Exercises Lecture 9

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Exercise 1

Suppose that $\varepsilon_1, \varepsilon_2, \cdots$, is a Gaussian white noise process with mean 0 and variance 1, and u_t and y_t are stationary processes such that

$$u_t = \sigma_t \varepsilon_t \quad \text{where} \quad \sigma_t^2 = 2 + 0.3u_{t-1}^2,$$

and

$$y_t = 2 + 0.6y_{t-1} + u_t.$$

1. What type of process is u_t?
2. What type of process is y_t?
3. Is u_t Gaussian? If not, does it have heavy or lighter tails than a Gaussian distribution? Explain.
4. What is the ACF of u_t?
5. What is the ACF of u_t^2?
6. What is the ACF of y_t?

Exercise 2

Consider the daily log returns on the SP500 index (GSPC).

Hint to data acquisition: Several data can be downloaded from for example "yahoo" or other sources. To download `SP500` you can run

```
library(quantmod)
getSymbols("^GSPC", from="2005-01-01", to="2014-12-31", src="yahoo")
```

```
## [1] "^GSPC"
```

Hints to the following: GARCH type models may be analyzed using the `rugarch` and `quantmod` packages. A GARCH type model is estimated using something like the following:

```
library(rugarch)
sp500 = xts( diff( log( GSPC[,6] ) )[-1] )
y = as.numeric(sp500)
result_object <-  ugarchspec(variance.model = list(model="sGARCH",
                        garchOrder=c(1,1)),
                        mean.model = list(armaOrder=c(2,0)),
                        distribution.model = "norm")
garch_est <- ugarchfit(spec=result_object, data=y)
```

1. Is there any serial correlation in the log returns of SP500 index? Why?
2. Is there any ARCH effect in the log returns of S&P 500 index? Why?
3. Specify and fit an ARCH model to the log returns of SP500 index. Write down the fitted model.
4. Is your fitted ARCH model stationary? Why?
5. Fit a GARCH(1,1) model for the log returns on the SP500 index using the Gaussian distribution for the innovations. Write down the fitted model.

6. Is the fitted GARCH model stationary? Why?
7. Make a Normal quantile plot for the standardized residuals. Use `qqnorm()` and `qqline()` in R.
8. Is the Gaussian distribution appropriate for the standardized innovations?
9. Plot the fitted conditional standard deviation process $\hat{\sigma}_t^2$ and comment.
10. Calculate the 1–10 step ahead forecasts from the end of the series for both the process y_t and the conditional variance using the `ugarchforecast()` function.

Exercise 3

As mentioned in the slides and elsewhere the GARCH-M model directly incorporates volatility as a regression variable. The parameter associated to volatility included in the regression represents the risk premium, or reward for additional risk. Modern portfolio theory states that increased volatility leads to increased risk, requiring larger expected returns. The presence of volatility as a statistically significant predictor of returns is one of the primary contributors to serial correlation in historic return series. The dataset `GPRO.csv` contains the adjusted daily closing price of GoPro stock from June 26, 2014 to January 28, 2015.

1. Fit a GARCH-M model to the GoPro stock returns [Hint: In `ugarchspec` you can in `mean.model` specify `archm = TRUE`]

2. Write out the fitted model. The risk premium parameter is equal to `archm` in the R output.

3. Test the one-sided hypothesis that the risk premium parameter is greater than zero versus the alternative that it is equal to zero. Is the risk premium significant?

Exercise 4 (SFM exercise 13.6)

For an ARCH(1) process, show that

$$\mathbf{E}[\sigma^2_{\mathbf{t+s}} \,|\, \mathcal{F}_{\mathbf{t-1}}] = \frac{1-\alpha_1^s}{1-\alpha_1}\alpha_0 + \alpha_1^s \sigma_t^2, \qquad s \geq 1.$$

[Hint: Use induction on s.] Interpret this result.

Exercise 5 (Model Identifiability, SFM exercise 13.8 b)

Show that the GARCH(1,1) process

$$\sigma_t^2 = 1 + \frac{1}{4}\varepsilon_{t-1}^2 + \frac{1}{2}\sigma_{t-1}^2$$

and the GARCH(2,2) process

$$\sigma_t^2 = \frac{5}{4} + \frac{1}{4}\varepsilon_{t-1}^2 + \frac{1}{16}\varepsilon_{t-2}^2 + \frac{1}{4}\sigma_{t-1}^2 + \frac{1}{8}\sigma_{t-2}^2$$

are equivalent, i.e. the given relationships are satisfied by the same process σ_t^2. [Hint: Use recursion on the GARCH(1,1) process]

What is noteworthy about the polynomials $p(z) = \alpha_1 z + \alpha_2 z^2$ and $q(z) = 1 - \delta_1 z - \delta_2 z^2$ for the second process? Recast your observation as ahypothesis concerning the identifiability of a general GARCH(p, q) process.