

# The non-reciprocal Ising model

Yael Avni,<sup>1</sup> Michel Fruchart,<sup>2</sup> David Martin,<sup>3</sup> Daniel Seara,<sup>1</sup> and Vincenzo Vitelli<sup>1,4</sup>

<sup>1</sup>University of Chicago, James Franck Institute, 929 E 57th Street, Chicago, IL 60637

<sup>2</sup>Gulliver, ESPCI Paris, Université PSL, CNRS, 75005 Paris, France

<sup>3</sup>University of Chicago, Kadanoff Center for Theoretical Physics and Enrico Fermi Institute, 933 E 56th St, Chicago, IL 60637

<sup>4</sup>University of Chicago, Kadanoff Center for Theoretical Physics, 933 E 56th St, Chicago, IL 60637

Systems with non-reciprocal interactions generically display time-dependent states. These are routinely observed in finite systems, from neuroscience to active matter, in which globally ordered oscillations exist. However, the stability of these uniform non-reciprocal phases in noisy spatially-extended systems, their fate in the thermodynamic limit, and the criticality of the corresponding phase transitions are not fully understood. Here, we address these questions by introducing a non-reciprocal generalization of the Ising model and study its critical behavior by means of numerical and analytical approaches. While the mean-field equations predict three stable homogeneous phases (disordered, ordered and a time-dependent swap phase), our large scale numerical simulations reveal a more complex picture. Static order is destroyed in any finite dimension due to the growth of rare droplets. The swap phase is destroyed by fluctuations in two dimensions through the proliferation of discrete spiral defects, but stabilized in three dimensions where non-reciprocity changes the universality class from Ising to XY giving rise to robust spatially-distributed clocks.

Non-reciprocal interactions naturally arise in out-of-equilibrium systems [1–22] ranging from neuroscience [23–25] and social networks [26, 27] to ecology [28–30] and open quantum systems [31–33]. A generic feature of these systems is the emergence of many-body limit cycles – time-dependent states arising from non-mutual interactions between constituents. Such states are routinely observed in experiments and simulations of finite systems. A single limit cycle oscillator subject to noise eventually forgets the initial phase [34–41]: its temporal correlations decay with time very much like density fluctuations decay with spatial separation in a liquid (Fig. 1a, left). In many-body systems, however, coherent oscillations may be restored over arbitrarily long periods [42–61]: temporal correlations persist (without any periodic drive externally imposing a phase) very much like spatial correlations do in a crystal (Fig. 1a, right). In this Letter, we ask: Can many-body limit cycles survive fluctuations in spatially-extended, locally-coupled non-reciprocal systems of arbitrary size? If so, what are the critical exponents of the resulting phase transitions? We address these questions by introducing a non-reciprocal generalization of what is perhaps the most paradigmatic statistical mechanical system: the Ising model.

*Dynamics without global optimisation.*— In equilibrium, the dynamics of Ising spins is described by kinetic Ising models in which spins  $\sigma_n$  tend to minimize a global potential, and hence interact reciprocally [62, 63]. To account for non-reciprocal interactions, we instead assume that each spin  $n$  tends to minimize its own *selfish energy*  $E_n$ , and choose the probability of flipping  $\sigma_n$  at each time step according to the generalized Glauber rule

$$p(\sigma_n \rightarrow -\sigma_n) = \frac{1}{2} [1 - \tanh(\Delta E_n / (2k_B T))] \quad (1)$$

where  $\Delta E_n$  is the change in selfish energy between both configurations,  $T$  is the temperature, and  $k_B$  the Boltz-

mann constant. Even though we can still define different potential functions for each spin, the dynamics cannot be derived from a single potential function. This formulation encompasses several non-reciprocal dynamics used among the physical sciences [64–72].

*The non-reciprocal Ising model.*— We split the spins  $\sigma_n \equiv \sigma_i^\alpha$  into two species, labeled by Greek indices  $\alpha = A, B$ , located on sites  $i$  of a  $d$ -dimensional cubic lattice of linear size  $L$  (Fig. 1b). While spins of the same species tend to align with their neighbors, spins of different species interact in a non-reciprocal manner: spins  $A$  tend to align with spins  $B$ , while spins  $B$  tend to anti-align with spins  $A$ . This is captured by the selfish energy

$$E_i^\alpha = -J \sum_{j \text{ nn of } i} \sigma_i^\alpha \sigma_j^\alpha - K \varepsilon_{\alpha\beta} \sigma_i^\alpha \sigma_i^\beta, \quad (2)$$

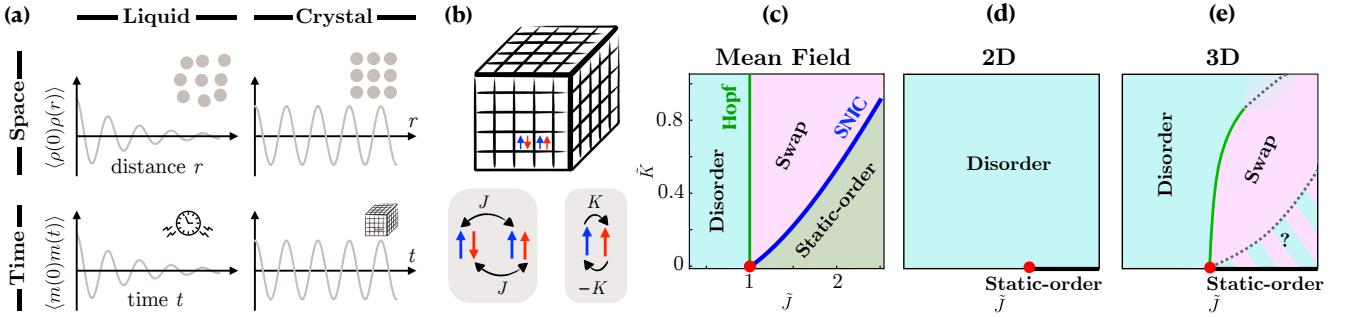
where the sum runs over nearest neighbors of  $i$ . Non-reciprocal interactions only occur on the same site ( $\varepsilon_{\alpha\beta}$  is the Levi-Civita symbol; reciprocal onsite interactions are considered in SM). As we show in the SM, detailed balance is broken whenever  $K \neq 0$ . While minimalistic, this model contains features arising in systems ranging from coupled micromechanical oscillators [66] to models of the human brain [64].

*Mean-field equation.*— Within the mean-field approximation and upon rescaling of time and space, the dynamics of the average magnetization  $m_\alpha(\vec{r}, t)$  is (SM)

$$\partial_t m_\alpha = -m_\alpha + \tanh \left[ \tilde{J} m_\alpha + \tilde{K} \varepsilon_{\alpha\beta} m_\beta + D \nabla^2 m_\alpha \right] \quad (3)$$

where  $D \equiv J/(k_B T)$ ,  $\tilde{K} \equiv K/(k_B T)$  and  $\tilde{J} \equiv 2dJ/(k_B T)$ .

The phase diagram in Fig. 1c shows the stable homogeneous solutions of Eq. (3) as a function of the couplings  $\tilde{J}$  and  $\tilde{K}$  (SM). When non-reciprocal interactions are turned off ( $\tilde{K} = 0$ ), a pitchfork bifurcation at  $\tilde{J}_c \equiv 1$  (red point in Fig. 1c) separates a disordered phase (in



**Fig. 1. Non-reciprocal Ising model.** (a) Comparison of noisy and robust clocks with different types of spatial order. (b) Schematic drawing of the non-reciprocal Ising model. The model includes two species per site with two interaction types: 1) reciprocal nearest-neighbors interaction of strength  $J$  between same-species spins and 2) non-reciprocal same-site interaction of strength  $K$  between spins of different species. (c) Mean-field phase diagram, (d) schematic 2D phase diagram and (e) schematic 3D phase diagram. The three phases shown on the diagrams are (i) disorder, (ii) swap and (iii) static order. The lines separating the phases are Hopf bifurcation (thin green), SNIC bifurcation (thick blue) and two yet undetermined transition lines (dashed grey). Red dot is the Pitchfork bifurcation.

blue) from a ferromagnetic phase (in green), as in the usual Ising model. When non-reciprocal interactions are present ( $K \neq 0$ ), a time-dependent oscillatory state that we dub “swap phase” arises (in pink). This limit cycle state where both species interchange their magnetizations repeatedly is separated from the disordered state by a Hopf bifurcation at  $\tilde{J}_c = 1$  (green line), and from the ferromagnetic phase by another bifurcation known as a saddle-node on an invariant circle (SNIC) bifurcation [73, 74] (blue line  $\tilde{K}(\tilde{J})$ ). The shape of the limit cycle in  $(m_A, m_B)$  space evolves from a circle near the Hopf bifurcation (as in the complex Ginzburg-Landau (CGL) equation [75]) to a square near the SNIC bifurcation (SM). The oscillation period remains finite at the Hopf bifurcation, and it diverges at the SNIC bifurcation [73].

*Monte-Carlo simulations.*— To go beyond the mean-field description, we perform large-scale Monte-Carlo simulations of the non-reciprocal Ising model in both two and three dimensions (see SM for details on the update rules; different rules do not change the results qualitatively). Unless mentioned otherwise, we initialize the system in an ordered state where all spins of the same species are either up or down, and let the system evolve until a steady-state is reached.

The qualitative results of our simulations are summarized in Fig. 1d-e. In 2D (Fig. 1d), any amount of non-reciprocity destroys all order. In 3D (Fig. 1e), our simulations suggest that the swap phase survives the thermodynamic limit, while the static ferromagnetic phase is eventually destroyed by any amount of non-reciprocity.

In order to distinguish the different possible phases, we introduce the synchronization order parameter [42]

$$R \equiv \langle r \rangle_{t,\Omega} \quad \text{with} \quad r \equiv \frac{1}{L^d} \left| \sum_j e^{i\theta_j} \right| = \sqrt{\frac{M_A^2 + M_B^2}{2}} \quad (4)$$

as well as the phase space angular momentum [76]

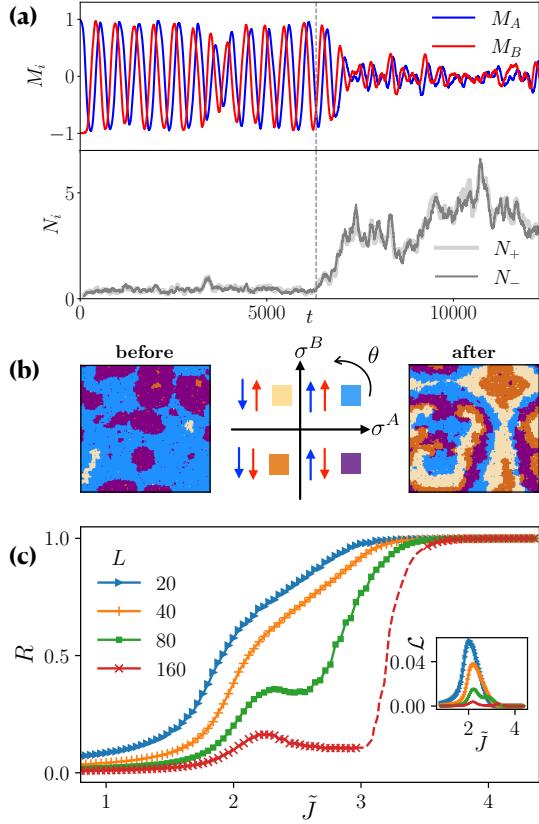
$$\mathcal{L} = \langle M_B \partial_t M_A - M_A \partial_t M_B \rangle_{t,\Omega} \quad (5)$$

where  $\theta_j \equiv \tan^{-1}(\sigma_j^A / \sigma_j^B)$  is the angle on the  $(\sigma_j^A, \sigma_j^B)$  plane (Fig. 2),  $M_\alpha = \sum_j \sigma_j^\alpha / L^d$  are the total magnetizations and the average  $\langle \dots \rangle_{t,\Omega}$  is over time and realizations. The quantity  $R$  is zero if the system is disordered and nonzero in both the swap and static ordered phases, while  $\mathcal{L}$  is zero in the disordered and static order phases and non-zero in the swap phase.

*Destruction of the swap phase in 2D by spiral defects.*— In small 2D systems, simulations show states in which  $M_A$  and  $M_B$  oscillate in time (like in the mean-field swap phase) with a large amplitude. However, oscillations are irregular, and typically occur through nucleation of droplets of opposite magnetization by the “unsatisfied” species, in alternating order (see Movie 1). As the size  $L$  increases, it becomes apparent that this oscillatory state is a transient, eventually destabilized by proliferation of spiral defects, similar to those observed in the CGL equation [44, 75, 77–85]. This is evidenced in Fig. 2a-b, which shows the emergence of defects in the angle  $\theta$  and the corresponding drop in the magnetizations  $M_A$  and  $M_B$  after an oscillatory transient (see also Movie 1). This behavior is reproduced in a noisy version of Eq. (3) (SM).

Quantitatively, the absence of a swap phase in the thermodynamic limit can be seen from the behavior of the order parameter  $R$  as a function of  $\tilde{J}$  and fixed  $\tilde{K}$  for different system sizes (Fig. 2c). In the intermediate region ( $1.5 \lesssim \tilde{J} \lesssim 3$ ) corresponding to the unstable swap phase (with nonzero  $\mathcal{L}$ , see inset), the order parameter  $R$  goes to zero as  $L$  increases, indicating the destruction of the swap phase. In addition, the putative critical  $\tilde{J}$  marking the transition from disorder to swap depends on the system size, signifying the absence of a well-defined phase transition in the thermodynamic limit.

*Existence of a stable swap phase in 3D and critical*



**Fig. 2. Destruction of the swap phase by spiral defects in 2D.** (a) The Magnetizations (top) and the number of + and - defects  $N_+$  and  $N_-$  averaged over 100 time steps (bottom) as a function of time, with simulation parameters:  $\tilde{J} = 2.8$ ,  $\tilde{K} = 0.3$ , and  $L = 150$ . We identify defects as adjacent 2 by 2 sites with a cycle of  $\uparrow\uparrow\rightarrow\downarrow\downarrow\rightarrow\downarrow\downarrow\rightarrow\uparrow\uparrow$ , either clockwise (+ defects) or anti-clockwise (- defects). The onset of defects, marked by a dashed grey line, reduces the magnetizations to zero. (b) Snapshots of  $\theta$ , before and after the proliferation of spiral defects. (c) Synchronization order parameter  $R$  as a function of  $\tilde{J}$  for  $\tilde{K} = 0.3$ , for different linear system size  $L$ . Phase space angular momentum  $\mathcal{L}$  is shown in the inset. Dashed red line shows upper bound for  $R$  for points that did not converge during simulation running time.

*exponents.*— In 3D, we find that a stable swap phase does exist. Its behavior is demonstrated in Fig. 3a where we show a kymograph of the  $\theta$ -field of a single 1D row in the 3D lattice as well as plot the total magnetizations and  $r$  as a function of time. The oscillations are spatially homogeneous and  $M_A(t)$  and  $M_B(t)$  have a fixed period and phase shift. The coherence time of individual spins is very long (inset of Fig. 3a; compare with Fig. 1a).

Simulations with varying system sizes reveal a well-defined phase transition between a disordered phase with  $R = 0$  to a swap phase with non-zero  $R$ . Figure. 3b shows a color map of  $R$  in the  $(\tilde{K}, \tilde{J})$  space, for  $L = 320$ , while Fig. 3c shows a cut of the color map for  $\tilde{K} = 0.3$ , for different  $L$ . Unlike in 2D (Fig. 2c), here there is a critical  $\tilde{J}$ , below which  $R \sim L^{-d/2}$  and above which it converges

to a non-zero value (Fig. 3d).

Figure. 3e shows that the phase-space trajectories become circular when approaching the phase transition line, supporting a CGL-like behavior. Renormalization group studies based on the  $\epsilon$ -expansion suggest that the CGL equation has the same universality class as the XY model [86, 87]. Using finite-size scaling (SM) we determine the critical exponents:  $\nu = 0.666 \pm 0.030$ ,  $\gamma = 1.312 \pm 0.059$ , and  $\beta = 0.343 \pm 0.015$ , which are in good agreement with 3D XY critical exponents [88, 89], more so than with 3D Ising critical exponents [88] corresponding to the  $\tilde{K} = 0$  case, see inset in Fig. 3c.

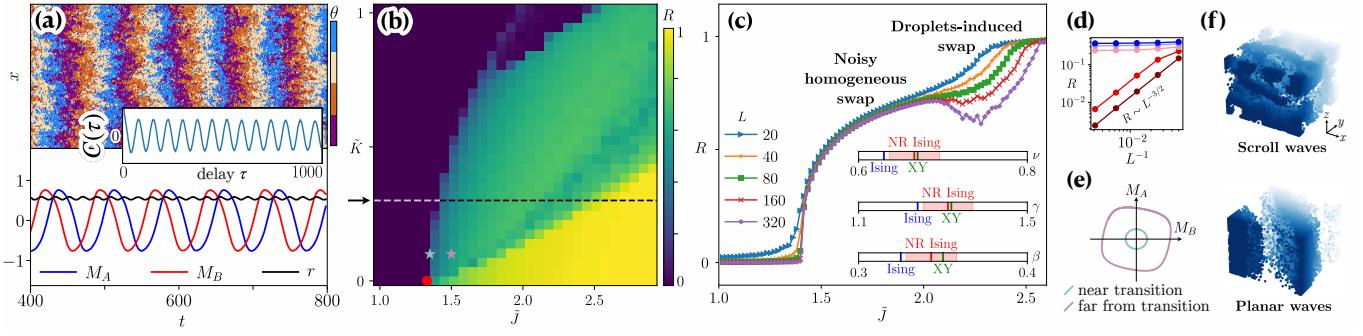
A phase transition similar to the one in Fig. 2c is observed for a fixed yet small  $\tilde{J}$  cut (SM):  $R$  is zero at large  $\tilde{K}$  and becomes finite below some critical  $\tilde{K}$ . This is in contrast with the mean-field prediction, in which the transition line from disordered to swap is independent of  $\tilde{K}$ . For a larger  $\tilde{J}$ , a different behavior is observed: the swap phase is abruptly destabilized by *scroll waves* (3D analog of spiral waves) [90–94] beyond some  $\tilde{K}(\tilde{J})$  threshold (SM). Within the size limitations of our simulations, it is unclear what part of the swap phase is destructed by scroll-waves in the thermodynamic limit and what type of phase transition is associated with their appearance (Fig. 1e).

Finally, we note that with random initial conditions, the system can coarsen into long-lived scroll waves and planar waves (Fig. 3f), even when ordered initial conditions would lead to global oscillations (SM and Movie 2).

*Destruction of static-order phase by droplet growth in any finite dimension.*— In finite systems, a ferromagnetic-like state with static order is observed both in 2D and 3D. Even in the usual Ising model, fluctuations can lead to transitions between the two equilibrium states. However, the characteristic time between transitions increases with system size. When non-reciprocity is turned on, droplet growth (see Fig. 4a) leads to a regime of droplet-induced oscillations where this characteristic time, which can be seen as an oscillation period, converges to a finite value as system size increases (Fig. 4b). We interpret this as the destruction of static order by non-reciprocity. This is further supported by Figs. 2c and 3c (right parts) showing that the transition point between static order and oscillations in finite systems (indicated by a sharp change in the slope) increases with size.

The regime of droplet-induced oscillations occurring at high  $\tilde{J}/\tilde{K}$  is unstable in 2D due to spirals (Fig. 2). In 3D, it exhibits large structures with out-of-phase regions, in contrast with the more homogeneous swap phase occurring at lower  $\tilde{J}$  (SM). Moreover, in this regime the degree of synchronization in 3D does not converge with system size (Fig. 3c). It is therefore unclear whether the resulting phase is disordered or oscillating, but the static order can be ruled out (Fig. 1e).

To see why non-reciprocity allows droplets to grow, in contrast with the equilibrium Ising model where they tend to shrink, assume first that the system is in the

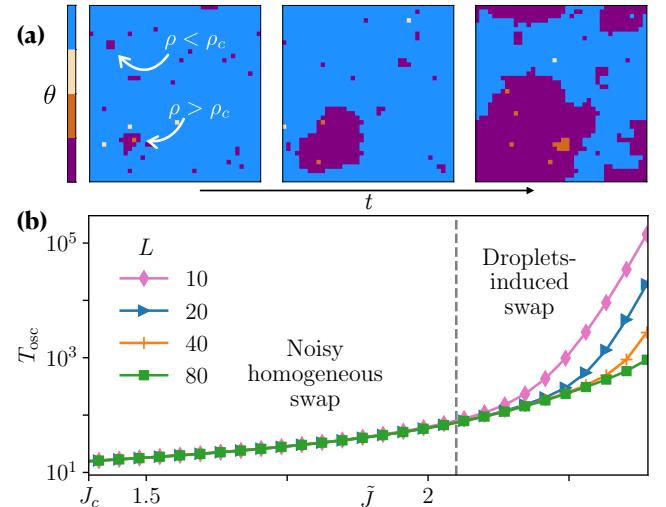


**Fig. 3. Stability of the swap phase in 3D and critical exponents.** (a) Time evolution of a stable swap phase with  $\tilde{J} = 1.5$ ,  $\tilde{K} = 0.1$  and  $L = 160$ . Top: a kymograph of  $\theta$  for a single 1D row in the 3D system. Bottom:  $M_A$ ,  $M_B$  and  $r$  as a function of time. Inset: single-spin autocorrelation function,  $C(\tau) \equiv \langle \sigma_i^A(t) \sigma_i^A(t + \tau) \rangle_t$ . (b) Colormap of  $R$  as a function of  $\tilde{J}$  and  $\tilde{K}$  for  $L = 320$ . Red dot is the pitchfork bifurcation. (c)  $R$  as a function of  $\tilde{J}$  for  $\tilde{K} = 0.3$  (dashed line in panel b) shown for different linear system size  $L$ . Inset: critical exponents of the 3D non-reciprocal Ising model (NR Ising) for  $\tilde{K} = 0.1$  extracted from finite-size scaling analysis along with their standard deviation represented by a semi-transparent red rectangular, and compared with the 3D Ising model and the 3D XY model. (d)  $M_B$  as a function of  $M_A$  for the points marked by stars in panel b. (e)  $R$  vs.  $L^{-1}$  for  $\tilde{K} = 0.3$  and  $\tilde{J} = 1.38, 1.40, 1.42, 1.43, 1.45$  from bottom to top on a log-log scale. (f) Simulation snapshots of the 3D system showing scroll waves and planar waves. Sites in  $\uparrow$  state are shown in blue while other sites are not shown. System parameters:  $\tilde{J} = 2.28$ ,  $\tilde{K} = 0.3$ ,  $L = 80$ .

static-order phase, so most spins in both lattices are up [95]. Since the system is static, it can be mapped into two equilibrium Ising models with opposite magnetic fields, with energy  $E = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i$  where  $H \approx K$  for A-spins and  $H \approx -K$  for B-spins. While sub-system A is in a global minimum in the effective equilibrium system, sub-system B is in a metastable state, as it prefers a state with opposite magnetization. Sub-system B then transitions to its global minimum by nucleating droplets larger than a critical size  $\rho_c$  above which the magnetic field overcomes surface tension and makes the droplet grow [95, 96] (Fig. 4a). After a droplet has expanded beyond  $\rho_c$ , the stable sub-system becomes metastable and nucleates droplets, and so on.

Why is the 3D swap phase not destroyed by a similar proliferation of droplets? This can be traced to the fact that the swap phase spontaneously breaks a continuous symmetry (continuous time translation invariance), while the static-order phase spontaneously breaks a discrete symmetry (up/down). As a consequence, the swap phase does not support well-defined droplet excitations that could destroy the order: the domain walls are progressively blurred as time evolves (SM and Refs. [44, 57]). Indeed, Fig. 4b shows that the period of oscillation depends continuously on parameters and is therefore in general incommensurate with the Monte-Carlo update time, excluding that only a discrete-time symmetry is broken.

We have shown that a non-reciprocal generalization of the 3D Ising model can act as a stable spatially-distributed clock characterized by well-defined critical properties reminiscent of time-crystals [90, 97–103] and biological criticality [104, 105].



**Fig. 4. Destruction of static order by droplet growth.** (a) Snapshots of the  $\theta$  field in a 2D system 20 (left) 100 (middle) and 150 (right) MC sweeps after initialization in an ordered state. The largest droplet at  $t = 20$  is shown to expand in later times and take over the entire system, while the second largest droplet, which did not exceed the critical droplet size,  $\rho_c$ , shrunk and disappeared in later times. System parameters:  $\tilde{J} = 2.8$ ,  $\tilde{K} = 0.3$ ,  $L = 40$ . (b) Oscillation period of the total magnetizations  $T_{\text{osc}}$  in the 3D system as a function of  $\tilde{J}$  for fixed  $\tilde{K} = 0.3$ .  $T_{\text{osc}}$  is measured as twice the average time between two subsequent sign flips of  $M_A$ . The period is finite at the critical point  $\tilde{J}_c$ , supporting a Hopf-like behavior.

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## I. SUPPLEMENTAL MATERIAL

### A. The non-reciprocal Ising model

In this section, we introduce the non-reciprocal (NR) Ising model in detail. The state of the system is described by a vector

$$\vec{\sigma} = \{\sigma_1^A, \dots, \sigma_{L^d}^A, \sigma_1^B, \dots, \sigma_{L^d}^B\} \quad (6)$$

in which each spin  $\sigma_i^\alpha = \pm 1$ . Greek indices  $\alpha = A, B$  label species, while Latin indices  $i = 1, L^d$  label the site.

We consider transition rates given by the generalized Glauber dynamics (Eq. (1) of the main text)

$$w(F_i^\alpha \vec{\sigma} | \vec{\sigma}) = \frac{1}{2\tau} \left( 1 - \tanh \left( \frac{\Delta E_i^\alpha}{2k_B T} \right) \right) \quad (7)$$

where  $\tau$  is the Monte-Carlo-step time-scale and

$$\Delta E_i^\alpha = E_i^\alpha(F_i^\alpha \vec{\sigma}) - E_i^\alpha(\vec{\sigma}) \quad (8)$$

is the difference in selfish energy between the states. Here  $F_i^\alpha$  is a single-spin-flipping operator, *i.e.*, when acted on  $\vec{\sigma}$  it flips the  $\alpha$ -species spin on site  $i$  from  $\sigma_i^\alpha$  to  $-\sigma_i^\alpha$  while keeping all the other spins unchanged, and  $w(\vec{\sigma} | \vec{\sigma}')$  is the transition rate from configuration  $\vec{\sigma}$  to  $\vec{\sigma}'$ .

In the discrete-time Monte-Carlo simulations, we consider transition probabilities  $p(F_i^\alpha \vec{\sigma} | \vec{\sigma}) = \tau w(F_i^\alpha \vec{\sigma} | \vec{\sigma})$ . Depending on the update algorithm, sites may be updated at the same time or not, see section I G for details.

In the continuous-time limit, the probability  $P(\vec{\sigma}, t)$  to observe a specific configuration of spins  $\vec{\sigma}$  at time  $t$

evolves according to the master equation

$$\begin{aligned} \partial_t P(\vec{\sigma}, t) &= \sum_{i,\alpha} -w(F_i^\alpha \vec{\sigma} | \vec{\sigma}) P(\vec{\sigma}, t) \\ &\quad + \sum_{i,\alpha} w(\vec{\sigma} | F_i^\alpha \vec{\sigma}) P(F_i^\alpha \vec{\sigma}, t). \end{aligned} \quad (9)$$

### B. Broken detailed balance

A Markov chain with rates  $w$  is said to satisfy detailed balance if it has a stationary distribution  $p_s$  such that

$$\frac{w(\vec{\sigma} \rightarrow \vec{\sigma}')}{w(\vec{\sigma}' \rightarrow \vec{\sigma})} = \frac{p_s(\vec{\sigma}')}{p_s(\vec{\sigma})}. \quad (10)$$

Let us assume that our kinetic model has an (unknown) stationary distribution  $p_s(\vec{\sigma})$ . Consider a loop  $\vec{\sigma}_1 \rightarrow \vec{\sigma}_2 \rightarrow \dots \rightarrow \vec{\sigma}_n$  in state space, as well as the same loop travelled in opposite direction. Then, detailed balance implies

$$\frac{w(\vec{\sigma}_1 \rightarrow \vec{\sigma}_2)}{w(\vec{\sigma}_2 \rightarrow \vec{\sigma}_1)} \cdot \frac{w(\vec{\sigma}_2 \rightarrow \vec{\sigma}_3)}{w(\vec{\sigma}_3 \rightarrow \vec{\sigma}_2)} \cdots \frac{w(\vec{\sigma}_n \rightarrow \vec{\sigma}_1)}{w(\vec{\sigma}_1 \rightarrow \vec{\sigma}_n)} = 1. \quad (11)$$

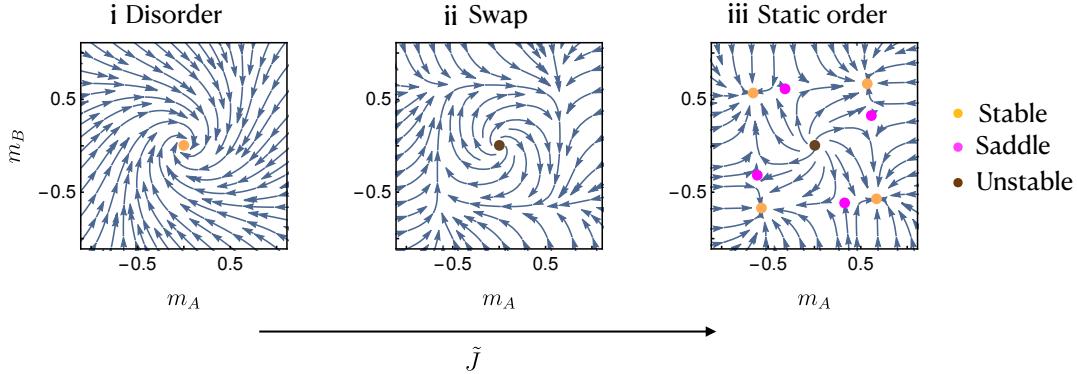
If we find a cycle such that this ratio is not unity, then detailed balance cannot be satisfied. Here, relevant cycles consist in flipping the spins  $(\sigma_i^A, \sigma_i^B)$  of a fixed site  $i$  according to  $\uparrow\uparrow \rightarrow \uparrow\downarrow \rightarrow \downarrow\downarrow \rightarrow \downarrow\uparrow \rightarrow \uparrow\uparrow$ , while keeping all the other spins unchanged. Using Eqs. (2) of the main text and (7) and assuming for simplicity that all the spins are up except for  $i$ -sites, we obtain

$$\begin{aligned} w(\uparrow\uparrow \rightarrow \uparrow\downarrow) &= f \left( \tilde{J} - \tilde{K} \right) \\ w(\uparrow\downarrow \rightarrow \downarrow\downarrow) &= f \left( \tilde{J} - \tilde{K} \right) \\ w(\downarrow\downarrow \rightarrow \downarrow\uparrow) &= f \left( -\tilde{J} - \tilde{K} \right) \\ w(\downarrow\uparrow \rightarrow \uparrow\uparrow) &= f \left( -\tilde{J} - \tilde{K} \right) \\ w(\uparrow\downarrow \rightarrow \uparrow\uparrow) &= f \left( -\tilde{J} + \tilde{K} \right) \\ w(\downarrow\downarrow \rightarrow \uparrow\downarrow) &= f \left( -\tilde{J} + \tilde{K} \right) \\ w(\downarrow\uparrow \rightarrow \downarrow\downarrow) &= f \left( \tilde{J} + \tilde{K} \right) \\ w(\uparrow\uparrow \rightarrow \downarrow\uparrow) &= f \left( \tilde{J} + \tilde{K} \right) \end{aligned} \quad (12)$$

with  $f \equiv (1 - \tanh x) / (2\tau)$ . Since  $(1 + \tanh(x)) / (1 - \tanh(x)) = e^{2x}$  we find that the forward and reverse cycles satisfy

$$\frac{w(\uparrow\uparrow \rightarrow \uparrow\downarrow) w(\uparrow\downarrow \rightarrow \downarrow\downarrow) w(\downarrow\downarrow \rightarrow \downarrow\uparrow) w(\downarrow\uparrow \rightarrow \uparrow\uparrow)}{w(\uparrow\downarrow \rightarrow \uparrow\uparrow) w(\downarrow\downarrow \rightarrow \uparrow\downarrow) w(\downarrow\uparrow \rightarrow \downarrow\downarrow) w(\uparrow\uparrow \rightarrow \downarrow\uparrow)} = e^{8\tilde{K}} \quad (13)$$

which means that detailed balance is indeed broken in our model whenever  $\tilde{K} \neq 0$ .



**Fig. 5. Mean-field flow in  $(m_A, m_B)$  space.** Arrows correspond to the vector  $\vec{v} = (\partial_t m_A, \partial_t m_B)$  calculated from Eq. (21). Reciprocal coupling between same-species nearest neighbours is (i)  $\tilde{J} = 0.8$ , (ii),  $\tilde{J} = 1.3$ , and (iii)  $\tilde{J} = 1.8$ , all with non-reciprocal coupling  $\tilde{K} = 0.4$ , representing the disorder, swap, and static-order phases, respectively.

### C. Derivation of the mean-field equation

The mean-field equation that corresponds to the non-reciprocal Ising model is obtained as follows. The average magnetization of a single spin at time  $t$  is

$$\langle \sigma_i^\alpha \rangle(t) = \sum_{\vec{\sigma}} \sigma_i^\alpha P(\vec{\sigma}, t). \quad (14)$$

Using Eq. (9) and changing summation variables we obtain the dynamics

$$\frac{d}{dt} \langle \sigma_i^\alpha \rangle = -2 \langle \sigma_i^\alpha w(F_i^\alpha \vec{\sigma} | \vec{\sigma}) \rangle. \quad (15)$$

By substituting the selfish energy (Eq. (2) of the main text) we obtain

$$\begin{aligned} \tau \frac{d}{dt} \langle \sigma_i^\alpha \rangle &= -\langle \sigma_i^\alpha \rangle \\ &+ \left\langle \tanh \left[ \frac{J}{k_B T} \sum_{j \text{ nn of } i} \sigma_j^\alpha + \frac{K}{k_B T} \varepsilon_{\alpha\beta} \sigma_i^\beta \right] \right\rangle \end{aligned} \quad (16)$$

where we used the identity  $x \tan(xa) = \tan(a)$  for  $x = \pm 1$ . Applying the mean-field approximation,  $\langle g(\vec{\sigma}) \rangle = g(\langle \vec{\sigma} \rangle)$  for any function  $g$ , and defining the average magnetization  $\langle \sigma_i^\alpha \rangle = m_i^\alpha$ , we obtain the equation,

$$\partial_t m_i^\alpha = -\frac{m_i^\alpha}{\tau} + \frac{1}{\tau} \tanh \left( \frac{J \sum_{j \text{ nn of } i} m_j^\alpha + K \varepsilon_{\alpha\beta} m_i^\beta}{k_B T} \right). \quad (17)$$

We now assume that the average magnetization changes over length-scales much larger than the lattice spacing, and write it as a continuum variable,  $m_i^\alpha = m_\alpha(\vec{r}, t)$ . We obtain the mean-field equation,

$$\partial_t m_\alpha = -\frac{m_\alpha}{\tau} + \frac{1}{\tau} \tanh \left( \frac{2d J m_\alpha + K \varepsilon_{\alpha\beta} m_\beta + J \ell^2 \nabla^2 m_\alpha}{k_B T} \right) \quad (18)$$

where  $\ell$  is the lattice spacing. Finally, upon rescaling time by  $\tau$  and space by  $\ell$  we obtain Eq. (3) of the main text.

In the  $m_\alpha \ll 1$  limit and for small gradients, the rescaled mean-field equation becomes

$$\begin{aligned} \partial_t m_\alpha &= -(1 - \tilde{J}) m_\alpha + \tilde{K} \varepsilon_{\alpha\beta} m_\beta + D \nabla^2 m_\alpha \\ &- \frac{1}{3} \left( \tilde{J} m_\alpha + \tilde{K} \varepsilon_{\alpha\beta} m_\beta \right)^3. \end{aligned} \quad (19)$$

For  $K = 0$ , Eq. (19) reduces to the standard mean-field equation associated with  $\phi^4$  Ginzburg-Landau theory. For any  $K \neq 0$  the mean-field equation can not be derived from a Hamiltonian dynamics nor a gradient descent, which corresponds to its far-from-equilibrium nature. Moreover, due to the lack of rotational symmetry in the cubic term of Eq. (19), our mean-field equation can not be mapped into the CGL equation [75, 106]

$$\partial_t \psi = a\psi + b\nabla^2 \psi + c|\psi|^2 \psi \quad (20)$$

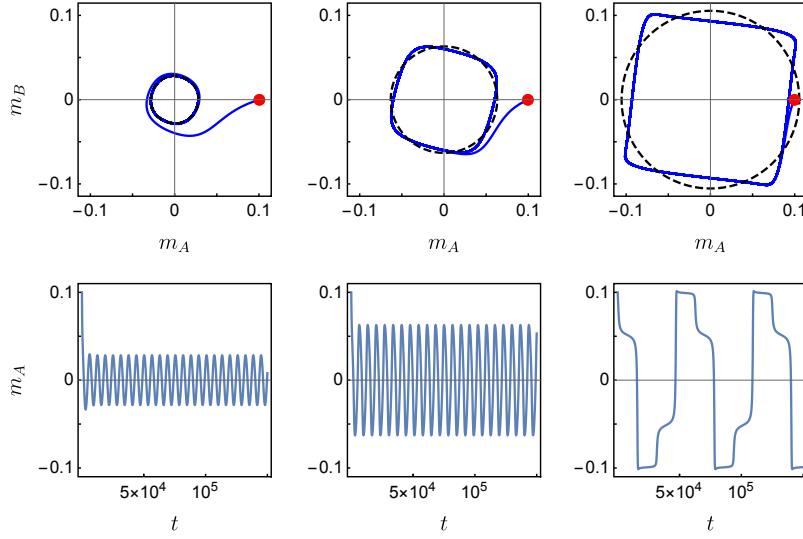
where  $\psi(\vec{r}, t)$  is a complex field and  $a, b$  and  $c$  are complex numbers. It can, however, become CGL-like in special limits as discussed in Sec. ID.

### D. Analysis of the bifurcations

A linear stability analysis shows that any homogeneous solution,  $m_\alpha(\vec{r}, t) \equiv m_\alpha(t)$ , that constitutes a stable fixed point of the spatially homogeneous equation (the  $D = 0$  case) remains stable in the inhomogeneous equation (with  $D > 0$ ). Therefore, in what follows we analyze the spatially homogeneous mean-field equation,

$$\begin{aligned} \partial_t m_\alpha &= -(1 - \tilde{J}) m_\alpha + \tilde{K} \varepsilon_{\alpha\beta} m_\beta \\ &- \frac{1}{3} \left( \tilde{J} m_\alpha + \tilde{K} \varepsilon_{\alpha\beta} m_\beta \right)^3. \end{aligned} \quad (21)$$

Figure 5 shows the phase space velocity field  $\vec{v} = (\partial_t m_A, \partial_t m_B)$ , according to Eq. (21), in the order, swap



**Fig. 6. Swap phase close and far from the Hopf bifurcation.** Numerical solution of the spatially homogeneous mean-field dynamics, Eq. (21). Upper line is  $m_B$  vs.  $m_A$  and lower line is  $m_A$  vs.  $t$ . Left:  $\tilde{J} = 1.0002$ , Center:  $\tilde{J} = 1.001$ , Right:  $\tilde{J} = 1.0028$ . In all plots  $\tilde{K} = 0.001$ . Red dot is the initial state and dashed black line is the steady state predicted from Eq. (26), valid in the  $\tilde{J} - 1 \ll 1$  limit.

and static order phases. The fixed points are obtained by numerically solving the  $\partial_t m_A = \partial_t m_B = 0$  equations. The stability of the fixed points shown in the figure is determined by the eigenvalues of the Jacobian matrix  $\mathcal{J}$  of the linearized equation  $\partial_t \delta \vec{m} = \mathcal{J} \delta \vec{m}$  where linearization is performed around the fixed point. This is also the analysis used to produce the mean-field phase diagram in Fig. 1c in the main text, but for the full non-linear homogeneous equation (Eq. (3) of the main text with  $D = 0$ ).

### 1. Pitchfork bifurcation

For  $\tilde{K} = 0$ , Eq. (21) leads to the standard pitchfork bifurcation at  $\tilde{J}_{PF} = 1$ . The point  $m_A = m_B = 0$  becomes unstable and four stable fixed points  $m_A = \pm[3(\tilde{J}-1)/\tilde{J}^3]^{1/2}$ ,  $m_B = \pm[3(\tilde{J}-1)/\tilde{J}^3]^{1/2}$  emerge, corresponding to the ordered phase.

### 2. Hopf bifurcation

For  $\tilde{K} \neq 0$ , the eigenvalues of  $\mathcal{J}$  at  $m_A = m_B = 0$  are complex,  $-1 + \tilde{J} \pm i\tilde{K}$ . This means that there is a Hopf bifurcation at  $\tilde{J}_H = 1$ : the  $m_A = m_B = 0$  point become unstable and a stable limit cycle emerges, corresponding to the swap phase.

To understand the limit cycle, it is convenient to use polar coordinates,  $m_A = r \cos \theta$  and  $m_B = r \sin \theta$ . Note that the radial coordinate  $r$  is different from the order parameter defined in Eq. (4) of the main text by a factor of  $\sqrt{2}$  and by the latter being global rather than local. The angular coordinate  $\theta$ , however, is the continuous

mean-field version of the  $\theta_i$  field defined in the main text. Equation. (21) becomes

$$\begin{aligned} \partial_t r &= -\left(1 - \tilde{J}\right)r - \frac{1}{4}\tilde{J}^3 \left[1 + \eta^2 + \left(\frac{1}{3} - \eta^2\right) \cos(4\theta) + \left(\eta - \frac{1}{3}\eta^3\right) \sin(4\theta)\right]r^3 \\ \partial_t \theta &= -\tilde{K} + \frac{1}{4}\tilde{J}^3 \left[\eta + \eta^3 + \left(\frac{\eta^3}{3} - \eta\right) \cos(4\theta) + \left(\frac{1}{3} - \eta^2\right) \sin(4\theta)\right]r^2 \end{aligned} \quad (22)$$

where we defined  $\eta \equiv \tilde{K}/\tilde{J}$ .

Unlike the CGL equation, here, the limit cycle leads to unharmonic oscillations, where both  $r$  and  $\theta$  depend on  $\theta$ . However, very close to the bifurcation, *i.e.*  $\tilde{J} - 1 \ll 1$ , the oscillations are harmonic. This can be seen as follows. The equation for  $\theta(t)$  in this limit is

$$\partial_t \theta = -\tilde{K} + \mathcal{O}(\tilde{J} - 1) \quad (23)$$

where we used the fact that  $r^2 \sim 1 - \tilde{J}$ .  $\theta$  then obeys the dynamics  $\theta = \theta_0 + \tilde{K}t$ , and the equation for  $r$  becomes

$$\begin{aligned} \partial_t r &= -\left(1 - \tilde{J}\right)r - \frac{1}{4}\tilde{J}^3 \left[1 + \eta^2 + \left(\frac{1}{3} - \eta^2\right) \cos(4\tilde{K}t) - \left(\eta - \frac{1}{3}\eta^3\right) \sin(4\tilde{K}t)\right]r^3 \end{aligned} \quad (24)$$

where we have set  $\theta_0$  to zero. Equation (24) can be solved analytically. In steady state ( $t \rightarrow \infty$  limit), the solution is

$$r(t) = 2\sqrt{\frac{\tilde{J}-1}{\tilde{J}^3(1+\eta^2)}} \times \frac{1}{\sqrt{1 + \frac{(\tilde{J}-1)[(3\eta^2-1+(1+3\eta^2-2\eta^4)\tilde{J})\cos(4\tilde{K}t)-\eta(\eta^2-3+(5\eta^2+1)\tilde{J})\sin(4\tilde{K}t)]}{3(1+\eta^2)[1-2\tilde{J}+(4\eta^2+1)\tilde{J}^2]}}}. \quad (25)$$

Taking the limit  $\tilde{J}-1 \ll 1$  once more we arrive at

$$r(t) = 2\sqrt{\frac{\tilde{J}-1}{\tilde{J}^3(1+\eta^2)}}. \quad (26)$$

which is a circular motion of constant angular velocity in phase space. This harmonic solution can be rationalized by looking at the two times scales that correspond to the angular and radial motion close to the bifurcation (described by Eqs. (23) and (24), respectively). The time scale for the angular motion is  $T_\theta \sim \tilde{K}^{-1}$ . The time scale of the radial motion, which determines the time it takes  $r(t)$  to relax to a new minimum when  $\theta$  changes, is found by linearizing the equation for  $r$  around 0 (since  $r$  is small). This yields  $T_r \sim (\tilde{J}-1)^{-1}$ . The two time scales,  $T_\theta$  and  $T_r$ , can be viewed as "oscillation time-scale" and "alignment time-scale", respectively. Since  $T_r \gg T_\theta$ , the angular motion is much faster and there is not enough time for  $r$  to reach a new minimum when  $\theta$  changes. Thus, only the  $\theta$ -averaged value of the right hand side of the  $r(t)$  equation matters. Our result is supported by Fig. 6 that shows typical phase-space trajectories in the swap phase, with different  $\tilde{J}-1$  values, from numerical solutions of the mean-field equation. It is seen that as  $\tilde{J}-1$  increases, the motion loses its rotational symmetry and becomes more "square-like".

We have shown that any  $\theta$  dependence in Eq. (22) can be averaged out due to slow  $r$  dynamics, compared to the  $\theta$  dynamics, near the Hopf bifurcation. Thus, in this limit Eq. (22) is akin to the CGL equation (Eq. 20), which in polar coordinates ( $\psi = re^{i\theta}$ ) and without spatial variation reads

$$\begin{aligned} \partial_t r &= a_R r + c_R r^3 \\ \partial_t \theta &= a_I + c_I r^2 \end{aligned} \quad (27)$$

where subscript "R" and "I" refer to real and imaginary parts, respectively. This supports the conjecture that the order-to-swap phase transition in our model should have the same universality class as the Hopf bifurcation in the CGL equation.

### 3. Infinite-period (SNIC) bifurcation

Starting from a point in the swap phase and increasing  $\tilde{J}$ , the system goes through an infinite period bifurcation, where four stable fixed points and four saddle fixed points emerge from the limit cycle (Fig. 5). In order to remain

in the limit of small  $r$  (which is a condition for deriving Eq. (21) we assume again  $\tilde{J}-1 \ll 1$ . As the infinite period bifurcation occurs when  $\tilde{K}$  and  $\tilde{J}-1$  are comparable (see Fig. 1c in the main text), we assume here that  $\tilde{K} \ll 1$ , as well. In this limit we obtain

$$\begin{aligned} \partial_t r &= (\tilde{J}-1)r - \frac{\tilde{J}^3}{4} \left(1 + \frac{1}{3}\cos(4\theta)\right)r^3 \\ \partial_t \theta &= -\tilde{K} + \frac{1}{12}\sin(4\theta)r^2. \end{aligned} \quad (28)$$

Let us assume that we are in a stationary state,  $\partial_t r = 0$  and  $\partial_t \theta = 0$ , with non-zero  $r$ . Then,

$$r = 2\sqrt{\frac{\tilde{J}-1}{\tilde{J}^3(1+\frac{1}{3}\cos(4\theta))}} \quad (29)$$

and the equation for  $\theta$  is

$$\partial_t \theta = -\tilde{K} + \frac{\sin(4\theta)}{3+\cos(4\theta)} (\tilde{J}-1). \quad (30)$$

Since  $\sin(4\theta)/(3+\cos(4\theta))$  is bounded between  $-1/(2\sqrt{2})$  and  $1/(2\sqrt{2})$ , a stationary state can only exist if  $\tilde{K} > (\tilde{J}-1)/(2\sqrt{2})$ . Therefore, the SNIC bifurcation line is

$$\tilde{K}_{\text{SNIC}} = \frac{\tilde{J}_{\text{SNIC}}-1}{2\sqrt{2}}. \quad (31)$$

This line coincides with thick blue line plotted in Fig. 1c in the main text, which was obtained numerically, in the  $\tilde{J}-1 \ll 1$  limit.

Finally we can obtain the stable points that emerge at the SNIC bifurcation from the angles that extremize  $\sin(4\theta)/(3+\cos(4\theta))$ . Those are

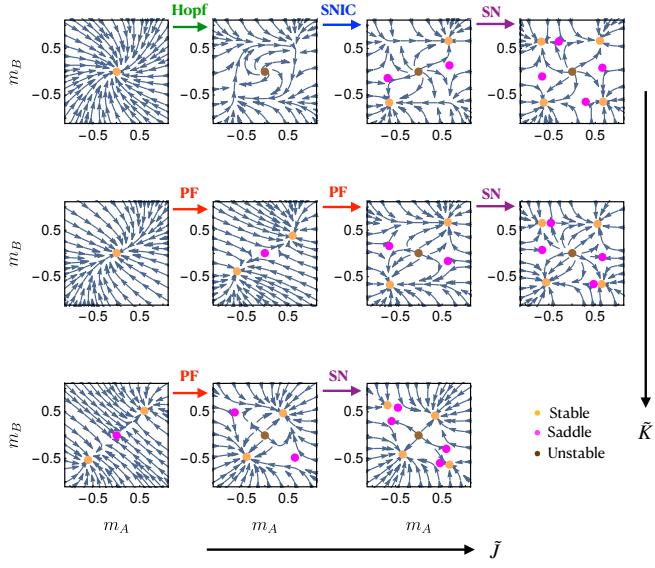
$$\theta_n = \left[ -\tan^{-1}(2\sqrt{2}) + \pi(2n+1) \right] / 4 \quad (32)$$

for  $n = 0, 1, 2, 3$ , and their corresponding  $r$  can be obtained from Eq. (29), leading to

$$(m_A, m_B) = 3 \left( \frac{\tilde{J}-1}{2\tilde{J}^3} \right)^{1/2} (\cos(\theta_n), \sin(\theta_n)). \quad (33)$$

### E. Reciprocal onsite interactions

Our minimal model assumes that the interactions between the two spin species are purely non-reciprocal (see



**Fig. 7. Bifurcations with both reciprocal and non reciprocal interactions between the two spin species.** Flow in  $(m_A, m_B)$  space is shown for  $\tilde{K}_- = 0.2$  and from top to bottom:  $\tilde{K}_+ = 0.1, 0.3$  and  $1.1$ . In each row,  $\tilde{J}$  is increased from left to right to capture the different phases. First row:  $\tilde{J} = 0.4, 1.2, 1.7$ . Second row:  $\tilde{J} = 0.4, 0.9, 1.4$ . Third row:  $\tilde{J} = 0.1, 2.3, 2.8$ . The arrows mark the bifurcations that connect every two configurations: Hopf, saddle-node on an invariant circle (SNIC), saddle-node (SN) or pitchfork (PF).

Eq. (2) of the main text). Here, we address the scenario where the inter-species interactions contain both reciprocal and non reciprocal components. We modify the energy by writing

$$E_i^\alpha = -J \sum_{j \text{ nn of } i} \sigma_i^\alpha \sigma_j^\alpha - K_{\alpha\beta} \sigma_i^\alpha \sigma_i^\beta \quad (34)$$

where

$$K_{\alpha\beta} = \begin{pmatrix} 0 & K_+ + K_- \\ K_+ - K_- & 0 \end{pmatrix}. \quad (35)$$

Note that by replacing  $K_+$  with zero and  $K_-$  with  $K$  we recover Eq. (2) of the main text.

Repeating the same analysis as before to derive the MF equation we obtain

$$\begin{aligned} \partial_t m_\alpha &= -(1 - \tilde{J})m_\alpha + \tilde{K}_{\alpha\beta}m_\beta + D\nabla^2 m_\alpha \\ &\quad - \frac{1}{3} (\tilde{J}m_\alpha + \tilde{K}_{\alpha\beta}m_\beta)^3, \end{aligned} \quad (36)$$

where  $K_{\alpha\beta} = \tilde{K}_{\alpha\beta}/k_B T$ .

The bifurcations described by Eq. (36) with  $D = 0$  upon changing  $\tilde{J}$  can have distinct behaviors, depending on the value of  $\tilde{K}_+$  and  $\tilde{K}_-$ . The eigenvalues  $\lambda_1$  and  $\lambda_2$  of the Jacobian matrix  $\mathcal{J}$  at  $m_A = m_B = 0$  are

$$\lambda_{1,2} = -1 + \tilde{J} \pm \sqrt{\tilde{K}_+^2 - \tilde{K}_-^2} \quad (37)$$

Thus, for  $\tilde{K}_+^2 - \tilde{K}_-^2 < 0$  ( $|\tilde{K}_+| < |\tilde{K}_-|$ ) the system is disordered ( $m_\alpha = 0$ ) for  $\tilde{J} < 1$ , and goes through a Hopf bifurcation at  $\tilde{J} = 1$  to a limit cycle. Note that unlike the  $K_+ = 0$  case studied throughout most of the paper, here the limit cycle is not rotationally-symmetric even in the  $\tilde{J} - 1 \ll 0$  limit. This is easily seen when writing the homogeneous MF equation in polar coordinate

$$\begin{aligned} \partial_t r &= - (1 - \tilde{J})r + \tilde{K}_+ \sin(2\theta)r + \mathcal{O}(r^3) \\ \partial_t \theta &= -\tilde{K}_- + \tilde{K}_+ \cos(2\theta) + \mathcal{O}(r^2) \end{aligned} \quad (38)$$

where the leading order terms in  $r$  are  $\theta$ -dependent.

Increasing  $\tilde{J}$  leads to addition bifurcations (first row in Fig. 7): first, a SNIC bifurcation where the limit cycle is destroyed by the emergence of two (rather than 4 for the  $K_+ = 0$  case) pairs of stable and saddle points, and then, a saddle-node bifurcation when two additional pairs of stable and saddle points emerge. We note that this behavior is observed for  $\tilde{K}_+$  and  $\tilde{K}_-$  much smaller than 1 (0.1 and 0.2 respectively in the Fig. 7). For larger values, a saddle-node bifurcation can occur such that the ferromagnetic phase coexists with the limit cycle (this scenario is explored in [69]).

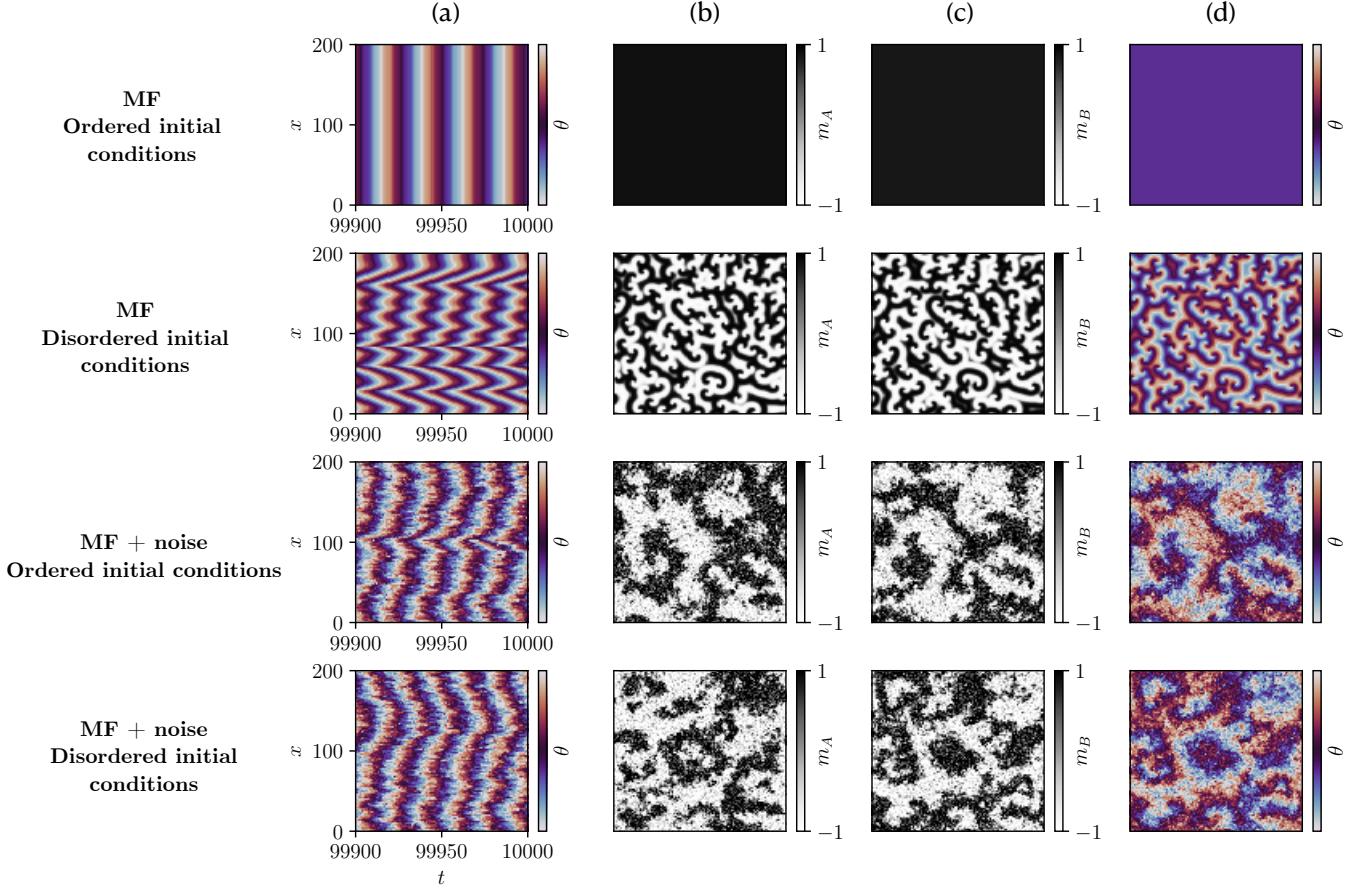
For  $0 < \tilde{K}_+^2 - \tilde{K}_-^2 < 1$  the system is disordered for  $\tilde{J} < 1 - \sqrt{\tilde{K}_+^2 - \tilde{K}_-^2}$  and goes through a pitchfork bifurcation with two stable fixed points at  $\tilde{J} = 1 - \sqrt{\tilde{K}_+^2 - \tilde{K}_-^2}$ , with no oscillatory phase (second row in Fig. 7). Finally, for  $\tilde{K}_+^2 - \tilde{K}_-^2 > 1$  there are two stable fixed points already at  $\tilde{J} = 0$  (second row in Fig. 7) and thus the only stable homogeneous phase is the static order phase (although  $\tilde{J}$  must be larger than zero for coarse-graining to take place). We note that the absence of oscillations for  $|\tilde{K}_+| > |\tilde{K}_-|$  is due to the fact that the two species, while having non-reciprocal interactions, “agree” with each other on whether they like to align or anti-align.

## F. Simulations of the mean-field equation in 2D

We simulate the mean-field equation on a grid (Eq. (17)) in 2D in the regime that corresponds to a stable swap phase. Results are shown in Fig. 8. Depending on initial conditions, the system can end up in different states. An initially ordered state leads to a stable swap phase where the  $\theta$  field changes periodically and homogeneously (Fig. 8, first row), while an initially randomly disordered state reaches a stable state with spiral defects, whose position barely changes over time (Fig. 8, second row).

When noise is added to the mean-field equation, the situation changes. We write the equation,

$$\partial_t m_i^\alpha = -\frac{m_i^\alpha}{\tau} + \frac{1}{\tau} \tanh \left( \frac{J \sum_{j \text{ nn of } i} m_j^\alpha + K \varepsilon_{\alpha\beta} m_i^\beta}{k_B T} \right) + \eta_i^\alpha. \quad (39)$$



**Fig. 8. The effect of added noise on the mean-field equation in 2D.** Simulations of Eq. (39) in 2D comparing the cases of no added noise with added noise and ordered vs. disordered initial conditions. (a) Kymographs of the  $\theta$  field for a single row, (b-d) snapshots of the  $m_A$ ,  $m_B$ , and  $\theta$  fields, respectively at  $t = 10000$  after initialization at  $t = 0$ . Mean-field without added noise results in a stable swap phase for ordered initial conditions and an inhomogeneous spiral phase for disordered initial conditions. With enough added noise, the swap phase is unstable even with ordered initial conditions and spiral defects are formed for both initialization types. System parameters are:  $\tilde{J} = 2$ ,  $\tilde{K} = 0.8$ ,  $\tau = 1$ . Simulation time step is  $\Delta t = 0.05$ , and system linear size  $L = 200$ . Noise amplitude is  $\zeta = 0.6$  for the case of added noise. At  $t = 0$ ,  $m_A, m_B = 1 \pm 0.1$  for ordered and  $m_A, m_B = \pm 0.1$  for disordered initial conditions, where the sign is chosen randomly for each lattice point and spin type.

where  $\eta_i^\alpha$  is white noise satisfying  $\langle \eta_i^\alpha \rangle = 0$  and  $\langle \eta_i^\alpha(t) \eta_j^\beta(t') \rangle = \zeta \delta_{\alpha\beta} \delta_{ij} \delta(t-t')$ . Above some critical  $\zeta$ , we get that even if the system is initialized in an ordered state, after a long time, spiral defect will form (see Fig. 8, third row). In that sense, the mean-field equation with noise exhibits the same qualitative behavior as the exact dynamics, where the swap phase is unstable in 2D.

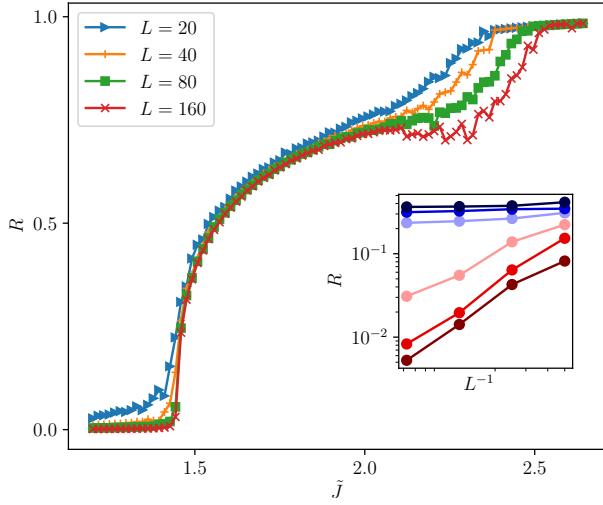
#### G. Details of the Monte-Carlo simulations

In our 2D simulations, we use the standard Glauber algorithm in which at each Monte Carlo (MC) step a spin  $\sigma_i^\alpha$  is chosen at random and then flipped with probability  $p_i^\alpha$ . Throughout the paper, time is measured in simulation sweeps, where a single sweep is defined as  $2 \times L^d$  (number of spins) MC steps.

Since the three dimensional simulations are much more

costly, we use a checker-board algorithm. In the checker-board algorithm non-interacting spins are flipped in parallel [107]. Here, this means that we consider whether  $x_1 + x_2 + \dots + x_d + \alpha$  is odd/even, in which  $x_n$  is the integer position in the  $n$ -th dimension of the hypercubic lattice, and  $\alpha = 0, 1$  labels the two species. This allows us both to parallelize the simulation and avoid random number generation (except for the acceptance probability). While different update rules may change the dynamics even at steady state, we verified they do not lead to qualitative changes. That way, the critical point  $\tilde{J}_c$ , for some fixed  $\tilde{K}$ , may be altered, but not the existence / destabilization of a phase transition. As an example, we show in Fig. 9 that the phase transition in 3D is observed with random sequential updates (standard Glauber algorithm), as it was observed with the checker-board algorithm which was used to produced Fig. 3c in the main text. We have also performed sim-

ulations with synchronous updates [108] (where all the spins are updated at the same time based on the previous configuration) and with Metropolis algorithm instead of Glauber and did not find qualitative differences related to phase transitions. For synchronous updates, fluctuations were considerably larger due to checkerboard patterns (in which each spin is anti-aligned with its same-species neighbours) appearing and disappearing on the lattice.



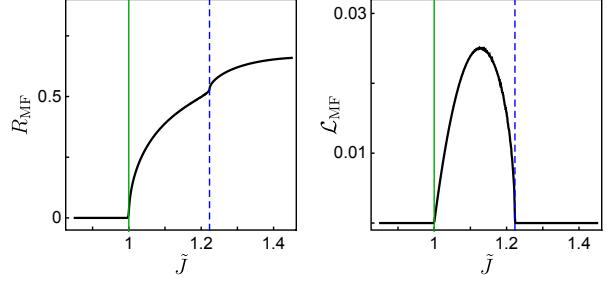
**Fig. 9. Qualitative agreement between two Monte-Carlo update rules.** Phase transition in 3D with random spin update rule instead of the checker-board update which is used in Fig. 3 in the main text. Result are shown for  $\tilde{K} = 0.3$  and are qualitatively the same as in Fig. 3 in the main text although exact  $\tilde{J}_c$  value varies.

## H. The behavior of the order parameters in the mean-field equation

Our analysis of possible phase transitions in the Monte-Carlo simulations is based on a calculation of the synchronization order parameter  $R$  and the phase space angular momentum  $\mathcal{L}$  defined in Eq. (4) and (5) of the main text. It is instructive to compare the behavior of  $R$  and  $\mathcal{L}$  as obtained from the mean-field dynamics to that of the Monte-Carlo simulations. In Fig. 10a,  $R$ , calculated from a numerical solution of Eq. (21), is plotted as a function of  $\tilde{J}$  for a fixed value of  $\tilde{K}$ . It is shown that  $R$  increases from zero to non-zero values when crossing the Hopf bifurcation at  $\tilde{J} = 1$ . When crossing the SNIC bifurcation, there is a sharp change in the slope: the slope increases when approaching the bifurcation and decreases in the static order regime. This change in slope is observed for finite-sized 2D and 3D systems (see Fig. 2c and Fig. 3c in the main text), although it is more pronounced in the 3D case.

Figure 10b shows  $\mathcal{L}$  as a function of  $\tilde{J}$  for a fixed value of  $\tilde{K}$ .  $\mathcal{L}$  is identically zero in the disordered and static-order phases, but deviates from zero in the swap phase.

This should be expected as the swap phase is the only phase which breaks time-reversal symmetry at the macroscopic scale.



**Fig. 10. Mean-field order parameters.** The synchronization order parameters  $R$  (left) and phase space angular momentum  $\mathcal{L}$  (right) as a function of  $\tilde{J}$  and for fixed  $\tilde{K} = 0.1$ , calculated from numerical solutions of Eq. (21) in steady state.

## I. Time correlations in the 0D noisy mean-field equation and the 3D non-reciprocal Ising model

Figure 11 compares the time autocorrelation function in the 0D noisy mean-field equation

$$\partial_t m_\alpha = -m_\alpha + \tanh\left(\tilde{J}m_\alpha + \tilde{K}\varepsilon_{\alpha\beta}m_\beta\right) + \eta_\alpha \quad (40)$$

and the Monte-Carlo simulations of the 3D non-reciprocal Ising model. The noisy mean-field has a decaying correlation function (with oscillations corresponding to the deterministic limit cycle), in contrast with the 3D non-reciprocal Ising model in which the memory of the initial phase persists for arbitrarily long times (when  $L \rightarrow \infty$ ), leading to oscillations with a constant amplitude in the correlation function.

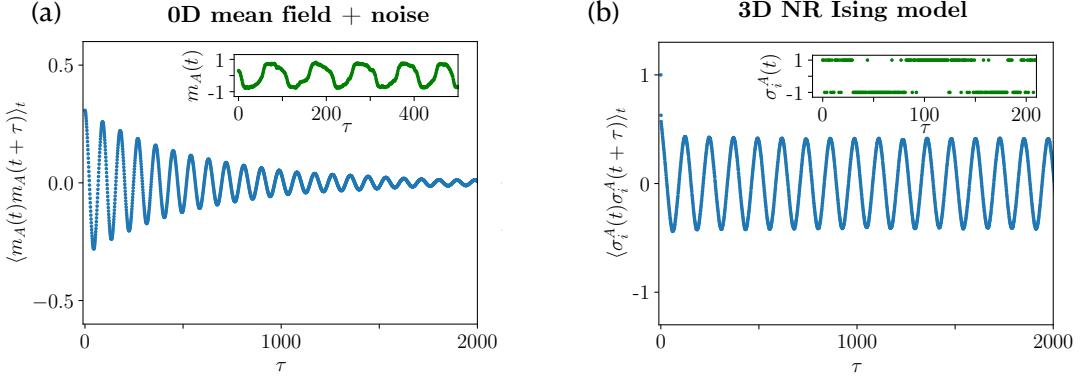
## J. Calculation of the critical exponents

According to the theory of finite-size scaling, in a finite system the order parameter  $R$  and the susceptibility  $\chi$ , defined as  $\chi = L^d (\langle r^2 \rangle - \langle r \rangle^2)$ , satisfy the following relations in the vicinity of the critical point  $\tilde{J}_c$  (assuming fixed  $\tilde{K}$ ) [109],

$$R = L^{-\beta/\nu} \hat{R}\left(L^{1/\nu}(\tilde{J} - \tilde{J}_c)\right) \quad (41)$$

$$\chi = L^{\gamma/\nu} \hat{\chi}\left(L^{1/\nu}(\tilde{J} - \tilde{J}_c)\right).$$

Here,  $\hat{\chi}(x)$  and  $\hat{R}(x)$  are called the scaling functions and  $\nu$ ,  $\gamma$  and  $\beta$  are the critical exponents  $\xi \sim |\tilde{J} - \tilde{J}_c|^{-\nu}$  where  $\xi$  is the correlation length,  $\chi \sim |\tilde{J} - \tilde{J}_c|^{-\gamma}$ , and



**Fig. 11.** Time correlations in the noisy mean-field equation and the 3D Ising model. (a) Auto-correlation function of  $m_A$  in the zero-dimensional MF equation with added noise (Eq. (39)). Parameters:  $\tilde{J} = 1.2$ ,  $\tilde{K} = 0.12$ ,  $\zeta = 0.245$ ,  $dt = 0.01$ . (b) Auto-correlation function of a single spin  $\sigma_i^A$  in the 3D non-reciprocal Ising model. Parameters:  $\tilde{J} = 1.6$ ,  $\tilde{K} = 0.1$ ,  $L = 80$ . Averages in (a) and (b) are taken over time for one realization shown in the insets.

$R \sim |\tilde{J} - \tilde{J}_c|^\beta$  (for  $\tilde{J} > \tilde{J}_c$ ). In what follows we assume that the finite-size scaling theory holds out of equilibrium, as is often done [110].

We first find the critical point  $\tilde{J}_c$  using the Binder Cumulant [111], defined as  $U_L = 1 - \langle r^4 \rangle / (3\langle r^2 \rangle^2)$ . Away from the critical point, the value of  $U_L$  depends on system size. However, at the critical point, in which correlation length exceeds the linear system size  $L$ ,  $U_L$  approaches a fixed value  $U^*$  independent of  $L$ . Hence, the critical point is found from the intersection point of all the  $U_L$  vs.  $\tilde{J}$  curves plotted for different system sizes. Figure 12a shows that indeed all the curves cross at a single point ( $\tilde{J}_c = 1.3461$  for  $\tilde{K} = 0.1$ ).

Next, we plot the susceptibility and  $R$  as a function of  $\tilde{J}$ , see Fig. 12b-c. For each system size  $L$ , we find the maximum of  $\chi$ ,  $\chi_{\max}(L)$ , the value of  $\tilde{J}$  at the maximum,  $\tilde{J}_{\max}(L)$ , and the value of  $R$  at the critical point  $\tilde{J}_c$ . We then use the finite-size scaling relations (which can be derived from Eq. (41) [109]),

$$\begin{aligned} \tilde{J}_{\max}(L) - \tilde{J}_c &= aL^{-1/\nu} \\ \chi_{\max}(L) &= bL^{\gamma/\nu} \\ R(L, \tilde{J} = \tilde{J}_c) &= cL^{-\beta/\nu} \end{aligned} \quad (42)$$

and perform linear fits by taking the logarithm of both equation sides. The procedure outlined here yields the critical exponents. Finally, we estimate the error using the bootstrap method [109, 112] by repeating the same procedure for multiple samples. Each sample is constructed by drawing samples randomly, one at a time, from our data sample, and returning them to the data sample after they have been chosen. The standard deviation of the critical exponents is estimated by the standard deviation calculated from 5000 bootstrap resamplings of the data. Our final result cited in the main text is  $\nu = 0.666 \pm 0.059$ ,  $\gamma = 1.312 \pm 0.030$ , and  $\beta = 0.343 \pm 0.015$ . This should be compared to the critical exponents of the 3d XY and 3d Ising model, which

are  $\nu_{XY} = 0.67$ ,  $\gamma_{XY} = 1.32$ ,  $\beta_{XY} = 0.35$  [88, 89], and  $\nu_I = 0.63$ ,  $\gamma_I = 1.24$ ,  $\beta_I = 0.32$  [88]. We note that in equilibrium, the critical exponents are related to each other by the hyperscaling relation

$$\gamma = d\nu - 2\beta. \quad (43)$$

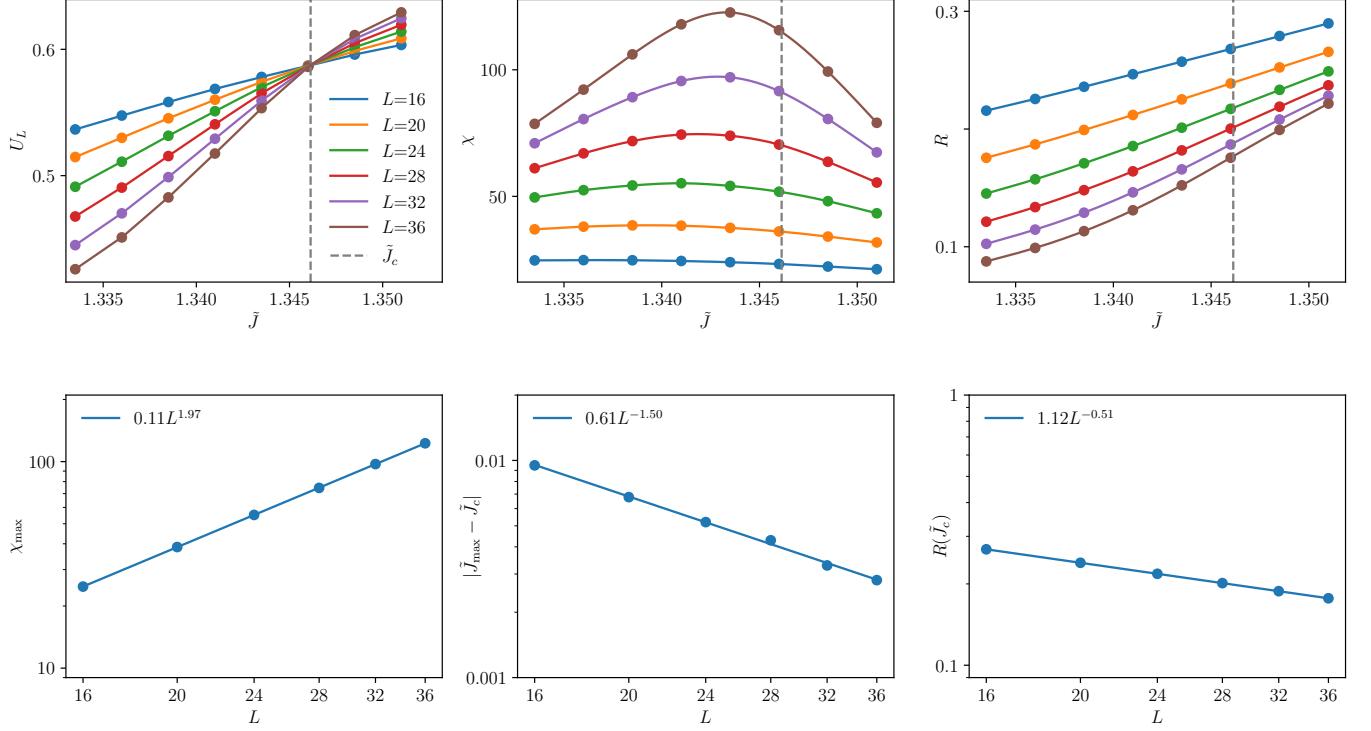
We have indeed obtained this relation, even though our system is not in equilibrium.

## K. Long-lived metastable states: scroll waves and planar waves

Most of our simulations were carried with ordered initial conditions as it takes the system less time to reach a steady state that way. In Fig. 13a we compared our findings for the 3D behavior shown in Fig. 3c in the main text with those obtained with disordered initial conditions. The averages in the plot are taken over a single realization and over a long time. In both the swap phase ( $1.4 \lesssim \tilde{J} \lesssim 2.3$ ) and static order phase ( $\tilde{J} \gtrsim 2.3$ ) we see that for some of the realizations the system reaches the same steady state as with ordered initialization (see Fig. 3c in the main text) with global oscillations / global static order. However, sometimes the measured  $R$  is much lower than that obtained with ordered initialization. This occurs because the system “gets stuck” in a meta-stable state that contains scroll waves or planar waves, shown in Fig. 13b and 13c.

The observed scroll waves are an extension of spiral waves to 3 dimensions having line defects instead of point defects [90], see Movie 2. Such waves have been observed in nature for example in Dictyostelium slugs [113] and are also known to emerge as a possible solution of the CGL equation [114].

The planar waves obtained in the non-reciprocal Ising model are analogous to the equilibrium Ising model in which the system often reaches stripes or minimal



**Fig. 12. Calculation of the critical exponents of the 3D non-reciprocal Ising model.** We set  $\tilde{K} = 0.1$ . (a-c) The Binder cumulant  $U_L$ , the susceptibility  $\chi$ , and the order parameter  $R$ , respectively, as a function of  $\tilde{J}$ , shown for the linear system sizes:  $L = 16, 20, 24, 28, 32, 36$ . Vertical dashed grey line is plotted at the critical point,  $\tilde{J}_c = 1.3461$ . Each point is calculated by averaging over  $10^6$  independent samples of  $r$ . (d-f)  $\chi_{\max}$ ,  $|\tilde{J}_{\max} - \tilde{J}_c|$  and  $R(\tilde{J}_c)$ , respectively, as a function of  $L$ , shown on a log-log scale. Dots are data extracted from analyzing (a-c) and full lines are best linear fits.

surfaces in the final stages of the coarsening dynamics [115, 116]. Here, however, the stripes moves in a particular direction that guarantees, for each site, the  $\uparrow\uparrow \rightarrow \downarrow\downarrow \rightarrow \downarrow\downarrow \rightarrow \downarrow\uparrow \rightarrow \uparrow\uparrow$  time-cycle. By that, the system exhibits broken time translation symmetry, broken spatial translational symmetry in one direction, and two broken rotational symmetries.

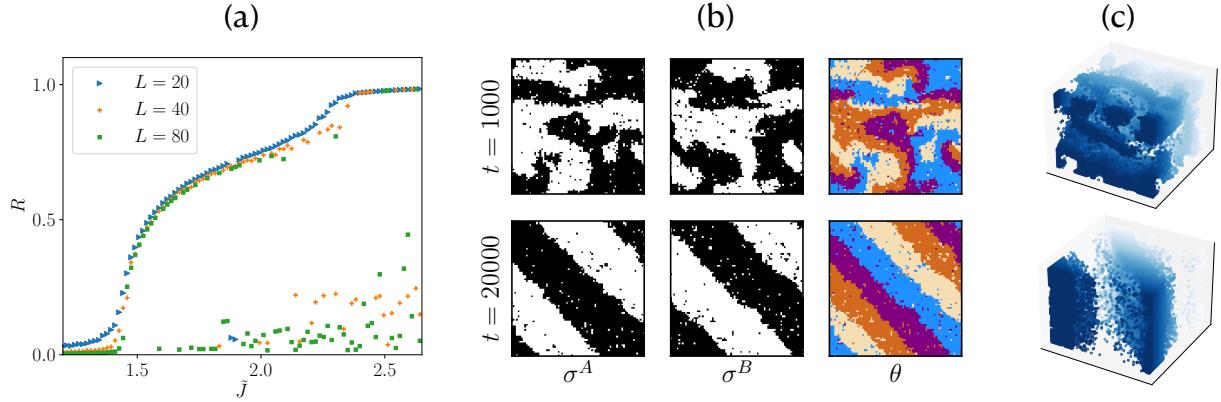
While the system can stay in scroll-waves or planar waves state for a very long time, it is likely that it can escape such states and reach global oscillations. This is supported by 13b: in a single simulation realization, initialized in a disordered state, the system escapes from a scroll-waves state to planar-waves state. Similarly, in Movie 2 we show a case in which the system escapes from a scroll-waves state to swap. The reverse, *i.e.* a homogeneous initialization in the swap phase leading to scroll waves or planar waves, has not been observed in the regime shown in Fig. 3c in the main text.

## L. High non-reciprocity behavior

In the  $\tilde{K} \rightarrow \infty$  limit, long-range order is unstable and  $R$  converges to zero. That is because each spin primarily responds to the local force by the other species and disregards alignment with its neighbours.

The phase transition from disorder (high  $\tilde{K}$ ) to swap (lower  $\tilde{K}$ ) depends on the nearest-neighbors interaction strength,  $\tilde{J}$ . This can be seen already from Fig. 3b in the main text where for  $\tilde{J}$  values which are closer to the pitchfork bifurcation value ( $\tilde{J} \approx 1.33$ ), the transition seems much smoother than for higher  $\tilde{J}$  values. This is further investigated in Fig. 14. One can see that for  $\tilde{J} = 1.5$ , the phase transition plot converges to a Hopf bifurcation as  $L$  increases:  $R$  and  $\mathcal{L}$  are zero beyond some critical  $\tilde{K}_c$  and converge to a non-zero value below this point, similarly to Fig. 3c in the main text except that  $\tilde{K}$  is varied and not  $\tilde{J}$ . For  $\tilde{J} = 2$ , the plot does not converge to the standard Hopf bifurcation. A clear critical point cannot be identified due to a sharp and non converging decrease in  $R$  as  $L$  increases (see  $0.8 \lesssim \tilde{K} \lesssim 1.1$  region in Fig. 14b).

The sharp decrease in  $R$  shown in the right part of Fig. 14b is caused by the emergence of noisy scroll-waves (Fig. 14c-d) that destructs the order. This is slightly reminiscent of the destruction of order in 2D due to spiral defects (Fig. 2 in the main text), except that here, it appears that the destruction only occurs for a limited range of  $\tilde{K}$  values whereas smaller  $\tilde{K}$  values are protected. Due to the size limitation of our simulations we cannot prove what portion of the 3D swap phase is destructed by the scroll waves or what kind of phase transition is associated



**Fig. 13. Random initial conditions lead to long-lived scroll waves and planar waves.** (a)  $R$  as a function of  $\tilde{J}$  shown for different linear system size  $L$ , and for  $\tilde{K} = 0.3$ .  $R$  is measured by averaging a single realization over time, with random initial conditions for the spins (as opposed to ordered initial conditions which were used throughout this work). (b) Snapshots from different times in a simulation with random initial conditions showing  $A$ -spins,  $B$ -spins and the  $\theta$  variable on a 2D slice of the 3D system. System parameters:  $\tilde{J} = 2.28$ ,  $\tilde{K} = 0.3$  and  $L = 80$ . (c) A 3D image of the snapshots in (b), where sites in  $\uparrow\uparrow$  state are marked in blue while the rest of the sites are not shown. Here, we have used a random update scheme (spins are chosen at random).

with this destruction.

### M. Droplet dynamics

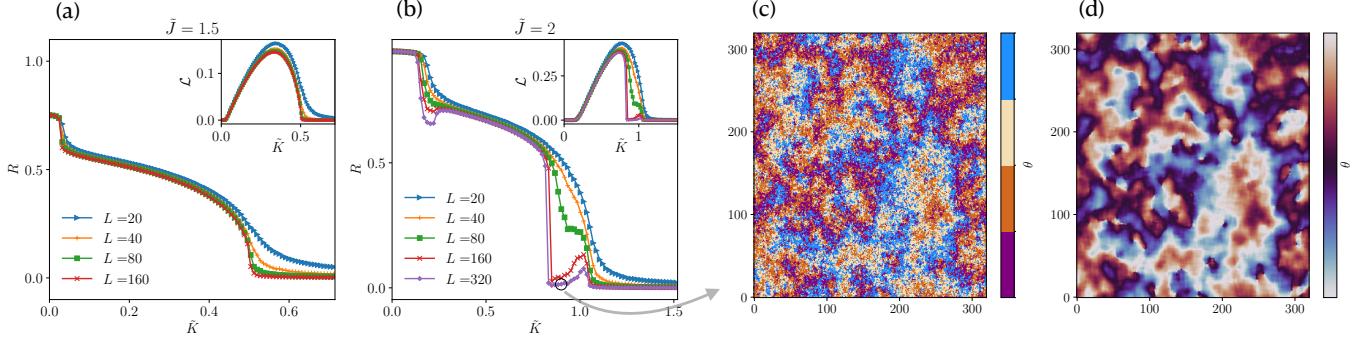
For both the 2D and 3D system, and for fixed system size, we observe a region in the  $(\tilde{J}, \tilde{K})$  space where the magnetization of each species changes signs periodically through the nucleation of droplets. As  $\tilde{J}$  ( $\tilde{K}$ ) increases (decreases), the critical droplet size increases, and as a result the droplets become more sparse and the distance they travel before colliding with each other increases. This trend continues until  $\tilde{J}$  ( $\tilde{K}$ ) becomes too large (small) such that the finite-sized system seems static. This is demonstrated in Fig. 15 where snapshots of the  $\theta$  field, as well as the magnetizations of the two species as a function of time, are shown for a 3D system with linear size  $L = 80$ , and for increasing values of  $\tilde{J}$ . For small  $\tilde{J}$  in the swap phase, there are no apparent droplets but rather noisy patterns without a well-defined shape (out-most left in Fig. 15). For higher  $\tilde{J}$  values we observe droplets that drive the swap (middle snapshots in Fig. 15). For an even larger  $\tilde{J}$ , no droplets are observed and the magnetizations do not change over a long period of time (out-most right in Fig. 15).

In Figure 16 we demonstrate the fate of the static order phase in thermodynamic limit by showing snapshots of the spins at different times for increasing system sizes, in a 2D system. For  $L = 10$ , both species do not change their magnetization. There are attempts of B-spins to nucleate droplets, but those do not reach the critical droplet size and hence shrink and disappear. For  $L = 40$ , the system is no longer static: a handful of B-spins droplets (one clearly observed in the figure) surpass the critical droplet

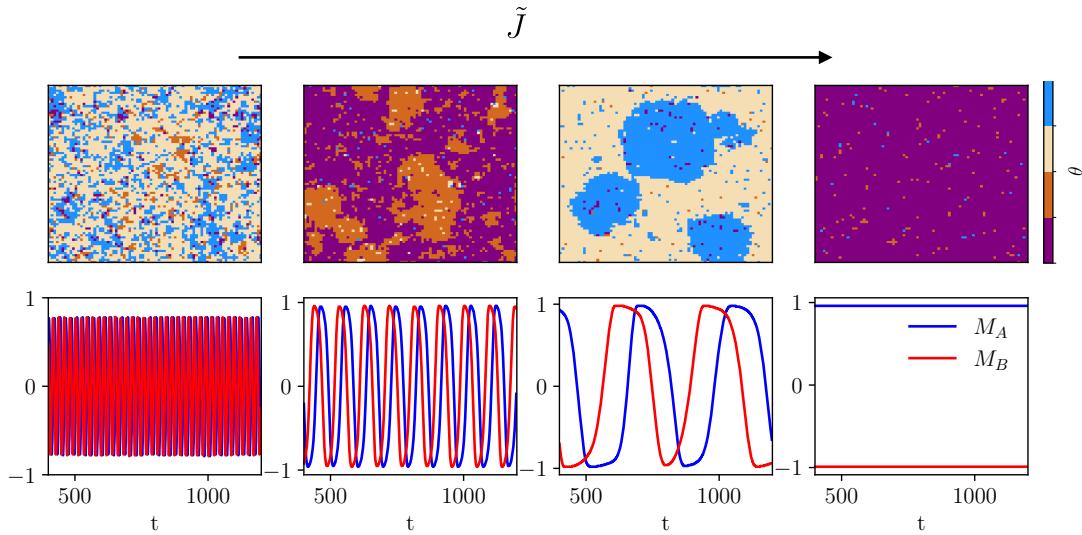
size, grow and flip the magnetization. Then, the same occurs for the A-spins, and so on, leading to an oscillatory behavior of  $M_A(t)$  and  $M_B(t)$ . When  $L = 200$ , multiple droplets ( $\gg 1$ ) drive the magnetization flipping of each species, making the  $M_A(t)$  and  $M_B(t)$  curves smooth at all times, *i.e.* there is no finite period of time in which the magnetization is constant and the system is static. We note that the figure shows a 2D system but a similar behavior is observed in 3D.

The droplet regime is characterized by a distinct behavior of the oscillation period as a function of system size. In Fig. 17, the oscillation period in 3D,  $T_{\text{osc}}$ , is shown as a function of  $\tilde{J}$  and a fixed  $\tilde{K}$ , for increasing  $L$  values. The oscillation period is measured as twice the average time between two subsequent sign flips of  $M_A$ . In the non-droplet regime, *i.e.* small  $\tilde{J}$ ,  $T_{\text{osc}}$  is independent of system size while in the droplet-induced swap regime (large  $\tilde{J}$ )  $T_{\text{osc}}$  decreases dramatically as  $L$  increases. In the large  $\tilde{J}$  limit, droplets are so rare that each magnetization flip is driven by a single droplet. In this regime, the droplet can emerge in any lattice position at a given time step, thus  $T_{\text{osc}}$  scales with  $L^{-3}$  [95, 96] (see bottom panel of Fig. 17).

The different oscillation regimes share similarities with the different transition mechanisms from a metastable to a stable state in the Ising model, studied in [96] by initializing the system magnetized opposite to an applied field  $H$ . Four regimes are identified, depending on the field strength and system size. The “strong-field” regime in which “the droplet picture is inappropriate”, and three other regimes which are classified using droplet theory by the relation between three length scales: the critical droplet radius  $\rho_c$ , the mean droplet separation  $\rho_0$ , and the linear system size  $L$ . These are the multi-droplet



**Fig. 14. 3D phase transition with fixed  $\tilde{J}$  and varied  $\tilde{K}$ .** (a-b)  $R$  vs.  $\tilde{K}$  for  $\tilde{J} = 1.5$  and  $\tilde{J} = 2$ . phase space angular momentum is shown in the inset. (c) Snapshot of a 2D slice from a 3D simulation with the parameters  $\tilde{J} = 2$ ,  $\tilde{K} = 0.9$ ,  $L = 320$ , marked by a black hollow circle on panel b. (d) The same snapshot coarse-grained over  $10 \times 10$  lattice sites. The snapshots show the destruction of order by scroll-waves (appearing as spiral defects due to the 2D projection).



**Fig. 15. Transition from swap to static order for fixed  $L$  in 3D.** Top: 2D-slice simulation snapshots of the  $\theta$  field for varying values of  $\tilde{J}$ . Bottom: The corresponding total magnetizations  $M_A$  and  $M_B$  as a function of time measured in simulation sweeps. From left to right:  $J = 1.6, 2.2, 2.3, 2.5$ . In all the snapshots  $\tilde{K} = 0.3$  and  $L = 80$ .

regime,

$$L \gg \rho_0 \gg \rho_c, \quad (44)$$

the single-droplet regime,

$$\rho_0 \gg L \gg \rho_c \quad (45)$$

and the “coexistence region”

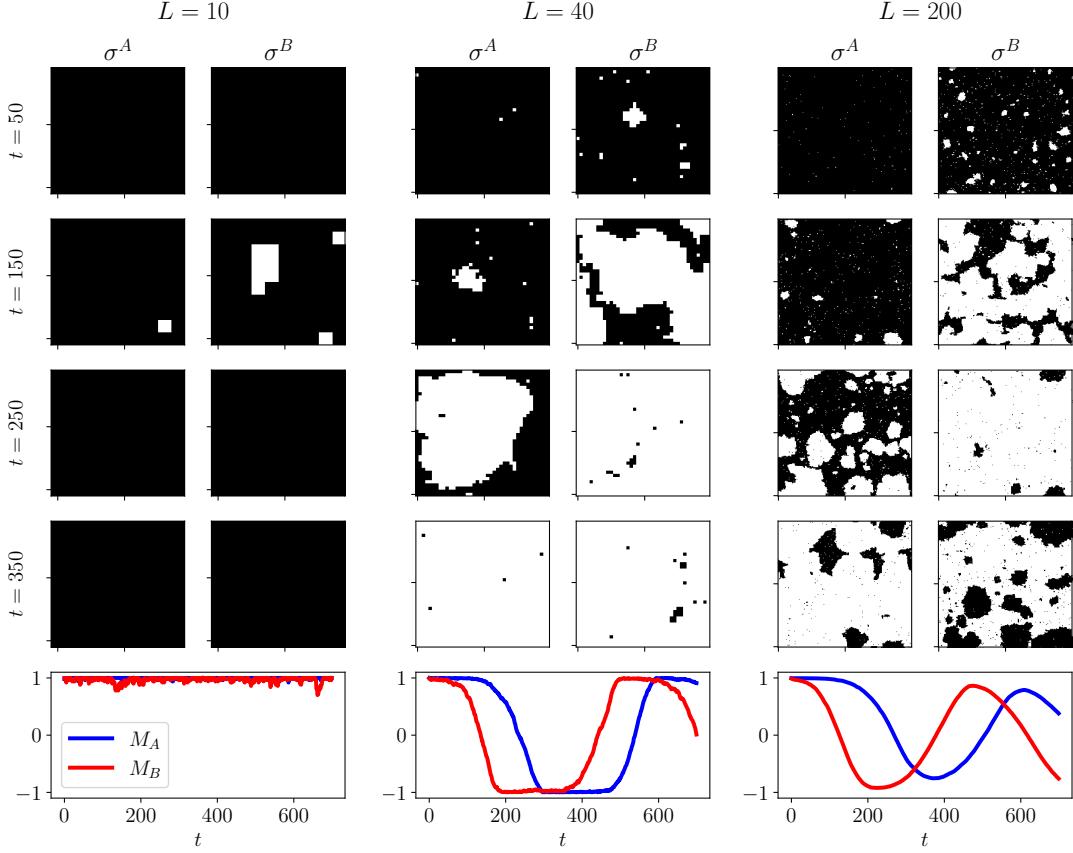
$$\rho_0 \gg \rho_c \gg L \quad (46)$$

where the system behaves as if there is no applied field. The average metastable lifetime calculated in [96] is shown to be independent of system size in strong field and multi-droplet regime yet highly dependent on system size in the single-droplet and coexistence regime, in a way reminiscent of Fig. 17. We note that the analogy

drawn here is not perfect because non-reciprocal interactions are different from an externally applied field in that they change once the magnetization flips.

#### N. Characterization of the decreased synchronization in the droplet regime

Figure 3c displays a non-monotonic behavior of the synchronization order parameter in 3D, observed only for large system size. For  $L = 320$ , Fig. 3c in the main text shows that  $R$  increases monotonically beyond the Hopf bifurcation until at  $\tilde{J} \sim 2.05$  it starts decreasing. It then reaches a local minimum at  $\tilde{J} \sim 2.25$  and increases until saturation at  $\tilde{J} \sim 2.55$ . A similar behavior is observed in the left part Fig. 14b, where  $R$  is non-monotonic as a function of  $\tilde{J}$ .



**Fig. 16. Instability of the static order phase due to droplets nucleation in the thermodynamic limit.** Top 4 rows: snapshots of  $\sigma^A$  and  $\sigma^B$  for  $\tilde{J} = 2.8$  and  $\tilde{K} = 0.3$ , for a 2D system. The snapshots are taken shortly after initialization ( $t = 0$ ) in an ordered state of all spins up, and time increases from top to bottom. Bottom: the corresponding total magnetizations,  $M_A$  and  $M_B$ , as a function of time. From left to right:  $L = 10, 40, 200$ .

The non-monotonicity occurs when the swap phase transitions from noisy oscillations (demonstrated in outmost left panel in Fig. 15) to droplet-induced oscillations (two middle panels in Fig. 15). The mechanism under which global synchronization decreases with increased  $\tilde{J}$  (or decreased  $\tilde{K}$ ) in the droplet regime is shown in Fig. 18. Simulation snapshots (Figure 18a) reveal that while the system is more correlated over short distances at larger  $\tilde{J}$  values, it loses *global order* due to large regions that are out-of-phase with the rest of the system (see bottom right panel in Fig. 18a). This can be quantified by a size dependent synchronization parameter,

$$R(\ell) \equiv \langle r(\ell) \rangle \quad \text{with} \quad r(\ell) \equiv \frac{1}{\ell^d} \left| \sum_j e^{i\theta_j} \right| \quad (47)$$

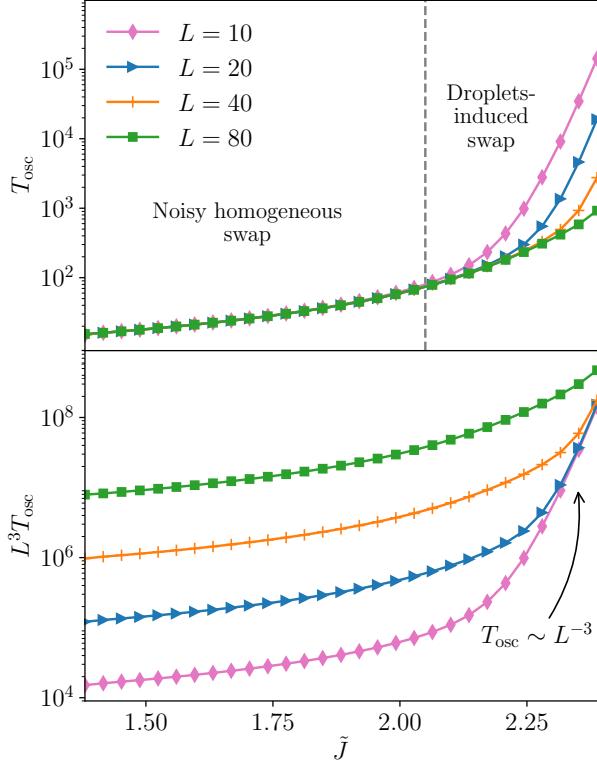
where the sum is over all sites inside a box of linear size  $\ell \leq L$ , and the average is taken with respect to different boxes. The size dependent synchronization parameter is shown in Fig. 18b, for squares instead of cubes to match with the 2D snapshots. For small  $\ell$ ,  $R(\ell)$  is larger for the higher  $\tilde{J}$  value while for large  $\ell$ ,  $R(\ell)$  is larger for the smaller  $\tilde{J}$  value. Moreover, while the  $R(\ell)$  behavior in the noisy oscillations regime decays exponentially,

in the droplet regime it is not and the second derivative of  $R(\ell)$  changes signs. This means that in the droplet-swap regime, the system cannot be described by a single correlation length, and instead has multiple length scales related to the droplet parameters such as the critical droplet size, the typical distance between droplets, and the correlation length inside a single droplet.

While the synchronization seems to decrease in the thermodynamic limit in the droplet-induced swap regime, it is unclear to what extent, and whether it remains oscillatory or is ultimately deemed disordered ( $R = 0$ ). If the latter is correct, it is unclear what is the type of phase transition that occurs and where lies the critical line between swap and disorder in this droplet regime (Fig. 1e in the main text).

## O. The possibility of stable time-periodic states in the thermodynamic limit

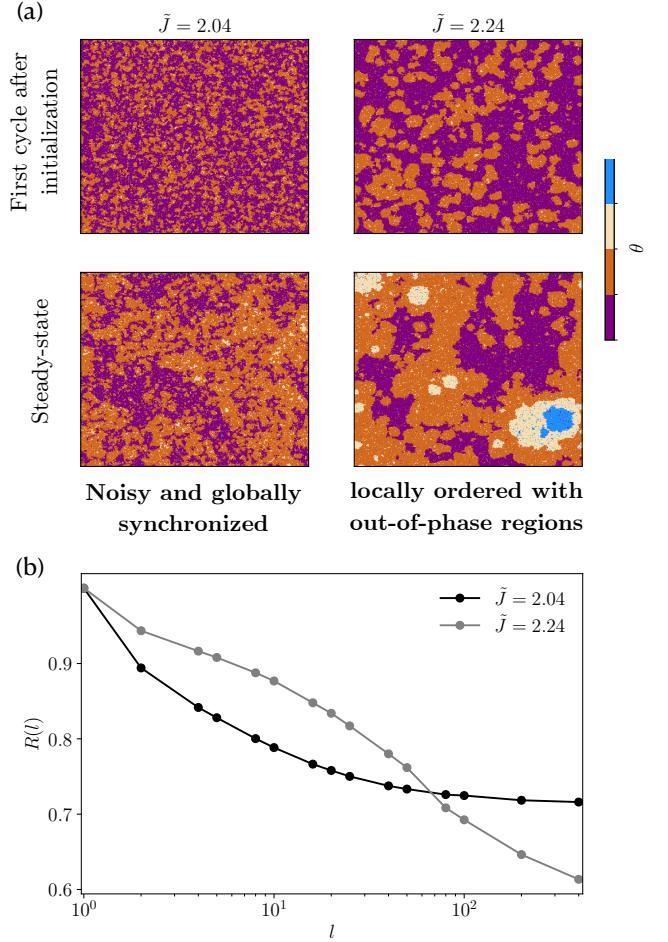
The question of whether time-periodic states of noisy many-body systems with short-range interactions can be stable in the thermodynamic limit is a longstanding



**Fig. 17. Behavior of the oscillation period in the swap phase.** Top: the oscillation period of the total magnetizations  $T_{\text{osc}}$  in the 3D system, on a logarithmic scale, as a function of  $\tilde{J}$  for fixed  $\tilde{K} = 0.3$  and different system sizes. Bottom: the same with a rescaled oscillation period,  $L^3 T_{\text{osc}}$ . Dashed grey line in the top panel separates a regime in which the oscillations are not driven by well-defined droplets from a regime in which they do.

one [42–59]. When the period of the oscillations is *commensurate*, i.e., the oscillation period is a multiple of the discrete time-unite, it was argued in [43] that long-range ordered oscillations are destabilized by the growth of out-of-phase droplets. That is because flat domain walls are expected to have a non-vanishing velocity (unlike equilibrium), indicating that a fluctuation that produces a large enough droplet, can expand rather than shrink. Note that these potential droplets are *not* the droplets that destabilize the static-order phase (Fig. 4 in the main text) or the ones observed in the droplet-induced-swap regime (Fig. 15). Instead, these are droplets of a homogeneously oscillating phase, immersed in an oscillatory background.

In the *incommensurate* case, *i.e.* when the oscillation period is not a multiple of the discrete time-unite, domain walls smear out (as in the XY model) and thus the argument does not apply [44, 57]. This difference is related to the distinction between commensurate states, that break a *discrete* time-translation symmetry, and incommensurate states that break a *continuous* time-translation symmetry. In the case of continuous symmetry breaking, do-



**Fig. 18. Decrease in global order in the droplet-induced swap regime.** (a) Snapshots of 2D slices taken from a 3D simulation with  $\tilde{K} = 0.3$  and  $L = 400$ , comparing two different  $\tilde{J}$  values,  $\tilde{J} = 2.04$  (left) and  $\tilde{J} = 2.24$  (right). The top snapshots are taken shortly after initialization while the bottom snapshots are taken after 10000 simulation time steps. (b) The synchronization parameter  $R(l)$  defined in Eq. (47), measured at steady state as a function of  $l$ , for the two  $\tilde{J}$  values in (a). Averages are taken over  $(L/l)^2$  squares and 1000 time steps.

main walls are not well defined.

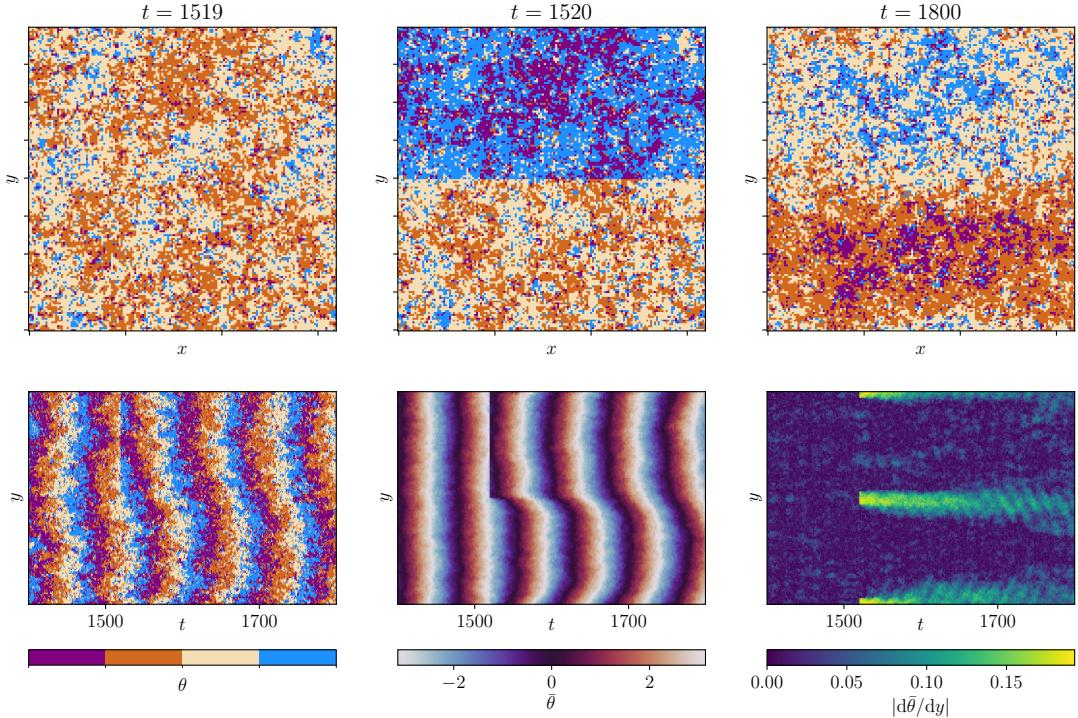
In the non-reciprocal Ising model studied here, the period of the oscillations changes continuously with the interaction parameters, as is demonstrated in Fig. 17. This puts our model in the latter category of incommensurate oscillations, which means that, while stable oscillations are not guaranteed, they are not ruled out by the droplet argument in Ref. [43]. Figure 19 further exemplifies this: it shows an analysis of a 3D simulation in which the system is perturbed from a stable swap state by introducing a flat domain wall. As time evolves, the domain wall is shown to widen and dissipate, supporting the picture of no steady domain-walls between oscillating states.

Could there be other mechanisms that would preclude stable ordered oscillations from existing in the thermo-

dynamic limit? As shown in Fig. 1e in the main text, the phase diagram has a complicated structure which has not been revealed in full: in the large  $\tilde{J}$  limit, synchronization decreases, possibly to zero, due to the difficulty of droplet-induced swap to synchronize. In the high  $\tilde{K}$  regime, scroll waves completely destabilize the long-range order. Both mechanisms are capable of destroying a large portion of the stable swap phase regime, and there is also a possibility that they would meet in the high  $\tilde{J}$  and high

$\tilde{K}$  regime, leading to a compact swap phase in  $(\tilde{J}, \tilde{K})$  space rather than an infinite one. Due to the limitations of our simulations, these interesting possibilities lie beyond the scope of this work, which has focused mostly on the behavior at, and close to, the Hopf bifurcation, where the swap phase appears to be stable and critical exponents can be calculated.

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**Fig. 19. Dissipation of domain walls between homogeneously oscillating states.** (a-c) Snapshots of  $\theta$  in 2D slices taken from a 3D simulation in which at  $t = 1520$ , all A-spins located at  $y > L/2$  flip their magnetization. System parameters are:  $\tilde{J} = 1.5$ ,  $\tilde{K} = 0.1$  and  $L = 160$ . (d) Kymograph of  $\theta$  for a single 1D row along the  $y$ -axis. (e) Kymograph of  $\theta$  averaged over the  $x$  axis in a 2D slice on the  $xy$  plane. This average is defined by the relation  $\frac{1}{L} \sum_x e^{i\theta} \equiv Ae^{i\bar{\theta}}$ . (f) Kymograph of  $|d\bar{\theta}/dy|$ , defined by the discrete symmetric derivative  $d\bar{\theta}/dy \equiv \left| \left( e^{i\bar{\theta}_{y+5}} - e^{i\bar{\theta}_{y-5}} \right) / 10 \right|$ . The smearing out of the region in which  $|d\bar{\theta}/dy| > 0$  indicates the dissipation of the wall boundaries.

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