

Numerical Methods for Natural Sciences: Foundation Topics

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Course material: <https://github.com/raghurama123/nm2024>

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References

Textbook

- David G. Moursund, Charles S. Duris, *"Elementary Theory and Application of Numerical Analysis"*, Dover Publishers (1988).

General References

- Samuel D. Conte, Carl de Boor, *"Elementary Numerical Analysis: An Algorithmic Approach"*, McGraw-Hill (1981).
- Lars Elden, Linde Wittmeyer-Koch and Hans Bruun Nielsen, *"Introduction to Numerical Computing"*, Overseas Press (2006).
- Anthony J. Pettofrezzo, *"Introductory Numerical Analysis"*, Dover Publishers (1984).
- W. Boehm, H. Prautzsch, *"Numerical Methods"*, Universities Press (2003).
- Richard L. Burden, J. Douglas Faires, *"Numerical Analysis"*, Cengage Learning (2011).
- Ward Cheney, David Kincaid, *"Numerical Methods and Computing"*, Cengage Learning (2013).
- Anne Greenbaum, Timothy P. Chartier, *"Numerical Methods"*, Princeton University Press (2012).
- William Bo Rothwell, *"Linux for Developers"*, Pearson (2018). *See the chapters about GitHub.*
- Numpy and Scipy Documentation, <https://docs.scipy.org/doc/>

Chapter 1: Solutions of equations by fixed-point iteration

Some definitions

Program, algorithm, elementary operation

- A *program* is a set of algorithms along with statements for user interaction (*i.e.* input/output). An *algorithm* is a set of pre-defined operations to convert an input to an output.
- For a given problem, there may not be a unique way to write a program, or the algorithms involved, or even the elementary operations involved (that comprise the algorithms).

Types of Numerical Methods: Direct, Iterative, and Heuristic

- *Direct methods* are predefined recipes (*i.e.* fixed algorithms with fixed elementary steps) for solving a problem. In this case, the error in the final result is only due to finite computer *precision* (*i.e.* rounding-off the numbers involved).
 - The standard formula for finding the root of a quadratic equation is a direct method.
 - It is often the case that for a given problem, a direct method may not exist (either it has not been found, or it may not even exist mathematically). A theorem in algebra says that there is no direct-method for finding a root of a polynomial of degree ≥ 5 .
 - We will later see that Gaussian elimination is a direct method for solving systems of linear equations.
- *Iterative methods* require an initial guess for the final solution of a problem. This value will be given as an input to an algorithm which will be repeated until its input and output are the same.
 - Newton-Raphson method for finding the solution of non-linear equations is an iterative method.
 - Iterative methods can be shown (using a theorem) that a solution (if it exists) can be found as a limit.
- *Heuristic methods* are methods developed based on experience. These methods are not guaranteed to give a solution.
 - Simplex method (Nelder-Mead method) for optimization is a heuristic method.

Square root by fixed-point iteration

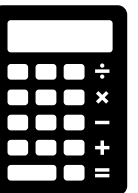
Square root of a real number

- The square root of a number A can be determined using the formula $g(x) = \frac{1}{2} \left(x + \frac{A}{x} \right)$, and using it successively. The procedure involves starting with an initial value x_0 and determining $g(x_0)$ using the formula. Now, one sets $g(x_0)$ as x_1 , and continue the process until $g(x_n)$ (the output to the formula) is the same as x_n (the input).

Example

- Find the square of 5 by applying fixed-point iteration. Let's begin with $x_0 = 1$.

n	x_n	$g(x_n) = (x_n + 5/x_n)/2$
0	1.0	3.0
1	3.0	2.333...
2	2.333...	2.238...
		...
		2.236068



- You can repeat these steps by a manual calculation with the help of a pocket-calculator, smart phone, or a computer.
- Now, you are ready to try these steps in Python. Try the notebook (*Chapter01_Fixed_Point_Iteration.ipynb*) provided in the course repository¹.

¹ <https://github.com/raghurama123/nm2024>

for loop

- You can refine the simple steps given in the notebook into a neat program as follows.

```
# Number, whose square root we want to find
A=5

# Maximum number of steps
MaxIter=4

# Start with a guess
xold=1

# Iterate
for n in range(MaxIter):
    xnew=g(xold,A)
    print(n,xold,xnew)
    xold=xnew
```

```
0 1 3.0
1 3.0 2.3333333333333335
2 2.3333333333333335 2.238095238095238
3 2.238095238095238 2.2360688956433634
```

Self-study

- Learn about Python's built-in function **range**, and **for** loops in Python.
- Learn what a Python module is. In the following, we are calling a procedure (**sqrt**) from a module (**numpy**). Learn about how to use an alias for a module or a procedure while importing in your code.

```
import numpy
print(numpy.sqrt(5))
```

```
2.23606797749979
```

Pretty print

- You should always use *formatted strings* to display any output.

```
# Number, whose square root we want to find
A=5

# Maximum number of steps
MaxIter=4

# Start with a guess
xold=1

# Iterate
for n in range(MaxIter):
    xnew=g(xold,A)
    fstr = "{:5d} {:15.8f} {:15.8f}".format(n, xold, xnew)
    print(fstr)
    xold=xnew
```

0	1.00000000	3.00000000
1	3.00000000	2.33333333
2	2.33333333	2.23809524
3	2.23809524	2.23606890

- How does this code differ from the one given in the previous page?
- You may also try the following two lines to print the output in the same format.

```
output = "{val1:5d} {val2:15.8f} {val3:15.8f}"
print(output.format(val1=n, val2=xold, val3=xnew))
```

- In this code, we have limited the number of iterations to a fixed number. Suppose we do not know how many iterations we will require. How will you modify this code by introducing a **while** loop?

Fixed-point iteration: Statement, Algorithm, and Theorem

Statement of the method

- The solution of $f(x) = 0$, can be determined by rewriting the equation in the form $x = g(x)$ and beginning with an initial value x_0 . Then, under certain conditions, the sequence $x_1 = g(x_0), x_2 = g(x_1), \dots$ will converge to one of the solutions of $x = g(x)$, which we will denote as $x^* = \lim_{n \rightarrow \infty} x_n$.

Algorithm

1. Define the function $g(x)$.
2. Initialize x to a real number, let's call it x_0 .
3. Generate the sequentially improved estimates for the root through the formula $x_{n+1} = g(x_n)$.
4. Stop when $|x_{n+1} - x_n|$ is below a threshold and return x_n as the solution x^* .

Theorem

- **Theorem 1:** The equation $x = g(x)$ has a solution for some value of x , which we call $x^* \in [a, b]$, provided $g(x) \in C([a, b])$ has its range contained in the same closed interval $[a, b]$.
 - The theorem does not tell us if we will have one solution (a unique solution) or the equation has multiple solutions.
- **Theorem 2:** If the same $g(x)$ (satisfying all conditions stated above) further satisfies $|g^{(1)}(x)| \leq k$ for some constant $0 \leq k < 1$ in the open interval (a, b) , then $x = g(x)$ has exactly one root in $[a, b]$.
 - The previous theorem stated that the domain of $g(x)$ is $[a, b]$ (and its range is also contained in this interval). Hence, if $x_0 \in [a, b]$, then $g[x_0] \in [a, b]$, and by induction $x_1, x_2, \dots, x^* \in [a, b]$.
 - In both the theorems, instead of the closed interval $[a, b]$, we can consider a symmetric interval $[x^* - \delta, x^* + \delta]$, where x^* is a solution of the $x = g(x)$, and $\delta > 0$.