Comp Stats
SS

Exercise 3, Heston model

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)},$$

$$dV_t = (\alpha - \lambda V_t) dt + \sigma_v \sqrt{V_t} dW_t^{(2)},$$

$$\langle W^{(1)}, W^{(2)} \rangle_t = \rho t, \quad \rho \leq |1|$$

1/ Show that for all $\rho \leq |1|$ the process

$$W_t^{(1)} := \sqrt{1-\rho^2} B_t^{(1)} + \rho B_t^{(2)}.$$ is a Brownian motion.

We use Levys characterization, so we need to show

① $P(W_0^{(1)} = 0) = 1$

② $W_t^{(1)}$ is a continous martingale

③ $\langle W_t^{(1)} \rangle = t$, almost surely w.r.t. $\mathbb{P}$

For ① we know

$$W_0^{(1)} = \sqrt{1-\rho^2} B_0^{(1)} + \rho B_0^{t} = 0, \text{ with probability } 1$$

For ② we know that the Brownian motions are continous martingales

For ③, consider

$$\langle W_t^{(1)} \rangle = \langle \sqrt{1-\rho^2} B_t^{(1)} + \rho B_t^{(2)} \rangle = \langle \sqrt{1-\rho^2} B_t^{(1)} \rangle + \langle \rho B_t^{(2)} \rangle$$

$$= (\sqrt{1-\rho^2})^2 \langle B_t^{(1)} \rangle + (\rho)^2 \langle B_t^{(2)} \rangle = (1 - \rho^2 + \rho^2) t = t \; \blacktriangle$$

Exercise 3

2) Use 1) to verify that $W_t^{(1)}$, and $W_t^{(2)} := B_t^{(2)}$

Satisfy $\langle W^{(1)}, W^{(2)} \rangle_t = \rho t$

$$\left\langle \sqrt{1-\rho^2} B^{(1)} + \rho B^{(2)}, B^{(2)} \right\rangle_t = \left\langle \sqrt{1-\rho^2} B^{(1)}, B^{(2)} \right\rangle_t + \left\langle \rho B^{(2)}, B^{(2)} \right\rangle_t$$

$$= \sqrt{1-\rho^2} \langle B^{(1)}, B^{(2)} \rangle_t + \rho \langle B^{(2)}, B^{(2)} \rangle_t = 0 + \rho t,$$

where we used that $B^{(1)}$ and $B^{(2)}$ were independent, and $\langle B^{(2)}, B^{(2)} \rangle_t = \langle B_t^{(2)} \rangle = t$.

Exercise 4:

Find functions $b: \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma: \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ such that

$$\left.\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t}\, dW_t^{(1)}, \quad S_0 \geq 0 \\ dV_t &= (\alpha - \lambda V_t) dt + \sigma_V \sqrt{V_t}\, dW_t^{(2)}, \quad V_0 > 0, \lambda, \alpha, \sigma_V \geq 0 \end{aligned}\right\} (3)$$

can be written as:

$$dY_t = b(Y_t) dt + \sigma(Y_t) dB_t, \quad Y_0 = (S_0, V_0)^T, \quad Y_t = \begin{bmatrix} S_t \\ V_t \end{bmatrix}$$

First we rewrite (3)

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t}\, d\left(\sqrt{1-\rho^2} B_t^{(1)} + \rho B_t^{(2)}\right)$$

$$= \mu dt + \sqrt{(1-\rho^2) V_t}\, dB_t^{(1)} + \sqrt{V_t} \rho\, dB_t^{(2)}$$

$$dV_t = (\alpha - \lambda V_t) dt + \sigma_V \sqrt{V_t}\, dB_t^{(2)}$$

$$\Rightarrow b(x) = \begin{bmatrix} \mu x_1 \\ \alpha - \lambda x_2 \end{bmatrix}, \quad \sigma(x) = \begin{bmatrix} \sqrt{(1-\rho^2) x_2}\, x_1 & \sqrt{x_2} \rho x_1 \\ 0 & \sigma_V \sqrt{x_2} \end{bmatrix}$$