# Optimization with PDEs

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# Chapter 1

# Introduction

## 1.0 Preliminaries

### Modalities

- Oral exams, 10.02. or 24.03.; register online until 12.12. by katrin.kopp@uni-hamburg.de
- Office hours on website
- Exercises every second week; half of points needed; attendance on 5 or more exercise classes required; exercises have to be presented ; register in Moodle for exercises

### Motivation

Given a function

$$f : X \to \mathbb{R}$$

such that $X$ is an (in general, infinite-dimensional) Banach space, we want to minimize $f(x)$ for $x \in X^{\mathrm{ad}}$, the set of admissible values.

**Example.** There are plenty of real-world applications:

- modelling of elastic bodies, damage calculation
- optimal design (buildings, ships, aircrafts, ...)
- optimal control (trajectories of spacecrafts, ...)
- inverse problems (seismic tomography, MRT, ...)

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**Theorem 1.1.** Let

$$f : X^{\mathrm{ad}} \subseteq \mathbb{R}^n \to \mathbb{R}$$

be a continuous function and $X^{\mathrm{ad}}$ be bounded and closed. Then, $\min_{x \in X^{\mathrm{ad}}} f(x)$ is attained.

**Proof.** Take a minimizing sequence $(x_k)_{k\in\mathbb{N}} \in X^{\mathrm{ad}}$, i.e. a sequence with

$$f(x_k) \to \inf_{x\in X^{\mathrm{ad}}} f(x).$$

Since $(x_k)$ is bounded, there exists a convergent subsequence $(x_n) \to x$. Now $x$ is a solution of our problem since:

1.
$$f(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} f(x_k) = \inf_{x\in X^{\mathrm{ad}}} f(x)$$
since $f$ is continuous.

2. Since $X^{\mathrm{ad}}$ is closed, $x \in X^{\mathrm{ad}}$.

□

**Example.** Consider the optimization problem

$$\min \int_{-1}^{1} (x(s) - x^d(s))^2 \, ds$$

s.t. $x \in \mathcal{C}([-1,1])$, $-1 \le x(s) \le 1 \, \forall s \in [-1,1]$ with

$$x^d(s) = \begin{cases} -1 & s < 0 \\ 1 & s \ge 0 \end{cases}.$$

1.
$$X^{\mathrm{ad}} = v \in \mathcal{C}([-1,1]) \mid -1 \le v(s) \le 1 \forall s \in [-1,1]$$

2. $X^{\mathrm{ad}}$ is closed: $v_k \to v$ in $\mathcal{C}$, so $v_k(s) \to v(s) \forall s \in [-1,1]$ since $v_k(s) \in [-1,1]$ is closed, so $v(s) \in [-1,1]$.

3. $f$ is continuous, so if $v_k \to v$ in $\mathcal{C}$, we have

$$\begin{aligned}
|f(v_k) - f(v)| &= \int_{-1}^{1} (v_k(s)^2 - 2v_k(s)x^d(s) + x^d(s)^2 - (v(s)^2 - 2v(s)x^d(s) + x^d(s)^2)) \, ds \\
&= \left| \int_{-1}^{1} v_k(s)^2 - v(s)^2 - 2(v_k(s) - v(s))x^d(s) \, ds \right| \\
&= 2 \cdot 2 \max(v_k(s) - v(s)) + 4\|x^d\|_\infty \max |v_k(s) - v(s)| \to 0
\end{aligned}$$

. However, there is no solution potential minimizing sequence

$$x_n(s) = \begin{cases} 1 & s > \frac{1}{n} \\ n \le & -\frac{1}{n} \le s \le \frac{1}{n} \\ -1 & s < \frac{-1}{n} \end{cases}.$$

4. To see this, consider

$$\begin{aligned}
0 \le f(x_n) &= \int_{-\frac{1}{k}}^{0} (ns+1)^2 \, ds + \int_{0}^{\frac{1}{n}} (ns-1)^2 \, ds \\
&= 2 \cdot \int_{0}^{\frac{1}{n}} (ks-1)^2 \, ds \le \frac{2}{n} \to 0
\end{aligned}$$

but

$$f(x) = 0 \iff (x(s) - x^d(s))^2 = 0 \forall s \implies x = x^d \notin \mathcal{C}.$$

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It can be even worse:

**Example.** The Lavrentier phenomenon is concerned with the functional

$$\mathcal{J}[u] = \int_0^1 (u(t)^3 - t)^2 u'(t)^6 \, dt \to \min$$

s.t. $u(0) = 0$ and $u(1) = 1$.

1. $u \in \mathcal{C}^1((0,1))$ leads to $v(t) = \sqrt[3]{t}$ with $\mathcal{J}[v] = 0$.
2. $u \in \mathcal{C}^{0,1}((0,1))$ but $\inf \mathcal{J}[v] \geq c > 0$.

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## 1.1 Existence of Solutions

Consider a problem $(P)$ consisting of finding $\min f(x)$ s.t. $\sigma(x) \in K$ under the following assumptions $(A1)$:

1. $f : X \to \mathbb{R}$, $X$ is Banach and $f$ is lower semi-continuous.
2. $\sigma : X \to Z$, $Z$ is Banach and $\sigma$ is continuous.
3. $K \subset Z$ closed and convex

### Existence by Compactness

**Theorem 1.2** (Existence by Compactness)**.** Assuming $(A1)$, assume also that there is $x^0 \in X^{\text{ad}}$ such that

$$\mathscr{L}_f(x^0) := \{x \in X^{\text{ad}} \mid f(x) \leq f(x^0)\}$$

is compact and non-empty. Then, $(P)$ has a solution.

**Proof.** Let $(x_k)_k \in X^{\text{ad}}$ be a minimizing sequence such that $f(x_k) \to \inf_{x \in X^{\text{ad}}} f(x)$. W.l.o.g., we can assume $x_k \in \mathscr{L}_f(x^0)$, so $x_k \to \overline{x} \in \mathscr{L}_f(x^0)$ since we have a convergent subsequence. Hence,

$$f(\overline{x}) = f(\lim x_k) \leq \liminf_{k \to \infty} f(x_k) = \inf_{x \in X^{\text{ad}}} f(x),$$

and therefore $\overline{x} \in X^{\text{ad}}$ (lower subcontinuity). □