Selfstudy 1

Introduction

This selfstudy assignment should be answered using a single Rmd file which can be run on one of the university servers js{01,02,03,04} in a resonable amount of time. Please add student name in the yaml header (if you are a group of students working together you are allowed to only hand in one answer with all your names on it).

To embed Rcpp code in your Rmd file use the chunk option `engine='Rcpp'` as below:

```
#include <Rcpp.h>
// [[Rcpp::export]]
int fibonacci(const int x) {
    if (x == 0 || x == 1) return(x);
    return (fibonacci(x - 1)) + fibonacci(x - 2);
}
```

```
fibonacci(4)
```

```
## [1] 3
```

To complete the exercises you need both to fill in a bit of LaTeX math formulas and some R and C++ code.

Monte Carlo integration

Problem A: Gaussian tail probability

Consider the tail probability $P(X > 4)$ where $X \sim N(0, 1)$.

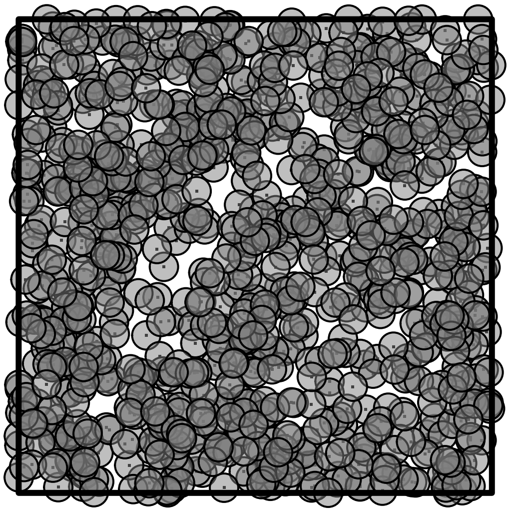
- Express the probability as an integral of an indicator function over the real line and provide a simple Monte Carlo (MC) estimate of the probability along with the MC standard error based on $100,000$ simulations.
- Make a change of variable $u = 1/x$ and express the integral with respect to the uniform distribution on $(0, 1/4)$ and use this to obtain a more accurate MC estimate (report both MC estimate and MC std. error).

Problem B: Complicated regions of integration

- Suppose we have a collection of (possibly overlapping) discs with centers in $[0, 1]^2$ and we wish to estimate the fraction of the unit square covered by these.

Give a MC estimate and MC std. error for the fraction covered by the following discs:

```
set.seed(42)
n <- 1000
centers <- data.frame(x=runif(n), y=runif(n))
r <- .03
par(pty = "s")
plot(centers, pch = ".", axes = FALSE, xlab = "", ylab = "")
symbols(centers, circles = rep(r, n), inches = FALSE, bg = gray(.5, .5), add = TRUE)
rect(0,0,1,1, lwd = 3)
```



- If you implement this with a for loop I suggest trying to do it in C++.
- Consider whether sorting the centers according to e.g. x-coordinate can be used to speed up calculations, and possibly implement this to see if it works.
- Explain how considering a single circle of radius 0.5 centered in (0.5,0.5) could be used to estimate $\pi = 3.14...$ using MC.

Problem C: Combining MC estimates

- Assume we have k MC estimates $m_i = 1/n_i \sum_{j=1}^{n_i} y_i$ with std. errors se_i based on sample size n_i, $i = 1, \ldots, k$ which are all independent estimates of the same mean value. How can you construct a combined estimate and standard error based on the total sample size $n_\bullet = \sum_{j=1}^{k} n_i$?
- Use `mclapply()` (or another parallel processing function) to run 10 independent MC estimates of the integral in problem A and use the formula derived above to combine these to a single MC estimate and MC std. error.

Rcpp

Problem D: Vector auto-regressive process

Consider a vector auto-regressive process of order 1, VAR(1) in $\mathbb{R}^d$ with $d \times d$ lag parameter matrix A:

$$x_t = Ax_{t-1} + e^t$$

where e_t is a process of iid multivariate Gaussian variables $e^t \sim N_d(\mu, \Sigma)$.

Given the starting state $x_0 = 0$ we wish to simulate a realization of $\{x_t\}_{t=1}^n$.

- Verify that the following R function indeed generates an $d \times n$ matrix containing `n` multivariate Gaussian vectors with mean vector `mu` and covariance matrix `Sigma` (simply check that the empirical covariance matrix is close to the given one for a specific 2x2 case of your choice):

```r
mvrnormR <- function(mu, Sigma, n) {
  d <- length(mu)
  stopifnot(nrow(Sigma) == d)
  std_norm <- matrix(rnorm(n*d), nrow = d, ncol = n)
  mu + t(chol(Sigma)) %*% std_norm
}
```

- Use `RcppArmadillo` to implement this as a new function `mvrnormcpp()` and benchmark the two functions for the following input (check that they approximately have the same distribution – you can only use `all.equal()` to check if you have managed to use the same random numbers in both cases):

```
d <- 2
Sigma <- matrix(c(2,1,1,4), nrow = 2, ncol = 2)
n <- 1000
set.seed(42)
mu <- c(10, 20)
```

Simulation of VAR(1) is obtained with the R function `simR()` below. Implement this as a C++ function `simcpp()` which calls `mvrnormcpp()` you made before and benchmark the C++ and R implementations. We will use `A <- matrix(c(0.7,0.2,0.2,0.7),2,2)` as the coefficient matrix.

```
simR <- function(A, Sigma, n) {
  mu <- rep(0, nrow(Sigma))
  errors <- mvrnormR(mu, Sigma, n)
  simdata <- matrix(0, length(mu), n)
  for (col in 2:ncol(errors)) {
    simdata[,col] = A %*% simdata[,(col-1)] + errors[,col]
  }
  return(simdata)
}
```