

Diagonalisation - The Road to Infinities, Truth and Gödel

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12 November 2025

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Lecture I: Infinities

Summary: We introduce number, one-to-one correspondence, and cardinality. We show how infinite sets like the naturals, evens, and rationals can be the same size. We present Cantor's diagonal argument and Russell's paradox.

No one shall expel us from the paradise that Cantor has created for us.

– David Hilbert

It is not obvious how to explain the concept of number to someone who doesn't already know what numbers are.

You know what the *names* of the numbers are. You also know that there can be many names for the same number: the Roman 'XXV' is another name for the Arabic '25'.¹ We claim that imparting these names is not the same as imparting the concept of number. If we are asked "What is a planet?" and we reply with a list of names for planets, we have not been of much help unless the named entities were already clearly understood. Maybe we are allowed to assume that our questioner has a good grasp of 'Earth', 'Mars', 'Venus', etc. — and we can answer "What is a planet?" by saying "Earth and Mars and Venus and ... are planets". But if our questioner has no such grasp of the named entities, we must provide a more fundamental explanation than this list of names.

Is any such fundamental explanation possible for the concept of number?

Yes. A simple idea does much of the work: *matching*. Imagine laying out two collections in parallel and pairing each item from the first collection with exactly one item from the second, leaving nothing unmatched on either side. If this can be done, we say the two collections are the *same size*. This way of thinking needs no numerals at all. If we have a given collection of sheep and a given collection of trees, and we can tie exactly one sheep to one tree with no leftover trees or sheep, then there are as many sheep as there are trees.

I.1: Number via matching

Same size (or “one-to-one correspondence”): Two collections A and B are of equivalent size if their members can be paired up perfectly: each member of A goes with

¹ Not all names for the numbers are equally useable: try calculating $XXV \times XXXII$ by hand without first converting the Roman numerals into Arabic. In this lecture course you will learn some unusual new names for the numbers; these names will have their uses.

one and only one member of B , and each member of B goes with one and only one member of A . No leftovers.

Here is how we can check that this behaves like an honest notion of equivalence of size. Equivalence is famously characterised by the three properties (reflexivity, symmetry, transitivity) which are all met:

- **Reflexivity:** Any collection can be matched with itself (pair each item to itself). Obvious, but reassuring.
- **Symmetry:** If A matches B , then B matches A (just reverse the pairings).
- **Transitivity:** If A matches B and B matches C , then A matches C (follow the lines in two hops).

We now introduce some technical terms for what we just discussed. What we have referred to as a perfect pairing is what mathematicians call a *bijection*. When a perfect pairing exists between two collections, mathematicians say that they are *equinumerous*.

DEFINITION I.1 (BIJECTION)

A bijection between two collections A and B is simply a perfect pairing between their members: a way to match each item of A with exactly one item of B , and each item of B with exactly one item of A . No item is left unmatched, and no item is used twice. When such a pairing exists, we say that A and B are *in bijection* (alternatively: *in one-to-one correspondence*).

DEFINITION I.2 (EQUINUMEROSITY)

Two collections A and B are equinumerous if there is a perfect pairing between them: each item of A is matched with exactly one item of B , and each item of B is matched with exactly one item of A . No leftovers and no repeats. (Notation: $A \sim B$)

With these two concepts defined, we are closer to exhibiting the concept of number without any vicious circularity. The trick is to deploy this notion of *size* that is revealed by our practice of matching. Clearly, not all collections are of the same size. So there are different sizes. We might then explain—to some entity that has no prior conception of numbers—that numbers are simply the different sizes that there can be. The technical term for size is *cardinality*.

When a collection can be matched with a finite list like

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \text{ and so on,}$$

we say it has a cardinality of 1, or 2, or 3, and so on.

In this usage, the numeral “3” is just a widely agreed label for the cardinality that all collections equinumerous with the set $\{1, 2, 3\}$. Let’s call “3” a cardinal number. The label is a convenience and we do not depend on it to explain *three-ness*; likewise for every other possible cardinal number. They emerge, ultimately, from the idea of matching.

I.2: First encounters with infinity

A collection is *infinite* if it cannot be matched with any finished list $\{1, 2, \dots, n\}$. One striking sign of being infinite is this: an infinite collection can sometimes be matched with a proper *part* of itself. That would never happen for a finite collection, but for infinite ones it can—and will.

Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the natural numbers. This is an infinite collection. We will meet three surprises.

I.2.1: The even naturals are the same size as the naturals

Let $E = \{0, 2, 4, 6, \dots\}$ be the even numbers. The even numbers, of course, are wholly contained within the collection of natural numbers. Now here is a perfect pairing:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 2 & 4 & 6 & 8 & \dots \end{array}$$

Each natural number is paired with its double. No even number is left out, and no two naturals share the same even partner. So \mathbb{N} and E are the same size. This already shows infinite size behaves unlike finite size: a whole can be the same size as a proper part. **The even naturals have the same cardinality as the naturals.**

I.2.2: The integers are the same size as the naturals

What if we compared the natural numbers to the collection of integers (i.e. positive *and* negative numbers)? The integers might seem to be ‘double’ the size of the naturals. Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. And yet it turns out that we can pair \mathbb{N} with \mathbb{Z} by “zig–zagging” out from 0:

$$\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

Every integer appears exactly once, so again the sizes match. **The integers have the same cardinality as the naturals.**

I.2.3: The rationals are the same size as the naturals

A rational number is a fraction p/q where p is an integer and q is a positive whole number, written in lowest terms (so $1/2$ rather than $2/4$). To see that the rationals can be listed in a single line like the naturals, imagine a grid:

$$\begin{array}{ccccccccc} \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \cdots \\ \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \cdots \\ \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Now sweep through this grid along diagonals (first the short diagonal, then the next, and so on), and *skip repeats* like $2/2$ (which is the same as $1/1$). In this way you eventually meet every rational number, exactly once. So the rationals are the same size as the naturals, despite feeling “denser”.

In each of the three demonstrations above, we took an infinite collection and showed that it actually has the same cardinality as the natural numbers by arranging every element of the set in a single infinite line that mimics the natural number line. We now introduce the technical term for such collections:

DEFINITION I.3 (COUNTABLE INFINITY)

Any collection that can be listed in a single infinite line—first, second, third, and

so on—is called *countably infinite*. We have just seen that \mathbb{N} , the even numbers in \mathbb{N} , \mathbb{Z} (the integers), and \mathbb{Q} (the rationals) are all countably infinite.

I.3: Cantor's diagonal argument: there are bigger infinities

So far, various infinite sets have turned out to be countably infinite—that is, to have the same cardinality as the natural numbers. We might wonder at this point: do *all* infinities have this cardinality?

They do not. Cantor's great discovery was that there are *different sizes* of infinity. In particular, the real numbers—the points on a line—form a larger infinity than the natural numbers.

Decimal expansions. Every real number between 0 and 1 has a decimal form

$$x = 0.d_1d_2d_3\dots, \quad d_k \in \{0, 1, \dots, 9\}.$$

A few numbers have two forms (e.g. $0.5000\dots = 0.4999\dots$). To avoid ambiguity, we agree to *never* use the version that ends with endless 9s.

THEOREM I.4 (CANTOR'S THEOREM)

The real numbers between 0 and 1 are not listable as x_1, x_2, x_3, \dots ; i.e. they are *uncountable*.

PROOF Imagine, for contradiction, that we *have* listed all reals in $[0, 1]$:

$$x_1, \ x_2, \ x_3, \ \dots$$

Write their decimals as $x_i = 0.a_{i1}a_{i2}a_{i3}\dots$ The underlined entries mark the *diagonal* positions $a_{11}, a_{22}, \dots, a_{66}$.

	1st digit	2nd digit	3rd digit	4th digit	5th digit	6th digit	...
x_1	<u>a_{11}</u>	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	...
x_2	a_{21}	<u>a_{22}</u>	a_{23}	a_{24}	a_{25}	a_{26}	...
x_3	a_{31}	a_{32}	<u>a_{33}</u>	a_{34}	a_{35}	a_{36}	...
x_4	a_{41}	a_{42}	a_{43}	<u>a_{44}</u>	a_{45}	a_{46}	...
x_5	a_{51}	a_{52}	a_{53}	a_{54}	<u>a_{55}</u>	a_{56}	...
x_6	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	<u>a_{66}</u>	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...

Building the “anti-diagonal” number. Define a new decimal

$$y = 0.b_1 b_2 b_3 b_4 b_5 b_6 \dots$$

by *changing each underlined diagonal digit*:

$$b_n = \begin{cases} 5 & \text{if } a_{nn} \neq 5, \\ 4 & \text{if } a_{nn} = 5. \end{cases}$$

Thus $b_n \neq a_{nn}$ for every n . Visually: b_1 disagrees with the $\underline{a_{11}}$, b_2 disagrees with the $\underline{a_{22}}$, \dots , b_6 disagrees with the $\underline{a_{66}}$, and so on down the diagonal.

Because y differs from x_n in its n -th digit, we have $y \neq x_n$ for *every* n . Yet y is still a real number in $[0, 1]$. So the list was not complete—a contradiction.

Takeaway. No list can capture all reals. The diagonal construction always finds a missing one, so

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \quad \text{but} \quad |\mathbb{R}| \text{ is strictly larger.}$$

I.4: Russell’s paradox and why we need rules for forming sets

A tempting but dangerous idea is: “For any clear condition, there is a set of all things that satisfy it.” Russell showed that this sweeping principle cannot be right.

In the late 19th century, many mathematicians and philosophers hoped to rebuild all of arithmetic on purely logical foundations (the “logicist” project). Cantor had just opened the door to talking rigorously about infinite collections, and people freely used an informal comprehension principle: for any sensible condition, there is a set of all things satisfying it. Warning signs appeared (for example, Burali–Forti’s 1897 observation that the “set of all ordinals” leads to trouble), but the principle still guided much routine reasoning.

Russell noticed in 1901 that the comprehension principle, taken at face value, collapses on itself.

Russell’s paradox. Consider the collection R of all collections that are *not* members of themselves. Is R a member of itself?

If you answer “yes”, then by definition R should *not* be a member of itself. If you answer “no”, then by definition R *is* a member of itself. Either way, contradiction.

A more homely version is the “barber” story: in a certain town, the barber shaves all

and only those who do not shave themselves. Does the barber shave himself? The story ties itself in a knot for the very same reason.

In 1902 Russell wrote to Frege about the problem, just as Frege was completing the second volume of his *Grundgesetze*; Frege added an appendix acknowledging that the paradox undermined his system. Russell's own response was to propose restrictions on formation of collections (type theory) and, later with Whitehead, to develop these ideas in *Principia Mathematica*.

The lesson of Russell's Paradox is that we must be careful about what we allow as a “set.” Modern set theories (such as the Zermelo–Fraenkel system) keep the fruitful parts of set talk while placing modest restrictions on how new sets may be formed. Within such systems, all the results above about sizes of infinity go through cleanly, and the paradoxes are blocked.

I.5: Takeaways

- The basic idea of number can be grounded in everyday *matching*: same number means a perfect pairing is possible.
- Infinite collections can match some of their own proper parts; this never happens with finite ones.
- Many familiar infinities (natural numbers, whole integers, rational numbers) are the same “listable” size: countably infinite.
- Cantor’s diagonal method shows that the real numbers form a strictly larger infinity: they cannot be captured by any list.
- Russell’s paradox warns that not every condition defines a legitimate set; some discipline is required.

References