Preface

This is my personal script for the lecture Riemannian and Lorentzian Geometry in the winter term of 25/26 at the University of Hamburg. The script mostly follows the lecture of Prof. Dr. Melanie Graf with occasional bits adapted form the avaliable literature. The layout is a personal adaption of Gilles Casel's layout. We will adapt most notations from **??** and **??** and use the Einstein summation convention throughout. The lecture first aims to fill some gaps often left in undergraduate differential geometry lectures, mainly the theorem of Hopf and Rinow. After that, we continue with some notions inherent to Lorentzian geometry before focussing our attention again on the (semi-)Riemannian case. Later, we will use Jacobi fields and do some comparison geometry. Unless clearly stated otherwise, we will work in a completely smooth category. For Lorentzian metrics, we choose the sign convention $(-, +, \dots, +)$.

Rasmus Raschke, December 2025

List of symbols

$M \pitchfork N$	Transverse intersection
$\Gamma(M)$	Space of smooth sections $\sigma : M \to TM$
$\Gamma_\gamma(M)$	Space of smooth sections
$\mathfrak{X}(M)$	Space of vector fields on M

Contents

CHAPTER ONE

Repetition

We start by listing some results that should already be known by the reader. In the following, we assume a basic understanding of smooth manifolds and Riemannian geometry.

1.1 Vector Fields and Flows

On a smooth manifold M, we consider vector fields as sections of the tangent projection $\pi : TM \to M$, i.e. a map

$$X : M \to TM$$

with $\pi \circ X = \mathrm{id}_M$.

Notation. We write a vector field X at $p \in M$ as X_p, while $X_p(f)$ is the vector field at p applied to a function f. We denote the space of sections of the tangent projection by $\Gamma(TM)$, and the space of smooth vector fields by $\mathfrak{X}(M)$. General (k,l)-tensor fields are denoted as $\mathcal{T}_l^k = \Gamma(T^{(k,l)}TM)$.

Definition 1.1 (Integral Curve). Given a manifold M and $V \in \mathfrak{X}(M)$, an **integral curve** of V is a smooth curve

$$\gamma : I \to M$$

such that for all $t \in I$,

$$\dot\gamma(t) = V_{\gamma(t)}$$

holds

Example 1.2. Consider the Euclidean plane $\mathbb{R}^2$ with standard coordinates.

- The coodinate vector field ∂_1 has straight lines

  $$\gamma(t) = (a + t, b)$$

  as integral curves for some $a, b \in \mathbb{R}$.

- The curl field $x^1\partial_2 - x^2\partial_1$ has counterclockwise traversed circles

  $$\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$$

  as integral curves.

Proposition 1.3. Let M be a manifold, $V \in \mathfrak{X}(M)$. For all $p \in M$ exists a unique maximal integral curve

$$\gamma_p : I_p \to M$$

of V with $\gamma_p(0) = p$.

Given a manifold M, we define a **flow domain** on M to be an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for each $p \in M$,

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\}$$

is an open interval.

Definition 1.4 (Flow). A **(local) flow** on M is a continuous local one-parameter group action

$$\theta : \mathcal{D} \to M$$

such that for all $p \in M$:

1. $\theta(0, p) := \theta_0(p) = \mathrm{id}_M(p) = p$, and
2. if $s \in \mathcal{D}^{(p)}$, $t + s \in \mathcal{D}^{(\theta_s(p))}$: $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ holds.

A flow gives rise to a family of curves

$$\theta^{(p)} : \mathcal{D}^{(p)} \to M$$

defined by $\theta^{(p)}(t) = \theta_t(p)$. An **infinitesimal generator** of a flow θ is then a vector field $V \in \mathfrak{X}(M)$ with

$$V_p = \left.\frac{d}{dt}\right|_{t=0} \theta^{(p)}(t)$$

for all p in the domain of θ. On the other hand, the $\theta^{(p)}$-curves are integral curves of V. We call a flow θ **maximal** if the flow domain $\mathcal{D}$ of θ is maximal. We call a flow **global** if $\mathcal{D} = \mathbb{R} \times M$.

Theorem 1.5 (Fundamental Theorem of Flows). Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. Then there exists a unique maximal flow

$$\Theta : \mathcal{D} \to M$$

with infinitesimal generator V and the following properties:

1. For all $p \in M$, $\Theta^{(p)}$ is the unique maximal integral curve of V starting at p.
2. For $s \in \mathcal{D}^{(p)}$, we have $\mathcal{D}^{(\theta_s(p))} = \{t - s \mid t \in \mathcal{D}^{(p)}\}$.
3. For each $t \in \mathbb{R}$, the set $M_t := \{p \in M \mid (t, p) \in \mathcal{D}\}$ is open in M, and $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse Θ_{-t}.

The flow of this theorem is called **flow of V**.

1.2 (Semi-)Riemannian Metrics

1.2.1 Linear Algebra

Definition 1.6 (Pseudo-Euclidean Scalar Product). Let V be a finite-dimensional real vector space. A **pseudo-Euclidean scalar product** on V is a map

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$$

which is:

1. Symmetric: $\langle u, v\rangle = \langle v, u\rangle$
2. Bilinear: $\langle \lambda u + v, w\rangle = \lambda\langle u, w\rangle + \langle v, w\rangle$
3. Non-degenerate: $v \mapsto \langle v, \cdot\rangle$ is an isomorphism $V \cong V^*$.

The **index** s of V is the number

$$s := \max\{\dim(W) \mid W \leq V : \langle\cdot,\cdot\rangle|_W \text{ negative definite}\}.$$

The pair $(V, \langle\cdot,\cdot\rangle)$ is called **pseudo-Euclidean vector space**.

We call this iso **musical isomorphism**.

The index can be easily calculated by choosing a basis (e_i), defining a matrix $A_{ij} := \langle e_i, e_j\rangle$, and determining the negative eigenvalues of A. This will be the index of V.

Example 1.7. The standard pseudo-Euclidean vector space is the $n-$dimensional space $\mathbb{R}^{n-s,s}$, which consists of the vector space $\mathbb{R}^n$ with scalar product

$$g_{ij} := diag\{\underbrace{-1, \cdots, -1}_{s \text{ times}}, \underbrace{1, \cdots, 1}_{n-s \text{ times}}\}.$$

We call $\mathbb{R}^{n,1}$ the $(n+1)-$dimensional **Minkowski space**.

Remark 1.8. Sylvester's theorem of inertia tells us that the important invariants for pseudo-Euclidean vector spaces are dimension and index. Every finite-dimensional vector space of dimension n and index s is isomorphic to $\mathbb{R}^{n-s,s}$.

Proposition 1.9. Let V be a pseudo-Euclidean vector space and $W \leq V$. Then the following are equivalent:

1. $(W^\perp)^\perp = W$ and $V = W \oplus W^\perp$
2. $W \cap W^\perp = \{0\}$
3. $\langle\cdot,\cdot\rangle|_{W\times W}$ is non-degenerate.

Proof. Corollary of the dimension formula. □

Proposition 1.10 (Parallelogram Law). If $\langle\cdot,\cdot\rangle$ is a pseudo-Euclidean scalar product, the **parallelogram law**

$$\langle v, w\rangle = \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$$

holds.

Definition 1.11 (Causality in Lorentzian Geometry)**.** If $(V, \langle\cdot,\cdot\rangle)$ is a Lorentzian vector space, we define:

1. $v \in V$ is **timelike** if $\|v\|^2 < 0$.
2. $v \in V$ is **spacelike** if $\|v\|^2 > 0$.
3. $v \in V$ is **null** or **lightlike** if $\|v\|^2 = 0$.
4. $v \in V$ is **causal** if it is time- or lightlike.
5. The zero vector is spacelike by definition.

We denote the space of timelike vectors by V^{tl}, the space of spacelike vectors by V^{sl}, the space of lightlike vectors by V^{null}, and the space of causal vectors by V^{causal}.

Proposition 1.12. Consider $\mathbb{R}^{n,1}$. Then:

1. The subspace of timelike vectors has two connected components.
2. Let v, w be lightlike. Then $\langle v, w\rangle = 0$ if and only if there is some $\lambda \in \mathbb{R}^*$ such that $v = \lambda w$.
3. If v, w are timelike with $\langle v, w\rangle < 0$, we have **reverse Cauchy-Schwarz**:
$$|\langle v, w\rangle| \geq \|v\|\|w\|$$
and **reverse triangle** identities:
$$\|v + w\| \geq \|v\| + \|w\|.$$

1.2.2 Semi-Riemannian Manifolds

Definition 1.13 (Semi-Riemannian Manifold)**.** A **semi-Riemannian metric** on a smooth manifold M is a smooth, covariant 2-tensor field $g \in T^2(M)$ such that for each $p \in M$ and all $U, V, W \in T_pM$, $\lambda, \mu \in \mathbb{R}$, the following is satisfied:

1. g has global signature (r, s).
2. $g_p(U, V) = g_p(V, U)$
3. $g_p(\lambda U + \mu V, W) = \lambda g_p(U, W) + \mu g_p(V, W) = g_p(W, \lambda U + \mu V)$
4. $g_p(U, U) = 0$ if and only if $U = 0$.

Hence, g_p is an inner product on each T_pM. The pair (M, g) is called **semi-Riemannian manifold** and s is the **index** of g. If $s = 0$, we call (M, g) **Riemannian**. If $s = 1$, we call it **Lorentzian**.

For convenience of notation, we will sometimes supress p and write $\langle\cdot,\cdot\rangle$ for $g_p(\cdot,\cdot)$.

Notation. Given local coordinates (U, x^i) on some neighbourhood U, g can be written as
$$g = g_{ij} dx^i \otimes dx^j,$$

where the g_{ij} are $(\dim M)^2$ smooth component functions given by $g_{ij}(p) = g_p(\partial_i|_p, \partial_j|_p)$. Interpreting these components as matrix components, one obtains a symmetric, non-singular matrix.

Example 1.14. The standard model for a semi-Riemannian manifold with index s is the space $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$. Given coordinates $(\xi^1, \dots, \xi^r, \tau^1, \dots, \tau^s)$, we define the semi-Riemannian standard metric to be

$$g^{(r,s)} = d\xi^1 \otimes d\xi^1 + \dots + d\xi^r \otimes d\xi^r + d\tau^1 \otimes d\tau^1 + \dots + d\tau^s \otimes d\tau^s.$$

For $s = 0$, we recover the **canonical Euclidean metric**

$$g_{\text{st}} = dx^1 \otimes dx^1 + \dots + dx^r \otimes dx^r = \delta_{ij} dx^i \otimes dx^j.$$

For $s = 1$, we obtain $r + 1$-dimensional **Minkowski space** with the **Minkowski metric**

$$\eta = -dt \otimes dt + dx^1 \otimes dx^1 + \dots + dx^r \otimes dx^r.$$

Example 1.15. Given Minkowski space $\mathbb{R}^{2,1}$ and $c \neq 0$, we define the smooth submanifold

$$S_c^\eta := \{(t, x) \in \mathbb{R}^{2,1} \mid \eta((t, x), (t, x)) = c\}.$$

The restriction of η induces a semi-Riemannian metric on S_c^η, turning it into a semi-Riemannian submanifold. For $c > 0$, we call $S_c^\eta = dS_3$ (3-dimensional) **de Sitter space**, and for $c < 0$, we call $S_c^\eta = AdS_3$ **anti-de Sitter space**. Anti-de Sitter space AdS_3 has two connected components which are model hyperbolic spaces.

Every smooth manifold can be endowed with a Riemannian metric:

Proposition 1.16. Every smooth manifold M is a Riemannian manifold.

Proof. Given a smooth manifold M, we can choose an atlas (φ_i, U_i) of M and a smooth partition of unity (ϱ_i) subordinate to the covering $\cup U_i = M$. On each coordinate patch U_i, we can use the euclidean metric g_{st} and define a metric

$$g_p := \sum_{i \in I} \varrho_i(p) \varphi_i^* g_{\text{st}}.$$

This metric is clearly symmetric and bilinear. Furthermore, the sum is finite since ϱ_i is a partition of unity, and non-degenerate as g_{st} is non-degenerate. □

This does not work in the semi-Riemannian case: Pulling back the standard semi-Riemannian metric of $\mathbb{R}^n$ can lead to a vanishing sum because the chart-wise metrics possibly attain negative values.

Definition 1.17 (Connection)**.** Given a smooth manifold M and a vector bundle $E \to M$, a **connection** or **covariant derivative** is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that:

1. For all $f_1, f_2 \in \mathcal{C}^\infty$, $X_1, X_2 \in \mathfrak{X}(M)$:
$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$
2. For all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$:
$$\nabla_X(\lambda_1 Y_1 + \lambda_2 Y_2) = \lambda_1 \nabla_X(Y_1) + \lambda_2 \nabla_X Y_2.$$
3. For all $f \in \mathcal{C}^\infty(M)$:
$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

Theorem 1.18 (Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a (semi)-Riemannian manifold. Then there exists a unique connection
$$\nabla : \mathfrak{X}(M) \times \Gamma(TM) \to \Gamma(TM)$$
which is:

1. metric with respect to g:
$$\nabla_X g_p(Y, Z) = g_p(\nabla_X Y, Z) + g_p(Y, \nabla_X Z)$$
2. symmetric:
$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Note that symmetry implies that ∇ is torsion-free since the torsion tensor is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

We call ∇ the **Levi-Civita-Connection**.

Proposition 1.19. The Levi-Civita-Connection admits the following forms:

1. **Koszul's formula**:
$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle && (1.1)\\ &- \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle && (1.2)\end{aligned}$$
2. The coefficients in local coordinates are the **Christoffel symbols**:
$$(\nabla_{\partial_i} \partial_j)^k = \Gamma^k_{ij} = \frac{g_{kl}}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$
3. Given a smooth local frame (E_i) and functions $\varepsilon^k_{ij} E_k$ given by $[E_i, E_j] = \varepsilon^k_{ij} E_k$, one has:
$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm}\varepsilon^m_{il} - g_{lm}\varepsilon^m_{ji} + g_{im}\varepsilon^m_{lj}).$$

If (E_i) is an orthonormal frame, this reduces to:

$$\Gamma^k_{ij} = \frac{1}{2}(\varepsilon^k_{ij} - \varepsilon^l_{ik} - \varepsilon^l_{jk}).$$

1.3 Curvature and Geodesics

Given a smooth curve

$$\gamma : I \to M,$$

we call a vector field $V : I \to TM$ a **vector field along** γ if $V(t) \in T_{\gamma(t)}M$ for all $t \in I$. We denote the space of vector fields along γ by $\mathfrak{X}(\gamma)$.

Definition 1.20 (Geodesic)**.** Let M be a smooth manifold and ∇ be a connection on TM. A smooth curve $\gamma : I \to M$ is called a **geodesic** if the acceleration $\nabla_{\frac{d}{dt}}\dot{\gamma}(t)$ vanishes for all $t \in I$. This is equivalent to the local **geodesic equation**

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0,$$

where x^i are the components of γ in some local coordinates.

Theorem 1.21 (Uniqueness and Maximality of Geodesics)**.** Let M be a smooth manifold and ∇ be a connection on TM. For each $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic

$$\gamma_v : I_v \to M$$

with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, defined on some open interval $I \ni 0$.

Remark 1.22. Some similarities to the fundamental theorem of flows emerge: Considering the open flow domain $\mathcal{D} := \cup_{v \in TM} I_v \times \{v\}$, we obtain a one-parameter group action

$$\vartheta : I \subseteq \mathbb{R} \times TM \to TM$$

given by $\vartheta_t(v) := \dot{\gamma_v}(t)$. This is called the **geodesic flow**. The geodesic flow is the maximal flow of the **geodesic spray**: Thinking of the tangent bundle TM as a manifold on its own, we can consider curves

$$\tilde{\gamma}_(p, v) : I \to TM$$

given by $\tilde{\gamma}_{(p,v)} := (\gamma_v, \dot{\gamma}(v))$, where $\gamma_v : I \to M$ is a geodesic in M with initial data (p, v). Then the geodesic spray is a vector field

$$G(t) := \nabla_{\frac{d}{dt}}\tilde{\gamma}_{(p,v)}.$$

Given a smooth manifold M and some $V \in \mathfrak{X}(\gamma)$ for some smooth curve γ, we call V **parallel along** γ if $\nabla_{\frac{d}{dt}}V \equiv 0$. In local coodinates, this reads as

$$\dot{V}^k(t) = -V^j(t)\dot{\gamma}^i(t)\Gamma^k_{ij}(\gamma(t)).$$

Theorem 1.23 (Existence and Uniqueness of Parallel Transport)**.** Given a smooth manifold M, a connection ∇ on TM, a smooth curve $\gamma : I \to M$ with $t_0 \in I$, and a vector $v \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V \in \mathfrak{X}(\gamma)$ with $V(t_0) = v$. We call V the parallel transport of v along γ and define for each $t_0, t_1 \in I$ the **parallel transport isomorphism**

$$P^{\gamma}_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M.$$

Definition 1.24 (Geodesic Completeness)**.** A geodesic $\gamma : I \to \mathbb{R}$ is called **complete** if $I = \mathbb{R}$. We call M **geodesically complete** if all geodesics for the Levi-Civita-Connection are complete.

We also have that if $\gamma_v : I \to M$ is a geodesic and $h : J \to I$ is a smooth reparametrization, then $\gamma_v \circ h$ is a geodesic if and only if h is affine.

Lemma 1.25 (Rescaling Lemma)**.** Let

$$\gamma_v : (a_v, b_v) \to M$$

be a geodesic and $C \neq 0, t_0 \in \mathbb{R}$. Then,

$$\tilde{\gamma} : \left(\frac{a_v}{C} - t_0, \frac{b_v}{C} - t_0\right) \to M$$

given by $\tilde{\gamma}(t) := \gamma_v(Ct + t_0)$ is also a geodesic.

1.4 The Exponential Map

The **domain of the exponential map** is a subset $\mathcal{E} \subseteq TM$ given by

$$\mathcal{E} := \{v \in TM \mid \gamma_v \text{ defined on interval containing } [0,1]\}.$$

Sometimes we restrict the map to $\mathcal{E}_p := \mathcal{E} \cap T_pM$ and write $\exp_p$.

Definition 1.26 (Exponential Map)**.** If M is a smooth manifold and $\mathcal{E}$ is an exponential domain, the **exponential map** $\exp : \mathcal{E} \to M$ is given by

$$\exp(v) := \gamma_v(t).$$

Proposition 1.27. The exponential map has the following properties:

1. $\mathcal{E} \subseteq TM$ is open, contains the image of the zero section, and each $\mathcal{E}_p$ is star-shaped at 0.
2. For each $v \in TM$, $\gamma_v(t) = \exp(tv)$ as long as one side is defined.
3. exp is smooth.
4. For all $p \in M$, the differential

$$(\exp_p)_{0,*} : T_0(T_pM) \cong T_pM \to T_pM$$

is the identity at 0.

Proposition 1.28 (Normal Neighbourhood)**.** Let (M, g) be a semi-Riemannian manifold. For all $p \in M$, there is an open neighbourhood U of p and a neighbourhood V of $0 \in TM$ such that $\exp_p : V \to U$ is an isomorphism. We call U a **normal coodinate neighbourhood**.

Normal neighbourhoods have very nice properties:

1. Normal charts around p are centered at p.
2. The metric coefficients at p are δ_{ij} in the Riemannian and $\pm\delta_{ij}$ in the semi-Riemannian case.e
3. Given $v = v^i\partial_i \in T_pM$, the geodesic with initial data (p, v) is given by $\gamma_v(t) = (tv^1, \dots, tv^n)$.
4. All christoffel symbols vanish at p.

Theorem 1.29 (Existence of Convex Neighbourhoods)**.** Given a semi-Riemannian manifold M, **convex neighbourhoods**, i.e. neighbourhoods which are normal for all points contained in them, form a neighbourhood basis for all $p \in M$.

Corollary 1.30. Given a convex neighbourhood U, all $p, q \in U$ are connected by a unique geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma = \gamma_{\exp_p^{-1}(q \exp_p^{-1}(q))}$.

Theorem 1.31 (Gauß' Lemma)**.** Let M be a semi-Riemannian manifold. For any $p \in M$, $x \in \mathcal{E}_p$ and $v, w \in T_pM$ such that $v = \lambda w$ for $\lambda \in \mathbb{R}$, we have

$$\langle (\exp_p(v))_{x,*}, (\exp_p(w))_{x,*} \rangle = \langle v, w \rangle.$$

1.5 Curvature

Definition 1.32 (Riemann Curvature Tensor)**.** Let M be a semi-Riemannian manifold and $X, Y, Z \in TM$. The **Curvature Tensor** is the $(1, 3)$-tensor field

$$R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$$

given by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Notation. There are many tensors derived from the curvature tensor:

1. The Riemann tensor itself has local form

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} - \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}.$$

   The map $Z \mapsto R(X, Y)Z$ is the **curvature endomorphism**.

2. The **Riemann tensor** is a $(0, 4)-$tensor field defined by $Riem := R^\flat = \langle R(X, Y)Z, W\rangle$.

3. The **Ricci curvature** is a $(0, 2)$-tensor field given by

$$Rc(X, Y) = tr(Z \mapsto R(Z, X)Y)$$

   with local form $R_{ij} = g^{km} R_{kijm}$.

4. The **scalar curvature** is given by

$$\kappa = tr\, Rc = g^{ij} R_{ij}.$$

CHAPTER TWO

Distances, Completeness and Causality Theory

CHAPTER THREE

Jacobi Fields

3.1 The Jacobi Equation

In this section we focus on semi-Riemannian manifolds.

Definition 3.1 (Variation through Geodesics). Let (M, g) be semi-Riemannian and $\gamma : I \to M$ be a geodesic. If K is another interval and

$$\Gamma : K \times I \to M$$

is a variation of γ such that $\gamma_s : I \to M$ is a geodesic for all $s \in K$, we call Γ a **variation through geodesics**.

Variation through geodesics give rise to several particular vector fields: The *variational field* of Γ is a vector field $J \in \mathfrak{X}(\Gamma)$ defined as

$$J(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s, t).$$

Furthermore, we define two accessory vector fields

$$T(s, t) := \frac{\partial \Gamma}{\partial t}$$

$$S(s, t) := \frac{\partial \Gamma}{\partial s}$$

which will be useful.

Lemma 3.2 (Symmetry Lemma). Let $\Gamma : K \times I \to M$ be a smooth family of curves. Then we have

$$\nabla_{\frac{d}{ds}} T = \nabla_{\frac{d}{dt}} S.$$

Proof. Take local coordinates (x^i) on some coordinate neighbourhood and write $\Gamma(s, t) = (\gamma^1(s, t), \dots, \gamma^n(s, t))$. We have $S = \frac{\partial \gamma^k}{\partial s} \partial_k$ and $T = \frac{\partial \gamma^k}{\partial t} \partial_k$. Calculating the left side directly yields:

$$\begin{aligned} \nabla_{\frac{d}{ds}} T = \nabla_{\frac{d}{ds}} \left(\frac{\partial \gamma^k}{\partial s} \partial_k \right) &= \frac{\partial^2 \gamma^k}{\partial t \partial s} \partial_k + \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial s} \nabla_{\partial_j} \partial_i \\ &= \left(\frac{\partial^2 \gamma^k}{\partial t \partial s} + \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial s} \Gamma^k_{ji} \right) \partial_k \end{aligned}$$

Exchanging $i \leftrightarrow j$ and using the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ yields the desired identity. $\square$

Lemma 3.3 (Curvature Lemma)**.** Let (M, g) be semi-Riemannian and $\Gamma : K \times I \to M$ be a smooth family of curves. Then for any $V \in \mathfrak{X}(\Gamma)$, we have

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} V - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} = R(S,T)V.$$

Proof. Take local coodinates (x^i) and write $\Gamma(s,t) = (\gamma^1(s,t), \dots, \gamma^n(s,t))$ as well as $V = V^i \partial_i$. We calculate the two left derivatives explicitly:

$$\nabla_{\frac{d}{dt}} V = \frac{\partial V^i}{dt} \partial_i + V^i \nabla_{\frac{d}{dt}} \partial_i$$

and

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} V = \left(\frac{\partial^2 V^i}{\partial s \partial t} + \frac{\partial V^i}{\partial t} \nabla_{\frac{d}{ds}} + \frac{\partial V^i}{\partial s} \nabla_{\frac{d}{dt}} + V^i \nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \right) \partial_i.$$

Exchanging $s \leftrightarrow t$ yields $\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} V$ and we see immediately that after subtraction, only the rightmost term remains:

$$\left(\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \right) V = V^i \left(\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \right) \partial_i.$$

Extending ∂_i and the covariant derivative, we can calculate:

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \partial_i = \nabla_{\frac{d}{ds}} \left(\frac{\partial \gamma^j}{\partial t} \nabla_{\partial_j} \partial_i \right) = \frac{\partial^2 \gamma^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i$$

and analogously

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \partial_i = \frac{\partial^2 \gamma^j}{\partial t \partial s} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial s} \frac{\partial \gamma^k}{\partial t} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i.$$

Exchanging $j \leftrightarrow k$ and subtracting cancels out the left term and we obtain:

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \partial_i - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \partial_i \partial_i = \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} (\nabla_{\partial_k} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_k}) \partial_i = R(S,T) \partial_i.$$

$\square$

We are now ready to consider the main theorem of this section:

Theorem 3.4 (Jacobi Equation)**.** Let (M, g) be semi-Riemannian, $\gamma : I \to M$ be a geodesic and $\Gamma : K \times I \to M$ be a variation of γ through geodesics with variational field J. Then J satisfies the **Jacobi Equation**:

$$\nabla^2_{\frac{d}{dt}} J + R(J, \dot\gamma)\dot\gamma = 0. \tag{3.1}$$

Proof. The fact that Γ is a variation through geodesics tells us that $\nabla_{\frac{d}{dt}} T \equiv 0$. Dervating once more yields $\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \equiv 0$. By the curvature and the symmetry lemma, we have

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} T = \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} T + R(S,T)T = \nabla^2_{\frac{d}{dt}} S + R(S,T)T.$$

At $s = 0$, we have $S(0,t) = \partial_s \Gamma_s(t) = J(t)$ and $T(0,t) = \partial_t \Gamma_0(t) = \dot\gamma(t)$ as claimed. $\square$

Proposition 3.5 (Existence and Uniqueness). Let (M, g) be a SRMF, $\gamma : I \to M$ be a geodesic, $t_0 \in I$ and $p := \gamma(t_0)$. For all $v, w \in T_pM$, there is a unique Jacobi field J along γ satisfying the initial data

$$J(t_0) = v,\, \nabla_{\frac{d}{dt}} J(t_0) = w.$$

Proof. Take $J \in \mathfrak{X}(\gamma)$ and choose a parallel ONF (E_i). We write $v = v^i E_i(t_0)$, $w = w^i E_i(t_0)$, $\dot\gamma(t) = \gamma^i(t) E_i(t)$ and $J = J^i(t) E_i(t)$. The Jacobi equation holds if and only if the following equation is satisfied:

$$\begin{aligned} 0 &= \nabla^2_{\frac{d}{dt}} J + R(J, \dot\gamma)\dot\gamma = \nabla^2_{\frac{d}{dt}} (J^i E_i) + R(J^j E_j, \dot\gamma^k E_k)\dot\gamma^l E_l \\ &= \ddot J^i E_i + J^j \dot\gamma^k \dot\gamma^l R(E_j, E_k) E_l \end{aligned}$$

Note that all terms of the form $\nabla_{\frac{d}{dt}} E_i$ in the left term vanish as the ONF is parallel w.r.t. ∇.

This yields a second-order system of n equations

$$\frac{d^2 J^i(t)}{dt^2} = -\dot J^j(t)\dot\gamma^k(t)\dot\gamma^l(t) R^i_{jkl}.$$

Substituting $W^i(t) := \dot J^i(t)$, this reduces to $2n$ first-order equations. The existence and uniqueness theorem of ODEs on manifolds thus guarantees that the claim holds. ☐

Definition 3.6 (Jacobi Field). Let (M, g) be a SRMF and $\gamma : I \to M$ be a geodesic. We call a vector field $J \in \mathfrak{X}(\gamma)$ a **Jacobi field** if it satiesfies the Jacobi equation. The space $\mathfrak{J}(\gamma) \subseteq \mathfrak{X}(\gamma)$ denotes the space of all Jacobi fields along γ.

Corollary 3.7. Let (M, g) be an n-dimensional SRMF and $\gamma : I \to M$ be any geodesic. Then $\mathfrak{J}(\gamma)$ is a $2n$-dimensional linear subspace of $\mathfrak{X}(\gamma)$.

Proof. Linearity of the Jacobi equation guarantees that $\mathfrak{J}(\gamma)$ is linear. Fixing some $p = \gamma(t_0)$, we have an isomorphism $\mathfrak{J}(\gamma) \cong T_pM \oplus T_pM$ given by the previous preposition as $J \mapsto (J(t_0), \nabla_{\frac{d}{dt}} J(t_0))$. ☐

Proposition 3.8. Let (M, g) be a SRMF and $\gamma : I \to M$ be a geodesic. If either

(i) M is complete, or

(ii) I is compact

then every Jacobi field along γ is the variation field of some variation of γ though geodesics.

Proof. Let $J \in \mathfrak{J}(\gamma)$. By translating, we can assume $0 \in I$ and set $p := \gamma(0)$, $v := \dot\gamma(0)$. This means we can write $\gamma(t) = \exp_p(tv)$ for all $t \in I$. By

The idea is the following: We define a small curve through a starting point on γ and use the assumptions to guarantee that (after eventually contracting the domain) exp can be used to define a variation through geodesics. After that, we use uniqueness of Jacobi fields to show that the variational vector field of this constructed variation has to agree with J.

construction of the tangent space, there is a small open interval $(-\varepsilon, \varepsilon)$ and a smooth curve $\sigma : (-\varepsilon, \varepsilon) \to M$ satisfying

$$\sigma(0) = p,\ \dot\sigma(0) = J(0).$$

Choose a vector field $V \in \mathfrak{X}(\sigma)$ with data

$$V(0) = w\, \nabla_{\frac{d}{ds}} V(0) = \nabla_{\frac{d}{dt}} J(0).$$

We want to define a variation through geodesics by

$$\Gamma(s,t) := \exp_{\sigma(s)}(tV(s)). \tag{3.2}$$

If M is geodesically complete, we can always define Γ on $(-\varepsilon, \varepsilon) \times I$. If I is compact, we can use that $\mathcal{E}_p$ is open and contains the compactum $\{(p, tv) \in TM \mid t \in I\}$. Therefore, we find some $\delta > 0$ such that Γ is definable on $(-\delta, \delta) \times M$.
Evaluating at $s = 0$ yields

$$\Gamma_0(t) = \exp_{\sigma(0)}(tV(0)) = \exp_p(tv) = \gamma(t),$$

which tells us that Γ is indeed a variation of γ: By definition of exp, it is also a variation thorugh geodesics with variational field $W(t) := \frac{\partial \Gamma}{\partial s}(0,t) \in \mathfrak{J}(\gamma)$. Now we match initial data:

1. $W(0) = \frac{\partial}{\partial s}\Gamma(0,0) = \dot\sigma(0) = J(0)$
2. We have $\frac{\partial}{\partial t}\Gamma_s(0) = V(s)\exp(0) = V(s)$. By applying the symmetry lemma, we obtain:

   $$\nabla_{\frac{d}{dt}} W(0) = \nabla_{\frac{d}{dt}} \frac{\partial}{\partial s}\Gamma(0,0) = \nabla_{\frac{d}{ds}} \frac{\partial}{\partial t}\Gamma(0,0) = \nabla_{\frac{d}{ds}} V(0) = \nabla_{\frac{d}{dt}} J(0).$$

   Since J and W have the same initial data, we can conclude by uniqueness that $J \equiv W$.

□

3.2 Jacobi Fields vanishing at a point

We turn our attention to Jacobi fields which vanish at some point $p \in M$.

Lemma 3.9. Let (M, g) be a SRMF, $I \ni 0$ an interval, $\gamma : I \to M$ be a geodesic, and $J \in \mathfrak{J}(\gamma)$ such that $J(0) = 0$. If M is geodesically complete or I is compact, J is the variation field of the geodesic variation

$$\Gamma(s,t) := \exp_p(t(v + sw)) \tag{3.3}$$

with $p = \gamma(0)$, $v = \dot\gamma(0)$, and $w = \nabla_{\frac{d}{dt}} J(0)$.

Proof. This follows directly from equation 3.2 by taking $\sigma(s) \equiv p$, and $V(s) = v + sw$. □

Proposition 3.10 (Jacobi Fields Vanishing at a Point). Let (M, g) be a SRMF, $p \in M$, $\gamma : I \to M$ be a geodesic with $0 \in I$ and $\gamma(0) = p$.

1. For every $w \in T_pM$, the Jacobi field with initial data $J(0) = 0$ and $\nabla_{\frac{d}{dt}} J(0) = w$ is given by
$$J(t) = (\exp_p)_{*,tv}(tw) \tag{3.4}$$
with $v = \dot\gamma(0)$ and $tw \in T_{tv}(T_pM) \cong T_pM$.
2. If (x^i) are normal coordinates on a normal neighbourhood completely containing $\operatorname{im}\gamma$ with $w = w^i \partial_i|_0$, J admits the form
$$J(t) = tw^i \partial_i|_{\gamma(t)}. \tag{3.5}$$

Proof.

1. Restricting to a compact subinterval of I, we can use equation 3.3 and calculate the pushforward:
$$\frac{\partial}{\partial s}\Gamma(0, t) = \big(\exp_p(t(v + sw))\big)_{*,s=0} = (\exp_p)_{*,tv} \circ (tw) = (\exp_p)_{*,tv}(tw).$$
As all $t \in I$ are contained in such compact subintervals, this holds for all t.
2. Given normal coordinates (U, x^i), the exponential map is the identity in coordinates, so we get $\Gamma(s, t) = t(v^i + sw^i)\partial_i$. We can calculate directly:
$$\frac{\partial}{\partial s}\Gamma(0, t) = tw^i \partial_i|_{\gamma(t)}.$$

□

This makes it possible to reach all vectors in a normal neighbourhood with Jacobi fields:

Corollary 3.11. Let (M, g) be a SRMF, $p \in M$ and U be a normal neighbourhood centered at p. For any $q \in U \setminus \{p\}$, every vector $v \in T_qM$ is the value of a Jacobi field J vanishing at p along a radial geodesic.

Proof. Take normal coordinates (U, x^i) and $q = (q^1, \ldots, q^n) \in U \setminus \{p\}$ as well as $w = w^i \partial_i|_q \in T_qM$. The radial geodesic $\gamma(t) := (tq^1, \ldots, tq^n)$ has endpoints $\gamma(0) = p$ and $\gamma(1) = q$. By the previous preposition,
$$J(t) := tw^i \partial_i|_{\gamma(t)}$$
is a Jacobi field satisfying $J(0) = 0$ adn $J(1) = w$. □

3.3 Normal and Tangential Jacobi Fields

CHAPTER FOUR

Comparision Geometry

As a preliminary, we construct a special function on normal neighbourhoods: Given a Riemannian manifold (M, g) and a normal chart (U, x^i) around some $p \in M$. Define the **radial distance function** $r : U \to \mathbb{R}$ by $r(q) = d(p, q) > 0$. If the coodinates are centered at p, we have the explicit form

$$r(x) = \sqrt{(x^1)^2 + \cdots + (x^n)^2},$$

and on $U \setminus \{p\}$ we get the **radial vector field**

$$\partial_r = \frac{x^i}{r(x)} \partial_i.$$

This is smooth on $U \setminus \{p\}$ and independent of choice of charts.

Given a time-oriented Lorentzian manifold, we define the radial distance function by the proper time $\tau : I^U(p) \to \mathbb{R}$ with $\tau(q) := \tau_U(p, q) > 0$.

Definition 4.1 (Hessian). Let (M, g) be a SRMF and $u : M \to \mathbb{R}$ smooth. The **covariant Hessian** is the $(0, 2)$-tensor

$$\mathrm{Hess}_u(X, Y) := (\nabla^2 u)(X, Y) = \nabla_X(\nabla_Y u) - \nabla_{\nabla_X Y} u.$$

The **Hessian operator** is the $(1, 1)$-tensor

$$\mathcal{H}_u := \mathrm{Hess}_u^\sharp .$$

In local coordinates (x^i), the Hessian has the (familiar) form

$$\mathrm{Hess}_u = (\partial_j \partial_i u - \Gamma^k_{ij} \partial_k u)\, dx^i \otimes dx^j .$$

Remark 4.2. The Hessian is symmetric if ∇ is the Levi-Civita-Connection (or, more generally, if and only if ∇ is torsion-free):

$$\begin{aligned}
\mathrm{Hess}_u(X, Y) &= Y(X(u)) - du(\nabla_Y X) = (XY - [X, Y])u - du(\nabla_Y X) \\
&= X(Y(u)) - du(\nabla_Y X + [X, Y]) \\
&= X(Y(u)) - du(\nabla_X Y - T(X, Y)) \\
&= \mathrm{Hess}_u(Y, X) + T(X, Y)(u)
\end{aligned}$$

Unraveling the definition of $\mathcal{H}_u$, we get

$$g(\mathcal{H}_u(X), Y) = \mathrm{Hess}_u(X, Y) = \mathrm{Hess}_u(Y, X) = g(X, \mathcal{H}_u(Y)),$$

and hence $\mathcal{H}_u$ is self-adjoint.

With the radial distance function defined above, we obtain the Hessian operator $\mathcal{H}_r$ associated to that function.

Notation. Given a normal neighbourhood U, we denote the geodesic spheres of radius r_0 and proper time τ_0, respectively, by

$$\mathbb{S}_{r_0} := \{x \in U \mid r(x) = r_0\}$$

and

$$\mathbb{S}_{\tau_0} := \{x \in U \mid \tau(x) = \tau_0\}.$$

The Gauß' Lemma immediately tells us that $T_q\mathbb{S}_{r(q)} = \partial_r|_q^\perp \subseteq T_qM$ and similarly for τ. Denote the projection on the radial tangent by

$$\pi_q^r : T_qM \to T_q\mathbb{S}_{r(q)}.$$

Lemma 4.3. Let (M, g) be a Riemannian or time-oriented Lorentzian manifold, (U, x^i) be a normal chart around $p \in M$ and let r be the radial distance function. Then we have for all $q \in U \setminus \{p\}$:

1. $\mathcal{H}_r(\partial_r|_q) = 0$ and $\mathcal{H}_\tau(\partial_\tau|_q) = 0$
2. For all $X \in T_qM$ with $X \perp \partial_r|_q$:

   $$\mathcal{H}_r(X) = \pi_q(\nabla_X(\operatorname{grad} r)),$$

   and similarly for $X \perp \partial_\tau|_q$:

   $$\mathcal{H}_\tau(X) = \pi_q(\nabla_X(\operatorname{grad} \tau)) = \pi_q^{\mathbb{H}}(\nabla_X \operatorname{grad}(\tau)),$$

   where $\pi_q^{\mathbb{H}} : T_qM \to \partial_\tau|_q^\perp \cong T\mathbb{H}^{n-1}$ denotes the orthogonal projection

   $$\pi_q^{\mathbb{H}}(Y) = Y + g(Y, \partial_\tau)\partial_\tau.$$

Proof. We proof the Lorentzian case:

1. Let $Y \in T_qM$ be arbitrary. We have:

   $$\begin{aligned} g(\mathcal{H}_\tau(\partial_\tau|_q), Y) &= g(\partial_\tau|_q, \mathcal{H}_\tau(Y)) = g(\partial_\tau|_q, -\nabla_Y \operatorname{grad}(\tau)) \\ &= g(\nabla_Y \partial_\tau, \partial_\tau|_q) = \frac{1}{2}Y_q(g(\partial_\tau, \partial_\tau)) = 0, \end{aligned}$$

   where we used symmetry of g and the fact that $-\operatorname{grad}(\tau) = \partial_\tau$ since the metric is $-d\tau^2 + \tilde{g}$.

2. This follows since the previous remark showed $\mathcal{H}_\tau(X) = \nabla_X \operatorname{grad}(\tau)$ and $\mathcal{H}_\tau(X) \perp \partial_\tau$ follows from the upper equation.

□

Note that $g|_{TH_{\tau_0} \times TH_{\tau_0}}$ is Riemannian, so the restriction is self-adjoint for a positive-definite inner product even in the Lorentzian case.

Remark 4.4. For any τ_0 such that

$$H_{\tau_0} := \left\{q \in I_U^+(p) \mid \tau_U(p, q) = \tau_0\right\} \neq \emptyset,$$

$\mathcal{H}_\tau$ restricts to a self-adjoint linear operator

$$\mathcal{H}_\tau : TH_{\tau_0} \to TH_{\tau_0}$$

and is given by the Weingarten map $\mathcal{H}_\tau(X) = -\nabla_X N$ where $N = \partial_\tau$ is the future-pointing unit normal to H_{τ_0}.

Proposition 4.5 (The Hessian and Jacobi Fields). Let $(\mathcal{L}, g)$ be a LMF, $p \in \mathcal{L}$ and $U \subseteq \mathcal{L}$ be a normal neighbourhood around p. Set $\tau := \tau_U(p, \cdot)$. Let $\gamma : [0, b] \to U$ be a future-directed timelike unit-speed geodesic starting at p and let $J \in \mathfrak{J}^\perp(\gamma)$ such that $J(0) = 0$. Then

$$\nabla_{\frac{d}{dt}} J(\tau_0) = -\mathcal{H}_\tau(J(\tau_0))$$

for all $\tau_0 \in (0, b]$.

Proof. Let $v := \dot\gamma(0)$ and $w := \nabla_{\frac{d}{dt}} J(0)$. By equation 3.3, we have a variation through geodesics

$$\Gamma_s(t) = \exp_{\sigma(s)}(tU(s))$$

where $\sigma(s)$ is any curve with $\sigma(0) = p$ and $\dot\sigma(0) = J(0)$. The vector field U satisfies $U(0) = 0$ and $\nabla_{\frac{d}{ds}} U(0) = w$ and the variational field of Γ is $\frac{\partial}{\partial s}\big|_{s=0} = J$. Since $J(0) = 0$, choose $\sigma(s) \equiv p$. Since $w \perp v$ (J is normal), we have

$$v \in H := \{X \in T_p\mathcal{L} \mid g_p(X, X) = -1\}.$$

Note that $g_p \cong \eta_p$.

Hence, $w \in T_v H$. Since $U : I \to T_pM = T_{\sigma(s)}M$ with $\nabla_{\frac{d}{ds}} U(0) \in T_v H$ and $U(0) = v \in H$, we can choose U such that $U(s) \in H$ for all s. Therefore, each curve $t \mapsto \Gamma_s(t)$ is a future-directed timelike unit-speed geodesic starting at p, implying

$$\partial_t \Gamma_{s_0}(t_0) = \partial_\tau|_{\Gamma(s_0, t_0)} \tag{4.1}$$

Now we compute:

$$\nabla_{\frac{d}{dt}} J = \nabla_{\frac{d}{dt}} \frac{\partial}{\partial s}\bigg|_{s=0} \Gamma = \nabla_{\frac{d}{ds}} \left(\frac{\partial}{\partial t}\Gamma\right)(0, t) = \nabla_{\frac{d}{ds}} (\partial_\tau \circ \Gamma)(0, t),$$

where we used equation 4.1. Since ∂_τ is a smooth vector field on $I_U^+(p)$ (which is an open neighbourhood of $\Gamma_0(\tau_0)$ for any $\tau_0 \in (0, b]$). This implies

$$\nabla_{\frac{d}{ds}} (\partial_\tau \circ \Gamma)(0, t) = \nabla_{\partial_s \Gamma|_{s=0}} \partial_\tau = \nabla_J \partial_\tau = -\nabla_J \operatorname{grad}(\tau) = \mathcal{H}_\tau(J)$$

using $J \perp \dot\gamma = \partial_\tau$. □

Recall the previously defined function

$$s_c(t) := \begin{cases} R\sinh(\frac{t}{R}) & c < 0,\ R := \frac{1}{\sqrt{c}} \\ t & c = 0 \\ R\sin(\frac{t}{R}) & c > 0,\ R := \frac{1}{\sqrt{c}}. \end{cases} \tag{4.2}$$

Proposition 4.6 (Sectional Curvature and Hessian)**.** Let $(\mathcal{L}, g)$ be a LMF, $p \in \mathcal{L}$ and U be a normal neighbourhood of p. Then $(\mathcal{L}, g)$ has constant sectional curvature c on $I_U^+(p)$ if and only if

$$\mathcal{H}_\tau = -\frac{s'_{-c} \circ \tau}{s_{-c} \circ \tau} \cdot \pi^H$$

on $I_U^+(p)$.

Proof. $(\Rightarrow)$: Let $q \in I_U^+(p)$, $\gamma : [0, b] \to M$ be the future-directed timelike unit-speed geodesic from p to q, and let $\{E_1 = \dot{\gamma}, E_2, \ldots, E_n\}$ be an ONF along γ. The proposition about Jacobi fields in constant curvature spaces tells us that

$$J_i := s_{-c} E_i \; i \geq 2$$

are normal Jacobi fields along γ with $J_i(0) = 0$. Compute

$$s'_{-c} E_i = \nabla_{\frac{d}{dt}} J_i = -\mathcal{H}_\tau(J_i) = -s_{-c}\mathcal{H}_\tau(E_i),$$

where we used proposition 3.5. Evaluation at $t = b$ and $s_{-c}(b)^{-1}$ noting $s_{-c}(b) \neq 0$ because $\gamma(b) = q \in U$, which is normal. Hence q is not conjugate to p implying $J_i(b) \neq 0$. This yields

$$\mathcal{H}_\tau(E_i(b)) = -\frac{s'_{-c}(b)}{s_{-c}(b)} E_i(b),$$

but $E_i(b) = \pi^H(E_i(b))$ because it is already orthogonal to $\dot{\gamma}(b)$. For $i = 1$, we have

$$0 = \mathcal{H}_\tau(E_1(b)) = \pi^H(E_1(b))$$

because $E_1(b) = \dot{\gamma}(b)$.

$(\Leftarrow)$: Let $q \in I_U^+(p)$ and $\gamma : [0, b) \to U$ be the corresponding geodesic. From theorem 2.23, we know that

$$\varphi : I_U^+(p) \to \mathbb{R}_+ \times \mathbb{H}^{n-1}$$

given by $\varphi(q) = (\tau(q), \xi(q))$ is a diffeomorpism onto its image. If we can show that it is an isometry between g and $g_c := -d\tau^2 + s_{-c}(\tau)^2 g_{\mathbb{H}^{n-1}}$, then g has constant sectional curvature. Following the proof of 2.23, we immediately get 1 and 2a. In 2b, the constant curvature assumption only entered via Jacobi fields, more precisely via 2.20 yielding: Any $J \in \mathfrak{J}^\perp(\gamma)$ with $J(0) = 0$ is given by

$$J(t) = k s_{-c}(t) E(t)$$

for some $k \in \mathbb{R}$ and $E \perp \dot{\gamma}$ as well as parallel. It only remains to show this in our setting. Let $J \in \mathfrak{J}^\perp(\gamma)$. We have

$$\nabla_{\frac{d}{dt}} J = \frac{s'_{-c}}{s_{-c}} J$$

on $(0, b]$. By the product rule, $\frac{1}{s_{-c}} J$ is parallel along γ on $(0, b]$. Define $\tilde{E}(t) := \frac{1}{s_{-c}} J$ and let $E = \frac{\tilde{E}}{\|\tilde{E}\|}$. Then E is a parallel unit VF orthogonal $\dot{\gamma}$ and $J = \|\tilde{E}\| s_{-c} E$. Set $k := \|\tilde{E}\|$, which is constant since