Chapter 1

Functions

1.1 Basics

A *function* is a map which sends each element of a given set to a specific element in some (other) given set. For instance, the operation of adding 1 defines a function: each number n is mapped to a unique number $n + 1$.

More generally, functions may take pairs, triples, etc., as inputs and returns some kind of output. Many functions are familiar to us from basic arithmetic. For instance, addition and multiplication are functions. They take in two numbers and return a third.

In this mathematical, abstract sense, a function is a *black box*: what matters is only what output is paired with what input, not the method for calculating the output.

Definition 1.1 (Function). A *function* $f\colon A \to B$ is a mapping of each element of A to an element of B.

We call A the *domain* of f and B the *codomain* of f. The elements of A are called inputs or *arguments* of f, and the element of B that is paired with an argument x by f is called the *value of f* for argument x, written $f(x)$.

The *range* $\operatorname{Range}(f)$ of f is the subset of the codomain consisting of the values of f for some argument; $\operatorname{Range}(f) = \{f(x) : x \in A\}$.

The diagram in Figure 1.1 may help to think about functions. The ellipse on the left represents the function's *domain*; the ellipse on the right represents the function's *codomain*; and an arrow points from an *argument* in the domain to the corresponding *value* in the codomain.

Example 1.2. Multiplication takes pairs of natural numbers as inputs and maps them to natural numbers as outputs, so goes from $\mathbb{N} \times \mathbb{N}$ (the domain) to $\mathbb{N}$ (the codomain). As it turns out, the range is also $\mathbb{N}$, since every $n \in \mathbb{N}$ is $n \times 1$.

Example 1.3. Multiplication is a function because it pairs each input—each pair of natural numbers—with a single output: $\times\colon \mathbb{N}^2 \to \mathbb{N}$. By contrast, the square root

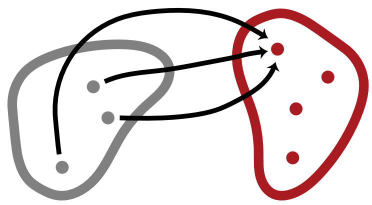

Figure 1.1: A function is a mapping of each element of one set to an element of another. An arrow points from an argument in the domain to the corresponding value in the codomain.

operation applied to the domain $\mathbb{N}$ is not functional, since each positive integer n has two square roots: $\sqrt{n}$ and $-\sqrt{n}$. We can make it functional by only returning the positive square root: $\sqrt{\ } : \mathbb{N} \to \mathbb{R}$.

Example 1.4. The relation that pairs each student in a class with their final grade is a function—no student can get two different final grades in the same class. The relation that pairs each student in a class with their parents is not a function: students can have zero, or two, or more parents.

We can define functions by specifying in some precise way what the value of the function is for every possible argment. Different ways of doing this are by giving a formula, describing a method for computing the value, or listing the values for each argument. However functions are defined, we must make sure that for each argment we specify one, and only one, value.

Example 1.5. Let $f : \mathbb{N} \to \mathbb{N}$ be defined such that $f(x) = x + 1$. This is a definition that specifies f as a function which takes in natural numbers and outputs natural numbers. It tells us that, given a natural number x, f will output its successor $x + 1$. In this case, the codomain $\mathbb{N}$ is not the range of f, since the natural number 0 is not the successor of any natural number. The range of f is the set of all positive integers, $\mathbb{Z}^+$.

Example 1.6. Let $g : \mathbb{N} \to \mathbb{N}$ be defined such that $g(x) = x + 2 - 1$. This tells us that g is a function which takes in natural numbers and outputs natural numbers. Given a natural number n, g will output the predecessor of the successor of the successor of x, i.e., $x + 1$.

We just considered two functions, f and g, with different *definitions*. However, these are the *same function*. After all, for any natural number n, we have that $f(n) = n + 1 = n + 2 - 1 = g(n)$. Otherwise put: our definitions for f and g specify the same mapping by means of different equations. Implicitly, then, we are relying upon a principle of extensionality for functions,

$$\text{if } \forall x\, f(x) = g(x), \text{ then } f = g$$

provided that f and g share the same domain and codomain.

Example 1.7. We can also define functions by cases. For instance, we could define $h : \mathbb{N} \to \mathbb{N}$ by

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

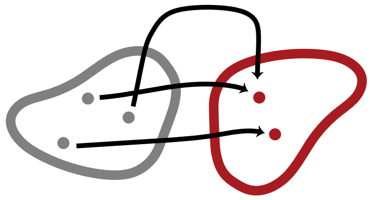

Figure 1.2: A onto function has every element of the codomain as a value.

Figure 1.3: A one-one function never maps two different arguments to the same value.

Since every natural number is either even or odd, the output of this function will always be a natural number. Just remember that if you define a function by cases, every possible input must fall into exactly one case. In some cases, this will require a proof that the cases are exhaustive and exclusive.

1.2 Kinds of Functions

It will be useful to introduce a kind of taxonomy for some of the kinds of functions which we encounter most frequently.

To start, we might want to consider functions which have the property that every member of the codomain is a value of the function. Such functions are called onto, and can be pictured as in Figure 1.2.

Definition 1.8 (Onto function). A function $f : A \to B$ is *onto* iff B is also the range of f, i.e., for every $y \in B$ there is at least one $x \in A$ such that $f(x) = y$, or in symbols:

$$(\forall y \in B)(\exists x \in A) f(x) = y.$$

We call such a function a surjection from A to B.

If you want to show that f is a surjection, then you need to show that every object in f's codomain is the value of $f(x)$ for some input x.

Note that any function *induces* a surjection. After all, given a function $f : A \to B$, let $f' : A \to \text{Range}(f)$ be defined by $f'(x) = f(x)$. Since $\text{Range}(f)$ is *defined* as $\{f(x) \in B : x \in A\}$, this function f' is guaranteed to be a surjection

Now, any function maps each possible input to a unique output. But there are also functions which never map different inputs to the same outputs. Such functions are called one-one, and can be pictured as in Figure 1.3.

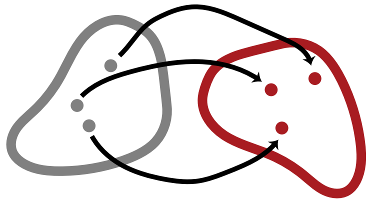

Figure 1.4: A bijective function uniquely pairs the elements of the codomain with those of the domain.

Definition 1.9 (One-one function). A function $f\colon A \to B$ is *one-one* iff for each $y \in B$ there is at most one $x \in A$ such that $f(x) = y$. We call such a function an injection from A to B.

If you want to show that f is an injection, you need to show that for any elements x and y of f's domain, if $f(x) = f(y)$, then $x = y$.

Example 1.10. The constant function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = 1$ is neither one-one, nor onto.

The identity function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x$ is both one-one and onto.

The successor function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$ is one-one but not onto.

The function $f\colon \mathbb{N} \to \mathbb{N}$ defined by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

is onto, but not one-one.

Often enough, we want to consider functions which are both one-one and onto. We call such functions bijective. They look like the function pictured in Figure 1.4. Bijections are also sometimes called *one-to-one correspondences*, since they uniquely pair elements of the codomain with elements of the domain.

Definition 1.11 (Bijection). A function $f\colon A \to B$ is *bijective* iff it is both onto and one-one. We call such a function a bijection from A to B (or between A and B).