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Diagonalisation

The Road to Infinities, Truth and Gödel

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Summary: We situate Hilbert's program historically, introduce the provability predicate, prove Gödel's First Incompleteness Theorem (noting the second), and discuss their genuine and alleged philosophical consequences.

Quote

"We must not believe those, who today, with philosophical bearing and deliberative tone, prophesy the fall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion none whatever in natural science. In opposition to the foolish ignorabimus our slogan shall be: We must know. We will know."

— David Hilbert

Lecture 4 Summary

- Introduce Hilbert's Programme.
- Define the Provability Predicate.
- Prove Gödel's First Incompleteness Theorem.
- State Gödel's Second Incompleteness Theorem and show how it entails that a consistent system can never prove its own consistency.

Roadmap

Motivating Hilbert's Programme

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- Turn away from naive mathematics, make the language formal and all our assumptions explicit.
- Introduce formal languages, axiomatic system and a formal definition of a proof.
- If anything went wrong, we could pinpoint where it went wrong.

Hilbert's Programme

- Hilbert wanted a formal axiomatic system T such that we could prove that:
  - T is *consistent*.
  - T is *complete*.
  - In T, we should be able to make mathematics *decidable*.

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- To narrow the scope, he focused on arithmetic.
- Hilbert was a finitist, so the consistency proof had to be done in a finitist manner.
- That means that we would need a $T' \subset T$ where T' is "finitary" such that T' can prove that T is consistent.

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- Hilbert was a finitist, so the consistency proof had to be done in a finitist manner.
- That means that we would need a $T' \subset T$ where T' is "finitary" such that T' can prove that T is consistent.
- Today we will see that Gödel ruins the first two things on Hilbert's wish-list, the last was even more ambitious and does not even hold for first-order predicate logic.

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- Let $q = \ulcorner \# \psi_1 \# \ldots \# \psi_n \# \urcorner$, then q is *witnessing* the proof of φ.
- In other words, there is a $q \in \mathbb{N}$ such that q is *witnessing* the proof of φ.

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- In other words, there is a $q \in \mathbb{N}$ such that q is *witnessing* the proof of φ.
- This, however, is in the meta-language, we would like to say this in the object language.

Motivating the Provability Predicate pt.2

Two levels to keep separate.

- *Meta-level* (informal): "p is the code of a T-proof of the formula coded by q."
- *Object-level* (inside arithmetic): a *single formula of arithmetic* that is true of exactly those pairs (p, q) with that meta-property.

Motivating the Provability Predicate pt.3

Hence, we want a predicate $proof_T(p, q)$ such that for all $\mathcal{L}$-formulas φ:

$$T \vdash Proof_T(\overline{\ulcorner \varphi \urcorner}, \overline{q}) \text{ just in case } q \text{ is the Gödel code of a } T\text{-proof of } \varphi.$$

$$T \vdash \neg Proof_T(\overline{\ulcorner \varphi \urcorner}, \overline{q}) \text{ just in case } q \text{ is not the Gödel code of a } T\text{-proof of } \varphi.$$

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If we then let $Prov_T(p) = \exists x Proof_T(p, x)$, we then get for all $\mathcal{L}$-formulas φ:

$$T \vdash Prov_T(\overline{\ulcorner\varphi\urcorner}) \text{ just in case } T \vdash \varphi.$$

A Rough Construction of the Provability Predicate pt.1

- There is a $\mathcal{L}$-formula $Form_{\mathcal{L}}(x)$ such that $T \vdash Form_{\mathcal{L}}(\overline{p})$ just in case p is the Gödel code of a $\mathcal{L}$-formula.

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- Using $Form_{\mathcal{L}}(x)$ we can define the formula $proof\_form(x)$ such that $T \vdash proof\_form(\overline{q})$ just in case $q = \ulcorner \#\psi_1\# \ldots \#\psi_n\# \urcorner$ for some $\mathcal{L}$-formulas $\psi_1, \ldots, \psi_n$.

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- For q such that $T \vdash proof\_form(\overline{q})$, we can then also define the function $proof\_len$ and the formula $is\_in\_proof(p, q, i)$ such that:
  - $T \vdash proof\_len(\overline{q}) = \overline{n}$ just in case the length of the proof that q is the Gödel code for is n, and
  - $T \vdash is\_in\_proof(\overline{p}, \overline{q}, \overline{i})$ just in case $q = \ulcorner \# \psi_1 \# \ldots \# \psi_n \# \urcorner$ and $p = \ulcorner \psi_i \urcorner$ for $i \leq n$.

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- We can also define $is\_axiom(x)$ such that $T \vdash is\_axiom_T(\overline{p})$ just in case p is the Gödel code of a T-axiom.

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- We can also define $is\_axiom(x)$ such that $T \vdash is\_axiom_T(\overline{p})$ just in case p is the Gödel code of a T-axiom.
- Similarly, we can define $inference\_rule(x, y, z)$ such that $T \vdash inference\_rule(\overline{p}, \overline{q}, \overline{r})$ just in case either:
  - $p = \ulcorner \varphi \to \psi \urcorner$, $p = \ulcorner \varphi \urcorner$ and $r = \ulcorner \psi \urcorner$ for some $\mathcal{L}$-formulas, or
  - $p = q = \ulcorner \varphi \urcorner$ and $r = \ulcorner \forall v_i \varphi \urcorner$ for a $\mathcal{L}$-formula φ.

A Rough Construction of the Provability Predicate pt.2

We can now formally verify that q is the Gödel code of a proof. Let $is\_proof(q)$ be defined as

$$
\begin{aligned}
&proof\_form(q) \wedge \forall x \leq q \forall i \leq proof\_len(q)(is\_in\_proof(x, p, i) \rightarrow \\
&\qquad (is\_axiom_T(x) \vee \\
&\exists y \leq q \exists z \leq q \exists j < i \exists k < i(is\_in\_proof(y, p, j) \wedge is\_in\_proof(z, p, k) \wedge inference\_rule(y, z, x)))).
\end{aligned}
$$

A Rough Construction of the Provability Predicate pt.3

- Define $Proof_T(p, q) = is\_proof(q) \wedge is\_in\_proof(p, q, proof\_len(q))$.

A Rough Construction of the Provability Predicate pt.3

- Define $Proof_T(p, q) = is\_proof(q) \wedge is\_in\_proof(p, q, proof\_len(q))$.
- Then define $Prov_T(p) = \exists q Proof_T(p, q)$.

Roadmap

The Gödel Sentence

- Diagonal lemma yields G_T with

$$G_T \longleftrightarrow \neg Prov_T(\overline{\ulcorner G_T \urcorner}).$$

The Gödel Sentence

- Diagonal lemma yields G_T with

$$G_T \longleftrightarrow \neg Prov_T(\overline{\ulcorner G_T \urcorner}).$$

- G_T asserts of itself that it is not provable in T.
  - $T \vdash G_T$ just in case $T \vdash Prov_T(\overline{\ulcorner G_T \urcorner})$.

Why $T \nvdash G_T$ (consistency)

- Suppose for contradiction that $T \vdash G_T$, then there is a proof $\varphi_1, \ldots, \varphi_n$ of G_T.

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- Hence $T \vdash \mathit{proof}_T(\overline{\ulcorner G_T \urcorner}, \overline{\ulcorner \# \varphi_1 \# \ldots \# \varphi_n \# \urcorner})$ and so $T \vdash \mathit{Prov}_T(\overline{\ulcorner G_T \urcorner})$.

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- But $T \vdash G_T \leftrightarrow \neg \mathit{Prov}_T(\overline{\ulcorner G_T \urcorner})$, so from $T \vdash G_T$ one deduces $T \vdash \neg \mathit{Prov}_T(\overline{\ulcorner G_T \urcorner})$.

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- Hence contradiction; consistency of T implies $T \nvdash G_T$.

ω-consistency

Definition (ω-inconsistency)

A theory T is ω-inconsistent just in case there is a formula $\varphi(v_i)$ such that $T \vdash \exists x \varphi(x)$ but for any numeral $\overline{k}$ we have that $T \vdash \neg\varphi(\overline{k})$.

We say that a theory is ω-*consistent* just in case it is not ω-inconsistent. Note that if T is ω-consistent, then it is consistent.

Why $T \nvdash \neg G_T$ (ω-consistency) pt.1

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  - And so $T \vdash G_T$, which contradicts $T \vdash \neg G_T$ and the consistency assumption.
  - This means that k is *not* the Gödel code of a T-proof of G_T, and so $T \vdash \neg Proof_T(\overline{\ulcorner G_T \urcorner}, \overline{k})$

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- Fix a numeral $\overline{k}$ and suppose that T proves $Proof_T(\overline{\ulcorner G_T \urcorner}, \overline{k})$.
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  - And so $T \vdash G_T$, which contradicts $T \vdash \neg G_T$ and the consistency assumption.
  - This means that k is *not* the Gödel code of a T-proof of G_T, and so $T \vdash \neg Proof_T(\overline{\ulcorner G_T \urcorner}, \overline{k})$
- Hence, for any numeral $\overline{k}$ we have that T proves $\neg Proof_T(\overline{\ulcorner G_T \urcorner}, \overline{k})$
- This contradicts T being ω-consistent, and so T cannot prove $\neg G_T$.

Rosser's improvement (only consistency)

- Rosser constructs R_T so that mere consistency (not ω-consistency) suffices to show T neither proves nor refutes R_T.
- Idea: compare lengths of purported proofs and refutations; the Rosser predicate prevents the "phantom witness" maneuver.

Gödel's First Incompleteness Theorem

Theorem (Gödel's First Incompleteness Theorem)

Let T be effectively axiomatized, sufficiently strong to express elementary arithmetic, and consistent. Then T is incomplete: there is a sentence (for instance G_T under ω-consistency, or Rosser's R_T under mere consistency) that T neither proves nor refutes.

The Truth of G_T

- The Gödel sentence is often called a true but unprovable sentence. The truth of G_T is often attributed to the fact that G_T is true in $\mathbb{N}$.
- Strengthening T can make some unprovable sentences provable, but new Gödel sentences appear.
- No single recursively enumerable T captures all arithmetical truth.

Roadmap

Hilbert's Second Hope

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- Let $Con(T) = \neg Prov_T(\overline{\ulcorner 1 = 0 \urcorner})$, then Hilbert's dream is that $T' \vdash Con(T)$.

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- With the provability predicate, we can formalise this. If T is consistent, then $T \nvdash 1 = 0$. Hence, $\neg Prov_T(\overline{\ulcorner 1 = 0 \urcorner})$ must hold.
- Let $Con(T) = \neg Prov_T(\overline{\ulcorner 1 = 0 \urcorner})$, then Hilbert's dream is that $T' \vdash Con(T)$.
- Gödel's Second Incompleteness Theorem (roughly): $T \nvdash Con(T)$.

Gödel's Second Incompleteness Theorem

Theorem (Gödel's Second Incompleteness Theorem)

Let T be a consistent effectively axiomatized theory of Arithmetic with its own Gödel sentence G_T, then for any $\mathcal{L}$-sentence φ, $T \vdash \neg Prov_T(\overline{\ulcorner \varphi \urcorner}) \rightarrow \neg Prov_T(\overline{\ulcorner G_T \urcorner})$.

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- This means that $T \vdash G_T$ by $T \vdash G_T \leftrightarrow \neg Prov_T(\overline{\ulcorner G_T \urcorner})$.

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- This means that $T \vdash G_T$ by $T \vdash G_T \leftrightarrow \neg Prov_T(\overline{\ulcorner G_T \urcorner})$.
- However, this contradicts Gödel's First Incompleteness Theorem, and so $T \nvdash Con(T)$.

Consequences and philosophical remarks

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  - $ZF \vdash Con(PA)$.
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- It is, however, not the case that the consistency of *PA* cannot be proven. You just cannot do it from *PA* itself (assuming *PA* is consistent).
    - $ZF \vdash Con(PA)$.
    - Gentzen gave a consistency proof using only the assumption that the ordinal ϵ_0 exists.
- So it is not the case that we have no reason to believe in the consistency of *PA*, we just cannot show if from finitistic means.

Concluding remarks

- We have used diagonalisation to prove different important results, both informally (Cantor and Russell) and formally (Tarski and Gödel).
- We leave you with the following unanswered question: are there true unprovable statements about arithmetic?