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Tuesday, August 12, 2025 2:07 PM

MA2014
Chapter 4...

Chapter 4

Series Solution of Differential Equations and Special Functions

4.1 Introduction

Many physical problems give rise to differential equations that are linear but have variable coefficients. Unfortunately, such equations do not permit a general solution in terms of known functions. Although numerical methods can be used to solve them, it is often more convenient to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations leads to special functions such as **Bessel's function, Legendre's polynomial, Laguerre's polynomial, Hermite's polynomial, and Chebyshev polynomials**. These special functions have numerous engineering applications.

Consider the following second-order ordinary differential equation with variable coefficients.

$$P_0(x)\,\frac{d^2y}{dx^2} + P_1(x)\,\frac{dy}{dx} + P_2(x)\,y = 0 \tag{4.1}$$

where $P_0(x)$,$P_1(x)$, and $P_2(x)$ are polynomials of x.

where $P_0(x)$, $P_1(x)$, and $P_2(x)$ are polynomials of x.

Definition 4.1. 1. *If $P_0(a) \neq 0$ then $x = a$ is called an* ***ordinary point*** *of (4.1), otherwise a is a* ***singular point****.*

2. *A singular point $x = a$ of (4.1) is called a* ***regular singular point****, if (4.1) is put in the form*

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{(x-a)}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

where $Q_1(x)$ and $Q_2(x)$ process derivatives of all orders in a neighborhood of a.

3. *A singular point that is not regular is called an* ***irregular singular point****.*

Example 4.2. *Identify singular points of*

$$x^2(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0.$$

Determine whether they are regular or irregular singular points.

4.2 Series Solutions

The general solution of a linear differential equation of second order will consist of two series, say y_1 and y_2. Then the general solution will be $y = ay_1 + by_2$ where a and b are arbitrary constants.

4.2.1 Solution about Ordinary Points

The solution for the equation (4.1) when $x = 0$ can be determined as follows.

Step 1: Assume its solution to be of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{4.2}$$

Step 2: Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (4.2) and substitute the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in (4.1).

Step 3: Equate to zero the coefficients of the various powers of x and determine a_2, a_3, $a_4 \cdots$ in terms of a_0 and a_1. The result obtained by equating the coefficient of x^n to zero

Step 3: Equate to zero the coefficients of the various powers of x and determine a_2, a_3, $a_4 \cdots$ in terms of a_0 and a_1. The result obtained by equating the coefficient of x^n to zero is called the **recurrence relation**.

Step 4: Substituting the values of a_2, a_3, $a_4 \cdots$ in (4.2), we get the desired series solution having a_0 and a_1 as its arbitrary constants.

Example 4.3. *Solve the following ODEs assuming they have a series solution.*

1. $\dfrac{d^2y}{dx^2} + x\ y = 0$
2. $y'' + xy' + y = 0$

Exercise 4.4. *Solve the following ODEs.*

1. $\dfrac{d^2y}{dx^2} - y = 0$
2. $\dfrac{d^2y}{dx^2} + x^2\ y = 0$

3. $(1 - x^2)\ \dfrac{d^2y}{dx^2} - 2x\ \dfrac{dy}{dx} + 2y = 0$

In the above example, we get found the solution about $x = 0$, which is still valid in the finite region around $x = 0$. We can also find out the series solution about a point other than $x = 0$, say about $x = a$. In this case, the validity of the series solution for equation (4.1) is determined by the following theorem.

Theorem 4.5. *If $x = a$ is an **ordinary point** of (4.1), its solution can be expressed in the form*

$$y = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - a)^n. \tag{4.3}$$

4.2.2 Solution about Singular Points

As we mentioned earlier, there are two types of singular points which are regular singular points and irregular singular points.

Theorem 4.6. *If $x = a$ is an **regular singular point** of (4.1), its solution can be expressed in the form*

$$y = (x - a)^m \left[a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \ldots\right]$$

expressed in the form

$$
\begin{aligned}
y &= (x-a)^m \left[a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots\right] \\
&= (x-a)^m \sum_{n=0}^{\infty} a_n (x-a)^n.
\end{aligned} \tag{4.4}
$$

Remark 4.7. *The series (4.3) and (4.4) are convergent at every point within the interval of convergence centered at a.*

Remark 4.8. *The solution for the case of irregular singular points is beyond the scope of this course.*

If $x = 0$ is a regualr singular point then the solution for $P_0 \dfrac{d^2y}{dx^2} + P_1 \dfrac{dy}{dx} + P_2\, y = 0$ is obtained as follows. This method is called **Frobenius method**.

Step 1: Assume its solution to be of the form

$$
y = x^m \left(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots\right) = x^m \sum_{n=0}^{\infty} a_n x^n. \tag{4.5}
$$

Step 2: Calculate $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from (4.5) and substitute the values of y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ in (4.1).

Step 3: Equate to zero the coefficients of the lowest degree terms in x. It gives a quadratic equation of m known as the **indicial equation**.

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Step 4: Thus, we will get two values for m. The series solution of (4.5) will depend on the nature of the roots of the indicial equation.

The indicial equation of a second-order ordinary differential equation (having $x = 0$ as its regular point) gives two roots, which may be:

- distinct and not differing by an integer,
- distinct and differing by an integer,
- equal.

Case 1: When the roots $m_1,\ m_2$ are distinct and not differing by an integer, the complete solution is

$$
y = c_1 (y)_{m_1} + c_2 (y)_{m_2}
$$

where $c_1,\ c_2$ are arbitrary real numbers.

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

where $c_1,\ c_2$ are arbitrary real numbers.

Case 2: When the roots of the indicial equation are equal, the complete solution is

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m_1}$$

where $c_1,\ c_2$ are arbitrary real numbers.

Case 3: When the roots $m_1,\ m_2$ of the indicial equation are distinct and differing by an integer and if some of the co-efficient of y series become infinite when $m = m_1$ then we replace a_0 by $b_0(m - m_1)$. Then we obtain the complete solution as

$$y = c_1 \left(\frac{\partial y}{\partial m} \right)_{m_1} + c_2 (y)_{m_2}$$

where $c_1,\ c_2$ are arbitrary real numbers.

Case 4: When the roots $m_1,\ m_2$ are distinct and differing by an integer, making some coefficient indeterminate then the complete solution is

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

where $c_1,\ c_2$ are arbitrary real numbers. Here, the coefficients do not become infinity when $m = m_1$ or m_2.

Example 4.9. *Solve in series the equation*

$$9x\,(1-x)\,\frac{d^2y}{dx^2} - 12\,\frac{dy}{dx} + 4\,y = 0.$$

Example 4.10. *Solve in series the equation*

$$x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\,y = 0.$$

Example 4.11. *Solve in series the equation*

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + (x^2 - 4)\,y = 0.$$

Exercise 4.12. *Solve in series the equation*

$$x\,(1-x)\,\frac{d^2y}{dx^2} - (1+3x)\,\frac{dy}{dx} - y = 0.$$

Example 4.13. *Solve in series the equation*

Example 4.13. *Solve in series the equation*

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (x^2+2)\, y = 0.$$

4.3 Bessel's Equation

Definition 4.14. *The differential equation*

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - n^2\right) y = 0$$

is called the ***Bessel's equation****. The solution of this equation is called* ***Bessel's function*** *of order n.*

Exercise 4.15. *Solve the Bessel's equation.*