Preface

This is my personal script for the lecture Riemannian and Lorentzian Geometry in the winter term of 25/26 at the University of Hamburg. The script mostly follows the lecture of Prof. Dr. Melanie Graf with occasional bits adapted form the avaliable literature. The layout is a personal adaption of Gilles Casel's layout. We will adapt most notations from [Lee13] and [Lee18] and use the Einstein summation convention throughout. The lecture first aims to fill some gaps often left in undergraduate differential geometry lectures, mainly the theorem of Hopf and Rinow. After that, we continue with some notions inherent to Lorentzian geometry before focussing our attention again on the (semi-)Riemannian case. Later, we will use Jacobi fields and do some comparison geometry. Unless clearly stated otherwise, we will work in a completely smooth category. For Lorentzian metrics, we choose the sign convention $(-,+,\dots,+)$.

Rasmus Raschke, December 2025

List of symbols

$M \pitchfork N$	Transverse intersection
$\Gamma(M)$	Space of smooth sections $\sigma : M \to TM$
$\Gamma_\gamma(M)$	Space of smooth sections along a curve.
$\mathfrak{X}(M)$	Space of vector fields on M.
$\mathfrak{X}(\gamma)$	Space of vector fields along a curve γ.
$\mathfrak{X}(\gamma)^\top$	Space of parallel vector fields.
$\mathfrak{X}(\gamma)^\perp$	Space of normal vector fields.
$\mathfrak{J}(\gamma)$	Space of Jacobi fields along a curve γ.
$\mathfrak{J}(\gamma)^\top$	Space of parallel Jacobi fields.
$\mathfrak{J}(\gamma)^\perp$	Space of normal Jacobi fields.
R, R_{ijk}^l	$(1,3)$-Riemann curvature tensor.
Rm	Riemann tensor
Ric	Ricci tensor
scal	Scalar curvature
sec	Sectional curvature.

Contents

CHAPTER ONE

Repetition

We start by listing some results that should already be known by the reader. In the following, we assume a basic understanding of smooth manifolds and Riemannian geometry.

1.1 Vector Fields and Flows

On a smooth manifold M, we consider vector fields as sections of the tangent projection $\pi : TM \to M$, i.e. a map

$$X : M \to TM$$

with $\pi \circ X = \mathrm{id}_M$.

Notation. We write a vector field X at $p \in M$ as X_p, while $X_p(f)$ is the vector field at p applied to a function f. We denote the space of sections of the tangent projection by $\Gamma(TM)$, and the space of smooth vector fields by $\mathfrak{X}(M)$. General (k,l)-tensor fields are denoted as $\mathcal{T}^k_l = \Gamma(T^{(k,l)}TM)$.

Definition 1.1 (Integral Curve). Given a manifold M and $V \in \mathfrak{X}(M)$, an **integral curve** of V is a smooth curve

$$\gamma : I \to M$$

such that for all $t \in I$,

$$\dot{\gamma}(t) = V_{\gamma(t)}$$

holds

Example 1.2. Consider the Euclidean plane $\mathbb{R}^2$ with standard coordinates.

- The coodinate vector field ∂_1 has straight lines

  $$\gamma(t) = (a + t, b)$$

  as integral curves for some $a, b \in \mathbb{R}$.

- The curl field $x^1\partial_2 - x^2\partial_1$ has counterclockwise traversed circles

  $$\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$$

  as integral curves.

Proposition 1.3. Let M be a manifold, $V \in \mathfrak{X}(M)$. For all $p \in M$ exists a unique maximal integral curve

$$\gamma_p : I_p \to M$$

of V with $\gamma_p(0) = p$.

Given a manifold M, we define a **flow domain** on M to be an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for each $p \in M$,

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\}$$

is an open interval.

Definition 1.4 (Flow)**.** A **(local) flow** on M is a continuous local one-parameter group action

$$\theta : \mathcal{D} \to M$$

such that for all $p \in M$:

1. $\theta(0,p) := \theta_0(p) = \mathrm{id}_M(p) = p$, and
2. if $s \in \mathcal{D}^{(p)}$, $t + s \in \mathcal{D}^{(\theta_s(p))}$: $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ holds.

A flow gives rise to a family of curves

$$\theta^{(p)} : \mathcal{D}^{(p)} \to M$$

defined by $\theta^{(p)}(t) = \theta_t(p)$. An **infinitesimal generator** of a flow θ is then a vector field $V \in \mathfrak{X}(M)$ with

$$V_p = \left.\frac{d}{dt}\right|_{t=0} \theta^{(p)}(t)$$

for all p in the domain of θ. On the other hand, the $\theta^{(p)}$-curves are integral curves of V. We call a flow θ **maximal** if the flow domain $\mathcal{D}$ of θ is maximal. We call a flow **global** if $\mathcal{D} = \mathbb{R} \times M$.

Theorem 1.5 (Fundamental Theorem of Flows)**.** Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. Then there exists a unique maximal flow

$$\Theta : \mathcal{D} \to M$$

with infinitesimal generator V and the following properties:

1. For all $p \in M$, $\Theta^{(p)}$ is the unique maximal integral curve of V starting at p.
2. For $s \in \mathcal{D}^{(p)}$, we have $\mathcal{D}^{(\theta_s(p))} = \{t - s \mid t \in \mathcal{D}^{(p)}\}$.
3. For each $t \in \mathbb{R}$, the set $M_t := \{p \in M \mid (t,p) \in \mathcal{D}\}$ is open in M, and $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse Θ_{-t}.

The flow of this theorem is called **flow of V**.

1.2 (Semi-)Riemannian Metrics

1.2.1 Linear Algebra

Definition 1.6 (Pseudo-Euclidean Scalar Product)**.** Let V be a finite-dimensional real vector space. A **pseudo-Euclidean scalar product** on V is a map

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$$

which is:

1. Symmetric: $\langle u, v\rangle = \langle v, u\rangle$
2. Bilinear: $\langle \lambda u + v, w\rangle = \lambda\langle u, w\rangle + \langle v, w\rangle$
3. Non-degenerate: $v \mapsto \langle v, \cdot\rangle$ is an isomorphism $V \cong V^*$.

The **index** s of V is the number

$$s := \max\{\dim(W) \mid W \leq V : \langle\cdot,\cdot\rangle|_W \text{ negative definite}\}.$$

The pair $(V, \langle\cdot,\cdot\rangle)$ is called **pseudo-Euclidean vector space**.

We call this iso **musical isomorphism**.

The index can be easily calculated by choosing a basis (e_i), defining a matrix $A_{ij} := \langle e_i, e_j\rangle$, and determining the negative eigenvalues of A. This will be the index of V.

Example 1.7. The standard pseudo-Euclidean vector space is the $n-$dimensional space $\mathbb{R}^{n-s,s}$, which consists of the vector space $\mathbb{R}^n$ with scalar product

$$g_{ij} := \mathrm{diag}\{\underbrace{-1, \cdots, -1}_{s \text{ times}}, \underbrace{1, \cdots, 1}_{n-s \text{ times}}\}.$$

We call $\mathbb{R}^{n,1}$ the $(n+1)-$dimensional **Minkowski space**.

Remark 1.8. Sylvester's theorem of inertia tells us that the important invariants for pseudo-Euclidean vector spaces are dimension and index. Every finite-dimensional vector space of dimension n and index s is isomorphic to $\mathbb{R}^{n-s,s}$.

Proposition 1.9. Let V be a pseudo-Euclidean vector space and $W \leq V$. Then the following are equivalent:

1. $(W^\perp)^\perp = W$ and $V = W \oplus W^\perp$
2. $W \cap W^\perp = \{0\}$
3. $\langle\cdot,\cdot\rangle|_{W\times W}$ is non-degenerate.

Proof. Corollary of the dimension formula. □

Proposition 1.10 (Parallelogram Law)**.** If $\langle\cdot,\cdot\rangle$ is a pseudo-Euclidean scalar product, the **parallelogram law**

$$\langle v, w\rangle = \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2)$$

holds.

Definition 1.11 (Causality in Lorentzian Geometry). If $(V, \langle\cdot,\cdot\rangle)$ is a Lorentzian vector space, we define:

1. $v \in V$ is **timelike** if $\|v\|^2 < 0$.
2. $v \in V$ is **spacelike** if $\|v\|^2 > 0$.
3. $v \in V$ is **null** or **lightlike** if $\|v\|^2 = 0$.
4. $v \in V$ is **causal** if it is time- or lightlike.
5. The zero vector is spacelike by definition.

We denote the space of timelike vectors by V^{tl}, the space of spacelike vectors by V^{sl}, the space of lightlike vectors by V^{null}, and the space of causal vectors by V^{causal}.

Proposition 1.12. Consider $\mathbb{R}^{n,1}$. Then:

1. The subspace of timelike vectors has two connected components.
2. Let v, w be lightlike. Then $\langle v, w\rangle = 0$ if and only if there is some $\lambda \in \mathbb{R}^*$ such that $v = \lambda w$.
3. If v, w are timelike with $\langle v, w\rangle < 0$, we have **reverse Cauchy-Schwarz**:
$$|\langle v, w\rangle| \geq \|v\|\|w\|$$
and **reverse triangle** identities:
$$\|v + w\| \geq \|v\| + \|w\|.$$

1.2.2 Semi-Riemannian Manifolds

Definition 1.13 (Semi-Riemannian Manifold). A **semi-Riemannian metric** on a smooth manifold M is a smooth, covariant 2-tensor field $g \in T^2(M)$ such that for each $p \in M$ and all $U, V, W \in T_pM$, $\lambda, \mu \in \mathbb{R}$, the following is satisfied:

1. g has global signature (r, s).
2. $g_p(U, V) = g_p(V, U)$
3. $g_p(\lambda U + \mu V, W) = \lambda g_p(U, W) + \mu g_p(V, W) = g_p(W, \lambda U + \mu V)$
4. $g_p(U, U) = 0$ if and only if $U = 0$.

Hence, g_p is an inner product on each T_pM. The pair (M, g) is called **semi-Riemannian manifold** and s is the **index** of g. If $s = 0$, we call (M, g) **Riemannian**. If $s = 1$, we call it **Lorentzian**.

For convenience of notation, we will sometimes supress p and write $\langle\cdot,\cdot\rangle$ for $g_p(\cdot,\cdot)$.

Notation. Given local coordinates (U, x^i) on some neighbourhood U, g can be written as
$$g = g_{ij} dx^i \otimes dx^j,$$

where the g_{ij} are $(\dim M)^2$ smooth component functions given by $g_{ij}(p) = g_p(\partial_i|_p, \partial_j|_p)$. Interpreting these components as matrix components, one obtains a symmetric, non-singular matrix.

Example 1.14. The standard model for a semi-Riemannian manifold with index s is the space $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$. Given coordinates $(\xi^1, \dots, \xi^r, \tau^1, \dots, \tau^s)$, we define the semi-Riemannian standard metric to be

$$g^{(r,s)} = d\xi^1 \otimes d\xi^1 + \cdots + d\xi^r \otimes d\xi^r + d\tau^1 \otimes d\tau^1 + \cdots + d\tau^s \otimes d\tau^s.$$

For $s = 0$, we recover the **canonical Euclidean metric**

$$g_{\mathrm{st}} = dx^1 \otimes dx^1 + \cdots + dx^r \otimes dx^r = \delta_{ij} dx^i \otimes dx^j.$$

For $s = 1$, we obtain $r + 1$-dimensional **Minkowski space** with the **Minkowski metric**

$$\eta = -dt \otimes dt + dx^1 \otimes dx^1 + \cdots + dx^r \otimes dx^r.$$

Example 1.15. Given Minkowski space $\mathbb{R}^{2,1}$ and $c \neq 0$, we define the smooth submanifold

$$S_c^\eta := \{(t, x) \in \mathbb{R}^{2,1} \mid \eta((t, x), (t, x)) = c\}.$$

The restriction of η induces a semi-Riemannian metric on S_c^η, turning it into a semi-Riemannian submanifold. For $c > 0$, we call $S_c^\eta = dS_3$ (3-dimensional) **de Sitter space**, and for $c < 0$, we call $S_c^\eta = AdS_3$ **anti-de Sitter space**. Anti-de Sitter space AdS_3 has two connected components which are model hyperbolic spaces.

Every smooth manifold can be endowed with a Riemannian metric:

Proposition 1.16. Every smooth manifold M is a Riemannian manifold.

Proof. Given a smooth manifold M, we can choose an atlas (φ_i, U_i) of M and a smooth partition of unity (ϱ_i) subordinate to the covering $\cup U_i = M$. On each coordinate patch U_i, we can use the euclidean metric g_{st} and define a metric

$$g_p := \sum_{i \in I} \varrho_i(p) \varphi_i^* g_{\mathrm{st}}.$$

This metric is clearly symmetric and bilinear. Furthermore, the sum is finite since ϱ_i is a partition of unity, and non-degenerate as g_{st} is non-degenerate. $\square$

This does not work in the semi-Riemannian case: Pulling back the standard semi-Riemannian metric of $\mathbb{R}^n$ can lead to a vanishing sum because the chart-wise metrics possibly attain negative values.

Definition 1.17 (Connection)**.** Given a smooth manifold M and a vector bundle $E \to M$, a **connection** or **covariant derivative** is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that:

1. For all $f_1, f_2 \in \mathcal{C}^\infty$, $X_1, X_2 \in \mathfrak{X}(M)$:
$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$
2. For all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$:
$$\nabla_X(\lambda_1 Y_1 + \lambda_2 Y_2) = \lambda_1 \nabla_X(Y_1) + \lambda_2 \nabla_X Y_2.$$
3. For all $f \in \mathcal{C}^\infty(M)$:
$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

Theorem 1.18 (Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a (semi)-Riemannian manifold. Then there exists a unique connection
$$\nabla : \mathfrak{X}(M) \times \Gamma(TM) \to \Gamma(TM)$$
which is:

1. metric with respect to g:
$$\nabla_X g_p(Y, Z) = g_p(\nabla_X Y, Z) + g_p(Y, \nabla_X Z)$$
2. symmetric:
$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Note that symmetry implies that ∇ is torsion-free since the torsion tensor is given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

We call ∇ the **Levi-Civita-Connection**.

Proposition 1.19. The Levi-Civita-Connection admits the following forms:

1. **Koszul's formula**:
$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \end{aligned}$$
2. The coefficients in local coordinates are the **Christoffel symbols**:
$$(\nabla_{\partial_i} \partial_j)^k = \Gamma^k_{ij} = \frac{g_{kl}}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$
3. Given a smooth local frame (E_i) and functions $\varepsilon^k_{ij} E_k$ given by $[E_i, E_j] = \varepsilon^k_{ij} E_k$, one has:
$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm}\varepsilon^m_{il} - g_{lm}\varepsilon^m_{ji} + g_{im}\varepsilon^m_{lj}).$$

If (E_i) is an orthonormal frame, this reduces to:

$$\Gamma_{ij}^k = \frac{1}{2}(\varepsilon_{ij}^k - \varepsilon_{ik}^l - \varepsilon_{jk}^l).$$

1.3 Curvature and Geodesics

Given a smooth curve

$$\gamma : I \to M,$$

we call a vector field $V : I \to TM$ a **vector field along** γ if $V(t) \in T_{\gamma(t)}M$ for all $t \in I$. We denote the space of vector fields along γ by $\mathfrak{X}(\gamma)$.

Definition 1.20 (Geodesic). Let M be a smooth manifold and ∇ be a connection on TM. A smooth curve $\gamma : I \to M$ is called a **geodesic** if the acceleration $\nabla_{\frac{d}{dt}}\dot{\gamma}(t)$ vanishes for all $t \in I$. This is equivalent to the local **geodesic equation**

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(x(t)) = 0,$$

where x^i are the components of γ in some local coordinates.

Theorem 1.21 (Uniqueness and Maximality of Geodesics). Let M be a smooth manifold and ∇ be a connection on TM. For each $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic

$$\gamma_v : I_v \to M$$

with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, defined on some open interval $I \ni 0$.

Remark 1.22. Some similarities to the fundamental theorem of flows emerge: Considering the open flow domain $\mathcal{D} := \cup_{v \in TM} I_v \times \{v\}$, we obtain a one-parameter group action

$$\vartheta : I \subseteq \mathbb{R} \times TM \to TM$$

given by $\vartheta_t(v) := \dot{\gamma_v}(t)$. This is called the **geodesic flow**. The geodesic flow is the maximal flow of the **geodesic spray**: Thinking of the tangent bundle TM as a manifold on its own, we can consider curves

$$\tilde{\gamma}_(p, v) : I \to TM$$

given by $\tilde{\gamma}_{(p,v)} := (\gamma_v, \dot{\gamma}(v))$, where $\gamma_v : I \to M$ is a geodesic in M with initial data (p, v). Then the geodesic spray is a vector field

$$G(t) := \nabla_{\frac{d}{dt}}\tilde{\gamma}_{(p,v)}.$$

Given a smooth manifold M and some $V \in \mathfrak{X}(\gamma)$ for some smooth curve γ, we call V **parallel along** γ if $\nabla_{\frac{d}{dt}}V \equiv 0$. In local coodinates, this reads as

$$\dot{V}^k(t) = -V^j(t)\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t)).$$

Theorem 1.23 (Existence and Uniqueness of Parallel Transport)**.** Given a smooth manifold M, a connection ∇ on TM, a smooth curve $\gamma : I \to M$ with $t_0 \in I$, and a vector $v \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V \in \mathfrak{X}(\gamma)$ with $V(t_0) = v$. We call V the parallel transport of v along γ and define for each $t_0, t_1 \in I$ the **parallel transport isomorphism**

$$P^{\gamma}_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M.$$

Definition 1.24 (Geodesic Completeness)**.** A geodesic $\gamma : I \to \mathbb{R}$ is called **complete** if $I = \mathbb{R}$. We call M **geodesically complete** if all geodesics for the Levi-Civita-Connection are complete.

We also have that if $\gamma_v : I \to M$ is a geodesic and $h : J \to I$ is a smooth reparametrization, then $\gamma_v \circ h$ is a geodesic if and only if h is affine.

Lemma 1.25 (Rescaling Lemma)**.** Let

$$\gamma_v : (a_v, b_v) \to M$$

be a geodesic and $C \neq 0, t_0 \in \mathbb{R}$. Then,

$$\tilde{\gamma} : \left(\frac{a_v}{C} - t_0, \frac{b_v}{C} - t_0\right) \to M$$

given by $\tilde{\gamma}(t) := \gamma_v(Ct + t_0)$ is also a geodesic.

1.4 The Exponential Map

The **domain of the exponential map** is a subset $\mathcal{E} \subseteq TM$ given by

$$\mathcal{E} := \{v \in TM \mid \gamma_v \text{ defined on interval containing } [0, 1]\}.$$

Sometimes we restrict the map to $\mathcal{E}_p := \mathcal{E} \cap T_pM$ and write $\exp_p$.

Definition 1.26 (Exponential Map)**.** If M is a smooth manifold and $\mathcal{E}$ is an exponential domain, the **exponential map** $\exp : \mathcal{E} \to M$ is given by

$$\exp(v) := \gamma_v(t).$$

Proposition 1.27. The exponential map has the following properties:

1. $\mathcal{E} \subseteq TM$ is open, contains the image of the zero section, and each $\mathcal{E}_p$ is star-shaped at 0.
2. For each $v \in TM$, $\gamma_v(t) = \exp(tv)$ as long as one side is defined.
3. exp is smooth.
4. For all $p \in M$, the differential

$$(\exp_p)_{0,*} : T_0(T_pM) \cong T_pM \to T_pM$$

is the identity at 0.

Proposition 1.28 (Normal Neighbourhood). Let (M,g) be a semi-Riemannian manifold. For all $p \in M$, there is an open neighbourhood U of p and a neighbourhood V of $0 \in TM$ such that $\exp_p : V \to U$ is an isomorphism. We call U a **normal coodinate neighbourhood**.

Normal neighbourhoods have very nice properties:

1. Normal charts around p are centered at p.
2. The metric coefficients at p are δ_{ij} in the Riemannian and $\pm\delta_{ij}$ in the semi-Riemannian case.
3. Given $v = v^i \partial_i \in T_pM$, the geodesic with initial data (p, v) is given by $\gamma_v(t) = (tv^1, \dots, tv^n)$.
4. All Christoffel symbols vanish at p.

Theorem 1.29 (Existence of Convex Neighbourhoods). Given a semi-Riemannian manifold M, **convex neighbourhoods**, i.e. neighbourhoods which are normal for all points contained in them, form a neighbourhood basis for all $p \in M$.

Corollary 1.30. Given a convex neighbourhood U, all $p, q \in U$ are connected by a unique geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma = \gamma_{\exp_p^{-1}(q \exp_p^{-1}(q))}$.

Theorem 1.31 (Gauß' Lemma). Let M be a semi-Riemannian manifold. For any $p \in M$, $x \in \mathcal{E}_p$ and $v, w \in T_pM$ such that $v = \lambda w$ for $\lambda \in \mathbb{R}$, we have

$$\langle (\exp_p(v))_{x,*}, (\exp_p(w))_{x,*} \rangle = \langle v, w \rangle.$$

1.5 Curvature

Definition 1.32 (Riemann Curvature Tensor). Let M be a semi-Riemannian manifold and $X, Y, Z \in TM$. The **Curvature Tensor** is the $(1,3)$-tensor field

$$R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$$

given by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Notation. There are many tensors derived from the curvature tensor:

1. The Riemann tensor itself has local form

$$\mathrm{R}_{ijk}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} - \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l}.$$

The map $Z \mapsto R(X, Y)Z$ is the **curvature endomorphism**.

2. The **Riemann tensor** is a $(0,4)-$tensor field defined by $\mathrm{Rm} := \mathrm{R}^\flat = \langle \mathrm{R}(X, Y)Z, W \rangle$.

3. The **Ricci curvature** is a $(0,2)$-tensor field given by

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto \mathrm{R}(Z, X)Y)$$

with local form $\mathrm{R}_{ij} = g^{km} \, \mathrm{R}_{kijm}$.

4. The **scalar curvature** is given by

$$\mathrm{scal} = \mathrm{tr}\,\mathrm{Ric} = g^{ij} \, \mathrm{R}_{ij} \, .$$

CHAPTER TWO

Distances, Completeness and Causality Theory

CHAPTER THREE

Jacobi Fields

3.1 The Jacobi Equation

In this section we focus on semi-Riemannian manifolds.

Definition 3.1 (Variation through Geodesics). Let (M, g) be semi-Riemannian and $\gamma : I \to M$ be a geodesic. If K is another interval and

$$\Gamma : K \times I \to M$$

is a variation of γ such that $\gamma_s : I \to M$ is a geodesic for all $s \in K$, we call Γ a **variation through geodesics**.

Variation through geodesics give rise to several particular vector fields: The *variational field* of Γ is a vector field $J \in \mathfrak{X}(\Gamma)$ defined as

$$J(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s, t).$$

Furthermore, we define two accessory vector fields

$$T(s, t) := \frac{\partial \Gamma}{\partial t}$$

$$S(s, t) := \frac{\partial \Gamma}{\partial s}$$

which will be useful.

Lemma 3.2 (Symmetry Lemma). Let $\Gamma : K \times I \to M$ be a smooth family of curves. Then we have

$$\nabla_{\frac{d}{ds}} T = \nabla_{\frac{d}{dt}} S.$$

Proof. Take local coordinates (x^i) on some coordinate neighbourhood and write $\Gamma(s, t) = (\gamma^1(s, t), \ldots, \gamma^n(s, t))$. We have $S = \frac{\partial \gamma^k}{\partial s} \partial_k$ and $T = \frac{\partial \gamma^k}{\partial t} \partial_k$. Calculating the left side directly yields:

$$\begin{aligned}
\nabla_{\frac{d}{ds}} T = \nabla_{\frac{d}{ds}} \left(\frac{\partial \gamma^k}{\partial s} \partial_k \right) &= \frac{\partial^2 \gamma^k}{\partial t \partial s} \partial_k + \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial s} \nabla_{\partial_j} \partial_i \\
&= \left(\frac{\partial^2 \gamma^k}{\partial t \partial s} + \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial s} \Gamma^k_{ji} \right) \partial_k
\end{aligned}$$

Exchanging $i \leftrightarrow j$ and using the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ yields the desired identity. $\square$

Lemma 3.3 (Curvature Lemma)**.** Let (M, g) be semi-Riemannian and $\Gamma : K \times I \to M$ be a smooth family of curves. Then for any $V \in \mathfrak{X}(\Gamma)$, we have

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} V - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} = R(S, T)V.$$

Proof. Take local coodinates (x^i) and write $\Gamma(s, t) = (\gamma^1(s, t), \ldots, \gamma^n(s, t))$ as well as $V = V^i \partial_i$. We calculate the two left derivatives explicitly:

$$\nabla_{\frac{d}{dt}} V = \frac{\partial V^i}{dt} \partial_i + V^i \nabla_{\frac{d}{dt}} \partial_i$$

and

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} V = \left(\frac{\partial^2 V^i}{\partial s \partial t} + \frac{\partial V^i}{\partial t} \nabla_{\frac{d}{ds}} + \frac{\partial V^i}{\partial s} \nabla_{\frac{d}{dt}} + V^i \nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \right) \partial_i.$$

Exchanging $s \leftrightarrow t$ yields $\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} V$ and we see immediately that after subtraction, only the rightmost term remains:

$$\left(\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \right) V = V^i \left(\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \right) \partial_i.$$

Extending ∂_i and the covariant derivative, we can calculate:

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \partial_i = \nabla_{\frac{d}{ds}} \left(\frac{\partial \gamma^j}{\partial t} \nabla_{\partial_j} \partial_i \right) = \frac{\partial^2 \gamma^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i$$

and analogously

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \partial_i = \frac{\partial^2 \gamma^j}{\partial t \partial s} \nabla_{\partial_j} \partial_i + \frac{\partial \gamma^j}{\partial s} \frac{\partial \gamma^k}{\partial t} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i.$$

Exchanging $j \leftrightarrow k$ and subtracting cancels out the left term and we obtain:

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \partial_i - \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} \partial_i \partial_i = \frac{\partial \gamma^j}{\partial t} \frac{\partial \gamma^k}{\partial s} (\nabla_{\partial_k} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_k}) \partial_i = R(S, T) \partial_i.$$

$\square$

We are now ready to consider the main theorem of this section:

Theorem 3.4 (Jacobi Equation)**.** Let (M, g) be semi-Riemannian, $\gamma : I \to M$ be a geodesic and $\Gamma : K \times I \to M$ be a variation of γ through geodesics with variational field J. Then J satisfies the **Jacobi Equation**:

$$\nabla^2_{\frac{d}{dt}} J + R(J, \dot{\gamma}) \dot{\gamma} = 0. \tag{3.1}$$

Proof. The fact that Γ is a variation through geodesics tells us that $\nabla_{\frac{d}{dt}} T \equiv 0$. Dervating once more yields $\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} \equiv 0$. By the curvature and the symmetry lemma, we have

$$\nabla_{\frac{d}{ds}} \nabla_{\frac{d}{dt}} T = \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{ds}} T + R(S, T)T = \nabla^2_{\frac{d}{dt}} S + R(S, T)T.$$

At $s = 0$, we have $S(0, t) = \partial_s \Gamma_s(t) = J(t)$ and $T(0, t) = \partial_t \Gamma_0(t) = \dot{\gamma}(t)$ as claimed. $\square$

Proposition 3.5 (Existence and Uniqueness). Let (M, g) be a SRMF, $\gamma : I \to M$ be a geodesic, $t_0 \in I$ and $p := \gamma(t_0)$. For all $v, w \in T_pM$, there is a unique Jacobi field J along γ satisfying the initial data

$$J(t_0) = v, \, \nabla_{\frac{d}{dt}} J(t_0) = w.$$

Proof. Take $J \in \mathfrak{X}(\gamma)$ and choose a parallel ONF (E_i). We write $v = v^i E_i(t_0)$, $w = w^i E_i(t_0)$, $\dot\gamma(t) = \gamma^i(t) E_i(t)$ and $J = J^i(t) E_i(t)$. The Jacobi equation holds if and only if the following equation is satisfied:

Note that all terms of the form $\nabla_{\frac{d}{dt}} E_i$ in the left term vanish as the ONF is parallel w.r.t. ∇.

$$\begin{aligned} 0 &= \nabla^2_{\frac{d}{dt}} J + R(J, \dot\gamma)\dot\gamma = \nabla^2_{\frac{d}{dt}} (J^i E_i) + R(J^j E_j, \dot\gamma^k E_k)\dot\gamma^l E_l \\ &= \ddot{J}^i E_i + J^j \dot\gamma^k \dot\gamma^l R(E_j, E_k) E_l \end{aligned}$$

This yields a second-order system of n equations

$$\frac{d^2 J^i(t)}{dt^2} = -\dot{J}^j(t)\dot\gamma^k(t)\dot\gamma^l(t) R^i_{jkl}.$$

Substituting $W^i(t) := \dot{J}^i(t)$, this reduces to $2n$ first-order equations. The existence and uniqueness theorem of ODEs on manifolds thus guarantees that the claim holds. $\square$

Definition 3.6 (Jacobi Field). Let (M, g) be a SRMF and $\gamma : I \to M$ be a geodesic. We call a vector field $J \in \mathfrak{X}(\gamma)$ a **Jacobi field** if it satiesfies the Jacobi equation. The space $\mathfrak{J}(\gamma) \subseteq \mathfrak{X}(\gamma)$ denotes the space of all Jacobi fields along γ.

Corollary 3.7. Let (M, g) be an n-dimensional SRMF and $\gamma : I \to M$ be any geodesic. Then $\mathfrak{J}(\gamma)$ is a $2n$-dimensional linear subspace of $\mathfrak{X}(\gamma)$.

Proof. Linearity of the Jacobi equation guarantees that $\mathfrak{J}(\gamma)$ is linear. Fixing some $p = \gamma(t_0)$, we have an isomorphism $\mathfrak{J}(\gamma) \cong T_pM \oplus T_pM$ given by the previous preposition as $J \mapsto (J(t_0), \nabla_{\frac{d}{dt}} J(t_0))$. $\square$

Proposition 3.8. Let (M, g) be a SRMF and $\gamma : I \to M$ be a geodesic. If either

(i) M is complete, or

(ii) I is compact

then every Jacobi field along γ is the variation field of some variation of γ though geodesics.

Proof. Let $J \in \mathfrak{J}(\gamma)$. By translating, we can assume $0 \in I$ and set $p := \gamma(0)$, $v := \dot\gamma(0)$. This means we can write $\gamma(t) = \exp_p(tv)$ for all $t \in I$. By

The idea is the following: We define a small curve through a starting point on γ and use the assumptions to guarantee that (after eventually contracting the domain) exp can be used to define a variation through geodesics. After that, we use uniqueness of Jacobi fields to show that the variational vector field of this constructed variation has to agree with J.

construction of the tangent space, there is a small open interval $(-\varepsilon, \varepsilon)$ and a smooth curve $\sigma : (-\varepsilon, \varepsilon) \to M$ satisfying

$$\sigma(0) = p,\, \dot{\sigma}(0) = J(0).$$

Choose a vector field $V \in \mathfrak{X}(\sigma)$ with data

$$V(0) = w\, \nabla_{\frac{d}{ds}} V(0) = \nabla_{\frac{d}{dt}} J(0).$$

We want to define a variation through geodesics by

$$\Gamma(s,t) := \exp_{\sigma(s)}(tV(s)). \tag{3.2}$$

If M is geodesically complete, we can always define Γ on $(-\varepsilon, \varepsilon) \times I$. If I is compact, we can use that $\mathcal{E}_p$ is open and contains the compactum $\{(p, tv) \in TM \mid t \in I\}$. Therefore, we find some $\delta > 0$ such that Γ is definable on $(-\delta, \delta) \times M$.
Evaluating at $s = 0$ yields

$$\Gamma_0(t) = \exp_{\sigma(0)}(tV(0)) = \exp_p(tv) = \gamma(t),$$

which tells us that Γ is indeed a variation of γ: By definition of exp, it is also a variation thorugh geodesics with variational field $W(t) := \frac{\partial \Gamma}{\partial s}(0,t) \in \mathfrak{J}(\gamma)$. Now we match initial data:

1. $W(0) = \frac{\partial}{\partial s}\Gamma(0,0) = \dot{\sigma}(0) = J(0)$
2. We have $\frac{\partial}{\partial t}\Gamma_s(0) = V(s)\exp(0) = V(s)$. By applying the symmetry lemma, we obtain:

   $$\nabla_{\frac{d}{dt}} W(0) = \nabla_{\frac{d}{dt}} \frac{\partial}{\partial s}\Gamma(0,0) = \nabla_{\frac{d}{ds}} \frac{\partial}{\partial t}\Gamma(0,0) = \nabla_{\frac{d}{ds}} V(0) = \nabla_{\frac{d}{dt}} J(0).$$

   Since J and W have the same initial data, we can conclude by uniqueness that $J \equiv W$.

□

3.2 Jacobi Fields vanishing at a point

We turn our attention to Jacobi fields which vanish at some point $p \in M$.

Lemma 3.9. Let (M, g) be a SRMF, $I \ni 0$ an interval, $\gamma : I \to M$ be a geodesic, and $J \in \mathfrak{J}(\gamma)$ such that $J(0) = 0$. If M is geodesically complete or I is compact, J is the variation field of the geodesic variation

$$\Gamma(s,t) := \exp_p(t(v + sw)) \tag{3.3}$$

with $p = \gamma(0)$, $v = \dot{\gamma}(0)$, and $w = \nabla_{\frac{d}{dt}} J(0)$.

Proof. This follows directly from equation 3.2 by taking $\sigma(s) \equiv p$, and $V(s) = v + sw$. □

Proposition 3.10 (Jacobi Fields Vanishing at a Point)**.** Let (M, g) be a SRMF, $p \in M$, $\gamma : I \to M$ be a geodesic with $0 \in I$ and $\gamma(0) = p$.

1. For every $w \in T_pM$, the Jacobi field with initial data $J(0) = 0$ and $\nabla_{\frac{d}{dt}} J(0) = w$ is given by
$$J(t) = (\exp_p)_{*,tv}(tw) \tag{3.4}$$
with $v = \dot\gamma(0)$ and $tw \in T_{tv}(T_pM) \cong T_pM$.
2. If (x^i) are normal coordinates on a normal neighbourhood completely containing $\operatorname{im}\gamma$ with $w = w^i\partial_i|_0$, J admits the form
$$J(t) = tw^i\partial_i|_{\gamma(t)}. \tag{3.5}$$

Proof.

1. Restricting to a compact subinterval of I, we can use equation 3.3 and calculate the pushforward:
$$\frac{\partial}{\partial s}\Gamma(0, t) = \big(\exp_p(t(v + sw))\big)_{*,s=0} = (\exp_p)_{*,tv}\circ(tw) = (\exp_p)_{*,tv}(tw).$$
As all $t \in I$ are contained in such compact subintervals, this holds for all t.
2. Given normal coordinates (U, x^i), the exponential map is the identity in coordinates, so we get $\Gamma(s, t) = t(v^i + sw^i)\partial_i$. We can calculate directly:
$$\frac{\partial}{\partial s}\Gamma(0, t) = tw^i\partial_i|_{\gamma(t)}.$$

□

This makes it possible to reach all vectors in a normal neighbourhood with Jacobi fields:

Corollary 3.11. Let (M, g) be a SRMF, $p \in M$ and U be a normal neighbourhood centered at p. For any $q \in U \setminus \{p\}$, every vector $v \in T_qM$ is the value of a Jacobi field J vanishing at p along a radial geodesic.

Proof. Take normal coordinates (U, x^i) and $q = (q^1, \ldots, q^n) \in U \setminus \{p\}$ as well as $w = w^i\partial_i|_q \in T_qM$. The radial geodesic $\gamma(t) := (tq^1, \ldots, tq^n)$ has endpoints $\gamma(0) = p$ and $\gamma(1) = q$. By the previous preposition,
$$J(t) := tw^i\partial_i|_{\gamma(t)}$$
is a Jacobi field satisfying $J(0) = 0$ adn $J(1) = w$. □

3.3 Normal and Tangential Jacobi Fields

We make the following observation: Given any SRMF and geodesic $\gamma : I \to M$, there are always geodesic variations of the form
$$\Gamma_s(t) := \gamma(\alpha(t)t)$$

for $\alpha : I \to \mathbb{R}$ affine, e.g. $\Gamma_s(t) = \gamma((1+s)t)$ and $\Gamma'_s(t) = \gamma(s+t)$. They give rise to Jacobi fields if I is compact or M is complete:

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma_s(t) = t\dot{\gamma}(t) =: J_0 \tag{3.6}$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma'_s(t) = \dot{\gamma}(t) =: J_1. \tag{3.7}$$

Those Jacobi fields are uninteresting, so we desire a splitting

$$\mathfrak{J}(\gamma) \cong \mathfrak{J}(\gamma)^\top \oplus \mathfrak{J}(\gamma)^\perp.$$

Given a vector field $V \in \mathfrak{X}(\gamma)$ along some curve γ, we call it **tangential** if $V(t) \in T_{\gamma(t)}M^\top$ at all $t \in I$ and **normal** if $V(t) \in T_{\gamma(t)}M^\perp$ for all $t \in I$.

Definition 3.12 (Tangential and Normal Jacobi Fields)**.** Let (M, g) be a SRMF and $\gamma : I \to M$ a geodesic. We call $J \in \mathfrak{J}(\gamma)$:

1. **tangential** if $J(t) \in T_{\gamma(t)}M^\top$ for all $t \in I$, i.e. $J(t) = f(t)\gamma(t)$ for some smooth $f : I \to \mathbb{R}$.
2. **normal** if $J(t) \in T_{\gamma(t)}M^\perp$ for all $i \in I$, i.e. $g_{\gamma(t)}(J(t), \dot{\gamma}(t)) = 0$.

We denote the space of tangential Jacobi fields by $\mathfrak{J}^\top(\gamma)$ and the space of normal Jacobi fields by $\mathfrak{J}^\perp(\gamma)$.

Lemma 3.13 (Tangential Jacobis are Uninteresting)**.** A smooth vector field $V \in \mathfrak{X}(\gamma)^\top$ along and tangential to a geodesic γ is a Jacobi field if and only if $J(t) = (at+b)\dot{\gamma}(t)$, i.e. $f(t) = at + b$ for some $a, b \in \mathbb{R}$.

Proof. Write $V(t) = f(t)\dot{\gamma}(t)$ and calculate:

$$\nabla_{\frac{d}{dt}} V(t) = \dot{f}(t)\dot{\gamma}(t) + f(t)\, \overset{\text{geodesic}}{\cancel{\nabla_{\frac{d}{dt}}\dot{\gamma}(t)}} = \dot{f}(t)\dot{\gamma}(t)$$

and

$$\nabla^2_{\frac{d}{dt}} V(t) = \ddot{f}(t)\dot{\gamma}(t).$$

The Riemann tensor vanishes:

$$R(V, \dot{\gamma})\dot{\gamma} = f(t)R(\dot{\gamma}, \dot{\gamma})\dot{\gamma} = 0.$$

Hence, the Jacobi equation is satisfied if and only if

$$\ddot{f}(t) = 0,$$

so by basic ODE theory, $f(t) = at + b$ for some $a, b \in \mathbb{R}$ if and only if V is a tangential Jacobi field. $\square$

Proposition 3.14 (Normality Criterion)**.** Let (M, g) be a SRMF, $\gamma : I \to M$ be a geodesic and $J \in \mathfrak{J}(\gamma)$. Then the following are equivalent:

(a) $J \in \mathfrak{J}(\gamma)^\perp$.

(b) J is orthogonal to $\dot{\gamma}$ at two distinct points $t_1, t_2 \in I$.

(c) J and $\nabla_{\frac{d}{dt}} J$ are orthogonal to $\dot{\gamma}$ at one point $t_0 \in I$.

Proof. Define the auxilliary function

$$f(t) := g_{\gamma(t)}(J(t), \dot{\gamma}(t))$$

expressing the tangential part of J. We have:

$$\dot{f}(t) = g_{\gamma(t)}(\nabla_{\frac{d}{dt}} J, \dot{\gamma}),$$

where the second term vanishes again as γ is a geodesic. Then, we have

$$\ddot{f}(t) = g_{\gamma(t)}(\nabla_{\frac{d}{dt}} J, \dot{\gamma}) = -g_{\gamma(t)}(R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma}) = g_{\gamma(t)}(R(\dot{\gamma}, \dot{\gamma})\dot{\gamma}, J) = 0$$

by the symmetries of R. Thus, $f(t) = at + b$ for some $a, b \in \mathbb{R}$. We see that (a) holds if and only if $f \equiv 0$. In that case, (b) and (c) hold, since $\nabla_{\frac{d}{dt}} J(t_0) \perp \gamma(t_0)$ is equivalent to $\dot{f}(t_0) = 0$. If (b) holds, we have $at_1 + b = 0$, so $b = -at_1$. We also have $at_2 + b = a(t_2 - t_1) = 0$, so $a = 0$ and hence $b = 0$, yielding $f \equiv 0$. If (c) holds, we have $\dot{f}(t_0) = a = 0$ and $f(t_0) = b = 0$, so again $f \equiv 0$. □

Corollary 3.15 (Orthogonal Decomposition)**.** Let (M, g) be a SRMF and $\gamma : I \to M$ be a non-null geodesic. Then:

- $\mathfrak{J}(\gamma)^\top$ is a 2-dimensional linear subspace of $\mathfrak{J}(\gamma)$.
- $\mathfrak{J}(\gamma)^\perp$ is a $(2n-2)$-dimensional linear subspace of $\mathfrak{J}(\gamma)$.

In addition, there is an orthogonal decomposition

$$\mathfrak{J}(\gamma) \cong \mathfrak{J}(\gamma)^\top \oplus \mathfrak{J}(\gamma)^\perp.$$

Proof. In corollary 3.7, we have already seen the isomorphism $\mathfrak{J}(\gamma) \cong T_{\gamma(t_0)}M \oplus T_{\gamma(t_0)}M$ at each $t_0 \in I$. The space $\mathfrak{J}(\gamma)^\top$ is by proposition 3.14 a preimage of the subspace

$$\left\{v, w \in T_{\gamma(t_0)}M \mid v, w \perp \gamma(\dot{t_0})\right\} \subseteq T_{\gamma(t_0)}M \oplus T_{\gamma(t_0)}M.$$

Since this subspace has dimension $2n-2$, this also holds for $\mathbb{J}(\gamma)^\top$. By lemma 3.13, we have the two Jacobi fields J_0 and J_1 from equations 3.6 and 3.7 which evidently are linearly independent for all t. Hence $\dim \mathfrak{J}(\gamma)^\perp \geq 2$. Since γ is non-null, we have $\mathfrak{J}(\gamma)^\top \cap \mathfrak{J}(\gamma)^\perp = \{0\}$ and hence $\dim \mathfrak{J}^\perp \leq 2$, so we conclude $\dim \mathfrak{J}(\gamma)^\perp = 2$. This yields the desired decomposition. □

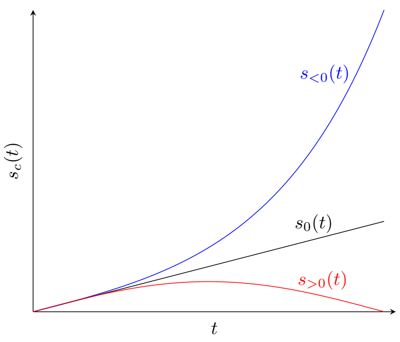
So even in the semi-Riemannian case, we have a splitting $J = J^\top + J^\perp$ as long as γ is not lightlike/null. In that lightlike case, the identity $\mathfrak{J}^\top \cap \mathfrak{J}^\perp = \{0\}$ does not hold.

3.4 Jacobi Fields in Constant Curvature Spaces

Our goal is to find a general formula of Jacobi fields in constant curvature spaces and use that to derive general results about the metric of such manifolds.

Lemma 3.16. A SRMF (M, g) having constant sectional curvature $c \equiv 1$ is equivalent to each of the following conditions:

1. For all $p \in M$ and $u, v \in T_pM$ such that $\operatorname{span}\{u, v\}$ is non-degenerate:
$$c = \frac{\operatorname{Rm}(u, v, v, u)}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}.$$
2. For all $p \in M$ and all $u, v \in T_pM$:
$$\operatorname{Rm}(u, v, v, u) = c(\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2).$$
3. For all $p \in M$ and orthonormal $e_1, e_2 \in T_pM$:
$$\operatorname{Rm}(e_1, e_2, e_2, e_1) = c\varepsilon_1\varepsilon_2$$
with $\varepsilon_i := \langle e_i, e_i\rangle$.
4. For all $p \in M$ and $u, v, w \in T_pM$:
$$\mathrm{R}(u, v)w = c(\langle v, w\rangle u - \langle u, w\rangle v). \tag{3.8}$$



Given $c \in \mathbb{R}$, consider the following function
$$s_c : \mathbb{R} \to \mathbb{R}$$
defined as
$$s_c(t) := \begin{cases} \frac{1}{\sqrt{c}}\sin(\sqrt{c}t) & c > 0 \\ t & c = 0 \\ \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}t) & c < 0 \end{cases} \tag{3.9}$$

Example 3.17. Let us consider several model manifolds for constant sectional curvature.

1. The n-dimensional Euclidean space $(\mathbb{R}^n, g_{\mathrm{st}})$ with the Euclidean metric and the n-dimensional Minkowski space $(\mathbb{R}^{1,n-1}, \eta)$ with the Minkowski metric have $c \equiv 0$. In polar coordinates, we have
$$g_{\mathrm{st}} = dr^2 + r^2\mathring{g}_{n-1} = dr^2 + s_0(r)^2\mathring{g}_{n-1}$$
and
$$\eta = -d\tau^2 + \tau^2\breve{g} = -d\tau^2 + s_0(\tau)^2\breve{g}_{n-1}.$$
2. The n-sphere $(\mathbb{S}^n, \mathring{g}_n)$ with the round metric has $c \equiv 1$. In polar coordinates, we have
$$\mathring{g}_n = d\theta^2 + \sin^2(\theta)d\varphi^2 = d\theta^2 + s_1(\theta)^2 d\varphi^2.$$
3. The hyperbolic n-space $(\mathbb{H}^n, \breve{g}_n)$ with the hyperbolic metric has $c \equiv -1$. In polar coordinates, we get
$$\breve{g}_n = d\xi^2 + \sinh(\xi)^2\mathring{g}_{n-1} = d\xi^2 + s_{-1}(\xi)^2\mathring{g}_{n-1}.$$

Proposition 3.18 (Jacobi Fields in Constant Curvature Spaces)**.** Let (M,g) be a SRMF with constant sectional curvature $c \in \mathbb{R}$ and let $\gamma : I \to \mathbb{R}$ be a unit-speed geodesic with $0 \in I$. Then some $J \in \mathfrak{X}(\gamma)^{\perp}$ with $J(0) = 0$ and $\nabla_{\frac{d}{dt}} J(0) \neq 0$ satisfies $J \in \mathfrak{J}(\gamma)$ if and only if

$$J(t) = k \cdot s_{\varepsilon c}(t) \cdot E(t) \tag{3.10}$$

for some $k \in \mathbb{R}$, $\varepsilon := \langle \dot\gamma, \dot\gamma \rangle$ and parallel $E \in \mathfrak{X}(\gamma)^{\perp}$ with $\|E\|^2 = 1$.

Proof. ($\Leftarrow$): We do one auxilliary computation beforehand: The second derivative of $s_\alpha(t)$ for some parameter $\alpha \in \mathbb{R}$ is:

$$\ddot s_\alpha(t) = \begin{cases} \frac{d}{dt}\cos(\sqrt{\alpha}t) = -\sqrt{\alpha}\sin(\sqrt{\alpha}t) & \alpha > 0 \\ \frac{d}{dt} 1 = 0 & \alpha = 0 \\ \frac{d}{dt}\cosh(\sqrt{-\alpha}t) = \sqrt{-\alpha}\sinh(\sqrt{-\alpha}t) & \alpha < 0 \end{cases} = -\alpha s_\alpha(t).$$

Directly calculating the Jacobi equation yields:

Note that in the $\alpha < 0$-case, we multiply with negative α in the end, so the sign switch is intentional and correct.

$$\begin{aligned} \nabla^2_{\frac{d}{dt}} J(t) - R(J,\dot\gamma)\dot\gamma &= \nabla^2_{\frac{d}{dt}}(k s_{\varepsilon c}(t) E(t)) + k s_{\varepsilon c}(t) R(E,\dot\gamma)\dot\gamma \\ &\overset{\text{Lemma 3.16}}{=} k \ddot s_{\varepsilon c}(t) E(t) + k s_{\varepsilon c}(t) \nabla_{\frac{d}{dt}}(\cancel{\nabla_{\frac{d}{dt}} E(t)}) \\ &\quad + k s_{\varepsilon c} c(\|\dot\gamma\|^2 E(t) - \cancel{\langle E, \dot\gamma\rangle}\dot\gamma) \\ &= -k c \varepsilon s_{\varepsilon c}(t) E(t) + k c \varepsilon s_{\varepsilon c}(t) E(t) = 0, \end{aligned}$$

where we used that E is parallel and normal and γ is unit-speed. ($\Leftarrow$) : The goal is to find k and E such that $J(0) = J_{k,E}(0)$ and $\nabla_{\frac{d}{dt}} J(0) = \nabla_{\frac{d}{dt}} J_{k,E}(0)$. Then we can apply uniqueness of Jacobi fields to obtain the claim.

1. $J(0) = J_{k,E}$ holds by assumption.
2. We have
$$\nabla_{\frac{d}{dt}} J(0) = k \dot s_{\varepsilon c}(0) E(0) = k E(0).$$
This agrees with our vector field if we choose $k := \|\nabla_{\frac{d}{dt}} J(0)\|$ and $E(0) := \frac{1}{k}\nabla_{\frac{d}{dt}} J(0)$ which is well-defined since $k \neq 0$ by assumption. We extend this by using the parallel transport $P_{\gamma(t)}$ along γ to define $E(t) := P_{\gamma(t)} E(0)$. We have already seen that
$$E(t) \perp \dot\gamma(t) \Leftrightarrow E(0) \perp \dot\gamma(0) \Leftrightarrow \nabla_{\frac{d}{dt}} J(0) \perp \dot\gamma(0),$$
and the last statement is part of our assumption.

$\square$

Remark 3.19.
1. If (M,g) is Riemannian (or Lorentzian and γ timelike), $J \in \mathfrak{J}(\gamma)^{\perp}$ already implies $\nabla_{\frac{d}{dt}} J(0) \perp \dot\gamma(0)$ and $\dot\gamma(0)^{\perp} \subseteq T_{\gamma(0)}M$ spacelike, so $\nabla_{\frac{d}{dt}} J(0) \neq 0$.
2. In constant curvature spaces, we have now two forms for Jacobi fields, $J(t) = t w^i \partial_i|_{\gamma(t)}$ and $J(t) = k s_{\varepsilon c}(t) E(t)$.

Given a time-oriented Lorentzian manifold, we define the radial distance function by the proper time $\tau : I^U(p) \to \mathbb{R}$ with $\tau(q) := \tau_U(p,q) > 0$. This yields a field ∂_τ.

Our goal is now to obtain the first general result with help of Jacobi fields. We will need the notion of a radial distance function and the associated vector field. Given a Riemannian manifold (M, g) and a normal chart (U, x^i) around some $p \in M$. Define the **radial distance function** $r : U \to \mathbb{R}$ by $r(q) = d(p,q) > 0$. If the coodinates are centered at p, we have the explicit form

$$r(x) = \sqrt{(x^1)^2 + \cdots + (x^n)^2},$$

and on $U \setminus \{p\}$ we get the **radial vector field**

$$\partial_r = \frac{x^i}{r(x)} \partial_i.$$

This is smooth on $U \setminus \{p\}$ and independent of choice of charts.

Definition 3.20 (Polar Coordinates)**.** In the following, $\mathring{g}$ is the round metric and $\breve{g}$ is the hyperbolic metric.

1. **Riemannian polar coordinates** are given by the map

$$\begin{aligned} \Phi_R : (\mathbb{R}_+ \times \mathbb{S}^{n-1}, dr^2 + r^2 \mathring{g}_{n-1}) &\to (\mathbb{R}^n \setminus \{0\}, g_{\mathrm{st}}) \\ (r, \theta) &\mapsto r \iota_{\mathbb{S}^{n-1}}(\theta), \end{aligned}$$

   where $\iota : \mathbb{S}^{n-1} \hookrightarrow (\mathbb{R}^n, g_{\mathrm{st}})$ is any isometric embedding with $\operatorname{im} \iota_{\mathbb{S}^{n-1}} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.

2. **Lorentian polar coordinates** are the map

$$\begin{aligned} \Phi_L : (\mathbb{R}_+ \times \mathbb{H}^{n-1}, -d\tau^2 + \tau^2 \breve{g}_{n-1}) &\to (I^+(0), \eta) \\ (\tau, \xi) &\mapsto \tau \iota_{\mathbb{H}^{n-1}}(\xi), \end{aligned}$$

   where $\iota : \mathbb{H}^{n-1} \hookrightarrow (\mathbb{R}^{1,n-1}, \eta)$ is any isometric embedding with $\operatorname{im} \iota_{\mathbb{H}^{n-1}} = \left\{x \in \mathbb{R}^{1,n-1} \mid \eta(x,x) = -1\right\}$.

Theorem 3.21 (Constant-Curvature Metrics in Normal Coordinates)**.** Let (M, g) be a RMF or a LMF with constant curvature $c \in \mathbb{R}$. Let U be a normal neighbourhood of $p \in M$ and $\psi = (x^i)$ be normal coordinates such that $\psi_* g_p = g_{\mathrm{st}}$. Let $r(q) = \|\exp_p^{-1}(q)\|$ resp. $\tau(q) = \|\exp_p^{-1}(q)\|$ be the radial distance function. Define angle functions $\theta(q) := \iota_{\mathbb{S}}^{-1}\left(\frac{\exp_p^{-1}(q)}{r(q)}\right)$ and $\xi(q) := \iota_{\mathbb{H}}^{-1}\left(\frac{\exp_p^{-1}(q)}{\tau(q)}\right)$. Then we have isometries

$$\begin{aligned} \varphi := \Psi_R^{-1} \circ \psi : U \setminus \{p\} &\to \mathbb{R}_+ \times \mathbb{S}^{n-1} \\ q &\mapsto (r(q), \theta(q)) \\ \varphi_L := \Phi_L^{-1} \circ \psi : I_U^+(p) &\to \mathbb{R}_+ \times \mathbb{H}^{n-1} \\ q &\mapsto (\tau(q), \xi(q)) \end{aligned}$$

onto their image between g and

$$g^c = dr^2 + s_c(r)^2 \mathring{g}_{n-1}$$

or

$$g^c = -d\tau^2 + s_c(\tau)^2 \breve{g}_{n-1},$$

respectively.

Proof. Our goal is to show that for all $q \in U \setminus \{p\}$ and all $w_1, w_2 \in T_{\varphi(q)}(\mathbb{R}_+ \times \mathbb{S}^{n-1})$,

$$g_q(\varphi^*(w_1), \varphi^*(w_2)) = g^c_{\varphi(q)}(w_1, w_2)$$

holds. We show this only for $w_1 = w_2 =: w$, since the general case follows by polarization. In the following, denote the Euclidean metric by $\langle\cdot,\cdot\rangle$. We split

$$T_{\varphi(q)}(\mathbb{R}_+ \times \mathbb{S}^{n-1}) \cong \operatorname{span}\{\partial_r|_{r(q)}\} \oplus T_{\theta(q)}\mathbb{S}^{n-1}$$

$$T_{\varphi(q)}(\mathbb{R}_+ \times \mathbb{H}^{n-1}) \cong \operatorname{span}\{\partial_\tau|_{\tau(q)}\} \oplus T_{\xi(q)}\mathbb{H}^{n-1}$$

and prove the claim in three parts:

1. $g_q(\varphi^*(\partial_r|_{r(q)}), \varphi^*(\partial_r|_{r(q)})) = g^c_{\varphi(q)}(\partial_r|_{r(q)}, \partial_r|_{r(q)}) = 1$

   $g_q(\varphi^*(\partial_\tau|_{\tau(q)}), \varphi^*(\partial_\tau|_{\tau(q)})) = -1$
2. $\forall w \in T_{\theta(q)}\mathbb{S}^{n-1} : g_q(\varphi^*(\partial_r|_{\varphi(r)}), \varphi^*(w)) = 0$
3. $\forall w \in T_{\theta(q)}\mathbb{S}^{n-1} : g_q(\varphi^*(w), \varphi^*(w)) = g^c_{\varphi(q)}(q)(w,w) = s_c(r(q))^2 \mathring{g}_{n-1}|_{\theta(q)}(w,w)$

   $g_q(\varphi^*(w), \varphi^*(w)) = s_{-c}(\tau(q))^2 \breve{g}_{n-1}|_{\xi(q)}(w,w)$

For 1 and 2, note that

$$(\Phi_R)_{*,(r(q),\theta(q))}(\partial_r) = (\Phi_R^{-1})^*(\partial_r|_{\varphi(q)}) = \iota(\theta(q)) \in T_{\psi(q)}\mathbb{R}^n.$$

Since $\psi(q) = r(q)\iota(\theta(q))$, this is radial, hence the Gauß lemma applies. Defining $v := (\Phi_R^{-1})^*(w)$, we get:

$$\begin{aligned}
g_q(\varphi^*(\partial_r|_{\varphi(q)}), \varphi^*(w)) &= g_q((\exp_p)_{*,\psi(q)}(\iota(\theta(q))), (\exp_p)_{*,\psi(q)}(v)) \\
&= g_p(\iota(\theta(q)), v) = \langle \iota(\theta(q)), v\rangle \\
&= \begin{cases} 1 & v = \iota(\theta(q)) \Leftrightarrow w = \partial_r|_{\varphi(q)} \\ 0 & v \perp \iota(\theta(q)) \ \underset{\Phi_R \text{ is isometry } g^c \leftrightarrow \langle\cdot,\cdot\rangle}{\Leftrightarrow} \ w \in T_{\theta(q)}\mathbb{S}^{n-1} \end{cases}.
\end{aligned}$$

This proves 1 and 2 simultaneously.

Note that we have not used our constant curvature assumption so far. This will become important in chapter 4.

Now we turn our attention to 3.: Let $w \in T_{\theta(q)}\mathbb{S}^{n-1}$ and set $v := (\Phi_R^{-1})^*(w) \in T_{\psi(q)}\mathbb{R}^n$. We have by definition

$$g^c_{\varphi(q)}(w,w) = s_c(r(q))^2 \mathring{g}_{n-1}|_{\theta(q)}(w,w) = \frac{s_c(r(q))^2}{r(q)^2}\langle v, v\rangle.$$

Let J be the unique Jacobi field along the unit-speed radial geodesic $\gamma : [0,b] \to U$ from p to q ($b = r(q)$) with $J(0) = 0$ and $J(b) = \varphi^*(w) \in T_qM$. We have $\dot\gamma(b) \propto \varphi^*_{r(q),\theta(q)}(\partial_r) \perp \varphi^*(w)$, so $J \in \mathfrak{J}(\gamma)^\perp$. This means we can write

This largely goes through for the Lorentzian case, just switch Φ_R for Φ_L and use $s_{-c}(t)$. Note that J is normal to a timelike geodesic, so the sign is still positive.

$$J(t) = k s_c(t) E(t)$$

with $E(b) \propto \varphi^*(w)$. We calculate explicitly:

$$k^2 = g_p(\nabla_{\frac{d}{dt}} J(0), \nabla_{\frac{d}{dt}} J(0))$$

and obtain

$$\|\varphi^*(w)\|^2_{g_q} = \|J(b)\|^2_{g_q} = k^2 s_c(b)^2 = s_c(r(q))^2 \|\nabla_{\frac{d}{dt}} J\|^2_{g_p}.$$

Now rescale γ to $[0,1] \to M$ and use $\varphi^*(w) = \psi^*(v) = v^i \partial_i|_q \in T_qM$ to get $J(t) = \frac{t}{b} v^i \partial_i|_{\gamma(t)}$. With this, we calculate

$$\|\nabla_{\frac{d}{dt}} J(0)\|^2_{g_p} = \|\frac{1}{b} v^i \partial_i|_{\gamma(0)}\|^2_{g_p} = \frac{1}{b^2}\langle v, v\rangle$$

With this, claim 3 follows. □

The Lorentzian case is a bit harder to phrase, but the idea still holds. The problem is that families of Riemannian metrics on $\mathbb{H}^{n-1}$ are generally not globally definable on $\mathbb{H}^{n-1}$ as $\psi(I^+_U(p))$ does not contain an entire $\mathbb{H}^{n-1}_\tau$, no matter how small τ is.

Corollary 3.22. Let (M, g) be a RMF, $p \in M$ and U a normal neighbourhood with normal coodinates ψ such that $\psi_* g_p = g_{\text{st}}$. Let $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq \psi(U)$. Then we find a family $\{h(r) \mid r \in (0, \varepsilon)\}$ of Riemannian metrics on $\mathbb{S}^{n-1}$ such that

$$\varphi : \Phi_R^{-1} \circ \psi : B^{d_g}_\varepsilon(p) \setminus \{p\} \to (0, \varepsilon) \times \mathbb{S}^{n-1}$$

given again by $\varphi(q) := (r(q), \theta(q))$ is a bijective isometry between g and $dr^2 + h(r)$.

Proof. We know that $B^{d_g}_\varepsilon(p) = \psi^{-1}(B_\varepsilon(0))$ by theorem **??**. Let $r \in (0, \varepsilon)$, $\theta \in \mathbb{S}^{n-1}$ and $w_1, w_2 \in T_\theta \mathbb{S}^{n-1}$. We define

$$h(r)(w_1, w_2) := g_{\varphi^{-1}(r,\theta)}(\varphi^{-1}_{*,(r,\theta)}(0\partial_r + w_1), \varphi^{-1}_{*,(r,\theta)}(0\partial_r + w_2)).$$

This is a smooth, symmetric, non-degenerate $(0, 2)$-tensor field on $\mathbb{S}^{n-1}$. φ being an isometry follows from the definition of $h(r)$, 1, and 2 of the previous proof.

□

Corollary 3.23 (Constant Curvature implies local Isometry)**.** All Riemannian and Lorentzian manifolds of constant curvature are locally isometric.

The Riemannian case is easier and can be found in [Lee18]. The theorem also holds for the semi-Riemannian case in general with a different proof, found in [ONe10].

Proof. We do the Lorentzian case. Given LMFs (M_1, g_1) and (M_2, g_2) with $p_i \in M_i$, we want to show that there are open neighbourhoods $U_i \subseteq M_i$ and a local isometry $\varphi : (U_1, g_1) \overset{\cong}{\to} (U_2, g_2)$. So let $\widetilde{U}_i$ be convex neighbourhoods of p_i, $p_i^- \ll_{\widetilde{U}_i} p_i$ and $\tau_{\widetilde{U}_1}(p_1^-, p_1) = \tau_{\widetilde{U}_2}(p_2^-, p_2) = \varepsilon$. Choose normal coordinates at p_i^- such that $\psi_i(p_i) = (\varepsilon, 0, \dots, 0)$. Consider polar-normal coordinates

$$\varphi_i : \widetilde{U}_i \to V_i \subseteq (0, \infty) \times \mathbb{H}^{n-1}.$$

They satisfy

$$\varphi_1(p_1) = \varphi_2(p_2) =: (\varepsilon, \xi_0) \in V_1 \cap V_2,$$

so $V_1 \cap V_2$ is an open neighbourhood of (ε, ξ_0). Setting $U_i := \varphi_i^{-1}(V_1 \cap V_2)$, we obtain the desired isometry

$$\varphi := \varphi_2^{-1} \circ \varphi_1 : U_1 \overset{\cong}{\to} U_2.$$

□

3.5 Conjugate Points

Definition 3.24 (Regular and Critical Points). Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. We call $p \in M$ a **regular point** of F if $F_{*,p} : T_pM \to T_{F(p)}N$ is surjective. Otherwise, we call p a **critical point**. We denote the critical points by $\operatorname{crit} F$.

Consider the exponential map

$$\exp_p : T_pM \to M$$

. The inverse function theorem guarantees that locally around all regular points, $\exp_p$ is a diffeomorphism. What happens at critical points?

Exercise. Consider $\mathbb{S}^2$. Given the unit circle $S_0 := \left\{ v \in T_p\mathbb{S}^2 \mid \|v\| = \pi \right\} \subseteq T_p\mathbb{S}^2$, the image, which is the antipodal point to p, consists only of critial points of $\exp_p$. At that point, several things happen:

1. Jacobi fields admit zeros: Given $J(t) = ks_1(t)E(t) = k\sin(t)E(t)$, we have $J(0) = J(\pi) = 0$.
2. All unit-speed geodesics originating at p meet at the antipodal point at $t = \pi$.
3. All these geodesics stop being minimizing after $t = \pi$.

The question is, which of these hold generally?

Definition 3.25 (Conjugate Points). Let (M, g) be a SRMF, $\gamma : I \to M$ be a geodesic, $a, b \in I$, and $\gamma(a) := p$, $\gamma(b) := q$. We call p **conjugate** to q along γ if there exists a non-zero Jacobi field $J \in \mathfrak{J}(\gamma)$ with $J(a) = J(b) = 0$. We call the dimension of the subspace of Jacobi fields vanishing at a and b the **order** or **multiplicity** of conjugacy.

Remark 3.26. Let γ be a timelike or spacelike geodesic. If there is a Jacobi field witnessing conjugacy, we can always choose a normal one satisfying the same condition since

$$J^\top = (kt + d)\dot\gamma(t),$$

which vanishes at most at one point. Hence the order of conjugacy is always $\leq \dim M - 1$.

Proposition 3.27 (Conjugacy and Critical Points). Let (M, g) be a SRMF, $p \in M$, $v \in \mathcal{E}_p$, and let $\gamma = \gamma_v : [0, 1] \to M$ be the unique geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Set $q := \gamma_v(1) = \exp_p(v)$. Then p is conjugate to q along γ if and only if $v \in \operatorname{crit}(\exp_p)$.

Proof. $(\Leftarrow)$: Let $v \in \operatorname{crit}(\exp_p)$. We find $w \in T_pM$ such that $(\exp_p)_{*,v}(w) = 0$ but $w \neq 0$. Take a Jacobi field with initial data $J(0) = 0$ and $\nabla_{\frac{d}{dt}} J(0) = w$. By lemma 3.9, we can write J as variation field of the geodesic variation

$$\Gamma(s,t) := \exp_p(t(v+sw)).$$

We have

$$J(1) = \left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s,1) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p(v+sw) = (\exp_p)_{*,v}(w) = 0.$$

$(\Rightarrow)$: Let J be a non-zero Jacobi field along γ with $J(0) = J(1) = 0$. Writing $w := \nabla_{\frac{d}{dt}} J(0) \neq 0$ and again J as variation field of

$$\Gamma(s,t) = \exp_p(t(v+uw))$$

yields $J(1) = 0 = (\exp_p)_{*,v}(w)$, so $(\exp_p)_{*,v}$ fails to be injective. $\square$

If (M,g) is a SRMF and $\gamma : I \to M$ is a geodesic, the **two-point boundary value problem** asks whether it is always possible to find a Jacobi field along γ with two prescribed boundary values $J(0) = v \in T_{\gamma(0)}M$ and $J(1) = w \in T_{\gamma(1)}M$.

Proposition 3.28 (The two-point Boundary Value Problem)**.** Let (M,g) be a SRMF and $\gamma : [a,b] \to M$ be a geodesic segment. The two-point boundary value problem is solvable if and only if $\gamma(a)$ is not conjugate to $\gamma(b)$.

Exercise. Prove the preceeding proposition.

Proposition 3.29 (Conjugacy and Sectional Curvature)**.** Let (M,g) be a RMF with non-positive sectional curvature. For any $p \in M$ and any geodesic $\gamma : [0,b] \to M$ originating at p, there are no points conjugate to p along γ.

Exercise. Prove the preceeding proposition.

3.6 The Second Variation Formula for Arc-Length

Theorem 3.30 (Second Variation Formula for Arc-Length)**.** Let $(M, \langle\cdot,\cdot\rangle)$ be a SRMF, $\gamma : [a,b] \to M$ be a unit-speed geodesic segment, and let

$$\Gamma : (-\delta,\delta) \times [a,b] \to M$$

be a continuous, fixed-endpoint variation of γ such that all curves $t \mapsto \Gamma_s(t)$ have the same causal character. Additionally, let there be a partition $\{t_i\}_{1\leq i\leq n}$ of $[a,b]$ such that $\Gamma|_{[t_i,t_{i+1}]}$ is smooth with $t_1 = a$

and $t_n = b$. Then the **second variation of arc-length** is given by

$$\frac{d^2}{ds^2}\bigg|_{s=0} L[\Gamma_s] = \varepsilon \int_a^b \Big(\mathrm{Rm}(V^\perp, \dot\gamma, V^\perp, \dot\gamma) + \langle \nabla_{\frac{d}{dt}} V^\perp, \nabla_{\frac{d}{dt}} V^\perp\rangle\Big)\, dt \tag{3.11}$$

where $\varepsilon = \langle\dot\gamma,\dot\gamma\rangle$, $V = \frac{\partial}{\partial s}\big|_{s=0}\Gamma$, and $V^\perp = V - \varepsilon\langle\dot\gamma,\dot\gamma\rangle\dot\gamma$.

Note that the sign change in [Lee18] originates from switching the second and third entry of the Riemann tensor.

Proof. The proof is done by direct calculation. We set $\frac{\partial}{\partial s}\big|_{s=0}\Gamma =: S(s,t)$ and $\frac{\partial}{\partial t}\big|_{t=0}\Gamma := T(s,t)$. We have

$$\begin{aligned}
\frac{d^2}{ds^2} L[\Gamma_s] &= \frac{d}{ds}\int_{t_i}^{t_{i+1}} \frac{\partial}{\partial s}\langle \dot\Gamma_s, \dot\Gamma_s\rangle dt = \frac{d}{ds}\int_{t_i}^{t_{i+1}} \frac{\varepsilon\langle \nabla_{\frac{d}{ds}} T, T\rangle}{\sqrt{\varepsilon\langle T,T\rangle}} dt \\
&= \varepsilon\int_{t_i}^{t_{i+1}} \frac{\partial}{\partial s}\frac{\langle \nabla_{\frac{d}{ds}} S, T\rangle}{\sqrt{\varepsilon\langle T,T\rangle}} dt \\
&= \varepsilon\int_{t_i}^{t_{i+1}} \left[\frac{\langle \nabla_{\frac{d}{ds}}\nabla_{\frac{d}{dt}} S, T\rangle + \langle \nabla_{\frac{d}{dt}} S, \nabla_{\frac{d}{ds}} T\rangle}{\sqrt{\varepsilon\langle T,T\rangle}} - \frac{\langle \nabla_{\frac{d}{dt}} S, T\rangle\langle \nabla_{\frac{d}{ds}} T, T\rangle}{(\varepsilon\langle T,T\rangle)^{\frac{3}{2}}}\right] dt \\
&= \varepsilon\int_{t_i}^{t_{i+1}} \left[\frac{\langle \nabla_{\frac{d}{ds}}\nabla_{\frac{d}{dt}} S, T\rangle + \langle \nabla_{\frac{d}{dt}} S, \nabla_{\frac{d}{dt}} S\rangle}{\sqrt{\varepsilon\langle T,T\rangle}} - \frac{\langle \nabla_{\frac{d}{dt}} S, T\rangle\langle \nabla_{\frac{d}{dt}} S, T\rangle}{(\varepsilon\langle T,T\rangle)^{\frac{3}{2}}}\right] dt
\end{aligned}$$

Evaluating at $s = 0$ yields $\frac{\partial}{\partial t}\Gamma(0,t) = \dot\gamma(t)$ and hence $\langle T,T\rangle = \varepsilon$. Also, we obtain $\nabla_{\frac{d}{ds}} S(0,t) = \nabla_{\frac{d}{dt}} V$. This yields

$$\frac{d^2}{ds^2}\bigg|_{s=0} L[\Gamma_s] = \varepsilon\int_{t_i}^{t_{i+1}} \Bigg[\underbrace{\langle \nabla_{\frac{d}{ds}}\nabla_{\frac{d}{dt}} S, \dot\gamma\rangle}_{=:A} + \underbrace{\langle \nabla_{\frac{d}{dt}} V, \nabla_{\frac{d}{dt}} V\rangle}_{=:B} - \underbrace{\varepsilon\langle \nabla_{\frac{d}{dt}} V, \dot\gamma\rangle^2}_{=:C}\Bigg] dt.$$

Term A can be simplified by lemma 3.3 to:

$$\begin{aligned}
A|_{s=0} &= \langle \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{ds}} S(0,t), \dot\gamma\rangle + \langle \mathrm{R}(S(0,t), T(0,t))S(0,t), \dot\gamma\rangle \\
&= \langle \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{ds}} S(0,t), \dot\gamma\rangle + \langle R(V, \dot\gamma)V, \dot\gamma\rangle \\
&= \mathrm{Rm}(V^\perp, \dot\gamma, V^\perp, \dot\gamma) + \langle \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{ds}} S(0,t), \dot\gamma\rangle
\end{aligned}$$

That the only contribution is from the normal part follows either by direct calculation or by using the symmetry $Rm(\cdot,\cdot,v,v) = \mathrm{Rm}(v,v,\cdot,\cdot) = 0$.

Term B can be written as

$$\langle \nabla_{\frac{d}{dt}} V, \nabla_{\frac{d}{dt}} V\rangle = \langle (\nabla_{\frac{d}{dt}} V)^\perp, (\nabla_{\frac{d}{dt}} V)^\perp\rangle + \langle \nabla_{\frac{d}{dt}} V, \dot\gamma\rangle^2\varepsilon.$$

A direct calculation shows that taking the normal part commutes with the covariant derivative:

$$\nabla_{\frac{d}{dt}} V^\perp = \nabla_{\frac{d}{dt}} V - \varepsilon\nabla_{\frac{d}{dt}}\langle V, \dot\gamma\rangle\dot\gamma = \nabla_{\frac{d}{dt}} V - \varepsilon\langle \nabla_{\frac{d}{dt}} V, \dot\gamma\rangle\dot\gamma = (\nabla_{\frac{d}{dt}} V)^\perp.$$

All other Leibniz rule terms vanish since γ is a geodesic segment.

This yields

$$\langle \nabla_{\frac{d}{dt}} V, \nabla_{\frac{d}{dt}} V\rangle = \langle \nabla_{\frac{d}{dt}} V^\perp, \nabla_{\frac{d}{dt}} V^\perp\rangle + \langle \nabla_{\frac{d}{dt}} V, \dot\gamma\rangle^2.$$

However, the rightmost term cancels with C. It remains to show that

$$\varepsilon\int_{t_i}^{t_{i+1}} \langle \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{ds}} S(0,t), \dot\gamma\rangle dt$$

vanishes when summing over i. We have:

$$\begin{aligned}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\langle\nabla_{\frac{d}{dt}}\nabla_{\frac{d}{ds}}S(0,t),\dot\gamma\rangle dt &= \sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\frac{d}{dt}\langle\nabla_{\frac{d}{ds}}S(0,t),\dot\gamma\rangle dt\\ &= \sum_{i=1}^{n}\Big(\langle\nabla_{\frac{d}{ds}}S(0,t_{i+1}),\dot\gamma(t_{i+1})\rangle-\langle\nabla_{\frac{d}{ds}}S(0,t_i),\dot\gamma(t_i)\rangle\\ &= \langle\nabla_{\frac{d}{ds}}S(0,b),\dot\gamma(b)\rangle-\langle\nabla_{\frac{d}{ds}}S(0,s),\dot\gamma(a)\rangle=0,\end{aligned}$$

where we used that Γ is a variation with fixed endpoints in the last step. □

There is also a slightly different version:

Proposition 3.31 (Second Variation for Submanifolds)**.** Let (M,g) be a RMF or LMF and Σ_1,Σ_2 be two spacelike submanifolds. Let $\gamma:[a,b]\to M$ be a timelike unit-speed geodesic segment meeting Σ_1 orthogonally at $t=a$ and Σ_2 orthogonally at $t=b$. Let Γ be a continuous variation of γ which is smooth on some partition of $[a,b]$, and with $\Gamma(s,a)\in\Sigma_1$ and $\Gamma(s,b)\in\Sigma_2$ for all s. Let II_i be the second fundamental form of Σ_i, and let $\varepsilon=\langle\dot\gamma,\dot\gamma\rangle$. Then the second variation of arc-length is given by

$$\begin{aligned}\frac{d^2}{ds^2}\bigg|_{s=0}L[\Gamma_s] = \int_a^b\Big[\langle\nabla_{\frac{d}{dt}}V^\perp,\nabla_{\frac{d}{dt}}V^\perp\rangle+\mathrm{Rm}(V^\perp,\dot\gamma,\dot\gamma,V^\perp)\Big]\,dt\\ +\langle\mathrm{II}_2(V(b),V(b)),\dot\gamma(b)\rangle-\langle\mathrm{II}_1(V(a),V(a)),\dot\gamma(a)\rangle\end{aligned}$$

Exercise. Prove the preceeding proposition.

The second variation behaves a bit like the Hessian in real analysis on $\mathbb{R}^n$: It tells us something about the maxima and minima of the length functional. We use this to define something akin to a Hessian for vector fields.

Notation. We denote the space of piecewise smooth vector fields along a curve γ by $\overline{\mathfrak{X}}$. If $\gamma:[a,b]\to M$, we denote the space of vector fields with $X(a)=X(b)=0$ by $\mathfrak{X}_0$.

Definition 3.32 (Index Form)**.** Let $(M,\langle\cdot,\cdot\rangle)$ be a SRMF and $\gamma:[a,b]\to M$ be a unit-speed geodesic segment. We define the **index form** to be a map

$$I:\overline{\mathfrak{X}}(\gamma)^\perp\times\overline{\mathfrak{X}}(\gamma)^\perp\to\mathbb{R}$$

given by

$$I(X,Y):=\varepsilon\int_a^b(\langle\nabla_{\frac{d}{dt}}X,\nabla_{\frac{d}{dt}}Y\rangle+\mathrm{Rm}(X,\dot\gamma,Y,\dot\gamma))dt.$$

Corollary 3.33 (Index Form and Extremizing Behaviour). Let (M, g) be a RMF and $\gamma : [a, b] \to M$ be a unit-speed geodesic segment. If there exists $V \in \overline{\mathfrak{X}}_0(\gamma)^\perp$ such that

$$I(V, V) < 0,$$

then γ cannot minimize the distance from $\gamma(a)$ to $\gamma(b)$.

Let $(\mathcal{L}, g)$ be a time-oriented LMF and $\gamma : [a, b] \to \mathcal{L}$ be a unit-speed future-directed timelike geodesic segment. If there exists a non-zero $V \in \overline{\mathfrak{X}}_0(\gamma)^\perp$ such that

$$I(V, V) > 0,$$

then γ cannot maximize the distance from $\gamma(a)$ to $\gamma(b)$.

Remark 3.34. We can also phrase this as if γ is minimizing from $\gamma(a)$ to $\gamma(b)$, then $I|_{\overline{\mathfrak{X}}_0(\gamma)^\perp \times \overline{\mathfrak{X}}_0(\gamma)^\perp}$ is positive semi-definite.

In the Lorentzian case, maximizing γ implies negative semi-definity.

Corollary 3.35 (Sectional Curvature and Submanifolds). Let (M, g) be a connected, complete RMF with positive sectional curvature. If Σ_1, Σ_2 are compact, totally geodesic submanifolds such that

$$\dim \Sigma_1 + \dim \Sigma_2 \geq \dim M,$$

then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$.

Exercise. Prove the preceeding corollary.

3.7 The Theorems of Bonnet-Myers and Hawking

The Theorem of Bonnet-Myers

Hawking's Singularity Theorem

Exercise Class 14.01.2026

Theorem 3.36 (Hawking's Singularity Theorem). Let $(\mathcal{L}, \langle\cdot,\cdot\rangle)$ be a globally hyperbolic Lorentzian n-manifold with $\mathrm{Ric}(X, X) \geq 0$ for all $X \in T_pM^{\mathrm{t.l.}}$ and $p \in M$. If there exists a spacelike hypersurface Σ such that:

(a) $J_-(\Sigma) \cap J_+(q)$ is compact for all $q \in J_-(\Sigma)$.

(b) There exists $\beta > 0$ such that $\langle H_p, N_p\rangle \geq \beta > 0$ for all $p \in \Sigma$, where N is the past-pointing timelike unit normal vector field to Σ and H_p is the mean curvature vector, i.e.

$$H_p := \frac{1}{n-1} \operatorname{tr}_\Sigma \mathrm{II}_p = \frac{1}{n-1} \sum_{i=i}^{n-1} \mathrm{II}_p(E_i, E_i)$$

for any orthonormal frame E_i of $T_p\mathcal{L}|_\Sigma$.

Then

$$\tau_\Sigma(q) = \sup\{\tau(q, x) \mid x \in \Sigma\} \leq \frac{1}{\beta}.$$

Proof. Let $q \in J^-(\Sigma)$. By assumption (a) and strong causality, which follows from global hyperbolicity by a previous exercise, there exists a future-directed timelike unit-speed geodesic $\gamma : [0, b] \to \mathcal{L}$ from q to Σ with $\dot\gamma(b) \perp T_{\gamma(b)}\Sigma$ and $b = L[\gamma] = \tau_\Sigma(q)$. Let $(E_i)_{1\leq i\leq n-1}$ be a parallel ONF for $\dot\gamma^\perp$ along γ. Consider vector fields $V_i := fE_i$ for some f with $f(0) = 0$ and $f(b) = 1$. Let Γ_i be a variation of γ with variational vector field V_i, and such that $\Gamma_i(s, 0) = q$ as well as $\Gamma_i(s, b) \in \Sigma$ for all s. This is possible since $V_i(0) = 0$ and $V_i(b) \in T_{\gamma(b)}\Sigma$. Since $L[\gamma] = \tau_\Sigma(q)$, proposition **??** yields

$$0 \leq \left.\frac{d^2}{ds^2}\right|_{s=0} L[\Gamma_{i,s}]$$

for all i. We sum over i to obtain:

$$\begin{aligned}
0 &\geq \sum_{i=1}^{n-1} \left.\frac{d^2}{ds^2}\right|_{s=0} L[\Gamma_{i,s}] \\
&= \varepsilon\left[\int_0^b \dot f^2(n-1) - \sum_{i=1}^{n-1}\mathrm{Rm}(fE_i,\dot\gamma,\dot\gamma,fE_i)\right]dt + \sum_{i=1}^{n-1}\langle \mathrm{II}(E_i,E_i),\dot\gamma(b)\rangle \\
&= -\int_0^b \dot f^2 dt + \sum_0^b \mathrm{Ric}(\dot\gamma,\dot\gamma)\cdot f^2 dt + \langle \underbrace{\sigma_{i=1}^{n-1}\mathrm{II}(E_i,E_i)}_{=(n-1)H}, \underbrace{-\dot\gamma(b)}_{=N_{\gamma(b)}}\rangle \\
&\geq -(n-1)\int_0^b (\dot f^2 + 0 + (n-1)\beta)dt = (n-1)\left(-\frac{1}{b}+\beta\right)
\end{aligned}$$

Here we see that $f(t) = \frac{t}{b}$ is a good choice.

Therefore, we get $0 \geq -\frac{1}{b} + \beta$, so

$$\frac{1}{\beta} \geq b = \tau_\Sigma(q)$$

. ∎

Exercise. An interesting example is $\mathcal{L} = (I^+(0), \eta)$ in Minkowski space where Σ is a hyperboloid.

Remark 3.37. The Riemannian version is a bit more difficult to phrase. If (M, g) is a RMF with boundary ∂M, this implies the existence of an inward-pointing smooth unit-normal vector field N along ∂M. Assume:

1. M is (metrically) complete.
2. $\mathrm{Ric} \geq 0$
3. $\langle H, N\rangle < \beta \leq 0$ on ∂M.

Then

$$d_{\partial M}(p) \leq \frac{1}{\beta}$$

for all $p \in M$. We can prove this as follows:

1. Show there exists a geodesic $\gamma : [0, b] \to M$ orthogonal to ∂M such that $\gamma(0) = p$ and $\gamma(b) \in \partial M$, so $L[\gamma] = d_{\partial M}(p)$.
2. For any $q \in \partial M$, there exists γ_q with $L[\gamma_q] = d(p, q)$. Hopf-Rinow tells us that if ∂M is compact, then

$$d_{\partial M}(p) = \inf\{d(p, q) \mid q \in \partial M\} = d(p, q_0)$$

for some $q_0 \in \partial M$. Take $\gamma = \gamma_{q_0}$. If ∂M is non-compact, consider $B_R(p)$. Then for all $q' \in \partial M \setminus B_R(p)$ we have $d(p, q') > R$. This yields

$$d_{\partial M}(p) = \left\{ d(p, \tilde{q}) \mid q \in \partial M \cap \overline{B_R(p)} \right\}.$$

However, $\partial M \cap \overline{B_R(0)}$ is compact by completeness and Hopf-Rinow.

3.8 The Min/Max-Behaviour of Geodesics

3.9 The Riemannian Cut Locus

CHAPTER FOUR

Comparision Geometry

4.1 Hessian Operators and the Riccati Equation

Definition 4.1 (Hessian). Let (M, g) be a SRMF and $u : M \to \mathbb{R}$ smooth. The **covariant Hessian** is the $(0, 2)$-tensor

$$\mathrm{Hess}_u(X, Y) := (\nabla^2 u)(X, Y) = \nabla_X(\nabla_Y u) - \nabla_{\nabla_X Y} u.$$

The **Hessian operator** is the $(1, 1)$-tensor

$$\mathcal{H}_u := \mathrm{Hess}_u^\sharp \,.$$

In local coordinates (x^i), the Hessian has the (familiar) form

$$\mathrm{Hess}_u = (\partial_j \partial_i u - \Gamma^k_{ij} \partial_k u)\, dx^i \otimes dx^j.$$

Remark 4.2. The Hessian is symmetric if ∇ is the Levi-Civita-Connection (or, more generally, if and only if ∇ is torsion-free):

$$\begin{aligned}
\mathrm{Hess}_u(X, Y) &= Y(X(u)) - du(\nabla_Y X) = (XY - [X, Y])u - du(\nabla_Y X) \\
&= X(Y(u)) - du(\nabla_Y X + [X, Y]) \\
&= X(Y(u)) - du(\nabla_X Y - T(X, Y)) \\
&= \mathrm{Hess}_u(Y, X) + T(X, Y)(u)
\end{aligned}$$

Unraveling the definition of $\mathcal{H}_u$, we get

$$g(\mathcal{H}_u(X), Y) = \mathrm{Hess}_u(X, Y) = \mathrm{Hess}_u(Y, X) = g(X, \mathcal{H}_u(Y)),$$

and hence $\mathcal{H}_u$ is self-adjoint.

With the radial distance function defined above, we obtain the Hessian operator $\mathcal{H}_r$ associated to that function.

Notation. Given a normal neighbourhood U, we denote the geodesic spheres of radius r_0 and proper time τ_0, respectively, by

$$S_{r_0} := \{x \in U \mid r(x) = r_0\}$$

and

$$H_{\tau_0} := \left\{x \in I_U^+(p) \mid \tau(x) = \tau_0\right\}.$$

The Gauß' Lemma immediately tells us that $T_q \mathbb{S}_{r(q)} = \partial_r|_q^\perp \subseteq T_q M$

and similarly for τ. Denote the projection on the radial tangent by

$$\pi_q^r : T_qM \to T_qS_{r(q)}.$$

Lemma 4.3. Let (M, g) be a Riemannian or time-oriented Lorentzian manifold, (U, x^i) be a normal chart around $p \in M$ and let r be the radial distance function. Then we have for all $q \in U \setminus \{p\}$:

1. $\mathcal{H}_r(\partial_r|_q) = 0$ and $\mathcal{H}_\tau(\partial_\tau|_q) = 0$
2. For all $X \in T_qM$ with $X \perp \partial_r|_q$:

$$\mathcal{H}_r(X) = \pi_q(\nabla_X(\operatorname{grad} r)),$$

and similarly for $X \perp \partial_\tau|_q$:

$$\mathcal{H}_\tau(X) = \pi_q(\nabla_X(\operatorname{grad} \tau)) = \pi_q^{\mathbb{H}}(\nabla_X \operatorname{grad}(\tau)),$$

where $\pi_q^{\mathbb{H}} : T_qM \to \partial_\tau|_q^\perp \cong T\mathbb{H}^{n-1}$ denotes the orthogonal projection

$$\pi_q^{\mathbb{H}}(Y) = Y + g(Y, \partial_\tau)\partial_\tau.$$

Proof. We proof the Lorentzian case:

1. Let $Y \in T_qM$ be arbitrary. We have:

$$\begin{aligned} g(\mathcal{H}_\tau(\partial_\tau|_q), Y) &= g(\partial_\tau|_q, \mathcal{H}_\tau(Y)) = g(\partial_\tau|_q, -\nabla_Y \operatorname{grad}(\tau)) \\ &= g(\nabla_Y \partial_\tau, \partial_\tau|_q) = \frac{1}{2} Y_q(g(\partial_\tau, \partial_\tau)) = 0, \end{aligned}$$

where we used symmetry of g and the fact that $-\operatorname{grad}(\tau) = \partial_\tau$ since the metric is $-d\tau^2 + \tilde{g}$.

2. This follows since the previous remark showed $\mathcal{H}_\tau(X) = \nabla_X \operatorname{grad}(\tau)$ and $\mathcal{H}_\tau(X) \perp \partial_\tau$ follows from the upper equation.

□

Note that $g|_{TH_{\tau_0} \times TH_{\tau_0}}$ is Riemannian, so the restriction is self-adjoint for a positive-definite inner product even in the Lorentzian case.

Remark 4.4. For any τ_0 such that $H_{\tau_0} \neq \emptyset$, $\mathcal{H}_\tau$ restricts to a self-adjoint linear operator

$$\mathcal{H}_\tau : TH_{\tau_0} \to TH_{\tau_0}$$

and is given by the Weingarten map $\mathcal{H}_\tau(X) = -\nabla_X N$ where $N = \partial_\tau$ is the future-pointing unit normal to H_{τ_0}.

Proposition 4.5 (The Hessian and Jacobi Fields)**.** Let $(\mathcal{L}, g)$ be a LMF, $p \in \mathcal{L}$ and $U \subseteq \mathcal{L}$ be a normal neighbourhood around p. Set $\tau := \tau_U(p, \cdot)$. Let $\gamma : [0, b] \to U$ be a future-directed timelike unit-speed geodesic starting at p and let $J \in \mathfrak{J}^\perp(\gamma)$ such that $J(0) = 0$. Then

$$\nabla_{\frac{d}{dt}} J(t) = -\mathcal{H}_\tau(J(t))$$

for all $\tau_0 \in (0, b]$.

In the Riemannian case, we have

$\nabla_{\frac{d}{dt}} J(t) = \mathcal{H}_r(J(t)).$

Proof. Let $v := \dot\gamma(0)$ and $w := \nabla_{\frac{d}{dt}} J(0)$. By equation 3.3, we have a variation through geodesics

$$\Gamma_s(t) = \exp_{\sigma(s)}(tU(s))$$

where $\sigma(s)$ is any curve with $\sigma(0) = p$ and $\dot\sigma(0) = J(0)$. The vector field U satisfies $U(0) = 0$ and $\nabla_{\frac{d}{ds}} U(0) = w$ and the variational field of Γ is $\frac{\partial}{\partial s}\big|_{s=0} = J$. Since $J(0) = 0$, choose $\sigma(s) \equiv p$. Since $w \perp v$ (J is normal), we have

Note that $g_p \cong \eta_p$.

$$v \in H := \{X \in T_p\mathcal{L} \mid g_p(X, X) = -1\}.$$

Hence, $w \in T_v H$. Since $U : I \to T_pM = T_{\sigma(s)}M$ with $\nabla_{\frac{d}{ds}} U(0) \in T_v H$ and $U(0) = v \in H$, we can choose U such that $U(s) \in H$ for all s. Therefore, each curve $t \mapsto \Gamma_s(t)$ is a future-directed timelike unit-speed geodesic starting at p, implying

$$\partial_t \Gamma_{s_0}(t_0) = \partial_\tau|_{\Gamma(s_0, t_0)} \tag{4.1}$$

Now we compute:

$$\nabla_{\frac{d}{dt}} J = \nabla_{\frac{d}{dt}} \frac{\partial}{\partial s}\bigg|_{s=0} \Gamma = \nabla_{\frac{d}{ds}} \left(\frac{\partial}{\partial t}\Gamma\right)(0, t) = \nabla_{\frac{d}{ds}} \left(\partial_\tau \circ \Gamma\right)(0, t),$$

where we used equation 4.1. Since ∂_τ is a smooth vector field on $I^+_U(p)$ (which is an open neighbourhood of $\Gamma_0(\tau_0)$ for any $\tau_0 \in (0, b]$). This implies

$$\nabla_{\frac{d}{ds}} \left(\partial_\tau \circ \Gamma\right)(0, t) = \nabla_{\partial_s \Gamma|_{s=0}} \partial_\tau = \nabla_J \partial_\tau = -\nabla_J \operatorname{grad}(\tau) = \mathcal{H}_\tau(J)$$

using $J \perp \dot\gamma = \partial_\tau$. $\square$

Recall the previously defined function 3.9.

Proposition 4.6 (Sectional Curvature and Hessian)**.** Let $(\mathcal{L}, g)$ be a LMF, $p \in \mathcal{L}$ and U be a normal neighbourhood of p. Then $(\mathcal{L}, g)$ has constant sectional curvature c on $I^+_U(p)$ if and only if

$$\mathcal{H}_\tau = -\frac{s'_{-c} \circ \tau}{s_{-c} \circ \tau} \cdot \pi^H$$

on $I^+_U(p)$.

Proof. $(\Rightarrow)$: Let $q \in I^+_U(p)$, $\gamma : [0, b] \to M$ be the future-directed timelike unit-speed geodesic from p to q, and let $\{E_1 = \dot\gamma, E_2, \dots, E_n\}$ be an ONF along γ. Proposition 3.18 tells us that

$$J_i := s_{-c} E_i,\ i \geq 2$$

are normal Jacobi fields along γ with $J_i(0) = 0$. Compute

$$s'_{-c} E_i = \nabla_{\frac{d}{dt}} J_i = -\mathcal{H}_\tau(J_i) = -s_{-c}\mathcal{H}_\tau(E_i),$$

where we used proposition 4.5 evaluating at $t = b$ and $s_{-c}(b)^{-1}$ noting $s_{-c}(b) \neq 0$ because $\gamma(b) = q \in U$, which is normal. Hence q is not conjugate to p, implying $J_i(b) \neq 0$. This yields

$$\mathcal{H}_\tau(E_i(b)) = -\frac{s'_{-c}(b)}{s_{-c}(b)} E_i(b),$$

but $E_i(b) = \pi^H(E_i(b))$ because it is already orthogonal to $\dot\gamma(b)$. For $i = 1$, we have

$$0 = \mathcal{H}_\tau(E_1(b)) = \pi^H(E_1(b))$$

because $E_1(b) = \dot\gamma(b)$.

$(\Leftarrow)$: Let $q \in I_U^+(p)$ and $\gamma : [0, b) \to U$ be the corresponding geodesic. From theorem 3.21, we know that

$$\varphi : I_U^+(p) \to \mathbb{R}_+ \times \mathbb{H}^{n-1}$$

given by $\varphi(q) = (\tau(q), \xi(q))$ is a diffeomorpism onto its image. If we can show that it is an isometry between g and $g_c := -d\tau^2 + s_{-c}(\tau)^2 g_{\mathbb{H}^{n-1}}$, then g has constant sectional curvature. Following the proof of theorem 3.21, we immediately get 1 and 2. In 3, the constant curvature assumption only entered via Jacobi fields, more precisely via proposition 3.18 yielding: Any $J \in \mathfrak{J}^\perp(\gamma)$ with $J(0) = 0$ is given by

$$J(t) = ks_{-c}(t)E(t)$$

for some $k \in \mathbb{R}$ and $E \perp \dot\gamma$ as well as parallel. It only remains to show this in our setting. Let $J \in \mathfrak{J}^\perp(\gamma)$. We have

$$\nabla_{\frac{d}{dt}} J = \frac{s'_{-c}}{s_{-c}} J$$

on $(0, b]$. By the product rule, $\frac{1}{s_{-c}} J$ is parallel along γ on $(0, b]$. Define $\tilde E(t) := \frac{1}{s_{-c}} J$ and let $E = \frac{\tilde E}{\|\tilde E\|}$. Then E is a parallel unit VF orthogonal $\dot\gamma$ and $J = \|\tilde E\| s_{-c} E$. Set $k := \|\tilde E\|$, which is constant since $\tilde E$ was parallel. It suffices to have this for $t \in (0, b]$ since this still gives $k = \|\nabla_{\frac{d}{dt}} J(0)\|$ by k constant. □

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Consider a LMF $(\mathcal{L}, g)$ with a normal neighbourhood U. The radial distance function $\tau := \tau_U(p, \cdot)$ is smooth on $I_U^+(p)$. Let $\gamma : [0, b] \to U$ be a timelike future-directed unit-speed geodesic starting at p. In this case, the Hessian operator restricts to

$$\mathcal{H}_\tau : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

which is a $(1, 1)$-tensor field along γ and $\mathcal{C}^\infty(I)$-linear. Therefore, we also get another $\mathcal{C}^\infty(I)$-linear $(1, 1)$-tensor field

$$\nabla_{\frac{d}{dt}} \mathcal{H}_\tau(X) := \nabla_{\frac{d}{dt}} (\mathcal{H}_\tau(X)) - \mathcal{H}_\tau\left(\nabla_{\frac{d}{dt}} X\right). \tag{4.2}$$

Definition 4.7 (Tidal Force Operator). Let (M, g) be a RMF or a LMF and let $\gamma : [0, b] \to M$ be a (timelike) geodesic. The **tidal force operator** is defined by

$$\begin{aligned} \mathrm{R}_{\dot\gamma} : \mathfrak{X}(\gamma) &\to \mathfrak{X}(\gamma) \\ X &\mapsto \mathrm{R}_{\dot\gamma}(X) := -\mathrm{R}(X, \dot\gamma)\dot\gamma. \end{aligned}$$

One sees that this is indeed $\mathcal{C}^\infty(I)$-linear, self-adjoint and $\operatorname{tr} \mathrm{R}_{\dot\gamma} = -\operatorname{Ric}(\dot\gamma, \dot\gamma)$. Comparing with the Jacobi equation also shows that

$$\mathrm{R}_{\dot\gamma}(J) = \nabla^2_{\frac{d}{dt}} J.$$

Remark 4.8. The tidal force operator restricts to a well-defined $\mathcal{C}^\infty(I)$-linear operator $\mathfrak{X}(\gamma)^\perp \to \mathfrak{X}(\gamma)^\perp$ and the trace of that restriction still yields $-\operatorname{Ric}(\dot\gamma,\dot\gamma)$ since $R_{\dot\gamma}(\dot\gamma) = 0$. This restriction is still self-adjoint with respect to $g|_{\mathfrak{X}(\gamma)^\perp\times\mathfrak{X}(\gamma)^\perp}$. This metric restriction is Riemannian even if the original g is Lorentzian.

Theorem 4.9 (Riccati Equation). Let (M, g) be a RMF or a LMF, U be a normal neighbourhood of some $p \in M$ and $\gamma : [0, b] \to U$ be a timelike future-directed radial unit-speed geodesic starting at p. Let r (τ) be the radial distance function on U $(I^+_U(p))$. Then, the Hessian operator satisfies the **Riccati equation** (REQ):

$$\nabla_{\frac{d}{dt}}\mathcal{H}_r + \mathcal{H}_r^2 - \mathrm{R}_{\dot\gamma} = 0 \tag{4.3}$$

$$\nabla_{\frac{d}{dt}}\widehat{\mathcal{H}} + \widehat{\mathcal{H}}^2 - \mathrm{R}_{\dot\gamma} = 0, \tag{4.4}$$

with $\widehat{\mathcal{H}} := -\mathcal{H}_\tau$ in the Lorentzian case, on $\gamma|_{(0,b]}$.

One can compare this with the classical scalar Riccati equation: Let $f : I \to \mathbb{R}$ and $\sigma : I \to \mathbb{R}$. Then the classical RQN is given by

$$f' + f^2 + \sigma = 0.$$

Note that this can only apply on the half-open interval $(0, b]$ since r and τ are undefined at 0.

Proof. We prove the Lorentzian case, the Riemannian case can be found in [Lee18]. Fix $\tau_0 \in (0, b]$. It suffices to check the (REQ) for $\partial_\tau|_{\tau_0}$ and $X \in T_{\gamma(\tau_0)}H_{\tau_0}$, i.e. $X \perp \partial_\tau|_{\tau_0}$. For ∂_τ, we already know $\mathcal{H}_\tau(\partial_\tau) = 0$, so $\mathcal{H}_\tau^2(\partial_\tau) = 0$. Furthermore, $\partial_\tau|_{\tau_0} = \dot\gamma(\tau_0)$, and hence $\mathrm{R}_{\dot\gamma}(\partial_\tau) = 0$. Lastly, we have

$$\nabla_{\frac{d}{dt}}\mathcal{H}_\tau(\partial_\tau) = \nabla_{\frac{d}{dt}}\Big(\overset{\text{vanishes on } (0,b]}{\cancel{\mathcal{H}_\tau(\partial_\tau)}}\Big) - \mathcal{H}_\tau\Big(\overset{\text{geodesic}}{\cancel{\nabla_{\frac{d}{dt}}\dot\gamma}}\Big) = 0,$$

and thus the REQ holds in this case. Now, let $X \perp \dot\gamma(\tau_0)$. By 3.11, we find $J \in \mathfrak{J}(\gamma)^\perp$ with $J(\tau_0) = X$ and $J(0) = 0$. This yields:

$$\begin{aligned}
\nabla_{\frac{d}{dt}}\mathcal{H}_\tau(X) &= \Big(\nabla_{\frac{d}{dt}}(\mathcal{H}_\tau(J)) - \mathcal{H}_\tau(\nabla_{\frac{d}{dt}}J)\Big)\Big|_{t=\tau_0} \\
&\overset{\text{Proposition 4.5}}{=} \Big(\nabla_{\frac{d}{dt}}\Big(-\nabla_{\frac{d}{dt}}J\Big) - \mathcal{H}_\tau(-\mathcal{H}_\tau(J))\Big)\Big|_{t=\tau_0} \\
&= -\nabla^2_{\frac{d}{dt}}J(\tau_0) + \mathcal{H}_\tau^2(X) = -\mathrm{R}_{\dot\gamma}(J(\tau_0)) + \mathcal{H}_\tau^2(X).
\end{aligned}$$

□

Theorem 4.10 (Matrix Riccati Comparison). Let $\operatorname{Sym}_{\mathbb{R}}(n) \subseteq \operatorname{Mat}_{\mathbb{R}}(n)$ be the space of symmetric real $n \times n$-matrices identified with self-adjoint endomorphisms of Euclidean $\mathbb{R}^n$. Let $H, \widetilde{H} : (a, b] \to \operatorname{Sym}_{\mathbb{R}}(n)$ be smooth such that the **matrix Riccati equations**

$$H' + H^2 + S = 0$$

and

$$\widetilde{H}' + \widetilde{H}^2 + \widetilde{S} = 0$$

are satisfied for some continuous $S, \widetilde{S} : [a, b] \to \operatorname{Sym}_{\mathbb{R}}(n)$ which are smooth on $(a, b]$. If

$$\widetilde{S} \geq S$$

We use the notion

$$A \geq B :\Leftrightarrow A - B \geq 0 \Leftrightarrow \langle (A - B)x, x\rangle$$

for all $x \in \mathbb{R}^n$. This means all eigenvalues of $A - B$ are non-negative.

on $[a, b]$ and
$$\lim_{t \searrow 0} \widetilde{H}(t) - H(t) \leq 0$$
(meaning the limit also exists in the first place), then
$$\widetilde{H} \leq H$$
on $(a, b]$.

Proof. See [Lee18]. □

Theorem 4.11 (Riccati Manifold Comparison)**.** Let (M, g) be a RMF or a LMF and $\gamma : [a, b] \to M$ be a timelike unit-speed geodesic segment. Suppose
$$\eta, \widetilde{\eta} : \mathfrak{X}(\gamma|_{(a,b]})^{\perp} \to \mathfrak{X}(\gamma|_{(a,b]})$$
are $\mathcal{C}^{\infty}(I)$-linear, self-adjoint with respect to the restricted metric and satisfy the REQs
$$\nabla_{\frac{d}{dt}} \eta + \eta^2 + \sigma = 0$$
and
$$\nabla_{\frac{d}{dt}} \widetilde{\eta} + \widetilde{\eta}^2 + \widetilde{\sigma} = 0$$
for some self-adjoint, $\mathcal{C}^{\infty}(I)$-linear $\sigma, \widetilde{\sigma} : \mathfrak{X}(\gamma)^{\perp} \to \mathfrak{X}(\gamma)^{\perp}$. If
$$\forall t \in [a, b] : \widetilde{\sigma}(t) \geq \sigma(t)$$
and
$$\lim_{t \searrow a} \widetilde{\eta}(t) - \eta(t) \leq 0$$
holds, then
$$\forall t \in [a, b] : \widetilde{\eta}(t) \leq \eta(t).$$

This roughly means that a comparision of RQNs and initial values leads to a comparison of solutions.

Proof. The goal is to utilize theorem 4.10. To this end, take a parallel ONF $(E_i) \cup \{\dot{\gamma}\}$ with $1 \leq i \leq n-1$ along γ. In this frame, we represent $\eta, \widetilde{\eta}, \sigma, \widetilde{\sigma}$ by symmetric, matrix valued maps
$$H, \widetilde{H} : [a, b] \to \mathrm{Sym}_{\mathbb{R}}(n-1)$$
and
$$S, \widetilde{S} : [a, b] \to \mathrm{Sym}_{\mathbb{R}}(n-1)$$
with components $\eta(t)(E_i) = H_i^j(t) E_j$ and analogously for the other operators. We have:
$$\begin{aligned}
\nabla_{\frac{d}{dt}} (\eta(E_i)) &= \nabla_{\frac{d}{dt}} \eta(E_i) + \eta \left(\nabla_{\frac{d}{dt}} E_i \right) = -\eta^2(E_i) - \sigma(E_i) \\
&= -\eta(H_i^j E_j) - S_i^k E_k = -H_i^j \eta(E_j) - S_i^k E_k \\
&= -H_i^j H_j^k E_k - S_i^k E_k
\end{aligned}$$
Without expanding, we obtain:
$$\begin{aligned}
\nabla_{\frac{d}{dt}} (\eta(E_i)) &= \nabla_{\frac{d}{dt}} (H_i^k E_k) \\
&= (H_i^k)' E_k + H_i^k \cancel{\nabla_{\frac{d}{dt}} E_k} = (H_i^k)' E_k.
\end{aligned}$$

Therefore,

$$H' + H^2 + S = 0$$

holds, with similar calculations estabilishing the same for $\widetilde{H}$ and $\widetilde{S}$. Clearly, $\widetilde{\sigma} \geq \sigma$ is equivalent to $\widetilde{S} \geq S$ and similar for the other condition. Theorem 4.10 is thus applicable and we conclude

$$\widetilde{H} \leq H \Leftrightarrow \widetilde{\eta} \leq \eta.$$

□

Remark 4.12. In the Riemannian case, we can do the same for

$$\eta, \widetilde{\eta} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma).$$

However, we mainly consider $\eta = \mathcal{H}_r$ and $\widetilde{\eta} \propto \pi^S$. Splitting $\mathfrak{X}(\gamma) \cong \operatorname{span}\{\dot{\gamma}\} \oplus \mathfrak{X}(\gamma)^\perp$, then we obtain the following matrix forms of $\mathcal{H}_r$ and π^S:

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{H}_r|_{\mathfrak{X}(\gamma)^\perp} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Therefore, the restriction to $\mathfrak{X}(\gamma)^\perp$ does not discard any information.

4.2 Hessian Comparison

In this section, we attempt to compare $\mathcal{H}_r$ with $\frac{s_c' \circ r}{s_c \circ r}\pi^S$ and $\mathcal{H}_\tau$ with $-\frac{s_{-c}' \circ \tau}{s_{-c} \circ \tau}\pi^H$. In the following, any inequalities for $\mathcal{H}_\tau$ in the Lorentzian case are to be understood for the restriction

$$\mathcal{H}_\tau|_{\mathfrak{X}(\gamma)^\perp} : \mathfrak{X}(\gamma)^\perp \to \mathfrak{X}(\gamma)^\perp.$$

Theorem 4.13 (Hessian Comparison)**.** Let (M, g) be a RMF or a LMF, $p \in M$, U be a normal neighbourhood of p and $r : U \to [0, \infty)$ ($\tau = \tau|_U(p, \cdot)$) be the radial distance function at p. Let $c \in \mathbb{R}$.

(a) If all timelike sectional curvatures of M are less or equal c, then

$$\mathcal{H}_r(q) \geq \frac{\dot{s}_c(r(q))}{s_c(r(q))}\pi_q^S$$

for

$$\begin{cases} \forall q \in U \setminus \{p\} & c \leq 0 \\ \forall q \in U \setminus \{p\} : r(q) < \pi R & c > 0 \end{cases}$$

with $R := \frac{1}{\sqrt{c}}$.

$\mathcal{H}_\tau|_q \geq -\frac{\dot{s}_{-c}(\tau(q))}{s_{-c}(\tau(q))}\pi_q^H$
Note this is equal to $\varphi^*(\mathcal{H}_\tau^c|_{\widetilde{q}})$ for $\widetilde{q} = \varphi(q) \in \mathbb{R} \times \mathbb{H}^{n-1}$ with metric g^c.

for all $q \in I_U^+(p)$

(b) If all sectional curvatures of M are greater or equal c, then

$$\mathcal{H}_r|_q \leq \frac{s_c'(r(q))}{s_c(r(q))}\pi_q^S$$

$$\mathcal{H}_\tau|_q \leq -\frac{\dot{s}_{-c}(\tau(q))}{s_{-c}(\tau(q))}\pi_q^H$$

for

$$\begin{cases} \forall q \in I_U^+(p) & c \geq 0 \\ \forall q \in I_U^+(p) : \tau(q) \leq \pi R & c < 0 \end{cases}.$$

for all $q \in U \setminus \{p\}$.

Proof. We prove the Lorentzian version. Let $q \in I^+_U(p)$ (with $\tau(q) < \pi R$ in case (b) if $c < 0$) and $\gamma : [0, b] \to U$ be the unit-speed timelike geodesic from p to q. Set $\xi(q) := \dot\gamma(0) \in \mathbb{H}^{n-1} \subseteq T_pM$. We want to apply Theorem 4.11 along γ to $\widehat{\mathcal{H}} := -\mathcal{H}_\tau \circ \gamma$, comparing it with $\widehat{\mathcal{H}}^c(t) := \frac{\dot s_{-c}(t)}{s_{-c}(t)} \pi^H_q = -\mathcal{H}^c_\tau((t, \xi(q)))$. Case (a) turns into $\widehat{\mathcal{H}}^c \geq \widehat{\mathcal{H}}$ while case (b) becomes $\widehat{\mathcal{H}}^c \leq \widehat{\mathcal{H}}$. We divide the proof in four steps:

1. Step: We show that, wherever defined, $\widehat{\mathcal{H}}^c$ satisfies the Riccati equation

$$\nabla_{\frac{d}{dt}} \widehat{\mathcal{H}}^c + (\widehat{\mathcal{H}}^c)^2 - R^c = 0$$

on $(0, b]$ for $R^c(t) := c\pi^H_{\gamma(t)}$. This follows directly by computation or by using the fact that $\widehat{\mathcal{H}}^c$ and R^c are pullbacks of the Hessian and the tidal force operator, respectively, in constant curvature spaces along a radial geodesic. This in turn follows from Proposition 4.6.

2. Step: We have $-R^c \leq -\mathrm{R}_{\dot\gamma}$ on $(0, b]$ in case (a) and $-R^c \geq -\mathrm{R}_{\dot\gamma}$ in case (b). To see this, let $t \in (0, b]$ and $w \in T_{\gamma(t)}H_t$. Then w is spacelike, so $\{\dot\gamma, w\}$ spans a timelike plane, implying

$$g(-\mathrm{R}_{\dot\gamma}(w), w) = \sec(\dot\gamma(t), w) g(w, w)(-1) = -\sec(\dot\gamma, w) g(w, w).$$

Therefore, we have

$$g(-\mathrm{R}_{\dot\gamma}(w), w) = \begin{cases} \geq -cg(w, w) = g(-R^c(w), w) & \text{(a)} \\ \leq -cg(w, w) = g(-R^c(w), w) & \text{(b)} \end{cases}.$$

3. Step: We continue by showing that $\lim_{t\to 0} \widehat{\mathcal{H}}^c(t) - \widehat{\mathcal{H}}(t)$ exists and is either non-negative in case (a) or non-positive in case (b). Direct computation shows $\widehat{\mathcal{H}}^c(t) = \frac{1}{t}\pi^H_{\gamma(t)} + \mathcal{O}$. Therefore, it suffices to show that also $\widehat{\mathcal{H}}(t) = \frac{1}{t}\pi^H_{\gamma(t)} + \mathcal{O}(t)$. Choose normal coodinates around p such that $\gamma(t) = (t, 0, \ldots, 0)$. In these coordinates, we have

Note that

$\pi^H_{\gamma(t)} = 0 \oplus \mathrm{id} : \mathrm{span}\{\dot\gamma(t)\} \oplus \dot\gamma(t)^\perp \to \mathrm{span}\{\dot\gamma(t)\} \oplus \dot\gamma(t)^\perp$

extends continuously at $t = 0$ to

$0 \oplus \mathrm{id} : \mathrm{span}\{\dot\gamma(0)\} \oplus \dot\gamma(0)^\perp \to \mathrm{span}\{\dot\gamma(0)\} \oplus \dot\gamma(0)^\perp$.

$$\tau(x) = \sqrt{-\eta(x, x)} = \sqrt{x_0^2 - \sum_{i=1}^{n-1} x_i^2},$$

$$g_{ij}(\gamma(t)) = \eta_{ij} + \mathcal{O}(t^2),$$

and

$$\Gamma^m_{jk}(\gamma(t)) = 0 + \mathcal{O}(t).$$

Calculating

$$\partial_m \tau = \frac{-\varepsilon_m x_m}{\tau} = \begin{cases} \frac{x_0}{\tau} & m = 0 \\ -\frac{x_m}{\tau} & 1 \leq m \leq n-1 \end{cases},$$

we see that ∂_m remains bounded as $t \to 0$ along γ since $\partial_m \tau_{\gamma(t)} = -1$ for $m = 0$ and otherwise, it vanishes. Furthermore, we get

$$\partial_j \partial_k \tau = -\varepsilon_j \varepsilon_k \frac{x_j x_k}{\tau^3} - \frac{\eta_{jk}}{\tau},$$

yielding

$$\partial_j \partial_k \tau|_{\gamma(t)} = -\frac{\delta_{j0}\delta_{k0} + \eta_{jk}}{t} = \mathcal{O}\left(\frac{1}{t}\right).$$

Putting everything together, we obtain

$$\begin{aligned} -\left(\mathcal{H}_\tau|_{\gamma(t)}\right)^i_k &= -g^{ij}(\partial_j\partial_k\tau - \Gamma^m_{jk}\partial_m\tau) \\ &= -\eta^{ij}(\partial_j\partial_k\tau|_\gamma - \mathcal{O}(t)) + \mathcal{O}(t^2)\mathcal{O}\left(\frac{1}{t}\right) \\ &= \begin{cases} 0 & i = 0 \text{ or } k = 0 \\ \frac{\delta^i_k}{t} + \mathcal{O}(t) & \text{else} \end{cases} \end{aligned}$$

So on $\dot\gamma(t)^\perp$, we have $\widehat{\mathcal{H}}(t) = \frac{1}{t}\,\mathrm{id} + \mathcal{O}(t)$. Now, Theorem 4.11 guarantees $\widehat{\mathcal{H}}^c \geq \widehat{\mathcal{H}}$ in case (a) and $\widehat{\mathcal{H}}^c \leq \widehat{\mathcal{H}}$ in case (b) for all $t \in (0, b]$, eventually with the restriction $t < \pi R$.

4. Step: In case (a), we still have to show that $-c > 0$ already forces $\tau(q) < \pi R$ for any $q \in I^+_U(p)$ to guarantee that $\widehat{\mathcal{H}}^c$ is well-defined along γ. We show that $b < \pi R$ automatically holds in this case. Since $\widehat{\mathcal{H}}$ is well-defined on $(0, b]$, the eigenvalues are bounded on this interval. But the inequality

$$\widehat{\mathcal{H}}(t) \leq \widehat{\mathcal{H}}^c(t) = \frac{\dot s_{-c}(t)}{s_{-c}(t)}\pi^H_{\gamma(t)} = \frac{1}{R}\frac{\cos(\frac{t}{R})}{\sin(\frac{t}{R})}\pi^H_{\gamma(t)} \overset{t\to\pi R}{\to} -\infty$$

forces the eigenvalues to diverge as $t \to \pi R$, so $b < \pi R$.

□

13.01.26

Theorem 4.14 (Lorentzian Jacobi Field Comparison)**.** Let (M, g) be a LMF, $\gamma : [0, b] \to M$ be a unit-speed timelike geodesic, and $J \in \mathfrak{J}(\gamma)^\perp$ with $J(0) = 0$.

1. If all timelike sectional curvatures are non-positive, then

$$\|J(t)\|_g \leq s_{-c}(t) \cdot \|\nabla_{\frac{d}{dt}} J(0)\|_g$$

for all $t \in [0, b_1]$ where

$$b_1 := \sup\{t \in [0, b] \mid \gamma(t) \text{ is not conjugate to } \gamma(0) \text{ along } \gamma\}.$$

2. If all timelike sectional curvatures are non-negative, then

$$\|J(t)\| \geq s_{-c}(t)\|\nabla_{\frac{d}{dt}} J(0)\|$$

for all $t \in [0, b_2]$ where

$$b_2 := \begin{cases} b & c \leq 0 \\ \min\{b, R\pi\} & c > 0 \end{cases}.$$

Proof. Let J be such a Jacobi field which is not the zero field.

1. Step: We show monotonicity. Let $b_0 := \min\{b_0, b_1\}$ and define

$$f : (0, b_0) \to \mathbb{R}$$

by

$$f(t) := \log(s_{-c}(t)^{-1}\|J(t)\|) = \log(\|J(t)\|) - \log(s_{-c}(t)).$$

Define $\widetilde{f}(t) := s_{-c}(t)^{-1}\|J(t)\|$, which is strictly positive on $(0, b_0)$ since $s_{-c}(t)$ and $J(t)$ are not zero by choice of b_0 and spacelike character of $J(t)$ (which is normal). Note that f is non-increasing resp. non-decreasing if and only if $\widetilde{f}$ is by monotonicity of log. Now compute

$$\dot{f}(t) = \frac{1}{\|J\|^2} g(\nabla_{\frac{d}{dt}} J, J) - \frac{\dot{s}_{-c}}{s_{-c}}.$$

Assume for now that $\gamma : [0, b_0] \to M$ is contained in a normal neighbourhood of $p := \gamma(0)$. Using $\widetilde{\mathcal{H}}(J) = \nabla_{\frac{d}{dt}} J$ (prop. 3.5) and Hessian comparison, we get

$$\dot{f}(t) = \frac{1}{\|J\|^2}(g(\widetilde{\mathcal{H}}(J), J) - g(\underbrace{\frac{\dot{s}_{-c}}{s_{-c}} J}_{=\widehat{\mathcal{H}}^c(J)}, J)) = \begin{cases} \leq 0 & \text{Case 1} \\ \geq 0 & \text{Case 2} \end{cases}.$$

2. Step: Now we treat the initial condition. We want to show that

$$\lim_{t \searrow 0} \frac{\|J(t)\|}{s_{-c}(t)} = \|\nabla_{\frac{d}{dt}} J(0)\|,$$

which is equivalent to

$$\lim_{t \searrow 0} \frac{\|J\|}{s_{-c}(t)} = \|\nabla_{\frac{d}{dt}} J(0)\| > 0.$$

The last inequality holds as J is normal. Applying l'Hospital's rule twice yields

$$\begin{aligned} \lim_{t \searrow 0} \frac{1}{s_{-c}^2}\|J\| &= \lim_{t \searrow 0} \frac{2g(\nabla_{\frac{d}{dt}} J, J)}{2s_{-c}\dot{s}_{-c}} \\ &= \lim_{t \searrow 0} \frac{\|\nabla_{\frac{d}{dt}} J\| +}{\dot{s}} \end{aligned}$$

□

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