Axiomatic Set Theory

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1 Introduction

In this essay, we will explore models of ZF and models of ZFC, and hence ZFC will be assumed throughout the essay. The aim of the first section is to show that if we assume the existence of a (strongly) inaccessible cardinal κ, then $V_\kappa \models ZF$. Most of the axioms are easy to verify, and the only real difficulty is with the axiom of Replacement. To get around the difficulty of showing that $V_\kappa \models$ Replacement directly, we will introduce another type of model H_λ, for which it is easy to show that Replacement holds for all regular cardinals. From there we will show that $V_\kappa = H_\kappa$ when κ is an inaccessible cardinal. We will round out the first section by showing that if $V_\lambda \models ZF$ then $L_\lambda \models ZFC$. From there we will note an easy Corollary, using the Downward Lowenheim-Skolem Theorem and Gödel's Condensation Lemma, to show under the assumption that $V_\lambda \models ZF$ we get an $\alpha < \omega_1$ for which $L_\alpha \models ZFC$[1].

In the second section we will introduce a semantic for second order logic, and show a variant of the downward Lowenheim-Skolem Theorem that presupposes the existence of a large cardinal.

2 Models of ZF and ZFC

The main aim for this section is to show that if κ is an inaccessible cardinal, then $V_\kappa \models ZF$. As explained in the introduction, we will introduce a new type of model, called H_λ, to make the proof of Replacement easier. H_λ will be the set of all sets whose

1 The author is inclined to mention that L_α will be a countable model of ZFC. Since ZFC is capable of constructing the real numbers, they will appear as a subset of L_α. This is a version of the famous Skolem Paradox.

transitive closure has cardinality less than λ. We let $Tc(x)$ denote the transitive closure of a set x. By Theorem 3.2.4 from the course notes we know that existence and uniqueness holds for the transitive closure. The first thing we need to show is that H_λ is indeed a set. Before we have shown this, writing $x \in H_\lambda$ is abuse of notation and simply means that $|Tc(x)| < \lambda$.

> LEMMA 2.1
> Let λ be an infinite cardinal, if $x \in H_\lambda$ then $x \in V_\lambda$.

PROOF By Lemma 10.1.5 from the course notes we know that $\{\aleph_\alpha : \alpha \in On\}$ is the set of all infinite cardinals. Hence, we do a proof by induction on α. For the base case, we know that H_ω is a set and that $H_\omega = V_\omega$ from exercise 7 of problem sheet 1. We quickly show the limit case before proceeding with the more lengthy successor case. Let $\gamma > \omega$ be a limit ordinal. Fix a set x and suppose that $|Tc(x)| < \aleph_\gamma$. Then $|Tc(x)| = \aleph_\alpha$ for some $\alpha < \gamma$, and so $|Tc(x)| < \aleph_{\alpha+1} < \aleph_\gamma$ as γ is a limit ordinal. By the induction hypothesis we get that $x \in V_{\aleph_{\alpha+1}}$ and so $x \in V_{\aleph_\gamma}$.

We now do the successor case by showing the contrapositive. Suppose the claim holds for a fixed α. That is, for any set x, if $x \notin V_{\aleph_\alpha}$ then $|Tc(x)| \geq \aleph_\alpha$. We now fix a set x and suppose that $x \notin V_{\aleph_{\alpha+1}}$. We need to show that $|Tc(x)| \geq \aleph_{\alpha+1}$. Now $x \notin V_{\aleph_{\alpha+1}}$ gives us that $x \notin V_{\aleph_\alpha}$ and so by the induction hypothesis we have that $|Tc(x)| \geq \aleph_\alpha$. Suppose for contradiction that $|Tc(x)| = \aleph_\alpha$. For any ordinal $\beta < \omega_{\alpha+1}$ we define $z_\beta = \{y : y \in Tc(x) \wedge y \in V_{\omega_\alpha+\beta}\}$. Let $M_\beta = Tc(x) \setminus z_\beta$. We fist observe that if there is a $\beta < \omega_{\alpha+1}$ such that M_β is empty then for all $y \in x$, $y \in z_\beta$ and so $y \in V_{\omega_\alpha+\beta}$. This gives us that $x \subseteq V_{\omega_\alpha+\beta}$ and so $x \in V_{\omega_\alpha+\beta+1}$. Now since $\beta < \omega_{\alpha+1}$, so $\omega_\alpha + \beta + 1 < \omega_{\alpha+1}$. Hence we get our desired contradiction by $x \in V_{\omega_{\alpha+1}}$. To achieve this, we will show that for any $\beta < \omega_{\alpha+1}$, if $M_\beta \neq \emptyset$ then $M_{\beta+1} \subsetneq M_\beta$. That is, each $M_{\beta+1}$ contains strictly less elements than M_β. We know that $M_0 \subseteq Tc(x)$, and so $|M_0| \leq \aleph_\alpha$. Hence, since the $\langle M_\beta : \beta < \omega_{\alpha+1}\rangle$ is a strictly decreasing sequence of sets, there has to be a $\delta < \omega_{\alpha+1}$ such that $M_\delta = \emptyset$. If not then $|M_0| \geq \aleph_{\alpha+1}$, which we know is not the case. So, all we have to do to finish this proof is fix $\beta < \omega_{\alpha+1}$ and find a $y \in M_\beta$ such that $y \notin M_{\beta+1}$. By the axiom of foundations there is a $y \in M_\beta$ such that $M_\beta \cap y = \emptyset$. We note that $y \in Tc(x)$ and so for all $t \in y$, $t \in Tc(x)$. Since $M_\beta \cap y = \emptyset$ we know that for all $t \in y$, $t \in z_\beta$. Hence, for all $t \in y$, $t \in V_{\omega_\alpha+\beta}$. This gives us that $y \subseteq V_{\omega_\alpha+\beta}$ and so $y \in V_{\omega_\alpha+\beta+1}$. This means that $y \in M_\beta$ but $y \notin M_{\beta+1}$ and so we get $M_{\beta+1} \subsetneq M_\beta$, as required.

We now get this immediate Corollary.

> COROLLARY 2.2 (TASK 1)
> For all infinite λ, H_λ is a set.

PROOF Let $\phi_\lambda(a)$ express in the language of set theory that $|Tc(a)| < \lambda$, then by the axiom of separation we get that $H_\lambda = \{x \in V_\lambda : \phi_\lambda(x)\}$ is a set.

We will now show that for a regular cardinal λ, $H_\lambda \models$ Replacement. Before we do this, we note that the assumption of λ being regular is necessary. One example to showcase why is to consider the function $f : \omega \to H_{\aleph_\omega}$ given by $n \to \omega_n$. We denote the image of f, which is $\{\omega_n : n \in \omega\}$, by z and note that although for each $n \in \omega$, $|Tr(\omega_n)| = |\omega_n| = \aleph_n < \aleph_\omega$ we see that $|Tc(z)| = |z| + \Sigma_{y \in z}|Tc(z)| = \aleph_0 + \Sigma_{n \in \omega}\aleph_n = \aleph_\omega$ and hence $z \notin H_{\aleph_\omega}$. Replacement therefore fails for singular cardinals. Before we can show that Replacement holds in H_λ when λ is regular, we need a helpful Lemma.

LEMMA 2.3
Suppose λ is a regular cardinal, and s a set such that $|s| < \lambda$ and for all $y \in s$, $|Tc(y)| < \lambda$, then $|Tc(s)| < \lambda$.

PROOF We notice that that $Tc(s) = s \cup (\bigcup_{y \in s} Tc(y))$. Hence $|Tc(s)| = |s| + \Sigma_{y \in s}|Tc(y)| \leq \lambda + \lambda * \lambda = \lambda$. Suppose for contradiction that $|Tc(s)| = \lambda$, then $\Sigma_{y \in s}|Tc(y)| = \lambda$ as the axiom of choice gives us that $|s| + \Sigma_{y \in s}|Tc(y)| = \max(|s|, \Sigma_{y \in s}|Tc(y)|)$ and $|s| < \lambda$. However, $\Sigma_{y \in s}|Tc(y)| = \lambda$ contradicts regularity of λ as $|s| < \lambda$ and for all $y \in s$, $|Tc(y)| < \lambda$. Hence $|Tc(s)| < \lambda$, as required.

We now show that $K_\lambda \models$ Replacement for regular cardinals.

THEOREM 2.4 (TASK 3)
Let λ be a regular cardinal, then $H_\lambda \models$ Replacement.

PROOF Fix a formula $\phi(x, y)$ and suppose that $H_\lambda \models \forall x, y, y'((\phi(x, y) \wedge \phi(x, y')) \to y = y')$. Fix $s \in H_\lambda$, we must find a $z \in H_\lambda$ such that $H_\lambda \models \forall y(y \in z \leftrightarrow \exists x \in s\phi(x, y))$. Let $\psi(x, y) = x \in H_\lambda \wedge y \in H_\lambda \wedge \phi^{H_\lambda}(x, y)$, then $V \models \forall x, y, y'((\psi(x, y) \wedge \psi(x, y')) \to y = y')$ by Lemma 4.1.6 from the course notes. Since $s \in V$ and $V \models$ Replacement, there is a $z \in V$ such that $V \models \forall y(y \in z \leftrightarrow \exists x \in s\psi(x, y))$. Clearly $|z| \leq |s| < \lambda$, and for all $y \in z$, $y \in H_\lambda$ as there is an $x \in s$ such that $\psi(x, y)$ holds, so by Lemma 2.3 we get that $z \in H_\lambda$ since λ is regular. As observed earlier H_λ is transitive, and $\forall y(y \in z \leftrightarrow \exists x \in s\psi(x, y))$ is Π_1 and hence downwards absolute, which gives us that $H_\lambda \models \forall y(y \in z \leftrightarrow \exists x \in s\psi(x, y))$.

We will now define a (strongly) inaccessible cardinal and show that if κ is strongly inaccessible, then $V_\kappa = H_\kappa$.

Definition 2.5 (Strongly Inaccessible)
A cardinal $\kappa > \aleph_0$ is (strongly) inaccessible just in case it is regular and for each cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.

Before we can show that for inaccessible cardinals κ, $H_\kappa = V_\kappa$, we need another helpful Lemma.

Lemma 2.6 (Task 4 i))
Suppose that κ is inaccessible, then for all $\alpha < \kappa$, $|V_\alpha| < \kappa$.

Proof We prove this by induction on α. The base case is trivial. Suppose the statement holds for α and that $\alpha + 1 < \kappa$. Let $\lambda = |V_\alpha|$, then by the induction hypothesis $\lambda < \kappa$. We know that $|V_{\alpha+1}| = 2^\lambda$, and so by assumption $|V_{\alpha+1}| < \kappa$. Suppose $\gamma < \kappa$ is a limit ordinal and suppose the claim holds for all $\alpha < \gamma$. We see that $|V_\gamma| = |\bigcup_{\alpha<\gamma} V_\alpha| = \Sigma_{\alpha<\gamma}|V_\alpha| < \kappa$ as κ is regular and $|V_\alpha| < \kappa$ by the induction hypothesis.

We now show that for inaccessible κ, $H_\kappa = V_\kappa$.

Lemma 2.7 (Task 4 ii))
Suppose that κ is inaccessible, then $V_\kappa = H_\kappa$.

Proof We know from Lemma 2.1 and Task 1 that $H_\kappa \subseteq V_\kappa$. Let $x \in V_\kappa$. Now κ is a cardinal and hence a limit ordinal, which means that there is an $\alpha < \kappa$ such that $x \in V_\alpha$. V_α is transitive, so $x \subseteq V_\alpha$. An inductive argument on the transitivity of V_α also shows us that $Tc(x) \subseteq V_\alpha$. Hence $|Tc(x)| < |V_\alpha| < \kappa$ by Lemma 2.6. This gives us $x \in H_\kappa$, as required.

We now need to verify that $V_\kappa \models ZF$ - Replacement. It is not always necessary to assume that κ is inaccessible to show that V_κ satisfy the other axioms. The next Lemma will show us two examples that for some axioms, any infinite cardinal λ works.

Lemma 2.8 (Task 2)
Fix an infinite cardinal λ, then $\langle V_\lambda, \in \rangle \models$ Power set, Separation.

Proof Power set: We have to show that $\langle V_\lambda, \in \rangle \models \forall x \exists y \forall t(t \in y \leftrightarrow \forall z(z \in t \to z \in x))$. Fix $a \in V_\lambda$, now λ is a cardinal, and so it is a limit ordinal. Hence there is a $\alpha < \lambda$ such that $a \in V_\alpha$. V_α is transitive, so $a \subseteq V_\alpha$, which means that for all $b \in P(a)$, $b \subseteq V_\alpha$. Hence, for all $b \in P(a)$, $b \in V_{\alpha+1}$ and so $P(a) \subseteq V_{\alpha+1}$. This gives us $P(a) \in V_{\alpha+2}$, and so $P(a) \in V_\lambda$. Hence, it suffices to show that $\langle V_\lambda, \in \rangle \models \forall t(t \in P(a) \leftrightarrow \forall z(z \in t \to z \in a))$. Now, clearly $\langle V, \in \rangle \models \forall t(t \in P(a) \leftrightarrow \forall z(z \in t \to z \in a))$, and since the relevant formula is Π_1, and therefore downwards absolute between the two transitive classes $V_\lambda \subset V$, we get the desired result.

Separation: Fix a formula in the language of set theory $\phi(x_1, \ldots, x_n, y)$. We must show that $\langle V_\lambda, \in \rangle \models \forall x_1 \ldots \forall x_n \forall u \exists z \forall y(y \in z \leftrightarrow (y \in u \wedge \phi(x_1, \ldots, x_n, y))$. So, fix $a_1, \ldots, a_n, a \in V_\lambda$, we aim to show that $\langle V_\lambda, \in \rangle \models \exists z \forall y(y \in z \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. By the axiom of separation there is a set b such that $b = \{y : y \in a \wedge \phi^{V_\lambda}(a_1, \ldots, a_n, y)\}$. Now, we know that λ is a limit ordinal, and so there is a ordinal $\alpha < \lambda$ such that $a \in V_\alpha$. V_α is transitive so $a \subseteq V_\alpha$. Clearly $b \subseteq a$, and so $b \subseteq V_\alpha$, which gives us that $b \in V_{\alpha+1}$. Hence $b \in V_\lambda$. It remains to fix $y \in V_\lambda$ and show that $\langle V_\lambda, \in \rangle \models y \in b \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. Suppose $\langle V_\lambda, \in \rangle \models y \in b$, then $y \in b$ by Σ_0 absoluteness and so $y \in a$ and $\phi^{V_\lambda}(a_1, \ldots, a_n, y)$ holds. By Lemma 4.1.6 from the course notes we get that $\langle V_\lambda, \in \rangle \models y \in a \wedge \phi(a_1, \ldots, a_n, y)$ and so $\langle V_\lambda, \in \rangle \models y \in b \to (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. Conversely, suppose that $\langle V_\lambda, \in \rangle \models y \in a \wedge \phi(a_1, \ldots, a_n, y)$, then by Lemma 4.1.6 from the course notes we get that $y \in a$ and $\phi^{V_\lambda}(a_1, \ldots, a_n, y)$ holds. Hence $y \in b$ and so by Σ_0 absoluteness $\langle V_\lambda, \in \rangle \models y \in b$. Hence $\langle V_\lambda, \in \rangle \models (y \in a \wedge \phi(a_1, \ldots, a_n, y)) \to y \in b$ and so $\langle V_\lambda, \in \rangle \models y \in b \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$, as required.

Combining all we have done so far, we show that if κ is inaccessible then $V_\kappa \models ZF$.

Theorem 2.9 (Task 4 iii))
Suppose that κ is inaccessible, then $V_\kappa \models ZF$.

Proof By Task 2 we have already seen that $V_\kappa \models$ Power set and Separation. Furthermore, from Task 3 and Lemma 2.7 we know that $V_\kappa \models$ Replacement. We now verify the rest. Before we do this, we recall that Σ_0-sentences are absolute between transitive classes. Hence, if we can show that V satisfies a Σ_0 sentence, then we know that V_κ satisfies the same sentence.

Extensionality: Fix $x, y \in V_\kappa$. $V \models x = y \leftrightarrow (\forall t \in x(t \in y) \wedge \forall t \in y(t \in x))$. This is Σ_0, and so $V_\kappa \models x = y \leftrightarrow (\forall t \in x(t \in y) \wedge \forall t \in y(t \in x))$. This in turn gives us that $V_\kappa \models x = y \leftrightarrow \forall t(t \in x \leftrightarrow t \in y)$, as required.

Empty set: We know that $\emptyset \in V_\kappa$. $V \models \forall y(y \notin \emptyset)$. This is Π_1 and so downwards absolute. Hence $V_\kappa \models \forall y(y \notin \emptyset)$, as required.

Pairing: Fix $x, y \in V_\kappa$. κ is a limit ordinal, and so there is $\alpha, \beta \in \kappa$ such that $x \in V_\alpha$ and $y \in V_\beta$. We may without loss of generality assume that $\alpha \leq \beta$. Hence $x \in V_\beta$ as well. By the pairing axiom $\{x, y\}$ exist, and clearly $\{x, y\} \subseteq V_\beta$. Hence $\{x, y\} \in V_{\beta+1}$ and so $\{x, y\} \in V_\kappa$ as $\beta + 1 < \kappa$. $V \models \forall t(t \in \{x, y\} \leftrightarrow (t = x \vee t = y))$, which is a Π_1 formula and hence downwards absolute. Hence $V_\kappa \models \forall t(t \in \{x, y\} \leftrightarrow (t = x \vee t = y))$, as required.

Union: Fix $x \in V_\kappa$, then $x \in V_\alpha$ for some $\alpha < \kappa$ as κ is a limit ordinal. V_α is transitive, and so $x \subseteq V_\alpha$. This means that for each $y \in x$, $y \in V_\alpha$. Another application of transitivity gives us that for each $y \in x$, $y \subseteq V_\alpha$. This gives us that $\cup x \subseteq V_\alpha$ as each $t \in \cup x$ is an element of some y who is an element of x, and so $t \in V_\alpha$ as $y \subseteq V_\alpha$. Hence $\cup x \in V_{\alpha+1}$ and so $\cup x \in V_\kappa$ as $\alpha+1 < \kappa$. Clearly $V \models \forall t(t \in \cup x \leftrightarrow \exists w \in x(t \in w))$, which is a Π_1 sentence and hence downwards absolute. This gives us $V_\kappa \models \forall t(t \in \cup x \leftrightarrow \exists w \in x(t \in w))$, as required.

Infinity: $\kappa > \omega + 1$, and so $\omega \in V_\kappa$. Now being ω is Σ_0 by the 28th clause of Theorem 7.1.11 from the course notes, and so since V thinks ω is ω then V_κ agrees[2].

Foundation: Fix $x \in V_\kappa$ and suppose $V_\kappa \models \exists z(z \in x)$. This is Σ_1, and so upwards absolute. Hence $V \models \exists z(z \in x)$. Since $V \models$ Foundation we get that $V \models \exists z(z \in x \wedge \forall y \in z(y \notin x))$. Let $z \in V$ be such that $V \models z \in x \wedge \forall y \in z(y \notin x)$. $z \in x$ means that $z \in V_\kappa$, and so since the sentence is Σ_0 we get that $V_\kappa \models z \in x \wedge \forall y \in z(y \notin x)$, as required.

We note that the only place we used the fact that κ was uncountable was when verifying the axiom of infinity[3]. This shows us that $V_\omega \models$ ZF $-$ Infinity. The last major thing we want to do in this section is to show that if λ is such that $V_\lambda \models ZF$, then $L_\lambda \models ZFC$. To get the Axiom of Choice, we will use that $ZF + V = L \vdash AC$, and so we first show that $L_\gamma \models V = L$ for any limit ordinal γ.

Lemma 2.10
Let γ be a limit ordinal, then $L_\gamma \models V = L$.

Proof Fix $a \in L_\gamma$, we must show that $L_\gamma \models \exists \alpha(On(\alpha) \wedge a \in L_\alpha)$. Now, γ is a limit ordinal, and so there is a $\beta < \gamma$ such that $a \in L_\beta$. We know that $L_\beta \in L_\gamma$, and so by the same absoluteness and transitivity arguments as in the proof of Theorem 7.1.16 from the course notes we get that $L_\gamma \models On(\beta) \wedge a \in L_\beta$, and the result follows.

From here we show that if $V_\lambda \models ZF$, then $L_\lambda \models ZFC$.

Lemma 2.11 (Task 5 i))
Suppose there is an ordinal λ such that $V_\lambda \models ZF$, then $L_\lambda \models ZFC$.

Proof It is easy to verify that λ has to be a limit ordinal, as, for example, the Power set axiom would not hold if λ weren't a limit ordinal. Furthermore $\lambda > \omega$ as V_ω doesn't satisfy the axiom of infinity.

2 This last sentence might sound very odd, but when I am saying that V thinks ω is ω, what I mean is that V thinks ω satisfies the definition of ω as given by the axiom of infinity, which the theorem tells us is Σ_0 and this is why V_κ agrees with V.

3 For Replacement we recall that $V_\omega = H_\omega$.

Extensionality: Fix $x, y \in L_\lambda$. $V_\lambda \models x = y \leftrightarrow (\forall t \in x(t \in y) \wedge \forall t \in y(t \in x))$. This is Σ_0, and so $L_\lambda \models x = y \leftrightarrow (\forall t \in x(t \in y) \wedge \forall t \in y(t \in x))$ as $L_\lambda \subseteq V_\lambda$ and both are transitive. This in turn gives us that $L_\lambda \models x = y \leftrightarrow \forall t(t \in x \leftrightarrow t \in y)$, as required.

Empty set: We know that $\emptyset \in L_\lambda$. $V_\lambda \models \forall y(y \notin \emptyset)$. This is Π_1 and so downwards absolute. Hence $L_\lambda \models \forall y(y \notin \emptyset)$, as required.

Pairing: Fix $x, y \in L_\lambda$. λ is a limit ordinal, and so there is $\alpha, \beta \in \kappa$ such that $x \in L_\alpha$ and $y \in L_\beta$. We may without loss of generality assume that $\alpha \leq \beta$. Hence $x \in L_\beta$ as well. By the pairing axiom $\{x, y\}$ exist, and clearly $\{x, y\} \subseteq L_\beta$. $\{x, y\}$ is also clearly a definable subset, and so $\{x, y\} \in L_{\beta+1}$. This gives us $\{x, y\} \in L_\lambda$ as $\beta + 1 < \lambda$. $V_\lambda \models \forall t(t \in \{x, y\} \leftrightarrow (t = x \vee t = y))$, which is a Π_1 formula and hence downwards absolute. Hence $L_\lambda \models \forall t(t \in \{x, y\} \leftrightarrow (t = x \vee t = y))$, as required.

Union: Fix $x \in L_\lambda$, then $x \in L_\alpha$ for some $\alpha < \lambda$ as λ is a limit ordinal. L_α is transitive, and so $x \subseteq L_\alpha$. This means that for each $y \in x$, $y \in L_\alpha$. Another application of transitivity gives us that for each $y \in x$, $y \subseteq L_\alpha$. This gives us that $\cup x \subseteq L_\alpha$ as each $t \in \cup x$ is an element of some y who is an element of x, and so $t \in L_\alpha$ as $y \subseteq L_\alpha$. Now $\cup x$ is a definable subset, and so $\cup x \in L_{\alpha+1}$. This gives us that $\cup x \in L_\lambda$ as $\alpha + 1 < \lambda$. Clearly $V_\lambda \models \forall t(t \in \cup x \leftrightarrow \exists w \in x(t \in w))$, which is a Π_1 sentence and hence downwards absolute. This gives us $L_\lambda \models \forall t(t \in \cup x \leftrightarrow \exists w \in x(t \in w))$, as required.

Separation Scheme: Fix a formula in the language of set theory $\phi(x_1, \ldots, x_n, y)$. We must show that $\langle L_\lambda, \in \rangle \models \forall x_1 \ldots \forall x_n \forall u \exists z \forall y(y \in z \leftrightarrow (y \in u \wedge \phi(x_1, \ldots, x_n, y))$. So, fix $a_1, \ldots, a_n, a \in L_\lambda$, we aim to show that $\langle L_\lambda, \in \rangle \models \exists z \forall y(y \in z \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. By the axiom of separation there is a set b such that $b = \{y : y \in a \wedge \phi^{L_\lambda}(a_1, \ldots, a_n, y)\}$. Now, we know that λ is a limit ordinal, and so there is a ordinal $\alpha < \lambda$ such that $a \in L_\lambda$. L_λ is transitive so $a \subseteq L_\alpha$. Clearly $b \subseteq a$, and so $b \subseteq L_\alpha$, which gives us that $b \in L_{\alpha+1}$ as b is a definable subset of a, and hence of L_α, by definition. Hence $b \in L_\lambda$. It remains to fix $y \in L_\lambda$ and show that $\langle L_\lambda, \in \rangle \models y \in b \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. Suppose $\langle L_\lambda, \in \rangle \models y \in b$, then $y \in b$ by Σ_0 absoluteness and so $y \in a$ and $\phi^{L_\lambda}(a_1, \ldots, a_n, y)$ holds. By Lemma 4.1.6 from the course notes we get that $\langle L_\lambda, \in \rangle \models y \in a \wedge \phi(a_1, \ldots, a_n, y)$ and so $\langle L_\lambda, \in \rangle \models y \in b \rightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$. Conversely, suppose that $\langle L_\lambda, \in \rangle \models y \in a \wedge \phi(a_1, \ldots, a_n, y)$, then by Lemma 4.1.6 from the course notes we get that $y \in a$ and $\phi^{L_\lambda}(a_1, \ldots, a_n, y)$ holds. Hence $y \in b$ and so by Σ_0 absoluteness $\langle L_\lambda, \in \rangle \models y \in b$. Hence $\langle L_\lambda, \in \rangle \models (y \in a \wedge \phi(a_1, \ldots, a_n, y)) \rightarrow y \in b$ and so $\langle L_\lambda, \in \rangle \models y \in b \leftrightarrow (y \in a \wedge \phi(a_1, \ldots, a_n, y))$, as required.

Replacement: Let $\varphi(x, y)$ be a formula in the language of set theory and assume that $L_\lambda \models \forall x, y, y'((\varphi(x, y) \wedge \varphi(x, y')) \rightarrow y = y')$. Furthermore, we fix $s \in L_\lambda$. We need to show that there is an $z \in L_\lambda$ such that $L_\lambda \models \forall y(y \in z \leftrightarrow \exists x \in s\varphi(x, y))$. This is equivalent to showing that $L_\lambda \models \forall y \in z \exists x \in s\varphi(x, y) \wedge \forall x \in s \exists y \in z\varphi(x, y)$. To do so we let $\psi(x, y) = x \in L_\lambda \wedge y \in L_\lambda \wedge \varphi^{L_\lambda}(x, y)$. It should be clear that $V_\lambda \models \forall x, y, y'((\psi(x, y) \wedge \psi(x, y')) \rightarrow y = y')$, and so we get that there is a $z \in V_\lambda$ such that $V_\lambda \models \forall y \in z \exists x \in s\psi(x, y) \wedge \forall x \in s \exists y \in z\psi(x, y)$. Now, λ is a limit ordinal, so there is an $\alpha < \lambda$ such that $z \in V_\alpha$. Hence, $V_\lambda \models \exists \alpha < \lambda \exists z \in V_\alpha(\forall y \in z \exists x \in s\psi(x, y) \wedge \forall x \in s \exists y \in z\psi(x, y))$. Now, by a combination of Δ_0 absoluteness and relativisation (Lemma 4.1.6 from the course notes), we get that $L_\lambda \models \exists \alpha < \lambda \exists z \in V_\alpha(\forall y \in z \exists x \in s\varphi(x, y) \wedge \forall x \in s \exists y \in z\varphi(x, y))$. Furthermore, as $L_\lambda \models V = L$, we get that $L_\lambda \models \exists\alpha < \lambda\exists z \in L_\alpha(\forall y \in z\exists x \in s\varphi(x, y) \wedge \forall x \in s\exists y \in z\varphi(x, y))$. Hence, there is an $\alpha < \lambda$ and a $z \in L_\alpha$ such that $L_\alpha \models \forall y \in z\exists x \in s\varphi(x, y) \wedge \forall x \in s\exists y \in z\varphi(x, y)$, which gives us $L_\lambda \models \forall y \in z\exists x \in s\varphi(x, y) \wedge \forall x \in s\exists y \in z\varphi(x, y)$, as required.

Power set: We fix $x \in L_\lambda$, then $x \in L_\alpha$ for some $\alpha < \lambda$ as λ is a limit ordinal. From the proof of Lemma 6.1.6 of the course notes we see that if we let $A = \{y \in L_\alpha : \langle L_\alpha, \in \rangle \models \forall v_2 \in y(v_2 \in x)\}$, then $A \in L_{\alpha+1}$. Furthermore, from the same proof, we see that $A = \mathcal{P}(x) \cap L$. Clearly $\alpha + 1 < \lambda$, and so $A \in L_\lambda$. We finish the proof by verifying that $L_\lambda \models \forall t(t \in A \leftrightarrow \forall z(z \in t \to z \in x))$. Fix $t \in L_\lambda$ and suppose that $t \in A$, then $L_\alpha \models \forall v_2 \in t(v_2 \in x)$, this is Δ_0 and so $L_\lambda \models \forall v_2 \in t(v_2 \in x)$. Hence $L_\lambda \models \forall t(t \in A \to \forall z(z \in t \to z \in x))$. Conversely, suppose $L_\lambda \models \forall z \in t(z \in x)$. Then by Δ_0 absoluteness and transitivity of L_α we get that $\forall z \in t, z \in L_\alpha$ as $x \in L_\alpha$. This means that $t \subseteq L_\alpha$. Now $t \in L_\lambda$, and so t is a definable subset whose elements are all in L_α. This means that $t \in L_{\alpha+1}$ and so $t \in A$. This is absolute and hence $L_\lambda \models \forall z(z \in t \to z \in x) \to t \in A$, and so we get $L_\lambda \models \forall t(t \in A \leftrightarrow \forall z(z \in t \to z \in x))$, as required.

Infinity: $V_\lambda \models$ Infinity, and so $\lambda > \omega$. This gives us that $\omega \in L_\lambda$. Furthermore, being ω is Δ_0 by clause 28th of Theorem 7.1.11 from the course notes, and so $L_\lambda \models$ Infinity.

Foundation: Fix $x \in L_\lambda$, then $x \in V_\lambda$ and so by foundation there is a $y \in x$ such that $V_\lambda \models y \in x \wedge x \cap y = \emptyset$. L_λ is transitive and so $y \in L_\lambda$. Furthermore, $x \cap y = \emptyset$ can be written as $\forall t \in x(t \notin y) \wedge \forall t \in y(t \notin x)$, and so by Δ_0 absoluteness we get that $L_\lambda \models y \in x \wedge x \cap y = \emptyset$, as required.

Axiom of Choice: We know that $V_\lambda \models ZF$ and that $V_\lambda \models V = L$. From Theorem 9.1.6 from the course notes we know that $ZF + V = L \vdash AC$. Hence $V_\lambda \models AC$.

Using the Downward Lowenheim-Skolem Theorem and Gödel's Condensation Lemma, we get the following Corollary.

Corollary 2.12
If there is an ordinal λ such that $V_\lambda \models ZF$ then there is an $\alpha < \omega_1$ such that $L_\alpha \models ZFC$.

Proof By the previous Lemma we know that $L_\lambda \models ZFC$. By the Downward Lowenheim-Skolem Theorem there a countable elementary substructure $\langle X, \in \rangle$ for L_λ (Lemma 12.1.8 in the course notes). Using the Condensation Lemma, Theorem 12.1.1 in the course notes, we know that there is a unique α and π such that $\alpha \leq \lambda$ and $\pi : X \sim L_\alpha$ is an $\in$-isomorphism. This gives us that $L_\alpha \models ZFC$. Lastly, we know that $|\alpha| = |L_\alpha| = |X| = \aleph_0$ and so $\alpha < \omega_1$, as required.

3 Downward Lowenheim-Skolem Theorem

The goal of this section is to show a variant of the Downward Lowenheim-Skolem Theorem for a second-order logic under the assumption of a certain large cardinal. The proof will relay heavily on Lemma 3.4, which states that there is a formula $Esub(\mathcal{A}, \mathcal{B})$ which expresses that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. Hence, most of this section will be devoting to defining $Esub$. It might therefore not be surprising that we will define a Gödel encoding on $\mathcal{L}^*$. For this purpose, we enumerate the primes such that p_1 is the first prime number, p_2 the second and so on. We also assume that we have enumerated the constants, the variable letters, the functions (of each arity), the predicates (of each arity) and the variable predicates (of each arity). Lastly, to fix some notation, if $\mathcal{A}$ is an $\mathcal{L}^*$-structure, then A denotes its domain.

Definition 3.1 (Gödel numbering of formulae)
We define $F : \omega^3 \to \omega$ by $F(n, m, l) = 2^n 3^m 5^l$. We write $[n, m, l]$ for $F(n, m, l)$, and define the *Gödel number* of a formulae or a term ϕ, denoted $\ulcorner \phi \urcorner$, recursively as follows:

For a constant c_i: $\ulcorner c_i \urcorner = [0, i, i]$
For a variable v_i: $\ulcorner v_i \urcorner = [1, i, i]$
For an n-ary function f_i: $\ulcorner f_i(t_1, \ldots, t_n) \urcorner = [2, i, p_1^{\ulcorner t_1 \urcorner} p_2^{\ulcorner t_2 \urcorner} \ldots p_n^{\ulcorner t_n \urcorner}]$
For an n-ary predicate P_i and terms $t_1, \ldots, t_n$: $\ulcorner P_i(t_1, \ldots, t_n) \urcorner = [3, i, p_1^{\ulcorner t_1 \urcorner} p_2^{\ulcorner t_2 \urcorner} \ldots p_n^{\ulcorner t_n \urcorner}]$
For an n-ary variable predicate S_i^n and terms $t_1, \ldots, t_n$: $\ulcorner S_i^n(t_1, \ldots, t_n) \urcorner = [4, i, p_1^{\ulcorner t_1 \urcorner} p_2^{\ulcorner t_2 \urcorner} \ldots p_n^{\ulcorner t_n \urcorner}]$
Negation: $\ulcorner \neg \phi \urcorner = [5, \ulcorner \phi \urcorner, \ulcorner \phi \urcorner]$
Disjunction: $\ulcorner \phi \wedge \psi \urcorner = [6, \ulcorner \phi \urcorner, \ulcorner \psi \urcorner]$
Universal Quantifier Variable Letter: $\ulcorner \forall v_i \phi \urcorner = [7, i, \ulcorner \phi \urcorner]$
Universal Quantifier Variable Predicate: $\ulcorner \forall S_m^n \phi \urcorner = [8, 2^n 3^m, \ulcorner \phi \urcorner]$

For the next definition, we need a subset of ω that gives us the Gödel numbers that corresponds to terms and variable letters. Hence, we let $\Omega = \{n \in \omega :$ there is a variable v_i such that $\ulcorner v_i \urcorner = n$ or there is a constant symbol c_i such that $\ulcorner c_i \urcorner = n$ or there is an m-ary function f_i and terms $t_1, \ldots, t_m$ such that $\ulcorner f_i(t_1, \ldots, t_m) \urcorner = n\}$.

Definition 3.2 (Evaluation functions in the language)
Given the Gödel encoding, we can now take a valuation v on an $\mathcal{L}^*$-structure $\mathcal{A}$ and define functions in the language of set theory that expresses v. We define $w^v : \Omega \to A$ and a set of functions $u_n^v : \omega \to \mathcal{P}(A^n)$ such that:

if c_i is a constant symbol, then $w^v(\ulcorner c_i \urcorner) = c^{\mathcal{A}}$,
if v_i is a variable letter, then $w^v(\ulcorner v_i \urcorner) = v(x)$,
if f is an n-ary function symbol, $t_1, \ldots, t_n$ are terms, then $w^v(\ulcorner f(t_1, \ldots, t_n) \urcorner) = f^{\mathcal{A}}(w^v(t_1), \ldots, w^v(t_n))$,

$u_n^v(m) = v(S_m^n)$

Hence, we can iterate over all evaluation functions on an $\mathcal{L}^*$-structure $\mathcal{A}$ by iterating over $A^\Omega \times (A^\omega)^\omega$, where X^Y denotes all the functions from Y to X. For convenience, we denote $A^\Omega \times (A^\omega)^\omega$ by Γ_A.

Because we can encode an evaluation function v as a pair $\langle w^v, (u_n^v)_{n\in\omega}\rangle$, we can define a function that maps an $\mathcal{L}^*$-structure and a formula to all those evaluation functions that makes the formula true in the structure.

Definition 3.3 (Satisfaction function)

Given the Gödel encoding, we define a satisfaction function $Sat(\mathcal{A}, \ulcorner\phi\urcorner)$ recursively as follows:

$$\begin{aligned}
Sat(\mathcal{A}, \ulcorner P_i(t_1,\dots,t_n)\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w(\ulcorner t_1\urcorner),\dots,w(\ulcorner t_n\urcorner)\rangle \in P_i^{\mathcal{A}}\}\\
Sat(\mathcal{A}, \ulcorner S_i^n(t_1,\dots,t_n)\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w(\ulcorner t_1\urcorner),\dots,w(\ulcorner t_n\urcorner)\rangle \in u_n(i)\}\\
Sat(\mathcal{A}, \ulcorner \neg\phi\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w,(u_n)_{n\in\omega}\rangle \notin Sat(\mathcal{A}, \ulcorner\phi\urcorner)\}\\
Sat(\mathcal{A}, \ulcorner \phi\wedge\psi\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w,(u_n)_{n\in\omega}\rangle \in Sat(\mathcal{A}, \ulcorner\phi\urcorner) \wedge \\
&\qquad \langle w,(u_n)_{n\in\omega}\rangle \in Sat(\mathcal{A}, \ulcorner\psi\urcorner)\}\\
Sat(\mathcal{A}, \ulcorner \forall v_i\phi\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w',(u_n)_{n\in\omega}\rangle \in Sat(\mathcal{A}, \ulcorner\phi\urcorner) \text{ for}\\
&\qquad \text{all } w' \text{ such that } w'(\ulcorner v_j\urcorner) = w'(\ulcorner v_j\urcorner) \text{ for all variable}\\
&\qquad \text{letters } v_j\text{, except possibly } v_i.\}\\
Sat(\mathcal{A}, \ulcorner \forall S_m^n\phi\urcorner) &= \{\langle w,(u_n)_{n\in\omega}\rangle \in \Gamma_A : \langle w,(u'_n)_{n\in\omega}\rangle \in Sat(\mathcal{A}, \ulcorner\phi\urcorner) \text{ for}\\
&\qquad \text{all } (u'_n)_{n\in\omega} \text{ such that } u_i = u'_i \text{ for } i \neq n \text{ and } u_n(r) = u'_n(r)\\
&\qquad \text{for all natural numbers } r\text{, except possibly } m.\}
\end{aligned}$$

Furthermore, we make $Sat(\mathcal{A}, x) = \emptyset$ if x is not the Gödel number of any formula.

The definition of elementary substructure given in the project is as follows. If $\mathcal{A}$ is a substructure of $\mathcal{B}$, then $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$ just in case for all $\mathcal{L}^*$-formula $\varphi(x_1,\dots,x_r,S_{m_1}^{n_1},\dots,S_{m_s}^{n_s})$, with only free variable letters $x_1,\dots,x_r$ and free variable predicates $S_{m_1}^{n_1},\dots,S_{m_s}^{n_s}$, $\mathcal{A} \models \varphi[a_1,\dots,a_n,A_{m_1}^{n_1},\dots,A_{m_s}^{n_s}]$ iff $\mathcal{B} \models \varphi[a_1,\dots,a_n,A_{m_1}^{n_1},\dots,A_{m_s}^{n_s}]$ for all $a_i \in A$ and $A_{m_j}^{n_j} \subseteq A^{n_j}$. With the semantics given in the project, we characterise this by saying that if $\mathcal{A}$ is a substructure of $\mathcal{B}$, then $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ just in case for all $\mathcal{L}^*$-formula φ, and all evaluations v of $\mathcal{A}$, $\mathcal{A}, v \models \varphi$ just in case $\mathcal{B}, v \models \varphi$. We are now ready to define *Esub*.

> LEMMA 3.4
> There is a formulae $Esub(\mathcal{A}, \mathcal{B})$ such that $V \models Esub(\mathcal{A}, \mathcal{B})$ just in case $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}^*$ structures and $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$.

PROOF It should be clear that there are formulae $Lstruc(\mathcal{A})$ stating that $\mathcal{A}$ is an $\mathcal{L}^*$-structure and $substruc(\mathcal{A}, \mathcal{B})$ stating that $\mathcal{A}$ is a substructure of $\mathcal{B}$. Now, stating that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ is equivalent to saying that for all formulae ϕ, for all evaluations v of $\mathcal{A}$, $\mathcal{A}, v \models \phi$ just in case $\mathcal{B}, v \models \phi$. Hence we let $Esub(\mathcal{A}, \mathcal{B}) = Lstruc(\mathcal{A}) \wedge Lstruc(\mathcal{B}) \wedge substruc(\mathcal{A}, \mathcal{B}) \wedge (\forall n \in \omega(\forall v \in \Gamma_A \to (v \in Sat(\mathcal{A}, n) \leftrightarrow v \in Sat(\mathcal{B}, n))))$.

We are now ready to state and prove a version of the Downward Lowenheim-Skolem Theorem for second order logic.

> THEOREM 3.5 (DOWNWARD LOWENHEIM-SKOLEM THEOREM FOR $\mathcal{L}^*$)
> Suppose that κ is a regular uncountable cardinal such that for all cardinals $\lambda > \kappa$, there exist a transitive class M and a class term $i : V \to M$ such that for all ordinals $\alpha < \kappa$, $i(\alpha) = \alpha$, such that $i(\kappa) > \lambda$, such that $V_\lambda \subseteq M$, and such that for all formulae $\phi(x_1, \dots, x_n)$ of the language of set theory, and sets $a_1, \dots, a_n$, $V \models \phi(a_1, \dots, a_n)$ just in case $M \models \phi(i(a_1), \dots, i(a_n))$. Then, if $\mathcal{A}$ is an $\mathcal{L}^*$ structure, it has an elementary substructure whose domain has size $< \kappa$.

PROOF Suppose there is such a κ, and let $\mathcal{A}$ be an $\mathcal{L}^*$-structure with domain A. If $|A| < \kappa$ then we are done, so suppose $|A| \geq \kappa$. Let λ be the smallest cardinal such that $\mathcal{A} \in H_\lambda$. Now $\kappa \leq |A| \leq |Tc(\mathcal{A})|$ and so $\kappa < \lambda$. By Lemma 2.1 we know that $\mathcal{A} \in V_\kappa$. Now, by assumption there is a transitive class M and a class term $i : V \to M$ such that for all ordinals $\alpha < \kappa$, $i(\alpha) = \alpha$, such that $i(\kappa) > \lambda$, such that $V_\lambda \subseteq M$, and such that for all formulae $\phi(x_1, \dots, x_n)$ of the language of set theory, and sets $a_1, \dots, a_n$, $V \models \phi(a_1, \dots, a_n)$ just in case $M \models \phi(i(a_1), \dots, i(a_n))$. Let $\gamma = |A|$, then we may without loss of generality assume that $A = \gamma$. We have that $\kappa \leq \gamma < \lambda$, and so $\mathcal{M} \models i(\kappa) \leq i(\gamma)$, which is absolute and so we get $\gamma < i(\gamma)$ as $\lambda < i(\kappa)$. This means that $\gamma \subseteq i(\gamma)$, and so it should be clear that $\mathcal{A}$ is an elementary substructure of $i(\mathcal{A})$. This gives us that $\mathcal{M} \models Esub(\mathcal{A}, i(\mathcal{A}))$. Now, let $\phi(\mathcal{A}, \lambda)$ express in the language of set theory that the domain of $\mathcal{A}$ has cardinality less than λ. We then get that $\mathcal{M} \models Esub(\mathcal{A}, i(\mathcal{A})) \wedge \phi(\mathcal{A}, \lambda)$. Since $\lambda < i(\kappa)$, we also get that $\mathcal{M} \models Esub(\mathcal{A}, i(\mathcal{A})) \wedge \phi(\mathcal{A}, i(\kappa))$. Using existential instantiation we get that $\mathcal{M} \models \exists x(Esub(x, i(\mathcal{A})) \wedge \phi(x, i(\kappa)))$. This gives us that $V \models \exists x(Esub(x, \mathcal{A}) \wedge \phi(x, \kappa))$, and so we have shown that $\mathcal{A}$ has an elementary substructure whose domain has size $< \kappa$.

4 Conclusion

In this essay we have seen how an inaccessible cardinal κ gives us a model of ZF and two different models of ZFC. Furthermore, we have also seen that the assumption of another large cardinal gives us a version of the Downwards Lowenheim-Skolem theorem for second-order logic.

References