Balls in Tubes: Differential equations

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1 Details of the numerical simulations

We write the position of the magnetic balls on the cylinder surface as,

$$\vec{r}_{1,2} = \begin{pmatrix} R^{Tube} \cos \beta_{1,2} \\ R^{Tube} \sin \beta_{1,2} \\ z_{1,2} \end{pmatrix} + \vec{o}_{1,2}, \tag{1}$$

where subindex represents the left and right ball. $\vec{o}_{1,2}$ defines an arbitrary origin of the tubes. We omit this index and reintroduce it when necessary. We define the normal vector $\hat{n}$ and the tangential vectors $\hat{e}_\beta$ and $\hat{e}_z$ as

$$\hat{n} = \begin{pmatrix} \cos \beta \\ \sin \beta \\ 0 \end{pmatrix}, \quad \hat{e}_\beta = \begin{pmatrix} -\sin \beta \\ \cos \beta \\ 0 \end{pmatrix}, \quad \hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

We introduce a rotation of the ball around an axis $\Delta\vec{\theta}$ by an angle of $|\Delta\vec{\theta}|$. Assuming that the ball does not slip, we can calculate the displacement $\Delta\vec{d}$ of a ball on two-dimensional sheet with the normal $\hat{n}$ is

$$\Delta\vec{d} = R \cdot \hat{n} \times \Delta\vec{\theta}, \tag{3}$$

where R is the radius of the ball. A small change $\Delta\beta$ and Δz on the cylinder surface result in a displacement,

$$\Delta\vec{d} = \Delta\beta \cdot \hat{e}_\beta + \Delta z \cdot \hat{e}_z. \tag{4}$$

Since $\hat{e}_\beta \cdot \hat{e}_z = 0$, it follows that,

$$\Delta\beta = (R \cdot \hat{n} \times \vec{\Delta}\theta) \cdot \hat{e}_\beta, \tag{5}$$

$$\Delta z = (R \cdot \hat{n} \times \vec{\Delta}\theta) \cdot \hat{e}_z. \tag{6}$$

Now taking the limit to infinitesimal small Δ and dividing by the time interval Δt, we obtain the differential equations

$$\partial_t \beta = (R \cdot \hat{n} \times \vec{\Omega}) \cdot \hat{e}_\beta, \tag{7}$$

$$\partial_t z = (R \cdot \hat{n} \times \vec{\Omega}) \cdot \hat{e}_z, \tag{8}$$

where $\vec{\Omega} = \partial_t \vec{\theta}$ is the angular velocity. The angular velocity can be calculated from the angular momentum $\vec{L}$. It intern is given by the following differential equation,

$$\vec{\Omega} = \frac{\vec{L}}{I}, \quad \partial_t \vec{L} = \vec{\tau} - \eta \vec{L}, \tag{9}$$

where $\vec{\tau}$ is the torque acting on the ball, I is the moment of inertia of the ball, and η is a damping parameter ($\eta > 0$). For a solid ball, $I = (2/5)m \cdot R^2$, with m being the mass of the ball. The magnetic moment $\vec{M}$ changes with the rotation of the ball, via $\partial_t \vec{M} = \vec{\Omega} \times \vec{M}$. To enforce a constant size of $\vec{M}$, we express $\vec{M}$ in spherical coordinates,

$$\vec{M} = \mu \begin{pmatrix} \sin\alpha \cos\varphi \\ \sin\alpha \sin\varphi \\ \cos\alpha \end{pmatrix}. \tag{10}$$

The differential equation for $\vec{M}$ results in the following equations for α and φ,

$$\partial_t \alpha = \Omega^x \cos\varphi - \Omega^y \sin\varphi, \tag{11}$$

$$\partial_t \varphi = \Omega^z - \cot\alpha(\Omega^x \cos\varphi + \Omega^y \sin\varphi). \tag{12}$$

Equations (7), (8), (9), (11), and (12) have to be solved, resulting in 7 coupled differential equations for each ball. These equations are coupled in the sense that the torque $\vec{\tau}$ depends on α, β, φ, and z. The torque on the ball 1 $\vec{\tau}_1$ is given by the magnetic torque $\vec{\tau}_1^{\mathrm{Mag}}$ and the frictional torque $\vec{\tau}_{\mathrm{Friction},1}$. The contribution to the frictional force can be split up in a part due to gravity $\vec{F}_{\mathrm{Gravity},1}$, and the part due to the magnetic attraction $\vec{F}_{\mathrm{Mag},1}$. The magnetic torque on the ball 1 is simply

$$\vec{\tau}_1^{\mathrm{Mag}} = \vec{M}_1(\alpha_1, \varphi_1) \times \vec{B}_2(\beta_1, \beta_2, z_1, z_2, \alpha_2, \varphi_2), \tag{13}$$

where $\vec{B}_2$ is the magnetic field of the ball 2 at the position of ball 1. It is given by,

$$\vec{B}_2(\ldots) = \frac{\mu_0}{4\pi} \left[\frac{3\vec{x}_{12} \cdot (\vec{M}_2 \cdot \vec{x}_{12})}{|\vec{x}_{12}|^5} - \frac{\vec{M}_2}{|\vec{x}_{12}|^3} \right], \tag{14}$$

with $\vec{x}_{12}$ being the relative distance between $\vec{r}_1$ and $\vec{r}_2$ (from Eq. (1)),

$$\vec{x}_{12} = \vec{r}_1 - \vec{r}_2. \tag{15}$$

For the frictional torque we consider the total force $\vec{F}_1 = \vec{F}_{\mathrm{Gravity},1} + \vec{F}_{\mathrm{Mag},1}$. The gravitational part for both balls is $mg \cdot \hat{e}_\downarrow$, where $\hat{e}_\downarrow$ is the direction of gravity. For tubes perpendicular to the ground the $\hat{e}_\downarrow = -\hat{e}_z$. The magnetic force on the ball 1 is,

$$\vec{F}_{\mathrm{Mag},1} = \vec{\nabla}(\vec{M}_1 \cdot \vec{B}_2) \tag{16}$$

$$= \frac{3\mu_0}{4\pi|\vec{x}_{12}|^5} \left[(\vec{M}_2 \cdot \vec{x}_{12})\vec{M}_1 + (\vec{M}_1 \cdot \vec{x}_{12})\vec{M}_2 + (\vec{M}_1 \cdot \vec{M}_2)\vec{x}_{12} - \frac{5(\vec{M}_1 \cdot \vec{x}_{12})(\vec{M}_2 \cdot \vec{x}_{12})}{|\vec{x}_{12}|^2} \vec{x}_{12} \right]. \tag{17}$$

Given the total force $\vec{F}_1$, we can calculate the normal and tangential parts of the force at the surface of the cylinder.

$$\vec{F}_1^{\text{Normal}} = (\vec{F}_1 \cdot \hat{n}_1)\hat{n}_1 \quad \text{and} \quad \vec{F}_1^{\text{Tangent}} = \vec{F}_1 - \vec{F}_1^{\text{Normal}}. \tag{18}$$

We calculate the frictional force as,

$$\vec{F}_1^{\text{Friction}} = -k|\vec{F}_1^{\text{Normal}}| \cdot \vec{F}_1^{\text{Tangent}}, \tag{19}$$

where k is the rolling friction coefficient. The frictional torque is then calculated as,

$$\vec{\tau}_1^{\vec{F}} = \begin{cases} \vec{0} & \text{if} \quad |\vec{F}_1^{\text{Friction}}| > |\vec{F}_1^{\text{Tangent}}|, \\ (R \cdot \hat{n}_1) \times (\vec{F}_1^{\text{Friction}} - \vec{F}_1^{\text{Tangent}}) & \text{else.} \end{cases} \tag{20}$$

The torques $\vec{\tau}_1^{\vec{F}}$ and $\vec{\tau}_1^{\text{Mag}}$ are added to result in the total torque on the ball 1 $\vec{\tau}_1$. Analog to this also the torque on the ball 2 can be calculated. The rotation of the tubes $\vec{\Omega}_{\text{Tube}}$ can be incorporated by adding the constant term $\vec{\Omega}_{\text{Tube}}$ to the equation Eq. (7). Since the magnetization rotates also when the tubes are rotated direction one has to also add $\vec{\Omega}_{\text{Tube}}$ from Eq. (12).

The full equations that have to be solved are,

$$\partial_t \vec{L}_1 = \vec{\tau}_1(\beta_1, \beta_2, z_1, z_2, \alpha_1, \alpha_2, \varphi_1, \varphi_2) - \eta \vec{L}_1, \tag{21}$$

$$\partial_t \vec{L}_2 = \vec{\tau}_2(\beta_1, \beta_2, z_1, z_2, \alpha_1, \alpha_2, \varphi_1, \varphi_2) - \eta \vec{L}_2, \tag{22}$$

$$\partial_t \beta_1 = \frac{1}{I}(R \cdot \hat{n}(\beta_1) \times \vec{L}_1) \cdot \hat{e}_\beta(\beta_1) + \vec{\Omega}_{\text{Tube},1}, \tag{23}$$

$$\partial_t \beta_2 = \frac{1}{I}(R \cdot \hat{n}(\beta_2) \times \vec{L}_2) \cdot \hat{e}_\beta(\beta_2) + \vec{\Omega}_{\text{Tube},2}, \tag{24}$$

$$\partial_t z_1 = \frac{1}{I}(R \cdot \hat{n}(\beta_1) \times \vec{L}_1) \cdot \hat{e}_z, \tag{25}$$

$$\partial_t z_2 = \frac{1}{I}(R \cdot \hat{n}(\beta_2) \times \vec{L}_2) \cdot \hat{e}_z, \tag{26}$$

$$\partial_t \alpha_1 = \frac{1}{I}(L_1^x \cos\varphi_1 - L_1^y \sin\varphi_1), \tag{27}$$

$$\partial_t \alpha_2 = \frac{1}{I}(L_2^x \cos\varphi_2 - L_2^y \sin\varphi_2), \tag{28}$$

$$\partial_t \varphi_1 = \frac{1}{I}(L_1^z - \cot\alpha_1(L_1^x \cos\varphi_1 + L_1^y \sin\varphi_1)) + \vec{\Omega}_{\text{Tube},1}, \tag{29}$$

$$\partial_t \varphi_2 = \frac{1}{I}(L_2^z - \cot\alpha_2(L_2^x \cos\varphi_2 + L_2^y \sin\varphi_2)) + \vec{\Omega}_{\text{Tube},2}. \tag{30}$$