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

Summary

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
7) Average value	$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

Summary

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$ $= F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$ $= f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Thank you!

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