Riemannian and Lorentzian Geometry

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Chapter 1

Introduction

1.0 Review of important topics

Definition 1.1 (Riemannian and Lorentzian Metrics)**.** Let $\nu \in \mathbb{N}$ with $0 \leq \nu \leq n$. A **semi-Riemannian metric** of **index** ν is a $(0,2)$-tensor field such that

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a symmetric non-degenerate bilinear form on T_pM with index ν. We say:

- $\nu = 0$: g is **Riemannian**.
- $\nu = 1$: g is **Lorentzian**.

In the Lorentzian case, we take the convention $(-,+,+,\dots)$.

Theorem 1.2 (Levi-Civita Connection)**.** Given a semi-Riemannian manifold (M,g), there exists exactly one connection ∇^g, called the **Levi-Civita connection**, such that:

- ∇^g is symmetric: $\nabla_X Y - \nabla_Y X = [X,Y]$
- ∇^g is compatible with g: $Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$
- The Koszul identity is satisfied.

Definition 1.3 (Geodesic)**.** A $\mathcal{C}^\infty$-curve $\gamma : (a,b) \to M$ is called a **geodesic** if $\dot\gamma$ is ∇^g*−parallel* along γ, i.e.

$$\ddot\gamma(t) = \nabla^g_{\frac{d}{dt}} \dot\gamma = 0.$$

In local coordinates, one finds the *geodesic equation*

$$0 = \frac{d^2}{dt^2}(x^i \circ \gamma) + (\Gamma^i_{kl} \circ \gamma)\frac{d}{dt}(x^k \circ \gamma)\frac{d}{dt}(x^l \circ \gamma).$$

Theorem 1.4 (Maximal Geodesic). For all $v \in TM$, there is exactly one geodesic

$$\gamma_v : I_v \to M$$

such that $\gamma_v(0) = \pi\|v\|$ and $\dot{\gamma}_v(0) = v$ and I_v is maximal.

Exponential Map

Definition 1.5 (Exponential Map). Define the (Riemannian) **exponential map**

$$\exp_p : \mathcal{D}_p \subseteq T_pM \to M$$

by

$$(p, v) \mapsto \exp_p(v) := \gamma_v(1)$$

where γ_v is the unique maximal geodesic.

We always have

$$\mathcal{D}_p \supseteq \{tv \in T_pM \mid t \in [0, 1]\}.$$

We can consider $s \mapsto \gamma_v(st)$ for fixed $t \in \mathbb{R}$. Then, we have

$$\dot{\gamma}_v(st) = t\dot{\gamma}_v(0) = tv = \dot{\gamma}_{tv}(s)$$

for all s, and therefore $\gamma_{tv}(s) = \gamma_v(ts)$. This yields the useful formula

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t).$$

Lemma 1.6. For all $p \in M$,

$$d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$$

is the identity $d(\exp_p)_0 = \mathrm{id}$ under the identification $\mathrm{id}(v^i \partial_{u_i}|_0) = v^i \partial_{x_i}|_p$.

Definition 1.7 (Normal Neighbourhood). An open set $U \ni p$ is a **normal neighbourhood** of p if there exists an open set $\tilde{U} \ni o_p \subseteq \mathcal{D}_p$ which is star-shaped such that

$$\exp_p|_{\tilde{U}} : \tilde{U} \to U$$

is a diffeomorphism.

Theorem 1.8 (Existence of Normal Neighbourhoods). For any $p \in M$, there is a normal neighbourhood around p.

Proof. Inverse function theorem. □

Definition 1.9 (Convex Neighbourhood). U is a **convex neighbourhood** if it is a normal neighbourhood for all $q \in U$.

Remark. If U is a normal neighbourhood of p, then for all $q \in U$ there is exactly one geodesic γ_{pq} in U from p to q, called **radial geodesic**.

Theorem 1.10 (Normal Coordinate Lines). For all $p \in M$ and a basis $\{v_1, \dots, v_n\}$ of T_pM exists a chart $(U, (x^1, \dots, x^n))$ such that:

1. U is a normal neighbourhood of p.
2. $\partial_i|_p = v$
3. $\Gamma^k_{ij} = 0$ for all i, j, k.

If the basis $\{v_1, \dots, v_n\}$ is orthonormal, we also have

$$g_{ij}(p) = \epsilon_i \delta_{ij}$$

and

$$\partial_k g_{ij}(p) = 0.$$

The chart $(U, (x^1, \dots, x^n))$ is called **normal coordinate chart**.

Chapter 2

Riemannian Geometry

In this chapter, we are concerned with Riemannian manifolds as metric spaces. The main goal is to prove the theorem of Hopf-Rinow.

Definition 2.1 (Regular Curve)**.** A piecewise $\mathcal{C}^1$-curve

$$\gamma : [a, b] \to M$$

is called **regular** if

$$\forall s \in [a, b] : \dot{\gamma}(s) \neq 0$$

and

$$\dot{\gamma}_{\pm}(t_i) \neq 0$$

at all $\mathcal{C}^1$-break-points.

Definition 2.2 (Arc-length)**.** Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \to M$ a (piecewise) $\mathcal{C}^1$-curve. The **arc-length** is defined to be the functional

$$L[\gamma] = \int_a^b \sqrt{|g(\dot{\gamma}(t), \dot{\gamma}(t))|}\, dt.$$

Remark.

1. In the Riemannian case, the $|\cdot|$ is redundant.
2. In semi-Riemannian geometry, there are curves with $L[\gamma] = 0$.
3. The arc-length functional is invariant under length parametrization.
4. If γ is regular, there exists a strictly monotonous reparametrization

   $$\varphi : [\tilde{a}, \tilde{b}] \to [a, b]$$

   such that $\tau := \gamma \circ \varphi$ satisfies $g(\dot{\tau}, \dot{\tau}) = 1$. This is a reparametrization by arc-length:

   $$L[\tau_{[\tilde{a}, s]}] = s - \tilde{a}$$

   for all $s \in [\tilde{a}, \tilde{b}]$.

Theorem 2.3 (Gauß' Lemma)**.** The exponential map is a radial isometry: For any $p \in M$, $x \in \mathcal{D}_p$ and $v, w \in T_x(T_pM) \cong T_pM$ with $v = \alpha x$ for some $\alpha \in \mathbb{R}$, the equations

$$g_{\exp_p(x)}(d(\exp_p(v))_x, d(\exp_p(w))_x) = g_p(v, w)$$

and

$$\dot{\gamma}(t) = \frac{d}{dt} \exp_p(tv)$$

hold.