LAKSHYA
JEE 2025

MATHEMATICS | **Lecture - 02**

CALCULUS

LIMIT, CONTINUITY & DIFFERENTIABILITY (LCD)

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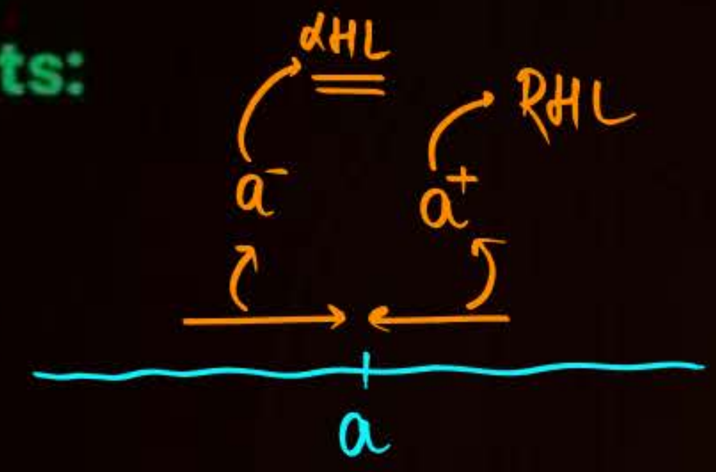
Topics *to be covered*

1. One Sided Limit
2. Indeterminate Form
3. Different Types of Questions
4.

LAST CLASS RECAP

Basics of Limits:

1. 

2. Limiting value

$$\boxed{\lim_{x\to a} f(x) = l}$$

l ↑ exact

3. ✓

4. Limit existing

$$\boxed{LHL = RHL = \text{finite}}$$

QUESTION

Which of the following functions represent identical graphs in $x-y$ plane $\forall\, x \in [-1, 1]$?

(A) $f_1(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ → Notes already done!

(B) $f_2(x) = \frac{\pi}{4} - \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}\right)$ → $\frac{\pi}{4} - \tan^{-1}\left(\frac{1-x^2}{\sqrt{1-x^4}}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$x^2 = \cos\theta$

+ve

$\theta \in [0, \frac{\pi}{2}]$

$= \frac{\pi}{4} - \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\pi}{4} - \frac{\theta}{2}$

$= \pi/4 - \frac{1}{2}\cos^{-1}x^2.$

(C) $f_3(x) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$ ✓

$\pi/4 - \frac{1}{2}\left(\frac{\pi}{2} - \sin^{-1}x^2\right)$

(D) $f_4(x) = \frac{1}{2}\sin^{-1}x^2$ ✓

CHALLENGER QUESTION

Let $f(x)=x^2-2ax+a-2$ and $g(x)=\left[2+\sin^{-1}\frac{2x}{1+x^2}\right]$. If the set of real values of 'a' for which $f(g(x))<0, \forall x\in R$ is (k_1,k_2), then find the value of $(10k_1-3k_2)$.

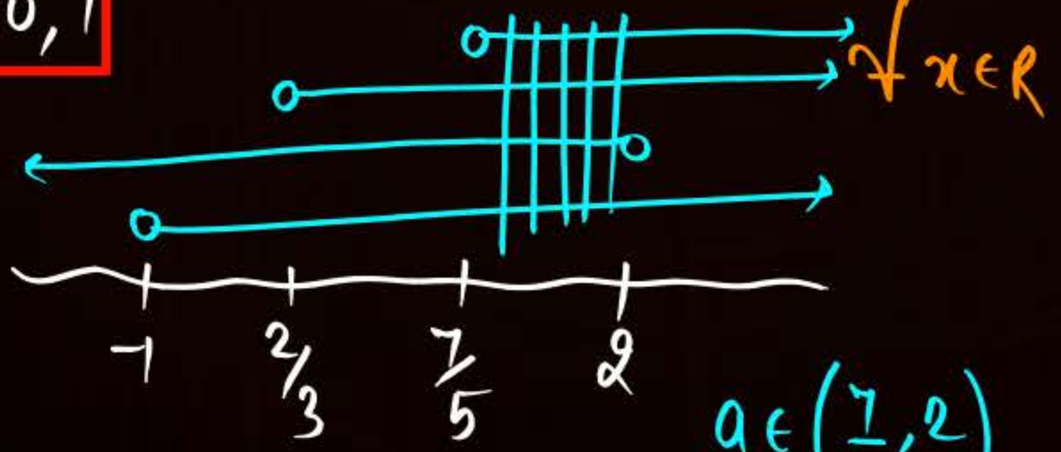
$\cancel{10}^{2}\times\frac{7}{\cancel{5}}-3\times2=14-6=8$

8

fnf.

[Note : $[k]$ denotes greatest integer less than or equal to k.]

$g(x)=2+\underbrace{\left[\overbrace{\sin^{-1}\frac{2x}{1+x^2}}^{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}\right]}_{\boxed{-2,-1,0,1}} \Rightarrow g(x)\in\{0,1,2,3\}$



$f(g(x))<0$

$f(0)<0 \Rightarrow a-2<0 \Rightarrow a<2$

$f(1)<0 \Rightarrow -a-1<0 \Rightarrow -1<a$

$f(2)<0 \Rightarrow 2-3a<0 \Rightarrow \frac{2}{3}<a$

$f(3)<0 \Rightarrow 7-5a<0 \Rightarrow \frac{7}{5}<a$

$a\in\left(\frac{7}{5},2\right)$

QUESTION [JEE (Adv.)-2018, Paper-1]

The number of real solutions of the equation $\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^{i}\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^{i} - \sum_{i=1}^{\infty}(-x)^{i}\right)$ lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is ________.

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

HW.

LIMIT, CONTINUITY & DIFFERENTIABILITY (LCD)

$\boxed{\text{LIMITS}}$

One-sided limit:

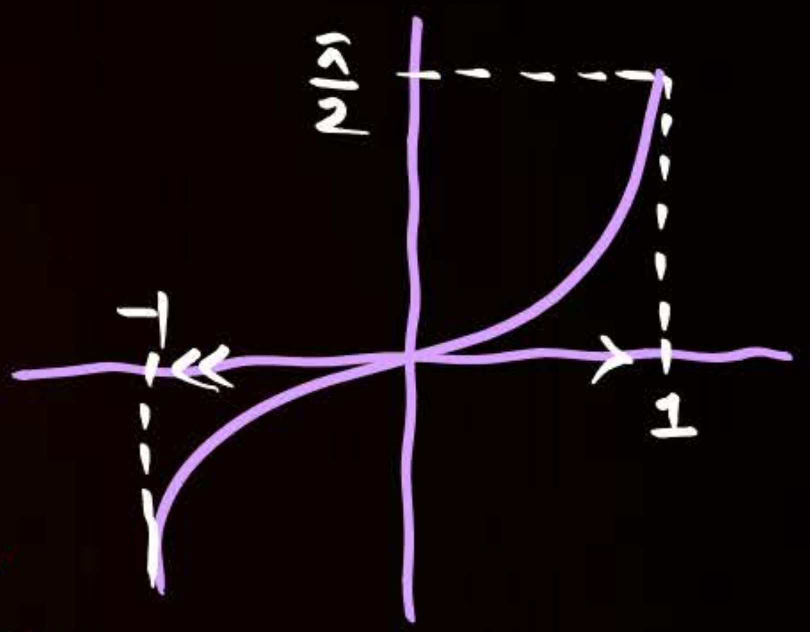
ex: $\lim_{x \to 1} \sin^{-1} x = ??$

$x \in [-1, 1]$

0.9999
0.9999
0.99

1^{-}

$LHL = \lim_{x \to 1^{-}} \sin^{-1} x = \frac{\pi}{2}$ finite



$RHL = \lim_{x \to 1^{+}} \sin^{-1} x$

1.01
1.001
1.0001
⋮

1^{+}

out of domain

ye calculate hi nhi होगा।

ex: $\lim_{x \to 0} \sqrt{x}$

$\boxed{LHL = \lim_{x \to 0^{-}} \sqrt{x}}$ ×

$RHL = \lim_{x \to 0^{+}} \sqrt{x} = 0.$

ONE SIDED LIMITS

At end points of intervals of domain only $\boxed{\text{ONE-SIDED}}$ limit is defined.

If f(x) is defined is $x \in [a, b]$ then:

(i) At $x = a$

RHL $= \lim\limits_{x \to a^+} f(x) = \text{finite}$

limit exists

(ii) At $x = b$

LHL $= \lim\limits_{x \to b^-} f(x) = \text{finite}$

Note:

Agar LHL ya RHL ko calculate krte waqt denominator mein $\boxed{\text{Exact 'zero',}}$ Root ke and $\boxed{\text{negative sign bne}}$ and kisi function ki personal domain distrub ho to smjh jana ki aapko vo calculate hi nhi krna tha. (ONE-SIDED LIMIT)

QUESTION

(i) $\lim_{x\to 0}\sqrt{x}$

$\lim_{x\to 0^+}\sqrt{x} = 0$

exists.

(ii) $\lim_{x\to(-1)}\overbrace{\cos^{-1}x}^{[-1,1]}$

$\lim_{x\to(-1)^+}\cos^{-1}x = \pi.$

Limit exists.

(iii) $\lim_{x\to 1}\frac{x}{[x]}$, [.] is GIF

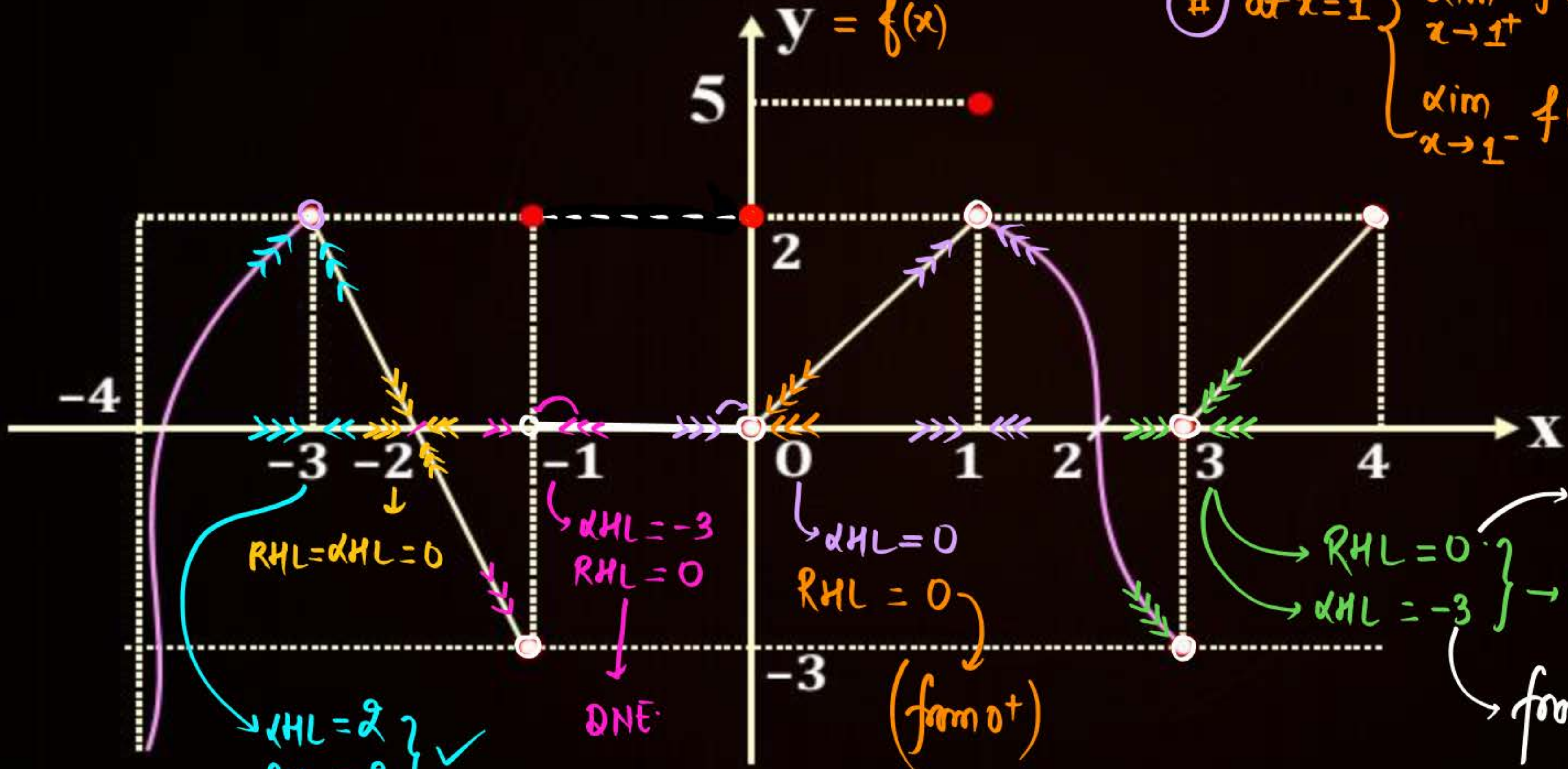
(#) old-IIT.

Limit exists & equal to '1'.

$RHL = \lim_{x\to 1^+}\frac{x}{[x]} = \frac{1}{1} = 1$ ✓

$$\boxed{LHL = \lim_{x\to 1^-}\frac{x}{[x]} = \frac{0.99}{0} = n.d.}$$ X

LIMITS BY GRAPH



At $x=-3$: $LHL = 2$, $RHL = 2$ } ✓

At $x=-2$: $RHL = LHL = 0$

At $x=-1$: $LHL = -3$, $RHL = 0$ → DNE.

At $x=0$: $LHL = 0$, $RHL = 0$ (from 0^+)

\# at $x=1$ $\left\{ \begin{array}{l} \lim\limits_{x\to 1^+} f(x) = 2 \\ \lim\limits_{x\to 1^-} f(x) = 2 \end{array} \right\} \lim\limits_{x\to 1} f(x) = 2$

from less than 2.

from less than 2.

At $x=3$: $\left. \begin{array}{l} RHL = 0 \\ LHL = -3 \end{array} \right\} \to$ DNE.

$RHL = 0$ → from more than 0. → 0^+

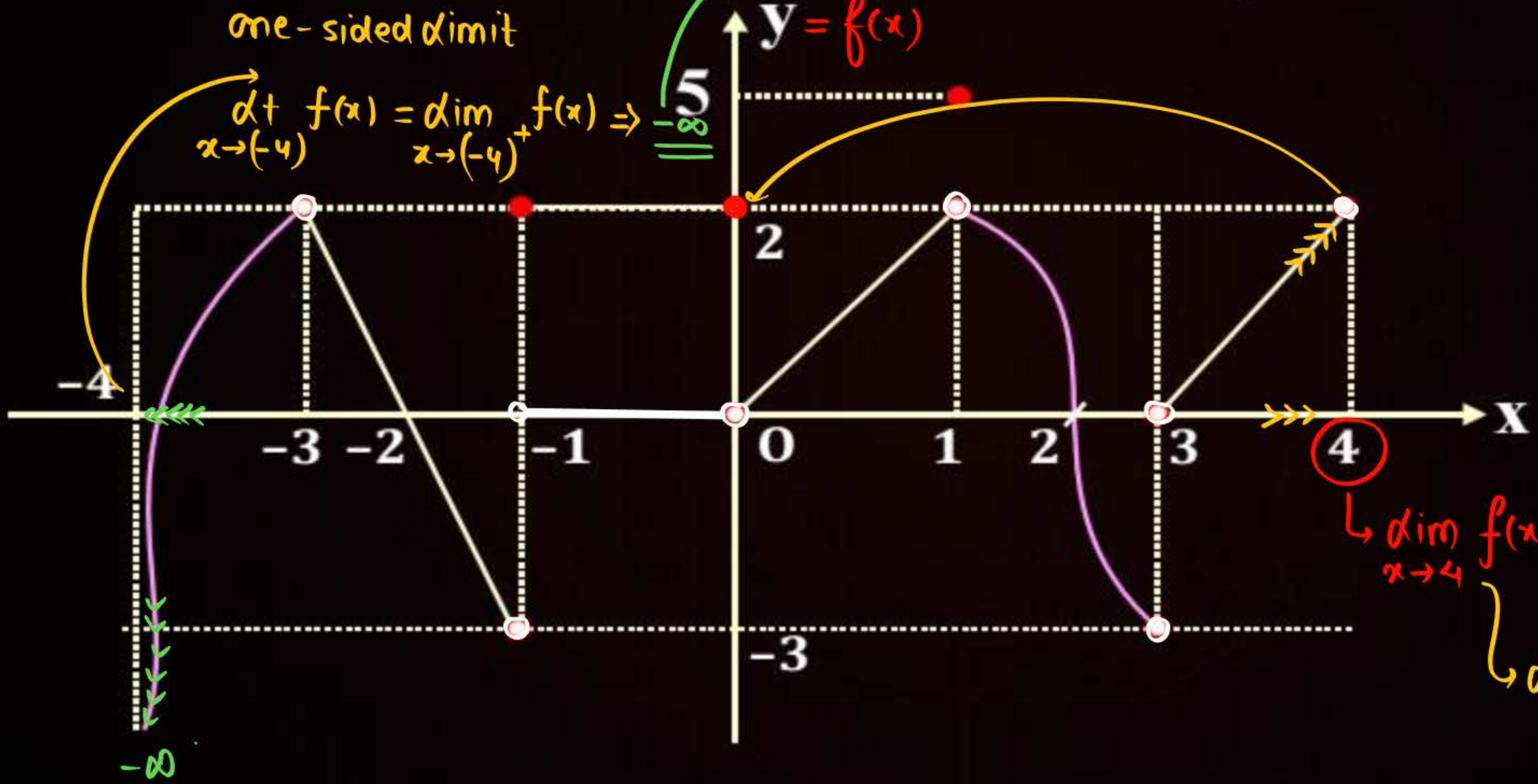
$LHL = -3$ → from more than (-3) → $(-3)^+$

LIMITS BY GRAPH

at $x=-4 \Rightarrow$ "$\underline{\underline{\text{LDE}}}$"

one-sided limit

$\lim_{x\to(-4)} f(x) = \lim_{x\to(-4)^+} f(x) \Rightarrow \underline{\underline{-\infty}}$

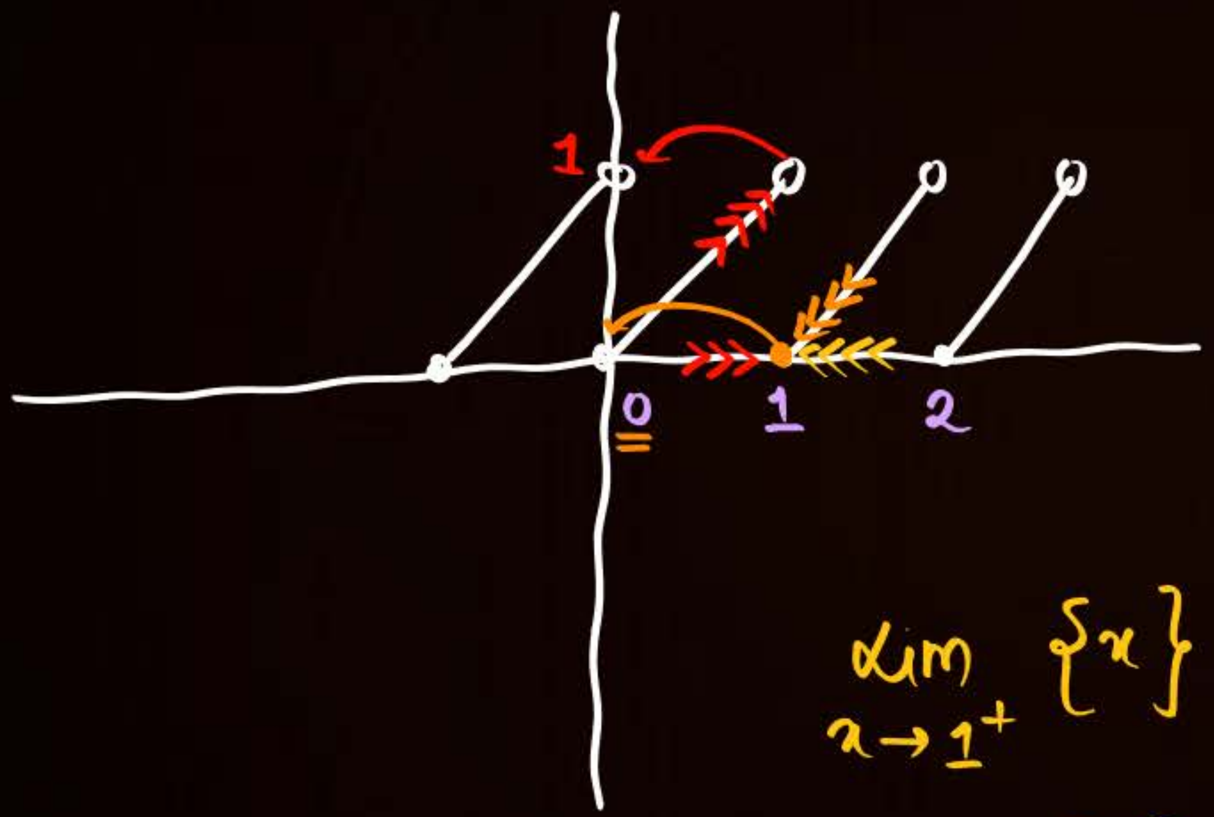


$\lim_{x\to 4} f(x) = \lim_{x\to 4^-} f(x) = 2$

limit exists.

ex:

$$\lim_{x \to 1} \{x\} = ??$$



$$\lim_{x \to 1^+} \{x\} = 0.$$

$$\lim_{x \to 1^-} \{x\} = 1.$$

QUESTION

$\cancel{2}\times 3 = \lambda\cancel{(2)}$

Refer the figure the value of λ for which

$2\left(\lim\limits_{x\to 0} f(x^3 - x^2)\right) = \lambda\left(\lim\limits_{x\to 0} f(2x^4 - x^5)\right)$ is

(A) 4/3

(B) 2

(C) 3

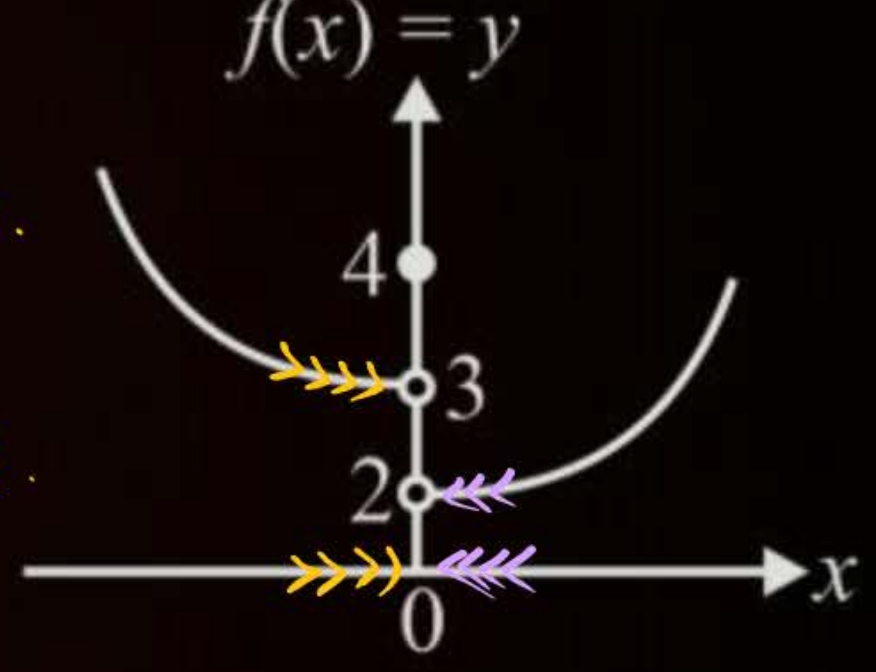
(D) 5

$$\lim_{x\to 0} f(x^3-x^2) \rightarrow \lim_{x\to 0^+} f(\underbrace{x^3-x^2}_{\to 0^-}) = f(0^-) = 3.$$

$$\lim_{x\to 0} f(x^3-x^2) \rightarrow \lim_{x\to 0^-} f(\underbrace{x^3-x^2}_{\to 0^-}) = f(0^-) = 3.$$

$$\lim_{x\to 0} f(2x^4-x^5) \rightarrow \lim_{x\to 0^+} f(0^+) = 2$$

$$\lim_{x\to 0} f(2x^4-x^5) \rightarrow \lim_{x\to 0^-} f(0^+) = 2$$



INDETERMINANT FORMS

7 Indeterminant Forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$

"chota sa badlav"

- $0 \times \infty$
- $\infty - \infty$

dummy form

- 1^{∞}
- 0^{0}
- ∞^{0}

Exponential

Only Indeterminate Form : $\boxed{\left(\dfrac{\text{zero approaching}}{\text{zero approaching}}\right)}$ $\equiv \left(\dfrac{0\uparrow}{0\uparrow}\right)$

NOTE :

(i) $\left(\dfrac{\text{Exact Zero}}{\text{zero approaching}}\right) = \dfrac{0}{0\uparrow} = 0.$

(ii) $\left(\dfrac{\text{Exact Zero}}{\text{Exact Zero}}\right) = \dfrac{0}{0} = \text{not defined}$

(iii) $\left(\dfrac{\text{zero approaching}}{\text{Exact Zero}}\right) = \dfrac{0\uparrow}{0} = \text{not defined}$

$$\lim_{x\to 3} \frac{\overset{\text{GIF}}{[x-3]}}{(x-3)}$$

$$\left.\begin{aligned} &\lim_{x\to 3^+} \frac{[x-3]}{(x-3)} = \frac{0}{0\uparrow} = 0. \\ &\lim_{x\to 3^-} \frac{[x-3]}{x-3} = \frac{-1}{0^-} = +\infty \end{aligned}\right\} \underline{\text{DNE}}.$$

#DHOKA!!

$\infty + \infty \equiv \infty$ $\quad \infty - 0 \equiv \infty$ $\quad \infty \times \infty \equiv \infty$

$(\infty)^{\infty}$ $\quad 0^{\infty} \equiv 0$ $\quad \frac{0}{\infty} \equiv 0$

$(0\uparrow)^{\boxed{??}} \to 0$

$(<1)^{\infty} \to 0.$

$(0\uparrow)^{0\uparrow}$

$(\boxed{??})^{0\uparrow} \to 1$

$(0\uparrow)^{\boxed{??}} \to 0$

opp.

NOTE NINE POINTS

1. For $\left(\frac{0}{0}\right)$ form use "factorsation".
2. For $\left(\frac{\infty}{\infty}\right)$ form, take highest power of variable common from numerator & denominator.

substitution

3. For $\boxed{(0 \times \infty)}$ form, put $\boxed{x = \frac{1}{t}}$ or slight change will convert form in $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$
4. For $\boxed{(\infty - \infty)}$ form, by substitution OR slight change they change to $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$.

ex :-

$$\lim_{x\to\infty}\left(\frac{x^2+3x}{5x^2-7}\right) \equiv \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty}\frac{\cancel{x^2}\left(1+\frac{3}{x}\right)}{\cancel{x^2}\left(5-\frac{7}{x^2}\right)} = \frac{1}{5}$$

Ans.

- L-H Rule

$$\lim_{x\to\infty}\left(\frac{2x+3}{10x}\right) \equiv \left(\frac{\infty}{\infty}\right)$$

L-H Rule.

$$\lim_{x\to\infty}\frac{2}{10} = \frac{1}{5}$$

NOTE NINE POINTS

Requirement $\equiv$ $\boxed{\text{differentiation}}$ ✓

5. For can also use **L-Hospital Rule** for $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form.

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}} = \boxed{\lim_{x\to a}\frac{f'(x)}{g'(x)}} = \lim_{x\to a}\frac{f''(x)}{g''(x)} \ldots\ldots$$

(Jab tak form gayab na ho jaye)

$\left(\frac{0\uparrow}{0\uparrow} \text{ or } \frac{\infty}{\infty}\right)$ $\left(\frac{0\uparrow}{0\uparrow} \text{ or } \frac{\infty}{\infty}\right)$

NOTE NINE POINTS

6. If square roots is present, use rationlisation.

7. If [x], {x}, |x|, sgn(x) & ***functions having different definition*** are present then LHL & RHL should be separately calculate.

8. $\boxed{\sqrt{x^2} = |x|}$

9. **Kisi bhi question mein sbse pehle form check kro!!**
Jab indeterminate forms bnegi tbhi one of the above method lgana hai otherwise direct answer aayega.

QUESTION

Evaluate: $\lim_{x\to 1} \frac{x^3 - x^2\log x + \log x - 1}{x^2 - 1} \equiv \frac{0}{0}$

M-I:

$$\lim_{x\to 1} \frac{(x^3-1) - \log x\,(x^2-1)}{(x-1)(x+1)}$$

$(x^3-1) = \cancel{(x-1)}(x^2+x+1)$, $(x^2-1) = \cancel{(x-1)}(x+1)$

$$\lim_{x\to 1} \frac{\overset{3}{(x^2+x+1)} - \overset{0}{\log x}\,\overset{2}{(x+1)}}{\underset{2}{(x+1)}} = \frac{3}{2}$$

M-II: "L-H Rule":-

$$\lim_{x\to 1} \left(\frac{3x^2 - \left(2x\ln x + x^2 \cdot \frac{1}{x}\right) + \frac{1}{x}}{2x} \right)$$

$$\frac{3 - \left(2\times 1\times \overset{0}{\cancel{0}} + 1\right) + 1}{2} = \frac{3}{2}$$

QUESTION [JEE Mains-2019]

$$\lim_{x\to\frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)} \equiv \left(\frac{0\uparrow}{0\uparrow}\right)$$

M-I:

$$\lim_{x\to\pi/4} \frac{\left(\frac{\cos^3 x}{\sin^3 x} - \frac{\sin x}{\cos x}\right)}{\frac{1}{\sqrt{2}}\cos x \cdot - \frac{1}{\sqrt{2}}\sin x} = \lim_{x\to\pi/4} \frac{\cos^4 x - \sin^4 x}{\frac{1}{\sqrt{2}}\cancel{(\cos x - \sin x)}\ \underbrace{\sin^3 x}_{\frac{1}{2\sqrt{2}}} \cdot \underbrace{\cos x}_{\frac{1}{\sqrt{2}}}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}} \times \frac{1}{4}} = 8$$

$$(\cos^2 x - \sin^2 x)(\underbrace{\cos^2 x + \sin^2 x}_{1})$$

$$\cancel{(\cos x - \sin x)}(\cos x + \sin x)$$

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

M-II: L-H Rule:

$$\lim_{x\to\pi/4} \frac{3(\cot x)^2(-\operatorname{cosec}^2 x) - (\sec^2 x)}{-\sin\left(x+\pi/4\right)} = \frac{3(1)^2(-2) - 2}{-1} = \frac{-6-2}{-1} = 8$$

Thoda Dhyaan rkhna:

Ex.: (i) $L = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{3x-6}$ $\left(\frac{\infty}{\infty}\right)$

Std-B

$\lim_{x\to-\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{6}{x}\right)} = \frac{1}{3}$ ✗

Std-A:-

$\lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(3-\frac{6}{x}\right)}$ (-ve)

$\lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{6}{x}\right)} = -\frac{1}{3}$ ✓

IIT ** (ii) $L = \lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x}$

$\sqrt{1-\cos 2x} = \sqrt{2\sin^2 x}$

$\lim_{x\to 0} \frac{\sqrt{2}\,|\sin x|}{x}$

$\lim_{x\to 0^+} \frac{\sqrt{2}\sin x}{x} \left(\frac{0}{0}\right) = \frac{\sqrt{2}\cos x}{1} = \sqrt{2}$

L-H Rule.

$\lim_{x\to 0^-} \frac{-\sqrt{2}\sin x}{x} = -\sqrt{2}$

LDE.

(iii) $L = \lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$

Thoda Dhyaan rkhna:

Ex.: (i) $L = \lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{3x-6}$

(ii) $L = \lim_{x \to 0} \frac{\sqrt{1-\cos 2x}}{x}$

(iii) $\boxed{L = \lim_{x \to 0} \frac{e^{1/x}-1}{e^{1/x}+1}}$

$\boxed{x \to 0 \Rightarrow \frac{1}{x} \to \infty}$ ×

$x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty$

$x \to 0^- \Rightarrow \frac{1}{x} \to -\infty$

$e^{\frac{1}{x}} \equiv e^{-\infty} \equiv \frac{1}{e^{\infty}} \equiv 0.$

$$\lim_{x \to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{\cancel{e^{\frac{1}{x}}}\left(1-e^{-\frac{1}{x}}\right)}{\cancel{e^{\frac{1}{x}}}\left(1+e^{-\frac{1}{x}}\right)} = \frac{1}{1} = 1$$

($e^{\frac{1}{x}} \to \infty$, $e^{-\frac{1}{x}} \to 0$)

$$\lim_{x \to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{0-1}{0+1} = -1$$

LHL ≠ RHL ⇒ limit DNE.

$\underline{\underline{\text{LDE}}}.$

QUESTION [JEE Mains-2023]

If $S = \left\{x \in \mathbb{R} : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}\right\}$

then $\sum_{x \in S}\left(\sin\left((x^2+x+5)\frac{\pi}{2}\right) - \cos\left((x^2+x+5)\pi\right)\right)$ is equal to

Hw.

Homework

Re-attempt all the Questions of Lecture.

DPP

Module: Chapter – ITF

Exercise (Prarambh) : COMPLETE

Exercise (Prabal) : COMPLETE

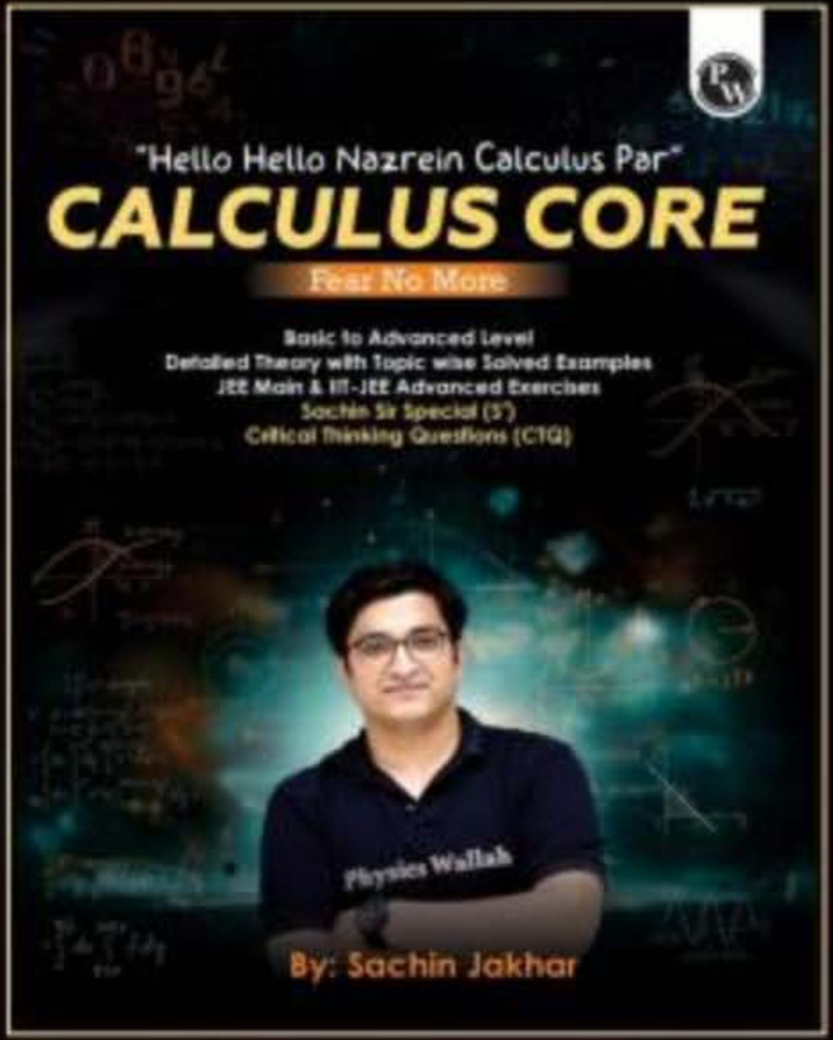
Exercise (Parikshit) : Ques: 1,5,7,9,11,23

NCERT:

INVERSE TRIGONOMETRIC FUNCTIONS

Exercise: 2.1 - COMPLETE

Exercise: 2.2 – Ques. 8 to 15



CALCULUS CORE

CHAPTER: ITF

DIBY : 3.3 COMPLETE

JEE MAINS : COMPLETE

It’s not about End Result,
It is all about JOURNEY

#futureIITians

THANK YOU