Riemannian and Lorentzian Geometry

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Chapter 1

Introduction

1.0 Review of important topics

1.0.1 Lecture 14.10.25

Definition 1.1 (Integral Curve)**.** Let $X \in \Gamma(TM)$ and $\gamma : I \to M$ be a smooth curve. We call γ **integral curve** of X if

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

Furthermore, γ is maximal if for all other integral curves $\omega : J \to M$ with $I \cap J \neq \emptyset$ and $\omega|_{I\cap J} = \gamma|_{I\cap J}$ we have $J \subseteq I$.

Theorem 1.2 (Existence of Integral Curves)**.** Let $X \in \Gamma(TM)$. For all $p \in M$ exists a unique maximal integral curve $\gamma_p : I_p \to M$ of X with $\gamma_p(0) = p$.

Definition 1.3 (Local Flow)**.** Let $X \in \Gamma(TM)$. A **local flow** of X is a smooth map

$$\Theta : I \times U \to M$$

where $I \subseteq \mathbb{R}$ is an open interval, $0 \in I$ and $U \subseteq M$ is open such that

$$\Theta(t,p) = \gamma_p(t),$$

where γ_p is the maximal integral curve of X with $\gamma_p(0) = p$ for all $(t,p) \in I \times U$.

Note. Similarly, a maximal flow for $X \in \Gamma(TM)$ is a smooth map $\Theta : \mathcal{D} \to M$ with

$$\mathcal{D} := \bigcup_{p\in M} I_p \times \{p\} \subseteq \mathbb{R} \times M$$

being open. We call $\mathcal{D}$ the **maximal domain**. It always containes $\{0\} \times M$. A **global flow** is then a unique maximal flow with $\mathcal{D} = \mathbb{R} \times M$.

Notation. To emphasize that Θ depends on X, we write Θ_X and $\mathcal{D}^X$. Fixing $t \in \mathcal{D}$, we write

$$\theta_t : \mathcal{D} \cap (\{t\} \times M) \to M$$

with $\theta_t(p) := \Theta(t, p)$.

Definition 1.4 (Riemannian and Lorentzian Metrics)**.** Let $\nu \in \mathbb{N}$ with $0 \leq \nu \leq n$. A **semi-Riemannian metric** of **index** ν is a $(0, 2)$-tensor field such that

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a symmetric non-degenerate bilinear form on T_pM with index ν. We say:

- $\nu = 0$: g is **Riemannian**.
- $\nu = 1$: g is **Lorentzian**.

In the Lorentzian case, we take the convention $(-, +, +, \dots)$.

Theorem 1.5 (Levi-Civita Connection)**.** Given a semi-Riemannian manifold (M, g), there exists exactly one connection

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

, called the **Levi-Civita connection**, such that:

- ∇^g is torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$
- ∇^g is compatible with g: $Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$
- The Koszul identity is satisfied:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

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Definition 1.6 (Riemann Curvature Tensor)**.** The **Riemann Curvature Tensor** is the $(1, 3)$-tensor field

$$R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$$

given by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Remark. In local coordinates (x^i), we have

$$R^l_{ijk} := (R(\partial_i, \partial_j)\partial_k)^l = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik}.$$

Definition 1.7 (Geodesic). A $\mathcal{C}^\infty$-curve $\gamma : (a, b) \to M$ is called a **geodesic** if $\dot{\gamma}$ is ∇^g*−parallel* along γ, i.e.

$$\ddot{\gamma}(t) = \nabla^g_{\frac{d}{dt}} \dot{\gamma} = 0.$$

In local coordinates, one finds the *geodesic equation*

$$0 = \frac{d^2}{dt^2}(x^i \circ \gamma) + (\Gamma^i_{kl} \circ \gamma) \frac{d}{dt}(x^k \circ \gamma) \frac{d}{dt}(x^l \circ \gamma).$$

Theorem 1.8 (Maximal Geodesic). For all $v \in TM$, there is exactly one geodesic

$$\gamma_v : I_v \to M$$

such that $\gamma_v(0) = \pi\|v\|$ and $\dot{\gamma}_v(0) = v$ and I_v is maximal.

Exponential Map

Definition 1.9 (Exponential Map). Define the (Riemannian) **exponential map**

$$\exp_p : \mathcal{D}_p \subseteq T_pM \to M$$

by

$$(p, v) \mapsto \exp_p(v) := \gamma_v(1)$$

where γ_v is the unique maximal geodesic.

We always have

$$\mathcal{D}_p \supseteq \{tv \in T_pM \mid t \in [0, 1]\}.$$

We can consider $s \mapsto \gamma_v(st)$ for fixed $t \in \mathbb{R}$. Then, we have

$$\dot{\gamma}_v(st) = t\dot{\gamma}_v(0) = tv = \dot{\gamma}_{tv}(s)$$

for all s, and therefore $\gamma_{tv}(s) = \gamma_v(ts)$. This yields the useful formula

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t).$$

Lemma 1.10. For all $p \in M$,

$$d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$$

is the identity $d(\exp_p)_0 = \mathrm{id}$ under the identification $\mathrm{id}(v^i \partial_{u_i}|_0) = v^i \partial_{x_i}|_p$.

Definition 1.11 (Normal Neighbourhood)**.** An open set $U \ni p$ is a **normal neighbourhood** of p if there exists an open set $\tilde{U} \ni o_p \subseteq \mathcal{D}_p$ which is star-shaped such that

$$\exp_p|_{\tilde{U}} : \tilde{U} \to U$$

is a diffeomorphism.

Theorem 1.12 (Existence of Normal Neighbourhoods)**.** For any $p \in M$, there is a normal neighbourhood around p.

Proof. Inverse function theorem. □

Definition 1.13 (Convex Neighbourhood)**.** U is a **convex neighbourhood** if it is a normal neighbourhood for all $q \in U$.

Remark. If U is a normal neighbourhood of p, then for all $q \in U$ there is exactly one geodesic γ_{pq} in U from p to q, called **radial geodesic**.

Theorem 1.14 (Normal Coordinate Lines)**.** For all $p \in M$ and a basis $\{v_1, \dots, v_n\}$ of T_pM exists a chart $(U, (x^1, \dots, x^n))$ such that:

1. U is a normal neighbourhood of p.
2. $\partial_i|_p = v$
3. $\Gamma^k_{ij} = 0$ for all i, j, k.

If the basis $\{v_1, \dots, v_n\}$ is orthonormal, we also have

$$g_{ij}(p) = \epsilon_i \delta_{ij}$$

and

$$\partial_k g_{ij}(p) = 0.$$

The chart $(U, (x^1, \dots, x^n))$ is called **normal coordinate chart**.

Chapter 2

Riemannian and Lorentzian Geometry

In this chapter, we are concerned with Riemannian manifolds as metric spaces. The main goal is to prove the theorem of Hopf-Rinow.

2.1 Riemannian Manifolds as Metric Spaces

Definition 2.1 (Regular Curve). A piecewise $\mathcal{C}^1$-curve

$$\gamma : [a, b] \to M$$

is called **regular** if

$$\forall s \in [a, b] : \dot{\gamma}(s) \neq 0$$

and

$$\dot{\gamma}_{\pm}(t_i) \neq 0$$

at all $\mathcal{C}^1$-break-points.

Definition 2.2 (Arc-length). Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \to M$ a (piecewise) $\mathcal{C}^1$-curve. The **arc-length** is defined to be the functional

$$L[\gamma] = \int_a^b \sqrt{|g(\dot{\gamma}(t), \dot{\gamma}(t))|}\, dt.$$

Remark.

1. In the Riemannian case, the $|\cdot|$ is redundant.
2. In semi-Riemannian geometry, there are curves with $L[\gamma] = 0$.
3. The arc-length functional is invariant under length parametrization.
4. If γ is regular, there exists a strictly monotonous reparametrization

$$\varphi : [\tilde{a}, \tilde{b}] \to [a, b]$$

such that $\tau := \gamma \circ \varphi$ satisfies $g(\dot{\tau}, \dot{\tau}) = 1$. This is a reparametrization by arc-length:

$$L[\tau_{[\tilde{a},s]}] = s - \tilde{a}$$

for all $s \in [\tilde{a}, \tilde{b}]$.

Theorem 2.3 (Gauß' Lemma). The exponential map is a radial isometry: For any $p \in M$, $x \in \mathcal{D}_p$ and $v, w \in T_x(T_pM) \cong T_pM$ with $v = \alpha x$ for some $\alpha \in \mathbb{R}$, the equations

$$g_{\exp_p(x)}(d(\exp_p(v))_x, d(\exp_p(w))_x) = g_p(v, w)$$

and

$$\dot{\gamma}(t) = \frac{d}{dt}\exp_p(tv)$$

hold.

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Theorem 2.4 (Minimizing Geodesic). Let (M, g) be a Riemannian manifold and U be a normal neighbourhood of $p \in M$. Then, γ_{pq} is the shortest curve from p tp q unique up to monotonically increasing, piecewise $\mathcal{C}^1$ reparametrization.

Proof. Let $\omega : [a, b] \to M$ be a piecewise $\mathcal{C}^1$ curve in U from p to q. W.l.o.g., $a = 0$, $b = 1$ and $\omega([0, 1]) \subseteq U \setminus \{p\}$. We can write

$$\omega(t) = \exp_p(R(t)v(t))$$

with $R(t) := |\exp_p^{-1}(\omega(t))|_{g_p}$ and $v(t) := \frac{\exp_p^{-1}(\omega(t))}{|\exp_p^{-1}(\omega(t))|_{g_p}}$ such that $v \in \mathbb{S}^{m-1}_{g_p} \subseteq T_pM$. Both R and v are piecewise $\mathcal{C}^1$ and $R(t) \in (0, \infty)$ for $t > 0$, since $\omega(t)$ does not meet p again. Away from the breakpoints, we have

$$\dot{\omega}(t) = d_{R(t)v(t)}\exp_p([R(t)v(t)]) = R(t)\underbrace{d_{R(t)v(t)}\exp_p(\dot{v}(t))}_{:=A} + \dot{R}(t)\underbrace{d_{R(t)v(t)}\exp_p(v(t))}_{:=B}.$$

With this, we can calculate

$$g(\dot{\omega}(t), \dot{\omega}(t)) = R^2(t)g(A, A) + R(t)\dot{R}(t)g(A, B) + \dot{R}(t)g(B, B)$$

and

$$g(B, B) = g(d_{R(t)v(t)}\exp_p(v(t)), d_{R(t)v(t)}\exp_p(v(t))) = g_p(v(t), v(t)) = 1,$$

where we used Gauß' Lemma.

Turning our attention to the second term, we obtain

$$\begin{aligned} g(A,B) &= g(d_{R(t)v(t)}\exp(v(t)), d_{R(t)v(t)}\exp_p(\dot{v}(t))) \\ &= g_p(v(t),\dot{v}(t)) = \frac{1}{2}\frac{d}{dt}g_p(v(t),v(t)) = 0. \end{aligned}$$

For the arc-length, this yields

$$\begin{aligned} L[\omega] &= \int_0^1 \sqrt{g(\dot{\omega}(t),\dot{\omega}(t))}\,dt && (2.1) \\ &\geq \int_0^1 \sqrt{\dot{R}^2(t)}\,dt \geq \int_0^1 \dot{R}(t)\,dt && (2.2) \\ &= R(1) - R(0) = |\exp_p^{-1}(q)|_{g_p} = L[\gamma_{pq}] && (2.3) \end{aligned}$$

where the last equation is left as an exercise. □

Remark. We actually have equality if:

1. If $d_{R(t)v(t)}\exp_p(\dot{v}(t)) = 0$, we have $\dot{v}(t) = 0$ and $v = \frac{\exp_p^{-1}(q)}{|\exp_p^{-1}(q)|_{g_p}}$, hence $\omega(t) = \exp_p\left(\frac{R(t)}{|\exp_p^{-1}(q)|_{g_p}}\exp_p^{-1}(q)\right)$.
2. If $\dot{R}(t) \geq 0$

With this result and the convex neighbourhood theorem, one obtains that any piecewise $\mathcal{C}^1$-curve minimizing L from p to q must be a broken geodesic.

Theorem 2.5 (Normal Basis)**.** Let (M,g) be a Riemannian manifold. Then, every point $p \in M$ has a basis of normal neighbourhoods $\{U_\epsilon\}$ of the form $U_\epsilon = \exp_p^{-1}(B_\epsilon(0))$ and such that for all $q \in U_\epsilon$, γ_{pq} is the shortest curve from p to q in M.

Proof. Idea: Show that any piecewise $\mathcal{C}^1$ shortest curve starting from o and leaving U_ϵ has $L \geq \epsilon$ ($\implies L < \epsilon$). □

Definition 2.6 (Induced Metric)**.** Let (M,g) be a Riemannian manifold. The **induced distance function** by g is given by

$$d_g(p,q) := \inf\{L_g[\gamma] \mid \gamma : [a,b] \to M \text{ pw. cont.}, \gamma(a) = p, \gamma(b) = q\}$$

for all $p,q \in M$. A piecewise $\mathcal{C}^1$-curve is **minimizing** if

$$d_g(p,q) = L[\gamma]..$$

Theorem 2.7 (Manifolds as Metric Spaces). Let (M, g) be a Riemannian manifold. The induced distance function

$$d_g : M \times M \to \mathbb{R}$$

is a distance function and the topology induced by d_g conincides with the topology of M.

Proof.

1. For finiteness, let γ be a $\mathcal{C}^1$ geodesic connecting p and q. We can cover the image of γ by a finite amount of sets and connect between the intersection points with geodesics.
2. For $d(p, q) \geq 0$, we start by showing that $d(p, q) = 0 \implies p = q$. If $p \neq q$, we can find a normal neighbourhood $U_\epsilon \ni p$ such that $q \notin U_\epsilon$ by the Hausdorff condition. Hence, $d(p, q) \geq \epsilon \neq 0$.
3. Next, we show symmetry. This is clear from the reparametrization invariance of L, using $t \mapsto \gamma(-t)$.
4. For the triangle equality, let $p, q, x \in M$. For any $\epsilon > 0$, choose γ_1, γ_2 such that

$$L[\gamma_1] \leq d(p, x) + \frac{\epsilon}{2}$$
$$L[\gamma_2] \leq d(x, q) + \frac{\epsilon}{2}.$$

Joining γ_1 and γ_2, we have $\gamma = \gamma_1 * \gamma_2$ with

$$d(p, q) \leq L[\gamma] = L[\gamma_1] + L[\gamma_2] \leq d(p, x) + d(x, q) + \epsilon$$

and $\epsilon > 0$ is arbitrary.

5. Lastly, we have to prove that the topologies agree. This means showing that

$$U_\epsilon = B^d_\epsilon(p) := \{q \in M \mid d(p, q) < \epsilon\}$$

for U_ϵ from theorem **??**. By the same theorem, U_ϵ is a basis, and by definition, so is $B^d_\epsilon(p)$. Now, we have:

- $$\forall q \in U_\epsilon \implies d(p, q) = L[\gamma_{pq}] < \epsilon \implies U_\epsilon \subseteq B^d_\epsilon(p)$$
- $\forall q \in B^d_\epsilon(p)$ there is a curve such that $\gamma(a) = p$, $\gamma(b) = q$ and $L[\gamma] < \epsilon$. But any curve leaving U_ϵ has length $L \geq \epsilon$, so $B^d_\epsilon(p) \subseteq U_\epsilon$.

□

Remark. Any Riemannian manifold is metrizable.

We will now consider another kind of length.

Definition 2.8 (Metric Arc-length)**.** Let (M, g) be a Riemannian manifold and $\gamma : [a, b] \to M$ be a $\mathcal{C}^0$ curve. The metric length is given by

$$L_d[\gamma] := \sup_{N \in \mathbb{N}} \sup \left\{ \sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i+1})) \mid a = t < \cdots < t_i < t_{i+1} < \cdots < t_N = b \right\}.$$

Theorem 2.9 (Geodesic equals Metric Length)**.** If (M, g) is a Riemannian manifold and $\gamma : [a, b] \to M$ is piecewise $\mathcal{C}^1$, then

$$L_d[\gamma] = L[\gamma].$$

Proof.

$$\sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i+1})) \leq L[\gamma|_{[t_i, t_{i+1}]}] \leq L[\gamma].$$

Taking the supremum, we have $L_d[\gamma] \leq L[\gamma]$. Now, we show that $L_d[\gamma] \geq L[\gamma]$. For this, we want to show that $t \mapsto L_d[\gamma|_{[0,t]}]$ is differentiable away from breakpoints with derivative $\|\dot{\gamma}(t)\|_g = \frac{d}{dt} L[\gamma|_{[0,t]}]$. So let $\delta > 0$ and consider

$$\frac{1}{\delta} d(\gamma(t), \gamma(t+\delta)) \leq \frac{1}{\delta} L_d[\gamma|_{[t,t+\delta]}] \leq \frac{1}{\delta} L[\gamma|_{[t,t+\delta]}] \xrightarrow{\delta \to 0} \|\dot{\gamma}\|.$$

It remains to show that $\frac{1}{\delta} d(\gamma(t), \gamma(t+\delta)) \to \|\dot{\gamma}(t)\|_g$. Let $\epsilon > 0$ and U be a normal neighbourhood of $\gamma(t)$. For δ small enough, $\gamma(t+\delta) \in U$ and $d(\gamma(t), \gamma(t+\delta)) = \| \exp^{-1}_{\gamma(t)}(\gamma(t+\delta))\|_g$. So

$$\begin{aligned}
\lim_{\delta \to 0^+} \frac{1}{\delta} d(\gamma(t), \gamma(t+\delta)) &= \lim_{\delta \to 0^+} \frac{1}{\delta} \| \exp^{-1}_{\gamma(t)}(\gamma(t+\delta))\|_g \\
&= \| \lim_{\delta \to 0^+} \frac{1}{\delta} \left(\exp^{-1}_{\gamma(t)}(\gamma(t+\delta)) - \exp^{-1}_{\gamma(t)}(\gamma(1)) \right) \|_g = \| \frac{d}{ds}\Big|_{s=t} \exp^{-1}_{\gamma(t)}(\gamma(s))\|_g \\
&= \| \underbrace{d_{\gamma(t)}(\exp^{-1}_{\gamma(t)})}_{=(d_0 \exp_{\gamma(t)})^{-1} = \mathrm{id}} (\dot{\gamma}(t))\|_g = \|\dot{\gamma}(t)\|_g
\end{aligned}$$

□

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Lemma 2.10. If $\gamma : [a, b] \to M$ is continuous from $p \in M$ to $q \in M$ and $L_d[\gamma] = d(p, q)$, then there exists an unbroken geodesic ω from p to q with the same image as γ and $L[\omega] = d(p, q)$.

Exercise. Prove the preceeding lemma.

Interlude: Extendibility of Geodesics

Note. In this interlude, we allow g to be semi-Riemannian.

Definition 2.11 (Continuously Extendibility). A curve $\gamma : [a, b) \to M$ is **continuously extendible** if there is some $q \in M$ with

$$q = \lim_{t \to b} \gamma(t)..$$

Lemma 2.12. A geodesic $\gamma : [a, b) \to M$ in a semi-Riemannian manifold (M, g) is *continuously extendible* if and only if it is extendible as a geodesic, i.e. there exists a geodesic $\overline{\gamma} : [a, b + \epsilon) \to M$ with $\overline{\gamma}|_{[a,b)} = \gamma$.

Proof. Assume $q = \lim_{t \to b} \gamma(t)$. There exists a convex neighbourhood U of q. We can find ϵ such that $\gamma([b - \epsilon, b)) \subseteq U$. Let $\overline{\gamma} : I \to M$ be the unique maximal geodesic with initial data $\overline{\gamma}(0) = \gamma(b - \epsilon)$ and $\dot{\overline{\gamma}}(0) = \exp^{-1}_{\gamma(b-\epsilon)}(q) \in T_{\gamma(b-\epsilon)}M$. Since $\overline{\gamma}(1) = q$, we can affinely reparametrize $\overline{\gamma}$ to obtain a geodesic $\omega : J \to M$ with $[a, b] \subseteq J$ and $\omega(b-\epsilon) = \gamma(b-\epsilon)$ and $\omega(b) = q$. The reparametrization is given by $\omega(t) = \overline{\gamma}(\frac{t-(b-\epsilon)}{\epsilon})$. So we have found a geodesic extending γ. □

Note. We can choose such $\overline{\gamma}$ since $\exp^{-1}_p$ is defined on a normal neighbourhood U of p and is a map $U \to T_pM$. Applying this to $p = \gamma(b-\epsilon)$, one gets a vector $v := \exp^{-1}_p(q) \in T_pM$. Then, let $\overline{\gamma}$ be the geodesic starting in p with initial velocity v. We have $\overline{\gamma}(1) = q$ since $\exp_p(v) := \overline{\gamma}_v(1)$ and

$$\overline{\gamma}_v(1) = \exp_p(v) = \exp_p(\exp^{-1}(q)) = q.$$

2.2 The Theorem of Hopf-Rinow

Theorem 2.13 (Hopf-Rinow). Let (M, g) be a Riemannian manifold. Then the following are equivalent:

1. The metric space (M, d_g) is complete.
2. (M, g) is geodesically complete.
3. There exists some $p \in M$ such that $\exp_p$ is defined on all of T_pM.[a]
4. The Heine-Borel property holds, i.e. a subset $A \subseteq M$ is compact if and only if it is bounded[b] and closed.

Each of these properties implies in addition: For all $p, q \in M$ there exists a *minimizing geodesic* γ from p to q with $L[\gamma] = d(p, q)$.

[a] This is equivalent to all geodesics starting at p being complete.
[b] In this case, this means that there is $C > 0$ such that $d(x, y) \leq C$ for all $x, y \in A$.

Lemma 2.14. Let $p \in M$, $q \in B_r(p) = \{x \in M \mid d(x,p) < r\}$. If $\overline{B_r(p)}$ is compact, then there exists a continuous curve γ from p to q with $L_d[\gamma] = d(p,q)$.

Exercise. Show that the lemma implies that there is a minimizing geodesic from p to q.

Intuition. The idea is to take a piecewise $\mathcal{C}^1$ family of curves γ_n with $L[\gamma_n] \to d(p,q)$ which yields that there exists a subsequence of γ_n converging nicely enough to some curve γ such that γ is continuous (Arzela-Ascoli) and such that

$$L_d[\gamma] \leq \liminf_{n\to\infty} L[\gamma_n] = d(p,q).$$

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Proof. Let γ_n be a sequence of piecewise geodesics from p to q, parametrized on $[0, d(p,q)]$ with $L[\gamma_n] \to d(p,q)$. Additionally, the parametrization should satisfy $\|\dot{\gamma}_n\| = C_n \in \mathbb{R}$ for all t. We have:

- $L[\gamma_n] = \int_0^{d(p,q)} C_n \, dt = d(p,q) \cdot C_n \implies C_n \to 1$ so the C_n are uniformly bounded.
- $d(p, \gamma_n(t)) \leq L[\gamma_n|_{[0,t]}] \leq L[\gamma_n] < R$ for n large enough

Hence, we can conclude $\mathrm{im}(\gamma_n) \subseteq \overline{B_r(p)}$ and *Arzela-Ascoli* yields that there exists a subsequence converging uniformly to a continuous curve

$$\gamma : [0, d(p,q)] \to \overline{B_r(p)} \subseteq M.$$

That $L_d[\gamma] = d(p,q)$ holds, follows from the claim that L_d is lower semi-continuous, i.e. if

$$\omega, \omega_n : [a,b] \to M$$

is a sequence of continuous curves with $\omega_n \to \omega$ pointwise, then

$$L_d[\omega] \leq \liminf_{n\to\infty} L_d[\gamma_n].$$

To show this, take some partition $(t_i)_{1\leq i\leq N}$ of $[a,b]$. Then:

$$L_d[\omega_n] \geq \sum_{i=1}^{N} d(\omega_n(t_i), \omega_n(t_{i+1})) \to \sum_{i=1}^{N} d(\omega(t_i), \omega(t_{i+1}))$$

since d is continuous. This holds for an arbitrary partition, so we can conclude

$$\liminf_{n\to\infty} L_d[\omega_n] \geq L_d[\omega].$$

□

Lemma 2.15. Let (M, g) be an Riemannian manifold. If there is some $p \in M$ such that $\exp_p$ is defined on all of T_pM, then every closed and bounded subset is compact and for all $q \in M$ exists a geodesic γ with $L[\gamma] = d(p, q)$.

Proof. It suffices to show that $\overline{B_r(p)}$ is compact for all $r > 0$ since closed subsets of compact sets are compact. Then, we can use the preceeding lemma to obtain the statement.
Define

$$R := \sup \left\{ r > 0 \mid \overline{B_r(p)} \text{ is compact} \right\}.$$

We have $R > 0$ since M is locally compact and want $R = \infty$. Assume $R < \infty$. We claim that $\overline{B_R(p)}$ is compact ($\iff$ $B_R(p)$ is relatively compact). Let (q_n) be a sequence in $B_R(p)$ and set $r_n := d(p, q_n) < R$. By the preceeding lemma, there exists a geodesic γ_n parametrized to unit speed from p to q_n with $L[\gamma_n] = d(p, q_n) = r_n$. W.l.o.g., let $r_n \to r_0$ for some $r_o \leq R$. Set $v_n := \dot{\gamma}_n(0) \in T_pM$. We have $g_p(v_n, v_n) = 1$. This implies that

$$\{v_n \mid n \in \mathbb{N}\} \subseteq \underbrace{\{v \in T_pM \mid g_p(v, v) = 1\}}_{\text{compact}} \subseteq T_pM.$$

Hence, there exists a subsequence (v_{n_k}) converging to some $v \in T_pM$ with $g_p(v, v) = 1$. By assumption, the maximal geodesic γ_v with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$ is defined on all of $\mathbb{R}$. Set

$$q := \gamma_v(r_0).$$

Then,

$$q_{n_k} = \gamma_{v_{n_k}}(r_{n_k}) = \exp_p\left(\frac{v_{n_k}}{r_{n_k}}\right) \to \exp_p\left(\frac{v}{r_0}\right) = \gamma_v(r_0) = q$$

by definition of q.
Now, we claim that if $\overline{B_R(p)}$ is compact, then there exists $\epsilon > 0$ such that $\overline{B_{R+\epsilon}(p)}$ is still compact. For all $q \in \overline{B_r(p)}$, there exists an ϵ_q such that $\overline{B_{\epsilon_q}(q)}$ is compact. Clearly,

$$\bigcup_{q \in \overline{B_R(p)}} B_{\epsilon_q}(q) = \overline{B_R(p)}.$$

Set $U := \cup_{1 \leq i \leq N} B_{\epsilon_{q_i}}(q_i)$. This is open and $\overline{U}$ is compact since the union is finite. Consider

$$\inf \left\{ d(x, M \setminus U) \mid x \in \overline{B_R(p)} \right\} = \min \left\{ d(x, M \setminus U) \mid x \in \overline{B_R(p)} \right\} \geq \epsilon_0 > 0$$

since $M \setminus U$ is closed, so $d(\cdot, M \setminus U)$ attains a minimum as a continuous function.

It remains to show that $\overline{B_{R+\epsilon}(p)} \subseteq U$. Let $y \in B_{R+\epsilon}(p)$. Then there exists a piecewise $\mathcal{C}^1$ curve ω from p to y with $L[\omega] < R + \epsilon$. Find a parametrization of ω such that $L[\omega|_{[0,t_0]}] < R$ and $L[\omega|_{[t_0,1]}] \leq \epsilon$. Then, $x := \omega(t_0)$ satisfies $x \in B_R(p)$ and $d(x,y) < \epsilon_0$ (since d is the infimum of lengths). This implies that y cannot lie in $M \setminus U$, since that would contradict $d(x,y) \geq \epsilon_0$ for all $x \in \overline{B_R(p)}$ and $\underline{y \in M \setminus} U$. Hence, $y \in U$. So we indeed have $B_{R+\epsilon_0}(p) \subseteq U$, therefore $\overline{B_{R+\epsilon_0}(p)} \subseteq \overline{U}$. Since $\overline{U}$ is compact, we are done. □

Now, we are able to prove the theorem of Hopf-Rinow.

Proof. ($1 \implies 2$): We know that a geodesic $\gamma : [a,b) \to M$ is extendable as a geodesic if and only if it extends continuously to b. Let $\gamma : I \to M$ be a geodesic with $I \neq \mathbb{R}$. W.l.o.g., $I = (a,b)$ with $b < \infty$. Set $C := \|\dot\gamma(t)\| \in \mathbb{R}$ by definition of a geodesic. We have

$$d(\gamma(t),\gamma(s)) \leq C \cdot |t-s| = L[\gamma|_{[\gamma(t),\gamma(s)]}].$$

Hence, the sequence $q_n := \gamma(b - \frac{1}{n})$ is a Cauchy sequence. By metric completeness, there is some $q = \lim q_n$. We have to show that $\lim_{s \to b} \gamma(s) = q$. We can estimate

$$d(\gamma(s),q) \leq d(\gamma(s),\gamma(b-\frac{1}{n})) + d(\gamma(b-\frac{1}{n}),q) \leq C \cdot \left| s - (b - \frac{1}{n}) \right| \overset{n\to\infty}{\to} |s-b|.$$

for arbitrary n. Therefore, $d(\gamma(s),q) \leq C \cdot |s-b|$, which goes to 0 as $s \to b$.
($2 \implies 3$) is trivial.
($3 \implies 4$) was proven in the preceeding lemma.
($4 \implies 1$): Let (x_n) be a Cauchy sequence. Then $\{x_n \mid n \in \mathbb{N}\}$ is bounded with

$$d(x,y) \leq \max\left\{\frac{1}{\epsilon}, d(x_i,x_j) + \frac{1}{\epsilon} \mid i,j \leq N_\epsilon\right\}.$$

By the Heine-Borel property, the closure is compact. Hence, there exists a subsequence converging to some $x \in M$. Since (x_n) is Cauchy, all of $x_n \to x$. □

2.3 Lorentzian Geometry

Definition 2.16 (Lorentzian Manifold). A Lorentzian manifold is a pair (M,η) where M is a smooth m-manifold and η is a non-degenerate, symmetric bilinear form with index $\nu = 1$.

Definition 2.17 (Types of Curves). A piecewise $\mathcal{C}^1$-curve

$$\gamma : [a, b] \to M$$

in a Lorentzian manifold M is called:

- **timelike** if $\dot{\gamma}(s)$ is timelike, i.e. $\eta_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s)) < 0$ for all s and at all breakpoints t_i we have $\eta_{\gamma(t_i)}(\dot{\gamma}_+(t_i), \dot{\gamma}_-(t_i)) < 0$.[a]
- **null** if $\dot{\gamma}(s)$ is null, i.e. $\eta(\dot{\gamma}(s), \dot{\gamma}(s)) = 0$ and $\dot{\gamma} \neq 0$.
- **spacelike** if $\dot{\gamma}(s)$ is spacelike, i.e. $\eta(\dot{\gamma}(s), \dot{\gamma}(s)) > 0$ or $\dot{\gamma} = 0$.
- **causal** if $\dot{\gamma}(t)$ is causal, i.e. timelike or null for all s.

[a] This means $\dot{\gamma}_\pm$ lie in the same connected component.