Quantitative Finance and Computational Statistics: Selfstudy 2

O. Sauri

March 30, 2021

In this selfstudy session we will concentrate on the risk-neutral Heston model, i.e. under the risk-neutral measure $\mathbb{Q}$ we have that

$$\begin{aligned}\frac{dS_t}{S_t} =& rdt + \sqrt{V_t}\left(\sqrt{1-\rho^2}dB_t^{(1)} + \rho dB_t^{(2)}\right), \ S_0 \geq 0. \\ dV_t =& (\alpha - \lambda V_t)dt + \sigma_V \sqrt{V_t} dB_t^{(2)}, \ V_0 > 0, \lambda, \alpha, \sigma_V \geq 0,\end{aligned}$$

with $|\rho| \leq 1$, $2\alpha > \sigma_V^2$, and $B^{(1)}$ and $B^{(2)}$ two independent $\mathbb{Q}$-Brownian motions. In what follows, we set the parameters to

$$r = 0, \alpha = 0.5, \lambda = 0.5, \sigma_V = 0.4, \rho = -0.7, S_0 = 1, V_0 = 0.3.$$

Exercise 1. In this exercise we will focus on the price-function associated to a call option with maturity time $T > 0$ and strike price $K > 0$, i.e. on the function

$$F(t, x, v) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left((K - S_T)^+ \middle| S_t = x, V_t = v\right).$$

1. Fix $n, m, q \in \mathbb{N}$. Let $0 \leq x_{min} < x_{max}$ and $0 \leq v_{min} \leq v_{max}$. For $i = 0, \ldots, n$, $j = 0, \ldots, m$, $k = 0, \ldots, q$ put
$$t_i = \Delta_n^{time} i; \ x_j = x_{min} + j\Delta_m^{price}; \ v_k = v_{min} + \Delta_q^{vol} k.$$
where $\Delta_n^{time} = \frac{T}{n}$, $\Delta_m^{price} = \frac{x_{max} - x_{min}}{m}$, and $\Delta_q^{vol} = \frac{v_{max} - v_{min}}{q}$. Using a Monte Carlo approach, write a code that approximates $F(t_i, x_j, v_k)$ for $i = 0, \ldots, n$, $j = 0, \ldots, m$, $k = 0, \ldots, q$.

2. Implement your previous code in the case when
$$(n, m, q, x_{min}, x_{max}, v_{min}, v_{max}, T, K) = (100, 10, 10, 0, 10, 0, 10, 1, 1).$$

   Remember that the more MC repetitions, the better the approximation becomes.

3. Use the Feynman-Kac Formula to show that the function $G(\tau, x, v) := F(T - \tau, x, v)$ satisfies the following PDE
$$-\partial_\tau G + rx\partial_x G + (\alpha - \lambda v)\partial_v G + \frac{1}{2}x^2 v\partial_{xx}G + \frac{1}{2}\sigma_V^2 v\partial_{vv}G + \rho\sigma_V vx\partial_{xv}G - rG = 0.$$
with initial condition $G(0, x, v) = (x - K)^+$.

4. Use that:
$$\begin{aligned}&F(t, 0, v) = 0; \\ &\frac{F(t, x, v)}{x} \to 1, \text{ as } x \uparrow +\infty;\end{aligned} \tag{1}$$
and that for all (t, x) the mapping $v \mapsto F(t, x, v)$ is increasing and
$$(x - e^{-r(T-t)}K)^+ \leq F(t, x, v) \leq x,$$
to impose conditions on the behavior of $G(\tau, 0, v)$, $G(\tau, x_{max}, v)$, $G(\tau, x, 0)$ and $G(\tau, x, v_{max})$.

5. Use the approximations (or finite differences)

$$\begin{aligned} f'(x) &= \frac{f(x+\Delta)-f(x)}{\Delta} + \mathrm{o}(\Delta), \\ f'(x) &= \frac{f(x+\Delta)-f(x-\Delta)}{2\Delta} + \mathrm{o}(\Delta^2), \\ f''(x) &= \frac{f(x+\Delta)-2f(x)+f(x-\Delta)}{\Delta^2} + \mathrm{o}(\Delta^2), \end{aligned} \tag{2}$$

to find an approximation of $\partial_\tau G$, $\partial_x G$,$\partial_v G$,$\partial_{xx} G$,$\partial_{vv} G$ and $\partial_{xv} G$ on the mesh described in 1. in order to create a recursive equation that approximates G in such a mesh. What you should be particularly aware of for this recursive equation?

6. Use the boundary conditions proposed in 4. to implement the finite-difference algorithm developed in 5. in order to approximate F via G in the case when

$$(n, m, q, x_l, s_u, v_l, v_u) = (100, 10, 10, 0, 10, 0, 10).$$

7. Plot together your approximations of F obtained in 2. and 6. as functions of x, v and t, respectively.

8. Use the put-call parity to approximate the price function of a put option with maturity time $T > 0$ and strike price $K > 0$ using your approximations obtained for F in the previous part.

Exercise 2. In this exercise we will use the realized volatility and the spot volatility estimators seen in class to estimate the integrate volatility

$$IV_t = \int_0^t V_s ds, \text{ and } V_t, \ \ t \le T.$$

1. Simulate a path of $(X_t := \log(S_t), V_t)$ via the Euler scheme for $\Delta'_n = 1/100000$.

2. Use your simulations and a Riemann-sum approximation to simulate IV_t.

3. From your simulated path, pick a subsample of X for $\Delta_n = 1/100, 1/1000, 1/1000, 1/100000$ and use

$$\begin{aligned} \hat{\theta}_t &:= \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2, \\ \hat{\sigma}_t^2 &:= \frac{1}{k_n \Delta_n} \sum_{i=1}^{k_n} \left(\Delta_{j+i}^n X\right)^2, \ \ (j-1)\Delta_n < t \le j\Delta_n, \end{aligned}$$

while for $T \ge t > (n-k_n)\Delta_n$ $\hat{\sigma}_t^2 = \hat{\sigma}^2_{(n-k_n)\Delta_n}$, to estimate, respectively, IV and V. Plot your estimates with their respectively confidence intervals.