# Classical Magnetic Systems

On the behaviour of macroscropic magnetized moments in dissipative environments

ELENA VEDMEDENKO, TIM MATTHIES, RASMUS RASCHKE

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Student Project
Hamburg, Germany September 2025

# Preface

This document contains our work regarding the novel spin revolution effect, first described by PD Dr. Elena Vedmedenko and Prof. Roland Wiesendanger[**Vedmedenko**]. The goal of this project is to further the understanding of the presented systems by investigating certain mathematical aspects of the system's description. Our hope is to procure either analytically solvable equations of motions or prove that such solutions do, in fact, not exist. We start by introducing the necessary mathematical principles rooted in the theory of manifolds, variational calculus and ordinary differential equations. From there on, we investigate various realizations of the spin revolution effect.

*Raschke, August 2025*

# Contents

# CHAPTER ONE

## Mathematical Introduction

We start with the necessary mathematical theory. If the reader already feels comfortable with the theory of smooth manifolds, variational calculus on aforementioned spaces, (Lie) groups as well as autonomous differential equations, they may skip this chapter. We assume basic knowledge of linear algebra and calculus.

## 1.1 Group Theory

We need some fundamental group theory. From here on, all vector spaces are finite-dimensional.

**Definition 1.1** (Group)**.** A **Group** is a pair $(G, \circ)$ consisting of a set $G$ and an operation $\circ$ such that the following axioms are satisfied:

$\forall a, b, c \in G : a \circ (b \circ c) = (a \circ b) \circ c$

- $\forall a \in G\, \exists a^{-1} \in G : a \circ a^{-1} = 1$
- $\exists e \in G\, \forall a \in G : e \circ a = a$

If the operation is commutative, we call the group **abelian**.

One should think of groups in terms of symmetries: A symmetry of an abstract object can be thought of as a collection of operations leaving said object invariant.

There are plenty of examples of groups important in physics:

**Example 1.2.** Let $V$ be a real vector space. The automorphisms $V \to V$ consitute a group under composition, the *general linear group* of dimension $n$, $\mathrm{GL}_n(V)$. There are several important subgroups of $\mathrm{GL}(\mathbb{R}^n)$ we will heavily use later. Let

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \to \mathbb{R}$$

be the euclidean scalar product on $\mathbb{R}^n$ and $T \in \mathrm{GL}(\mathbb{R}^n)$. We define the *orthogonal group* as

$$\mathrm{O}(n) := \{Q \in \mathrm{GL}(\mathbb{R}^n) \mid QQ^t op = Q^\top Q = \mathrm{id}\} \leq \mathrm{GL}(\mathbb{R}^n).$$

Further restricting our attention to orthogonal automorphisms with unit determinant yields the *special orthogonal group*

$$\mathrm{SO}(n) := \{R \in \mathrm{O}(n) \mid \det R = 1\} \leq \mathrm{O}(n)..$$

Note that every vector space over $\mathbb{R}$ has a basis. Choosing one and representing automorphisms as $n \times n$-matrices yields the usual group structure by matrix multiplication. This also clearly demonstrates that groups of linear maps cannot be abelian in general.

In physics, we are usually concerned with how a certain group acts on a physical system. For that, we need actions and representations.

**Definition 1.3** (Group Action)**.** Let $G$ be a group with identity $e$ and $M$ be a set. A **right-action** of $G$ on $M$ is a map

$$\alpha : G \times X \to X.$$

such that:

- $\alpha(e, x) = x$
- $\alpha(h, \alpha(g, x)) = \alpha(gh, x)$

is satisfied. We write $G \curvearrowright M$ and $\alpha(g, x) =: g.x$ for short.

**Definition 1.4** (Representation)**.** Let $V$ be a real vector space and $G$ be a group. A **real** $G$**-representation** is a group homomorphism

$$\rho : V \to \mathrm{GL}(V).$$

We call $V$ **representation space** and $\deg V$ the degree of the representation.

## 1.2 Configuration Manifolds and Lie groups

From now on, we use the Einstein summation convention.

**Definition 1.5** (Topological Manifold)**.** A **topological** $n$**-manifold** is a topological space $M$ such that:

- $M$ has the Hausdorff property.
- $M$ is second-countable.
- $M$ is locally euclidean: For every $p \in M$ there is an open neighbourhood $U \subseteq M$ of $p$ and a homeomorphism $\phi : U \to V$ such that $V \subseteq \mathbb{R}^n$ is open. We call $(\phi, U)$ a **chart** on $M$.

Since we want to apply the formalism in concrete examples, we will often work in local coordinates. Note that with the standard basis of $\mathbb{R}^n$, we can write a chart map as

$$\phi(p) = (\phi_1(p), \ldots, \phi_n(p)) =: (x^1(p), \ldots, x^n(p)).$$

where $x^i : M \to \mathbb{R}$ are the chart components. Since the charts are homeomorphisms, they can be locally inverted to a map $\phi^{-1} : V \to U$, in which case we call $\phi^{-1}$ a *local parametrization* of $M$.

**Example 1.6.**
- Trivially, the real euclidean space $\mathbb{R}^n$ is an $n$-dimensional smooth manifold with the identity id being a global chart.
- The $n$-sphere $\mathbb{S}^n$ is a smooth manifold with local charts given by projection onto the coordinate axes.

- The $n$-torus

$$T^n = \underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n \text{ times}}$$

  is a smooth $n$-manifold. In the case of $T^2$, one obtains local charts easily as the 2-torus is a surface of revolution.

We need the notion of smoothness on manifolds.

**Definition 1.7** (Smooth Manifold)**.** A **maximal atlas** for a topological manifold $M$ is a collection $\mathfrak{A}$ of charts of $M$ such that every $p \in M$ is contained in some chart and $\mathfrak{A}$ is not properly contained in some other atlas. We call $\mathfrak{A}$ smooth if for any two charts $(U, \phi), (V, \psi) \in \mathfrak{A}$ we have either $U \cap V = \emptyset$ or

$$\phi^{-1} \circ \psi : V \to U.$$

is a smooth ($\mathcal{C}^\infty$) map, called **chart transition map**. A **smooth manifold** is a pair $(M, \mathfrak{A})$ such that $M$ is a topological manifold and $\mathfrak{A}$ is a maximal smooth atlas.

Note that with this, we get a notion of smooth maps on abstract manifolds: If $M, N$ are smooth manifolds and $F : M \to N$ is a map, we call it smooth if for every $p \in M$ there is a chart $(U, \phi)$ with $p \in U$ and a smooth chart $(V, \psi)$ with $F(p) \in V$ such that

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

is smooth in the usual sense.

There are many examples important to physics:

**Example 1.8.** • The $\mathbb{R}^n$ obtains a smooth maximal atlas by means of the identity.

- The sphere $\mathbb{S}^2$ has a smooth atlas with two charts $\mathbb{S}^2 \setminus N$ and $\mathbb{S}^2 \setminus S$, where $N$ and $S$ are north and south pole, respectively. The chart map is given by the *stereographic projection* $\sigma : \mathbb{S}^2 \setminus N \to \mathbb{R}^2$.

- Any finite-dimensional real normed vector space $V$ is a smooth manifold. The choice of a basis determines an isomorphism $\mathbb{R}^n \cong V$ which we take as global chart. For any other basis, one obtains a basis transformation matrix which is linear, hence a $\mathcal{C}^\infty$ chart transition.

- Any open subset $U \subseteq \mathbb{R}^n$ is a smooth manifold on its own with the smooth atlas $\{U, \mathrm{id}_U\}$. We call such a manifold an *open submanifold* of $\mathbb{R}^n$.

With smooth maps on manifolds, we are already able to talk about curves or paths on such spaces. However, in physics one is especially interested in velocity and acceleration. For this, we will need two additional structures: The tangent bundle, which comes naturally with every smooth manifold, and a metric to measure distances and angles. We define a **curve** on a manifold $M$ to be a smooth map

$$\gamma : I \to M$$

where $I \subseteq \mathbb{R}$ is some interval.

**Definition 1.9** (Tangent Bundle)**.** Let $M$ be a smooth manifold and $p \in M$. We declare two smooth curves $\gamma_i : I \to M$, $i \in \{1, 2\}$ to be equivalent if for any chart $(U, \phi)$ of $p$ we have

$$\frac{d}{dt}(\phi \circ \gamma_1)\bigg|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2)\bigg|_{t=0}. \tag{1.1}$$

The space of equivalence classes of such curves is called **tangent space** at $p$ and denoted $T_pM$. The **tangent bundle** of $M$ is the disjoint union

$$TM := \coprod_{p \in M} T_pM.$$

Note that while the tangent space at a point on an $n$-manifold is an $n$-dimensional real vector space (and hence isomorphic to a copy of $\mathbb{R}^n$), the tangent bundle is a $2n$-dimensional space and indeed also a smooth manifold. However, it is not always possible to identify $TM \cong M \times \mathbb{R}^n$. A point in $TM$ is a pair $(p, v)$, where $p \in M$ and $v \in T_pM$. We also have a natural projection $\pi : TM \to M$, given by $\pi(p, v) = p$.

Tangent vectors act on smooth functions $f : M \to \mathbb{R}$ by

$$v(f) = \frac{d}{dt}\bigg|_{t=0} (f \circ \gamma)(t).$$

where $\gamma$ represents one path with $\dot{\gamma}|_{t=0} = v$. Using the fact that for every $p \in M$ there is a chart $(U, (x^1, \dots, x^n))$ centered around $p$, we are able to pull back the basis of $T_{\phi(p)}\mathbb{R}^n$ to $T_pM$. The basis of $T_{\phi(p)}\mathbb{R}^n$ is given by paths of the form $t \mapsto \phi(p) + te_i$, where $e_i$ is the $i$-th canonical basis vector of $\mathbb{R}^n$. Writing this basis as $(\partial_{x^1}, \dots, \partial_{x^n})$, we can pull back along $\phi^{-1}$ to obtain a basis of $T_pM$, defined as

Note that this gives rise to the usual partial derivative of a function $g : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\frac{\partial g}{\partial x^i} |_y := \frac{d}{dt} g(y + te_i).$$

$$\partial_i|_p := \phi^{-1}(\phi(p) + te_i).$$

This expression acts on a smooth function as above by

$$\partial_i|_p f = \frac{\partial (f \circ \phi^{-1})}{\partial x^i}\bigg|_{\phi(p)}.$$

**Definition 1.10** (Differential)**.** Let $F : M \to N$ be a smooth map between smooth manifolds. The **differential** of $F$ at $p \in M$ is a linear map

$$dF_p : T_pM \to T_{F(p)}N$$

given by

$$dF_p([\gamma]) = [F \circ \gamma].$$

where $[\gamma]$ is an equivalence class of curves through $p$.

Given local charts $(U, (x^1, \dots, x^m))$ around $p$ and $(V, y^1, \dots, y^n)$ around $F(p)$, the differential can be expressed in local coordinates as

$$dF_p(\partial_i) = \partial_i F^j|_p \hat{\partial}_j|_{F(p)}.$$

with $F^j = y^j \circ F$ and $\hat{\partial}_j$ being a basis vector of $T_{F(p)}N$.

## 1.3 Variational Calculus

## 1.4 Ordinary Differential Equations

CHAPTER TWO

# Magnetic Ball on an Incline

## 2.1 Two-Dimensional System

We start our investigation by considering a very simplified model of a rigid two-dimensional disk with a constant magnetic moment rolling on an inclined plane without slipping.