Preface

This thesis is on Morse theory, the study of nice real valued functions on manifolds, called Morse functions. While elementary, they provide great insight in the structure of manifolds, eventually leading to some of the most important theorems in differential topology. The goal of this thesis is to prove one of them, namely the generalized Poincaré conjecture in higher dimensions, stating that a homotopy sphere is a topological sphere.

This thesis could not have been written without the help of many people.

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List of symbols

$M \pitchfork N$	Transverse intersection
$\langle \cdot, \cdot \rangle$	Riemannian metric on a manifold or, Inner product on space of critical points $\langle c, d \rangle = \delta_{cd}$
$N \cdot N'$	Intersection number of two manifolds
A	A $\mathbb{Z}$-module, i.e. an abelian group
B^n	Closed disk of dimension n
$C_k(f, \mathbb{Z})$	Free module over $\mathbb{Z}$ generated by index k critical points of f, i.e. the space of formal sums of index k critical points
$C_k(f, \mathbb{Z}_2)$	Vector space over $\mathbb{Z}_2$ generated by the index k critical points of the Morse function f
$\operatorname{codim} N$	Codimension of N
$\dim N$	Dimension of N
$\operatorname{Crit}_k f$	Critical points of f of index k
$\operatorname{Crit} f$	Critical points of f
$C^\infty(M, N)$	Smooth maps from M to N
$\partial_{X,k}$	Morse differential associated to pseudo-gradient X
$[\partial_k]$	Matrix of the Morse differential $\partial_k : C_k \to C_{k-1}$
D^n	Open disk of dimension n
$\operatorname{grad} f$	Gradient of f, i.e. $(df)^\sharp$
$HM(M; \mathbb{Z}_2)$	Morse homology of a manifold M with coefficients in $\mathbb{Z}_2$
$HM(M; \mathbb{Z})$	Morse homology of a manifold M with coefficients in $\mathbb{Z}$
$HM(C_\bullet(f), \partial_X)$	Morse homology of Morse function f and pseudo-gradient X
$H_k(M, N)$	Singular homology of M relative to N
$H_k(M; \mathbb{Z})$	Singular homology of M over $\mathbb{Z}$
$H_k(M; \mathbb{Z}_2)$	Singular homology M over $\mathbb{Z}_2$
$\operatorname{Ind} a$	Index of critical point a

$\mathcal{L}(p, q)$	Moduli space of unbroken trajectories between p and q, i.e. $\mathcal{M}(p, q)/\mathbb{R}$, where $\mathbb{R}$ acts by time translations
$\overline{\mathcal{L}}(p, q)$	Space of broken and unbroken trajectories between p and q, the compactification of $\mathcal{L}(p, q)$
M	A smooth manifold
$\mathcal{M}(p, q)$	Set of all points on trajectories following a pseudo-gradient from p to q, $W^u(p) \pitchfork W^s(q)$
$N_X(p, q)$	Signed number of trajectories of X connecting p to q
$n_X(p, q)$	Number of trajectories of X connecting p to q
$\pi_k(M)$	Homotopy group of a manifold
$r_0(A)$	Free rank of a $\mathbb{Z}$-module, i.e. $\dim_{\mathbb{Q}} A \otimes \mathbb{Q}$
$r_p(A)$	p-torsion rank of a $\mathbb{Z}$-module, i.e. cardinality of a maximal set of independent elements of order p^k for some k
$r_t(A)$	Total torsion rank of a $\mathbb{Z}$-module, i.e. $\sum r_t$
$r(A)$	Total rank of a $\mathbb{Z}$-module, i.e. $r_t(A) + r_0(A)$
$S^s(p)$	Stable sphere associated to a critical point p, alternatively called the belt sphere
$S^u(p)$	Unstable sphere associated to a critical point p, alternatively called the attachment sphere
S^n	Sphere of dimension n
$W^s(p)$	Stable manifold of a critical point p
$W^u(p)$	Unstable manifold of a critical point p
$\overline{W}^u(p)$	Compactification of the unstable manifold associated to a critical point p
X	Pseudo-gradient vector field

Contents

CHAPTER ONE

Preamble