

Classical Magnetic Systems

On the behaviour of macroscopic magnetized moments in dissipative
environments

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Preface

This document contains our work regarding the novel spin revolution effect, first described by PD Dr. Elena Vedmedenko and Prof. Roland Wiesendanger[**Vedmedenko**]. The goal of this project is to further the understanding of the presented systems by investigating certain mathematical aspects of the system's description. Our hope is to procure either analytically solvable equations of motions or prove that such solutions do, in fact, not exist. We start by introducing the necessary mathematical principles rooted in the theory of manifolds, variational calculus and ordinary differential equations. From there on, we investigate various realizations of the spin revolution effect.

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CHAPTER ONE

Mathematical Introduction

We start with the necessary mathematical theory. If the reader already feels comfortable with the theory of smooth manifolds, variational calculus on aforementioned spaces, (Lie) groups as well as autonomous differential equations, they may skip this chapter. We assume basic knowledge of linear algebra and calculus.

1.1 Group Theory

We need some fundamental group theory. From here on, all vector spaces are finite-dimensional.

Definition 1.1 (Group). A **Group** is a pair (G, \circ) consisting of a set G and an operation \circ such that the following axioms are satisfied:

- $\forall a, b, c \in G : a \circ (b \circ c) = (a \circ b) \circ c$
- $\bullet \forall a \in G \exists a^{-1} \in G : a \circ a^{-1} = 1$
- $\bullet \exists e \in G \forall a \in G : e \circ a = a$

If the operation is commutative, we call the group **abelian**.

One should think of groups in terms of symmetries: A symmetry of an abstract object can be thought of as a collection of operations leaving said object invariant.

There are plenty of examples of groups important in physics:

Example 1.2. Let V be a real vector space. The automorphisms $V \rightarrow V$ constitute a group under composition, the *general linear group* of dimension n , $GL_n(V)$. There are several important subgroups of $GL(\mathbb{R}^n)$ we will heavily use later. Let

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \rightarrow \mathbb{R}$$

be the euclidean scalar product on \mathbb{R}^n and $T \in GL(\mathbb{R}^n)$. We define the *orthogonal group* as

$$O(n) := \{Q \in GL(\mathbb{R}^n) \mid QQ^t = Q^T Q = \text{id}\} \leq GL(\mathbb{R}^n).$$

Further restricting our attention to orthogonal automorphisms with unit determinant yields the *special orthogonal group*

$$SO(n) := \{R \in O(n) \mid \det R = 1\} \leq O(n)..$$

Note that every vector space over \mathbb{R} has a basis. Choosing one and representing automorphisms as $n \times n$ -matrices yields the usual group structure by matrix multiplication. This also clearly demonstrates that groups of linear maps cannot be abelian in general.

In physics, we are usually concerned with how a certain group acts on a physical system. For that, we need actions and representations.

Definition 1.3 (Group Action). Let G be a group with identity e and M be a set. A **right-action** of G on M is a map

$$\alpha : G \times X \rightarrow X.$$

such that:

- $\alpha(e, x) = x$
- $\alpha(h, \alpha(g, x)) = \alpha(gh, x)$

is satisfied. We write $G \curvearrowright M$ and $\alpha(g, x) =: g.x$ for short.

Definition 1.4 (Representation). Let V be a real vector space and G be a group. A **real G -representation** is a group homomorphism

$$\rho : G \rightarrow \text{GL}(V).$$

We call V **representation space** and $\deg V$ the degree of the representation.

1.2 Differential Geometry

Smooth Manifolds and Tangent Spaces

From now on, we use the Einstein summation convention.

Definition 1.5 (Topological Manifold). A **topological n -manifold** is a topological space M such that:

- M has the Hausdorff property.
- M is second-countable.
- M is locally euclidean: For every $p \in M$ there is an open neighbourhood $U \subseteq M$ of p and a homeomorphism $\phi : U \rightarrow V$ such that $V \subseteq \mathbb{R}^n$ is open. We call (ϕ, U) a **chart** on M .

Since we want to apply the formalism in concrete examples, we will often work in local coordinates. Note that with the standard basis of \mathbb{R}^n , we can write a chart map as

$$\phi(p) = (\phi_1(p), \dots, \phi_n(p)) =: (x^1(p), \dots, x^n(p)).$$

where $x^i : M \rightarrow \mathbb{R}$ are the chart components. Since the charts are homeomorphisms, they can be locally inverted to a map $\phi^{-1} : V \rightarrow U$, in which case we call ϕ^{-1} a *local parametrization* of M .

Example 1.6. • Trivially, the real euclidean space \mathbb{R}^n is an n -dimensional smooth manifold with the identity id being a global chart.

- The n -sphere \mathbb{S}^n is a smooth manifold with local charts given by projection onto the coordinate axes.

1.2. DIFFERENTIAL GEOMETRY

- The n -torus

$$T^n = \underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n \text{ times}}$$

is a smooth n -manifold. In the case of T^2 , one obtains local charts easily as the 2-torus is a surface of revolution.

We need the notion of smoothness on manifolds.

Definition 1.7 (Smooth Manifold). A **maximal atlas** for a topological manifold M is a collection \mathfrak{A} of charts of M such that every $p \in M$ is contained in some chart and \mathfrak{A} is not properly contained in some other atlas. We call \mathfrak{A} smooth if for any two charts $(U, \phi), (V, \psi) \in \mathfrak{A}$ we have either $U \cap V = \emptyset$ or

$$\phi^{-1} \circ \psi : V \rightarrow U.$$

is a smooth (\mathcal{C}^∞) map, called **chart transition map**. A **smooth manifold** is a pair (M, \mathfrak{A}) such that M is a topological manifold and \mathfrak{A} is a maximal smooth atlas.

Note that with this, we get a notion of smooth maps on abstract manifolds: If M, N are smooth manifolds and $F : M \rightarrow N$ is a map, we call it smooth if for every $p \in M$ there is a chart (U, ϕ) with $p \in U$ and a smooth chart (V, ψ) with $F(p) \in V$ such that

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is smooth in the usual sense.

There are many examples important to physics:

Example 1.8. • The \mathbb{R}^n obtains a smooth maximal atlas by means of the identity.

- The sphere \mathbb{S}^2 has a smooth atlas with two charts $\mathbb{S}^2 \setminus N$ and $\mathbb{S}^2 \setminus S$, where N and S are north and south pole, respectively. The chart map is given by the *stereographic projection* $\sigma : \mathbb{S}^2 \setminus N \rightarrow \mathbb{R}^2$.
- Any finite-dimensional real normed vector space V is a smooth manifold. The choice of a basis determines an isomorphism $\mathbb{R}^n \cong V$ which we take as global chart. For any other basis, one obtains a basis transformation matrix which is linear, hence a \mathcal{C}^∞ chart transition.
- Any open subset $U \subseteq \mathbb{R}^n$ is a smooth manifold on its own with the smooth atlas $\{U, \text{id}_U\}$. We call such a manifold an *open submanifold* of \mathbb{R}^n .

With smooth maps on manifolds, we are already able to talk about curves or paths on such spaces. However, in physics one is especially interested in velocity and acceleration. For this, we will need two additional structures: The tangent bundle, which comes naturally with every smooth manifold, and a metric to measure distances and angles. We define a **curve** on a manifold M to be a smooth map

$$\gamma : I \rightarrow M$$

where $I \subseteq \mathbb{R}$ is some interval.

Definition 1.9 (Tangent Bundle). Let M be a smooth manifold and $p \in M$. We declare two smooth curves $\gamma_i : I \rightarrow M$, $i \in \{1, 2\}$ to be equivalent if for any chart (U, ϕ) of p we have

$$\left. \frac{d}{dt}(\phi \circ \gamma_1) \right|_{t=0} = \left. \frac{d}{dt}(\phi \circ \gamma_2) \right|_{t=0}. \quad (1.1)$$

The space of equivalence classes of such curves is called **tangent space** at p and denoted $T_p M$. The **tangent bundle** of M is the disjoint union

$$TM := \coprod_{p \in M} T_p M.$$

Note that while the tangent space at a point on an n -manifold is an n -dimensional real vector space (and hence isomorphic to a copy of \mathbb{R}^n), the tangent bundle is a $2n$ -dimensional space and indeed also a smooth manifold. However, it is not always possible to identify $TM \cong M \times \mathbb{R}^n$. A point in TM is a pair (p, v) , where $p \in M$ and $v \in T_p M$. We also have a natural projection $\pi : TM \rightarrow M$, given by $\pi(p, v) = p$.

Tangent vectors act on smooth functions $f : M \rightarrow \mathbb{R}$ by

$$v(f) = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t).$$

Note that this gives rise to the usual partial derivative of a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by setting

$$\frac{\partial g}{\partial x^i} \Big|_y := \frac{d}{dt} g(y + te_i).$$

where γ represents one path with $\dot{\gamma}|_{t=0} = v$. Using the fact that for every $p \in M$ there is a chart $(U, (x^1, \dots, x^n))$ centered around p , we are able to pull back the basis of $T_{\phi(p)} \mathbb{R}^n$ to $T_p M$. The basis of $T_{\phi(p)} \mathbb{R}^n$ is given by paths of the form $t \mapsto \phi(p) + te_i$, where e_i is the i -th canonical basis vector of \mathbb{R}^n . Writing this basis as $(\partial_{x^1}, \dots, \partial_{x^n})$, we can pull back along ϕ^{-1} to obtain a basis of $T_p M$, defined as

$$\partial_i|_p := \phi^{-1}(\phi(p) + te_i).$$

This expression acts on a smooth function as above by

$$\partial_i|_p f = \left. \frac{\partial(f \circ \phi^{-1})}{\partial x^i} \right|_{\phi(p)}.$$

Definition 1.10 (Differential). Let $F : M \rightarrow N$ be a smooth map between smooth manifolds. The **differential** of F at $p \in M$ is a linear map

$$dF_p : T_p M \rightarrow T_{F(p)} N$$

given by

$$dF_p([\gamma]) = [F \circ \gamma].$$

where $[\gamma]$ is an equivalence class of curves through p .

Given local charts $(U, (x^1, \dots, x^m))$ around p and $(V, (y^1, \dots, y^n))$ around $F(p)$, the differential can be expressed in local coordinates as

$$dF_p(\partial_i) = \partial_i F^j|_p \hat{\partial}_j|_{F(p)}.$$

with $F^j = y^j \circ F$ and $\hat{\partial}_j$ being a basis vector of $T_{F(p)} N$. We conclude this section with the definition of smooth vector fields.

Definition 1.11 (Vector Fields). Let M be a smooth manifold. A **smooth vector field** is a smooth section

$$V : M \rightarrow TM$$

such that $\pi \circ V = \text{id}_M$ with the natural projection $\pi : TM \rightarrow M$. The space of smooth sections on M is denoted $\mathfrak{X}(M)$.

Once more, we desire a local coordinate expression. If $(U, (x^1, \dots, x^n))$ is a local chart, we can use the basis of $T_p M$ defined above for all $p \in U$ to write

$$X_p = X^i(p) \partial_i|_p$$

where $X^i : M \rightarrow \mathbb{R}$ are the component functions of X . If a collection of vector fields (X_1, \dots, X_n) spans $T_p M$ locally or globally, we call it a **local or global frame**. Directing our attention one last time to curves, we can now define curves whose velocity vectors form a given vector field.

Definition 1.12 (Integral Curves). Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. An **integral curve** of V is a smooth curve $\gamma : I \rightarrow M$ such that

$$\frac{d}{dt} \gamma(t) = V_{\gamma(t)}$$

for all $t \in I$.

We need this notion of walking along the path prescribed by a vector field later for Hamiltonian phase flows. In general, a flow is the rigorous version of this intuitive idea of following a field:

Definition 1.13 (Flow). Suppose M is a smooth manifold with $V \in \mathfrak{X}(M)$ such that a unique global integral curve of V exists for all $p \in M$. Denoting the integral curve at p by $\gamma_p : \mathbb{R} \rightarrow M$, a **global flow** is a continuous action of \mathbb{R} on M :

$$\begin{aligned} \theta : M \times \mathbb{R} &\rightarrow M \\ (p, t) &\mapsto \gamma_p(t). \end{aligned}$$

Note that vector fields also act naturally on smooth functions $f : M \rightarrow \mathbb{R}$ by means of partial derivation:

$$(Xf)(p) = X_p f = X^i(p) \partial_i|_p(f).$$

Intuitively, one should think of $t \in \mathbb{R}$ as time. A global flow then shifts the whole manifold along some vector field, e.g. the torus rotates under the global flow of a radial vector field.

Riemannian Metrics

To measure distances and angles, we need additional structure on our smooth manifolds. Dual to the tangent bundle of M , we introduce the **cotangent bundle**

$$T^*M := \coprod_{p \in M} (T_p M)^*$$

which is, as usual in linear algebra, given by the space of linear forms

$$\omega : T_p M \rightarrow \mathbb{R}.$$

The basis dual to $(\partial_1, \dots, \partial_n)$ will be denoted (dx^1, \dots, dx^n) .

Definition 1.14 (Riemannian Manifolds). A **Riemannian metric** on a smooth manifold M is a smooth, symmetric and positive-definite bilinear form

$$g : TM \times TM \rightarrow \mathbb{R}.$$

The pair (M, g) is called **Riemannian manifold**.

This also yields the usual notions obtained with a scalar product on \mathbb{R}^n : We have a norm on each $T_p M$ defined by

$$\|v\|_g := \sqrt{\langle v, v \rangle}$$

and an angle given by

$$\cos \angle(v, w) := \frac{\langle v, w \rangle}{\|v\|_g \|w\|_g} \in [0, \pi].$$

Since each g_p is non-degenerate, we even have the well-known musical isomorphisms $T_p M \cong T_p^* M$. If $X = X^i \partial_i$ is a vector field, we identify it with $X^\flat = g_{ij} X^i dx^j$. A covector field $\omega = \omega_i dx^i$ is identified with $\omega^\sharp = g^{ij} \omega_j \partial_i$ where $g^{ij} = g_{ij}^{-1}$.

In local coordinates at some $p \in M$, we can use the canonical basis of $T_p M$ to write $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ as

$$g_p(v, w) = g_p(v^i \partial_i|_p, w^j \partial_j|_p) =: \langle v^i \partial_i|_p, w^j \partial_j|_p \rangle = v^i w^j g_{ij}$$

where $g_{ij} = g_p(\partial_i|_p, \partial_j|_p)$ denotes the matrix representation of the metric g . The most important definition arising from the existence of a metric function is in our case the arc length.

Definition 1.15 (Length). Let (M, g) be a smooth Riemannian manifold and $\gamma : I \rightarrow M$ be a smooth curve. The **length** of γ is given by the functional

$$L[\gamma] := \int_I \|\partial_t \gamma(t)\|_g dt.$$

We will now connect those concepts with physics' formalism.

1.3 Lagrangian Mechanics

Definition 1.16 (Configuration Manifold). The **configuration manifold** Q of a system of N point masses consists of all possible positions of those N masses. We call $\dim Q$ the **degrees of freedom**.

Configuration Manifold

Example 1.17. • The configuration manifold of N free point masses in Euclidean 3-space is given by \mathbb{R}^{3N} as each particle is free to move in three spacial directions independently.

- The two-dimensional double pendulum has the configuration manifold $Q = \mathbb{S}^1 \times \mathbb{S}^1$ as each point can move in one copy of \mathbb{S}^1 . Note that $\mathbb{S}^1 \times \mathbb{S}^1 \cong T$, so we can think of Q as a 2-torus.
- A rigid rod in Euclidean 2-space is free to rotate around an axis at its end. The configuration manifold is $Q = \mathbb{R}^2 \times \mathbb{S}^1$ since we need two coordinates (q^1, q^2) to describe the position of the axis of rotation and one angle $q^3 \in [0, 2\pi]$ to describe the rotation itself.

From classical physics we know very well that positions are not sufficient to characterize a system fully, one also needs velocities.

Definition 1.18 (Velocity Phase Space). Let Q be the configuration manifold of a physical system. The **velocity phase space** of Q is the tangent bundle TQ .

In physical notation, we write a local coordinate system of TQ as $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$.

1.3. LAGRANGIAN MECHANICS

With this complete description of our system, we are ready to define the fundamental quantities needed to solve for the equations of motion.

Definition 1.19 (Kinetic and Potential Energy). Let TQ be the velocity phase space of a physical system Q . The **kinetic energy** is a quadratic form given locally at $q \in Q$ by

$$T : T_q Q \rightarrow \mathbb{R}$$

$$v \mapsto \frac{\|v\|^2}{2}$$

and the **potential energy** is some smooth function

$$U : Q \rightarrow \mathbb{R}.$$

Unless specified otherwise, we use the Euclidean metric for \mathbb{R}^{3N} . This means that every submanifold Q is endowed with the restricted Euclidean metric, hence kinetic energy is well-defined.

Note that in the examples above, many systems are constrained to move on some m -dimensional submanifold Q of \mathbb{R}^{3N} . We call such constraints **holonomic** if Q is given by $3N - m$ independent equations of the type

$$f_k(x^1, \dots, x^{3N}) = 0$$

with $1 \leq k \leq 3N - m$, (x^1, \dots, x^{3N}) being global coordinates of \mathbb{R}^{3N} and f being some smooth function in those coordinates.

Definition 1.20 (Lagrangian). Let $Q \subset \mathbb{R}^{3N}$ be the configuration m -manifold of N point masses m_1, \dots, m_N with $3N - m = k$ holonomic constraints. The **Lagrangian** is a smooth function

$$\mathcal{L} : TQ \rightarrow \mathbb{R}. \quad (1.2)$$

We call \mathcal{L} natural if it can be written as

$$\mathcal{L} = T - U$$

for some kinetic energy T and some potential energy U .

With this, we formulate our two main theorems:

Theorem 1.21 (Lagrangian System of Motion). Let Q be a configuration space, TQ be its velocity phase space, and $\mathcal{L} : TQ \rightarrow \mathbb{R}$ be a Lagrangian. A curve $\gamma : I \rightarrow Q$ is a motion of Q if it is an extremal of the **action functional**

$$S[\gamma] = \int_I \mathcal{L}(\dot{\gamma}) dt. \quad (1.3)$$

This should be understood as the principal axiom classical mechanical motion seems to obey to our best understanding. By methods of variational calculus, one shows that this is equal to the following theorem:

Theorem 1.22 (Euler-Lagrange). In the situation above, let $(q^1, \dots, q^m) = y(t)$ denote a local coordinate description of a point on a motion of Q and $\mathcal{L} \equiv \mathcal{L}(q, \dot{q})$ denote the Lagrangian in local coordinates. Then the evolution of $\gamma(t)$ with t is such that the **Euler-Lagrange equations**

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad (1.4)$$

are satisfied.

1.4 Ordinary Differential Equations

CHAPTER TWO

Magnetic Ball on an Incline

2.1 Two-Dimensional System

We start our investigation by considering a very simplified model of a rigid two-dimensional disk with a constant magnetic moment rolling on an inclined plane without slipping.