Fundamental Operations in Artificial Neurons: Dot Product, Matrix Multiplication, Linear Layers, and Activation Functions

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Artificial neurons form the backbone of modern deep learning systems. This chapter explores the foundational mathematical operations that underpin artificial neurons: the dot product, matrix multiplication, linear layers, and activation functions. We extend the analysis with illustrative diagrams and a section on the computational flow of neural layers. These operations, while simple individually, enable the construction of powerful neural architectures capable of modeling highly complex functions.

1 Introduction

Artificial Neural Networks (ANNs) are inspired by the structure and functioning of the biological brain. Their core processing unit is the **artificial neuron**, which computes a weighted sum of its inputs, adds a bias term, and applies a non-linear transformation to produce an output.

This chapter presents the key operations that define the behavior of artificial neurons, leading to the composition of multi-layer neural networks.

2 Dot Product

The dot product of two vectors $\mathbf{x}, \mathbf{w} \in \mathbb{R}^n$ is defined as:

$$\mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^{n} w_i x_i$$

In neural computation, this operation computes the weighted sum of input features. It serves as the linear core of the neuron's function.

3 Matrix Multiplication

In practice, multiple inputs and multiple neurons are processed simultaneously using matrix multiplication.

Let:

- $\mathbf{X} \in \mathbb{R}^{m \times n}$ be a matrix of m input samples (rows) each with n features,
- $\mathbf{W} \in \mathbb{R}^{n \times p}$ be the weight matrix for p neurons.

Then the output of the layer is:

$$\mathbf{Z} = \mathbf{X} \cdot \mathbf{W}$$

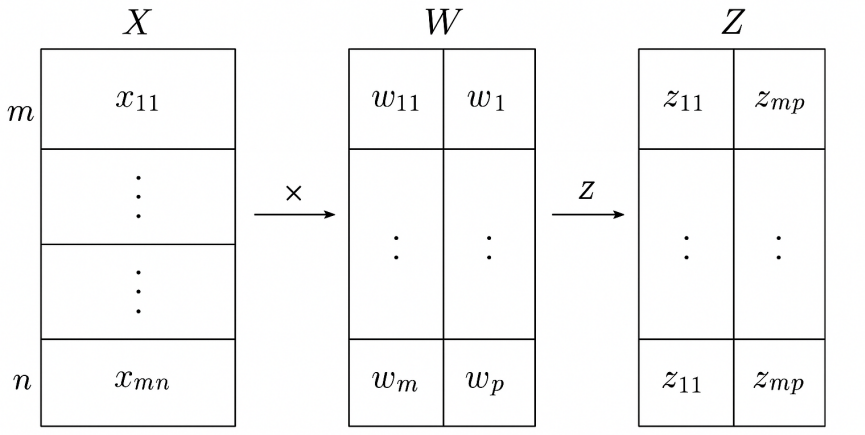

Figure 1: Illustration of matrix multiplication in a neural layer. Each output neuron receives a dot product between weights and input features.

4 Linear Layer (Affine Transformation)

A **linear layer** computes:

$$\mathbf{z} = \mathbf{W} \cdot \mathbf{x} + \mathbf{b}$$

where:

- $\mathbf{W}$: weights,
- $\mathbf{x}$: input vector,
- $\mathbf{b}$: bias vector.

This operation projects the input into a new feature space. The bias allows each output neuron to have an offset.

5 Activation Functions

After linear transformation, non-linear activation functions are applied to introduce expressiveness and prevent the network from collapsing into a purely linear system.

Common Activation Functions

- **Sigmoid:**

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

- **Tanh:**

$$\tanh(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

- **ReLU:**

$$\text{ReLU}(z) = \max(0, z)$$

Each function affects the optimization landscape and gradient behavior differently.

6 Computational Flow of an Artificial Neuron

Putting it all together, the output of a neuron is:

$$a = f(\mathbf{w} \cdot \mathbf{x} + b)$$

Or, in matrix form for a batch of inputs:

$$\mathbf{A} = f(\mathbf{X} \cdot \mathbf{W} + \mathbf{b})$$

7 Example Workflow

Assume a single-layer network with 3 inputs and 2 neurons:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

Then the output is:

$$\mathbf{z} = \mathbf{x} \cdot \mathbf{W} + \mathbf{b} \quad \Rightarrow \quad \mathbf{a} = f(\mathbf{z})$$

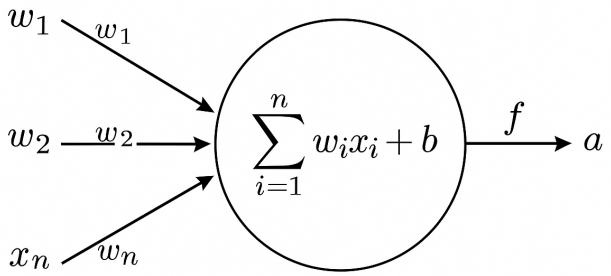

Diagram of an artificial neuron with weights, bias and activation function

Figure 2: A basic artificial neuron takes a vector input, computes a dot product with weights, adds a bias, and passes the result through an activation function.

8 Conclusion

This paper reviewed the basic mathematical operations behind artificial neurons:

- The dot product for weighted input summation,
- Matrix multiplication for parallel computation,
- Linear layers for affine transformations,
- Activation functions for non-linear modeling.

Together, these components form the backbone of deep learning models, enabling them to represent and learn complex functions.

References

1. Goodfellow, I., Bengio, Y., & Courville, A. (2016). *Deep Learning.* MIT Press.
2. Bishop, C. M. (2006). *Pattern Recognition and Machine Learning.* Springer.
3. Nielsen, M. (2015). *Neural Networks and Deep Learning.* Determination Press.

9 Practice by Hand

This section includes beginner-friendly exercises that reinforce the theoretical concepts presented earlier. You are encouraged to solve them manually to gain intuitive understanding.

9.1 Dot Product Examples

Example 1: Let $\mathbf{a} = \begin{bmatrix}3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}5\end{bmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = 3 \times 5 = 15$$

Example 2: Let $\mathbf{a} = \begin{bmatrix}1 & 3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}4 \\ 1\end{bmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = (1 \times 4) + (3 \times 1) = 4 + 3 = 7$$

9.2 Matrix Multiplication Examples

Example 1: Let

$$\mathbf{A} = \begin{bmatrix}2 & 4 & -6 \\ 3 & 4 & 4\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}4 \\ -1\end{bmatrix}$$

Since dimensions are incompatible, this example may be meant for vector-matrix dot application or has a typo. Let's move to the next valid example.

Example 2: Let

$$\mathbf{A} = \begin{bmatrix}2 & -3 \\ 1 & 5\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}-1 & 2 \\ 1 & 1\end{bmatrix}$$

Then,

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix}(2)(-1) + (-3)(1) & (2)(2) + (-3)(1) \\ (1)(-1) + (5)(1) & (1)(2) + (5)(1)\end{bmatrix} = \begin{bmatrix}-2 - 3 & 4 - 3 \\ -1 + 5 & 2 + 5\end{bmatrix} = \begin{bmatrix}-5 & 1 \\ 4 & 7\end{bmatrix}$$

9.3 Linear Layer Example

Let

$$\mathbf{X} = \begin{bmatrix}3 \\ 1\end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix}2 & 3 & 4 \\ 1 & -1 & 3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}2 & 2 & 2\end{bmatrix}$$

Then the output $\mathbf{Y}$ is:

$$\mathbf{Y} = \mathbf{W}^T \cdot \mathbf{X} + \mathbf{b} \Rightarrow \begin{aligned} Y_1 &= (2)(3) + (1)(1) + 2 = 6 + 1 + 2 = 9 \\ Y_2 &= (3)(3) + (-1)(1) + 2 = 9 - 1 + 2 = 10 \\ Y_3 &= (4)(3) + (3)(1) + 2 = 12 + 3 + 2 = 17 \end{aligned}$$

9.4 Activation Function: ReLU Practice

Apply ReLU to the following inputs:

$$\text{ReLU}(x) = \max(0, x)$$

Input (x)	Output (ReLU(x))
-5	0
-4	0
-3	0
-2	0
-1	0
0	0
1	1
2	2
3	3
4	4
5	5

9.5 Artificial Neuron Example

Given:

$$\mathbf{X} = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$$

Compute each neuron's linear and ReLU-activated output:

$$Y_1 = \text{ReLU}(2 \cdot 5 + 1 \cdot 2 + 1) = \text{ReLU}(10 + 2 + 1) = \text{ReLU}(13) = 13$$
$$Y_2 = \text{ReLU}(1 \cdot 5 + (-2) \cdot 2 - 1) = \text{ReLU}(5 - 4 - 1) = \text{ReLU}(0) = 0$$
$$Y_3 = \text{ReLU}((-1) \cdot 5 + 0 \cdot 2 - 1) = \text{ReLU}(-5 - 1) = \text{ReLU}(-6) = 0$$

$$\Rightarrow \mathbf{Y} = \begin{bmatrix} 13 & 0 & 0 \end{bmatrix}$$

These examples help in grasping the fundamentals of neural operations and provide a basis for building more complex models.