Classics in Magnetics

A Phenomenological Theory of Damping in Ferromagnetic Materials

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***Abstract*—In 1955, a phenomenological theory of ferromagnetism was well established and had been corroborated by a considerable amount of experimental data. However, there were problems in the phenomenological theory of the dynamics of the magnetization field. The Landau–Lifshitz equation for damping of the motion of the magnetization field could not account for the large noneddy-current damping in thin Permalloy sheets. The problem undertaken herein is a reformulation of the theory in a way that is more consistent with the theory of damping in other physical systems in order to be able to take large damping into account.**

***Index Terms*—Ferromagnetic damping, ferromagnetic materials, magnetic core memories, magnetic domains, magnetic losses, magnetic recording, magnetization processes.**

I. Introduction

TECHNOLOGICAL problems of interest for magnetic materials include reducing energy losses due to damping and developing materials with higher rates of remagnetization for weak driving fields. The latter problem was of particular interest in the 1950s because magnetic materials were used for random access memories in electronic computers. The rate of remagnetization was a major limiting factor for the computation rate.

The rate of remagnetization in a ferromagnet is determined by damping mechanisms. Damping involves loss of energy from the macroscopic motion of the local magnetization field by transfer of energy to microscopic thermal motion.[1] The mechanisms by which this occurs include coupling of the magnetization field to spin waves, eddy currents, and lattice vibrations, and the effects of polycrystalline structure, strains, and crystal defects such as voids, interstitial atoms, and "foreign" atoms. When the external magnetic field is not strong enough to eliminate all domain walls, the domain structure plays a dominant role in the damping, and the local rate of energy loss may vary by large amounts from point to point within a ferromagnet.

As of 1955, a number of different damping mechanisms had been studied [1]–[4], [6]–[18], [21]–[25], [27], [28], but (except for eddy currents in metallic ferromagnets) the dominant mechanisms had not yet been identified. Gaining an understanding of the different complex damping mechanisms requires a combination of theoretical calculations and experimental measurements. In order to identify and understand the dominant damping mechanisms, it is useful to have a phenomenological theory with parameters that provide a quantitative measure of the rate of energy loss and a summary means for comparing experimental data with theoretical calculations for different materials under different experimental conditions.

The Landau–Lifshitz phenomenological damping term in common use in 1955 [2] could be used when the damping was small, but encountered problems for large damping. The purpose of the research on which this paper is based was to develop a phenomenological theory of damping that could be used for measurements of rotational damping in thin ferromagnetic sheets, for which anomalous damping (damping that could not be attributed to eddy currents) was quite large [28].

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[1]The local magnetization field $\mathbf{M}(\mathbf{r})$, as defined herein for a phenomenological theory of damping in ferromagnetic materials, is the expectation value of the magnetic moment per unit volume due to the spins and orbital motion of unpaired electrons averaged over a few lattice cells. (The contribution from orbital motion is ignored in the theory and taken into account by adjusting the gyromagnetic ratio.) The number of cells must be large enough to smooth out microscopic fluctuations in the magnitude and orientation of the magnetization field at an atomic level but small enough to avoid smearing out the complicated domain structure. The local magnetization field should be distinguished from the average magnetization, $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$, used in the classical theory of magnetic materials in which the magnetization field used in Maxwell's equations is usually obtained by averaging the local magnetization field over the entire volume of a ferromagnetic object. Other averaging volumes could be useful for studying specific phenomena.

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The starting point is the phenomenological theory of an undamped and uncoupled magnetization field $\mathbf{M}(\mathbf{r})$[2]. The basis of this theory was given by Bloch [1]. Phenomenological equations of motion for a damped magnetization field were first formulated by Landau and Lifshitz [2]. The phenomenological theory for an undamped and uncoupled magnetization field is based on a classical variational procedure in which the equations of motion are derived from an expression for the potential energy of the magnetization field called the "domain energy." The phenomenological equations for an uncoupled and undamped magnetization field are reformulated in terms of functional analysis in Section II and used as the starting point, in Section III, for deriving a phenomenological equation of motion with a damping term that can be used when the damping is large.

II. Field Equations for an Undamped Magnetization Field

The equation for the rotational motion of a rigid body in classical mechanics is

$$d\mathbf{L}/dt = \mathbf{T} \tag{1}$$

where $\mathbf{L}$ is the angular momentum of the body and $\mathbf{T}$ is the torque acting on it. The angular momentum is defined as $\mathbf{L} = \mathbf{I}\cdot\boldsymbol{\omega}$, where $\mathbf{I} = [\mathbf{I_{ij}}]$ is the inertia tensor and $\boldsymbol{\omega} = [\omega_i]$ is a vector along the axis of rotation. The magnitude of $\boldsymbol{\omega}$ is $\omega = d\psi/dt$, where ψ is the angle between two lines perpendicular to the axis, one fixed in the body and the other fixed in space.

Equation (1) remains valid in quantum mechanics when $\mathbf{L}$ and $\mathbf{T}$ are reinterpreted as operators in a Hilbert space, and can be used for spin systems by replacing the angular momentum operator $\mathbf{L}$ with the operator $\mathbf{S}$ for the angular momentum associated with an electron spin

$$d\mathbf{S}/dt = \mathbf{T}. \tag{2}$$

Equation (2) can be used for a classical theory of a spin system by replacing the operators by expectation values of operators using appropriate quantum states or a density matrix. This presents a problem for a spin system when a Lagrangian formulation of the classical equations is used to introduce a damping term into the equations of motion for a spin system because a classical Lagrangian introduces dynamical variables that are not defined for a quantum mechanical spin operator.

We can circumvent this problem for an undamped magnetization field by converting (2) to an equation in which the only dynamical variable is the magnetic moment. The magnetic moment of an electron is related to the spin momentum by

$$\mathbf{M} = \gamma\mathbf{S} \tag{3}$$

where $\gamma < 0$ is the gyromagnetic ratio for an electron spin. Equation (3) remains valid in both classical and quantum mechanics, and does not introduce any undefined variables. The torque exerted on a magnetic moment $\mathbf{M}$ by a magnetic field $\mathbf{H}$ is

$$\mathbf{T} = \mathbf{M} \times \mathbf{H}. \tag{4}$$

Equations (2)–(4) give an equation of motion for the magnetic moment of an electron spin

$$d\mathbf{M}/dt = \gamma\mathbf{M} \times \mathbf{H}. \tag{5}$$

The applicability of (5) is not limited to the torque exerted by an external magnetic field. Any torque on a magnetic moment $\mathbf{M}$ can be written in the form of (5) if we define an "effective" magnetic field

$$\begin{aligned}\mathbf{H} &= -\left[\mathbf{i}\frac{\partial U(\mathbf{M})}{\partial M_x} + \mathbf{j}\frac{\partial U(\mathbf{M})}{\partial M_y} + \mathbf{k}\frac{\partial U(\mathbf{M})}{\partial M_z}\right] \\ &\equiv -\frac{\partial U(\mathbf{M})}{\partial \mathbf{M}}\end{aligned} \tag{6}$$

where $U(\mathbf{M})$ is the potential energy of the system with respect to the work done by rotating the moment against whatever forces are present. The potential energy for a magnetic moment $\mathbf{M}$ in a magnetic field $\mathbf{H}_e$ has the form $U_e = -\mathbf{M}\cdot\mathbf{H}_e$. The functional form for other torque-exerting fields is quite different (see the Appendix).

For a discrete set of magnetic moments, $\mathbf{M}_i, i = 1, 2, \ldots n$, the equations of motion become

$$d\mathbf{M}_i/dt = \gamma\mathbf{M}_i \times \mathbf{H}_i \tag{7}$$

where

$$\mathbf{H}_i = -\partial U(\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_n)/\partial \mathbf{M}_i \tag{8}$$

is the effective field acting on the ith moment and $U(\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_n)$ is a generalized potential energy that takes into account all interactions that can exert torque on the individual moments.

Equations (7) and (8), interpreted as classical equations for a system of discrete magnetic moments $\mathbf{M}_i$ at points $\mathbf{r}_i$ in space, can be transformed into classical field equations by introducing a continuous field $\mathbf{M}(\mathbf{r})$ and setting $\mathbf{M}(\mathbf{r}_i) = \mathbf{M}_i/\Delta\mathbf{r}_i$, where $\mathbf{r}_i$ is the location of a lattice cell within which an electron with an unpaired spin is localized and $\Delta\mathbf{r}_i$ is the volume of a lattice cell.[3] It is assumed that the field is smooth on an atomic scale, i.e., that it does not have any "wiggles" over distances smaller than a few lattice cells (other than discontinuities that may occur at crystal boundaries) or the wiggles can be smoothed out by averaging over several adjacent lattice cells. Substituting

[2]An uncoupled field is a field for which there is no coupling to other fields, such as the elastic strain field. An undamped field is a field for which internal energy losses (transfer of energy from macroscopic motion to microscopic thermal motion) and energy loss by transfer of energy to other fields are neglected.

[3]This step assumes a model for which the spins are in a lattice array and that electrons with unpaired spins can be assigned to localized orbitals in the lattice cells. The generalization of this model to amorphous solids becomes more complicated, but presents no conceptual problems. The model would not be applicable to electrons in conduction bands, for which the orbitals cannot be well-localized. However, the electrons with unpaired spins that contribute to the magnetization fields of ferromagnets are all in narrow energy bands corresponding to d and f orbitals for which a representation of the wave function for a crystal using localized orbitals is a good approximation. A rigorous justification of the model would require a general theory of localized orbitals in solids, which was not available in 1955.

$\mathbf{M}_i = \mathbf{M}(\mathbf{r}_i)\Delta\mathbf{r}_i$ into (7) and (8) and going to the limit $\Delta\mathbf{r}_i \to 0$, discrete arrays of magnetic moments become magnetization fields, sums become integrals, functions, $F(\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_n)$, of a discrete set of magnetic moments $\mathbf{M}_i$ become functionals, $F[\mathbf{M}(\mathbf{r})]$, of the continuous field $\mathbf{M}(\mathbf{r})$[4], and partial derivatives, $\partial U(\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_n)/\partial \mathbf{M}_i$, become functional derivatives, $\delta F[\mathbf{M}(\mathbf{r})]/\delta \mathbf{M}(\mathbf{r})$, defined by

$$\begin{aligned}\delta F[\mathbf{M}(\mathbf{r})] &= F[\mathbf{M}(\mathbf{r}) + \delta\mathbf{M}(\mathbf{r})] - F[\mathbf{M}(\mathbf{r})] \\ &\equiv \int \frac{\delta F[\mathbf{M}(\mathbf{r})]}{\delta \mathbf{M}(\mathbf{r})} \cdot \delta\mathbf{M}(\mathbf{r})\, d\mathbf{r} \end{aligned} \tag{9}$$

for an arbitrary infinitesimal vector field $\delta\mathbf{M}(\mathbf{r})$. This leads to a classical equation of motion for an undamped magnetization field

$$\partial\mathbf{M}(\mathbf{r}, t)/\partial t = \gamma \mathbf{M}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{10}$$

where

$$\mathbf{H}(\mathbf{r}, t) = -\frac{\delta U[\mathbf{M}(\mathbf{r}, t)]}{\delta \mathbf{M}(\mathbf{r}, t)} \tag{11}$$

is the effective field, and $U[\mathbf{M}(\mathbf{r}, t)]$ is the potential energy of the magnetization field, commonly referred to as the "domain energy."[5]

Experimental and theoretical studies of the properties of ferromagnets have led to identification of five different domain energy terms

$$\begin{aligned}U[\mathbf{M}(\mathbf{r})] = U_e[\mathbf{M}(\mathbf{r})] + U_d[\mathbf{M}(\mathbf{r})] + U_{\rm ex}[\mathbf{M}(\mathbf{r})] \\ + U_a[\mathbf{M}(\mathbf{r})] + U_{\rm me}[\mathbf{M}(\mathbf{r})] \end{aligned} \tag{12}$$

where U_e is the external field energy (interaction between the magnetization field and the external magnetic field), U_d is the demagnetization energy (interaction of the magnetization field with itself), $U_{\rm ex}$ is the exchange energy (potential energy associated with gradients in the orientation of the magnetization field), U_a is the anisotropy energy (dependence of the potential energy on the orientation of the magnetization field relative to the crystal axes), and $U_{\rm me}$ is the magnetoelastic energy (changes in the magnetization field energy introduced by strains in the crystal lattice).

[4]A functional $F[f(x)]$ differs from a function $f(x)$ in that a particular value of a function $f(x)$ depends on a particular value of the numerically valued independent variable x, whereas a particular value of a functional $F[f(x)]$ depends on the entire set values of the function $f(x)$, [19]. One may regard a functional as a generalization of the concept of a function, $F(f_1, f_2, \ldots f_n)$, of many independent variables f_i in which the discrete index, $i = 1, 2, 3, \ldots n$, is replaced by a continuous variable x.

[5]The term "domain energy" was used in 1955 in reference to the fact that the equilibrium magnetization field structures that correspond to minima in the potential energy functional exhibit a characteristic domain structure of regions of uniform magnetization separated by thin "domain walls" in which the magnetization field changes by 90° or 180° from one domain to another. The term "Stoner–Wohlfarth energy" is more commonly used today in recognition of the work of Stoner and Wohlfarth [29] in using the potential energy functional to elucidate the domain structure in ferromagnets.

The corresponding effective fields are

$$\begin{aligned}\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_e(\mathbf{r}, t) + \mathbf{H}_d(\mathbf{r}, t) + \mathbf{H}_{\rm ex}(\mathbf{r}, t) \\ + \mathbf{H}_a(\mathbf{r}, t) + \mathbf{H}_{\rm me}(\mathbf{r}, t). \end{aligned} \tag{13}$$

The first two fields in (13) are magnetic fields. The last three are effective fields that have quantum mechanical origins. Explicit functional forms for the domain energy and effective field terms are given in the Appendix.

III. Field Equations for a Damped Magnetization Field

Damping of a physical system is accompanied by a deceleration of the macroscopic motion and a transfer of the kinetic and potential energy associated with macroscopic motion to kinetic energy of microscopic thermal motion (heat energy). For a magnetization field, this can be through direct transfer of the energy of macroscopic motion to the energy of microscopic thermal motion in spin waves or by transfer of macroscopic energy of the magnetization field to other fields to which it is coupled, e.g., the eddy-current and strain fields. This transferred energy ends up as microscopic thermal motion in the form of spin waves, lattice vibrations (phonons), and thermal excitation of conduction electrons.

Details of the mechanisms for the transfer processes are too complex to be taken into account explicitly in the field equations. However, we can introduce into the field equations a phenomenological damping term that contains damping parameters that correspond to the rate of energy transfer and can be determined experimentally without knowing the details of the transfer mechanisms. The dependence of experimentally determined damping parameters on parameters that characterize different materials can often be used to identify the different mechanisms and how they might be controlled.

Damping of a physical system generates a force in opposition to the macroscopic driving force. When the two forces balance, the energy gain from the driving force is balanced by the energy loss from the damping force and a steady state is maintained. When the forces are not equal, energy is either gained (if the driving force is larger) or lost (if the damping force is larger), and the macroscopic motion either accelerates or decelerates. If the damping force always increases or decreases as the rates of change of the dynamical variables that characterize the macroscopic motion increase or decrease, then when the driving force is constant the rates of change of the dynamical variables will increase or decrease until the driving and damping forces are equal and a steady-state condition is attained. The simplest case, which commonly occurs when there are many different damping forces and resonance phenomena do not occur, is that the damping force is directly proportional to the rates of change of the macroscopic dynamical variables of the system. For a ferromagnet, the dynamical variables are the magnetization field $\mathbf{M}(\mathbf{r}, t)$.

A common way of introducing a damping term of this kind into classical equations of motion for a physical system is to use a Lagrangian formulation of the equations of motion and

add a velocity-dependent term derived from a quadratic function of the time derivatives of the dynamical variables called "Rayleigh's dissipation function" [26, pp. 21 and 341]. For a ferromagnet, for which the dynamical variables are the components of the magnetization field $\mathbf{M}(\mathbf{r},t)$, the added term in the equation of motion is $-\eta\partial\mathbf{M}(\mathbf{r},t)/\partial t$, where η is a damping parameter that is characteristic of the material. The equation of motion for an undamped magnetization field in Lagrangian form is[6]

$$\frac{d}{dt}\frac{\delta\mathcal{L}[\mathbf{M}(\mathbf{r},t),\dot{\mathbf{M}}(\mathbf{r},t)]}{\delta\dot{\mathbf{M}}(\mathbf{r},t)} - \frac{\delta\mathcal{L}[\mathbf{M}(\mathbf{r},t),\dot{\mathbf{M}}(\mathbf{r},t)]}{\delta\mathbf{M}(\mathbf{r},t)} = 0 \quad (14)$$

where

$$\mathcal{L}[\mathbf{M}(\mathbf{r},t),\dot{\mathbf{M}}(\mathbf{r},t)] = \mathcal{T}[\mathbf{M}(\mathbf{r},t),\dot{\mathbf{M}}(\mathbf{r},t)] - U[\mathbf{M}(\mathbf{r},t)] \quad (15)$$

is the Lagrangian, and $\mathcal{T}$ and U are the kinetic energy and potential energy, respectively, of the magnetization field.

Equation (14) can be converted into an equation for a damped magnetization field by adding a dissipative force

$$\delta\mathcal{R}[\dot{\mathbf{M}}(\mathbf{r},t)]/\delta\dot{\mathbf{M}}(\mathbf{r},t) \quad (16)$$

to the left side of (14) to obtain

$$\frac{d}{dt}\frac{\delta\mathcal{L}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} - \frac{\delta\mathcal{L}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\mathbf{M}} + \frac{\delta\mathcal{R}[\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} = 0 \quad (17)$$

where

$$\mathcal{R}[\dot{\mathbf{M}}(\mathbf{r},t)] = \frac{\eta}{2}\int \dot{\mathbf{M}}(\mathbf{r},t)\cdot\dot{\mathbf{M}}(\mathbf{r},t)\,d\mathbf{r} \quad (18)$$

is a Rayleigh dissipation functional. The parameter η quantifies the average damping throughout the sample. Equation (18) implies that the distribution of energy loss due to damping mechanisms (the distribution of heat energy generated locally by damping) is uniform. This is not, in fact, the case because local damping can be caused by a variety of nonuniform mechanisms: rapid spin reorientation in moving domain walls, random size and orientation of crystal grains, crystal defects, impurities, local strains, etc. A dissipation functional of the form

$$\mathcal{R} = \frac{1}{2}\sum_{i,j}\int\int\left[\frac{\partial\mathbf{M}_i(\mathbf{r},\mathrm{t})}{\partial t}\eta_{ij}(\mathbf{r},\mathbf{r}')\frac{\partial\mathbf{M}_j(\mathbf{r}',t)}{\partial t}\right]d\mathbf{r}d\mathbf{r}' \quad (19)$$

would take nonuniform damping into account; however, it is of little use because it is not possible to calculate or measure the matrix damping function $\eta_{ij}(\mathbf{r},\mathbf{r}')$ that replaces the single damping parameter η.

When measurements are done at low external fields, the damping parameter η will depend on the domain structure, which is dependent on the sample shape and size and the magnitude of the external field. In order to obtain a damping parameter that is characteristic of the ferromagnetic material, independent of the shape and size of the sample or the magnitude of the external field, it would be necessary to use an external magnetic field that is large enough for the condition $dM/dH_e \approx 0$ to be satisfied or a sample that was small enough to be a single domain.

It can be shown that the rate at which work is done by the system against the dissipative force (i.e., the rate at which energy is lost) is [26]

$$\frac{dW}{dt} = 2\mathcal{R}[\dot{\mathbf{M}}] = \eta\int\dot{\mathbf{M}}(\mathbf{r},t)\cdot\dot{\mathbf{M}}(\mathbf{r},t)\,d\mathbf{r}. \quad (20)$$

Substituting (15) into (17) in order to separate the kinetic and potential energy contributions and using

$$\delta U[\mathbf{M}]/\delta\dot{\mathbf{M}} = 0 \quad (21)$$

$$\delta\mathcal{R}[\dot{\mathbf{M}}]/\delta\dot{\mathbf{M}} = \eta\dot{\mathbf{M}} \quad (22)$$

and (11), we obtain

$$\frac{d}{dt}\frac{\delta\mathcal{T}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} - \frac{\delta\mathcal{T}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\mathbf{M}} + [-\mathbf{H}(\mathbf{r},t) + \eta\dot{\mathbf{M}}(\mathbf{r},t)] = 0. \quad (23)$$

At this point, we encounter the problem noted earlier that the kinetic energy $\mathcal{T}$ of a classical Lagrangian for a rotating object depends on dynamical variables that are not defined for quantum spin operators. An explicit expression for the first two terms in (23) would contain an azimuthal Eularian angle variable ψ and rate of rotation $d\psi/dt$ about the principal axis [26]. I was not able to derive an expression for the kinetic energy of a rotating body in classical mechanics that would correspond to the spin of an elementary particle in quantum mechanics in a way that made physical sense.[7] It is, however, possible to circumvent this problem by means of the following argument.

If we set $\eta = 0$ in (23), then it becomes an equation for an undamped magnetization field, and should, therefore, be equivalent to (10). We note that the damping term in (23) is an added "damping field" that can reduce the effective magnetic field and change the torque exerted on the magnetization field. It is reasonable to argue that, when $\eta \neq 0$, adding this same damping term to the effective field for the equation of motion for an undamped magnetization field given by (10) gives a valid equation

[6]A dot over a variable signifies a time derivative, e.g., $\dot{\mathbf{M}}(\mathbf{r},t) = \partial\mathbf{M}(\mathbf{r},t)/\partial t$.

[7]One can show that the inertial tensor must have a single nonzero term corresponding to the rotational inertia for rotation about the principal axis. I was unable to conceive of a physical object with an inertial tensor of this kind.

of motion for a damped magnetization field. This equation can be written as[8]

$$\frac{\partial \mathbf{M}(\mathbf{r},t)}{\partial t} = \gamma \mathbf{M}(\mathbf{r},t) \times \left[\mathbf{H}(\mathbf{r},t) - \eta \frac{\partial \mathbf{M}(\mathbf{r},t)}{\partial t} \right]. \quad (24)$$

It is of interest to compare (24) with the Landau–Lifshitz equation [2]

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma \mathbf{M} \times \mathbf{H} - \frac{\lambda}{M^2} \mathbf{M} \times \mathbf{M} \times \mathbf{H}. \quad (25)$$

Equations (24) and (25) can be rewritten as

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma \mathbf{M} \times \mathbf{H} - \frac{\alpha}{M} \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \quad (26)$$

and (substituting the equality $\mathbf{M} \times \mathbf{H} = \boldsymbol{\gamma}^{-1}[\partial \mathbf{M}/\partial t + (\lambda/M^2)\mathbf{M} \times \mathbf{M} \times \mathbf{H}]$ into the right-hand term in (25) and simplifying)

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma^* \mathbf{M} \times \mathbf{H} - \frac{\alpha}{M} \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \quad (27)$$

respectively, where

$$\alpha = \eta \gamma M = \lambda / \gamma M \quad (28)$$

[8]The derivation in the thesis of (24) differs from the foregoing derivation. The steps from the equation for a single magnetic moment, (5), to field equations, (10), were the same. It was pointed out in Appendix B of the thesis that (10) could also be derived from the principle of least action by defining the kinetic energy for an angular momentum field $\mathbf{M}(\mathbf{r},t)/\gamma$ and introducing $\partial M(\mathbf{r},t)/dt = 0$ (where M is the magnitude of $\mathbf{M}$) as a separate constraint. These steps, and the implicit assumption that the classical equations of motion could be interpreted as quantum operator equations (without attempting to specify the functional form of the classical kinetic energy term for spin systems), circumvented the problem of defining an inertial tensor and angular variables for spin rotations.

Although it was noted in Appendix B of the thesis that the damping term could be derived by introducing a Rayleigh dissipation functional into the Lagrangian formulation of the field equations, the dissipation functional was not used for the derivation in the thesis. Instead, a rather elaborate argument that led to the same equation for a damped magnetization field was used. The argument distinguished between macroscopic and microscopic (thermal) motion and introduced a free energy for the state of the macroscopic magnetization field and other macroscopic fields to which it might be coupled (such as the eddy current field and strain field). The free energy for the macroscopic fields was assumed to be the domain energy. The equilibrium states of the macroscopic fields become time dependent for a time-dependent external field. The magnitude of the magnetization field and the constants in the domain energy become temperature-dependent when the domain energy is interpreted as the Helmholtz free energy. It was assumed that the minima of this free energy determined the (time dependent) equilibrium states of the macroscopic fields, and that nonequilibrium macroscopic states would always decay monotonically to equilibrium states as a consequence of transfer of energy by coupling to the microscopic fields. If the deviations from equilibrium are small and the times to return to equilibrium (the decay times) are small, the rate of return to equilibrium will be exponential and can be fully characterized by the decay times and time derivatives of the macroscopic fields. This reasoning led to characterization of the effect of damping of the magnetization field by means of a single term, $\eta \partial \mathbf{M}(\mathbf{r},t)/\partial t$, subtracted from effective field defined by (13), where η is a linear function of the various decay times.

The foregoing derivation of the equation of motion for a damped magnetization field was long and not very useful (apart from providing independent justification for use of a Rayleigh dissipation functional) without detailed specification of the coupling of the magnetization field with the various macroscopic and microscopic fields. Hence, in this paper a short derivation based on a Rayleigh dissipation functional has been used.

$$\gamma^* = \gamma(1 + \alpha^2). \quad (29)$$

We observe that the damping terms in the two equations are identical, the only difference between the equations being that as the dimensionless damping parameter α increases in the Landau–Lifshitz form, the gyromagnetic ratio γ^* and, hence, the rate of precession of the spin also increases. The difference between the two equations is small when $\alpha^2 \ll 1$. This is the case for damping inferred from ferromagnetic resonance linewidth measurements for which the largest published value of the damping parameter was $\alpha \approx 0.1$ in 1955 [24]. The combined effect of the increase in the damping term $(\alpha/M)\mathbf{M} \times \partial \mathbf{M}/\partial t$ and the gyromagnetic ratio γ^* (which increases the precession rate) in (27) can be seen most clearly in (25), where the added term is a torque perpendicular to the plane of $\mathbf{M}$ and $\mathbf{H}$ that rotates the magnetization field $\mathbf{M}$ directly toward the effective magnetic field $\mathbf{H}$ without directly affecting the precession rate, thereby causing the magnetization field to move directly into alignment with the effective field more and more rapidly until, for sufficiently large values of α, it moves in before precession can occur. In the modified form, (23) or (26), the Rayleigh damping field $\eta(\partial \mathbf{M}/\partial t)$ introduces a small torque field $(\alpha/M)\mathbf{M} \times \partial \mathbf{M}/\partial t$ proportional to the damping parameter α that rotates the magnetization field $\mathbf{M}$ toward the effective field $\mathbf{H}$ without reducing the torque field $\gamma \mathbf{M} \times \mathbf{H}$ that causes precession, thereby causing the magnetization to spiral in toward the effective field without increasing the precession rate.

APPENDIX
DOMAIN ENERGY AND EFFECTIVE FIELD

The energy of interaction with the external field is

$$U_e = -\int \mathbf{M} \cdot \mathbf{H}_e \, d\mathbf{r} \quad (A1)$$

where $\mathbf{H}_e$ is the externally applied field

$$\mathbf{H}_e = -\delta U_e / \delta \mathbf{M}. \quad (A2)$$

The demagnetizing energy is

$$U_d = -\frac{1}{2} \int \mathbf{M} \cdot \mathbf{H}_d \, d\mathbf{r} \quad (A3)$$

where

$$\mathbf{H}_d = \nabla \int \frac{\nabla \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r}' + \nabla \int \frac{\mathbf{n} \cdot (\mathbf{M}_1 - \mathbf{M}_2)}{|\mathbf{r} - \mathbf{r}'|} \, dS'. \quad (A4)$$

The surface integral in over all surfaces (with normal $\mathbf{n}$ directed toward the side labeled 2) across which $\mathbf{M}$ changes discontinuously from $\mathbf{M}_1$ to $\mathbf{M}_2$. The demagnetizing field $\mathbf{H}_d$ is itself a functional of the magnetization field. It can be shown to satisfy the relation

$$\mathbf{H}_d = -\delta U_d / \delta \mathbf{M}. \quad (A5)$$

The exchange energy and exchange fields are

$$U_{\rm ex} = (A/M^2)\int [(\nabla M_x)^2 + (\nabla M_y)^2 + (\nabla M_z)^2]\, d\mathbf{r} \tag{A6}$$

$$\mathbf{H}_{\rm ex} = -\delta U_{\rm ex}/\delta \mathbf{M} = (2A/M^2)\nabla^2 \mathbf{M} \tag{A7}$$

where A is the exchange stiffness constant.

The first-order anisotropy energies for cubic crystals (with the rectangular areas parallel to the cubic axes) and hexagonal crystals (with the z axis along the hexagonal axis) are

$$U_a = -(K/2M^4)\int \left(M_x^4 + M_y^4 + M_z^4\right)\, d\mathbf{r},$$
$$\mathbf{H}_a = (2K/M^4)\left(\mathbf{i}M_x^3 + \mathbf{j}M_y^3 + \mathbf{k}M_z^3\right) \tag{A8}$$

for cubic crystals and

$$U_a = -(K/M^2)\int M_z^2\, d\mathbf{r},$$
$$\mathbf{H}_a = (2K/M^2)\mathbf{k}M_z \tag{A9}$$

for hexagonal crystals.

The magnetoelastic energy and field in a cubic crystal are

$$U_{\rm me} = M^{-2}\int \left(B_1 \sum_i M_i^2 e_{ii} + B_2 \sum_{i<j} M_i M_j e_{ij}\right) d\mathbf{r} \tag{A10}$$

$$\begin{aligned}\mathbf{H}_{\rm me} = M^{-2}\{&\mathbf{i}(2B_1 M_x e_{xx} + B_2 M_y e_{xy} + B_2 M_z e_{xz})\\ &+ \mathbf{j}(B_2 M_x e_{xy} + 2B_1 M_y e_{yy} + B_2 M_z e_{yz})\\ &+ \mathbf{k}(B_2 M_x e_{xz} + B_2 M_y e_{yz} + 2B_1 M_z e_{zz})\}\end{aligned} \tag{A11}$$

where B_1 and B_2 are the magnetoelastic coupling constants, related to the magnetostriction constants and the elastic moduli by

$$\lambda_{100} = -\frac{2}{3}B_1 \Big/ (c_{11} - c_{12}), \quad \lambda_{111} = -\frac{1}{3}B_2 \Big/ c_{44},$$
$$e_{ii} = \partial s_i/\partial x_i, \quad e_{ij} = \partial s_i/\partial x_j + \partial s_j/\partial x_i \tag{A12}$$

where $\mathbf{s}(\mathbf{r})$ is the displacement field of the lattice. For an isotropic material ($B_1 = B_2 = B, c_{44} = (c_{11} - c_{12})/2, \lambda_{100} = \lambda_{111} = \lambda_m$), the magnetoelastic energy reduces to

$$U_{\rm me} = BM^{-2}\int \left(\sum_{i,j} M_i M_j \partial s_i/\partial x_j\right) d\mathbf{r}. \tag{A13}$$

When the strain is caused by a uniform tensile stress T, (A13) reduces to the familiar form

$$U_{\rm me} = \frac{3}{2}\lambda_m T \int \sin^2\Theta\, d\mathbf{r} \tag{A14}$$

where Θ is the angle between the magnetization and direction of applied stress.

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The author also gratefully acknowledges the inspiration and guidance of Dr. Hans Ekstein. Thanks are also due to Dr. Joshua N. Goldberg for helpful discussions. The cooperation of the Armour Research Foundation was an important factor in the successful completion of the work.

This paper is a condensed version of chapters I–III of my Ph.D. thesis in which a new equation for the damped motion of the magnetization field in a ferromagnet was derived in order to resolve a problem in use of the original Landau–Lifshitz damping term for large damping. An application of the equation for measurements of rotational damping in thin permalloy sheets by Joseph Kelly in which the equation was derived by a transformation of the Landau–Lifshitz equation and an *ad hoc* choice of damping parameters was presented in a paper by Gilbert and Kelly at the first Magnetism and Magnetic Materials Conference, but the thesis and the physical basis for the proposed damping term (which were not fully worked out in the thesis) were never published. A number of errors and misconceptions in the thesis have been corrected and speculative material removed.

I am indebted to Dr. Vladimir L. Safonov for proposing that a paper presenting the key results of my thesis be included in the "Classics in Magnetics" series in IEEE TRANSACTIONS ON MAGNETICS. I am grateful for his competent work in selecting those parts of the thesis to include, in preparing the first draft of the paper, for correcting some of my errors, and his patience as I struggled to recall and understand what I had done in my thesis, which I had not looked at since 1956 when I moved to a different field of research.

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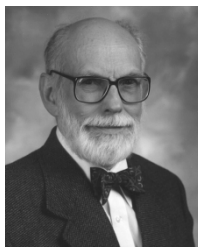

Thomas L. Gilbert was born in Topeka, KS, in 1922. He received the B.S. degree in physics, with honors, from the California Institute of Technology (Caltech), Pasadena, in 1944.

He spent the remainder of the World War II war years (1944–46) at Armour Research Foundation (ARF), since renamed Illinois Institute of Technology Research Institute (IITRI), where he worked under Marvin Camras, who contributed the key inventions (recording head design and use of high-frequency bias) that made recording on magnetic media feasible. He married Winifred Watt and started graduate school at Caltech in 1946, but returned to ARF in 1947 after his first daughter arrived. Around 1953, he and Joseph M. Kelly, an experimental physicist, were assigned to a research project for the National Security Agency to investigate anomalous damping in ferromagnetic materials. The theoretical work on this project led to a reformulation of the damping term in the Landau–Lifschitz equation and a Ph.D. thesis in theoretical physics granted by Illinois Institute of Technology in 1956. The thesis was never published. In 1956, he joined the research staff of Argonne National Laboratory, where he remained until his retirement at the end of 1987. During that time, he worked on problems in the electronic structure of atoms, small molecules, and defects in crystals and (from 1979 to 1988) on problems concerning the environmental risks of radioactive waste. During that time, his main contributions were the Adams–Gilbert equation for localized orbitals in polyatomic systems and the soft-sphere model for the repulsive interaction between closed-shell atoms and ions. On retirement, he accepted a position as Adjunct Professor of Religion and Science Studies at the Lutheran School of Theology at Chicago and Director of the Epic of Creation Program for the Zygon Center for Religion and Science, where he has been involved in developing a program that would provide seminary students with a better understanding of science and how to deal with the tensions between religion and science in a way that respects the contributions and traditions of both to the human drama.