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Diagonalisation

Infinities, Truth, Gödel's Theorems

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Lecture Series Overview: Lectures 1 and 2

- **Lecture 1: Infinities.**
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  We'll show that rigorous formal theories can, in a precise and controlled sense, "talk about themselves". No tricks of language, no ambiguity. Just maths.

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  We discuss their implications for the scope and limits of mathematical theories and mathematical reasoning.

Lecture 1 Summary

- Introduce number via *matching* (one-to-one correspondence) and *cardinality*.
- Show that $\mathbb{N}$, the evens, the integers, and the rationals are all the same size (*countably infinite*).
- Present Cantor's diagonal argument: the reals are *uncountable*.
- Explain Russell's paradox and why set formation needs rules.

Paradise

No one shall expel us from the paradise that Cantor has created for us.

— David Hilbert

Roadmap

What is a number?

Names like *twenty-five*, *25*, or *XXV* label the same thing, but names alone don't explain the concept of number.[1]

A more basic route: matching. Lay out two collections in parallel and pair items perfectly. If this can be done with no leftovers, they are the *same size*.

[1] Try computing XXV × XXXII without converting to Arabic numerals. Some names are handier than others!

Same size = one-to-one correspondence

Sameness of size

Two collections A and B are the same size if their members can be paired up perfectly (*bijection*). No item is paired twice; none is left out.

Definitions

- **Bijection**: perfect pairing.
- **Equinumerous**: $A \sim B$ iff a bijection exists.
- **Cardinality**: the *size* revealed by matching.

Checking that equinumerosity behaves like equivalence

Is equinumerosity a good way to talk about equivalent size? Well, equivalence famously has three properties. Let's check that equinumerosity respects these three properties:

- **Reflexive**: Any collection matches itself (pair each item to itself).
- **Symmetric**: If A matches B, then B matches A (reverse the pairings).
- **Transitive**: If A matches B and B matches C, then A matches C (follow the lines in two hops).

Numbers as sizes (cardinalities)

If a collection matches a finite list $\{1\}$, $\{1, 2\}, \ldots, \{1, \ldots, n\}$, we say it has cardinality $1, 2, \ldots, n$.

The numeral "3" is a convenient *label* for the common size of all collections equinumerous with $\{1, 2, 3\}$. We write that $|\{1, 2, 3\}| = 3$. Ultimately, numbers emerge from matching.

Roadmap

Finite vs. Infinite

Infinite collection

A collection is *infinite* if it cannot be matched with any finite list $\{1, 2, \ldots, n\}$.

We will now see that an infinite collection may be matched with a *proper part* of itself. This is impossible for finite collections.

Evens are the same size as naturals

Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $E = \{0, 2, 4, 6, \dots\}$. Pair each natural with its double:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 2 & 4 & 6 & 8 & \cdots \end{array}$$

No leftovers and no repeats $\Rightarrow \mathbb{N} \sim E$.

Integers are the same size as naturals

List integers by "zig–zagging" from 0:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

Every integer appears exactly once $\Rightarrow \mathbb{N} \sim \mathbb{Z}$.

Rationals are the same size as naturals

A rational is a reduced fraction p/q with $p \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$. Arrange as a grid and sweep along diagonals, skipping repeats (e.g. $2/2 = 1/1$):

$$\begin{array}{cccccc}
\frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\frac{0}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Hence $\mathbb{Q} \sim \mathbb{N}$.

Countable infinity

Definition (Countably infinite)

A collection is *countably infinite* if its elements can be listed in a single infinite line: first, second, third, ...

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Theorem

We have $\mathbb{N} \sim E \sim \mathbb{Z} \sim \mathbb{Q}$. In other words, the integers and the rational numbers are both countably infinite.

Roadmap

Decimal expansions in $(0,1)$

Every real number $x \in (0,1)$ has a decimal form $x = 0.d_1 d_2 d_3 \ldots$, with digits $d_k \in \{0, \ldots, 9\}$.

A few have two forms (e.g. $0.5000\ldots = 0.4999\ldots$); to avoid ambiguity, agree *never* to use the endlessly repeating-9 form.

The reals are not countable

Theorem (Cantor)

The real numbers in $(0,1)$ cannot be listed as $x_1, x_2, x_3, \ldots$. Equivalently, $(0,1)$ is uncountable.

Proof idea: build an anti-diagonal

Assume for contradiction we have a list with decimals $x_i = 0.a_{i1}a_{i2}a_{i3}\ldots$.

	1st	2nd	3rd	4th	5th	6th	$\cdots$
x_1	$\underline{a_{11}}$	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
x_2	a_{21}	$\underline{a_{22}}$	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$
x_3	a_{31}	a_{32}	$\underline{a_{33}}$	a_{34}	a_{35}	a_{36}	$\cdots$
x_4	a_{41}	a_{42}	a_{43}	$\underline{a_{44}}$	a_{45}	a_{46}	$\cdots$
x_5	a_{51}	a_{52}	a_{53}	a_{54}	$\underline{a_{55}}$	a_{56}	$\cdots$
x_6	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	$\underline{a_{66}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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x_4	a_{41}	a_{42}	a_{43}	$\underline{a_{44}}$	a_{45}	a_{46}	$\cdots$
x_5	a_{51}	a_{52}	a_{53}	a_{54}	$\underline{a_{55}}$	a_{56}	$\cdots$
x_6	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	$\underline{a_{66}}$	$\cdots$
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Build $y = 0.b_1b_2b_3\ldots$ by changing the diagonal digits: set

$$b_n = \begin{cases} 5, & a_{nn} \neq 5, \\ 4, & a_{nn} = 5. \end{cases}$$

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$$b_n = \begin{cases} 5, & a_{nn} \neq 5, \\ 4, & a_{nn} = 5. \end{cases}$$

Then y differs from x_n in the n-th place for every n, so y is not on the list—contradiction.

Consequences of diagonalization

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \quad \text{but} \quad |\mathbb{R}| \text{ is strictly larger.}$$

No list can capture all reals; the diagonal construction always finds a missing one.

Roadmap

Unrestricted set formation is dangerous

Late 19th/early 20th century: widespread use of an informal comprehension principle—"for any clear condition, there is a set of all things satisfying it."

Russell's paradox

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There is no set containing all sets.

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Either way: contradiction.

$\square$

Russell's Paradox - Corollary

Corollary

It is not the case that for any clear condition there is a set satisfying it.

Aftermath and the modern view

- Frege's attempt to reconstruct arithmetic from logic hindered. Russell proposed type-theoretic restrictions; later developed with Whitehead in *Principia Mathematica*.
- Modern set theories (e.g. Zermelo–Fraenkel, ZF/ZFC) keep fruitful set talk while restricting formation to block paradoxes.

Takeaways

- Number can be grounded in everyday *matching*: ‘same number’ means ‘a perfect pairing is possible’.
- Infinite collections can match proper parts; finite collections cannot.
- Many familiar infinities ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}$) are the same size: countably infinite.
- Cantor’s diagonal argument shows $\mathbb{R}$ is strictly larger than $\mathbb{N}$.
- Russell’s paradox warns that not every condition defines a legitimate set; rules are required.