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THE DIFFERENCE BETWEEN GILBERT'S AND LANDAU-LIFSHITZ'S EQUATIONS

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Abstract—The difference between the Gilbert equation and the Landau–Lifshitz equation is discussed. An interpretation is presented that both equations assume viscous type damping, and the damping action is isotropic in the Gilbert equation, but is strongly anisotropic in the Landau–Lifshitz equation. It seems more reasonable to assume the Gilbert equation physically, if we do not know anything about the detailed mechanism of the damping of a problem.

1. INTRODUCTION

It is well known that there are two basic kinetic equations for the movement of the magnetization of a ferromagnet. These are the Landau–Lifshitz equation,[1] and the Gilbert equation,[2] as shown below.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\gamma'(M\times H)-\gamma\frac{M}{M}\times\left(\frac{M}{M}\times H\right) \qquad (1)$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\gamma(M\times H)+\alpha\frac{M}{M}\times\frac{\mathrm{d}M}{\mathrm{d}t} \qquad (2)$$

Here, the second term represents the damping of the motion. Although the first term can be derived quantum mechanically, there is no intrinsic general derivation for the second term.

Equation (1) was presented by Landau–Lifshitz[1] and equation (2) was presented by Gilbert.[2,3]† Gilbert thought that equation (2) should be considered a more fundamental equation than equation (1). He emphasized that in a case of eddy-current damping equation (2) is a direct result of Lagrangian analysis. His paper, however, is not well explained and the physical meaning of the two equations was not made clear. Kittel has employed the form of equation (1) in his analysis of the heavy damping phenomena of rare earth iron garnets in ferromagnetic resonance experiments.[4] He mentions the Gilbert form of damping in that paper and concludes that equation (2) cannot describe the heavy damping phenomena intrinsically. His paper, however, does not clearly explain the reason for the intrinsic inadequateness of the Gilbert equation. One of the reasons mentioned is the lack of success of the equation for the description of the experimental results on rare earth iron garnets. This evidence, however, is not conclusive since there are alternative explanations[5] for this experimental result.

Kikuchi[6] has pointed out mathematically that, in the rotational switching phenomena of spherical single domain ferromagnets, the Gilbert equation gives the highest switching speed when α is equal to one, but there is no limit for the speed as a function of γ in Landau–Lifshitz equation. Gyorgy found experimentally[7] that in the range‡ of the switching which follows after the wall displacement range as a function of applied field, the switching coefficient

$$S_\omega = \frac{\mathrm{d}\tau_s}{\mathrm{d}(1/H)} \qquad (3)$$

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† Suhl and Walker have suggested this equation in *Bell Syst. tech. J.* **33,** 568 (1954).

‡ This region is described by him as a nonuniform rotational range.

of rectangular hysteresis loop ferromagnets, both for thin films and ferrites, is in good agreement with the minimum switching coefficient that can be expected from the Gilbert equation, i.e. with that which is obtained by putting α equal to one. Although, since we have deduced another form of description of this phenomenon,[8] the experimental results of the rotational switching phenomena do not necessarily indicate the intrinsic adequateness of the Gilbert equation, this is another field of interest in which the difference between the two equations may possibly become quite important.

In these connections, we feel that the past interpretations of the two equations are quite uncertain. Here we hope to present our own interpretation.

2. MATHEMATICAL EQUIVALENCE

It is already well known that the two equations are mathematically equivalent when we use the following transfer relations.

$$\gamma' = \frac{\gamma}{1+\alpha^2} \tag{4}$$

$$\lambda = \gamma M \frac{\alpha}{1+\alpha^2} \tag{5}$$

or

$$\gamma = \gamma'\left(1+\frac{\lambda^2}{M^2\gamma'^2}\right) \tag{6}$$

$$\alpha = \frac{\lambda}{M\gamma'}. \tag{7}$$

It is noticed that these are all one-to-one relations. The relation of equations (4) and (5) are shown in Fig. 1. It may be noted that λ has the same value for two different values of α, but γ' does not have this character.

Mathematically, the Landau–Lifshitz equation is more convenient because it has no time derivative on the right-hand side and when we define θ and ϕ as is shown in Fig. 2, it is easily deduced that

$$\frac{d\phi}{dt} = \gamma' H \tag{8}$$

$$\frac{d\theta}{dt} = -\lambda\frac{H}{M}\sin\theta \tag{9}$$

Thus, mathematically, the two principal components of the movement are separately represented by the two constants of the Landau–Lifshitz equation. Therefore this equation seems more convenient as a phenomenological equation of the problem.

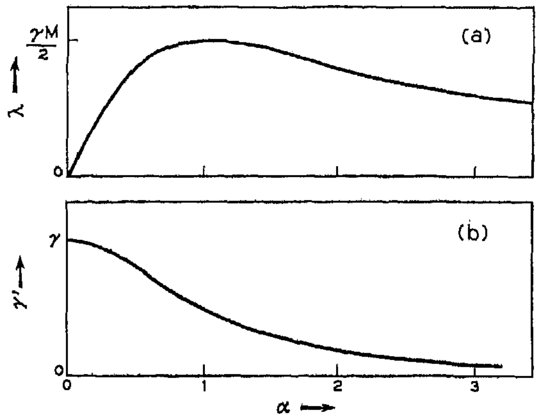

FIG. 1. Constant λ or γ' in the Landau–Lifshitz equation as a function of α of the Gilbert equation.

The corresponding expressions of the Gilbert equation are obtained by transforming γ' and λ in equations (8) and (9) into expressions in terms of γ and α with the aid of equations (4) and (5). The relation between γ' and λ and γ, α is already illustrated in Fig. 1.

Now, the two equations are quite different physically when γ' or γ is regarded as a true gyromagnetic constant

$$\frac{e}{2mc}\cdot g' \tag{10}$$

and the second term is assumed to represent the energy dissipation of the magnetic system to other systems which are discarded in the equation. This is the main subject of the following discussion.

It is, however, also well known that, even in this case, the two equations are practically equivalent when the dissipation of the energy is very small. Mathematically, when

$$\alpha = \frac{\lambda}{M\gamma'} \ll 1 \tag{11}$$

and α^2 can be neglected, equations (4), (5), (6), and (7), become

$$\gamma' = \gamma \tag{12}$$

$$\lambda = \gamma M \alpha \tag{13}$$

so that the two equations become equivalent physically as well. Since, in most cases, equation (11) can be assumed, the difference between the two equations is usually not significant.

of ferromagnets where the magnetic field applied is much smaller than the exchange field. Of course it is possible to have long wavelength spin waves which destroy the constancy of $|M|$ without consuming much exchange energy. In this case, however, only a limitation is required for the size of the region where the equations can be applied to a certain small unit where M can be assumed uniform.

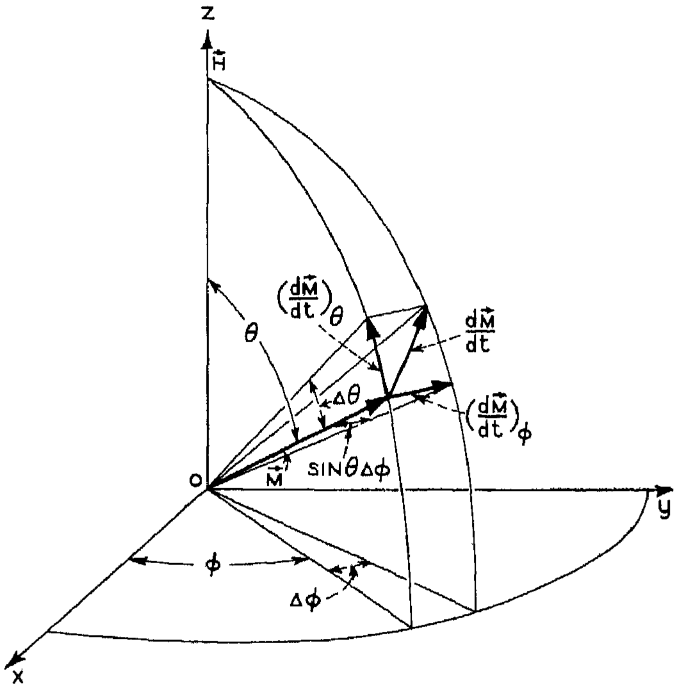

FIG. 2. Definitions of θ, ϕ, $(dM/dt)_\theta$ and $(dM/dt)_\phi$.

3. BASIC ASSUMPTIONS OF THE TWO EQUATIONS

There are several assumptions present as a basis of the two equations.

The first is the constancy of the absolute magnitude of the magnetization, $|M|$. The basis of this assumption is the presence of a strong exchange field between every neighboring unit component of M. Since we know that even if a considerable amount of spin waves is excited, still $|M|$ can be assumed unchanged up to a first order of the amplitudes of spin waves; this assumption is quite reasonable in the usual phenomena

The second assumption is the isotropic nature of the ferromagnet. If an anisotropic nature of the crystal is to be taken into account then γ or γ' and damping constant α or γ must be a function of the direction of M referred to the crystal axes. It is noticed, however, that even under this assumption, the crystal can still be anisotropic because of the presence of the applied field H. This is the case for the Landau–Lifshitz equation as is shown next.

The third and most important assumption that we have concluded is the assumption of the presence of viscous type damping. By viscous damping we mean that the damping force at each instant is only a function of the location and the rate of movement of M at that instant and is a negative linear function of the rate of the movement. In a magnetic system, the magnitude of the direct force operating is implicit in general and this is represented by an equivalent magnetic field. In the GILBERT equation,[2] it is obvious that this damping field is

$$-\frac{\alpha}{\gamma}\frac{1}{M}\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} \tag{14}$$

and obviously is proportional to and opposes the velocity of the motion, $\mathrm{d}\boldsymbol{M}/\mathrm{d}t$. In the Landau–Lifshitz equation, equation (1) can be transformed into

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\gamma'(\boldsymbol{M}\times\boldsymbol{H})+\frac{\lambda}{\gamma' M}\frac{\boldsymbol{M}}{M}\times\left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\phi} \tag{15}$$

where $(\mathrm{d}\boldsymbol{M}/\mathrm{d}t)_{\phi}$ is the ϕ component of $\mathrm{d}\boldsymbol{M}/\mathrm{d}t$, as has been shown in Fig. 2. Therefore in this case the damping field is

$$-\frac{\lambda}{\gamma'^2 M^2}\left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\phi} \tag{16}$$

and is always proportional to ϕ, i.e. the precessional component of $\mathrm{d}\boldsymbol{M}/\mathrm{d}t$. This means that a viscous type damping is operating for the movement along ϕ direction but is not operating for the movement along θ direction. Therefore, the Landau–Lifshitz equation represents a motion of the magnetization in an extremely highly anisotropic medium for damping. According to the second assumption, this highly anisotropic damping must be introduced merely from the presence of magnetic field H, since the material itself is assumed to be isotropic.

4. GENERAL EXPRESSION FOR THE TWO EQUATIONS

Based on the discussion of the previous Section, the following general expression is obtained for the dynamical equation of the magnetic system. It is

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\gamma^{+}(\boldsymbol{M}\times\boldsymbol{H})+\alpha_{\phi}\frac{\boldsymbol{M}}{M}\times\left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\phi} + \alpha_{\theta}\frac{\boldsymbol{M}}{M}\times\left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)_{\theta} \tag{17*}$$

Here the second term represents the viscous type damping for the ϕ component and the third term that for the θ component. If we put

$$\alpha_{\phi} = \alpha_{\theta} = \alpha. \tag{18}$$

Equation (17) coincides with the Gilbert equation and if we put

$$\alpha_{\theta} = 0$$
$$\alpha_{\phi} = \frac{\lambda}{\gamma^{+}M}. \tag{19}$$

Equation (17) coincides with the Landau–Lifshitz equation. As is the case of the Gilbert equation, equation (17) again can be reduced to a Landau–Lifshitz form mathematically as

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\frac{\gamma^{+}}{1+\alpha_{\theta}\alpha_{\phi}}(\boldsymbol{M}\times\boldsymbol{H}) - \frac{M\alpha_{\phi}\gamma^{+}}{1+\alpha_{\theta}\alpha_{\phi}}\left[\frac{\boldsymbol{M}}{M}\times\left(\frac{\boldsymbol{M}}{M}\times\boldsymbol{H}\right)\right] \tag{20}$$

Therefore

$$\gamma' = \frac{\gamma^{+}}{1+\alpha_{\theta}\alpha_{\phi}} \tag{21}$$

$$\lambda = \frac{\gamma^{+}M\alpha_{\phi}}{1+\alpha_{\theta}\alpha_{\phi}}. \tag{22}$$

It is noticed that in the other extreme case where

$$\alpha_{\phi} = 0 \tag{23}$$

Equation (20) reduces to the original Zeeman term only. The damping is inactive in this case.

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* An equation with combined damping terms:

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\gamma(\boldsymbol{M}\times\boldsymbol{H})+\frac{\boldsymbol{M}}{M}\times\left(\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}\right)-\lambda\frac{\boldsymbol{M}}{M}\times\left(\frac{\boldsymbol{M}}{M}\times\boldsymbol{H}\right)$$

was suggested by L. R. WALKER in private discussion.

5. DISCUSSION

As is shown, the Landau–Lifshitz equation describes one of the extreme cases in which no resistance is present for the movement of magnetization along θ direction. This is so unusual that we feel that the Gilbert equation that assumes isotropic damping is the more normal expression which should be taken as a zero order approximation, so far as we know nothing about the detailed mechanism of the damping. The Landau–Lifshitz equation can then regarded as a simplified form of the Gilbert equation which is derived mathematically by eliminating the time derivative term in the right-hand side of the Gilbert equation, and employing the meaning of the constants γ' and λ as given in equations (4) and (5). In this connection it is interesting to illustrate the vector relations of the two equations on a plane which is normal to M. They are shown in Figs. 3(a) and (b). In the Landau–Lifshitz equation, the damping torque

$$-\lambda\frac{M}{M}\times\left(\frac{M}{M}\times H\right) \tag{24}$$

always operates at a right angle to the driving torque

$$-\gamma'(M\times H) \tag{25}$$

The actual movement, therefore, of the magnetization dM/dt is always larger than the driving and damping torques. Further, this equation allows the possibility that the damping torque can become even larger than the driving torque when

$$\lambda > \gamma'M \sim 10^{10}\ \mathrm{sec}^{-1} \tag{26}$$

Since, in this magnetic system, movement is always a result of the presence of a certain torque and the presence of a damping torque is a result of the presence of a driving torque, the fact mentioned seems quite unusual. It is noticed, however, that this is not illogical since this torque does not do any more work than that which is allowed by the energy conservation law. Obviously this unusual situation is a result of the mentioned extreme condition implied for damping.

In Fig.2(b), it is shown that the damping torque

$$\alpha\frac{M}{M}\times\frac{dM}{dt} \tag{27}$$

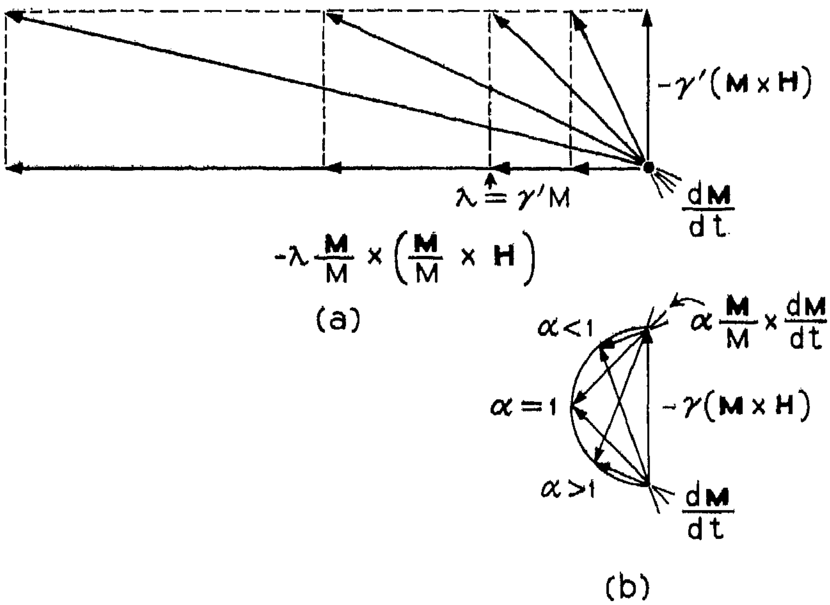

FIG. 3. Vector relations in the Landau–Lifshitz and the Gilbert equations, illustrated on the normal plane to the magnetization M.

of the Gilbert equation is always at right angles to the movement of magnetization dM/dt and less than the driving torque

$$-\gamma(M \times H) \tag{28}$$

As a result, the movement of magnetization dM/dt is always less than the driving torque $-\gamma(M \times H)$. This is quite normal.

In the case of rotational switching of a spherical single domain ferromagnet, the employment of the Gilbert equation gives an essential limitation for the speed as a function of damping constant α, as is discussed by Kikuchi, Gyorgy *et al.* and is shown here in equations (9) and (5). We feel this is a correct conclusion for normal ferromagnets. It is emphasized, however, that if we could find a material which has a quite anisotropic damping mechanism in which α_θ is quite small and α_ϕ is quite large, then equations (9) and (22) say that this material can exhibit a very high speed switching without limitation. This theoretical standpoint seems somewhat analogous to the case of the ferroxplana problem. It is, however, noticed again that this theory can be applicable only in the case of ideal spherical single domain ferromagnets and not the cases of the switching of polycrystalline ferrites or thin films.[8]

The assumption of viscous type damping is not a unique mechanism of damping. We are familiar with frictional type damping in classical dynamics, and also nonlinear damping effects in fluid dynamics. In the case of damping by spin-wave generation, we know also that many nonlinear effects are introduced by the mutual interactions of spin waves and the uniform mode. Therefore, the applicability of equation (17) depends always on whether in the physical situation of the problem viscous type damping can be assumed or not. It is noted that so far as a linear relation can be assumed between the damping field and the velocity of the movement of the magnetization, the equation can be reduced always to a Landau–Lifshitz form. (This is equivalent to regarding α_ϕ and α_θ in equation (17) as tensors which operate on $(dM/dt)_\phi$ or $(dM/dt)_\theta$.)

As mentioned in Section 2, the difference of the two equations can be appreciable only in the case where the damping term is quite larg We feel so far that there is no experimental evidence that can differentiate between the two equations clearly, except the case of eddy-current damping of Gilbert. In this connection the experiment of magnetic damping of rare earth garnets seems quite interesting. But in this case again, if we made the experiment on single crystals, the anisotropies of γ, α_θ and α_ϕ referred to the crystal axes and to the direction of the applied field, may be found to be significant. We feel, therefore, that the main purpose of this paper is to make clear the physical situation of the two equations. Practical application of this paper should be made carefully, taking into account many additional factors.

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