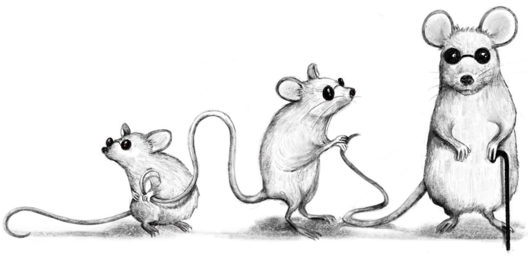
How to Tell Your Mice Apart

An Introduction to $0^{\#}$ and Mice.

Rasmus Moldstad Bakken



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Abstract

This thesis aims to be an accessible introduction to $0^\#$ and mice. Although defining the two concepts will be the main focus of the thesis, it will also try to outline where these two concepts fit into a larger story of fine structure theory and large cardinal axioms. The motivating results for $0^\#$, pronounced "zero sharp", will be showing that if there is a measurable cardinal then $0^\#$ exists and showing that if $0^\#$ exist then $V \neq L$. This will give us an apparent dichotomy between fine structure provided by L on the one hand and large cardinal axioms on the other hand. In an attempt to resolve this dichotomy, we will use a measurable cardinal to create mice. Then we will see how defining the Core Model as the union of all mice will allow us to get a new inner model that extends L, but still keeps several important fine structure results like the Axiom of Choice, the Generalised Continuum Hypothesis, $\diamondsuit$ and $\square$.

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1 Introduction

From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does.

– David Hilbert

The aim of this thesis is to give an accessible introduction to $0^\#$ and mice. These have both highly technical definitions, so major parts of this thesis is spent on developing the machinery for definition 2.4.9 and definition 3.2.25. These two concepts play an integral part in a larger story about large cardinal axioms and fine structure theory. A brief account of parts of this story will be given below to serve as motivation for this thesis.

Gödel showed that the Axiom of Choice (AC) and the Generalised Continuum Hypothesis (GCH) are consistent with ZF by defining his Constructible Universe L and showing that $ZF + V = L \vdash AC + GCH$[5]. This is one out of several nice results about L that may be used in favour of accepting the $V = L$ axiom. There is another group of axioms that may be added to ZF called *large cardinal* axioms. Roughly, a large cardinal is a cardinal that ZF cannot prove exist but we believe that the existence of the cardinal is consistent with ZF. The first large cardinal typically encountered is the (strongly) inaccessible cardinal. It can be shown that if λ is an inaccessible cardinal then $V_\lambda \models ZF$, which means that $ZF+$ "there is an inaccessible cardinal" $\vdash ZF$ is consistent. Hence, by Gödel's Second Incompleteness Theorem, $ZF \nvdash$ "there is an inaccessible cardinal", assuming that ZF is in fact consistent. A natural question then arises in how well the $V = L$ axiom and the large cardinal axiom mix. The answer is that if there is a measurable cardinal, which is a large cardinal not very high up in the large cardinal hierarchy, then $V \neq L$. In other words, there are no measurable cardinals in L. Showing this will be the main goal of Chapter 2. To do so, we will define $0^\#$ and show the following two implications. Firstly, if there is a measurable cardinal then $0^\#$ exists. Secondly, if $0^\#$ exist then $V \neq L$.

At this point the reader may feel unhappy that they are left with a choice, either they like the fine structure provided by L or they like large cardinals and the consequences of them. It turns out that it is possible to (almost) have your cake and eat it too, and spelling this out is the goal Chapter 3. Assuming there is a measurable cardinal, one can construct mice. Very informally speaking, a mouse is a model $\mathcal{M}$ that looks like L_α but it believes that is has a measurable cardinal. Using what the mouse thinks is a measurable cardinal, we can iteratively create new mice from the old ones all the way through the ordinals. Furthermore, this iterative process is done in such a way that it preserves a lot of the fine structure qualities from the mouse. We can then define the core model K as the union of all mice. K has many of the same fine structure properties as L, such as the Axiom of Choice and the Generalised Continuum Hypothesis both being true in K. Hence, K may serve as a way of keeping much of the fine structure from L in the presence of certain large cardinal axioms.

1.1 Notation

We briefly fix some notation. $\alpha, \beta, \gamma, \delta$ are ordinals unless otherwise indicated. κ and λ are cardinals unless otherwise indicated. We denote the proper class of ordinals by Ω and so $\alpha < \Omega$ indicates that α is an ordinal.

For a map $f : X \to Y$, we let $f"(X') = \{y \in Y : \exists x \in X' \ f(x) = y\}$ and $f^{-1}(Y') = \{x \in X : f(x) \in Y'\}$ for $X' \subseteq X$ and $Y' \subseteq Y$[1].

For a model, we distinguish between the domain of the model M and the model itself $\mathcal{M}$. The exceptions to this rule are V, L, H, J and S. Furthermore, if $\mathcal{M}$ is a model, $m_1, \ldots, m_n \in M$ and $\varphi(v_1, \ldots, v_n)$ is a formula then $\mathcal{M} \models \varphi[m_1, \ldots, m_n]$ means that $\varphi(v_1, \ldots, v_n)$ is true in $\mathcal{M}$ when applied to $\langle m_1, \ldots, m_n \rangle$.

If $\tau(v_1, \ldots, v_n)$ is a term, then $\tau^{\mathcal{M}}(m_1, \ldots, m_n)$ and $\tau_{\mathcal{M}}(m_1, \ldots, m_n)$ both denote the relativisation of $\tau(m_1, \ldots, m_n)$ to $\mathcal{M}$. Both will be used depending on what is convenient, but this should not cause any confusion.

1.2 Preliminaries

This thesis is aimed at graduate mathematics students who have taken an advanced course in axiomatic set theory. In particular, the thesis assumes familiarity with V and Gödel's L. Furthermore, the thesis assumes that the reader is familiar with the proof of $ZF + V = L \vdash AC + GCH$. The thesis also assumes familiarity with the Lévy Hierarchy, i.e. that the reader is familiar with the notation of Σ_n and Π_n sentences and formulae. We will assume ZFC through the thesis, unless otherwise specified, and so we will define a cardinal to be the least ordinal of a given cardinality.

We will now introduce fragments of ZFC called R, R^+, R^+_ω, and RA. We will also introduce other L like structures, called S, H and Jensen J. If J and S are unfamiliar to the reader then they may replaced it with L without losing much.

We start out by defining R, which is the weakest fragment of ZFC that we will consider.

1 Although the notation for the inverse image may be ambiguous with the image of an inverse function, the meaning should always be clear in this thesis.

Definition 1.2.1 (R [3] (Definition 1.1))
The theory R is the deductive closure of the following axioms:
(extensionality) $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$,
(foundation) $\forall x (x \neq \emptyset \leftrightarrow \exists y (y \in x \wedge x \cap y = \emptyset))$,
(pairing) $\forall x \forall y \exists z \forall t (t \in z \leftrightarrow (t = x \vee t = y))$,
(union) $\forall x \exists y \forall t (t \in y \leftrightarrow \exists z (z \in x \wedge t \in z))$, and
(Σ_0-closure) $\forall u \forall x_1 \ldots \forall x_k \forall w \exists y \forall z$
$(z \in y \leftrightarrow \exists t \in w (z = \{s \in u : \varphi(s, t, x_1, \ldots, x_k)\}))$ where φ is Σ_0 and $k \in \omega$.

Although the last axiom is an axiom schema, it can be shown that R is finitely axiomatisable (Lemma 1.9 [3]). In fact, using extensionality and foundation plus 17 axioms on the form:

$$\forall x \forall y \exists z F_i(x, y) = z, \tag{A_i}$$

where F_i is taken from the definition below, then we have our finite axiomatisation of R.

Definition 1.2.2 (Basis Functions [3] (Definition 1.3))
The *basis functions* F_i for $1 \leq i \leq 17$ are class functions defined by:

$F_1(x, y) = \{x, y\}$,
$F_2(x, y) = \bigcup x$,
$F_3(x, y) = x \setminus y$,
$F_4(x, y) = x \times y$,
$F_5(x, y) = \mathrm{dom}(x)$,
$F_6(x, y) = \{x''(z) : z \in y\}$, assuming x is a function,
$F_7(x, y) = \{\langle u, v, w\rangle : u \in x \wedge \langle v, w\rangle \in y\}$,
$F_8(x, y) = \{\langle u, v, w\rangle : \langle u, v\rangle \in x \wedge w \in y\}$,
$F_9(x, y) = \{\langle u, v\rangle \in x \times y : u = v\}$,
$F_{10}(x, y) = \{\langle u, v\rangle \in x \times y : u \in v\}$,
$F_{11}(x, y) = \langle x, y\rangle$,
$F_{12}(x, y) = \langle x, v, w\rangle$ if $y = \langle v, w\rangle$ and $\emptyset$ otherwise,
$F_{13}(x, y) = \langle u, y, v\rangle$ if $x = \langle u, v\rangle$ and $\emptyset$ otherwise,
$F_{14}(x, y) = \{\langle x, v\rangle, w\}$ if $y = \langle v, w\rangle$ and $\emptyset$ otherwise,
$F_{15}(x, y) = \{\langle u, y\rangle, v\}$ if $x = \langle u, v\rangle$ and $\emptyset$ otherwise,
$F_{16}(x, y) = \{\langle x, y\rangle\}$,
$F_{17}(x, y) = \{t : \langle y, t\rangle \in x\}$.

Using these basis functions, we can now define S.

> DEFINITION 1.2.3 (S [3] (DEFINITION 2.1))
> S is the class function given by $u \mapsto u \cup \bigcup_{i=1}^{17} F_i``(u \times u)$.

There is a natural way of restricting this to the ordinals so that we can talk about S_α for $\alpha < \Omega$. This is done iteratively through the ordinals, letting $S_{\alpha+1}$ be the closure of S_α under S and taking unions at the limit stage. Omitting some details, R^+ is roughly R together with the axiom saying that for all sets x there is an ordinal α such that $x \in S_\alpha$. For limit ordinals λ we get that $S_\lambda \models R^+$. From R^+ we can define R^+_ω as R^+ plus the axiom schema stating that every definable limit segment of ω is either 0 or ω. From here we make our last jump to RA.

> DEFINITION 1.2.4 (RA [3] (DEFINITION 3.1))
> RA is the theory R^+_ω together with the axiom of acceptability which states:
> Suppose λ is a limit ordinal and $\delta < \lambda$, then if there is some $a \subseteq \delta$ such that $a \in S_{\lambda+\omega} \setminus S_\lambda$ then for any $u \in S_{\lambda+\omega}$ there is $f = \langle f_\xi : \delta \leq \xi < \lambda \rangle \in S_{\lambda+\omega}$ such that $f_\xi : \xi \to \{\xi\} \cup (P(\xi) \cap u)$ is a surjective function.

The axiom of acceptability gives us a weak form of GCH as with it we can show that if $\alpha \geq \omega$ then either $P(\alpha)$ exist and $|P(\alpha)| = |\alpha|^+$ or $P(\alpha)$ is not a set and for all $u \subseteq P(\alpha)$, if u is a set then $|u| \leq |\alpha|$[2].

We can now define Jensen's J by $J_\alpha = S_{\omega\alpha}$. This definition is done in ZF, where we have the appropriate machinery for ordinal arithmetic. Suppose now that the sets $B_1, \ldots, B_k$ are given, then we define $F_{17+j}(x, y) = B_j \cap x$ where $1 \leq j \leq k$. This allows us to define the class function $S_{B_1,\ldots,B_k}$ by $u \mapsto u \cup \bigcup_{i=1}^{17+k} F_i``(u \times u)$ and relativise it in a similar fashion to get $S_\alpha^{B_1,\ldots,B_k}$ and $J_\alpha^{B_1,\ldots,B_k}$. Lastly, we define the transitive closure of x to be the smallest transitive set $Tc(x)$ containing x. This is well-defined and always exists, and so we can define $H_\lambda = \{x : |Tc(x)| < \lambda\}$. This is a well-defined set as it is not hard to show that $H_\lambda \subseteq V_\lambda$, with equality if λ is (strongly) inaccessible.

The last thing we want to look at here is the definition of $\models^n_{\Sigma_1}$. We call a function *rudimentary* just in case it is a composition of the basis functions. Similarly, we say that an n-ary fomula φ is *rudimentary* just in case there is a rudimentary function F such that $\varphi(x_1, \ldots, x_n)$ holds just in case $F(x_1, \ldots, x_n) \neq \emptyset$. It is not hard to show that every Σ_0 formula is rudimentary. So, for a Σ_0 formula $\varphi(y, x_1, \ldots, x_n)$ we denote its corresponding

2 This is in models of RA where the power set axiom may fail.

rudimentary function by $F_\varphi(y, x_1, \ldots, x_n)$. Then we say that $\models^n_{\Sigma_1} \exists y \varphi(y, x_1, \ldots, x_n)$ holds just in case $\exists y F_\varphi(y, x_1, \ldots, x_n) \neq \emptyset$. This extends in a natural way to arbitrary Σ_m, although we will only need the Σ_1 version in this thesis.

2 $0^{\#}$

Je ne puis;—malgré moi l'infini me tourmente.
I can't help it, the idea of the infinite torments me.

– Alfred de Musset, *L'Espoir en Dieu*

In this chapter we will work our way up to the definition of $0^{\#}$ and look at some initial consequences of the existence of $0^{\#}$. The main result we want to establish is that if there exist a measurable cardinal then $V \neq L$. The proof of this will consist of showing the following three statements:

(i) Every measurable cardinal is a Ramsey cardinal.
(ii) If there is a Ramsey cardinal then $0^{\#}$ exists.
(iii) If $0^{\#}$ exists then $V \neq L$.

Section 2.1 will serve as an introduction to measurable cardinals, and contains several important results used to show that measurable cardinals are Ramsey cardinals. Section 2.2 aims to show Ramsey's Theorem, which is crucial in the construction of indiscernibles. This is important to us as $0^{\#}$ is constructed from a certain L_α and a set of indiscernibles for L_α. Furthermore, the section introduces Ramsey cardinals and show that any measurable cardinal is Ramsey. Section 2.3 serves as a brief introduction to to the theory of Skolem hulls and Skolem functions. These are important as Skolem hulls allow us to characterise the models of the EM Blueprints, which are used in the definition of $0^{\#}$. Finally, section 2.4 defines indiscernibles, uses them to define EM blueprints which in turn is used to define $0^{\#}$. We then show that if there is a Ramsey cardinal, then $0^{\#}$ exists. Furthermore, we also show that if $0^{\#}$ exist then $P(\omega)^L$ is countable, which means that $V \neq L$. Hence, this allows us to conclude that if there is a measurable cardinal then $V \neq L$.

2.1 Measurable Cardinals

2.1.1 Definition through Ultrafilters

In this section we will introduce measurable cardinals and show several lemmata about them that will be important later on. There are several different equivalent definitions of measurable cardinals, one fittingly involving the notion of a measure, but for our purposes we are more interested in the definition using ultrafilters. Hence, we start out by defining filters and ultrafilters.

> DEFINITION 2.1.1 (FILTERS AND ULTRAFILTERS)
> A *filter* $\mathcal{F}$ on a set I is a collection of subsets of I such that:
> 1. If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.
> 2. If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$ then $Y \in \mathcal{F}$.
> 3. $\mathcal{F} \neq \emptyset$.
> 4. $\emptyset \notin \mathcal{F}$.
>
> An *ultrafilter* $\mathcal{U}$ on I is a filter on I such that for all $X \subseteq I$, $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

We say that a filter $\mathcal{F}$ over a set I is *principal* just in case there is an $Y \subseteq I$ such that any $X \in \mathcal{F}$, $Y \subseteq X$. In this case we say that $\mathcal{F}$ is *generated by* Y. An example of this is a filter $\mathcal{F}$ where $\{i\} \in \mathcal{F}$ for some $i \in I$[1]. This is because otherwise there is an $X \in \mathcal{F}$ such that $X \cap \{i\} = \emptyset$, which contradicts the first and the fourth clause of the definition. An easy inductive proof will show us that any filter $\mathcal{F}$ is closed under finite intersections. For an ultrafilter $\mathcal{U}$, we get a dual notion of this. For suppose we have an ultrafilter $\mathcal{U}$ over a set I, and we have $X_1, \ldots, X_n \subseteq I$ such that $\bigcup_{i=1}^n X_i \in \mathcal{U}$, then we can show that there has to be a $j \leq n$ such that $X_j \in \mathcal{U}$. To see why this is the case, suppose otherwise. Then, by $\mathcal{U}$ being an ultrafilter, we know that $I \setminus X_i \in \mathcal{U}$ for all $i \leq n$. As we have seen, $\mathcal{U}$ is closed under finite intersection, so $\bigcap_{i=1}^n I \setminus X_i \in \mathcal{U}$. Now, $\bigcap_{i=1}^n I \setminus X_i = I \setminus \bigcup_{i=1}^n X_i$, but this contradicts $\bigcup_{i=1}^n X_i \in \mathcal{U}$. So, if the union of a finite set is in the ultrafilter, then we can find an element of that set who is also in the ultrafilter. This property is called ω-*completeness*. We will now generalise this property to arbitrary cardinals.

> DEFINITION 2.1.2 (λ-COMPLETE ULTRAFILTER)
> An ultrafilter $\mathcal{U}$ is called λ-complete for some cardinal λ just in case for any $\gamma < \lambda$ and $\bigcup\{Y_\alpha : \alpha < \gamma\} \in \mathcal{U}$ there is a $\beta < \gamma$ such that $Y_\beta \in \mathcal{U}$.

It is through κ-complete ultrafilters that we will characterise measurable cardinals.

> DEFINITION 2.1.3 (MEASURABLE CARDINALS)
> A cardinal $\kappa > \omega$ is called *measurable* just in case there is a κ-complete ultrafilter $\mathcal{U}$ over κ.

1 This is in fact a principal ultrafilter. In fact, this is the only type of principal ultrafilters we have. Assume that $\mathcal{U}$ is a principal ultrafilter where for all $X \in \mathcal{U}$, $Y \subseteq X$ and $|Y| \geq 2$. Then let $i_1, i_2 \in Y$ for $i_1 \neq i_2$. We see that $\{i_1\} \notin \mathcal{U}$ as $Y \nsubseteq \{i_1\}$, but this gives us $I \setminus \{i_1\} \in \mathcal{U}$, which is a contradiction as $Y \nsubseteq I \setminus \{i_1\}$.

As we saw earlier, there is a dual notion of ω-completeness for ultrafilters. It turns out that for the κ-complete ultrafilters on measurable cardinals, the same dual notion holds there as well.

Lemma 2.1.4 (This is well known)
Suppose κ is a cardinal and $\mathcal{U}$ is an ultrafilter over κ. Then $\mathcal{U}$ is κ-complete just in case for any $\gamma < \kappa$, if $X_\alpha \in \mathcal{U}$ for all $\alpha < \gamma$, then $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{U}$.

Proof Suppose that $\mathcal{U}$ is κ-complete and that $\bigcap_{\alpha<\gamma} X_\alpha \notin \mathcal{U}$, then $\kappa \setminus \bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{U}$. We note that $\kappa \setminus \bigcap_{\alpha<\gamma} X_\alpha = \bigcup_{\alpha<\gamma}(\kappa \setminus X_\alpha)$. Hence, by κ-completeness there is an $\beta < \gamma$ such that $\kappa \setminus X_\beta \in \mathcal{U}$, which contradicts $X_\beta \in \mathcal{U}$.

Conversely, suppose that for any $\gamma < \kappa$, if $X_\alpha \in \mathcal{U}$ for all $\alpha < \gamma$, then $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{U}$ and assume that $\bigcup\{Y_\alpha : \alpha < \gamma\} \in \mathcal{U}$ for some $\gamma < \kappa$. Assume for contradiction that there is no $\beta < \gamma$ such that $Y_\beta \in \mathcal{U}$. We then get that $\kappa \setminus Y_\alpha \in \mathcal{U}$ for all $\alpha < \gamma$. This gives us that $\kappa \setminus \bigcup_{\alpha<\gamma} Y_\alpha = \bigcap_{\alpha<\gamma} \kappa \setminus Y_\alpha \in \mathcal{U}$, a contradiction.

In the beginning we will refer to this Lemma several times, but after a while we will use both notions of κ-completeness interchangeably. We observe one more useful results about filters before we move on to the next part of this section.

Lemma 2.1.5 ([1] (Proposition 4.2.2))
Let $\mathcal{F}$ be a filter over a set I of cardinality α. If $\mathcal{F}$ is α^+-complete, then $\mathcal{F}$ is principal.

Proof Let $E = \{I \setminus \{i\} : i \in I \text{ and } I \setminus \{i\} \in \mathcal{F}\}$. Since $|I| = \alpha$, $|E| \leq \alpha < \alpha^+$. $\mathcal{F}$ is α^+-complete and $E \subset \mathcal{F}$, so $\bigcap E \in \mathcal{F}$[2]. On the other hand, if $X \in \mathcal{F}$ then $\bigcap E \subseteq X$. To see why, if $i \notin X$, then $X \subseteq I \setminus \{i\}$, and so $I \setminus \{i\} \in \mathcal{F}$, which gives us $I \setminus \{i\} \in E$ and so $i \notin \bigcap E$. The contra positive gives us that if $i \in \bigcap E$ then $i \in X$ and so $\bigcap E \subseteq X$ and so $\mathcal{F}$ is a principal filter generated by $\bigcap E$.

2.1.2 Normal Ultrafilters and Ultraproducts

The aim of this subsection is to define normal ultrafilters and show two important lemmata about them. The first one says that any measurable cardinal has a normal ultrafilter over itself, and the second one is an important property about normal ultrafilters that we will use in the proof that any measurable cardinal is a Ramsey cardinal in section 2.2.2. We first define a normal ultrafilter.

2 If E is empty then $\bigcap E = I$ by convention.

Definition 2.1.6 (Normal Ultrafilter)
Let κ be an uncountable cardinal. Suppose $\mathcal{U}$ is a non-principal κ-complete ultrafilter on κ such that, for every function $f \in \kappa^\kappa$ if $\{\beta : f(\beta) < \beta\} \in \mathcal{U}$ then there is a $\gamma < \kappa$ such that $\{i \in \kappa : f(i) = \gamma\} \in \mathcal{U}$. Then $\mathcal{U}$ is called *normal*.

We now turn to the first aim of this subsection; showing that any measurable cardinal has a normal ultrafilter over itself. To do this, we will take the κ-complete ultrafilter $\mathcal{U}$ and define another ultrafilter $\mathcal{V}$ in terms of $\mathcal{U}$. To show that $\mathcal{V}$ is indeed normal we need to develop the notion of an ultraproduct, and show several lemmata concerning different types of ultraproducts.

Definition 2.1.7 (Ultraproduct)
Given a set I, a family of structures $(\mathcal{M}_i)_{i\in I}$ and an ultrafilter $\mathcal{U}$ on I, we define the ultraproduct $\Pi_{i\in I}\mathcal{M}_i/\mathcal{U}$ as the structure with the underlying set $\Pi_{i\in I}M_i/\mathcal{U}$, that is the set $\Pi_{i\in I}M_i$ modulo an equivalence relation $\sim$ where $(f) \sim (g)$ just in case $\{i \in I : f(i) = g(i)\} \in \mathcal{U}$. We denote the equivalence class of f by $[f]$. Each constant c, n-ary function f and m-ary predicate P is interpreted as follows:

1. c is interpreted as $[f]$ where $f(i)$ is the interpretation of c in $\mathcal{M}_i$.
2. $f([g_1], \dots, [g_n]) = [h]$ just in case $\{i \in I : \mathcal{M}_i \models f(g_1(i), \dots, g_n(i)) = h(i)\} \in \mathcal{U}$.
3. $\langle [g_1], \dots, [g_k]\rangle \in P$ just in case $\{i \in I : \mathcal{M}_i \models \langle g_1(i), \dots, g_k(i)\rangle \in P\} \in \mathcal{U}$.

I quote without proof a celebrated theorem of Łoś that we will use several times.

Theorem 2.1.8 (Łoś)
Let $\Pi_{i\in I}\mathcal{M}_i/\mathcal{U}$ be an ultraproduct, then for any formula $\phi(v_1, \dots, v_n)$:

$$\Pi_{i\in I}\mathcal{M}_i/\mathcal{U} \models \phi[[g_1], \dots, [g_n]] \quad \text{iff} \quad \{i \in I : \mathcal{M}_i \models \phi[g_1(i), \dots, g_n(i)]\} \in \mathcal{U}.$$

To simplify notation, if $\mathcal{M}$ is a structure, I is an indexing set such that $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$ and $\mathcal{U}$ is an ultrafitler on I, then we write $\Pi_{\mathcal{U}}\mathcal{M}$ for $\Pi_{i\in I}\mathcal{M}_i/\mathcal{U}$. Similarly, we write $\Pi_{\mathcal{U}}M$ for $\Pi_{i\in I}M_i/\mathcal{U}$ when looking at the domain of the structure.

The general idea for creating the desired normal ultrafilter $\mathcal{V}$ is to first let $\mathcal{M} = \Pi_{\mathcal{U}}\langle\kappa, <\rangle$ and then define $\mathcal{V} = \{X \subseteq \kappa : f^{-1}(X) \in \mathcal{U}\}$ where $[f]$ is the κ-th element of $\mathcal{M}$. To be able to do this, we will show that $\mathcal{M}$ is a well-ordered structure of order type greater than κ, and so that there is a κ-th element of $\mathcal{M}$. This is Lemma 2.1.11, and to be able to prove it we first need to prove Lemma 2.1.9 and Lemma 2.1.10.

> LEMMA 2.1.9 ([1] (PARAPHRASED FROM PROPOSITION 4.2.4))
> Let $\mathcal{M}$ be a structure of cardinality α and let $\mathcal{U}$ be an ultrafilter over some set I. The natural embedding $d : \mathcal{M} \to \Pi_{\mathcal{U}}\mathcal{M}; m \mapsto [f_m]$, where $f_m : I \to M; i \mapsto m$, is onto if and only if $\mathcal{U}$ is α^+-complete.

PROOF Suppose that $\mathcal{U}$ is α^+-complete. Let $[f] \in \Pi_{\mathcal{U}}\mathcal{M}$. Then f maps I into M. Since $|M| = \alpha$, the partition $I = \bigcup\{f^{-1}(m) : m \in M\}$ partitions I into fewer than α^+ parts. Clearly $I \in \mathcal{U}$, so by α^+-completeness there is an $m \in M$ such that $f^{-1}(m) \in \mathcal{U}$. In other words $\{i \in I : f(i) = m\} \in \mathcal{U}$, and so $[f] = d(m)$. This means that $d(\mathcal{M}) = \Pi_{\mathcal{U}}\mathcal{M}$ and so d is onto.

Conversely, suppose that d is onto. Let $Y = \bigcup_{\eta<\beta} X_\eta$ be a partition of Y into $\beta < \alpha^+$ parts for some $Y \in \mathcal{U}$. Since $|\beta| \leq \alpha = |M|$, we may renumber the sets X_η with indices from $N \subseteq M$. Hence $Y = \bigcup_{n\in N} X_n$. We define $f : Y \to M$ by $f(i) = n$ just in case $i \in X_n$. f can be extended arbitrarily to a function $f' : I \to N$ as any such functions will agree on Y and hence will be equivalent. We note that $[f'] \in \Pi_{\mathcal{U}}\mathcal{M}$, and so by d being onto there is an $m \in M$ such that $d(m) = [f']$. Hence $(f')^{-1}(m) \in \mathcal{U}$ and so $X_m \in \mathcal{U}$ as $(f')^{-1}(m) = X_m$.

> LEMMA 2.1.10 ([1] (PARAPHRASED FROM PROPOSITION 4.2.13 (II)))
> Let κ be a measurable cardinal and let $\mathcal{U}$ be a non-principal κ-complete ultrafilter over κ. Let $\mathcal{M} = \Pi_{\mathcal{U}}\langle\kappa, <\rangle$ and $d : \langle\kappa, <\rangle \to \mathcal{M}$ be the natural embedding, then for every $\gamma < \kappa$, $d(\gamma)$ is the γ-th element of $\mathcal{M}$.

PROOF Fix $\gamma < \kappa$, and suppose that $\mathcal{M} \models [f] < d(\gamma)$. By Łoś' Theorem there is a $U \in \mathcal{U}$ such that $\forall\alpha \in U\ f(\alpha) < \gamma$. If we can show that there is a $\delta < \gamma$ such that $\mathcal{M} \models [f] = d(\delta)$ then we have shown that $d(\gamma)$ must be the γ-th element of $\mathcal{M}$. By Łoś, this would mean that there is a $V \in \mathcal{U}$ such that $\forall\alpha \in V,\ f(\alpha) = \delta$. Suppose for contradiction this is not the case, then for all $\delta < \gamma$, let $V_\delta = \{\alpha \in U : f(\alpha) = \delta\}$. Clearly $U = \bigcup_{\delta<\gamma} V_\delta$, but this contradicts κ-completeness for $\mathcal{U}$ as $U \in \mathcal{U}$ and for all $\delta < \gamma$, $V_\delta \notin \mathcal{U}$. Hence there is a $\delta < \gamma$ such that $V_\delta \in \mathcal{U}$, which gives us that $\mathcal{M} \models [f] = \delta$ by Łoś.

> LEMMA 2.1.11 ([1] (PARAPHRASED PROPOSITION 4.2.13 (I)))
> Let κ be a measurable cardinal and let $\mathcal{U}$ be a non-principal κ-complete ultrafilter over κ. Let $\mathcal{M} = \Pi_{\mathcal{U}}\langle\kappa, <\rangle$, then $\mathcal{M}$ is a well-ordered structure of order type greater than κ.

Proof We know that $\langle \kappa, < \rangle$ is a total order, and so by Łoś Theorem $\mathcal{M}$ is a total order as well since being a total order is first order definable. In the presence of the axiom of choice, it is sufficient to show that $\mathcal{M}$ cannot have an infinite decreasing sequence of elements. Suppose there is a sequence $\langle f_n : n \in \mathbb{N} \rangle$ such that $\mathcal{M} \models [f_{n+1}] < [f_n]$ for all $n \in \mathbb{N}$. By Łoś there is a set $U_n \in \mathcal{U}$ for each $n \in \mathbb{N}$ such that for all $\alpha \in U_n$, $f_{n+1}(\alpha) < f_n(\alpha)$. Since $\mathcal{U}$ is κ-complete, $U = \bigcap_{n \in \mathcal{N}} U_n \in \mathcal{U}$, and so U is non-empty. This is a contradiction as for all $\alpha \in U$, $f_{n+1}(\alpha) < f_n(\alpha)$ for all $n \in \mathbb{N}$, and so U must be empty. Hence we have shown that $\mathcal{M}$ is a well-order and we now proceed to show that the order-type is greater than κ. The natural embedding d from Lemma 2.1.9 is injective and order preserving, and hence $\mathcal{M}$ has order type at least κ. Furthermore, $\mathcal{U}$ is non-principal, and so by Lemma 2.1.5 it is not κ^+-complete and so by Lemma 2.1.9 is not onto. Now, by Lemma 2.1.10, $d(\kappa)$ is an initial segment of $\mathcal{M}$, and so since d is not onto must it be the case that $\mathcal{M}$ is of order type greater than κ.

We are now ready to show the first main result of this subsection, namely that any measurable cardinal has a normal ultrafilter over itself.

Theorem 2.1.12 ([1] (Paraphrased from Proposition 4.2.20))
If κ is a measurable cardinal then there exists a normal ultrafilter over κ.

Proof κ is measurable, so let $\mathcal{U}$ be a non-principal κ-complete ultrafilter over κ. Let $\mathcal{M} = \Pi_{\mathcal{U}} \langle \kappa, < \rangle$. From Lemma 2.1.11 we know that $\mathcal{M}$ is well-ordered and of greater order type than κ, hence we can let $[f]$ be the κ-th element of $\mathcal{M}$. Recall that $f : \kappa \to \kappa$, and so we define $\mathcal{V} = \{X \subseteq \kappa : f^{-1}(X) \in \mathcal{U}\}$. We will proceed to show that $\mathcal{V}$ does the job. We first show that $\mathcal{V}$ is a filter:

1. Suppose $X, Y \in \mathcal{V}$, then $f^{-1}(X), f^{-1}(Y) \in \mathcal{U}$. Furthermore, we get that $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) \in \mathcal{U}$. Hence $X \cap Y \in \mathcal{V}$.
2. Suppose $X \in \mathcal{V}$ and $X \subseteq Y \subseteq \kappa$. Then $f^{-1}(X) \in \mathcal{U}$ and $f^{-1}(X) \subseteq f^{-1}(Y) \subseteq \kappa$ and so $f^{-1}(Y) \in \mathcal{U}$. This gives us $Y \in \mathcal{V}$.
3. We have $\kappa \in \mathcal{V}$ as $f^{-1}(\kappa) = \kappa \in \mathcal{U}$. Hence $\mathcal{V} \neq \emptyset$.
4. Suppose that $\emptyset \in \mathcal{V}$, then $f^{-1}(\emptyset) \in \mathcal{U}$. Recall that $f^{-1}(X) = \{i \in \kappa : f(i) \in X\}$, hence $f^{-1}(\emptyset) = \{i \in \kappa : f(i) \in \emptyset\} = \emptyset$. This gives us $\emptyset \in \mathcal{U}$, a contradiction.

Continuing, we show that $\mathcal{V}$ is an ultrafilter. Fix $X \subseteq \kappa$, then by $\mathcal{U}$ being an ultrafilter we have that $f^{-1}(X) \in \mathcal{U}$ or $\kappa \setminus f^{-1}(X) \in \mathcal{U}$. We note that $f^{-1}(\kappa \setminus X) = \kappa \setminus f^{-1}(X)$ as $f^{-1}(\kappa) = \kappa$. Hence we get that $X \in \mathcal{V}$ or $\kappa \setminus X \in \mathcal{V}$.

Turning to κ-completeness we fix $\gamma < \kappa$ and suppose that $X_\alpha \in \mathcal{V}$ for all $\alpha < \gamma$. Then $f^{-1}(X_\alpha) \in \mathcal{U}$ for all $\alpha < \gamma$ and so by κ-completeness and Lemma 2.1.4 we get that $f^{-1}(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} f^{-1}(X_\alpha) \in \mathcal{U}$. Hence we get $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{V}$ which makes $\mathcal{V}$ κ-complete by Lemma 2.1.4.

To show that $\mathcal{V}$ is non-principal we let $\bar{\alpha}$ denote the α-th element in $\mathcal{M}$ according to the well-ordering from Lemma 2.1.11. For each $\gamma < \kappa$, we know that $[f] = \bar{\kappa} \neq \bar{\gamma} = d(\gamma)$, where the last equality follows from Lemma 2.1.10. This means that $f^{-1}(\{\gamma\}) = \{i \in \kappa : f(i) = \gamma\} \notin \mathcal{U}$ and so $\{\gamma\} \notin \mathcal{V}$.

Finally, we fix a function $g \in \kappa^\kappa$ and assume that $X = \{\beta : g(\beta) < \beta\} \in \mathcal{V}$. Let $h = g \circ f$, then for all $\beta \in f^{-1}(X)$, $h(\beta) = g(f(\beta)) < f(\beta)$. Now $X \in \mathcal{V}$ and so $f^{-1}(X) \in \mathcal{U}$. We see that $f^{-1}(X) \subseteq \{\beta \in \kappa : h(\beta) < f(\beta)\}$ and so $\{\beta \in \kappa : h(\beta) < f(\beta)\} \in \mathcal{U}$, which gives us that $[h] < [f] = \bar{\kappa}$. This means that there is a $\gamma < \kappa$ such that $[h] = \bar{\gamma}$. Now, by Lemma 2.1.10 we have that $d(\gamma) = \bar{\gamma}$, and so $\{i \in \kappa : h(i) = \gamma\} \in \mathcal{U}$. We notice that $\{i \in \kappa : h(i) = \gamma\} = \{i \in \kappa : g(f(i)) = \gamma\} = f^{-1}(\{j \in \kappa : g(j) = \gamma\})$. Hence $f^{-1}(\{j \in \kappa : g(j) = \gamma\}) \in \mathcal{U}$ and so $\{j \in \kappa : g(j) = \gamma\} \in \mathcal{V}$, as required.

For the section main result of this subsection, which is a result about a certain property of normal ultrafilters, we need to introduce some new terminology. Firstly, if $\mathcal{F}$ is a filter on κ, and $S \subseteq \kappa$, then S is $\mathcal{F}$-*stationary* just in case $\forall Z \in \mathcal{F}$, $Z \cap S \neq \emptyset$. A good analogy is to think that if a filter $\mathcal{F}$ determines which sets are large, then a $\mathcal{F}$-stationary set is not small as it intersects all the large sets. Secondly, if X is a set, then we say that $f : X \mapsto \Omega$ is regressive just in case $f(\alpha) < \alpha$ for all ordinals $\alpha \in X \setminus \{0\}$.

We finish up this subsection by proving two lemmata, where the first will be used to prove the second one, which is the second main result of this subsection. Both of the lemmata could have been strengthened to if and only ifs, but for our purposes we only need one direction.

Lemma 2.1.13 ([7] (Paraphrased from Exercise 5.10))
Let $\mathcal{U}$ be an ultrafilter on κ, and suppose that for any $\mathcal{U}$-stationary sets X and regressive functions $f : X \mapsto \kappa$ there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is $\mathcal{U}$-stationary. Then, for any $\{X_\alpha : \alpha < \kappa\}$, where $X_\alpha \in \mathcal{U}$ for all $\alpha < \kappa$, its diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\} \in \mathcal{U}$.

Proof Suppose for a reductio that $X_\alpha \in \mathcal{U}$ for all $\alpha < \kappa$, yet $\Delta_{\alpha<\kappa} X_\alpha \notin \mathcal{U}$. Then we have that $\kappa \setminus \Delta_{\alpha<\kappa} X_\alpha \in \mathcal{U}$ and so $\kappa \setminus \Delta_{\alpha<\kappa} X_\alpha$ is $\mathcal{U}$-stationary. We now define a function f on $\kappa \setminus \Delta_{\alpha<\kappa} X_\alpha$ by mapping ξ to the least α such that $\xi \notin X_\alpha$. f is clearly regressive, as for all $\xi \in \kappa \setminus \Delta_{\alpha<\kappa} X_\alpha$ there is an $\alpha < \xi$ such that $\xi \notin X_\alpha$ because if not then $\xi \in \bigcap_{\alpha<\xi} X_\alpha$. However, for any $\alpha < \xi$, we see that $f^{-1}(\{\alpha\}) = \{\xi \in \kappa \setminus \Delta_{\alpha<\kappa} X_\alpha : \xi \notin X_\alpha\}$, and so $X_\alpha \cap f^{-1}(\{\alpha\}) = \emptyset$. Hence, for all $\alpha < \kappa$, $f^{-1}(\{\alpha\})$ is not $\mathcal{U}$-stationary, a contradiction.

We now show the second main result of this subsection, which will be crucial when showing that any measurable cardinal is a Ramsey cardinal.

Lemma 2.1.14 ([7] (Paraphrased from Exercise 5.11))
Let $\mathcal{U}$ be a normal ultrafilter on κ, then for any $\{X_\alpha : \alpha < \kappa\}$, where $X_\alpha \in \mathcal{U}$ for all $\alpha < \kappa$, its diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\} \in \mathcal{U}$.

Proof By Lemma 2.1.13 we only have to show that for any $\mathcal{U}$-stationary sets X and regressive functions $f : X \to \Omega$ there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is $\mathcal{U}$-stationary. Fix such an $\mathcal{U}$-stationary set X and such regressive function $f : X \to \Omega$. Without loss of generality, we may assume that $X \subseteq \kappa$. If $\kappa \setminus X \in \mathcal{U}$ then $(\kappa \setminus X) \cap X \neq \emptyset$, a contradiction. Hence, $X \in \mathcal{U}$. If $0 \notin X$ then $X = \{\beta \in X : f(\beta) < \beta\}$. On the other hand, if $0 \in X$ then $X = \{\beta \in X : f(\beta) < \beta\} \cup \{0\}$ and so either $\{\beta \in X : f(\beta) < \beta\} \in \mathcal{U}$ or $\{0\} \in \mathcal{U}$. The later would contradict $\mathcal{U}$ being non-principal, and so we get that regardless $\{\beta \in X : f(\beta) < \beta\} \in \mathcal{U}$. We now define $g \in \kappa^\kappa$ by $g(\alpha) = f(\alpha)$ if $\alpha \in X$ and $g(\alpha) = \alpha$ otherwise. We then see that $\{\beta < \kappa : g(\beta) < \beta\} = \{\beta \in X : f(\beta) < \beta\} \in \mathcal{U}$ and so by $\mathcal{U}$ being a normal ultrafilter we get that there is an $\eta < \kappa$ such that $\{i \in \kappa : g(i) = \eta\} \in \mathcal{U}$. If $\eta \in X$ then $\{i \in \kappa : g(i) = \eta\} = \{i \in X : f(i) = \eta\} \in \mathcal{U}$[3]. Otherwise, we get that $\{i \in \kappa : g(i) = \eta\} = \{i \in X : f(i) = \eta\} \cup \{\eta\}$. Again, either $\{i \in X : f(i) = \eta\} \in \mathcal{U}$ or $\{\eta\} \in \mathcal{U}$, but $\mathcal{U}$ is non-principal so we get $\{i \in X : f(i) = \eta\} \in \mathcal{U}$. Hence we always have $f^{-1}(\{\eta\}) = \{i \in X : f(i) = \eta\} \in \mathcal{U}$, which makes $f^{-1}(\{\eta\})$ $\mathcal{U}$-stationary.

2.2 Transfinite Combinatorics

The aim of this section is to show Ramsey's Theorem and to show that measurable cardinals are Ramsey cardinals. Ramsey's Theorem is a combinatorial result that will allow us to construct models with indiscernibles in section 2.4. Furthermore, in section 2.4 we will show that if Ramsey cardinals exist, then $0^\#$ exists. Hence, showing that measurable cardinals are Ramsey will make measurable cardinals a sufficient condition for the existence of $0^\#$.

2.2.1 Ramsey's Theorem

We begin by introducing the *ordinary partition relation*. If X is a set of ordinals and γ is an ordinal, then let $[X]^\gamma$ denote $\{Y \subset X : Y \text{ has order type } \gamma\}$. For ordinals $\alpha, \beta, \gamma, \delta$ we say that the ordinary partition relation holds, denoted $\beta \to (\alpha)^\gamma_\delta$, just in case for any $f : [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ that is *homogeneous* for f, which means that $|f"([H]^\gamma)| \leq 1$. We also let $[X]^{<\omega} = \bigcup_{n\in\omega}[X]^n$, and say that $\beta \to (\alpha)^{<\omega}_\delta$ holds just in case for any $f : [\beta]^{<\omega} \to \delta$ there is a $H \in [\beta]^\alpha$ such that $|f"([H]^n)| \leq 1$ for every $n \in \omega$.

To prove Ramsey's Theorem, which states that for any $n, m \in \omega$, $\omega \to (\omega)^n_m$, we will be using trees. A *tree* is a partially ordered set $\langle T, <_T\rangle$ such that for any $t \in T$, the set $\{u \in T : u <_T t\}$ is well-ordered by $<_T$. For any ordinal α, the α-th *level* of T is the set of all $t \in T$ such that $\{u \in T : u <_T t\}$ has order type α. The height of T is the smallest ordinal α such that the α-th level is empty. A *chain* of T is a linearly ordered subset. Although this is not standard, we will assume that the 0-th level consist of a

3 Unless $\eta = 0$, but then $\{i \in \kappa : g(i) = \eta\} = \{i \in X : f(i) = \eta\} \cup \{0\}$ and we do the same trick as before.

single element, called the *root*. We now prove a helpful Lemma relating ordinary partition relations to trees which we will use to prove Ramsey's Theorem.

LEMMA 2.2.1 ([7] (LEMMA 7.2))
Suppose that $2 \leq n < \omega$, σ is a cardinal, and $f : [\kappa]^n \to \sigma$. Then there is a tree $\langle \kappa, <_f \rangle$ such that:

1. If $\xi_1 <_f \xi_2$ then $\xi_1 < \xi_2$.
2. If $\xi_1 <_f \ldots <_f \xi_{n-1} <_f \delta <_f \eta$ then $f(\xi_1, \ldots, \xi_{n-1}, \delta) = f(\xi_1, \ldots, \xi_{n-1}, \eta)$.
3. For each $\alpha \leq \kappa$, the α-th level of the tree has cardinality at most $\sigma^{|\omega+\alpha|}$, and is finite if both σ and α are finite.

PROOF We define the tree by recursion. Initially, for the first n levels we say $0 <_f 1 <_f \ldots <_f n-1$. Suppose now that $n \leq \eta < \kappa$ and that $<_f\restriction_{(\eta\times\eta)}$ has been defined. We now choose a downward closed chain b of $\langle \eta, <_f\restriction_{(\eta\times\eta)} \rangle$ that is maximal with respect to the following property: if $\xi_1, \ldots, \xi_n \in b$ are such that $\xi_1 <_f \ldots <_f \xi_n$ then $f(\xi_1, \ldots, \xi_n) = f(\xi_1, \ldots, \xi_{n-1}, \eta)$. With this b and η we say that $\xi <_f \eta$ for all $\xi \in b$. We first show that such a chain is unique, hence the axiom of choice is not required for this proof.

Suppose there are two different such chains b_1 and b_2, and let $c = b_1 \cap b_2$. Further, we let γ_1 be the $<_f$ minimal element in $b_1 \setminus c$ and γ_2 the $<_f$ minimal element in $b_2 \setminus c$. WLOG we assume $\gamma_1 < \gamma_2$. Then, if $\xi_1, \ldots, \xi_{n-1}$ are all in c, with $\xi_1 <_f \ldots <_f \xi_{n-1}$, we get that $f(\xi_1, \ldots, \xi_{n-1}, \gamma_1) = f(\xi_1, \ldots, \xi_{n-1}, \eta) = f(\xi_1, \ldots, \xi_{n-1}, \gamma_2)$. This is contradictory, because for γ_2 to be put on top of c in the formation of b_2, c would have to be $<_f\restriction_{(\gamma_2\times\gamma_2)}$ maximal, something it cannot be as $\gamma_1 < \gamma_2$ and so γ_1 was already put on top of c in the formation of b_1. Hence there is a unique such b.

It should be clear that 1 and 2 holds, to complete the proof and establish 3 we do transfinite induction. The base case is taken care of, so for the successor case we assume that 3 holds for a fixed α. We first show that if ξ is at the α-th level, then ξ has at most $\sigma^{|\alpha|^{n-2}}$ immediate successors at the $(\alpha+1)$-st level. To see why, if $\eta_1 < \eta_2$ are both immediate successors of ξ, then there must be some $\xi_1 <_f \ldots <_f \xi_{n-1} \leq_f \xi$ such that $f(\xi_1, \ldots, \xi_{n-1}, \eta_1) \neq f(\xi_1, \ldots, \xi_{n-1}, \eta_2)$. If not then η_2 could not have been an immediate successor of ξ, as it would have been a successor of η_1. We can further see that $\xi_{n-1} = \xi$. This is because η_1 and η_2 are both immediate successors of ξ, so for all $\xi_1 <_f \ldots <_f \xi_{n-1} <_f \xi$ we have both $f(\xi_1, \ldots, \xi_{n-1}, \eta_1) = f(\xi_1, \ldots, \xi_{n-1}, \xi)$ and $f(\xi_1, \ldots, \xi_{n-1}, \eta_2) = f(\xi_1, \ldots, \xi_{n-1}, \xi)$, hence the disagreement between $f(\xi_1, \ldots, \xi_{n-1}, \eta_1)$ and $f(\xi_1, \ldots, \xi_{n-1}, \eta_2)$ must happen when $\xi_{n-1} = \xi$. There are at most $|\alpha|^{n-2}$ many $(n-2)$-tuples $\langle \xi_1, \ldots, \xi_{n-2} \rangle$ as ξ is at the α-th level. Each immediate successor η of ξ determines for each such tuple a value $f(\xi_1, \ldots, \xi_{n-2}, \xi, \eta) < \sigma$. Since each of these values must be different for each distinct successor, as we have just seen, there are at most $\sigma^{|\alpha|^{n-2}}$ immediate successors. By induction, the α-th level has cardinality at most $\sigma^{|\omega+\alpha|}$, and each element there has at most $\sigma^{|\alpha|^{n-2}}$ immediate successors, hence the cardinality of the $(\alpha+1)$-st level is at most $\sigma^{|\alpha|^{n-2}} * \sigma^{|\omega+\alpha|} \leq \sigma^{|\omega+\alpha+1|}$. Additionally, if α and σ are finite, then $\sigma^{|\alpha|^{n-2}}$ is finite. This gives us that the $(\alpha+1)$-st level is finite because the α-th level is finite by the induction hypothesis, and each element there has a finite amount of successors.

We now turn to the limit case. So let $\alpha > 0$ be a limit ordinal. We first show that if b is a maximal chain through the part of the tree below the α-th level, then b has at most one immediate successor at the α-th level. Suppose this was not the case, and $\eta_1 < \eta_2$ were both immediate successors. Then for any $\xi_1 <_f \ldots <_f \xi_{n-1}$ in b, we find a $\xi \in b$ such that $\xi_{n-1} <_f \xi$ as α is a limit ordinal, and we observe that we get the following equalities $f(\xi_1, \ldots, \xi_{n-1}, \eta_1) = f(\xi_1, \ldots, \xi_{n-1}, \xi)$ and $f(\xi_1, \ldots, \xi_{n-1}, \eta_2) = f(\xi_1, \ldots, \xi_{n-1}, \xi)$ as η_1 and η_2 are both immediate successors of b. This is contradictory however, as η_2 is an immediate successor to b, however b is not maximal in $<_f\restriction_{(\eta_2 \times \eta_2)}$ as $\eta_1 < \eta_2$ and η_1 is an immediate successor of b. In other words, η_2 could never have been put on top of b because η_1 had been put on top of b already. Thus the cardinality of the α-th level is at most the cardinality of the set of maximal chains through the part of the tree below the α-th level. By the induction hypothesis this is at most $\Pi_{\beta<\alpha}\sigma^{|\omega+\beta|} = \sigma^{|\alpha|} \leq \sigma^{|\omega+\alpha|}$. The last thing to note is that the statement about finite cardinalities holds trivially as α cannot be finite.

We can now show Ramsey's Theorem.

Theorem 2.2.2 (Ramsey's Theorem)
Let $n, m \in \omega$, then $\omega \to (\omega)^n_m$.

Proof We do a proof by the induction on n. For the inductive case to work, we need to assume that the statement holds for a fixed n such that $n+1 \geq 2$. Hence we will check the case of $n = 0$ and $n = 1$ explicitly first. Let $n = 0$ and fix $f : [\omega]^0 \to m$, now for any set X we have that $[X]^0 = \{\emptyset\}$ as $\emptyset$ is the only set with order type 0 and $\emptyset \subseteq X$ always holds. Hence, if we let $H = \omega$ then $H \in [\omega]^\omega$ and $|f"([H]^0)| = |f(\emptyset)| = 1$. In the second base case we let $n = 1$ and we fix $f : [\omega]^1 \to m$. Now, for any set X we have that $[X]^1 \cong X$ by the bijection $\{x\} \mapsto x$. Hence, we think of $f : \omega \to m$. Now, by the pigeonhole principle there is a $k < m$ such that $|f^{-1}(k)|$ is infinite. We find $H \subseteq f^{-1}(k)$ with order type ω. This gives us $H \in [\omega]^\omega$ with $|f"(H)| = 1$, as required.

We now turn to the inductive case. Assume the statement holds for a fixed $n \geq 1$ and consider $f : [\omega]^{n+1} \to m$. Since $n + 1 \geq 2$ we can invoke Lemma 2.2.1 to get a tree $\langle \omega, <_f \rangle$ with the properties outlined by the Lemma. It should be clear that every finite level is non-empty. This is because each level is of finite cardinality (due to the third clause of the Lemma) and so if we had an empty finite level then ω would be a finite union of finite sets, a contradiction. For any maximal chain C in $\langle \omega, <_f \rangle$ and we can define $g : [C]^n \to m$ by $g(\xi_1, \ldots, \xi_n) = f(\xi_1, \ldots, \xi_n, \delta)$ for $\xi_1 <_f \ldots <_f \xi_n$ and $\delta \in C \setminus (\xi_n + 1)$. This is well defined as for any two distinct $\delta, \delta' \in C \setminus (\xi_n + 1)$ we have $\xi_1 <_f \ldots <_f \xi_n <_f \delta <_f \delta'$ or $\xi_1 <_f \ldots <_f \xi_n <_f \delta' <_f \delta$, which in both cases gives us $f(\xi_1, \ldots, \xi_n, \delta) = f(\xi_1, \ldots, \xi_n, \delta')$ due to the second clause of Lemma 2.2.1. Now, we fix a maximal chain C of order type ω. It must exist as every finite level is non-empty. Now $\langle \omega, < \rangle$ and $\langle C, <_f \rangle$ are order isomorphic by an order isomorphism $\Phi : \langle \omega, < \rangle \to \langle C, <_f \rangle$. Hence, we can define $h : [\omega]^n \to m$ by $\langle i_1, \ldots, i_n \rangle \mapsto g(\Phi(i_1), \ldots, \Phi(i_n))$.

By the induction hypothesis there is a $H \in [\omega]^\omega$ such that $|h"([H]^n)| \leq 1$. Hence, if we let $\Phi^{-1}(H) = \{\Phi^-1(x) : x \in H\}$, then $|g"([\Phi^{-1}(H)]^n)| = |h"([H]^n)| \leq 1$. This gives us $|f"([\Phi^{-1}(H)]^{n+1})| \leq 1$ as for any $\xi_1, \dots, \xi_n, \xi_{n+1} \in C$ with $\xi_1 <_f \dots <_f \xi_n <_f \xi_{n+1}$ we get $f(\xi_1, \dots, \xi_n, \xi_{n+1}) = g(\xi_1, \dots, \xi_n)$. Lastly, $\langle H, < \rangle$ is order type ω, which means that $\langle \Phi^{-1}(H), <_f \rangle$ is order type ω. Further, as $\alpha <_f \beta$ gives us $\alpha < \beta$ we have that $\langle \Phi^{-1}(H), < \rangle$ is order type ω and so $\Phi^{-1}(H) \in [\omega]^\omega$. This gives us $\omega \to (\omega)^{n+1}_m$, as required.

2.2.2 Ramsey Cardinals

The aim of this subsection will be to show that any measurable cardinal is a Ramsey cardinal. A cardinal $\kappa > \omega$ is *Ramsey* just in case $\kappa \to (\kappa)^{<\omega}_2$. The proof of the following Theorem, which has the desired result for this subsection as an immediate Corollary, will benefit from a lot of the work done in section 2.1.2.

> THEOREM 2.2.3 ([7] (PARAPHRASED FROM THEOREM 7.17))
> Suppose that κ is measurable and $\mathcal{U}$ is a normal ultrafilter over κ. Then if $f : [\kappa]^{<\omega} \to \gamma$ where $\gamma < \kappa$, there is a set in $\mathcal{U}$ homogeneous for f.

PROOF We know that such an ultrafilter exists by Theorem 2.1.12. If for each $n \in \omega$ there is a set $X_n \in \mathcal{U}$ such that X_n is homogeneous for $f \restriction_{[\kappa]^n}$, then $\bigcap_{n\in\omega} X_n \in \mathcal{U}$ by Lemma 2.1.4. Furthermore, $\bigcap_{n\in\omega} X_n$ is homogeneous for f. Hence, it is sufficient to establish that for any $n \in \omega$ and any $g : [\kappa]^n \to \gamma$ with $\gamma < \kappa$, there is a set in $\mathcal{U}$ homogeneous for g. We show this claim by induction on n. For the base case, we fix a function $g : \kappa \to \gamma$ where $\gamma < \kappa$. Pick $x \in \gamma$, then exactly one of the two following possibilities has to be the case. Either $g^{-1}(\{x\}) \in \mathcal{U}$ or $\kappa \setminus g^{-1}(\{x\}) \in \mathcal{U}$. If the former is the case then we are done, so suppose the later is the case. We note that $\kappa \setminus g^{-1}(\{x\}) = \bigcup\{g^{-1}(\{y\}) : y \in \gamma \wedge y \neq x\}$. By κ-completeness there is a $y \in \gamma$ such that $y \neq x \wedge g^{-1}(\{y\}) \in \mathcal{U}$, and so the base case holds. For the inductive case, assume the claim holds for a fixed n and fix a function $g : [\kappa]^{n+1} \to \gamma$ for some $\gamma < \kappa$. For each $s \in [\kappa]^n$ we define $g_s : \kappa \to \gamma$ by $g_s(\beta) = g(s \cup \{\beta\})$ if $\beta > \max(s)$ and $g_s(\beta) = 0$ otherwise. By a similar argument as to the base case, we use κ-completeness to find a $\delta_s < \gamma$ and $Y_s \in \mathcal{U}$ such that $g"(Y_s) = \{\delta_s\}$ for each $s \in [\kappa]^n$. Let $h : [\kappa]^n \to \gamma$ be given by $s \mapsto \delta_s$, then by the induction hypothesis we get a $\delta < \gamma$ and $Z \in \mathcal{U}$ such that $h"([Z]^n) = \delta$. This means that for $s \in [Z]^n$, $\delta_s = \delta$. For each $\alpha < \kappa$, let $Z_\alpha = \bigcap\{Y_s : \max(s) \leq \alpha\}$. By κ-completeness $Z_\alpha \in \mathcal{U}$. $\mathcal{U}$ is normal, and so by Lemma 2.1.14 $\Delta_{\alpha<\kappa} X_\alpha \in \mathcal{U}$. Let $H = Z \cap \Delta_{\alpha<\kappa} X_\alpha$, clearly $H \in \mathcal{U}$. We finish the proof by verifying that $g"([H]^{n+1}) = \{\delta\}$. Suppose that $t \in [H]^{n+1}$, write $t = s \cup \{\beta\}$ where $\max(s) < \beta$. Then $g(t) = g_s(\beta) = \delta_s$ as $\beta \in Z_{\max(s)} \subseteq Y_s$. Lastly $s \in [Z]^n$, so $\delta_s = \delta$.

We immediately get the following Corollary.

COROLLARY 2.2.4 ([7] (COROLLARY 7.18))
Measurable cardinals are Ramsey.

2.3 Skolem terms and Skolem Hulls

This section will serve as a brief introduction to the theory of Skolem functions and Skolem hulls. This is done because Skolem hulls will allow us to characterise the models of the EM Blueprints, which are used in the definition of $0^{\#}$. We will see in this section how adding Skolem functions to a theory T will allow for quantifier elimination in the resulting theory T^+. Furthermore, we will also see how we can create elementary substructures of models of T^+. These elementary substructures are known as Skolem hulls. We start out by defining the procedure of adding Skolem functions to a theory T.

DEFINITION 2.3.1 (SKOLEMISATION [6])
Let T be a theory in a first-order language L. Then a skolemisation of T is a theory $T^+ \supseteq T$ in a first-order language $L^+ \supseteq L$ such that:

1. every L-structure which is a model of T can be expanded to a model of T^+, and
2. for every formula $\phi(x_1, \ldots, x_n, y)$ of L^+, with $n \geq 1$, there is a term $t_\phi(v_1, \ldots, v_n)$ of L^+ such that $T^+ \vdash \forall x_1, \ldots, \forall x_n(\exists y \phi(x_1, \ldots, x_n, y) \leftrightarrow \phi(x_1, \ldots, x_n, t_\phi(x_1, \ldots, x_n)))$.

We call the terms t_ϕ for *Skolem functions*. Further, we say that T has *Skolem functions* or that T is a *Skolem theory* if T is a skolemisation of itself. We now rigorously define a Skolem hull.

DEFINITION 2.3.2 (SKOLEM HULL [6])
Suppose T is a Skolem theory in a first order language L. Let $\mathcal{A}$ be a model of T, and let $X \subseteq A$. Then the *Skolem hull* of X is the smallest substructure $\langle X \rangle_{\mathcal{A}}$ of $\mathcal{A}$ such that X is a subset of the domain of $\langle X \rangle_{\mathcal{A}}$.

We will now see that Skolem theories are quite powerful as they allow for quantifier elimination. Furthermore, we will show that Skolem hulls in Skolem theories are always elementary equivalent to the original model.

Theorem 2.3.3 ([6] (Paraphrased from Theorem 3.1.1))
Suppose T is a theory in a first-order language L, and T has Skolem functions. Then

1. modulo T, each formula $\phi(\mathbf{x})$ of L, where $\mathbf{x}$ is non-empty, is equivalent to a quantifier-free formula $\phi^*(\mathbf{x})$ of L, and
2. if $\mathcal{A}$ is an L-structure where $\mathcal{A} \models T$, and $X \subseteq A$ such that the Skolem hull $\langle X \rangle_{\mathcal{A}}$ of X is non-empty, then $\langle X \rangle_{\mathcal{A}}$ is an elementary substructure of $\mathcal{A}$.

Proof The first clause we prove by induction on the complexity of $\phi(\mathbf{x})$. The base case is trivial, and for the inductive cases the only one we have to check is when $\phi(\mathbf{x}) = \exists y \psi(\mathbf{x}, y)$. In that case $T \vdash \forall \mathbf{x}(\exists y \psi(\mathbf{x}, y) \leftrightarrow \psi(\mathbf{x}, t_\psi(\mathbf{x})))$ as T is a Skolem theory. By the induction hypothesis, $T \vdash \forall \mathbf{x}(\psi(\mathbf{x}, t_\psi(\mathbf{x})) \leftrightarrow \phi^*(\mathbf{x}))$ for some quantifier-free formula $\phi^*(\mathbf{x})$. Hence, $T \vdash \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \phi^*(\mathbf{x}))$, as required. For the second clause we use the Tarski-Vaught criterion to show that $\mathcal{B} = \langle X \rangle_{\mathcal{A}}$ is an elementary substructure of $\mathcal{A}$. Hence, we fix a tuple $\mathbf{b} \in B^n$ and assume that $\mathcal{A} \models \exists y \phi[\mathbf{b}, y]$ for some L-formula $\phi(v_1, \dots, v_n, v_{n+1})$. Then, by T being a Skolem theory and $\mathcal{A} \models T$, there is a Skolem function t_ϕ such that $\mathcal{A} \models \phi[\mathbf{b}, t_\phi(\mathbf{b})]$. Now, $\mathcal{B}$ is closed under Skolem functions, hence $t_\phi(\mathbf{b}) \in B$. This is exactly what the Tarski-Vaught criterion requires, and the proof is completed.

Although the theories we will be interested in have Skolem functions, we survey for the sake of completeness a theorem on how to add Skolem functions into a language to create a Skolem theory.

Theorem 2.3.4 (Skolemisation theorem [6] (Paraphrased from Theorem 3.1.2))
Let L be an infinite first-order language. Then there is a first-order language $L^\Sigma \supseteq L$ and a set Σ of L^Σ sentences such that:

1. every L-structure $\mathcal{A}$ can be expanded to a model A^Σ of Σ,
2. Σ is a Skolem theory in L^Σ and,
3. $|L^\Sigma| = |L|$.

Proof For each formula $\phi(\mathbf{x}, y)$ of L, where $\mathbf{x}$ is not empty, we introduce a new function symbol $F_{\phi,\mathbf{x}}$ of the same arity as $\mathbf{x}$. The language L' will consist of L with these new function symbols added. The set $\Sigma(L)$ will consist of all the sentences $\forall \mathbf{x}(\exists y \phi(\mathbf{x}, y) \to \phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x})))$. We first prove that every L-structure $\mathcal{A}$ can be expanded into a model of $\Sigma(L)$. We expand it to an L'-structure $\mathcal{A}'$ as follows. Let $\phi(\mathbf{x}, y)$ be any L-formula where $\mathbf{x}$ is non-empty. Then let $\mathbf{b}$ be a tuple of elements from A, which we can do as A has to be non-empty. If there is an $a \in A$ such that $\mathcal{A} \models \phi[\mathbf{b}, a]$, then we put $F^{\mathcal{A}'}_{\phi,\mathbf{x}}(\mathbf{b}) = a$ for some such a. If there is no such element, then the choice for $F^{\mathcal{A}'}_{\phi,\mathbf{x}}(\mathbf{b})$ is arbitrary. It should be clear that $\mathcal{A}' \models \Sigma(L)$.

The theory is built by iterating the construction of $\Sigma(L)$ ω times. We define a chain of languages $(L_n : n \in \omega)$ and a chain of theories $(\Sigma_n : n \in \omega)$ by induction on n. Firstly, put $L_0 = L$ and take Σ_0 to be the empty theory. Then we define L_{n+1} to be $(L_n)'$ as above. Furthermore, we define $\Sigma_{n+1} = \Sigma_n \cup \Sigma(L_n)$. Finally, we define $L^\Sigma = \bigcup_{n\in\omega} L_n$ and $\Sigma = \bigcup_{n\in\omega} \Sigma_n$.

We now verify that the three clauses are met. For clause 1 we fix a L-structure $\mathcal{A}$ and expand it to an L^Σ-structure $\mathcal{A}^\Sigma$ as follows. Let $\phi(\mathbf{x}, y)$ be any L^Σ-formula where $\mathbf{x}$ is non-empty. Then we let $\mathbf{b}$ be a tuple of elements from A. If there is an $a \in A$ such that $\mathcal{A} \models \phi[\mathbf{b}, a]$, then we put $F_{\phi,\mathbf{x}}^{\mathcal{A}^\Sigma}(\mathbf{b}) = a$. If there is no such element, then the choice for $F_{\phi,\mathbf{x}}^{\mathcal{A}^\Sigma}(\mathbf{b})$ is arbitrary. It should be clear that $\mathcal{A}^\Sigma \models \Sigma$ as any formula in Σ is on the form $\forall\mathbf{x}(\exists y\phi(\mathbf{x}, y) \to \phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x})))$ where ϕ is an L^Σ-formula. For 2 we note that for any formula ϕ in the language of L^Σ lies in some L_n, and so $\forall\mathbf{x}(\exists y\phi(\mathbf{x}, y) \to \phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x}))) \in \Sigma_{n+1}$. This means that $\Sigma \vdash \forall\mathbf{x}(\exists y\phi(\mathbf{x}, y) \to \phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x})))$. Furthermore, $\forall\mathbf{x}(\phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x})) \to \exists y\phi(\mathbf{x}, y))$ is a tautology, and hence $\Sigma \vdash \forall\mathbf{x}(\exists y\phi(\mathbf{x}, y) \leftrightarrow \phi(\mathbf{x}, F_{\phi,\mathbf{x}}(\mathbf{x})))$. This makes Σ a Skolem theory in L^Σ. Clause 3 should be rather clear as $|L_n| = |L_{n+1}|$ and because $|L_0| \geq |\omega|$ then $|\bigcup_{n\in\omega} L_n| = |L_0|$.

> Corollary 2.3.5 ([6] (Paraphrased from Theorem 3.1.3))
> Let T be a theory in an infinite first-order language L. Then T has a skolemisation T^+ in a first-order language L^+ with $|L| = |L^+|$.

Proof Put $T^+ = T \cup \Sigma$.

2.4 Indiscernibles and $0^\#$

In this section we will introduce indiscernibles and define $0^\#$. We will also give a couple of sufficient conditions for the existence of $0^\#$ and then note some initial consequences from the assumption of the existence of $0^\#$.

2.4.1 Introducing Indiscernibles

In this subsection we will define indiscernibles and show how one might construct them. These are important to us as $0^\#$ will be defined using indiscernibles. Furthermore, we will see how the construction of indiscernibles in Theorem 2.4.2 benefits greatly from Ramsey's Theorem from section 2.2.1.

> Definition 2.4.1 (indiscernibles)
> Let $\mathcal{M}$ be a structure, and let $X \subseteq M$ be linearly ordered by $<$. Then we say that $\langle X, < \rangle$ is a set of *indiscernibles* for $\mathcal{M}$ just in case for every formula $\phi(v_1, \ldots, v_n)$

in the language of $\mathcal{M}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ such that $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ we have that $\mathcal{M} \models \phi[x_1, \ldots, x_n]$ iff $\mathcal{M} \models \phi[y_1, \ldots, y_n]$.

Armed with Ramsey's Theorem from section 2.2.1 we can prove a powerful theorem about how to construct indiscernibles for theories with models of infinite cardinality.

Theorem 2.4.2 ([7] (Paraphrased from Theorem 9.2))
Suppose that T is a theory with a model of infinite cardinality and $\langle X, < \rangle$ is a linearly ordered set. Then there is a model $\mathcal{M}$ of T such that $X \subseteq M$ and $\langle X, < \rangle$ is a set of indiscernibles for $\mathcal{M}$.

Proof We start by expanding the language of T by adding a new constant c_x for each $x \in X$. We let $T_1 = \{c_x \neq c_y : x, y \in X \text{ and } x \neq y\}$, $T_2 = \{\phi(c_{x_1}, \ldots, c_{x_n}) \leftrightarrow \phi(c_{y_1}, \ldots, c_{y_n}) : \phi(v_1, \ldots, v_n)$ is a formula in the language of $T, x_1, \ldots, x_n, y_1, \ldots y_n \in X, x_1 < \ldots < x_n,$ and $y_1 < \ldots < y_n\}$ and $\overline{T} = T \cup T_1 \cup T_2$. It is sufficient to show that $\overline{T}$ is satisfiable, which by the compactness theorem means that all we have to do is to show that any finite subset of $\overline{T}$ is satisfiable. Let $S \subseteq \overline{T}$ be finite and let $\mathcal{M}$ be an infinite model of T. We then let $\{m_i \in M : i \in \omega\}$ be a set of distinct elements of M. Let n be the number of new constants appearing among the members of S, we define f on $[\omega]^n$ by $f(i_1, \ldots, i_n) = \{\phi(v_1, \ldots, v_k) : k \leq n, \phi(c_{x_1}, \ldots, c_{x_k}) \leftrightarrow \phi(c_{y_1}, \ldots, c_{y_k}) \in S$ for some $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ such that $x_1 < \ldots < x_k, y_1 < \ldots < y_k,$ and $\mathcal{M} \models \phi[m_{i_1}, \ldots, m_{i_k}]\}$. Since S is finite the range of f is finite, and so we can apply Theorem 2.2.2 to find an $H \in [\omega]^\omega$ such that $|f"([H]^n)| \leq 1$. Consider $\{m_i : i \in H\} \subseteq \{m_i : i \in \omega\}$ and let $\Phi : \omega \to H$ be the order isomorphism between them. Then we interpret the first new constant in S (with respect to the order $c_x < c_y$ iff $x < y$) by $m_{\Phi(1)}$, the second by $m_{\Phi(2)}$ and so on up til the n-th constant that is interpreted by $m_{\Phi(n)}$. Suppose $\phi(v_1, \ldots, v_k)$ is a formula such that $\phi(c_{x_1}, \ldots, c_{x_k}) \leftrightarrow \phi(c_{y_1}, \ldots, c_{y_k}) \in S$, then clearly $k \leq n$. Furthermore, let $m_{i_1}, \ldots, m_{i_k}$ be the elements of M that interpret $c_{x_1}, \ldots, c_{x_k}$ and $m_{j_1}, \ldots, m_{j_k}$ be the elements of M that interpret $c_{y_1}, \ldots, c_{y_k}$. We know that $x_1 < \ldots < x_k$ and so we get $i_1 < \ldots < i_k$ and similarly $y_1 < \ldots < y_k$ gives us that $j_1 < \ldots < j_k$. Hence $f(i_1, \ldots, i_k, a_1, \ldots, a_{n-k}) = f(j_1, \ldots, j_k, a_1, \ldots, a_{n-k})$ where $\max(i_k, j_k) < a_1 < \ldots < a_{n-k}$ for $a_i \in H$. Suppose $\mathcal{M} \models \phi[m_{i_1}, \ldots, m_{i_k}]$, then $\phi(v_1, \ldots, v_k) \in f(i_1, \ldots, i_k, a_1, \ldots a_{n-k})$ and so $\phi(v_1, \ldots, v_k) \in f(j_1, \ldots, j_k, a_1, \ldots a_{n-k})$, which means that $\mathcal{M} \models \phi[m_{j_1}, \ldots, m_{j_k}]$. Conversely, suppose $\mathcal{M} \not\models \phi[m_{i_1}, \ldots, m_{i_k}]$, then $\phi(v_1, \ldots, v_k) \notin f(i_1, \ldots, i_k, a_1, \ldots a_{n-k})$ and so $\phi(v_1, \ldots, v_k) \notin f(j_1, \ldots, j_k, a_1, \ldots a_{n-k})$, which means that $\mathcal{M} \not\models \phi[m_{j_1}, \ldots, m_{j_k}]$. Hence $\mathcal{M} \models \phi[m_{i_1}, \ldots, m_{i_k}] \leftrightarrow \phi[m_{j_1}, \ldots, m_{j_k}]$. This gives us that $\mathcal{M} \models T_2 \cap S$. Furthermore, $\mathcal{M} \models T_1 \cap S$ holds as all new constant where chosen to be distinct from each other. We have $\mathcal{M} \models T$ by assumption, which gives us $\mathcal{M} \models S$, as required.

We proceed by proving another useful Theorem about indiscernibles and partitions. This Theorem will be used in section 2.4.3 when constructing $0^\#$.

Theorem 2.4.3 ([7] (Theorem 9.3))
For infinite limit ordinals α, $\kappa \to (\alpha)_2^{<\omega}$ iff for any structure $\mathcal{M}$ of a countable language with κ a subset of its domain, there is a set of indiscernibles $X \in [\kappa]^\alpha$ for $\mathcal{M}$.

Proof Fix a an infinite limit ordinal α and suppose that $\kappa \to (\alpha)_2^{<\omega}$. By assumption the language is countable, so we let $\{\phi_n : n \in \omega\}$ enumerate the formulas of the language in a way such that ϕ_n has $k(n)$ free variables where $k(n) \leq n$. This is possible as there is more than one formula with n free variables for a given n. Now, we define $f : [\kappa]^{<\omega} \to 2$ by $f(\xi_1, \ldots, \xi_n) = 0$ if $\mathcal{M} \models \phi_n[\xi_1, \ldots, \xi_{k(n)}]$ and $f(\xi_1, \ldots, \xi_n) = 1$ otherwise. By assumption we have a set $H \in [\kappa]^\alpha$ such that $|f"([H]^n)| \leq 1$ for each $n \in \omega$. Now, fix a formula ϕ_m and any $\xi_1, \ldots, \xi_{k(m)}, \xi'_1, \ldots, \xi'_{k(m)} \in H$ such that $\xi_1 < \ldots < \xi_{k(m)}$ and $\xi'_1 < \ldots < \xi'_{k(m)}$. Then, by the homogeneity of H, $f(\xi_1, \ldots, \xi_{k(m)}) = f(\xi'_1, \ldots, \xi'_{k(m)})$. Hence, $\mathcal{M} \models \phi_m[\xi_1, \ldots, \xi_{k(m)}]$ iff $\mathcal{M} \models \phi_m[\xi'_1, \ldots, \xi'_{k(m)}]$. This gives us $\mathcal{M} \models \phi_m[\xi_1, \ldots, \xi_{k(m)}] \leftrightarrow \phi_m[\xi'_1, \ldots, \xi'_{k(m)}]$, and so $H \in [\kappa]^\alpha$ is a set of indiscernibles for $\mathcal{M}$.

Conversely, fix $f : [\kappa]^{<\omega} \to 2$ and consider the structure $\mathcal{M} = \langle \kappa, \in, f \restriction_{[\kappa]^n} \rangle_{n\in\omega}$. By assumption there is a set $X \in [\kappa]^\alpha$ of indiscernibles for $\mathcal{M}$. Hence, for any $n \in \omega$, $\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n \in X$, where $\xi_1 < \ldots < \xi_n$ and $\xi'_1 < \ldots < \xi'_n$, we have that $\mathcal{M} \models f \restriction_{[\kappa]^n} (\xi_1, \ldots, \xi_n) = 0$ iff $\mathcal{M} \models f \restriction_{[\kappa]^n} (\xi'_1, \ldots, \xi'_n) = 0$ and so $\mathcal{M} \models f \restriction_{[\kappa]^n} (\xi_1, \ldots, \xi_n) = f \restriction_{[\kappa]^n} (\xi'_1, \ldots, \xi'_n)$. Hence, $|f"([X]^n)| = 1$ for all $n \in \omega$, and so $\kappa \to (\alpha)_2^{<\omega}$.

2.4.2 Introducing the EM Blueprint

In this subsection we introduce the *EM* blueprint, which we will use to define $0^\#$. We also prove several important results about certain *EM* blueprints. Before we can define an *EM* blueprint, we need some notation. If $\mathcal{L}_\in$ is the language of set theory, then we let $\mathcal{L}^*_\in = \mathcal{L}_\in \cup \{c_k : k \in \omega\}$, where c_k is a constant.

Definition 2.4.4 (EM blueprint [7])
An *EM blueprint* is the theory in $\mathcal{L}^*_\in$ of some structure $\langle L_\delta, \in, x_k \rangle_{k\in\omega}$ where $\delta > \omega$ is a limit ordinal and $\{x_k : k \in \omega\}$ is a set of indiscernibles for $\langle L_\delta, \in \rangle$.

For a theory T in $\mathcal{L}^*_\in$, let T^- denote its restriction to $\mathcal{L}_\in$. Now, the first important result we will look at concerns itself with models of *EM* blueprints and certain Skolem hull within these models. To get there, we first need to define Skolem terms in the language of set theory. We will do this within L, and we can therefore exploit the well-ordering of L given by the next Theorem.

Theorem 2.4.5 ([7] (Paraphrased from Theorem 3.3))
There is a formula $\phi_0(v_0, v_1)$ in the language of set theory that defines in L a well-ordering $<_L$ of L such that for any limit ordinal $\delta > \omega$, any $y \in L_\delta$, and any x,

$$x <_L y \quad \text{iff} \quad x \in L_\delta \wedge \langle L_\delta, \in \rangle \models \phi_0[x, y]$$

We now define the relevant Skolem terms.

Definition 2.4.6 (Canonical Skolem Term)
Given a formula $\phi(v_0, v_1, \ldots, v_m)$ in the language of set theory, and $\phi_0(v_0, v_1)$ from Theorem 2.4.5, we define the *canonical Skolem term* t_ϕ as follows:

$$\begin{aligned} t_\phi(v_1, \ldots, v_m) = v_0 \quad \text{iff} \quad & (\forall v_{m+2} \neg \phi(v_{m+2}, v_1, \ldots, v_m) \wedge v_0 = \emptyset) \vee \\ & (\phi(v_0, v_1, \ldots, v_m) \wedge \forall v_{m+1}(\phi_0(v_{m+1}, v_0) \rightarrow \neg \phi(v_{m+1}, v_1, \ldots, v_m))). \end{aligned}$$

In other words, the Skolem term $t_\phi(v_1, \ldots, v_m)$ gives us the least v_0 such that $\phi(v_0, v_1, \ldots, v_m)$ holds according to the well order $<_L$ that is given by $\phi_0(v_0, v_1)$. We now use these Skolem terms to characterise Skolem hulls within structures that are elementarily equivalent to fragments of L.

Lemma 2.4.7 (This is well known)
Let $\mathcal{M} = \langle M, E \rangle$ be elementarily equivalent to some $\langle L_\delta, \in \rangle$ for some limit ordinal $\delta > \omega$. Then $\{t_\phi^{\mathcal{M}} : \phi \text{ is an } \mathcal{L}_\in \text{ formula}\}$ is a complete set of Skolem functions for $\mathcal{M}$ and for any $X \subseteq M$, the Skolem hull of X in $\mathcal{M}$ has domain $\{t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) : \phi \text{ is an } \mathcal{L}_\in \text{ formula and } x_1, \ldots, x_n \in X\}$.

Proof The fact that $\{t_\phi^{\mathcal{M}} : \phi \text{ is an } \mathcal{L}_\in \text{ formula }\}$ is a complete set of Skolem functions for $\mathcal{M}$ is trivial. To show that the Skolem hull of X in $\mathcal{M}$ has domain $\{t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) : \phi \text{ is an } \mathcal{L}_\in \text{ formula and } x_1, \ldots, x_n \in X\}$ we first show that $\mathcal{N} = \langle N, \in \rangle$ is elementarily equivalent to $\mathcal{M}$, where $N = \{t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) : \phi \text{ is an } \mathcal{L}_\in \text{ formula and } x_1, \ldots, x_n \in X\}$. Secondly, we show that for any $\mathcal{N}' = \langle N', \in \rangle$, where $\mathcal{N}'$ is elementarily equivalent to $\mathcal{M}$ and $X \subseteq N'$, we have that $N \subseteq N'$.

To establish that $\mathcal{N}$ and $\mathcal{M}$ are elementarily equivalent we do a proof by induction on complexity for ϕ. The base case and most of the inductive cases are trivial, so we will only verify $\phi(v_1, \ldots, v_n) = \exists x\psi(x, v_1, \ldots, v_n)$. Fix $x_1, \ldots, x_n \in N$ and suppose that $\mathcal{N} \models \phi[x_1, \ldots, x_n]$, then it is straight forward to see that $\mathcal{M} \models \phi[x_1, \ldots, x_n]$. Conversely, suppose that $\mathcal{M} \models \phi[x_1, \ldots, x_n]$, then $\mathcal{M} \models \exists x\psi[x, x_1, \ldots, x_n]$. Now, for each $x_j \in N$, there is a tuple $\mathbf{x}^j$ of elements of X, an $\mathcal{L}_\in$ formula ψ_j such that $x_j = t^{\mathcal{M}}_{\psi_j}(\mathbf{x}^j)$[4]. Hence, if we let $\psi'(x, \mathbf{x}^1, \ldots, \mathbf{x}^n) = \phi(x, t^{\mathcal{M}}_{\psi_1}(\mathbf{x}^1), \ldots, t^{\mathcal{M}}_{\psi_n}(\mathbf{x}^n))$, then $\mathcal{M} \models \exists x\psi'[x, \mathbf{x}^1, \ldots, \mathbf{x}^n]$. This gives us $\mathcal{M} \models \psi'[t^{\mathcal{M}}_{\psi'}(\mathbf{x}^1, \ldots, \mathbf{x}^n), \mathbf{x}^1, \ldots, \mathbf{x}^n]$. The crucial part here is that $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are all elements of X, and so we get that $t^{\mathcal{M}}_{\psi'}(\mathbf{x}^1, \ldots, \mathbf{x}^n) \in N$. Hence $\mathcal{N} \models \psi'[t_{\psi'}(\mathbf{x}^1, \ldots, \mathbf{x}^n), \mathbf{x}^1, \ldots, \mathbf{x}^n]$, which gives us that $\mathcal{N} \models \exists x\psi'[x, \mathbf{x}^1, \ldots, \mathbf{x}^n]$. Again, we then get that $\mathcal{N} \models \exists x\psi[x, x_1, \ldots, x_n]$ and so $\mathcal{N} \models \phi[x_1, \ldots, x_n]$. This finishes the proof that $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$.

We now prove that it is the smallest such substructure that contains X in its domain. Suppose that $\mathcal{N}'$ is an elementarily equivalent substructure of $\mathcal{M}$ and fix $t^{\mathcal{M}}_{\phi}(x_1, \ldots, x_n) \in N$. Now, by assumption the domain of $\mathcal{N}'$ contains X, and since $\mathcal{M} \models \exists x(t_\phi(x_1, \ldots, x_n) = x)$ we get that $\mathcal{N}' \models \exists x(t_\phi(x_1, \ldots, x_n) = x)$. This gives us that $N \subseteq N'$, and so $\mathcal{N}$ is the smallest such substructure that contains X in its domain.

Armed with this characterisation of the Skolem hulls, we can now show an important Lemma about *EM* blueprints. The second clause of this Lemma combined with the previous Lemma will be used to show that if $0^\#$ exists then $V \neq L$. $0^\#$ will be an *EM* blueprint with certain nice properties, and so the previous Lemma and the following Lemma will allow us to say that $P(\omega)^L = \{t_\varphi(z_1, \ldots, z_n)\}$ where $z_1 < \ldots < z_n$ is a set of indiscernible from $0^\#$. Because $0^\#$ has some nice properties, it turns out we only need a countable set of indiscernibles to capture all of $P(\omega)^L$, and since there are countable Skolem terms we see that $P(\omega)^L$ is countable. Hence $0^\#$ gives us that $V \neq L$ as $P(\omega) \neq P(\omega)^L$. This will all be shown rigoursly in section 2.4.4.

Lemma 2.4.8 ([7] (Lemma 9.4))
Suppose that T is an EM blueprint. Then for any $\alpha \geq \omega$ there is a model $\mathcal{M} = \mathcal{M}(T, \alpha)$ of T^- unique up to isomorphism such that:

1. There is a set X of ordinals in the sense of $\mathcal{M}$ of order type α (under $\mathcal{M}$'s ordinal ordering) that constitutes a set of indiscernibles for $\mathcal{M}$. Moreover, for any formula $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}_\in$, an increasing n-tuple from X satisfies ϕ in $\mathcal{M}$ exactly when $\phi(c_0, \ldots, c_{n-1}) \in T$.
2. The Skolem hull of X in $\mathcal{M}$ is again $\mathcal{M}$.

Proof We first show existence, then uniqueness. We know that T is the theory of $\langle L_\delta, \in, x_k\rangle_{k\in\omega}$ where x_k is an ordinal. We then fix a set X of ordinals with order type α such that $\{x_k : k \in \omega\} \subseteq X$. Now, by definition there is at least one infinite model of T (namely $\langle L_\delta, \in, x_k\rangle_{k\in\omega}$) and so by theorem 2.4.2 there is a model $\mathcal{M}'$ of T such that X is in the domain of $\mathcal{M}'$ and X constitutes a set of indiscernibles for $\mathcal{M}'$. We recall from the proof of theorem 2.4.2

4 We are here using j as a superscript for a tuple of variables $\mathbf{x}$, not as an exponent.

that we constructed $\mathcal{M}'$ as a model for $T \cup \{c_x \neq c_y : x, y \in X \text{ and } x \neq y\} \cup \{\phi(c_{x_1}, \ldots, c_{x_n}) \leftrightarrow \phi(c_{y_1}, \ldots, c_{y_n}) : \phi(v_1, \ldots, v_n)$ is a formula in the language of $T, x_1, \ldots, x_n, y_1, \ldots y_n \in X, x_1 < \ldots < x_n,$ and $y_1 < \ldots < y_n\}$. Now, to complete the proof that $\mathcal{M}'$ satisfies 1, we fix an $\mathcal{L}_\in$ formula $\phi(v_1, \ldots, v_n)$ and we fix $y_1 < \ldots < y_n$ all in X. Assume that $\phi(c_0, \ldots, c_{n-1}) \in T$. By $\phi(c_0, \ldots, c_{n-1}) \in T$ and $\mathcal{M}' \models T$ we get that $\mathcal{M}' \models \phi[x_0, \ldots, x_{n-1}]$ as $c_k^{\mathcal{M}} = x_k$. Since X is a set of indiscernibles for $\mathcal{M}'$ we get that $\mathcal{M} \models \phi[y_1, \ldots, y_n]$. Conversely, if $\phi(c_0, \ldots, c_{n-1}) \notin T$ then $\neg\phi(c_0, \ldots, c_{n-1}) \in T$ as T is a complete theory. This gives us $\mathcal{M}' \not\models \phi[y_1, \ldots, y_n]$ by a similar argument.

Instead of showing that $\mathcal{M}'$ satisfy 2, we let $\mathcal{M} = \langle X \rangle_{\mathcal{M}'}$. As we have seen, an *EM* blueprint has Skolem functions and so $\mathcal{M}'$ and $\mathcal{M}$ are elementary equivalent. This ensures that $\mathcal{M}$ satisfies 1. Furthermore, this gives us that $\mathcal{M} = \langle X \rangle_{\mathcal{M}'} = \langle X \rangle_{\mathcal{M}}$ and so 2 holds as well.

We now turn to the second part of the claim and prove uniqueness up to isomorphism. Suppose now that there are two such models $\mathcal{M}$ and $\mathcal{N}$ with corresponding set of indiscernibles X and Y. Since both X and Y have order type α we have an order isomorphism $h : X \to Y$. Now, by the second clause we know that $\langle X \rangle_{\mathcal{M}} = \mathcal{M}$ and that $\langle Y \rangle_{\mathcal{N}} = \mathcal{N}$. Using Lemma 2.4.7 we get that $M = \{t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) : \phi$ is an $\mathcal{L}_\in$-formula and $x_1, \ldots, x_n \in X\}$ and $N = \{t_\phi^{\mathcal{N}}(y_1, \ldots, y_n) : \phi$ is an $\mathcal{L}_\in$-formula and $y_1, \ldots, y_n \in Y\}$. Hence, it is sufficient to show that the map $h' : M \to N$ given by $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) \mapsto t_\phi^{\mathcal{N}}(h(x_1), \ldots, h(x_n))$ is an isomorphism. We first show that the map is well-defined. Suppose $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = t_{\phi'}^{\mathcal{M}}(x_1', \ldots, x_m')$, then we need to show that $t_\phi^{\mathcal{N}}(h(x_1), \ldots, h(x_n)) = t_{\phi'}^{\mathcal{N}}(h(x_1'), \ldots, h(x_m'))$. Now, $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = t_{\phi'}^{\mathcal{M}}(x_1', \ldots, x_m')$ gives us that $\mathcal{M} \models \exists x\phi[x, x_1, \ldots, x_n]$ iff $\mathcal{M} \models \exists x\phi'[x, x_1', \ldots, x_m']$[5]. We note for any increasing set of $x_1, \ldots, x_n$ from X and any $\mathcal{L}_\in$-formula $\psi(v_1, \ldots, v_n)$ we have that $\mathcal{M} \models \psi[x_1, \ldots, x_n]$ iff $\psi(c_0, \ldots, c_{n-1}) \in T$ iff $\mathcal{N} \models \psi[h(x_1), \ldots, h(x_n)]$ as h is order preserving. We may without loss of generality assume that $x_1, \ldots, x_n$ and $x_1', \ldots, x_m'$ are in increasing order[6]. Hence we get that $\mathcal{M} \models \exists x\phi[x, x_1, \ldots, x_n]$ iff $\mathcal{N} \models \exists x\phi[x, h(x_1), \ldots, h(x_n)]$ and $\mathcal{N} \models \exists x\phi'[x, h(x_1'), \ldots, h(x_m')]$ iff $\mathcal{M} \models \exists x\phi'[x, x_1', \ldots, x_m']$. This gives us $\mathcal{N} \models \exists x\phi[x, h(x_1), \ldots, h(x_n)]$ iff $\mathcal{N} \models \exists x\phi'[x, h(x_1'), \ldots, h(x_m')]$. If $\mathcal{N} \not\models \exists x\phi[x, h(x_1), \ldots, h(x_n)]$, then $t_\phi^{\mathcal{N}}(h(x_1), \ldots, h(x_n)) = \emptyset = t_{\phi'}^{\mathcal{N}}(h(x_1'), \ldots, h(x_m'))$. On the other hand, if $\mathcal{N} \models \exists x\phi[x, h(x_1), \ldots, h(x_n)]$ then $\mathcal{M} \models \phi[x^*, x_1, \ldots, x_n]$ for some $x^* = t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = t_{\phi'}^{\mathcal{M}}(x_1', \ldots, x_m')$. Again, we may without loss of generalisation assume that $x^* < x_1$ and $x^* < x_1'$. This gives us that $\mathcal{N} \models \phi[h(x^*), h(x_1), \ldots, h(x_n)]$ and $\mathcal{N} \models \phi'[h(x^*), h(x_1'), \ldots, h(x_m')]$. Assume for contradiction that there is a y such that $\mathcal{N} \models y <_L h(x^*)$ and $\mathcal{N} \models \phi[y, h(x_1), \ldots, h(x_n)]$ or $\mathcal{N} \models \phi'[y, h(x_1'), \ldots, h(x_m')]$. Then we would get $\mathcal{M} \models \phi[x, x_1, \ldots, x_n]$ or $\mathcal{M} \models \phi'[x, x_1', \ldots, x_m']$ for some x such that $h(x) = y$. This gives us that $\mathcal{M} \models x <_L x^*$ which contradicts $x^* = t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = t_{\phi'}^{\mathcal{M}}(x_1', \ldots, x_m')$ as the canonical skolem term picks out the $<_L$ minimal element which satisfies the existential formula. Hence we get that $t_\phi^{\mathcal{N}}(h(x_1), \ldots h(x_n)) =$

5 Technically this is not true if $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = \emptyset = t_{\phi'}^{\mathcal{M}}(x_1', \ldots, x_m')$ as we could have $\mathcal{M} \not\models \exists x\phi[x, x_1, \ldots, x_n]$ and $\mathcal{M} \models \phi'[\emptyset, x_1', \ldots, x_m']$, however we still get that $\mathcal{N} \not\models \exists x\phi[x, h(x_1), \ldots, h(x_n)]$ and $\mathcal{N} \models \phi'[\emptyset, h(x_1'), \ldots, h(x_m')]$. Hence we get $t_\phi^{\mathcal{N}}(h(x_1), \ldots, h(x_n)) = \emptyset = t_{\phi'}^{\mathcal{N}}(h(x_1'), \ldots, h(x_m'))$ and so the claim still holds.

6 If not then we define $\psi(v, v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \phi(v, v_1, \ldots, v_n)$ for some permutation σ such that $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ is in increasing order. It is straight forward to see that $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n) = t_\psi^{\mathcal{M}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, and so we proceed the argument by considering $t_\psi^{\mathcal{M}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ instead of $t_\phi^{\mathcal{M}}(x_1, \ldots, x_n)$. A similar approach works for ϕ'.

$h(x^*) = t^{\mathcal{N}}_{\phi'}(h(x'_1), \dots h(x'_m))$.

Now, to show injectivity suppose $h'(t^{\mathcal{M}}_{\phi}(x_1, \dots, x_n)) = h'(t^{\mathcal{M}}_{\phi'}(x'_1, \dots, x'_m))$ for some fixed $t^{\mathcal{M}}_{\phi}(x_1, \dots, x_n)$ and $t^{\mathcal{M}}_{\phi'}(x'_1, \dots, x'_m)$. This means that $\mathcal{N} \models \exists x \phi[x, h(x_1), \dots, h(x_n)]$ iff $\mathcal{N} \models \exists x \phi'[x, h(x'_1), \dots, h(x'_m)]$[7]. Similarly here, we may assume that $x_1, \dots, x_n$ and $x'_1, \dots, x'_m$ are in increasing order. Hence we get that $\mathcal{M} \models \exists x \phi[x, x_1, \dots, x_n]$ iff $\mathcal{N} \models \exists x \phi[x, h(x_1), \dots, h(x_n)]$ and $\mathcal{N} \models \exists x \phi'[x, h(x'_1), \dots, h(x'_m)]$ iff $\mathcal{M} \models \exists x \phi'[x, x'_1, \dots, x'_m]$. This gives us $\mathcal{M} \models \exists x \phi[x, x_1, \dots, x_n]$ iff $\mathcal{M} \models \exists x \phi'[x, x'_1, \dots, x'_m]$. If $\mathcal{M} \not\models \exists x \phi[x, x_1, \dots, x_n]$ then $t^{\mathcal{M}}_{\phi}(x_1, \dots, x_n) = \emptyset = t^{\mathcal{M}}_{\phi'}(x_1, \dots, x_m)$. On the other hand, if $\mathcal{M} \models \exists x \phi[x, x_1, \dots, x_n]$ then $\mathcal{N} \models \phi[y^*, h(x_1), \dots, h(x_n)]$ for some $y^* \in Y$ such that $t^{\mathcal{N}}_{\phi}(h(x_1), \dots, h(x_n)) = t^{\mathcal{N}}_{\phi'}(h(x'_1), \dots, h(x'_m)) = y^*$. Again, we may without loss of generality assume that $y^* < h(x_1)$ and $y^* < h(x'_1)$. This gives us $\mathcal{M} \models \phi[x^*, x_1, \dots, x_n]$ and $\mathcal{M} \models \phi'[x^*, x'_1, \dots, x'_m]$ for some $x^* \in X$ such that $h(x^*) = y^*$. Suppose for contradiction that there is an x such that $\mathcal{M} \models x <_L x^*$ and $\mathcal{M} \models \phi[x, x_1, \dots, x_n]$ or $\mathcal{M} \models \phi'[x, x'_1, \dots, x'_m]$. Then we get $\mathcal{N} \models \phi[h(x), h(x_1), \dots, h(x_n)]$ or $\mathcal{N} \models \phi'[h(x), h(x'_1), \dots, h(x'_m)]$. This gives us $\mathcal{N} \models h(x) <_L h(x^*) = y^*$, which contradicts $t^{\mathcal{N}}_{\phi}(h(x_1), \dots, h(x_n)) = t^{\mathcal{N}}_{\phi'}(h(x'_1), \dots, h(x'_m)) = y^*$. Hence we get that $t^{\mathcal{M}}_{\phi}(x_1, \dots, x_n) = x^* = t^{\mathcal{M}}_{\phi}(x'_1, \dots, x'_m)$.

Now, for surjectivity, fix $t^{\mathcal{N}}_{\phi}(y_1, \dots, y_n) \in N$. Since h is an isomorphism there are $x_i \in X$ such that $h(x_i) = y_i$ for $1 \leq i \leq n$. This gives us $h'(t^{\mathcal{M}}_{\phi}(x_1, \dots, x_n)) = t^{\mathcal{N}}_{\phi}(h(x_1), \dots, h(x_n)) = t^{\mathcal{N}}_{\phi}(y_1, \dots, y_n)$, as required.

2.4.3 Introducing $0^\#$

In this subsection we will define $0^\#$, and work our way up to sufficient conditions for the existence of $0^\#$.

> Definition 2.4.9 ($0^\#$ [7])
> For a given EM blueprint T, we say that it satisfies:
> (I) iff $\mathcal{M}(T, \alpha)$ is well-founded for every $\alpha < \omega_1$;
> (II) iff for any n-ary Skolem term t, T contains the sentence: $t(c_0, \dots, c_{n-1}) < \Omega \to t(c_0, \dots, c_{n-1}) < c_n$; and
> (III) iff for any $(m+n+1)$-ary Skolem term t, T contains the sentence: $t(c_0, \dots, c_{n+m}) < c_m \to t(c_0, \dots, c_{n+m}) = t(c_0, \dots, c_{m-1}, c_{k_1}, \dots, c_{k_{n+1}})$ for any $k_1 < \dots < k_{n+1}$ with $m \leq k_1$.
> Then we say that $0^\#$ is the unique EM blueprint satisfying $(I) - (III)$.

The rest of this subsection is going to be spent looking at under which conditions $0^\#$ exists, i.e. under which conditions there is a unique EM blueprint satisfying $(I) - (III)$. The uniqueness part is taken care of by clause (e) from Lemma 2.4.20, which states that if there is such an EM blueprint, then it is unique.

7 We have the same caveat and the same solution as when we showed that the map was well-defined.

We start with clause (I). To motivate why we are interested in well-founded $\mathcal{M}(T,\alpha)$, we note that the following two Theorems gives us an interesting Corollary.

Theorem 2.4.10 (The Mostowski Collapse [7] (Paraphrased from Theorem 0.4))
Suppose that $\langle M, E, \dots \rangle$ is a (possibly proper class) structure with E a binary relation on M satisfying:

1. E is well-founded, i.e., there is no chain $\langle m_i : i \in \omega \rangle$ such that for all $i \in \omega$ $m_{i+1} \, E \, m_i$;
2. $\langle M, E \rangle$ is extensional, i.e. if $a, b \in M$ and $x \, E \, a$ iff $x \, E \, b$ for all $x \in M$, then $a = b$; and
3. E is set-like, i.e. $\{x : x \, E \, a\}$ is a set for every $a \in M$.

Then there is a unique isomorphism $\pi : \langle M, E, \dots \rangle \to \langle \overline{M}, \in, \dots \rangle$ where $\overline{M}$ is transitive.

Theorem 2.4.11 ([7] (Paraphrased from 3.3))
There is a sentence σ_0 of $\mathcal{L}_\in$ such that for any transitive class N, $\langle N, \in \rangle \models \sigma_0$ iff $N = L$ or $N = L_\delta$ for some limit ordinal $\delta > \omega$.

Corollary 2.4.12
Let T be an EM blueprint and suppose that $\mathcal{M}(T,\alpha)$ is well-founded. Then $\mathcal{M}(T,\alpha) \cong \langle L_\delta, \in \rangle$ for some limit ordinal $\delta > \omega$.

Hence, we are interested in figuring out when $\mathcal{M}(T,\alpha)$ is well-founded. When this is the case we will assume that $\mathcal{M}(T,\alpha)$ is equal to the corresponding $\langle L_\delta, \in \rangle$.

An initial response to such an inquiry could be that surely $\mathcal{M}(T,\alpha)$ is always well-founded as it is a model of T^- where T is the theory of some $\langle L_\delta, \in, x_k \rangle_{k\in\omega}$. However, being well-founded is not a property of first order logic, and hence not a sentence in T. So, a priori, $\mathcal{M}(T,\alpha)$ does not have to be well-founded. The next Lemma justifies why (I) only needs $\mathcal{M}(T,\alpha)$ to be well-founded for $\alpha < \omega_1$.

Lemma 2.4.13 ([7] (Lemma 9.5))
Suppose that T is an EM blueprint. Then $\mathcal{M}(T,\alpha)$ is well-founded for every α just in case $\mathcal{M}(T,\alpha)$ is well-founded for every $\alpha < \omega_1$.

Proof One direction is immediate. For the other direction we assume that $\mathcal{M}(T,\alpha)$ is ill-founded for some α and proceed to show that there is a $\beta < \omega_1$ such that $\mathcal{M}(T,\beta)$ is ill-founded. Now, assume that that $\mathcal{M}(T,\alpha) = \langle M, E\rangle$ is ill-founded and let $\langle a_i \in M : i \in \omega\rangle$ be a sequence such that $a_{i+1} \, E \, a_i$ for every $i \in \omega$. We know from the second clause of lemma 2.4.8 that each a_i is on the form $t_\phi^{\langle M,E\rangle}(x_1, \ldots, x_j)$ for some Skolem term t and indiscernibles $x_1, \ldots, x_j$. We therefore let Y be the countable set consisting of the set of indiscernibles involved in these terms. Let $\mathcal{N} = \langle Y\rangle_{\langle M,E\rangle}$, then we know that $\mathcal{N} \cong \langle M, E\rangle$, and so $\mathcal{N}$ is ill-founded. Furthermore, by the second clause of 2.4.8, we know that $\mathcal{N} \cong \mathcal{M}(T,\beta)$ where β is the order type of Y. As Y is countable we have that $\beta < \omega_1$, which concludes the proof.

We now prove that a certain partition relation is all we need to satisfy (I). We will show later that the existence of a Ramsey cardinal, and hence the existence of a measurable cardinal by Corollary 2.2.4, is enough to satisfy the assumption of the partition relation holding.

Lemma 2.4.14 ([7] (Lemma 9.6))
Suppose there is a κ such that $\kappa \to (\omega_1)_2^{<\omega}$. Then there is an EM blueprint T such that $\mathcal{M}(T,\alpha)$ is well-founded for every $\alpha < \omega_1$.

Proof We know from Theorem 2.4.3 that L_κ has a set $X \in [\kappa]^{\omega_1}$ of indiscernibles. Let T be the corresponding EM blueprint. Fix $\alpha < \omega_1$ and let $X_\alpha \subset X$ be the first α indiscernibles. Now $\mathcal{M}(T,\alpha)$ is elementarily equivalent to L_κ, and so $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)} \cong \langle X_\alpha\rangle_{L_\kappa}$, which makes $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)}$ well-founded. By the second clause of Lemma 2.4.8 we know that $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)} \cong \mathcal{M}(T,\alpha)$, and so $\mathcal{M}(T,\alpha)$ is well-founded.

From here we can conclude that under the hypothesis of $\kappa \to (\omega_1)_2^{<\omega}$, there is an EM blueprint T such that $\mathcal{M}(T,\omega_1)$ is well-founded. Hence $\mathcal{M}(T,\omega_1) \cong \langle L_\delta, \in\rangle$ for some limit ordinal δ. This means that L_δ has a set X of indiscernibles of order type ω_1. Under this assumption we now fix some notation. Let ρ be the least limit ordinal such that L_ρ has a set of ordinals indiscernibles of order type ω_1. Further, let H be such a set of indiscernibles for L_ρ with the least possible ω-th element and lastly let T_0 be the corresponding EM blueprint. We now check that $(I) - (III)$ holds for T_0.

Lemma 2.4.15 (This is well known)
Let T be a EM blueprint such that $T = T_0$, then $\mathcal{M}(T,\alpha)$ is well-founded for all $\alpha < \omega_1$.

Proof Fix $\alpha < \omega_1$, and let $X_\alpha \subset H$ be the first α indiscernibles. Now, $\mathcal{M}(T,\alpha)$ is elementarily equivalent to L_ρ, so $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)} \cong \langle X_\alpha\rangle_{L_\rho}$, which makes $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)}$ well-founded. By the second clause of Lemma 2.4.8 we know that $\langle X_\alpha\rangle_{\mathcal{M}(T,\alpha)} \cong \mathcal{M}(T,\alpha)$, and so $\mathcal{M}(T,\alpha)$ is well-founded.

We now move on to (II).

Lemma 2.4.16 ([7] (Paraphrased from Lemma 9.8))
Let T be a EM blueprint such that $T = T_0$, then for any n-ary Skolem term t, T contains the sentence $t(c_0, \ldots, c_{n-1}) < \Omega \to t(c_0, \ldots, c_{n-1}) < c_n$.

Proof Assume to the contrary that $t(c_0, \ldots, c_{n-1}) < \Omega \wedge c_n \leq t(c_0, \ldots, c_{n-1})$ is in T_0 for some t. Let $z_0 < \ldots < z_{n-1}$ be the first n members of our fixed set H of indiscernibles for L_ρ. Further, set $\overline{H} = H \setminus \{z_0, \ldots, z_{n-1}\}$ and $\delta = t^{L_\rho}(z_0, \ldots, z_{n-1}) < \rho$. We know that $z_n \leq \delta$, and so by indiscernibility, $z^* \leq \delta$ for all $z^* \in H$. Hence $\overline{H} \subseteq \delta$. Now suppose δ is not a limit ordinal, then $\delta = \overline{\delta} + k$ for $k \in \omega$ for some limit ordinal $\overline{\delta}$. There must be at least one element of $\overline{H}$ below $\overline{\delta}$, if not there is an infinite increasing sequence of ordinals between $\overline{\delta}$ and $\overline{\delta} + k$. Let $z^* \in \overline{H}$ be such an element. By indiscernibility and the fact that $\overline{\delta}$ is definable from $z_0, \ldots, z_{n-1}$ we get $z' < \overline{\delta}$ for all $z' \in \overline{H}$ and so $\overline{H} \subseteq \overline{\delta}$. Hence, we may without loss of generality assume that δ is a limit ordinal.

We will now show that $\overline{H}$ is a set of indiscernibles for L_δ, which will contradict the minimality of ρ as $\overline{H}$ also has order type ω_1. Suppose that $x_1 < \ldots < x_m$ and $y_1 < \ldots < y_m$ are all in $\overline{H}$. Then for any formula $\phi(v_1, \ldots, v_m)$ we know that $L_\delta \models \phi[x_1, \ldots, x_m]$ is equivalent to $L_\rho \models \phi^{L_\delta}[x_1, \ldots, x_m]$ where the relativization is possible as $L_\delta \subseteq L_\rho$ are transitive. We know that L_δ is definable in L_ρ from δ, and so $L_\rho \models \phi^{L_\delta}[x_1, \ldots, x_m]$ is equivalent to $L_\rho \models \overline{\phi}[x_1, \ldots, x_m, z_0, \ldots, z_{n-1}]$ for some $\overline{\phi}$. Analogously we have that $L_\delta \models \phi[y_1, \ldots, y_m]$ is equivalent to $L_\rho \models \overline{\phi}[y_1, \ldots, y_m, z_0, \ldots, z_{n-1}]$, and so we get that $L_\delta \models \phi[x_1, \ldots, x_m]$ iff $L_\delta \models \phi[y_1, \ldots, y_m]$ from the indiscernibility of H for L_ρ.

Hence we have verified that T_0 satisfies (I) and (II) and move on to verify that it satisfies (III).

Lemma 2.4.17 ([7] (Paraphrased from 9.10))
Suppose that T is an EM blueprint such that $T = T_0$. Then for any $(m+n+1)$-ary Skolem term t, T contains the sentence $t(c_0, \ldots, c_{m+n}) < c_m \to t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1})$.

Proof We assume that $t(c_0, \ldots, c_{m+n}) < c_m$ is in T_0 and aim to show that $t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1})$ is in T_0. Now, let $H = \bigcup\{s_\xi : \xi < \omega_1\}$ be a disjoint partition into sets consisting of consecutive elements of H such that $|s_0| = m$, $|s_\eta| = n+1$ and $\max(s_\xi) < min(s_\eta)$ for $0 \leq \xi < \eta < \omega_1$. We recall that H is the set of indiscernibles for L_ρ of order type ω_1 that has the least possible ω-th element. We let $t(s_0, s_\xi)$ denote $t^{L_\rho}(x_0, \ldots, x_{m-1}, y^\xi_0, \ldots, y^\xi_n)$ for $0 < \xi < \omega_1$ where $s_0 = \langle x_0, \ldots, x_{m-1}\rangle$ and $s_\xi = \langle y^\xi_0, \ldots, y^\xi_n\rangle$ in increasing order. It suffices by indiscernibility to derive a contradiction from the assumption that $t(s_0, s_\xi) \neq t(s_0, s_\eta)$ for some $0 < \xi < \eta < \omega_1$.

If $t(s_0, s_\xi) > t(s_0, s_\eta)$ then we find a sequence of ordinals $\{\eta_n : n \in \omega\}$ where $\eta_0 = \eta$, $\eta_n < \eta_{n+1} < \omega_1$. By indiscernibility we get that $t(s_0, s_{\eta_n}) > t(s_0, s_{\eta_{n+1}})$ for all $n \in \omega$, which gives us an ω decreasing sequence of ordinals, a contradiction.

Now suppose $t(s_0, s_\xi) < t(s_0, s_\eta)$, then we get that for all $\xi_1, \xi_2 \in \omega_1$, $t(s_0, s_{\xi_1}) < t(s_0, s_{\xi_2})$ iff $\xi_1 < \xi_2$ by a simple indiscernibility argument. We will then show that the set $\{t(s_0, s_\xi) : 0 < \xi < \omega_1\}$ is a set of indiscernibles for L_ρ with a smaller ω-th element than H, which will be our contradiction and give us the desired result. We first show that $\{t(s_0, s_\xi) : 0 < \xi < \omega_1\}$ is a set of indiscernibles. Note that for any $\mathcal{L}_\in$-formula $\phi(v_1, \ldots, v_n)$ there is another formula ϕ^* such that $\langle L_\rho, \in\rangle \models \phi[t(s_0, s_{\xi_1}), \ldots, t(s_0, s_{\xi_n})] \leftrightarrow \phi^*[s_0, s_{\xi_1}, \ldots, s_{\xi_n}]$. This is because t^{L_ρ} is expressible in the language. Suppose that we have $t(s_0, s_{\xi_1}) < \ldots < t(s_0, s_{\xi_n})$ and $t(s_0, s_{\eta_1}) < \ldots < t(s_0, s_{\eta_n})$ for some $\xi_i, \eta_i \in \omega_1$. Then we have that $s_{\xi_1} < \ldots < s_{\xi_n}$ and $s_{\eta_1} < \ldots < s_{\eta_n}$ and so we get that $\langle L_\rho, \in\rangle \models \phi^*[s_0, s_{\eta_1}, \ldots, s_{\eta_n}] \leftrightarrow \phi^*[s_0, s_{\xi_1}, \ldots, s_{\xi_n}]$. This gives us $\langle L_\rho, \in \rangle \models \phi[t(s_0, s_{\xi_1}), \ldots, t(s_0, s_{\xi_n})] \leftrightarrow \phi[t(s_0, s_{\eta_1}), \ldots, t(s_0, s_{\eta_n})]$. Hence $\{t(s_0, s_\xi) : 0 < \xi < \omega_1\}$ is a set of indiscernibles for L_ρ. Now, we know that if we let y_0^ω be the first element of s_ω, then y_0^ω is the ω-th element of H. Furthermore, by assumption we have that $t(s_0, s_\omega) < y_0^\omega$. Hence, the ω-th element of $\{t(s_0, s_\xi) : 0 < \xi < \omega_1\}$ is smaller than the ω-th element of H, a contradiction.

We can easily see that for any $k_1 < \ldots < k_{n+1}$, with $m \leq k_1$, that T_0 also contains $t(c_0, \ldots, c_{m+n}) < c_m \to t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{k_1}, \ldots, c_{k_{n+1}})$ by appealing to a simple indiscernible argument. Hence T_0 satisfies (III).

We now build up to Lemma 2.4.20, as the last clause will give us uniqueness of any EM blueprint that satisfies $(I) - (III)$. We need two more lemmata before we are ready to prove Lemma 2.4.20.

> Lemma 2.4.18 ([7] (Paraphrased from Lemma 9.9))
> Let T be an EM blueprint, then for any n-ary Skolem term t, T contains the sentence $t(c_0, \ldots, c_{n-1}) < \Omega \to t(c_0, \ldots, c_{n-1}) < c_n$ just in case for any infinite limit ordinal α, the set of indiscernibles corresponding to $\mathcal{M}(T, \alpha)$ is cofinal in the ordinals of the structure.

Proof Fix T and suppose that for any n-ary Skolem term t, T contains the sentence $t(c_0, \ldots, c_{n-1}) < \Omega \to t(c_0, \ldots, c_{n-1}) < c_n$. Let α be a limit ordinal and fix any ordinal β in $\mathcal{M}(T, \alpha)$. We need to find an indiscernible γ of $\mathcal{M}(T, \alpha)$ such that $\beta < \gamma$. We know that $\beta = t^{\mathcal{M}(T,\alpha)}(x_0, \ldots, x_n)$ for some Skolem term t and indiscernibles $x_0 < \ldots < x_n$ by the second clause of Lemma 2.4.8 and Lemma 2.4.7. Let $y_i = c_i^{\mathcal{M}(T,\alpha)}$ for $0 \leq i \leq n$, then $y_0 < \ldots < y_n$ and so $\mathcal{M}(T, \alpha) \models \forall z(z \in t[x_0, \ldots, x_n] \leftrightarrow z \in t[y_0, \ldots, y_n])$. By extentionality we get that $\mathcal{M}(T, \alpha) \models \beta = t[y_0, \ldots, y_n]$. Furthermore, by assumption we have that $\mathcal{M}(T, \alpha) \models t[y_0, \ldots, y_n] < \gamma$ where $\gamma = c_{n+1}^{\mathcal{M}(T,\alpha)}$ is some ordinal.

Conversely, suppose that for any infinite limit ordinal α, the set of indiscernibles corresponding to $\mathcal{M}(T, \alpha)$ is cofinal in the ordinals of the structure. Then, we inductively define the interpretation of the constants c_i amongst the indiscernibles. The first pick of c_0 is arbitrary, so suppose we have picked out indiscernibles for $c_0 < \ldots < c_n$, then let $X_n = \{(t(c_0, \ldots, c_n))^{\mathcal{M}(T,\alpha)} : t \text{ is a Skolem term and } (t(c_0, \ldots, c_n))^{\mathcal{M}(T,\alpha)} < \Omega\}$ and $\beta_n = \sup(X_n)$. Then β_n is an ordinal in $\mathcal{M}(T, \alpha)$, and so since the indiscernibles are cofinal in the ordinals of the structure we have an indiscernible γ_n such that $\beta_n < \gamma_n$. Hence we let $c_{n+1}^{\mathcal{M}(T,\alpha)} = \gamma_n$. It should be clear that the claim follows.

Fixing some further notation, for an EM blueprint T that satisfies $(I) - (III)$ and any ordinal α, we let $\{\iota_\xi^{T,\alpha} : \xi < \alpha\}$ denote the increasing sequence of indiscernibles for $\mathcal{M}(T, \alpha)$.

Lemma 2.4.19 ([7] (Lemma 9.11))
If T is an EM blueprint satisfying $(I) - (III)$ and $\omega \leq \alpha < \beta$ with α a limit ordinal, then the Skolem hull of $\{\iota_\xi^{T,\beta} : \xi < \alpha\}$ in $\mathcal{M}(T, \beta)$ is L_ι, where $\iota = \iota_\alpha^{T,\beta}$. Consequently, $\mathcal{M}(T, \alpha) = L_\iota$ and $\iota_\xi^{T,\alpha} = \iota_\xi^{T,\beta}$ for every $\xi < \alpha$.

Proof Let $\mathcal{N}$ be the stated Skolem hull. It suffices to show that $\Omega^{\mathcal{N}} = \iota_\alpha^{T,\beta}$, as the second sentence follows from the definition and uniqueness of $\mathcal{M}(T, \alpha)$. If σ is an ordinal in $\mathcal{N}$, then we have some Skolem term t and ordinals $\xi_0 < \ldots < \xi_{n-1} < \alpha$ such that $\sigma = t^{\mathcal{M}(T,\beta)}(\iota_{\xi_0}^{T,\beta}, \ldots, \iota_{\xi_{n-1}}^{T,\beta})$ by the second clause of Lemma 2.4.8 and Lemma 2.4.7. Since α is a limit ordinal there is a ξ_n such that $\xi_{n-1} < \xi_n < \alpha$ and so by (II) we get that $t^{\mathcal{M}(T,\beta)}(\iota_{\xi_0}^{T,\beta}, \ldots, \iota_{\xi_{n-1}}^{T,\beta}) < \iota_{\xi_n}^{T,\beta} < \iota_\alpha^{T,\beta}$ and so $\Omega^{\mathcal{N}} \subseteq \iota_\alpha^{T,\beta}$. Conversely, suppose $\tau < \iota_\alpha^{T,\beta}$, then $\tau = u^{\mathcal{M}(T,\beta)}(\iota_{\xi_0}^{T,\beta}, \ldots, \iota_{\xi_k}^{T,\beta})$ for some k and ordinals $\xi_0 < \ldots < \xi_k < \beta$. Now, we rename our ordinals from $\xi_0 < \ldots < \xi_k$ to $\zeta_0 < \ldots < \zeta_{m-1} < \eta_0 < \ldots < \eta_n$ such that $\zeta_{m-1} < \alpha \leq \eta_0$. α is a limit ordinal so $\zeta_{m-1} < \zeta_{m-1} + 1 < \ldots < \zeta_{m-1} + n + 1 < \alpha$. We see that $u^{\mathcal{M}(T,\beta)}(\iota_{\zeta_0}^{T,\beta}, \ldots, \iota_{\zeta_{m-1}}^{T,\beta}, \iota_{\eta_0}^{T,\beta}, \ldots, \iota_{\eta_n}^{T,\beta}) < \iota_{\eta_0}^{T,\beta}$ and so by (III) we get that $u^{\mathcal{M}(T,\beta)}(\iota_{\zeta_0}^{T,\beta}, \ldots, \iota_{\zeta_{m-1}}^{T,\beta}, \iota_{\eta_0}^{T,\beta}, \ldots, \iota_{\eta_n}^{T,\beta}) = u^{\mathcal{M}(T,\beta)}(\iota_{\zeta_0}^{T,\beta}, \ldots, \iota_{\zeta_{m-1}}^{T,\beta}, \iota_{\zeta_{m-1}+1}^{T,\beta}, \ldots, \iota_{\zeta_{m-1}+n+1}^{T,\beta})$ and so $\tau < \Omega^{\mathcal{N}}$. Hence $\Omega^{\mathcal{N}} = \iota_\alpha^{T,\beta}$, as required.

Hence for any EM blueprint T satisfying $(I) - (III)$ and any ξ we can unambiguously set $\iota_\xi^T = \iota_\xi^{T,\alpha}$ for some limit ordinal $\alpha > \xi$. Hence, we fix the following notation: $I^T = \{\iota_\xi^T : \xi \in \Omega\}$. This is a class of indiscernibles for the models of the EM blueprints. We are now ready to prove the Lemma that will gives us uniqueness of an EM blueprint that satisfy $(I) - (III)$.

Lemma 2.4.20 ([7] (Lemma 9.12))
Suppose that T is an EM blueprint satisfying $(I) - (III)$. Then:
(a) If $\xi < \zeta$ then $L_{\iota_\xi^T} \prec L_{\iota_\zeta^T}$.
(b) $|\iota_\xi^T| = |\xi| + \aleph_0$ for every ξ.

(c) I^T is a closed unbounded class of ordinals.

(d) For any cardinal $\lambda > \omega$, $\iota^T_\lambda = \lambda \in I^T$ and so $\mathcal{M}(T, \lambda) = L_\lambda$.

(e) T is the only EM blueprint satisfying $(I) - (III)$.

Proof (a) For infinite limit ordinals $\xi' < \zeta'$ the results follows directly from Lemma 2.4.19. This is because $\mathcal{M}(T, \zeta') = L_{\iota^T_{\zeta'}}$ and the Skolem hull of $\{\iota^T_\xi : \xi < \xi'\}$ in $\mathcal{M}(T, \zeta')$ is $L_{\iota^T_{\xi'}}$. Hence, for a sufficiently large $\beta \in \Omega$, we get that $\langle L_{\iota^T_\beta}, \in\rangle \models L_{\iota^T_{\xi'}} \prec L_{\iota^T_{\zeta'}}$. So, by an indiscernibility argument, we also get $\langle L_{\iota^T_\beta}, \in \rangle \models L_{\iota^T_\xi} \prec L_{\iota^T_\zeta}$ for arbitrary $\xi < \zeta$.

(b) Suppose α is an infinite limit ordinal, then by Lemma 2.4.19 we know that $L_{\iota^T_\alpha}$ is the Skolem hull in itself of $\{\iota^T_\xi : \xi < \alpha\}$. Hence $|\iota^T_\alpha| = |L_{\iota^T_\alpha}| = |\{\iota^T_\xi : \xi < \alpha\}| = |\alpha|$. Hence the claim holds for all limit ordinals, and we proceed to show that it holds for all infinite successor ordinals. Now, for any infinite successor ordinal ξ, we find two limit ordinals α, α' such that $\alpha' < \xi < \alpha$ and $|\alpha'| = |\alpha|$. This gives us $|\xi| = |\alpha| = |\alpha'|$ straight away. Furthermore, $\alpha' < \xi < \alpha$ gives us $\iota^T_{\alpha'} < \iota^T_\xi < \iota^T_\alpha$. Since α' and α are limit ordinals, we get that $|\iota^T_{\alpha'}| = |\alpha'| = |\alpha| = |\iota^T_\alpha|$. This gives us $|\iota^T_\xi| = |\xi|$, as required.

(c) Unbounded: Fix an ordinal α, we need to find some $\xi \in \Omega$ such that $\alpha < \iota^T_\xi$. Let κ be the smallest cardinal such that $\alpha < \kappa$. We know by (b) that $|\iota^T_\kappa| = |\kappa|$, and so since κ is a cardinal we get that $\kappa \leq \iota^T_\kappa$. Hence we get that $\alpha < \iota^T_\kappa$.
Closed: Fix α and suppose that $\sup(I^T \cap \alpha) = \alpha$. This means that $\sup(\{\iota^T_\xi < \alpha : \xi \in \Omega\}) = \alpha$. Let β be the smallest ordinal such that $\xi < \beta$ implies that $\iota^T_\xi < \alpha$. Then $\alpha = \sup(\{\iota^T_\xi : \xi < \beta\})$. There are two cases. If α is a successor ordinal then $\sup(\{\iota^T_\xi : \xi < \beta\}) = \max(\{\iota^T_\xi : \xi < \beta\}) = \alpha$, and so there is a $\xi' < \beta$ such that $\iota^T_{\xi'} = \alpha$. Hence $\alpha \in I^T$. If α is a limit ordinal then we see that β must be a limit ordinal. Hence, by Lemma 2.4.18 we know that $\{\iota^T_\xi : \xi < \beta\}$ is cofinal in ι^T_β. So, for all $\gamma < \iota^T_\beta$ there is a $\xi < \beta$ such that $\gamma < \iota^T_\xi$, which gives us that $\gamma < \alpha$. Hence, $\iota^T_\beta \leq \alpha$. Conversely, if $\gamma < \alpha$ there is a $\xi < \beta$ such that $\gamma < \iota^T_\xi$, or else γ would be the supremum of $\{\iota^T_\xi : \xi < \beta\}$. This gives us $\gamma < \iota^T_\beta$ and so $\alpha \leq \iota^T_\beta$. Hence $\alpha = \iota^T_\beta \in I^T$.

(d) Let $\lambda > \omega$ be a cardinal. We know that $|\iota^T_\lambda| = |\lambda| + \aleph_0 = \lambda$ by (b). We also know from (b) that for $\beta < \lambda$, $|\iota^T_\beta| = |\beta| + \aleph_0 < \lambda$ as λ is a cardinal. Now I^T is unbounded in λ, so $\sup(I^T \cap \lambda) = \lambda$, and so by I^T being closed we get that $\lambda \in I^T$. If $\iota^T_\lambda \neq \lambda$ then $\lambda < \iota^T_\lambda$ as λ is a cardinal. This means that there is a $\beta < \lambda$ such that $\iota^T_\beta = \lambda$, but from (b) we would then get that $|\beta| + \aleph_0 = |\iota^T_\beta| = \lambda$, contradicting the fact that λ is a cardinal. Hence $\iota^T_\lambda = \lambda \in I^T$. $\mathcal{M}(T, \lambda) = L_\lambda$ follows from Lemma 2.4.19 and the fact that any cardinal is a limit ordinal.

(e) Let T and T' be two EM blueprints satisfying $(I)-(III)$. Then $T = Th(\langle L_\delta, \in, x_k\rangle_{k\in\omega})$ and $T' = Th(\langle L_{\delta'}, \in, y_k\rangle_{k\in\omega})$ for some limit ordinals δ and δ'. By (d) we know that $M(T, \omega_\omega) = \langle L_{\omega_\omega}, \in\rangle$ and so $\langle L_{\omega_\omega}, \in\rangle \models T^-$. Furthermore, for any $\mathcal{L}_\in$ formula $\phi(v_1, \ldots, v_n)$, we know that $\langle L_{\omega_\omega}, \in\rangle \models \phi[\iota^T_{\omega_1}, \ldots, \iota^T_{\omega_n}]$ iff $\phi(c_0, \ldots, c_{n-1}) \in T$ as I^T is a class of indiscernibles. Hence, $\langle L_{\omega_\omega}, \in, \iota^T_{\omega_{n+1}}\rangle_{n\in\omega} \models T$. By (d) again we know that $\iota^T_{\omega_{n+1}} = \omega_{n+1}$, and so $\langle L_{\omega_\omega}, \in, \omega_{n+1}\rangle_{n\in\omega} \models T$. A similar argument shows that $\langle L_{\omega_\omega}, \in, \omega_{n+1}\rangle_{n\in\omega} \models T'$, and so since T and T' are complete theories we get that $T = T' = Th(\langle L_{\omega_\omega}, \in, \omega_{n+1}\rangle_{n\in\omega})$, as required.

We can now easily observe the following Lemma as under the assumption that $\kappa \to (\omega_1)_2^{<\omega}$, we know that T_0 is the unique EM blueprint that satisfies $(I) - (III)$.

LEMMA 2.4.21
Suppose there is a κ such that $\kappa \to (\omega_1)_2^{<\omega}$, then $0^{\#}$ exists.

We can therefore quickly see that a measurable cardinal will get the job done.

COROLLARY 2.4.22
Suppose there exist a measurable cardinal. Then $0^{\#}$ exists.

PROOF By Corollary 2.2.4, measurable cardinals are Ramsey. Hence, if κ is a measurable cardinal then $\kappa \to (\kappa)_2^{<\omega}$. Now, fix $f : [\kappa]^{<\omega} \to 2$, then there is an $H \in [\kappa]^\kappa$ such that $|f"([H]^{<\omega})| \leq 1$. Let $H' = \{y \subseteq \beta : \exists z \in H$ so that y is an initial segment of z with order type $\omega_1\}$, this is well-defined as $\omega_1 < \kappa$. Then $f"([H']^{<\omega}) \subseteq f"([H]^{<\omega})$, and so $|f"([H']^{<\omega})| \leq 1$. Hence, $\kappa \to (\omega_1)_2^{<\omega}$ and so $0^{\#}$ exists.

2.4.4 Initial Consequences of $0^{\#}$

In this subsection we will look at two initial consequences of $0^{\#}$. The first is that $V \neq L$ and the second is that $L \models$ "every uncountable cardinal is inaccessible". To prove the first, we show that if $0^{\#}$ exists then $P(\omega)^L$ is countable, and so $P(\omega)^L \neq P(\omega)$ which gives us $V \neq L$. For the second result, we show that L thinks that $\aleph_1$ is regular and that $\aleph_\omega$ is a strong limit cardinal. From the existence of $0^{\#}$ we get I^T, which is a set of indiscernibles for L that contains all uncountable cardinals. Hence L will think that all uncountable cardinals are inaccessible. We first show that we can find $P(\omega)^L$ in L_δ for any $\delta \geq \omega_1$.

LEMMA 2.4.23
If $\delta \geq \omega_1$ is a limit ordinal, then $P(\omega)^L \subseteq L_\delta$.

PROOF It is sufficient to show that $P(\omega)^L \subseteq L_{\omega_1}$. We know that $\langle L, \in \rangle \models$ Powerset, and hence for any $a \subseteq \omega$ there is an α such that $a \in L_\alpha$. By downwards Loweheim-Skolem, there is a countable elementarily equivalent submodel $\langle X, \in \rangle$ of $\langle L_\alpha, \in \rangle$. Taking the Mostowski collapse of $\langle X, \in \rangle$ we get a $\beta < \omega_1$ (as it has to be countable) such that $\langle X, \in \rangle \cong \langle L_\beta, \in \rangle$. Hence $a \in L_\beta$ and so $a \in L_{\omega_1}$. This gives us that $P(\omega)^L \subseteq L_{\omega_1}$.

We now show that under the assumption of (I), $P(\omega)^L$ is countable.

LEMMA 2.4.24 ([7] (LEMMA 9.7))
If there is an EM blueprint T such that $\mathcal{M}(T, \alpha)$ is well-founded for every $\alpha < \omega_1$, then $P(\omega)^L$ is countable.

PROOF Suppose there is such a T. Then we know that $\mathcal{M}(T, \omega_1) \cong \langle L_\delta, \in \rangle$ for some limit ordinal $\delta \geq \omega_1$ by Lemma 2.4.13 and Corollary 2.4.12. By Lemma 2.4.23 we know that $P(\omega)^L \subseteq L_\delta$. Hence, if $a \in P(\omega)^L$ then $a = t^{L_\delta}(x_0, \ldots, x_n)$ for some Skolem term t and indiscernibles $x_0 < \ldots < x_n$ by the second clause of Lemma 2.4.8 and Lemma 2.4.7. Let $\langle z_i : i \in \omega \rangle$ be the increasing enumeration of the first ω indiscernibles. Since $x_0 < \ldots < x_n$ and $z_0 < \ldots < z_n$ are indiscernibles we get that $\langle L_\delta, \in \rangle \models k \in t^{L_\delta}(x_0, \ldots, x_n) \leftrightarrow k \in t^{L_\delta}(z_0, \ldots, z_n)$ for every $k \in \omega$. By the axiom of extensionality we get that $\langle L_\delta, \in \rangle \models a = t^{L_\delta}(z_0, \ldots, z_n)$ and so since L is transitive we have that $a = t^{L_\delta}(z_0, \ldots, z_n)$. There are countably many Skolem terms, as the language is countable, and so $\{t^{L_\delta}(z_0, \ldots, z_n) : t \text{ is a Skolem term}\}$ is countable. Hence $P(\omega)^L$ is countable.

We then get the immediate corollary.

COROLLARY 2.4.25
Suppose $0^\#$ exists, then $V \neq L$.

PROOF If $0^\#$ exists then there is an EM blueprint satisfying (I). By Lemma 2.4.24 we then get that $P(\omega)^L$ is countable, and so $P(\omega)^L \neq P(\omega)$. Hence, $V \neq L$ as required.

We now turn to the second consequence of $0^\#$, which is that $L \models$ "every uncountable cardinal is inaccessible". To show that $L \models$ "$\aleph_1$ is a regular cardinal", we will first show that being a regular cardinal is Π_1.

LEMMA 2.4.26 (THIS IS WELL-KNOWN)
Being a regular cardinal is Π_1.

PROOF To express that x is a cardinal is clearly Π_1, as you say that x is an ordinal, which is Δ_0, and that for any $y \in x$ there is no bijection between them. To express that x is regular you say that for every set z, if $|z| < x$ and for all $t \in z$ $|t| < x$ then $|\bigcup z| < x$. This is also clearly Π_1.

We now show the second consequence of $0^\#$.

Lemma 2.4.27 (This is well known)
Suppose $0^{\#}$ exists. Then $L \models$ "every uncountable cardinal is strongly inaccessible".

Proof From the existence of $0^{\#}$, Lemma 2.4.20 d) gives us that all uncountable cardinals are elements of I^T, which is a set of indiscernibles for L. We know that $\aleph_1$ is a regular cardinal, and since the statement is Π_1 by Lemma 2.4.26, and so downwards absolute, $L \models \aleph_1$ is a regular cardinal. Hence, for any uncountable cardinal κ, $L \models$ "κ is a regular cardinal". Furthermore, $L \models$ "$\aleph_\omega$ is a strong limit cardinal". To see why, $L \models GCH$, and so for $n < \omega$, $L \models 2^{\aleph_n} = \aleph_{n+1} < \aleph_\omega$. Hence, for any uncountable cardinal κ, $L \models$ "κ is a strong limit cardinal". This gives us that for any uncountable cardinal, $L \models$ "κ is a strongly inaccessible cardinal".

3 Mice

There is one historical point on which I can offer no help: people occasionally ask why mice are so-called. I am afraid that neither Jensen nor I can remember why, but plausible explanations would be welcomed.

– A.J. Todd, *The Core Model*

In this chapter we will introduce the concept of a mouse. We start out by looking at baby mice in section 3.1. They are easier to work with, and the iteration process is very similar to the one for mice. They fall short of mice because we are not guaranteed that the iterative process preserves all the required fine structure that we want the core model to have. In section 3.2 we first introduce the fine structure theory which is necessary to define a mouse, and then we show how this fine structure is preserved in the iterative process for mice. This allows us to define the core model K as the union of all mice in section 3.3. In that section we will state without proof some of the nice results about K that have been the motivation for spending so much time working our way up to the definition of a mouse. The idea is that K will allow us to have larger cardinals than L but also keep a lot of the fine structure from L.

3.1 Baby mice

In this section we are going to work our way up to the definition and existence of a baby mouse. Section 3.1.1 will show how a non-trivial elementary embedding $j : V \to L$ give rise to a measurable cardinal κ through a κ-complete ultrafilter $\mathcal{U}_j$. This ultrafilter will motivate the definition of a baby premouse, which section 3.1.2 will show us how we can turn that into a baby mouse.

Informally, a baby premouse is a model $\mathcal{M} = \langle L_\lambda, \in, \mathcal{U} \rangle$ which believes that $\mathcal{U}$ is a κ-complete ultrafilter over κ, where κ is the largest cardinal in L_λ, and hence it believes that it has a measurable cardinal. Furthermore, under certain nice conditions, a baby premouse allows us to iteratively create new baby premice in the sense that we can define a baby premouse in terms of the ultraproduct $\Pi_{i \in \kappa} L_\lambda / \mathcal{U}$. We then define a baby mouse to be a baby premouse where we can do this iteration Ω many times. The last thing we show is that if we have a non-trivial elementary embedding $j : L \to L$, then a baby mouse exists.

3.1.1 Elementary Embeddings and Critical Points

We saw earlier that if there is a measurable cardinal, then $0^{\#}$ exists and so $V \neq L$. One of the reasons we care about mice is that they are L-like constructions which are compatible with measurable cardinals. The way we are going to construct the baby mice will rely heavily on non-trivial elementary embeddings on L. We say that $j : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* just in case it is an injective map such that for all formulae $\varphi(v_1, \ldots, v_n)$ and all $m_1, \ldots, m_n \in M$ we have that $\mathcal{M} \models \varphi(m_1, \ldots, m_n)$ just in case $\mathcal{N} \models \varphi(j(m_1), \ldots, j(m_n))$. Furthermore, we say that j is *non-trivial* just in case it is not the identity map.

Now, suppose we have such a non-trivial elementary embedding $j : V \to \mathcal{M}$, where $\mathcal{M}$ is a transitive (class) model of ZFC that has all the ordinals[1]. These models are called *inner models*. We then see that if α is an ordinal then $j(\alpha)$ is an ordinal. We also see that if $\alpha < \beta$ then $(j(\alpha) < j(\beta))^{\mathcal{M}}$ which in turn gives us $j(\alpha) < j(\beta)$ by Δ_0 absoluteness and j being an elementary embedding. Hence $\alpha > j(\alpha)$ would give us an infinite descending chain of ordinals, which contradicts the axiom of foundation, and so we see that for all ordinals α, $\alpha \leq j(\alpha)$. The fist thing we will show is that there must be an ordinal δ such that $\delta < j(\delta)$. The least such δ is called the *critical point* of j, which we denote by $crit(j)$. We will then show that δ must in fact be a measurable cardinal, and hence we see that if there is a non-trivial elementary embedding $j : V \to \mathcal{M}$, then there is a measurable cardinal.

> Lemma 3.1.1 ([7](Proposition 5.1))
> Let $\mathcal{M}$ be an inner model and $j : V \to \mathcal{M}$ be a non-trivial elementary embedding, then there is an ordinal δ such that $\delta < j(\delta)$.

Proof Let x be of the least rank such that $j(x) \neq x$ and set $\delta = rank(x)$. We see that $x \subseteq j(x)$ because if $y \in x$ then $rank(y) < rank(x)$ and so $j(y) = y$. Hence $y \in x$ gives us $j(y) \in j(x)$ by j being an elementary embedding. This, in turn, gives us $y \in j(x)$. Now, since $x \neq j(x)$, there must be a $z \in j(x) \setminus x$. If $rank(j(x)) \leq \delta$ then $rank(z) < \delta$ and so $j(z) = z$. This would give us $j(z) \in j(x)$ which would give us $z \in x$, a contradiction. Hence $rank(j(x)) > \delta$. We also have that $rank(j(x)) = j(\delta)$, which gives us the desired result of $j(\delta) > \delta$.

We now show that the critical point of j is a measurable cardinal.

1 L is the minimal inner model. However, you could never have a non-trivial elementary embedding $j : V \to L$ as then you can show that $V = L$ and so you get a non-trivial elementary embedding $j : V \to V$, which is impossible.

LEMMA 3.1.2 ([7] (THEOREM 5.6))
The critical point of a non-trivial elementary embedding $j : V \to \mathcal{M}$ for an inner model $\mathcal{M}$ is a measurable cardinal.

PROOF Set $\delta = crit(j)$. We see that $\delta > \omega$ as every ordinal $\leq \omega$ is definable and j is elementary. We let $\mathcal{U} = \{X \subseteq \delta : \delta \in j(X)\}$ and aim to show that $\mathcal{U}$ is a δ-complete non-principal ultrafilter over $\mathcal{U}$. Clearly $\delta \in \mathcal{U}$ as $\delta < j(\delta)$. Furthermore, $\emptyset \notin \mathcal{U}$ as $j(\emptyset) = \emptyset$. We also see that if $X \subseteq Y \subseteq \delta$ and $X \in \mathcal{U}$ then $\delta \in j(X) \subseteq j(Y)$ which gives us $Y \in \mathcal{U}$. $\mathcal{U}$ is also clearly non-principal as for $\alpha < \delta$, $j(\{\alpha\}) = \{\alpha\}$. To check that $\mathcal{U}$ is an ultrafilter we fix $X \subseteq \delta$ and observe that $j(\delta) = j(X) \cup j(\delta \setminus X)$. So, since $\delta \in j(\delta)$, we get that $\delta \in j(X)$ or $\delta \in j(\delta \setminus X)$. Hence, the last thing to verify is that $\mathcal{U}$ is δ-complete[2]. Suppose that $\gamma < \delta$ and $\chi \in \mathcal{U}^\gamma$. This gives us that $\delta \in \bigcap_{\alpha<\gamma} j(\chi(\alpha))$. Now, $j(\alpha) = \alpha$ for all $\alpha \leq \gamma$ and so $j(\chi)$ is a function with domain γ such that $j(\chi)(\alpha) = j(\chi(\alpha))$ for all $\alpha < \gamma$. Hence we get that $j(\bigcap_{\alpha<\gamma} \chi(\alpha)) = \bigcap_{\alpha<\gamma} j(\chi(\alpha)) = \bigcap_{\alpha<\gamma} j(\chi)(\alpha)$. This, in turn, gives us that $\bigcap_{\alpha<\gamma} \chi(\alpha) \in \mathcal{U}$.

Hence we see that a non-trivial elementary embedding $j : V \to \mathcal{M}$ entails that $V \neq L$. When constructing the baby mouse we will work with a non-trivial elementary embedding $j : L \to L$ instead. In a similar fashion to Lemma 3.1.1 one can show that j must have a critical point. Hence, we can also define $\mathcal{U}_j = \{X \subseteq \delta : \delta \in j(X)\}$ where $\delta = crit(j)$. It turns out that $\mathcal{U}_j$ is not an ultrafilter over $P(\delta)$. However, with some modification to the proof of 3.1.2 one can show that $\mathcal{U}_j$ is a non-principal ultrafilter over $P(\delta) \cap L$. Furthermore, we can also show that $\mathcal{U}_j$ is L-δ-complete in the sense that for any $\gamma < \kappa$ and $\chi \in (\mathcal{U}_j)^\gamma \cap L$ we have that $\bigcap_{\alpha<\gamma} \chi(\alpha) \in \mathcal{U}_j$. It turns out that this is all we need for the baby mouse construction.

Now, in the same fashion that we relativised being κ-complete to L, we can also relativise being normal to L. We will show that $\mathcal{U}_j$ is L-normal in the sense that for any function $f \in \kappa^\kappa \cap L$, if $\{\beta : f(\beta) < \beta\} \in \mathcal{U}_j$ then there is a $\gamma < \kappa$ such that $\{i \in \kappa : f(i) = \gamma\} \in \mathcal{U}_j$. However, before we do this we note that we have shown that if there is a measurable cardinal then $0^\#$ exist. We will now quickly sketch how the existence of $0^\#$ gives us a non-trivial embedding $j : L \to L$. Indeed, if $0^\#$ exists then we have the a proper class of indiscernibles $I^T = \{\iota^T_\xi : \xi \in \Omega\}$. We note that ι^T_0 is not definable from the other elements of I^T. Let X be the proper class of elements of L that are definable from parameters from $I^T \setminus \{\iota^T_0\}$ and let $\pi : L \cong X$ be the isomorphism given by the Mostowski Collapse. Then π is a non-trivial elementary embedding from L to L ([9] Lemma 2.12). It is worth noting that the existence of a measurable cardinal is a strictly strong assumption than the existence of $0^\#$. This is because if κ is a measurable cardinal then $V_\kappa \models$ "$0^\#$ exists", and so $ZFC+$ "there is a measurable cardinal" $\vdash Con(ZFC+$

2 Which, of course, also gives us that $\mathcal{U}$ is closed under binary intersections.

"$0^\#$ exists"). The claim then follows by Gödel's Second Incompleteness Theorem. We now proceed by showing that $\mathcal{U}_j$ is L-normal.

> Lemma 3.1.3 ([9](Paraphrased From Lemma 2.3))
> Suppose that $j : L \to L$ is a non-trivial elementary embedding with critical point κ, then $\mathcal{U}_j$ is L-normal.

Proof Let $f \in \kappa^\kappa \cap L$ and suppose that $S = \{\beta : f(\beta) < \beta\} \in \mathcal{U}_j$. We then know that $\kappa \in j(S)$ and so we let $\alpha = j(f)(\kappa)$. Since f is regressive we know that $\alpha < \kappa$ and so $j(\alpha) = \alpha$. This allows us to define $S' = f^{-1}(\alpha)$. We now aim to show that $S' \in \mathcal{U}_j$. Now, $S' = f^{-1}(\alpha)$ gives us that $j(S') = j(f)^{-1}(j(\alpha))$ and so we get that $j(S') = j(f)^{-1}(\alpha)$. Hence we have that $\kappa \in j(S')$ which gives us $S' \in \mathcal{U}_j$, as required.

Hence, we have shown that $\mathcal{U}_j$ is a non-principal, L-normal, L-κ-complete ultrafilter on $\kappa \cap L$. We will now use the ultrafilter $\mathcal{U}_j$ to create an ultraproduct over L, which we denote by $Ult(L,\mathcal{U}_j)$[3]. As L is a proper class, and not a set, this will be done with some care. We will use something known as *Scott's Trick* to achieve this, which is taken from [7] (chapter 5). We recall from definition 2.1.7 that if $\mathcal{U}$ is an ultrafilter over I then elements of $\Pi_{i\in I}\mathcal{M}_i/\mathcal{U}$ are equivalence classes of the form $[f] = \{g | g : I \to \bigcup_{i\in I}\mathcal{M}_i, g(i) \in \mathcal{M}_i \text{ and } \{i \in I : f(i) = g(i)\} \in \mathcal{U}\}$. Now, if we replace $\bigcup_{i\in I}\mathcal{M}_i$ with L then $[f]$ becomes a proper class. Scott's trick lets us avoid this by defining $[f]^0 = \{g | g \in [f] \wedge g \in L \wedge \forall h(h \in [f] \to rank(g) \leq rank(h))\}$. In other words, we only consider the functions in L of minimal rank. It is important that the functions are in L because $\mathcal{U}_j$ is only defined on $P(\kappa) \cap L$. This trick makes $[f]^0$ a set, and so the domain of $Ult(L,\mathcal{U}_j)$ will be the class $\Pi_\kappa L/\mathcal{U}_j = \{[f]^0 | f : \kappa \to L\}$. For the membership relation we define $[f]^0 E_{\mathcal{U}_j} [g]^0$ just in case $\{i \in \kappa : f(i) \in g(i)\} \in \mathcal{U}_j$. Thus, the ultraproduct is defined by $Ult(L,\mathcal{U}_j) = \langle \Pi_\kappa L/\mathcal{U}_j, E_{\mathcal{U}_j}\rangle$. Defining the maps $\iota : L \to Ult(L,\mathcal{U}_j)$ and $k : Ult(L,\mathcal{U}_j) \to L$ by $x \mapsto [\alpha \mapsto x]^0$ and $[f]^0 \mapsto j(f)(\kappa)$ respectively we get the following commuting diagram[9].

$$\begin{tikzcd} L \arrow[r, "j"'] \arrow[dr, "\iota"] & L \\ & Ult(L,\mathcal{U}_j) \arrow[u, "k"] \end{tikzcd}$$

We cite without proof that $\iota : L \to Ult(L,\mathcal{U}_j)$ and $k : Ult(L,\mathcal{U}_j) \to L$ are elementary embeddings[9]. Now, because k is an elementary embedding from $Ult(L,\mathcal{U}_j)$ to L we may

3 We use this notation to distinguish it from the normal ultraproduct, which is a set and not a proper class.

conclude that $Ult(L,\mathcal{U}_j)$ is well-founded. Hence, using the Mostowski Collapse, Theorem 2.4.10, there is a unique isomorphism $\pi : \langle M, \in\rangle \to Ult(L,\mathcal{U}_j)$ where M is transitive class. We therefore define $i = \iota \circ \pi^{-1}$. When taking the ultraproduct of L_λ over an appropriate ultrafilter, it turns out that $\Pi_{i\in\kappa} L_\lambda/\mathcal{U}$ is isomorphic to L_δ for some limit ordinal δ. This result is due to Gödel's Condensation Lemma. We will here also make a similar distention between $\iota : L_\lambda \to \Pi_{i\in\kappa} L_\lambda/\mathcal{U}$ and $i : L_\lambda \to L_\delta$ when $\Pi_{i\in\kappa} L_\lambda/\mathcal{U} \cong L_\delta$.

Lemma 3.1.4
Let λ be an infinite cardinal, then $L_\lambda = H^L_\lambda$.

Proof We know that $L_\lambda \models V = L$, and so from $H_\lambda \subseteq V_\lambda$ we get that $H^L_\lambda \subseteq L_\lambda$. Conversely, we know that L_λ is transitive and $|L_\lambda| = \lambda$, and so we get that $L_\lambda \subseteq H^L_\lambda$.

For a cardinal λ, we say that an ultrafilter $\mathcal{U}$ is L_λ-*amenable* just in case whenever $A \in L_\lambda$ we have that $\mathcal{U} \cap A \in L_\lambda$.

Lemma 3.1.5 ([9] (Lemma 2.5))
Suppose there is a non-trivial elementary embedding $j : L \to L$, and let $\kappa = crit(j)$. If $\lambda = (\kappa^+)^L$ then

1. $\mathcal{U}_j$ is an ultrafilter on $L_\lambda \cap P(\kappa)$,
2. $\mathcal{U}_j$ is L_λ-normal,
3. $\mathcal{U}_j$ is L_λ-κ-complete,
4. $\mathcal{U}_j$ is L_λ-amenable.

Proof For clause 1, 2 and 3 we see that $\mathcal{U}_j \in H^L_{\lambda^+}$ as $|\mathcal{U}_j| = \lambda$ since $L \models$ GCH. This gives us $\mathcal{U}_j \subseteq L_\lambda$ by Lemma 3.1.4, and so the clauses holds. For clause 4 we see that it is sufficient to show that if $\alpha < \lambda$ then $\mathcal{U}_j \cap L_\alpha \in L_\lambda$. We fix $\alpha < \lambda$ and note that $|L_\alpha|^L = |\alpha| \leq \kappa$. Hence, we can use κ to enumerate in L the elements of $P(\kappa) \cap L_\alpha$. We denote this enumeration by $\langle A_\eta : \eta < \kappa\rangle$. We then let $\langle B_\eta : \eta < j(\kappa)\rangle = j(\langle A_\eta : \eta < \kappa\rangle)$. This gives us that $\{B_\eta \cap \kappa : \eta < \kappa \wedge \kappa \in B_\eta\} = \mathcal{U}_j$. To see why, if $\eta < \kappa$ and $\kappa \in B_\eta$ then $A_\eta \subseteq B_\eta$ as for all $\alpha \in A_\eta$, $j(\alpha) = \alpha \in B_\eta$. This gives us $B_\eta \subseteq j(B_\eta)$ and so $\kappa \in j(B_\eta)$. Additionally, we have that $\kappa \in j(\kappa)$, and so we get that $\kappa \in j(B_\eta \cap \kappa)$. Furthermore, $B_\eta \cap \kappa \subseteq \kappa$, and so $B_\eta \cap \kappa \in \mathcal{U}_j$. Conversely, if $A \in \mathcal{U}_j$ then $A = A_\eta$ for some $\eta < \kappa$. Since $B_\eta \cap \kappa = j(A_\eta) \cap \kappa = A$ we get that $A \in \{B_\eta \cap \kappa : \eta < \kappa \wedge \kappa \in B_\eta\}$. Hence it follows that $\mathcal{U}_j \cap L_\alpha \in L_\kappa$.

We end the subsection by formally defining a baby premouse.

DEFINITION 3.1.6 (ACTIVE BABY PREMOUSE [9] (DEFINITION 2.6))
A structure $\mathcal{M}$ is an *active baby premouse* if and only if for some $\kappa < \lambda$ and ultrafilter $\mathcal{U}$,

1. $\mathcal{M} = \langle L_\lambda, \in, \mathcal{U} \rangle$,
2. $\langle L_\lambda, \in \rangle \models \mathrm{ZF} \setminus \text{Power Set}$,
3. $\mathcal{M} \models \kappa$ is the largest cardinal,
4. $\mathcal{M} \models \mathcal{U}$ is a κ-complete non-principal normal ultrafilter over κ,
5. $\mathcal{M}$ is amenable,
6. If $\Pi_{i\in\kappa} L_\lambda/\mathcal{U}$ is well-founded then $\lambda = (\kappa^+)^{\Pi_{i\in\kappa} L_\lambda/\mathcal{U}}$.

If we assume that there is a non-trivial elementary embedding $j : L \to L$ then we say that $\langle L_\lambda, \in, \mathcal{U}_j \rangle$ is the active baby premouse *derived from* j, where $\kappa = crit(j)$ and $\lambda = (\kappa^+)^L$. To verify that $\langle L_\lambda, \in, \mathcal{U}_j \rangle$ is indeed an active baby premouse we note that all clauses but the third follows directly from Lemma 3.1.5 and the fact that $(\kappa^+)^{\Pi_{i\in\kappa} L_\lambda/\mathcal{U}} = (\kappa^+)^L$. The third clause follows immediately from $H^L_\lambda = L_\lambda$, which is Lemma 3.1.4.

3.1.2 From Baby Premouse to Baby Mouse

We ended the last section by defining an active baby premouse. In this subsection we will describe what makes a baby premouse a baby mouse as well. It is an iterative process which we will describe in some detail. As mentioned in the introduction of this section, we want to iterate the construction of a new baby premouse from an old one Ω many times. We will first spend quite a bit of time explaining the successor stage, before we have a look at the limit stage in this iterative process.

Let $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$ be an arbitrary active baby premouse, where $\mathcal{U}_0$ is an ultrafilter over κ_0, and assume that $\Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ is well-founded. We then let L_{λ_1} be the result of applying Gödel's Condensation Lemma to $\Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ and $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ be the ultraproduct embedding. We state without proof that $i_{0,1}$ is an elementary embedding.

LEMMA 3.1.7 ([9])
Let $\mathcal{M} = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$ be a baby premouse, $\Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ be well-founded, $L_{\lambda_1} = \Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ and $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ be the ultraproduct embedding. Then $i_{0,1}$ is an elementary embedding in the language of set theory $\mathcal{L}_\in$.

We will now use $i_{0,1}$ to turn L_{λ_1} into a baby premouse. Let $\kappa_1 = i_{0,1}(\kappa_0)$, then $L_{\lambda_1} \models \kappa_1$ is the largest cardinal. Furthermore, we let $\mathcal{U}_1 = \bigcup_{\alpha<\lambda_0} i_{0,1}(\mathcal{U}_0 \cap L_\alpha)$. This is well-defined as we have seen that for $\alpha < \lambda_0$, $\mathcal{U}_0 \cap L_\alpha \in L_{\lambda_0}$. So, our new baby premouse will be defined as $\mathcal{M}_1 = \langle L_{\lambda_1}, \in, \mathcal{U}_1 \rangle$. Now, $i_{0,1} : \mathcal{M}_0 \to \mathcal{M}_1$ is a priori no longer an elementary embedding. It is, however, not hard to show that $i_{0,1} : \mathcal{M}_0 \to \mathcal{M}_1$ is an elementary embedding with respect to Σ_0 formulae.

To show that $\mathcal{M}_1$ is a baby premouse will take several steps. The first thing we will show is that $i_{0,1}$ is an cofinal embedding in the sense that $\sup(i_{0,1}\text{"}(\lambda_0)) = \lambda_1$. Then we will show that this makes $i_{0,1}$ an elementary embedding with respect to Σ_1 sentences. Lastly, we will show that baby premousehood is preserved upwards by cofinal Σ_1-embeddings, and hence conclude that $\mathcal{M}_1$ is indeed a baby premouse.

Before we do all of this, we note some easy calculations. Firstly, because $\lambda_0 = (\kappa_0^+)^{L_{\lambda_1}}$ we have that $L_{\lambda_1} \models \lambda_0 = \kappa_0^+$. Furthermore, $L_{\lambda_1} \models i_{0,1}(\kappa_0)$ is the largest cardinal. Hence $L_{\lambda_1} \models \kappa_0 < i_{0,1}(\kappa_0)$. It turns out that κ_0 is the critical point for $i_{0,1}$. To see why, we fix $\alpha < \kappa_0$ and aim to show that $i_{0,1}(\alpha) = \alpha$. To do this, we suppose that $\Pi_{i \in \kappa_0} L_{\lambda_0}/\mathcal{U}_0 \models [g] < \Omega \wedge [g] < \iota_{1,0}(\alpha)$. By Łoś theorem we know that $\{\beta < \kappa_0 : g(\beta) < \alpha\} \in \mathcal{U}_0$. For each $\delta < \alpha$, let $A_\delta = \{\beta < \kappa_0 : g(\beta) = \delta\}$. We see that $\{\beta < \kappa_0 : g(\beta) < \alpha\} = \bigcup_{\delta < \alpha} A_\delta$, and so by $\mathcal{U}_0$ being κ_0-complete in L_{λ_0} we get that there is a $\delta < \alpha$ such that $\{\beta < \kappa_0 : g(\beta) = \delta\} \in \mathcal{U}_0$. Hence $[g]$ is the ordinal δ, and so $\iota_{0,1}(\alpha) \leq \alpha$ as all ordinal less than $\iota_{0,1}(\alpha)$ are less than α. Of course, this gives us that $\iota_{0,1}(\alpha) = \alpha$, and so κ_0 is the critical point.

Secondly, we see that $\mathcal{M}_0 \models \kappa_0$ is a measurable cardinal, which means that $\mathcal{M}_0 \models \kappa_0$ is inaccessible. L_{λ_0} is a reduct of $\mathcal{M}_0$, and so $L_{\lambda_0} \models \kappa_0$ is inaccessible. This means that $L_{\lambda_1} \models \kappa_1$ is inaccessible. Now, we have already seen that $L_{\lambda_1} \models \kappa_0 < \kappa_1$ and so we get $L_{\lambda_1} \models \kappa_0^+ < \kappa_1$. This gives us that $L_{\lambda_1} \models \lambda_0 < \kappa_1$ as $(\kappa_0^+)^{L_{\lambda_1}} = \lambda_0$. We are now ready to show the first Lemma that we will use when showing that $\mathcal{M}_1$ is a baby premouse.

Lemma 3.1.8 ([9])
Let $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$ be an arbitrary baby premouse such that $\Pi_{i \in \kappa_0} L_{\lambda_0}/\mathcal{U}_0$ is well-founded. Let $L_{\lambda_1} \cong \Pi_{i \in \kappa_0} L_{\lambda_0}/\mathcal{U}_0$ and $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ be the ultraproduct embedding, then $\sup(i_{0,1}\text{"}(\lambda_0)) = \lambda_1$.

Proof We know that $L_{\lambda_0} \cap \Omega = \lambda_0$ and $L_{\lambda_1} \cap \Omega = \lambda_1$, and so clearly for $\alpha < \lambda_0$ we have $i(\alpha) < \lambda_1$. Hence, it only remains to show that for any $\beta < \lambda_1$ there is an $\alpha < \lambda_0$ such that $i(\alpha) > \beta$. Let $f \in \Pi_{i \in \kappa_0} L_{\lambda_0}/\mathcal{U}_0$, then by replacement in L_{λ_0} there is a $\gamma \in L_{\lambda_0}$ such that $\gamma > \sup(\text{ran } f)$. Let $g : \kappa_0 \to \kappa_0$ be given by $i \mapsto \gamma$, then $\Pi_{i \in \kappa_0} L_{\lambda_0}/\mathcal{U}_0 \models [f] < [g]$. This gives us that $[f] < \iota_{0,1}(\gamma)$ and so the claim follows[4].

It is now rather straightforward to show that $i_{0,1} : \mathcal{M}_0 \to \mathcal{M}_1$ is Σ_1 elementary.

4 Recall the convention of ι denoting the map from L to its ultraproduct whilst i is the composition with the Mostowski Collapse when the ultraproduct is well-founded.

LEMMA 3.1.9 ([9])
Let $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$ be an arbitrary baby premouse such that $\Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ is well-founded. Let $L_{\lambda_1} = \Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ and $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ be the ultraproduct embedding, then $i_{0,1} : \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle \to \langle L_{\lambda_1}, \in, \mathcal{U}_1 \rangle$ is Σ_1 elementary.

PROOF Let $\varphi(v_1, v_2)$ be a Σ_0 formulae, we need to show that for any $x \in \mathcal{M}_0$, $\mathcal{M}_0 \models \exists y\varphi[x, y]$ just in case $\mathcal{M}_1 \models \exists y\varphi[i_{0,1}(x), y]$. Suppose $\mathcal{M}_0 \models \exists y\varphi[x, y]$, then there is some $y \in \mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi[x, y]$. This gives us $\mathcal{M}_1 \models \varphi[i_{0,1}(x), i_{0,1}(y)]$ as $\varphi(v_1, v_2)$ is Σ_0 and so $\mathcal{M}_1 \models \exists\varphi[i_{0,1}(x), y]$. Conversely, suppose $\mathcal{M}_1 \models \exists\varphi[i_{0,1}(x), y]$. Then there is a $y \in \mathcal{M}_1$ such that $\mathcal{M}_1 \models \varphi[i_{0,1}(x), y]$. Now, let $\alpha = \text{rank}(y)$. From Lemma 3.1.8 we know that $i_{0,1}$ is confial, and so there is a $\beta \in \mathcal{M}_0$ such that $i_{0,1}(\beta) > \alpha$. Hence, we get that $\mathcal{M}_1 \models \exists y \in L_{i_{0,1}(\beta)}\varphi[i_{0,1}(x), y]$. This is Σ_0, and so $\mathcal{M}_0 \models \exists y \in L_\beta\varphi[x, y]$, which in turn gives us $\mathcal{M}_0 \models \exists y\varphi[x, y]$.

We now show that $i_{0,1}$ preserves baby premousehood, and hence we can conclude that $\mathcal{M}_1$ is a baby premouse.

LEMMA 3.1.10 ([9])
Let $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$ be an arbitrary baby premouse such that $\Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ is well-founded. Let $L_{\lambda_1} = \Pi_{i\in\kappa_0} L_{\lambda_0}/\mathcal{U}_0$ and $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ be the ultraproduct embedding, then $\mathcal{M}_1 = \langle L_{\lambda_1}, \in, \mathcal{U}_1 \rangle$ is a baby premouse.

PROOF From Lemma 3.1.7 we know that $i_{0,1} : L_{\lambda_0} \to L_{\lambda_1}$ is an elementary embedding, and hence $\langle L_{\lambda_1}, \in \rangle \models \text{ZF} \setminus \text{Power Set}$. Similarly, $L_{\lambda_1} \models \kappa_1$ is the largest cardinal, and so $\mathcal{M}_1 \models \kappa_1$ is the largest cardinal. We now proceed to verify that $\mathcal{M}_1 \models \mathcal{U}_1$ is a κ_1-complete non-principal normal ultrafilter over κ_1.

$\mathcal{U}_1$ is a filter: It is clear that $\mathcal{U}_1$ does not contain the empty set and is non-empty. We now verify that $\mathcal{U}_1$ is closed under supersets. We know that $\mathcal{M}_0 \models \forall A \in \mathcal{U}_0 \forall B(A \subseteq B \subseteq \kappa_0 \to B \in \mathcal{U}_0)$ and so $\mathcal{M}_1 \models \forall A \in i_{0,1}(\mathcal{U}_0)\forall B(A \subseteq B \subseteq \kappa_1 \to B \in i_{0,1}(\mathcal{U}_0))$. To see why, we note that the formula in question is a Π_1 formula and so if it was not true in $\mathcal{M}_1$ then the negation would be true in $\mathcal{M}_1$ which is Σ_1 and so $i_{0,1}$ being Σ_1 elementary would make the negation true in $\mathcal{M}_0$, which is a contradiction. Hence we see that if $A \in \mathcal{U}_1$ and $A \subseteq B \subseteq \kappa_1$ then $B \in i_{0,1}(\mathcal{U}_0)$. Furthermore, $B \in L_\beta$ for some $\beta < \lambda_1$, and since $\sup(i_{0,1}"(\lambda_0)) = \lambda_1$ we know there is a $\beta' < \lambda_0$ such that $\beta < i_{0,1}(\beta') < \lambda_1$. This gives us that $B \in i_{0,1}(L_{\beta'})$ and so we get that $B \in \mathcal{U}_1$. We don't have to check that $\mathcal{U}_1$ is closed under binary intersections as κ_1-completeness is a stronger condition.

$\mathcal{U}_1$ is an ultrafilter: Fix $A \subseteq \kappa_1$. We need to show that either $A \in \mathcal{U}_1$ or $\kappa_1 \setminus A \in \mathcal{U}_1$. We know that there is an $A' \subseteq \kappa_0$ such that $i_{0,1}(A') = A$, and so by $\mathcal{U}_0$ being an ultrafilter we have either $A' \in \mathcal{U}_0$ or $\kappa_0 \setminus A' \in \mathcal{U}_0$. WLOG assume $A \in \mathcal{U}_0$. This gives us $A \in i_{0,1}(\mathcal{U}_0)$. Furthermore, $A' \in L_{\lambda_0}$ gives us an $\alpha < \lambda_0$ such that $A' \in L_\alpha$. Hence we have that $A \in i_{0,1}(\mathcal{U}_0 \cap L_\alpha)$ and so $A \in \mathcal{U}_1$.

$\mathcal{U}_1$ is non-principal: Suppose for contradiction that $\mathcal{U}_1$ is principal. Then $\mathcal{M}_1 \models \exists\alpha < \kappa_1 \forall\beta < \lambda_0 \forall A \in i_{0,1}(\mathcal{U}_0 \cap L_\beta)(\alpha \in A)$. Now, we know that $\mathcal{M}_0 \models \mathcal{U}_0$ is non-principal, and so $\mathcal{M}_0 \models \forall\alpha < \kappa_0 \exists A \in \mathcal{U}_0(\alpha \notin A)$. This is a Σ_0 formulae, and so $\mathcal{M}_1 \models \forall\alpha < \kappa_1 \exists A \in i_{0,1}(\mathcal{U}_0)(\alpha \notin A)$, which contradicts $\mathcal{M}_1 \models \exists\alpha < \kappa_1 \forall\beta < \lambda_0 \forall A \in i_{0,1}(\mathcal{U}_0 \cap L_\beta)(\alpha \in A)$.

$\mathcal{U}_1$ is normal: We know that $\mathcal{M}_0 \models \mathcal{U}_0$ is normal, and so $\mathcal{M}_0 \models \forall f \in \kappa_0^{\kappa_0}(\{\beta < \kappa_0 : f(\beta) < \beta\} \in \mathcal{U}_0 \to \exists\alpha < \kappa_0(f^{-1}(\alpha) \in \mathcal{U}_0))$. This is a Σ_0 sentence, and so $\mathcal{M}_1 \models \forall f \in \kappa_1^{\kappa_1}(\{\beta < \kappa_1 : f(\beta) < \beta\} \in i_{0,1}(\mathcal{U}_0) \to \exists\alpha < \kappa_1(f^{-1}(\alpha) \in i_{0,1}(\mathcal{U}_0)))$. This clearly gives us that $\mathcal{M}_1 \models \forall f \in \kappa_1^{\kappa_1} \forall\alpha < \lambda_0(\{\beta < \kappa_1 : f(\beta) < \beta\} \in i_{0,1}(\mathcal{U}_0 \cap L_\alpha) \to \exists\alpha < \kappa_1(f^{-1}(\alpha) \in i_{0,1}(\mathcal{U}_0 \cap L_\alpha)))$ and so $\mathcal{M}_1 \models \mathcal{U}_1$ is normal.

$\mathcal{U}_1$ is κ_1-complete: We know that $\mathcal{M}_0 \models \mathcal{U}_0$ is κ_0-complete. Hence $\mathcal{M}_0 \models \forall\alpha < \kappa_0 \forall f \in (\mathcal{U}_0)^\alpha(\bigcap_{\gamma<\alpha} f(\gamma) \in \mathcal{U}_0)$. This is a Σ_0 sentence and hence we get that $\mathcal{M}_1 \models \forall\alpha < \kappa_1 \forall f \in (i_{0,1}(\mathcal{U}_0))^\alpha(\bigcap_{\gamma<\alpha} f(\gamma) \in i_{0,1}(\mathcal{U}_0))$. This, in turn, gives us that $\mathcal{M}_1 \models \forall\alpha < \kappa_1 \forall f \in (\bigcup_{\beta<\lambda_0} i_{0,1}(\mathcal{U}_0 \cap L_\beta))^\alpha(\bigcap_{\gamma<\alpha} f(\gamma) \in \bigcup_{\beta<\lambda_0} i_{0,1}(\mathcal{U}_0 \cap L_\beta))$ and so $\mathcal{M}_1 \models \mathcal{U}_1$ is κ_1-complete.

We now show that $\mathcal{M}_1$ is amenable. We need to show that if $A \in L_{\lambda_1}$ then $\mathcal{U}_1 \cap A \in L_{\lambda_1}$. Fix $A \in L_{\lambda_1}$, then $A \in L_\alpha$ for some $\alpha < \lambda_1$ as λ_1 is a limit ordinal. Now, $i_{0,1}$ is cofinal by Lemma 3.1.8, and so there is a $\beta < \kappa_0$ such that $i(\beta) > \alpha$. This gives us that $A \in L_{i_{0,1}(\beta)}$ and so there is an $A' \in L_\beta$ such that $i_{0,1}(A') = A$. This means that $A' \in L_{\lambda_0}$ as $\beta < \kappa_0 < \lambda_0$ and so $\mathcal{U}_0 \cap A' \in L_{\lambda_0}$ since $\mathcal{M}_0$ is amenable. This gives us that $i_{0,1}(\mathcal{U}_0 \cap A') = i_{0,1}(\mathcal{U}_0) \cap A \in L_{\lambda_1}$ and so we get that $i_{0,1}(\mathcal{U}_0 \cap L_\alpha) \cap A \in L_{\lambda_1}$. Hence we get that $\mathcal{U}_1 \cap A \in L_{\lambda_1}$, as desired.

Lastly, we show that if $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1$ is well-founded then $(\kappa_1^+)^{\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1} = \lambda_1$. Now, $L_{\lambda_1} \models \kappa_1$ is the largest cardinal, and so for all ordinals $\kappa_1 \leq \gamma < \lambda_1$, $L_{\lambda_1} \models \exists f : \kappa_1 \to \gamma$ and f is a bijection. Hence, $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1 \models \exists f : \kappa_1 \to \gamma$ and f is a bijection for all ordinals $\kappa_1 \leq \gamma < \lambda_1$. So, if we can show that $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1 \models \kappa_1 < |\lambda_1|$ then it follows that $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1 \models \kappa_1^+ = \lambda_1$, which is what we want to show. Hence, suppose for contradiction that $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1 \models \exists f : \kappa_1 \to \lambda_1$ and f is a bijection. f induces a well-ordering of κ_1 in order-type λ_1. Now, $f = [(f_\alpha)_{\alpha<\kappa_1}]$, and so we can define, in L_{λ_1}, a well-ordering of κ_1 as follows. For $\beta, \gamma < \kappa_1$, let $\beta \leq \gamma$ just in case $\{\alpha < \kappa_1 : f_\alpha(\beta) \leq f_\alpha(\gamma)\} \in \mathcal{U}_1$. This is a well-ordering of κ_1 in L_{λ_1} of order-type λ_1, which is impossible. Hence, no such f can exists and $\Pi_{i\in\kappa_1} L_{\lambda_1}/\mathcal{U}_1 \models \kappa_1^+ = \lambda_1$, as required.

Assuming that each step is well-founded, it should be clear that we can iterate this process any finite number of times. We want, however, to iterate this through all the ordinals. Hence, we need to discuss what happens at the limit ordinal stage. Let γ be a limit ordinal, and assume that we have a baby premouse $\mathcal{M}_\alpha$ for each $\alpha < \gamma$ in the iteration stage. Composing maps when appropriate, we get a *commutative system* $\langle(\mathcal{M}_\alpha)_{\alpha<\gamma}, (i_{\alpha,\beta})_{\alpha\leq\beta<\gamma}\rangle$, which means that (i) the maps $i_{\alpha,\beta}$ are Σ_0 embeddings for all $\alpha \leq \beta < \gamma$, (ii) $i_{\alpha,\delta} = i_{\beta,\delta} \circ i_{\alpha,\beta}$ for all $\alpha \leq \beta \leq \delta < \gamma$ and (iii) $i_{\alpha,\alpha}$ is the identity on $\mathcal{M}_\alpha$ for all $\alpha < \gamma$[5]. Given such a commutative system, we can define a direct limit on the commutative system, which in our present case will end up being the desired premouse $\mathcal{M}_\gamma$.

5 Note that the objects in the commutative system need not be baby premice, the definition is general for any kind of models.

Definition 3.1.11 (Direct limit [3] (Definition 6.2))
Let $\langle(\mathcal{M}_\alpha)_{\alpha<\gamma},(i_{\alpha,\beta})_{\alpha\le\beta<\gamma}\rangle$ be a commutative system, then $\langle\mathcal{M}_\gamma,(i_{\alpha,\gamma})_{\alpha<\gamma}\rangle$ is a *direct limit* just in case:

(i) $i_\alpha:\mathcal{M}_\alpha\to\mathcal{M}_\gamma$ is a Σ_0 embedding for all $\alpha<\gamma$,

(ii) $i_\beta\circ i_{\alpha,\beta}=i_\alpha$ for all $\alpha\le\beta<\gamma$, and

(iii) for all $m\in M_\gamma$ there is an $\alpha<\gamma$ and $m'\in M_\alpha$ such that $m=i_\alpha(m')$.

We will first show that such a limit always exist, then show that there is a cofinal Σ_1 embedding to the direct limit and hence conclude that it is also a baby premouse.

Lemma 3.1.12 ([3](Special case of Lemma 6.4))
Let γ be a limit ordinal, and suppose that there is a commutative system of baby premice $\langle(\mathcal{M}_\alpha)_{\alpha<\gamma},(i_{\alpha,\alpha'})_{\alpha\le\alpha'<\gamma}\rangle$, then there is a direct limit $\mathcal{M}_\gamma$ of $\langle(\mathcal{M}_\alpha)_{\alpha<\gamma},(i_{\alpha,\alpha'})_{\alpha\le\alpha'<\gamma}\rangle$.

Proof We call a function $f:\gamma\to\bigcup_{\alpha<\gamma}L_{\lambda_\alpha}$ convergent just in case there is a $\beta<\gamma$ such that for all δ with $\beta\le\delta<\gamma$ $f(\delta)=i_{\beta,\delta}(f(\beta))$[6]. Let $M^*=\{f|f:\lambda\to\bigcup_{\alpha<\gamma}L_{\lambda_\alpha}$ is convergent$\}$. For $f,g\in M^*$ we say that $f\sim g$ just in case there is a $\beta<\gamma$ such that for all δ if $\beta\le\delta<\gamma$ then $f(\delta)=g(\delta)$. This is clearly an equivalence relation. We then let $M=M^*/\sim$. If $f\in M^*$ then we denote its equivalence class in M by $[f]$. We define $E\subseteq M\times M$ by $[f]\ E\ [g]$ just in case $\exists\beta<\gamma\forall\delta(\beta\le\delta<\gamma\to f(\delta)\in g(\delta))$. For the maps, we let $\tau^\alpha_x:\gamma\to\bigcup_{\alpha<\gamma}L_{\lambda_\alpha}$ be given by $\beta\mapsto\emptyset$ if $\beta<\alpha$ and $\beta\mapsto i_{\alpha,\beta}(x)$ otherwise where $x\in L_{\lambda_\alpha}$. We now define $i_\alpha(x)=[\tau^\alpha_x]$ and proceed to show that $\langle\langle M,E\rangle,(i_\alpha)_{\alpha<\gamma}\rangle$ is a direct limit.

We first verify that the i_β's commute with $(i_{\alpha,\alpha'})_{\alpha\le\alpha'<\gamma}$. If $\alpha\le\beta\le\delta<\gamma$ then $\tau^\alpha_x(\delta)=i_{\alpha,\delta}(x)$ and $\tau^\beta_{i_{\alpha,\beta}(x)}(\delta)=i_{\beta,\delta}(i_{\alpha,\beta}(x))=i_{\alpha,\delta}(x)$. Hence we see that $\tau^\alpha_x\sim\tau^\beta_{i_{\alpha,\beta}(x)}$, which in other words means that $i_\beta(i_{\alpha,\beta}(x))=i_\alpha(x)$.

We now verify that $M=\{i_\alpha(x):\alpha<\gamma\wedge x\in L_{\lambda_\alpha}\}$. Let $f\in M^*$, then because f is convergent we know that there is a $\beta<\gamma$ such that $\beta\le\delta<\gamma$ implies that $f(\delta)=i_{\beta,\delta}(f(\beta))$. Then we let $x=f(\beta)$ and observe that $f\sim\tau^\beta_x$. This means that $[f]=i_\beta(x)$.

The last thing we verify is that $i_\alpha:L_{\lambda_\alpha}\to M$ is a Σ_0 embedding. We do so carefully by induction on Σ_0 formulae.

$x=y$: If $L_{\lambda_\alpha}\models x=y$ then clearly $M\models i_\alpha(x)=i_\alpha(y)$. Conversely, if $M\models i_\alpha(x)=i_\alpha(y)$ then $\tau^\alpha_x\sim\tau^\alpha_y$ and so $i_{\alpha,\beta}(x)=i_{\alpha,\beta}(y)$ for some $\alpha\le\beta<\gamma$, but this gives us that $x=y$ as $i_{\alpha,\beta}$ is injective.

$x\in y$: If $L_{\lambda_\alpha}\models x\in y$ then for all $\alpha\le\beta<\gamma$ we have that $L_{\lambda_\beta}\models i_{\alpha,\beta}(x)\in i_{\alpha,\beta}(y)$ and so $[\tau^\alpha_x]\ E\ [\tau^\alpha_y]$. This in turn gives us $M\models i_\alpha(x)\ E\ i_\alpha(y)$. Conversely, if $M\models i_\alpha(x)\ E\ i_\alpha(y)$ then there is a β such that $\alpha\le\beta<\gamma$ and $L_{\lambda_\beta}\models i_{\alpha,\beta}(x)\in i_{\alpha,\beta}(y)$, which gives us $L_{\lambda_\alpha}\models x\in y$ as $i_{\alpha,\beta}$ is an elementary embedding.

6 By convention $\mathcal{M}_\alpha=\langle L_{\lambda_\alpha},\in,\mathcal{U}_\alpha\rangle$ with largest cardinal κ_α.

$\varphi(x) = \neg\psi(x)$: $L_{\lambda_\alpha} \models \varphi[x]$ just in case $L_{\lambda_\alpha} \not\models \psi[x]$ just in case $M \not\models \psi[i_\alpha(x)]$ just in case $M \models \varphi[i_\alpha(x)]$, where the second just in case is the induction hypothesis.

$\varphi(x) = \varphi_1(x) \wedge \varphi_2(x)$: $L_{\lambda_\alpha} \models \varphi[x]$ just in case $L_{\lambda_\alpha} \models \varphi_1[x]$ and $L_{\lambda_\alpha} \models \varphi_2[x]$ just in case $M \models \varphi_1[i_\alpha(x)]$ and $M \models \varphi_2[i_\alpha(x)]$ just in case $M \models \varphi[i_\alpha(x)]$, where the second just in case is the induction hypothesis.

$\varphi(x,z) = \exists y \in z\psi(x,y)$: If $L_{\lambda_\alpha} \models \varphi[x,z]$ then $L_{\lambda_\alpha} \models \exists y \in z\psi[x,y]$ and so $M \models \exists i_\alpha(y) \in i_\alpha(z)\psi[i_\alpha(x), i_\alpha(y)]$ by the induction hypothesis and the $x \in y$ case from earlier. This, of course, gives us that $M \models \varphi[i_\alpha(x), i_\alpha(z)]$. Conversely, if $M \models \varphi[i_\alpha(x), i_\alpha(z)]$ then $M \models \exists y \in i_\alpha(z)\psi[i_\alpha(x), y]$ and so $M \models \exists i_\beta(y') \in i_\alpha(z)\psi[i_\alpha(x), i_\beta(y')]$ for $i_\beta(y') = y$. We can fix β such that $\alpha \leq \beta$, and so we get that $L_{\lambda_\beta} \models \exists i_\beta(y') \in i_{\alpha,\beta}(z)\psi[i_{\alpha,\beta}(x), i_\beta(y')]$. This is, of course, just $L_{\lambda_\beta} \models \varphi[i_{\alpha,\beta}(x), i_{\alpha,\beta}(z)]$, and so we get $L_{\lambda_\alpha} \models \exists\varphi[x,z]$ by lemma 3.1.7 and the fact that the composition of two elementary embedding is an elementary embedding.

Hence, we see that $\langle\langle M, E\rangle, (i_\alpha)_{\alpha<\gamma}\rangle$ is a direct limit. If we assume that the direct limit is well-founded, we can define $\mathcal{M}_\gamma = \langle L_{\lambda_\gamma}, \in, \mathcal{U}_\gamma\rangle$ where $\langle L_{\lambda_\gamma}, \in\ \rangle$ is the result of applying Gödel's Condensation Lemma to $\langle M, E\rangle$ and $\mathcal{U}_\gamma = \bigcup_{\alpha<\gamma} i_\alpha(\mathcal{U}_\alpha)$. To conclude that $\mathcal{M}_1$ is a baby premouse we will show that i_α is both cofinal and an Σ_1 embedding, which will suffice in a similar fashion to the successor case. The Σ_1 embedding uses essentially the same proof as the $\exists y \in z\psi(x,z)$ clause of the Σ_0 embedding, so we skip this. For showing that i_α is cofinal we fix $x \in M$ and see that $x = i_\beta(y)$ for some $\beta < \gamma$ and $y \in L_{\lambda_\beta}$, which we again fix such that $\alpha \leq \beta$. We know that $i_{\alpha,\beta}$ is cofinal, so there is a $z \in L_{\lambda_\alpha}$ such that $y \in i_{\alpha,\lambda}(z)$. This gives us that $x \in i_\alpha(z)$, and hence i_α is cofinal. We are now ready to define a baby mouse, which we will do in two steps.

Definition 3.1.13 ([9])
An active baby premouse $\mathcal{M}$ is γ-*iterable* just in case letting $\mathcal{M}_0 = \mathcal{M}$ in the construction above, for all $\beta < \gamma$;
if $\beta = \alpha + 1$ then $\Pi_{i\in\kappa_\alpha} L_{\lambda_\alpha}/\mathcal{U}_\alpha$ is well-founded, and
if β is a limit ordinal then the direct limit of $\langle(\mathcal{M}_\alpha)_{\alpha<\beta}, (i_{\alpha,\alpha'})_{\alpha\leq\alpha'<\beta}\rangle$ is well-founded.

Definition 3.1.14 (Active Baby Mouse [9])
$\mathcal{M}$ is an active baby mouse just in case $\mathcal{M}$ is an Ω-iterable baby premouse.

We finish up this subsection by showing that if $j : L \to L$ is a non-trivial elementary embedding, then we have an active baby mouse.

Theorem 3.1.15 ([9](Lemma 2.10))
If $j : L \to L$ is a non-trivial elementary embedding and $\mathcal{M}$ is the active baby premouse derived from j, then $\mathcal{M}$ is Ω-iterable.

Proof Let $\mathcal{U}_0 = \mathcal{U}_j$ and $i_{0,1} : L \to Ult(L,\mathcal{U}_0)$ be the ultraproduct map. We will show something stronger, namely that $\langle L, \in, \mathcal{U}_0 \rangle$ is γ-iterable for every $\gamma < \Omega$[7]. This will be shown by induction on γ. As we have seen so far, the base case and the successor case are sufficiently similar so that we only need to consider the base case. We recall that we have the following commuting diagram, where i and k are both elementary embeddings.

$$\begin{tikzcd}
L \arrow[rr, "j"'] \arrow[drr, "\iota"] & & L \\
& & Ult(L,\mathcal{U}_0) \arrow[u, "k"]
\end{tikzcd}$$

From the fact that $k : Ult(L,\mathcal{U}_0) \to L$ is an elementary embedding we can conclude that $Ult(L,\mathcal{U}_0)$ is well-founded. This completes the successor case. Before we move on to the limit case we observe that we can take the Mostowski collapse of $Ult(L,\mathcal{U}_0)$, as it is well-founded, and get a transitive class model of $ZFC + V = L$. It is a theorem due to Gödel that this transitive class model has to be L, and so we see that $Ult(L, \mathcal{M}_0) = L$ ([9]).

Let γ be a limit ordinal, we are going to consider an arbitrary continuous chain $\langle X_\alpha : \alpha < \gamma \rangle$ of proper class elementary substructures of L such that for all $\alpha < \gamma$, $\{x \in L : \delta < \gamma \to i_{0,\delta}(x) = x\} \cup \{\kappa_\beta : \beta < \alpha\} \subset X_\alpha$. That is, each X_α is a proper class substructure of L, and for $\alpha \leq \beta < \gamma$ we have that $X_\alpha \preceq X_\beta$. Letting $\rho_\alpha : X_\alpha \to L$ be the elementary embedding, then $X_\alpha \models V = L$ and so ρ_α"$(X_\alpha) = L$. Hence ρ_α is an isomorphism, and we let $\pi_\alpha : L \to X_\alpha$ denote its inverse. For $\alpha \leq \beta < \gamma$ we define $\sigma_{\alpha,\beta} = \rho_\beta \circ \pi_\alpha$. It should be clear that the following diagram commutes:

$$\begin{tikzcd}
X_0 \arrow[r, "id"] & \dots \arrow[r, "id"] & X_\alpha \arrow[r, "id"] & X_{\alpha+1} \arrow[r, "id"] & \dots \arrow[r, "id"] & \bigcup_{\beta<\gamma} X_\beta \\
L \arrow[u, "\pi_0"] \arrow[r] & \dots \arrow[r] & L \arrow[u, "\pi_\alpha"] \arrow[r, "\sigma_{\alpha,\alpha+1}"'] & L \arrow[u, "\pi_{\alpha+1}"] \arrow[r] & \dots \arrow[r] & \lim(\langle L \rangle_{\beta<\gamma}, \langle \sigma_{\beta',\beta} \rangle_{\beta' \leq \beta < \gamma}) \arrow[u]
\end{tikzcd}$$

We will now show that $\sigma_{\alpha,\beta} = i_{\alpha,\beta}$, where $i_{\alpha,\alpha+1} : L \to Ult(L,\mathcal{U}_\alpha) \cong L$. This will ensure that $\mathcal{M}_\gamma$ is well-founded, and conclude our proof. To do so, we will define a sequence of maps $\sigma_\alpha : L \to L$ for $\alpha < \gamma$ as follows. σ_0 will be the identity, $\sigma_{\alpha+1}$ will be given by $i_{\alpha,\alpha+1}(f)(\kappa_\alpha) \mapsto \sigma_{\alpha,\alpha+1}(f)(\kappa_\alpha)$ and for a limit ordinal $\delta < \gamma$ and any $\alpha < \delta$ we define σ_δ by $i_{\alpha,\delta}(x) \mapsto \sigma_{\alpha,\delta}(x)$. We aim to show that every map in the sequence is an isomorphism,

7 Applying Scott's tricks when appropriate.

which will force them all to the the identity map and hence we will be able to conclude that $i_{\alpha,\beta} = \sigma_{\alpha,\beta}$. We first need to check that $(\sigma_\alpha)_{\alpha<\gamma}$ are well-defined. To do so, we will show that $i_{\alpha,\beta}(a) = \sigma_{\alpha,\beta}(a)$ for $a \subseteq \kappa_\alpha$. This will allow us to observe that $a \in \mathcal{U}_\alpha$ just in case $\kappa_\alpha \in \sigma_{\alpha,\beta}(a)$, and hence we will know that $(\sigma_\alpha)_{\alpha<\gamma}$ are well-defined. The calculations leading up to this will be done in several steps.

We firstly observe that for $a \subseteq \kappa_0$, $\pi_0(a) \cap \kappa_0 = a$, which gives us that $i_{0,\alpha}(a) = i_{0,\alpha}(\pi_0(a) \cap \kappa_0) = \pi_0(a) \cap \kappa_\alpha$ as $\pi_0(a) \in X$.

Then we see that $\kappa_i \subseteq X_i$. To see why, suppose $\beta < \kappa_\alpha$. Then we might write $\beta = i_{0,\alpha}(f)(\eta_1, \ldots, \eta_p)$ for some $f : \kappa_0^p \to \kappa_0$ and $\eta_1 < \ldots < \eta_p < \alpha$. By previous work we know that $i_{0,\alpha}(f) = \pi_0(f) \cap \kappa_\alpha^p$, and so $\beta = \pi_0(f)(\eta_1, \ldots, \eta_p)$, which is definitely in X_α.

We now observe that $X_\alpha \cap \kappa_\beta = \kappa_\alpha$ for all $\alpha \leq \beta < \gamma$. If this was not the case then we would have a $\delta < \kappa_\beta$ such that $\delta \in X_\alpha$ but $\delta > \kappa_\alpha$. We then let μ be the largest ordinal such that $\kappa_\mu \leq \delta$. Clearly $\mu \geq \alpha$ and $\delta < \kappa_{\mu+1}$. This gives us that $i_{\mu,\mu+1}(\delta) \geq i_{\mu,\mu+1}(\kappa_\mu) = \kappa_{\mu+1} > \delta$. This, however, contradicts the fact that $i_{\mu,\mu+1}(\delta) = \delta$, which we know must be the case as $\delta \in X_\alpha$.

We are now going to show that $\sigma_{\alpha,\beta}(\kappa_\alpha) = \kappa_\beta$. Suppose for contradiction that $\sigma_{\alpha,\beta}(\kappa_\alpha) < \kappa_\beta$. We know that $\pi_\alpha(\kappa_\alpha) = \pi_\beta(\sigma_{\alpha,\beta}(\kappa_\alpha))$ from our commuting diagram, and so we get that $\pi_\alpha(\kappa_\alpha) = \sigma_{\alpha,\beta}(\kappa_\alpha))$ as $\sigma_{\alpha,\beta}(\kappa_\alpha) < \kappa_\beta$. Hence, $\pi_\alpha(\kappa_\alpha) < \kappa_\beta$. This gives us $\pi_\alpha(\kappa_\alpha) \in X_\alpha \cap \kappa_\beta$, and so $\pi_\alpha(\kappa_\alpha) < \kappa_\alpha$ by the previous paragraph, which is a contradiction. Suppose now for contradiction that $\sigma_{\alpha,\beta}(\kappa_\alpha) > \kappa_\beta$. We know that $\sigma_{\alpha,\beta} = \rho_\beta \circ \pi_\alpha$, and so we get that $\rho_\beta \circ \pi_\alpha(\kappa_\alpha) > \kappa_\beta$, which of course gives us that $\pi_\alpha(\kappa_\alpha) > \pi_\beta(\kappa_\beta)$. We know that $\pi_\alpha(\kappa_\alpha)$ is the smallest member of X_α that is greater than κ_α, and we are going to use this to get our contradiction. Let $\pi_\beta(\kappa_\beta) = t(\eta_1, \ldots, \eta_p, x)$ for $\eta_1 < \ldots < \eta_p < \kappa_\beta$ and $x \in X$. Then we get that $L \models \exists \xi_1 < \ldots < \xi_p < \kappa_\beta(\kappa_\beta < t(\xi_1, \ldots, \xi_p, x) < \pi_\alpha(\kappa_\alpha))$. Applying $(i_{\alpha,\beta})^{-1}$ to this gives us that $L \models \exists \xi_1 < \ldots < \xi_p < \kappa_\alpha(\kappa_\alpha < t(\xi_1, \ldots, \xi_p, x) < \pi_\alpha(\kappa_\alpha))$. Now, each $\xi_n < \kappa_\alpha$, and so since $\kappa_\alpha \subseteq X_\alpha$ we get that $t(\xi_1, \ldots, \xi_p, x) \in X_\alpha$. This contradicts $\pi_\alpha(\kappa_\alpha)$ being the smallest element of X_α greater than κ_α.

We now show that for $a \subseteq \kappa_\alpha$, $i_{\alpha,\beta}(a) = \sigma_{\alpha,\beta}(a)$, which will gives us that $a \in \mathcal{U}_\alpha$ just in case $\kappa_\alpha \in \sigma_{\alpha,\beta}(a)$. We know that $a = \pi_\alpha(a) \cap \kappa_\alpha$ and so we get that $i_{\alpha,\beta}(a) = i_{\alpha,\beta}(\pi_\alpha(a) \cap \kappa_\alpha) = \pi_\alpha(a) \cap \kappa_\beta = \rho_\beta \circ \pi_\alpha(a) \cap \kappa_\beta = \sigma_{\alpha,\beta}(a)$. The last equality uses the fact that $\sigma_{\alpha,\beta}(\kappa_\alpha) = \kappa_\beta$.

Hence, we know that $(\sigma_\alpha)_{\alpha<\gamma}$ are well-defined, and so we finish up the proof by verifying that σ_α is an isomorphism for the successor case and the limit case. For the successor case, $\sigma_{\alpha+1}$ is clearly an elementary embedding, so we just have to verify that it is a surjective map. Fix $x \in L$, then $\pi_{\alpha+1}(x) = t^L(\kappa_{\eta_1}, \ldots, \kappa_{\eta_p}, \kappa_\alpha, x^*)$ where $\eta_1 < \ldots < \eta_p < \alpha$ and $x^* \in X$. This gives us that $x = \rho_{\alpha+1}(t^L(\kappa_{\eta_1}, \ldots, \kappa_{\eta_p}, \kappa_\alpha, x^*))$, and so $x = \rho_{\alpha+1} \circ \pi_\alpha(f)(\kappa_\alpha)$ for $f(\xi) = t^L(\kappa_{\eta_1}, \ldots, \kappa_{\eta_p}, \xi, x^*)$. This gives us that $x = \sigma_{\alpha,\alpha+1}(f)(\kappa_\alpha)$, and so $\sigma_{\alpha+1}$ is surjective.

Let $\delta < \gamma$ be a limit ordinal, then we observe that σ_δ is an elementary embedding. Now, if $x \in L$ then we again see that $\pi_\delta(x) = t^L(\kappa_{\eta_1}, \ldots, \kappa_{\eta_p}, x^*)$ where $\eta_1 < \ldots < \eta_p$ and $x^* \in X$. Hence we see that $\pi_\delta(x) \in X_\beta$ for some $\beta < \delta$, and so the induction hypothesis gives us that σ_δ is surjective.

We have now seen that $(\sigma_\alpha)_{\alpha<\gamma}$ are all isomorphisms, which means that they are all the identity. This gives us that $i_{\alpha_\beta} = \sigma_{\alpha,\beta}$ for all $\alpha \leq \beta < \gamma$, and hence $\mathcal{M}_\gamma$ is well-founded, as required.

3.2 Mice

In this section we will define premice and see how they turn into mice. The iteration process will be rather similar to the one turning baby premice into baby mice. Before we can define a premouse we need to introduce some fine structure theory. The mice will be able to preserve the fine structure that we are going to introduce in the iteration process, which is what will give the Core Model its desired properties.

3.2.1 Fine Structure Theory

In this subsection we will introduce some fine structure theory. This is an interesting and broad subject of study in its own right, but since we are mainly interested in using it to define premice and mice in the next subsection the introduction will be brief. Furthermore, some results here will be quoted without proof as developing the appropriate machinery to prove them is beyond the scope of this thesis.

The fine structure definitions are rather technical, and so we give some informal intuition before listing the formal definition. Let $\mathcal{M} = \langle J_\alpha, \in, \mathcal{U} \rangle$, then the *projectum* $\rho_\mathcal{M}$ is intuitively the least γ in $\mathcal{M}$ such that there is some $A \subseteq \gamma$ which is Σ_1 definable over $\mathcal{M}$ but is not a member of $\mathcal{M}$. One can prove that if $\rho_\mathcal{M} \leq \kappa$, where $\mathcal{M} \models \mathcal{U}$ is a κ-complete ultrafilter over κ, then there is a surjective function $f : \kappa \to \mathcal{M}$ which is Σ_1 definable over $\mathcal{M}$. Hence, if $\rho_\mathcal{M} \leq \kappa$ then $\mathcal{M}$ can be coded by a subset of κ. If we have an enumeration $(\varphi_i)_{i \in \omega}$ of our Σ_1 formulae then this subset can be written down as $A_\mathcal{M} = \{\langle i, u \rangle : i \in \omega \wedge u < \kappa \wedge \mathcal{M} \models^2_{\Sigma_1} \varphi_i(u, p)\}$, where the p is defined later on. The point being that if we let $\mathcal{N} = \langle J_\beta, \in, \mathcal{V} \rangle$ be the next step in the iteration process, with map $j : \mathcal{M} \to \mathcal{N}$, then we can prove that $j(A_\mathcal{M}) \cap \kappa = A_\mathcal{M}$ and so $A_\mathcal{M} \in \mathcal{N}$. This means that $\mathcal{M} \in \mathcal{N}$ by reconstructing it from $A_\mathcal{M}$. Hence, by being careful with ensuring that $\rho_\mathcal{M} \leq \kappa$, we can preserve a lot of the fine structure in the iterative process.

> Definition 3.2.1 (The Projectum [3] (Definition 3.5))
> The *projectum* ρ is defined by $\omega\rho = \{\alpha < \Omega :$ If $A \subseteq \Omega$ is Σ_1 definable then $A \cap \alpha$ is a set.$\}$

In the previous definition it is important to keep in mind that we will always look at the projectum in some model J_α, and not in V or L where it would be Ω. The same is true for the next couple of definitions.

> Definition 3.2.2 ([3] (Definition 3.12))
> P denotes $\{p \in [\Omega]^{<\omega}$: There is a $A \subseteq \Omega$ that is Σ_1 definable with parameters p such that there is no x with $A \cap \omega\rho = A \cap x.\}$.

Definition 3.2.3 ([3] (Definition 3.23))
We define ρ_n,PA_n, V^{np}, A^{np}, and T^{np} by induction as follows:

- $A^0 = T^0 = PA_0 = \emptyset$, $\rho_0 = \Omega$, and $V^0 = V$,
  Suppose now that PA_n has been defined and that for all $p \in PA_n$, V^{np}, T^{np}, A^{np} and PA_n have been defined. Then:
- $PA_{n+1} = \{\langle p_1, \ldots, p_{n+1}\rangle : \langle p_1, \ldots, p_n\rangle \in PA_n \wedge p_{n+1} \in (\omega\rho_n)^{<\omega}\}$,
- $\rho^{n+1} = \bigcap\{$ projecta of $V^{np} : p \in PA_n\}$
- $T^{n+1,p} = \{\langle i, x\rangle : i \in \omega \wedge x \in H_{\omega\rho^{n+1}} \wedge V^{np'} \models^2_{\Sigma_1} \varphi_i(x, p'')\}$, where $p' = \langle p'_1, \ldots, p'_n\rangle$ and $p = \langle p'_1, \ldots, p'_n, p''\rangle \in PA_{n+1}$,
- $A^{n+1,p} = T^{n+1,p} \cap (\omega \times (\omega\rho^{n+1})^{<\omega})$, and
- $V^{n+1,p} = J^{A^{n+1,p}}_{\rho^{n+1}}$.

Now if $\mathcal{M}$ is a transitive model of RA then we can define the projectum within $\mathcal{M}$, and hence we can relativise all of the definitions above to $\mathcal{M}$. When we relativise V^{np} to a model $\mathcal{M} \models RA$ we denote it by $\mathcal{M}^{np}$ instead of $V^{np}_{\mathcal{M}}$ for simplicity.

It is not hard to show that $R^+ \vdash AC$ ([3] Corollary 2.24). This gives us a well-order of S_α, and using this we can create a skolem functions h_n such that $\langle i, x_1, \ldots, x_n\rangle \mapsto y$ if y is the least element of S, according to our well-ordering, such that $R_i(x_1, \ldots, x_n, y)$ and $\langle i, x_1, \ldots, x_n\rangle \mapsto \emptyset$ otherwise. We here assume that we have an indexing over the n-ary relations. We let $h^* = \bigcup_{n\in\omega} h_n$ and define $h(X) = h^*``(\omega \times X^{<\omega})$. We call h the *canonical Skolem function.* We use this function to give a notion of soundness that we want to preserve in the mice iteration.

Definition 3.2.4 (p-soundness [3] (Definition 4.17))
Let $\mathcal{M} \models RA$, we define p-soundness recursively. Suppose $p \in PA^{\mathcal{M}}_1$, then $\mathcal{M}$ is p-sound just in case $\mathcal{M} \models V = h(\omega\rho \cup p)$ where h is the canonical Skolem function.

Suppose now that p-soundness is defined for all $p \in PA^{\mathcal{M}}_n$. Let $\langle p_1, \ldots, p_{n+1}\rangle \in PA^{\mathcal{M}}_{n+1}$, then $\mathcal{M}$ is $\langle p_1, \ldots, p_{n+1}\rangle$-sound just in case $\mathcal{M}$ is $\langle p_1, \ldots, p_n\rangle$-sound and $\mathcal{M}^{n,\langle p_1 \ldots p_n\rangle}$ is p_{n+1}-sound.

When defining mice, we will be interested in standard models. We end this subsection by defining standard models, and surveying some results about them.

Definition 3.2.5 (Standard model of R [3] (Definition 4.1))
A model $\mathcal{M} = \langle M, E, A_1, \ldots, A_n\rangle$ of R is standard just in case

(i) if M' is an initial segment of M, that is if $x \in M'$ and yEx then $y \in M'$, and $E \cap (M')^2$ is well-founded then M' is transitive and $E \cap (M')^2 = \in \cap (M')^2$, and

(ii) $\omega + 1$ is an initial segment of M.

We note that M is an initial segment of M, and so any standard model $\mathcal{M} = \langle M, E \rangle$ has to be transitive and $E = \in$. We now show a Mostowski Collapse like Lemma for standard models.

Lemma 3.2.6 ([3] (Lemma 4.3))
Let $\mathcal{M} = \langle M, E \rangle$ and $z = \{n : \mathcal{M} \models n \in \omega\}$. Suppose $\mathcal{M} \models R$ and $E \cap z^2$ is well-founded, then $\mathcal{M}$ is isomorphic to a standard model of R.

Proof Let $T(x) = \{z : zEy_n$ where $n \in \omega$ and $\mathcal{M} \models y_n = \cup^n x\}$ and set $Z = \{x \in \mathcal{M} : T(x)^2 \cap E$ is well-founded$\}$.

We first observe that Z is an initial segment of $\mathcal{M}$. To see why, let $x \in Z$ and suppose that yEx then $\mathcal{M} \models y \subseteq \cup x$ and so for $n \in \omega$ $\mathcal{M} \models \cup^n y \subseteq \cup^{n+1} x$. This gives us that $T(y) \subseteq T(x)$ and so $T(y)^2 \cap E$ is well-founded, which means that $y \in Z$.

We now show that $Z^2 \cap E$ is well-founded. Let $A \subseteq Z$, where $A \neq \emptyset$. We need to find an E-minimal element of A. Let $x \in A$ and let $A' = T(x) \cap A$. If $A' = \emptyset$ then $yEx \Rightarrow y \in T(x) \Rightarrow y \notin A$, and so x is E-minimal in A. Otherwise, if $A' \neq \emptyset$ let y be the E-minimal element of A'. We know that there is one as $x \in Z$ so $T(x)^2 \cap E$ is well-founded. If $t \in A$ is such that tEy then $t \in T(x)$ and so $t \in A'$, contradicting y being E-minimal in A'. Hence A has an E-minimal element and so $Z^2 \cap E$ is well-founded.

We now go on to show that $\langle z, E \cap Z^2 \rangle$ is extensional, which will allow us to use the Mostowski collapse. Let $x, y \in Z$ where $x \neq y$, then $\mathcal{M} \models \exists t(tEx \wedge \neg tEy)$. From tEx we get that $t \in Z$ and so $\langle Z, E \cap Z^2 \rangle \models \exists t(tEx \wedge \neg tEy)$. Hence, by the Mostowski collapse there is a transitive Z' and a π such that $\pi : \langle Z', \in \rangle \cong \langle Z, E \cap Z^2 \rangle$. We let $M' = Z' \cup (M \setminus Z)$, with the caveat that if $Z' \cap (M \setminus Z) \neq \emptyset$ then we use a disjoint isomorphic copy of $(M \setminus Z)$. We now define $\pi^* : M' \to M$ by $x \mapsto \pi(x)$ if $x \in Z'$ and $x \mapsto x$ otherwise. We also say that $xE'y$ just in case $(x, y \in Z' \wedge x \in y) \vee (x \in Z' \wedge y \notin Z') \vee (x, y \notin Z' \wedge xEy)$.

The proof is completed by showing that $\langle M', E' \rangle$ is standard. Suppose N is an initial segment of M' and that $E' \cap N^2$ is well-founded. We then let $N' = \pi^{*}{}''(N)$ and observe that N' is a well-founded initial segment of M'. Indeed, if $x \in N$, $zE\pi^*(x)$ and $z = \pi^*(z^*)$ then $z^*E'x$ and so $z^* \in N$, which in turn gives us that $z \in N'$. Now, suppose $x \in N'$, then $zEy \Rightarrow z \in N'$ for $y \in T(x)$, and so $T(x)^2 \cap E$ is well-founded. This gives us that $x \in Z$. Thus we have shown that $N' \subseteq Z$, which gives us that $N \subseteq Z'$, and so N is transitive and $E' \cap N^2 = \in \cap N^2$.

For the second clause we have that $E \cap z^2$ is well-founded by the assumption of the Lemma, so $\pi : \omega \cong z$. If $\mathcal{M} \models$ "ω exists", then $\mathcal{M}' \models$ "ω exists". We also have that $E' \cap (\omega \cup \{\omega\})^2$ is well-founded, and so ω is ω in the sense of $\mathcal{M}'$. This gives us that $\omega+1$ is an initial segment of $\mathcal{M}'$.

Now if we have two standard models $\mathcal{M}$ and $\mathcal{N}$ of R and a Σ_0 elementary embedding $\pi : \mathcal{M}^{np'} \to \mathcal{N}^{np}$ where $\mathcal{M}$ is p'-sound then there is a unique $\pi^* \supseteq \pi$ such that $\pi^*(p') = p$ and $\pi^* \restriction_{M^{mp'_m}} : \mathcal{M}^{mp'_m} \to \mathcal{N}^{mp_m}$ is a Σ_0 embedding for all $m \leq n$ where $p' = \langle p'_m, p''_{m+'}, \ldots, p''_n \rangle$ and $p = \langle p_m, p^*_{m+1}, \ldots, p^*_n \rangle$. If π also is a cofinal map then we say that π^* is the *n-completion* of π to $\mathcal{M}$.

Lastly, we cite without proof the following result which will be very useful in the successor stage of the premice iteration.

LEMMA 3.2.7 ([3] (LEMMA 4.25))
Suppose $\mathcal{M}$ and $\mathcal{N}$ are standard models of R, and that $\mathcal{M}$ is p-sound. Then if $\pi : \mathcal{M}^{np} \to \mathcal{N}$ is a cofinal Σ_0 embedding then there are $\mathcal{N}'$ and p', unique up to isomorphism, such that $\mathcal{N}'^{np'} = \mathcal{N}$ and $\mathcal{N}'$ is p'-sound.

3.2.2 From Premouse to mouse

We start this subsection by defining a premouse. We will then spend the rest of this subsection developing the necessary machinery for the iteration that will turn a premouse into a mouse. This will be done in two steps. We will first ignore all the fine structure theory that we developed in the previous subsection, and spell out an iteration process given by a map $\to_{\mathcal{U}}$ that preserves premousehood. This iteration process should remind us of the iteration process we had with the baby premice. Then we will narrow the scope from premice to n-suitable premice, which are premice that have certain nice fine structure properties, and generalise the iteration process from $\to_{\mathcal{U}}$ to $\to^n_{\mathcal{U}}$. This will ensure that the nice fine structure properties are also preserved.

DEFINITION 3.2.8 (PREMOUSE [3] (DEFINITION 5.27))
A model $\mathcal{M}$ is a *premouse* just in case:

1. $\mathcal{M} = \langle M, E, \mathcal{U}, A_1, \ldots, A_n \rangle$,
2. $\mathcal{M} \models R^+$,
3. $\mathcal{M}$ is standard,
4. $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ.

We say that κ is the critical point of $\mathcal{M}$. Furthermore, if $\mathcal{M} = \langle M, E, \mathcal{U} \rangle$ then we say that $\mathcal{M}$ is a *pure* premouse.

We will now start developing the machinery to iterate the premice in the same fashion that we iterated the baby premice.

Definition 3.2.9 ([3] (Definition 5.11))
Suppose $\mathcal{M} \models R + AC$ is standard, and suppose that $j : \mathcal{M} \to \mathcal{N}$ is Σ_0 elementary with critical point κ. Furthermore, suppose $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ, then $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$ means:

(i) for all $x \in N$, there is $f \in M$ of the form $f : \kappa \to M$ such that $\mathcal{N} \models x = j(f)(\kappa^*)$ where $\mathcal{N} \models \kappa^* = \sup(j"(\kappa))$, and

(ii) for all $x \in (P(\kappa))^{\mathcal{M}}$, $x \in \mathcal{U}$ just in case $\mathcal{N} \models \kappa^* \in j(x)$.

$\mathcal{N}$ is then called the *ultrapower* of $\mathcal{M}$ by $\mathcal{U}$.

This should reminds us of the ultraproducts we were forming in the iterative process of the baby premice. We will now show that this iterative process is possible, and that it is unique. We start with the existence claim.

Lemma 3.2.10 (Existence [3] (Lemma 5.16))
Suppose $\mathcal{M} \models R + AC$ and suppose $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ, then there are $\mathcal{N}$ and j such that $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$.

Proof Let $T^* = \{f \in M : f : \kappa \to M\}$. Then for $f, g \in T^*$ we say that $f \sim g$ just in case $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{U}$. This is clearly an equivalence relation, and so we let $T = T^*/\sim$. We now say that $[f]E[g]$ just in case $\{\xi : f(\xi) \in g(\xi)\} \in \mathcal{U}$. Lastly, for predicates A_k, we say that $\langle [f_1], \ldots, [f_n] \rangle \in A_k$ just in case $\{\xi : \langle f_1(\xi), \ldots, f_n(\xi) \rangle \in A_k\} \in \mathcal{U}$.

We first observe that for any Σ_0 formula φ, we have that $\langle T, E, A_1, \ldots, A_k \rangle \models \varphi[[f_1], \ldots, [f_n]]$ just in case $\{\xi : \mathcal{M} \models \varphi[f_1(\xi), \ldots, f_n(\xi)]\} \in \mathcal{U}$. The proof of this is done by induction on the complexity of φ, and the only step that is not immediate is the bounded quantifier step. Hence, let $\varphi(v_1, \ldots, v_n) = \exists v_{n+1} \in v_n \psi(v_1, \ldots, v_{n-1}, v_{n+1})$ and suppose $\langle T, E, A_1, \ldots, A_k \rangle \models \exists [f_{n+1}] \in [f_n] \psi[[f_1], \ldots, [f_{n-1}], [f_{n+1}]]$. Then we get that $\{\xi : \mathcal{M} \models \psi[f_1(\xi), \ldots, f_{n-1}(\xi), f_{n+1}(\xi)] \wedge f_{n+1}(\xi) \in f_n(\xi)\} \in \mathcal{U}$ and so we have that $\{\xi : \mathcal{M} \models \exists v_{n+1} \in f_n(\xi) \psi[f_1(\xi), \ldots, f_{n-1}(\xi), v_{n+1}]\} \in \mathcal{U}$. Conversely, suppose $\{\xi : \mathcal{M} \models \exists v_{n+1} \in f_n(\xi) \psi[f_1(\xi), \ldots, f_{n-1}(\xi), v_{n+1}]\} \in \mathcal{U}$. Let $g : \kappa \to M$ be given by $\xi \mapsto \{x \in f_n(\xi) : \mathcal{M} \models \psi[f_1(\xi), \ldots, f_{n-1}(\xi), x]\}$. g is a set in M, and since $\mathcal{M} \models AC$ there is a $h \in M$ such that $\mathcal{M} \models h(\xi) \in g(\xi)$ whenever $g(\xi) \neq \emptyset$. We can easily ensure that $h : \kappa \to M$. Furthermore, $\{\xi : g(\xi) \neq \emptyset\} \in \mathcal{U}$ and so $\{\xi : h(\xi) \in g(\xi)\} \in \mathcal{U}$. This gives us that $\{\xi : h(\xi) \in f_n(\xi) \wedge \psi[f_1(\xi), \ldots, f_{n-1}(\xi), h(\xi)]\} \in \mathcal{U}$, and so $\langle T, E, A_1, \ldots, A_k \rangle \models \exists [f_{n+1}] \in [f_n] \psi[[f_1], \ldots, [f_{n-1}], [f_{n+1}]]$. This finishes the induction step, and so the subclaim holds.

We define $c_x : \kappa \to M$ by $\xi \mapsto x$ for all $x \in M$ and let $j(x) = [c_x]$. It is straight forward to see that if $\mathcal{N} = \langle T, E, A_1, \ldots, A_k \rangle$ then $j : \mathcal{M} \to \mathcal{N}$ is Σ_0 elementary.

We now show that κ is the critical point of j. Firstly, we show that j"(κ) is an initial segment of $\Omega^{\mathcal{N}}$. Hence suppose that $x = [f]$ and that $\mathcal{N} \models x < j(\beta)$. Then $\mathcal{N} \models [f] < [c_\beta]$ and so $\{\xi : f(\xi) < \beta\} \in \mathcal{U}$. For $\alpha < \beta$ we let $X_\alpha = \{\xi : f(\xi) = \alpha\}$. Clearly $\bigcup_{\alpha<\beta} X_\beta = \{\xi : f(\xi) < \beta\}$, and so by κ-completeness there is an $\alpha < \beta$ such that $X_\alpha \in \mathcal{U}$. This gives us that $\mathcal{N} \models [f] = [c_\alpha]$ and so $x = j(\alpha)$. This makes j"(κ) an initial segment of $\Omega^{\mathcal{N}}$.

We now show that if ι is the identity map on κ, then $[\iota] = \kappa^* = j$"(κ). If $\beta < \kappa$ then $\{\xi : \beta < \xi\} \in \mathcal{U}$ and so $[c_\beta] < [\iota]$. Suppose $\mathcal{N} \models [f] < [\iota]$, then $\{\xi : f(\xi) < \xi\} \in \mathcal{U}$. Let $Y_\alpha = \{\xi : f(\xi) \neq \alpha\}$. If $\xi \in \bigcap_{\alpha<\xi} Y_\alpha$ then $f(\xi) \geq \xi$, and so $\{\xi : \xi \in \bigcap_{\alpha<\xi} Y_\alpha\} \notin \mathcal{U}$, which gives us that there is an α such that $Y_\alpha \notin \mathcal{U}$ and so $[f] = [c_\alpha]$. This gives us that $[\iota] = \kappa^*$. Finally we observe that $[\iota] < [c_\kappa]$ as $\{\xi : \xi < \kappa\} \in \mathcal{U}$.

We observe that for any $[f] \in \mathcal{N}$, we have that $\mathcal{N} \models [f] = [c_f]$ as $\{\xi : f(\xi) = f(\xi)\} \in \mathcal{U}$. Lastly, we note that for any $x \in (P(\kappa))^{\mathcal{M}}$ we have that $x \in \mathcal{U}$ just in case $\{\xi : \xi \in x\} \in \mathcal{U}$ just in case $\mathcal{N} \models \kappa^* \in [c_x]$. Hence $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$, as desired.

Having finished the existence part of the proof, we move on to the uniqueness part.

Lemma 3.2.11 (Uniqueness [3] (Lemma 5.14))
Suppose $\mathcal{M} \models R$, $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ, and further suppose there are $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$ and $j' : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}'$. Then there is an isomorphism $\sigma : \mathcal{N} \cong \mathcal{N}'$ such that $\sigma \circ j = j'$ and $\sigma(\kappa^*) = \kappa'$ where $\mathcal{N} \models \kappa^* = \sup(j$"$(\kappa))$ and $\mathcal{N}' \models \kappa' = \sup(j'$"$(\kappa))$.

Proof We observe that $\mathcal{N} \models j(f)(\kappa^*) = j(f')(\kappa^*)$ just in case $\mathcal{N} \models \kappa^* \in \{\xi < j(\kappa) : j(f)(\xi) = j(f')(\xi)\}$ just in case $\mathcal{N} \models \kappa^* \in j(\{\xi < \kappa : \mathcal{M} \models f(\xi) = f'(\xi)\})$ just in case $\{\xi < \kappa : \mathcal{M} \models f(\xi) = f'(\xi)\} \in \mathcal{U}$, where the last just in case follows from the second clause of the definition of $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$. Now, a similar argument can be made for $\mathcal{N}'$ with respect to j' and κ'. Hence, we get that $\mathcal{N} \models j(f)(\kappa^*) = j(f')(\kappa^*)$ just in case $\mathcal{N}' \models j'(f)(\kappa') = j'(f')(\kappa')$. By a similar line of reasoning we also get that $\mathcal{N} \models j(f)(\kappa^*) \in j(f')(\kappa^*)$ just in case $\mathcal{N}' \models j'(f)(\kappa') \in j'(f')(\kappa')$ and $\mathcal{N} \models A_k(j(f)(\kappa^*))$ just in case $\mathcal{N}' \models A_k(j'(f)(\kappa'))$. Hence, if we define $\sigma : \mathcal{N} \to \mathcal{N}'$ by $j(f)(\kappa^*) \mapsto j'(f)(\kappa')$, then we see that σ is an isomorphism. We also see that $\sigma(j(x)) = \sigma(j(c_x)(\kappa^*)) = j'(c_x)(\kappa') = j'(x)$, where $c_x : \kappa \to M$ is given by $\xi \mapsto x$. Hence $\sigma \circ j = j'$, as required.

We also get an easy corollary, which establishes a sort of universal property for the ultrapower.

Corollary 3.2.12 ([3] (Corollary 5.15))
Suppose $\mathcal{M} \models R$, $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ, and further suppose there are j and $\mathcal{N}$ such that $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$. If $j' : \mathcal{M} \to \mathcal{N}'$ with $crit(j') = \kappa$ and $x \in \mathcal{U}$ just in case $\kappa' \in j'(x)$ then there is a Σ_0 elementary embedding $\sigma : \mathcal{N} \to \mathcal{N}'$ such that $\sigma \circ j = j'$.

Now, it remains to show that if $\mathcal{M}$ is a premouse, and $\mathcal{M} \to_{\mathcal{U}} \mathcal{N}$ then $\mathcal{N}$ is a premouse. This will allow us to iteratively create new premice any finite number of times. This is not difficult as there is a Π_1 formula $\varphi(\kappa)$ such that $\langle M, E, \mathcal{U}\rangle \models \varphi(\kappa)$ just in case $\langle M, E, \mathcal{U}\rangle \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ. It is not hard to show that $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$ is cofinal, and hence Σ_1 elementary, and so if $\mathcal{N}$ is not a premouse then $\mathcal{M}$ is not a premouse. We can therefore iteratively create new premice from ultrapowers any finite number of times. Just as we did with the baby premice, we want to extend this through the ordinals, and so we need to deal with the limit ordinal case. It turns out that if we index the ultrapower maps $j_{n,n+1} : \mathcal{M}_n \to_{\mathcal{U}_n} \mathcal{M}_{n+1}$, and allow for composition $j_{n,m} : \mathcal{M}_n \to \mathcal{M}_m$ whenever $n \leq m$, then we can take the same direct limit as we did in the baby premice case. We summarise all the work done so far in the following definition and lemmata.

Definition 3.2.13 ([3] (Definition 6.1 and Definition 6.9))
Suppose $\mathcal{M} = \langle M, E, \mathcal{U}, A_1, \dots, A_k\rangle \models R + AC$ and that $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ. A pair $\langle (\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ is an *iterated ultrapower of $\mathcal{M}$ by $\mathcal{U}$* just in case for all $\alpha \leq \beta < \gamma < \Omega$:

(i) $\langle (\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ is a commutative system,
(ii) $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{U}_0 = \mathcal{U}$,
(iii) $\langle M_\alpha, E, \mathcal{U}_\alpha, A_1, \dots, A_k\rangle \models R + AC$,
(iv) $j_{\alpha,\alpha+1} : \langle M_\alpha, E, \mathcal{U}_\alpha\rangle \to_{\mathcal{U}} \langle M_{\alpha+1}, E, \mathcal{U}_{\alpha+1}\rangle$, and
(v) $\langle \mathcal{M}_\lambda, (j_{\alpha,\lambda})_{\alpha<\lambda}\rangle$ is a direct limit of $\langle (\mathcal{M}_\alpha)_{\alpha<\lambda}, (j_{\alpha,\beta})_{\alpha\leq\beta<\lambda}\rangle$ whenever λ is a limit ordinal.

We get the following lemmata immediately, which outline existence and uniqueness.

Lemma 3.2.14 ([3] (Lemma 6.10))
Suppose $\mathcal{M} = \langle M, E, \mathcal{U}, A_1, \dots, A_k\rangle \models R+AC$ and $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ. Then there exists an iterated ultrapower of $\mathcal{M}$ by $\mathcal{U}$.

Lemma 3.2.15 ([3] (Lemma 6.11))
Suppose $\mathcal{M} = \langle M, E, \mathcal{U}, A_1, \dots, A_k\rangle \models R + AC$ and that $\mathcal{M} \models \mathcal{U}$ is a normal κ-complete ultrafilter on κ. If $\langle (\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ and $\langle (\mathcal{M}'_\alpha)_{\alpha<\Omega}, (j'_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ are both iterated ultrapowers of $\mathcal{M}$ by $\mathcal{U}$ then there are isomorphism $\sigma_\alpha : \mathcal{M}_\alpha \to \mathcal{M}'_\alpha$ such that for all $\alpha \leq \beta < \Omega$:

(i) $\sigma_0 = id \restriction_{\mathcal{M}}$,

(ii) $\sigma_\beta \circ j_{\alpha,\beta} = j'_{\alpha,\beta} \circ \sigma_\alpha$, and
(iii) $\sigma_\beta(j_{0,\alpha}(\kappa)) = j'_{0,\alpha}(\kappa)$.

For a premouse $\mathcal{M}$ we say that $\mathcal{M}$ is *iterable* by $\mathcal{U}$ provided each $\mathcal{M}_\alpha$ is well-founded and standard.

Up to this point, none of the fine structure introduced in the previous subsection have been used. We want to restrict the premice under consideration to premice with certain nice fine structure properties, and we want the iteration process to preserve these nice fine structure properties. Hence, we will now define an n-suitable premouse, which is a premouse that has a lot of the nice fine structure properties that we want, and create a similar iteration process as the one we have considered so far. This will be the iteration process for which we want each iterate to be well-founded, and when that is the case then we (almost) say that we have a mouse.

Definition 3.2.16 ([3] (Definition 9.9))
Suppose $\mathcal{M} \models \mathrm{RA}$. Then we say that $\mathcal{M}$ is *strongly acceptable above* γ just in case for every limit ordinal $\lambda \geq \gamma$ and every $a \subseteq \delta$, if $a \in S^{\mathcal{M}}_{\lambda+\omega} \setminus S^{\mathcal{M}}_{\lambda}$ then $S^{\mathcal{M}}_{\lambda+\omega} \models |\lambda| \leq \max(\gamma, \delta)$

Definition 3.2.17 (n-suitable premouse [3] (Definition 4.30 and Definition 9.12))
A premouse $\mathcal{M}$ is *n-suitable* just in case:
(i) $\mathcal{M} \models \mathrm{RA}$,
(ii) $\mathcal{M}$ is strongly acceptable over κ, where κ is the critical point of $\mathcal{M}$,
(iii) $\kappa \in \omega\rho^n_{\mathcal{M}}$, and
(iv) $\mathcal{M}$ is p-sound for some $p \in PA^{\mathcal{M}}_n$.

We will now introduce the new iteration map $\to^n_{\mathcal{U}}$ that will preserve n-suitableness as well as premicehood.

Definition 3.2.18 ([3] (Definition 9.13))
Suppose $\mathcal{M} = \langle M, \in, \mathcal{U}, A_1, \ldots, A_k \rangle$ is an n-suitable premouse. Then $j : \mathcal{M} \to^n_{\mathcal{U}} \mathcal{N}$ means that for some $p \in PA^{\mathcal{M}}_n$ where $\mathcal{M}$ is p-sound, $j \restriction_{M^{np}} : \mathcal{M}^{np} \to_{\mathcal{U}} \mathcal{N}^{nj(p)}$ and j is the n-completion of $j \restriction_{M^{np}}$ to $\mathcal{M}$.

We see that $\to^n_{\mathcal{U}}$ generalises $\to_{\mathcal{U}}$ as $\to^0_{\mathcal{U}}$ is the same as $\to_{\mathcal{U}}$. We now show that we can have the same iterative process for $\to^n_{\mathcal{U}}$ as we had for $\to_{\mathcal{U}}$. We first show that $\to^n_{\mathcal{U}}$ preserves n-suitableness.

Lemma 3.2.19 ([3] (Lemma 9.16))
Let $\mathcal{M}$ be an n-suitable premouse and suppose that $j : \mathcal{M} \to^n_{\mathcal{U}} \mathcal{N}$, then $\mathcal{N}$ is a n-suitable premouse.

Proof $\mathcal{N}$ is clearly a premouse. j is cofinal and a Σ_0 embedding, which makes it a Σ_1 embedding. That suffices to show that $\mathcal{N} \models RA$ and that $\mathcal{N}$ is strongly acceptable above $j(\kappa)$. $\mathcal{M}$ is p-sound for some $p \in PA^{\mathcal{M}}_n$, which makes $\mathcal{N}$ $j(p)$-sound for $j(p) \in PA^{\mathcal{N}}_p$. We also clearly have that $j(p) \in \omega\rho^n_{\mathcal{N}}$.

We now move on to the successor step of the iterative process.

Lemma 3.2.20 ([3] (Lemma 9.15))
Suppose $\mathcal{M}$ is an n-suitable premouse, then there are j and $\mathcal{N}$ such that $j : \mathcal{M} \to^n_{\mathcal{U}} \mathcal{N}$.

Proof By Lemma 3.2.10 we know that there is an $\mathcal{N}$ and a j such that $j : \mathcal{M} \to_{\mathcal{U}} \mathcal{N}$. Let $p \in PA_n$ be such that $\mathcal{M}$ is p-sound. We then see that $j \restriction_{M^{np}} : \mathcal{M}^{np} \to \mathcal{N}$ is a cofinal Σ_0 embedding, and so by Lemma 3.2.7 we know that there are $\mathcal{N}^*$ and p^*, unique up to isomorphism, such that $(\mathcal{N}^*)^{np^*} = j"(\mathcal{M}^{np})$ where $\mathcal{N}^*$ is p^*-sound. This gives us $j : \mathcal{M} \to^n_{\mathcal{U}} \mathcal{N}^*$, as required.

We now prepare for the limit case.

Lemma 3.2.21 ([3] (Paraphrased from Lemma 9.18))
Suppose $\langle(\mathcal{M}_\alpha)_{\alpha<\lambda}, (j_{\alpha,\beta})_{\alpha\leq\beta<\lambda}\rangle$ is a commutative system where λ is a limit ordinal such that $\mathcal{M}_0$ is an n-suitable premouse that is p-sound for $p \in PA^{\mathcal{M}_0}_n$. Let $\langle\mathcal{M}_\lambda, (j_{\alpha,\lambda})_{\alpha<\lambda}\rangle$ be the direct limit of the commutative system and for $\alpha \leq \beta < \lambda$ suppose that each $j_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the n-completion of $j'_{\alpha,\beta} = j_{\alpha,\beta} \restriction_{M^{nj_{0,\alpha}(p)}_\alpha}$. Let $j'_\alpha = j_\alpha \restriction_{M^{nj_{0,\alpha}(p)}_\alpha}$, then $\langle\mathcal{M}^{nj_0(p)}_\lambda, (j'_\alpha)_{\alpha<\lambda}\rangle$ is a direct limit of $\langle(\mathcal{M}^{nj_{0,\alpha}(p)}_\alpha)_{\alpha<\lambda}, (j'_{\alpha,\beta})_{\alpha\leq\beta<\lambda}\rangle$ and j_α is the n-completion of j'_α.

Proof It is sufficient to show this for $n = 1$ as the general case is sufficiently similar. We start by showing that $\langle \mathcal{M}_\lambda^{nj_0(p)}, (j'_\alpha)_{\alpha<\lambda} \rangle$ is a direct limit of $\langle (\mathcal{M}_\alpha^{nj_{0,\alpha}(p)})_{\alpha<\lambda}, (j'_{\alpha,\beta})_{\alpha\leq\beta<\lambda} \rangle$. The only non-trivial part is to show that for every $x \in M_\lambda^{nj_0(p)}$ there is an $\alpha < \lambda$ and $x' \in M_\alpha^{nj_{0,\alpha}(p)}$ such that $j'_\alpha(x') = x$. Fix $x \in M_\lambda^{nj_0(p)}$, then there is an $\alpha < \lambda$ and $x' \in M_\alpha$ such that $j_\alpha(x') = x$. We then see that $\mathcal{M}_\lambda \models |Tc(x)| \in \omega\rho^n_{\mathcal{M}_\lambda}$. We let $\gamma = \{y \in \mathcal{M}_\lambda : \mathcal{M}_\lambda \models y \in j_\alpha(\delta), \alpha < \lambda \wedge \delta \in \omega\rho_{\mathcal{M}_\delta}\}$ and aim to show that $\mathcal{M} \models V = h(\gamma \cup j_0(p))$, where h is the canonical Skolem function. This will give us $\omega\rho_{\mathcal{M}_\lambda} \subseteq \gamma$, which in turn gives us that $\mathcal{M}_\lambda \models Tc(x) \in \gamma$. Indeed, we see that if $y \in M_\lambda$ then $y = j_\beta(y')$ for some $\beta < \lambda$ and $y' \in M_\beta$ because $\mathcal{M}_\lambda$ is a direct limit. We know that $\mathcal{M}_\beta \models y' = h^*(i, \langle \delta_1, \ldots, \delta_k, j_{0,\alpha}(p) \rangle)$ for $\delta_1, \ldots, \delta_k \in \omega\rho_{\mathcal{M}_\beta}$. This gives us that $\mathcal{M}_\lambda \models y = h^*(i, \langle j_\beta(\delta_1), \ldots, j_\beta(\delta_k), j_0(p) \rangle)$ and so $\mathcal{M} \models V = h(\gamma \cup j_0(p))$. Hence we see that $\omega\rho_{\mathcal{M}_\lambda} \subseteq \gamma$, which gives us that $\mathcal{M}_\lambda \models Tc(x) \in \gamma$. We can without loss of generality assume that $\mathcal{M}_\alpha \models Tc(x') \in \omega\rho_{\mathcal{M}_\alpha}$ and so $x' \in M^{nj_{0,\alpha}(p)}$.

For the second part of the claim it is sufficient to show that $\mathcal{M}_\lambda$ is p-sound. Suppose for contradiction that it is not. Then $\omega\rho_{\mathcal{M}_\lambda} \neq \gamma$, which means that there is an $\alpha < \lambda$ and a $\delta \in \omega\rho_{\mathcal{M}_\alpha}$ such that $j_\alpha(\delta) \in \gamma \setminus \omega\rho_{\mathcal{M}_\lambda}$. Let A be the Σ_1 definable set over $\mathcal{M}_\lambda$ with parameters q such that $A \cap j_\alpha(\delta) \notin M_\lambda$. We may assume that $q = j_\alpha(q^*)$ for some $q^* \in M_\alpha$. If we let A^* have the same Σ_1 definition over $\mathcal{M}_\alpha$ with parameter p^* then we get that $\mathcal{M}_\alpha \models \forall\beta(\beta \in A^* \cap \delta \leftrightarrow \varphi(\beta, p^*))$, which is Π_2. Now, j_α is the 1-completion of j'_α, which is Σ_1 elementary. It is not too hard to show that this makes j_α Σ_2 elementary. This gives us that $\mathcal{M}_\lambda \models \forall\beta(\beta \in j_\alpha(A^*) \cap j_\alpha(\delta) \leftrightarrow \varphi(\beta, p))$ because the negation is a Σ_2 sentence, and then the Σ_2 embedding would give us that the negation is true for $\mathcal{M}_\alpha$, a contradiction. However, this gives us that $A \cap j_\alpha(\delta) = j_\alpha(A^* \cap \delta)$, which contradicts our initial assumption that $A \cap j_\alpha(\delta) \notin M_\lambda$.

We are now ready to put everything together. For n-suitable premice, we will have a concept of n-iterated ultrapowers. The essential difference between them and iterated ultrapowers is that they preserve n-suitableness. We then get the same existence and uniqueness results that we got for iterated ultrapowers. The proof of the uniqueness claim is essentially the same, so we will not cover that here. For the proof of the existence claim we have already laid the ground work for the successor case and the limit case by the two previous lemmata.

Definition 3.2.22 ([3] (Definition 9.19))
Suppose $\mathcal{M}$ is an n-suitable premouse. Then $\langle (\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega} \rangle$ is an n-iterated ultrapower of $\mathcal{M}$ just in case:

(i) $\langle (\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega} \rangle$ is a commutative system,

(ii) each $\mathcal{M}_\alpha = \langle M_\alpha, \in, \mathcal{U}_\alpha, A^1_\alpha, \ldots, A^k_\alpha \rangle$ is n-suitable,

(iii) $\mathcal{M}_0 = \mathcal{M}$,

(iv) $j_{\alpha,\alpha+1} : \mathcal{M}_\alpha \to^n_{\mathcal{U}_\alpha} \mathcal{M}_{\alpha+1}$,

(v) $\langle \mathcal{M}_\lambda, (j_{\alpha,\lambda})_{\alpha<\lambda} \rangle$ is a direct limit of $\langle (\mathcal{M}_\alpha)_{\alpha<\lambda}, (j_{\alpha,\beta})_{\alpha\leq\beta<\lambda} \rangle$ whenever λ is a limit ordinal.

Lemma 3.2.23 ([3] (Lemma 9.20))
If $\mathcal{M}$ is an n-suitable premouse then $\mathcal{M}$ has an n-iterated ultrapower.

Lemma 3.2.24 ([3] (Lemma 9.21))
Suppose $\langle(\mathcal{M}_\alpha)_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ and $\langle(\mathcal{M}'_\alpha)_{\alpha<\Omega}, (j'_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ are n-iterated ultrapowers of $\mathcal{M}$. Then there are isomorphisms $\sigma_\alpha : \mathcal{M}_\alpha \to \mathcal{M}'_\alpha$ such that for all $\alpha \leq \beta < \Omega$:

(i) $\sigma_\beta \circ j_{\alpha,\beta} = j'_{\alpha,\beta} \circ \sigma_\alpha$, and

(ii) If κ_α and κ'_α are the criticial points of $\mathcal{M}_\alpha$ and $\mathcal{M}'_\alpha$ respectively, then $\sigma_\beta(\kappa_\alpha) = \kappa'_\alpha$.

There are two more definitions that we will make before we end the section by defining a mouse. Firstly, if we have an n-suitable premouse $\mathcal{M}$ then we know that $\kappa \in \omega\rho^n_{\mathcal{M}}$, where κ is the critical point. We want to have a notion of the largest n for which this is the case, and so we call an n-suitable premouse for *critical* just in case $\kappa \notin \omega\rho^{n+1}_{\mathcal{M}}$. For a critical premouse $\mathcal{M}$, we denote this n by $n(\mathcal{M})$. For an $n(\mathcal{M})$-suitable critical premouse $\mathcal{M}$ we call the $n(\mathcal{M})$-iterated ultrapower of $\mathcal{M}$ the *mouse iteration* of $\mathcal{M}$.

Secondly, suppose $\langle(\mathcal{M})_{\alpha<\Omega}, (j_{\alpha,\beta})_{\alpha\leq\beta<\Omega}\rangle$ is an n-iterated ultrapower of $\mathcal{M}$, then we call each $\mathcal{M}_\alpha$ an n-iterate of $\mathcal{M}$. Furthermore, we say that a premouse $\mathcal{M}$ is *n-iterable* just in case each n-iterate of $\mathcal{M}$ is well-founded.

We are now ready to define a mouse. It turns out that the existence of a mouse has the same sufficient condition as the existence of a baby mouse. The proof is analogous, although certain changes has to be done. In fact, the details for the proof of the limit case in Theorem 3.1.15 is actually taken from chapter 12 of [3], which shows how a non-trivial elementary embedding $j : L \to L$ is sufficient for the existence of a mouse. We end the section by defining a mouse.

Definition 3.2.25 (mouse [3] (Definition 9.25))
Suppose $\mathcal{M}$ is a critical and $n(\mathcal{M})$-iterable pure premouse then $\mathcal{M}$ is a mouse.

3.3 The Core Model

When defining K, which we will do in terms of mice, we will assume that $0^\#$ exists and hence that there are mice. There are other definitions of K that do not rely on mice.

The definition given in [3] is such that if $0^\#$ does not exist, and hence there are no mice, then $K = L$. Setting up the appropriate machinery to make the same definition will take us outside the scope of this thesis, and hence we will make the inelegant definition that if $0^\#$ does not exist then we define $K = L$.

> Definition 3.3.1 ([3] (Definition 14.3 and Lemma 14.14))
> Suppose $0^\#$ exist, then the *Core Model* is defined as $K = \bigcup\{M : \mathcal{M} \text{ is a mouse.}\}$.

If $K = L$ then it is straightforward to see that $K \models ZFC + GCH$, but it turns out that this result also holds if $0^\#$ exists.

> Lemma 3.3.2 ([3] (Corollary 14.16))
> $K \models ZFC + GCH$.

There are other nice results about L that also transfer to K, and we will end this subsection, and consequently this thesis, but outlining two of them and having a quick look at why we might care about these results. The two results of interest are $\diamondsuit$ and $\square_\kappa$. Before we can state $\diamondsuit$ we need a bit of terminology. A set $C \subseteq \omega_1$ is *closed* just in case for every $0 < \alpha < \omega_1$, if $\sup(C \cap \alpha) = \alpha$ then $\alpha \in C$. We also say that C is *unbounded* just in case for any $\alpha < \omega_1$ there is a $\beta \in C$ such that $\alpha < \beta$. We then say that C is a *club* set just in case it is both closed and unbounded. Intuitively, club sets are deemed large sets. A set $S \subseteq \omega_1$ is called *stationary* just in case it intersects all the club sets. Under the same intuition, a stationary set is a set that is not small as it intersects all the large sets.

> Definition 3.3.3 ($\diamondsuit$ [3] (Definition 12.22))
> $\diamondsuit$ is the following statement: there exists a sequence of sets $(S_\alpha)_{\alpha<\omega_1}$ such that for every $x \subseteq \omega_1$ the set $\{\alpha < \omega_1 : x \cap \alpha = S_\alpha\}$ is stationary.

> Lemma 3.3.4 ([3] (Lemma 17.23))
> $K \models \diamondsuit$.

The second statement of interest is $\square_\kappa$.

Definition 3.3.5 ($\Box_\kappa$ [3] (Definition 17.24))
For a cardinal κ, $\Box_\kappa$ is the statement: there is a sequence C_λ for a limit ordinal $\lambda < \kappa^+$ such that:

(i) C_λ is closed unbounded in λ,
(ii) if $cf(\lambda) < \kappa$ then $|C_\lambda| < \kappa$, and
(iii) if γ is a limit point of C_λ then $C_\gamma = \gamma \cap C_\lambda$.

Here being closed unbounded in λ means that we swap ω_1 with λ in the previous definition of club sets.

Lemma 3.3.6 ([3] (Lemma 17.25))
$K \models \Box_\kappa$ for all κ.

One reason we might care about $\diamondsuit$ is that it allows for easier constructions in topology. A question raised in 1951 by the mathematician C.H. Dowker was if there is a normal topological space X such that $X \times [0, 1]$ is not normal [4]. Such a space is called a *Dowker* space. It was conjectured that such a space could not exist, but this conjecture was proven false by M.E. Rudin who created a Dowker space in 1971 [8]. The only problem with the construction was that the space had cardinality $\aleph_\omega^{\aleph_0}$. Hence, it was very natural to ask whether there were Dowker spaces of a more manageable cardinality. This is where $\diamondsuit$ becomes useful, as it allows us to create Dowker spaces with a smaller cardinality. P. De Caux showed that if we assume $\diamondsuit$ then it is possible to create a Dowker space of cardinality $\aleph_1$ [2]. Hence, under the assumption of $\diamondsuit$, and so inside K, creating certain nice topological spaces is a lot easier due to the fine structure that is kept from L.

4 Conclusion and Further Work

To Infinity and Beyond!

– Buzz Lightyear, *Toy Story*

This thesis has served as an introduction to $0^\#$ and mice. The first part of the thesis aimed to show that if there is a measurable cardinal then $V \neq L$. From there, we have used the non-elementary embedding to create mice. This has allowed us to define the Core Model K as the union of all mice, which we have observed is very L-like in the sense that a lot of nice results about L are also true in K. An intuitive reason for why this is the case is that the mice iteration preserves a lot of the fine structure form L. As briefly sketched, this is because we can encode the mouse by using the projectum and when the mouse is n-suitable this is preserved into the next step in the iteration.

As one would expect, there are generalisations of the Core Model. One thing not covered in the main material is that although the Core Model has Ramsey cardinals, which L does not, there are no measurable cardinals in K [3]. Hence, there has been an endeavour to generalise the Core Model to allow for larger cardinals. For this to be done, the first thing we need to do is to generalise our definition of a core model. Instead of defining it through mice, it will be defined as a *canonical inner model* of the form $L[\mathbf{E}]$ where $\mathbf{E}$ is a *coherent sequence of extenders*. It is not terribly important what an extender is, but it generalises the idea of a non-principal ultrafilter. Depending on the choice of extender, we can create core models which allow for measurable cardinals. In fact, Steel created a core model that allows for Woodin cardinals, which are cardinals further up in the large cardinal hierarchy than measurable cardinals [10]. Hence, there is a lot of working being done in creating new core models that allows for larger cardinals whilst keeping as much as possible from the fine structure of L.

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