Gödel's Incompleteness Theorems

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1 Introduction

Peano Arithmetic (PA) was introduced as a first order set of axioms which aim was to axiomatise *true arithmetic*, the set of all true statements about the natural numbers. Gödel showed that PA is incomplete, and hence it cannot fulfill its intended task. This essay we will explore different ways of strengthening PA to get it closer to true arithmetic. The first two ways will be by adding extra axioms to PA. We will first add an axiom schema called the uniform reflection principle, which intuitively says that if you can prove every instance of a statement, then every instance of the statement is true. Another axiom we will add is a consistency axiom. The axiom says that there is something that the system cannot prove, which intuitively means that the system is consistent as an inconsistent system can prove everything. Both these ways of adding new axioms will be done at different levels, for the reflection principle we will vary which formulae we include in the axioms schema and for the consistency statement we will vary how many iteration of consistency we add. Both these system will be compared against each other, and we will show that at the same level the reflection principle entail the consistency axiom. We will then show that both of them fail to be Σ_2 complete. This means that although both extensions are non-conservative over PA, i.e. they can prove more things that PA can, they do not give us any more degrees of completeness.

The last way of strengthening PA that we will explore is to approximate true arithmetic through ZFC. The idea is that true arithmetic can be approximated axiomatically by looking at all the sentences ZFC can prove to be true about the natural numbers. We will show that this gives us a consistent but incomplete system that is at least as strong as PA with the added consistency axioms iterated to arbitrary finite degree.

2 Uniform Reflection and Iterated Consistency

In this section we will define and examine URP_{Π_n} and Con^n_{PA}, which are two non-conservative extensions of PA. We will then show that $URP_{\Pi_n} \vdash Con^n_{PA}$ and that $Con^1_{PA} \vdash URP_{\Pi_1}$. To achieve the latter, we will show that $Con^1_{PA} \vdash Con_{PA}$ and quote a result from the lecture notes which says that $PA \cup \{Con_{PA}\} \vdash URP_{\Pi_1}$. We will show that this is the best that we can do, in the sense that for $n \geq 2$ we show that $Con^n_{PA} \nvdash URP_{\Pi_n}$. Then we will finish up the section by showing that neither URP_{Π_n} nor Con^n_{PA} is Σ_{n+1} complete for $n \geq 1$. We first define URP_{Π_n}.

Definition 2.1 (The Uniform Reflection Principle)
Let Φ be a class of formulae, then URP_Φ is PA plus the following set of axioms $\{\forall n Pr_{PA}(\overline{\ulcorner F(\dot{n})\urcorner}) \to \forall n F(n) : F(n) \in \Phi\}$ where $\forall n Pr_{PA}(\overline{\ulcorner F(\dot{n})\urcorner})$ abbreviates $\forall n Pr_{PA}(\overline{\ulcorner \forall v_1(v_1 =\urcorner}{}^\frown\,\overline{n}{}^\frown\overline{\ulcorner \to F(v_1)\urcorner})$.

Definition 2.2
$\forall x Tr(\overline{\ulcorner F(\dot{x})\urcorner}) := \forall x Tr(\overline{\ulcorner \forall v_1(v_1 =\urcorner}{}^\frown x {}^\frown\overline{\ulcorner \to F(v_1))\urcorner})$

Defining Con^n_{PA} is slightly easier. We let $Con^0_{PA} = PA$ and define $Con^{n+1}_{PA} = Con^n_{PA} \cup \{\exists v_1 \neg Pr_{Con^n_{PA}}(v_1)\}$. Intuitively, Con^1_{PA} is saying that PA is consistent because in an inconsistent system everything can be proved. We will show later that this is in fact equivalent to the normal Con_{PA} statement, which is $Pr_{PA}(\overline{\ulcorner \neg\overline{0} = \overline{0}\urcorner})$. In fact, we will show something stronger. If we let $T_0 = PA$ and $T_{n+1} = T_n \cup \{Con_{T_n}\}$, then we will show that Con^n_{PA} and T_n are provably equivalent. Before we do all of that, we are going to show that $URP_{\Pi_n} \vdash Con^n_{PA}$. To do this, we will first show that Con^n_{PA} is a set of assumptions definable in Σ_{n+1}. This will gives us that $Pr_{Con^n_{PA}}$ is a Σ_{n+1} statement, which is crucial for the proof of $URP_{\Pi_n} \vdash Con^n_{PA}$.

Lemma 2.3
Con^n_{PA} is a set of assumptions definable in Σ_{n+1}.

Proof We do a proof by induction. For the base case we know that $Con^0_{PA} = PA$, which is definable in Δ_1 by Theorem 4.3.1 from the course notes. So, in particular, Con^0_{PA} is Σ_1 definable. For the inductive step we assume that Con^n_{PA} is Σ_{n+1} definable. Now, $Con^{n+1}_{PA} = Con^n_{PA} \cup \{\exists v_1 \neg Pr_{Con^n_{PA}}(v_1)\}$. We know that Con^n_{PA} is Σ_{n+1} definable, which makes $Pr_{Con^n_{PA}}(x)$ a Σ_{n+1} formula. Hence $\exists v_1 \neg Pr_{Con^n_{PA}}(v_1)$ is provably Σ_{n+2}, and so Con^{n+1}_{PA} is Σ_{n+2} definable.

We are now ready to show that $URP_{\Pi_n} \vdash Con^n_{PA}$.

LEMMA 2.4
For any $n \geq 1$, $URP_{\Pi_n} \vdash Con^n_{PA}$.

PROOF We prove this by induction on n. For the base case we have that $Con^1_{PA} = PA \cup \{\exists v_1 \neg Pr_{PA}(v_1)\}$. By Theorem 7.2.4 from the course notes $URP_{\Pi_1} \vdash PA \cup \{\neg Pr_{PA}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})\}$ and so $URP_{\Pi_1} \vdash PA \cup \{\exists v_1 \neg Pr_{PA}(v_1)\}$. For the inductive step, assume the claim holds for a fixed n. Then clearly $URP_{\Pi_{n+1}} \vdash Con^n_{PA}$, hence we must only verify that $URP_{\Pi_{n+1}} \vdash \exists v_1 \neg Pr_{Con^n_{PA}}(v_1)$. This will be done by showing that $URP_{\Pi_{n+1}} \vdash \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$. By Lemma 2.3 Con^n_{PA} is Σ_{n+1} definable, so $Pr_{Con^n_{PA}}(x)$ is Σ_{n+1} by Lemma 4.2.2 from the course notes. Now, $Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}) = \exists y \; proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, y)$, and so $\forall y \; \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, y)$ is Π_{n+1}. Now, $URP_{\Pi_{n+1}} \vdash \neg \overline{0} = \overline{0} \to \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, y)$ as it is a tautology. Hence, by the two first provability rules and modus ponens, we get that $URP_{\Pi_{n+1}} \vdash Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}) \to Pr_{Con^n_{PA}}(\overline{\ulcorner \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, \dot{y}) \urcorner})$. Now, we fix a model $\mathcal{M}$ of $URP_{\Pi_{n+1}}$ and show that $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$. Suppose that there is a $m \in \mathcal{M}$ such that $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, \dot{m}) \urcorner})$, then $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$. If, on the other hand, there is no $m \in \mathcal{M}$ such that $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, \dot{m}) \urcorner})$, then $\mathcal{M} \models \forall x Pr_{Con^n_{PA}}(\overline{\ulcorner \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, \dot{x}) \urcorner})$. Now, $\forall x \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, x)$ is Π_{n+1}, and so clearly $\neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, x)$ is Π_{n+1}. Hence, using the uniform reflection principle, we get that $\mathcal{M} \models \forall x \neg proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, x)$. This, of course, gives us that $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$. Hence, we have shown that for any model $\mathcal{M}$ of $URP_{\Pi_{n+1}}$, $\mathcal{M} \models \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$, and so by Gödel's Completeness Theorem we get that $URP_{\Pi_{n+1}} \vdash \neg Pr_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner})$.

We now show that T_n is equivalent with Con^n_{PA}. This will immediately give us that $Con^1_{PA} \vdash URP_{\Pi_1}$ as we know that $PA \cup \{Con_{PA}\} \vdash URP_{\Pi_1}$ from the course notes. The crutch of the following proof will be to show that $PA \vdash \exists x \; proof_{PA}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, x) \to \forall y \exists z \; proof_{PA}(y, z)$. We will do this by using Gödel's Completeness Theorem, hence we will fix a model $\mathcal{M} \models PA \cup \{\exists x \; proof_{PA}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, x)\}$ and show that $\mathcal{M} \models \forall y \exists z \; proof_{PA}(y, z)$. The basic idea is to take the non-standard proof x of $\neg 0 = 0$ and replace any instance of $\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}$ by y. This will give us a non-standard proof z of y.

LEMMA 2.5
For any n, $Con^n_{PA} \vdash\dashv T_n$.

Proof $T_n \vdash Con^n_{PA}$ is clear. We will prove $Con^n_{PA} \vdash T_n$ by induction on n. The base case is trivial as $Con^0_{PA} = PA = T_0$. For the inductive step we assume that $Con^n_{PA} \vdash T_n$ for a fixed n. We are going to show that $Con^n_{PA} \vdash \exists x \neg Pr_{Con^n_{PA}}(x) \to Con_{T_n}$ by showing the contrapositive; $Con^n_{PA} \vdash \neg Con_{T_n} \vdash \forall x Pr_{Con^n_{PA}}(x)$. We have that $Con^n_{PA} \vdash\dashv T_n$, and so we only need to show that $Con^n_{PA} \vdash \neg Con_{Con^n_{PA}} \vdash \forall x Pr_{Con^n_{PA}}(x)$. We fix a model $\mathcal{M}$ of Con^n_{PA} and assume that $\mathcal{M} \models \neg Con_{Con^n_{PA}}$. This gives us an $m \in \mathcal{M}$ such that $\mathcal{M} \models proof_{Con^n_{PA}}(\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}, m)$. We then fix $n \in \mathcal{M}$ and aim to show that $\mathcal{M} \models Pr_{Con^n_{PA}}(n)$. We recall from the lecture notes that there is an Δ_0 formulae kPn which says that k is a substring of n. We see that $PA \vdash \forall b \forall a \leq b(aPb \to \exists x, y \leq b(b = ax + y \wedge \neg aPy))$, which is essentially saying that if a is a substring of b, then we can split b into two substrings b_1 and b_2 such that all instances of a is in b_1. We note that $\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner} Pm$, and so there are $x, y \leq m$ such that $\mathcal{M} \models m = x\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner} + y \wedge \neg \overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner} Py$. We then let $m' = xn + y$ and argue that $\mathcal{M} \models proof_{Con^n_{PA}}(n, m')$. The idea is that m is a non-standard proof of $\neg 0 = 0$, and so if we replace any instance of $\overline{\ulcorner \neg \overline{0} = \overline{0} \urcorner}$ with n in m, $\mathcal{M}$ will still think that m' is a proof in Con^n_{PA}. Furthermore, m' is on the form $l^\frown \ulcorner \# \urcorner {}^\frown n^\frown \ulcorner \# \urcorner$ for some $l \in \mathcal{M}$, and so $\mathcal{M}$ thinks that m' is a proof of n. Hence we get that $\mathcal{M} \models \forall x Pr_{Con^n_{PA}}(x)$ and so, by Gödel's Completeness Theorem, $PA \vdash \neg Con_{PA} \to \forall x Pr_{PA}(x)$.

As mentioned earlier, we get the following Corollary immediately.

> **Corollary 2.6**
> $Con^1_{PA} \vdash URP_{\Pi_1}$.

Before we move on to show that we have not gained any degrees of completeness, we show that the previous Corollary is the best we are going to get. In other words, we will show that for $n \geq 2$, $Con^n_{PA} \nvdash URP_{\Pi_n}$. To do this we need to add some new notation. We let $URP_{\Pi_n}(X)$ be the following set of axioms $\{\forall n Pr_X(\overline{\ulcorner F(\dot{n}) \urcorner}) \to \forall n F(n) : F(n) \in \Pi_n\}$. Hence $PA + URP_{\Pi_n}(PA) = URP_{\Pi_n}$. We first show a well know Lemma that shows how much more deductive power $URP_{\Pi_{n+1}}$ has than URP_{Π_n}.

> **Lemma 2.7**
> Let U be a finite Σ_n extension over PA, then $U \vdash URP_{\Pi_n} \leftrightarrow URP_{\Pi_n}(U)$.

Proof It is sufficient to derive $URP_{\Pi_n}(U)$ from $U + URP_{\Pi_n}$. Let ψ be the conjunction of the extra axioms for U, then ψ is clearly Σ_n. Hence, we fix a Π_n formula $F(n)$ and aim to show $U + URP_{\Pi_n} \vdash \forall n Pr_U(\overline{\ulcorner F(\dot{n}) \urcorner}) \to \forall n F(n)$. We note that $\psi \to F(n)$ is Π_n, and so we have that $U + URP_{\Pi_n} \vdash \forall n Pr_{PA}(\overline{\ulcorner \psi \to F(\dot{n}) \urcorner}) \to \forall n(\psi \to F(n))$. This gives us that $U + URP_{\Pi_n} \vdash \forall n Pr_U(\overline{\ulcorner F(\dot{n}) \urcorner}) \to (\forall n Pr_{PA}(\overline{\ulcorner \psi \to F(\dot{n}) \urcorner}) \to \forall n(\psi \to F(n)))$. Now, $U \vdash \psi$, and so $U + URP_{\Pi_n} \vdash \forall n Pr_U(\overline{\ulcorner F(\dot{n}) \urcorner}) \to \forall n F(n)$, as required.

We get the following Corollary immediately.

COROLLARY 2.8
Let S be a first order theory containing PA, then $S + URP_{\Pi_{n+1}}(S) \vdash Con(S + URP_{\Pi_n}(S))$, where we write $Con(S + URP_{\Pi_n}(S))$ instead of $Con_{S+URP_{\Pi_n}(S)}$ for better readability.

Using the previous Lemma and its Corollary, we are now ready to show that $Con^n_{PA} \nvdash URP_{\Pi_n}$ for $n \geq 2$.

LEMMA 2.9
Let $n \geq 2$, then $Con^n_{PA} \nvdash URP_{\Pi_n}$.

PROOF Fix $n \geq 2$. By Lemma 2.3 we know that Con^{n-1}_{PA} is Σ_n definable, and it is clearly a finite extension of PA as well. Hence, by Lemma 2.7 we get that $Con^{n-1}_{PA} \vdash URP_{\Pi_n} \leftrightarrow URP_{\Pi_n}(Con^{n-1}_{PA})$. Assume for contradiction that $Con^n_{PA} \vdash URP_{\Pi_n}$, then $Con^n_{PA} \vdash URP_{\Pi_n}(Con^{n-1}_{PA})$. Using Corollary 2.8 twice, which we can do as $n \geq 2$, we get that $Con^n_{PA} \vdash Con(Con(Con^{n-1}_{PA}))$, and so $Con^n_{PA} \vdash Con(Con^n_{PA})$, contradicting Gödel's Second Incompleteness Theorem.

We will now show that although both extensions are non-conservative, they are not Σ_n complete for any $n \geq 2$.

LEMMA 2.10
For any $n \geq 1$, URP_{Π_n} is not Σ_{n+1} complete.

PROOF We use a similar proof as in Theorem 4.3.1 from the course notes to argue that URP_{Π_n} is Δ_1 definable. There is an algorithm that decides whether or not a formula is an element of PA or not, as claimed in the proof of Theorem 4.3.1, and this algorithm can easily be extended to check whether a formula is an element of URP_{Π_n} or not. Hence, by Lemma 4.2.2 from the course notes, $Pr_{URP_{\Pi_n}}(x)$ is Σ_1 and so $Con_{URP_{\Pi_n}}$ is Π_1, which means that $Con_{URP_{\Pi_n}}$ is Σ_{n+1} for any $n \geq 1$. It follows from Gödel's Second Incompleteness Theorem that $URP_{\Pi_n} \nvdash Con_{URP_{\Pi_n}}$, which makes URP_{Π_n} not Σ_{n+1} complete as $\mathbb{N} \models Con_{URP_{\Pi_n}}$.

A simpler way of showing something weaker would be to note that if URP_{Π_n} were Σ_{n+1} complete for all n, then URP would have been complete, as it would have been Σ_n complete for all n, and this contradicts Gödel's First Incompleteness Theorem. We now note an easy Corollary.

Corollary 2.11
For any $n \geq 1$, Con^n_{PA} is not Σ_{n+1} complete.

Proof It there was an $n \geq 1$ such that Con^n_{PA} were Σ_{n+1} complete, then URP_{Π_n} would have been Σ_{n+1} complete as well due to $URP_{\Pi_n} \vdash Con^n_{PA}$.

3 PA in ZFC

In this section we will examine how ZFC can interpret PA. More concretely, we will let $\Theta = \{\varphi : ZFC \vdash (\omega, \in) \models \varphi\}$ and show that $\Theta \vdash Con^n_{PA}$ for all n. Furthermore, we will show that Θ is consistent and not complete under the assumption that ZFC is consistent.

Lemma 3.1
$\Theta \vdash Con^n_{PA}$ for all $n \in \omega$.

Proof For convenience, we will show that $\Theta \vdash T_n$ for all $n \in \omega$. This will be done by induction on n. Base case: $n = 0$. It should be clear that $\Theta \vdash PA$. Inductive case: assume $\Theta \vdash T_n$, we need to show that $\Theta \vdash Con_{T_n}$. By assumption, $ZFC \vdash (\omega, \in) \models T_n$, and so $ZFC \vdash Con_{T_n}$ as it shows that T_n has a model. We are here using the fact that ZFC can also prove that a set of first-order sentences is consistent just in case it has a model. From here we see that $ZFC \vdash (\omega, \in) \models Con_{T_n}$ as ZFC has just proved that there are no element of ω that codes a proof of a contradiction in T_n. Hence, we get that $\Theta \vdash Con_{T_n}$, as required.

Again, we get an immediate Corollary.

Corollary 3.2
$\Theta \vdash URP_{\Pi_1}$.

We now show that Θ is consistent and incomplete, given that ZFC is consistent. If ZFC is inconsistent, then Θ is inconsistent and (trivially) complete.

Lemma 3.3
If ZFC is consistent, then Θ is consistent.

PROOF Suppose for contradiction that $\varphi \in \Theta$ and $\neg\varphi \in \Theta$. Then $ZFC \vdash (\omega, \in) \models \varphi$ and $ZFC \vdash (\omega, \in) \models \neg\varphi$. We also get that $ZFC \vdash \neg(\omega, \in) \models \varphi$ as $ZFC \vdash (\omega, \in) \models \neg\varphi \leftrightarrow \neg(\omega, \in) \models \varphi$. Hence $ZFC \vdash \psi$ and $ZFC \vdash \neg\psi$ for $\psi := (\omega, \in) \models \varphi$, contradicting the consistency of ZFC.

> LEMMA 3.4
> If ZFC is consistent, then Θ is incomplete.

PROOF It follows from Gödel's Second Incompleteness Theorem that if ZFC is consistent then $ZFC + \neg Con_{ZFC}$ is consistent. Hence, we have a model $\mathcal{M}$ such that $\mathcal{M} \models ZFC + \neg Con_{ZFC}$. In other words, $\mathcal{M} \models (\omega, \in) \models \neg Con_{ZFC}$ and so $ZFC \nvdash (\omega, \in) \models Con_{ZFC}$ by Gödel's Completeness Theorem. This makes Θ incomplete as we have that $\mathbb{N} \models Con_{ZFC}$ under the assumption of consistency of ZFC.

4 Conclusion

We have seen in this essay different way of strengthening PA to get closer to axiomatising true arithmetic. The perhaps most intuitive strengthening, Con^n_{PA}, has also been the weakest since we have seen that $URP_{\Pi_n} \vdash Con^n_{PA}$ and $\Theta \vdash Con^n_{PA}$. We have also seen that non of our attempts have fully succeeded since none of them are complete, but they are all non-conservative over PA so they are all able to prove more than PA is.

References