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Chapter 2...

Chapter 2

Fourier Series

Many phenomena studied in engineering and science, such as the current and voltage in an alternating current circuit, the displacement, velocity, and acceleration of a piston, and many parameters in a vibrating system, are periodic. It is often desirable to represent such functions in infinite trigonometric series in order to solve such problems. The sum of these special trigonometric functions is called the **Fourier Series**.

We hope that such modeling provides adequate representation over the entire cycle of periodicity rather than the Taylor series representation's local nature.

First, let us recall what a periodic function is. $\sin(x+2\pi) = \sin x$

Definition 2.1. *A function $f(x)$ is said to be* ***periodic*** *if there exists $p > 0$ such that for each x in the domain of f, $f(x+p) = f(x)$. The smallest p satisfying the above relationship is called the* ***period*** *of $f(x)$.*

Any periodic function $f(x)$ whose period is 2π can be represented in the form

$$\boxed{f(x) = c_0 + \sum_{n=1}^{\infty} c_n \sin(nx + \alpha).}$$

Definition 2.2. *Suppose $f(x)$ is a periodic function whose period is 2π. The Fourier series for the function $f(x)$ in the interval $-\pi \le x \le \pi$ is given by*

- $\pi < x < \pi$ also get same coefficie
- nts.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right). \tag{2.1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \sin(nx + \alpha)$$

$f(x)$ periodic

period $= 2\pi$

∞

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n \sin(nx + \alpha) \quad \text{period} = 2\pi$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \left(\sin(nx)\cos\alpha + \cos(nx)\sin\alpha \right)$$

$$= C_0 + \sum_{n=1}^{\infty} \left(C_n \sin\alpha \cos(nx) + C_n \cos\alpha \sin(nx) \right)$$

choose $a_0 = 2C_0$, $a_n = C_n \sin\alpha$ $b_n = C_n \cos\alpha$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx.$$

Here a_0, a_n, b_n *are called* ***Fourier coefficients***.

The following definite integrals will be required to establish a_0, a_n and b_n.

- $\int_0^{2\pi} \sin nx \ dx = 0, \quad n \neq 0.$
- $\int_0^{2\pi} \cos nx \ dx = 0, \quad n \neq 0.$
- $\int_0^{2\pi} \sin nx \sin mx \ dx = 0, \quad n \neq m.$
- $\int_0^{2\pi} \cos nx \cos mx \ dx = 0, \quad n \neq m.$
- $\int_0^{2\pi} \sin nx \cos mx \ dx = 0, \quad n, m \neq 0.$
- $\int_0^{2\pi} \sin^2 nx \ dx = \pi, \quad n \neq 0.$
- $\int_0^{2\pi} \cos^2 nx \ dx = \pi, \quad n \neq 0.$

Example 2.3. *Find the Fourier series representing* $f(x) = x$ *in the interval* $0 < x < 2\pi$.

Example 2.4. *Obtain the Fourier series for* $f(x) = x + x^2$ *in the interval* $-\pi < x < \pi$. *Deduce that* $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$.

2.1 Conditions for Fourier Expansion

The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $f(x)$ to be equal to its Fourier series expansion at each point where f is continuous. Moreover, the behavior of the Fourier series at each point of discontinuity is determined as well.

2.1.1 Dirichlet conditions

Any periodic waveform of period $p = 2L$ can be expressed in a Fourier series if

- it has a finite number of discontinuities within the period,
- the function is absolutely integrable over the period (i.e. $\int_0^{2L} |f(x)| \ dx < \infty$), and
- it has a finite number of maxima and minima within the period.

The Fourier series for the function $f(x)$ exists when the above conditions are satisfied. Those conditions are called the **Dirichlet conditions**, which were named after Peter Gustav Lejeune Dirichlet.

Ex 2.3. $f(x) = x$ and $f(x+2\pi) = f(x)$

$0 < x < 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{2\pi}$$

$$= \underline{\underline{2\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x}\,\underset{dv}{\cos nx\,dx}$$

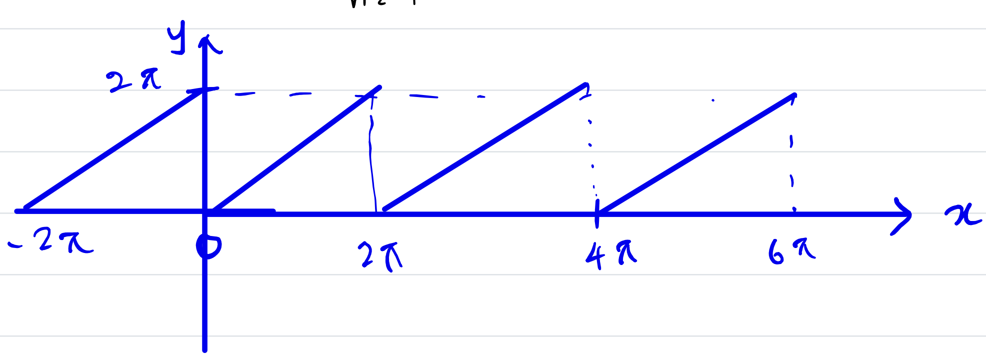
$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

$$= -\frac{2}{n}$$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(0\cos(nx) + -\frac{2}{n}\sin(nx)\right)$$

$$= \pi - \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(nx)\right)$$



Ex 2.4 $\quad f(x) = x + x^2$ and $f(x+2\pi) = f(x)$

$-\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n(\sin(nx)\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2\,dx$$

$$= \frac{1}{\pi}\left\{ \frac{x^2}{2}\Big|_{-\pi}^{\pi} + \frac{x^3}{3}\Big|_{-\pi}^{\pi} \right\}$$

$\pi^2 - \pi^2$

$$= \frac{1}{\pi}\left\{ 0 + \frac{\pi^3}{3} + \frac{\pi^3}{3} \right\}$$

$$= \frac{1}{\pi}\left(\frac{2\pi^3}{3} \right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos(nx)\,dx = (-1)^n \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin(nx)\,dx = (-1)^n\left(\frac{-2}{n}\right)$$

$$f(x) = x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n 4}{n^2}\cos(nx) + (-1)^n\left(\frac{-2}{n}\right)\sin(nx)\right)$$

Let $x = \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n 4}{n^2}\cos(n\pi) + \cancel{(-1)^n\left(\frac{-2}{n}\right)\sin n\pi}^{\,0}\right)$$

$$\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \quad \text{—— } (1)$$

Let $x = -\pi$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos(-n\pi)$$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \quad \text{—— } (2)$$

$$-\pi+\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \text{ — } (2)$$

(1) + (2) =>

$$\cancel{2}\pi^2 = \frac{\cancel{2}\pi^2}{3} + \cancel{2}\sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}} \qquad 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}$$

Generalized integration by parts

suppose $f(x)$ and $g(x)$ are 2 real valued functions which are integrable and differenciable at least n times. Then,

$$\int f^{(0)} g^{(n)}\,dx = f^{(0)} g^{n-1} - f^{(1)} g^{(n-2)} + f^{2} g^{n-3} - \dots + (-1)^{n-1} f^{(n-1)} g^{(0)} + (-1)^{n} \int f^{(n)} g^{(0)}\,dx$$

Ex $\int \underbrace{x^3}_{f(0)} \underbrace{\cos x}_{g^{(3)}}\,dx \qquad n=3$

$$f^{(0)} g^{(2)} - f^{(1)} g^{(1)} + f^{(2)} g^{(0)} + (-1)^3 \int f^{(3)} g^{(0)}\,dx$$

$$x^3 \sin x - 3x^2(-\cos x) + 6x(-\sin x) - \int 6(-\sin x)\,dx$$

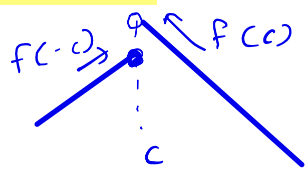
$$= x^3 \sin x + 3x^2 \cos x - 6x\sin x - 6\cos x + c$$

2.2 Discontinuous Functions

At a point of discontinuity, the Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

If f is discontinuous at $x = c$ then the value of $f(x)$ at c is given by

$$\boxed{\frac{f(c-) + f(c+)}{2}}.$$



where $f(c-)$ is the left-hand limit of f at c and $f(c+)$ is the right-hand limit of f at c.

Example 2.5. *Suppose $f(x) = x$ if $-\pi < x < \pi$ and $f(x) = f(x + 2\pi)$. Find the Fourier series expansion for $f(x)$ in the interval $-\pi < x < \pi$. Find the value of $f(\pi)$.*

2.3 Function Defined in More Than One Sub Range

A given function may consist of a finite number of different curves given by different equations. This type of function can also be expressed as a Fourier series.

For instance, if $f(x)$ is defined by

$$f(x) = \begin{cases} \rho(x) & \text{if } 0 < x < c\,, \\ \theta(x) & \text{if } c < x < 2\pi \end{cases} \tag{2.2}$$

where c is the point of discontinuity, then

$\leftarrow \rho(x)$ $\quad$ $\leftarrow \theta(x)$

$$a_0 = \frac{1}{\pi}\left[\int_0^c f(x)\,dx + \int_c^{2\pi} f(x)\,dx\right],$$

$$a_n = \frac{1}{\pi}\left[\int_0^c f(x)\cos nx\,dx + \int_c^{2\pi} f(x)\cos nx\,dx\right],$$

and

$$b_n = \frac{1}{\pi}\left[\int_0^c f(x)\sin nx\,dx + \int_c^{2\pi} f(x)\sin nx\,dx\right].$$

Example 2.6. *Find the Fourier series expansion for*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi\,, \\ -1 & \text{if } \pi < x < 2\pi. \end{cases}$$

Example 2.7. *Find the Fourier series expansion for*

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0\,, \\ x & \text{if } 0 < x < \pi. \end{cases}$$

Ex 2.5 $\quad f(x) = x \quad ; \quad -\pi < x < \pi \quad f(x) = f(x + 2\pi)$

Ex 2.5 $\quad f(x) = x \quad ; \; -\pi < x < \pi \quad f(x) = f(x+2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\,dx = 0$$

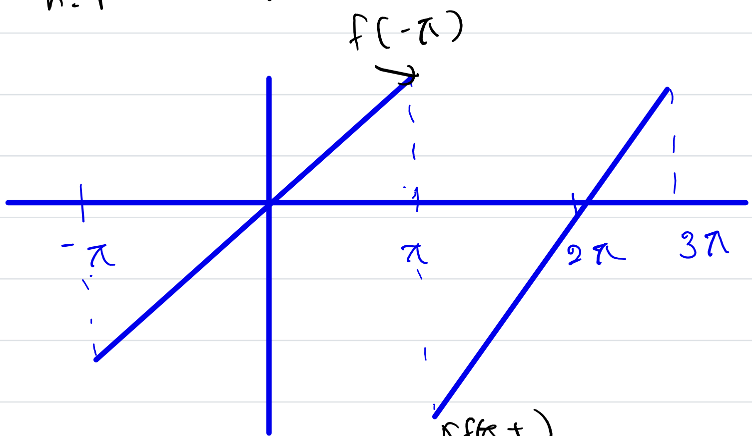
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{u}{x} \, \underset{dv}{\cos(nx)\,dx} = \frac{1}{\pi} x \frac{\sin(nx)}{n} \Bigg|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(nx)}{n} \times 1 \, dx$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\,dx = \frac{1}{\pi} \, -x\frac{\cos(nx)}{n} \Bigg|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} \, dx$$

$$= (-1)^{n+1} \frac{2}{n}$$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)$$



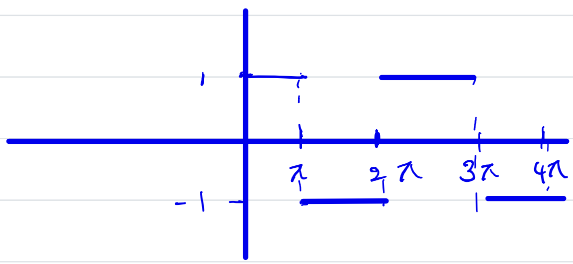
$f(x)$ is discontinuous at π

$$\therefore f(\pi) = \frac{f(\pi-) + f(\pi+)}{2}$$

$$= \frac{\pi + (-\pi)}{2}$$

$$= \underline{\underline{0}}$$

EX 2.6. $f(x) = \begin{cases} 1 & ; \; 0 < x < \pi \\ -1 & ; \; \pi < x < 2\pi \end{cases}$



$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} 1 \, dx + \int_{\pi}^{2\pi} -1 \, dx\right]$$

$$= \frac{1}{\pi}[\pi - \pi] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} 1 \cos(nx)\, dx + \int_{\pi}^{2\pi} -1 \cos(nx)\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} 1 \sin(nx)\, dx + \int_{\pi}^{2\pi} -1 \sin(nx)\, dx\right]$$

$\cos(n\pi) = (-1)^n$

$$b_n = \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right) \quad \text{or} \quad \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$f(x) = \sum^{\infty} \left[\frac{2}{} \left(1 + (-1)^{n+1}\right) \sin(nx) \right)$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \left(1 + (-1)^{n+1}\right) \sin(nx) \right)$$

2.4 Odd and Even Functions

Definition 2.8. *A function $f(x)$ is said to be an* **odd** *(or skew-symmetric) function if $f(-x) = -f(x)$ for each x in the domain of f.*

Eg:- $\sin x$, x
symmetric on x axis

If f is odd, the area under the curve,

$$\boxed{\int_{-\pi}^{\pi} f(x)\, dx = 0.}$$

Definition 2.9. *A function $f(x)$ is said to be an* **even** *(or symmetric) function if $f(-x) = f(x)$ for each x in the domain of f.*

Eg:- $\cos x$, x^2
symmetric on y axis

If f is even, the area under the curve,

$$\boxed{\int_{-\pi}^{\pi} f(x)\, dx = 2\int_{0}^{\pi} f(x)\, dx.}$$

Remark 2.10.
- *The graph of an odd function is symmetrical about the origin, whereas the graph of an even function is symmetrical about the y-axis.*
- *A product of two even or two odd functions is even.*
- *A product of even and odd functions is odd.*

2.4.1 Expansion of an odd function:

For an odd function $f(x)$ defined over the range $-\pi$ to π, we find that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0.$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function. Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)}\,\overset{\text{even}}{\cos nx}}_{\text{odd}}\, dx = 0.$$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an even function. Therefore, coefficients b_n are given by:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)}\,\overset{\text{odd}}{\sin nx}}_{\text{even}}\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx.$$

Example 2.11. *Suppose $f(x) = x$ if $-\pi < x < \pi$ and $f(x) = f(x + 2\pi)$. Find the Fourier series expansion for $f(x)$ in the interval $-\pi < x < \pi$.*

2.4.2 Expansion of an even function:

For an even function $f(x)$ defined over the range $-\pi$ to π, $f(x)\sin nx$ is an odd function. Thus we have for each $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = 0.$$

Since $f(x)$ is an even function,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ dx.$$

Since $\cos nx$ is even, $f(x)\cos nx$ is also an even function. Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\ dx.$$

Example 2.12. *Find the Fourier series expansion for $f(x) = x^2$ in the interval $-\pi < x < \pi$. Here $f(x) = f(x + 2\pi)$.*

Exercise 2.13. *Obtain Fourier series for the function $f(x)$ given by*

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{if } -\pi < x < 0\ , \\ 1 - \frac{2x}{\pi} & \text{if } 0 < x < \pi. \end{cases}$$

and $f(x) = f(x + 2\pi)$. Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

2.5 Half-range Series

If a function is defined over half the range, say 0 to π, instead of the full range from $-\pi$ to π, it may be expanded in a series of sine terms only or cosine terms only. The produced series is called a **half-range Fourier series**.

Conversely, the Fourier Series of an even or odd function can be analyzed using the half-range definition.

To get the series with only cosines terms, we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$. Then,

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ dx, \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\ dx, \quad \text{and} \quad b_n = 0.$$

To expand $f(x)$ as a sine series, we extend the function to the interval $(-\pi, \pi)$ as an odd function. Then,

$$b_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\ dx, \quad a_0 = 0, \quad \text{and} \quad a_n = 0.$$

Example 2.14. *Represent the following function by a Fourier sine series.*

$$f(x) = \begin{cases} x & \text{if } 0 < x < \dfrac{\pi}{2}\,, \\ \frac{\pi}{2} & \text{if } \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Example 2.15. *Find the Fourier cosine series for the function.*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \dfrac{\pi}{2}\,, \\ -x & \text{if } \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Exercise 2.16. *Expand $f(x)$ as the Fourier series of sine terms.*

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{if } 0 < x < \dfrac{\pi}{2}\,, \\ x - \frac{3}{4} & \text{if } \dfrac{\pi}{2} < x < \pi. \end{cases}$$

Exercise 2.17. *Expand $f(x) = \sin x$ in a Fourier cosine series in $0 < x < \pi$. Graph the Fourier series of the even periodic extension of $f(x)$.*

2.6 Fourier Series of General Period

So far, we only considered the functions whose period is 2π. But in many engineering problems, the period of the functions required to be expanded is not 2π but some other interval. In practice, it is often necessary to find a Fourier series of $f(x)$ defined over the interval $-l$ to l or 0 to $2l$. The Fourier series of $f(x)$ defined on $(-l, l)$ with period of $2l$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right). \tag{2.3}$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx,$$
$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx,$$
$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

Example 2.18. *Expand $f(x) = |x|$ as a Fourier series in the interval (-2,2).*

Exercise 2.19. *Determine the Fourier series of the waveform shown in Figures 2.1, 2.2 and 2.3.*

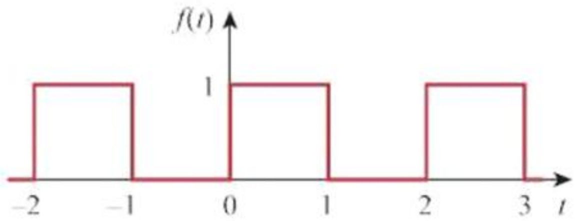


Figure 2.1

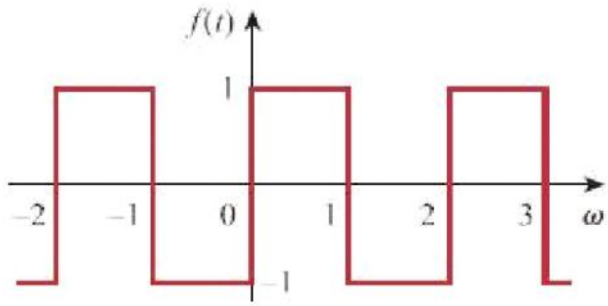


Figure 2.2

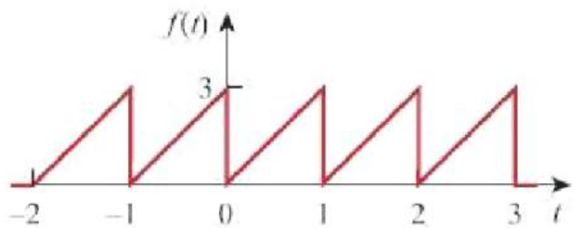


Figure 2.3

2.7 Parseval's Formula

Suppose $c > 0$ and $\int_{-c}^{c} |f(x)|\, dx < \infty$. Then,

$$\int_{-c}^{c} [f(x)]^2\, dx = c\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right). \tag{2.4}$$

The above formula is called **Parseval's Formula**.

If $f(x)$ is a half range cosine series in $0 < x < c$ then

$$\int_0^c [f(x)]^2 \, dx = \frac{c}{2}\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right).$$

If $f(x)$ is a half range sine series in $0 < x < c$ then

$$\int_0^c [f(x)]^2 \, dx = \frac{c}{2}\left(\sum_{n=1}^{\infty} b_n^2\right).$$

Example 2.20. *From the Fourier series expansion of $f(x) = x^2$ in $-\pi < x < \pi$, prove that*

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2.8 Complex Form of Fourier Series

The Fourier series of a periodic function $f(x)$ of period 2π, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

We know that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. Therefore, we can express $f(x)$ as

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{(inx)} + c_{-n} e^{(-inx)}\right),$$

where,

$$c_0 = \frac{a_0}{2}, c_n = \frac{a_n - ib_n}{2} \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2}$$

Further, the series can be compactly written as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx},$$

where,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Example 2.21. *Find the complex form of the Fourier series of*

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

Fourier Series: Exercises

1. Let m and n be two integers. Prove the following statements.
   - $\int_0^{2\pi} \sin nx \ dx = 0$ if $n \neq 0$.
   - $\int_0^{2\pi} \cos nx \ dx = 0$ if $n \neq 0$.
   - $\int_0^{2\pi} \sin nx \sin mx \ dx = 0$ if $n \neq m$.
   - $\int_0^{2\pi} \sin^2 nx \ dx = \pi$ if $n \neq 0$.
   - $\int_0^{2\pi} \cos nx \cos mx \ dx = 0$ if $n \neq m$.
   - $\int_0^{2\pi} \cos^2 nx \ dx = \pi$ if $n \neq 0$.
   - $\int_0^{2\pi} \sin nx \cos mx \ dx = 0$ if $n, m \neq 0$.

2. Suppose $f(x)$ is a 2π-periodic function. The Fourier series for the function $f(x)$ in the interval $-\pi \leq x \leq \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

   Show that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx,$$

   and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx.$$

3. State Dirichlet theorem.

4. Consider the following function.

$$f(x) = \begin{cases} \pi & \text{if } -\pi < x < 0\,, \\ x & \text{if } 0 < x < \pi. \end{cases}$$

   (a) Find the Fourier series of the function $f(x)$.

   (b) Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \frac{1}{(2n-1)^2} + \dots$.

5. Consider the function $g(x) = \cos x$ where $0 < x < \pi$.

   (a) Expand $g(x)$ in a Fourier sine series.

   (b) Expand $g(x)$ in a Fourier cosine series.

6. Consider the function $h(x) = 2x$ where $0 < x < 4$.

   (a) Find the Fourier sine series of $h(x)$.

   (b) Find the Fourier cosine series of $h(x)$.

7. Find the complex form of the Fourier series of

   (a) $f(x) = e^{2x}$ where $-\pi < x < \pi$.

   (b) $f(x) = \cos x$ where $-\pi < x < \pi$.