Multivariate Time Series, part 2

Nonstationary Dynamic Models: Cointegration

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(based on slides by J. Eduardo Vera-Valdés)

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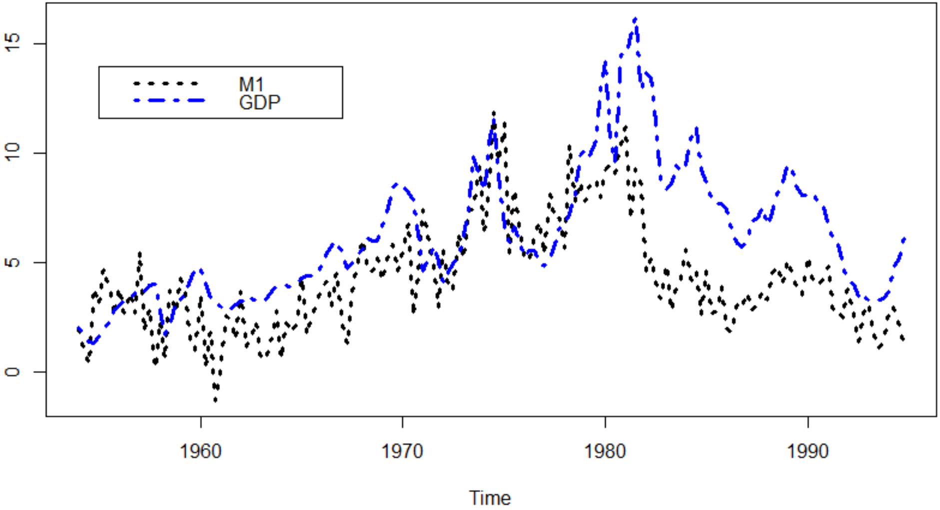
Introduction



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Introduction

Most modelling schemes require the data to be stationary.

It can be difficult to determine whether the data we have collected were generated by a stationary process.

A process with a trend, either deterministic or stochastic, is clearly nonstationary, its mean changes over time. Nonetheless, we may require long datasets to determine if there exists such a trend.

We consider **nonstationary** multivariate time series models: explaining one variable from its own past including current or lagged values of other variables.

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Spurious Regression

Whenever a variable with a unit root is used as a regressor, the standard assumptions that we have made are violated.

This has serious consequences for asymptotic analysis. It implies that the results on consistency and asymptotic normality are not applicable.

One manifestation of these deviations is the phenomenon of spurious regression.

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Spurious Regression

Let Y_t and X_t be two independent unit root processes and consider the regression

$$Y_t = \alpha + \beta X_t + U_t.$$

Given the independence, we would expect the estimates to be non-significant and the R-squared to be close to zero.

Nonetheless, the t-statistic diverges asymptotically, and the R-squared converges to a positive constant.

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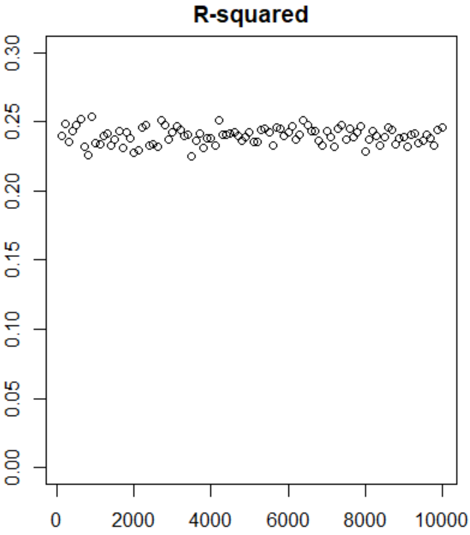
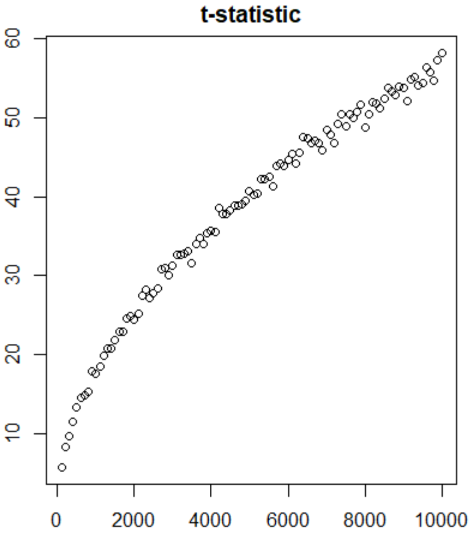
Spurious Regression





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Unit Root Tests

It is quite important to know whether a series has a unit root; thus, dozens of tests have been developed.

By far, the most used one is the one due to Dickey and Fuller. Assume we want to test whether Y_t has a unit root, we consider the regression

$$\Delta Y_t = \beta Y_{t-1} + U_t,$$

and we test for $H_0 : \beta = 0$ against $H_1 : \beta < 0$.

The test has a nonstandard distribution under H_0, and thus critical values have to be simulated.

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Unit Root Tests

There are many extensions to the Dickey-Fuller tests

$$\Delta Y_t = \alpha + \beta Y_{t-1} + \gamma_1 t + \gamma_2 t^2 + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + U_t.$$

- α, γ_1, and γ_2, account for constant, drift, and linear trend, respectively.
- δ_i account for autocorrelation.

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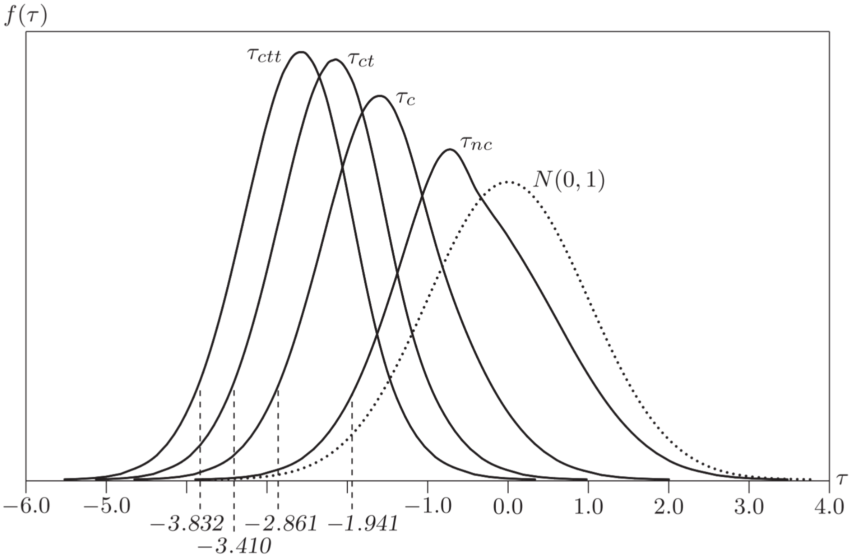
Unit Root Tests



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Unit Root Tests

Another relevant extension to the Dickey-Fuller test is the GLS modification, p. 613. The idea is to obtain higher power by estimating the deterministic regressors prior to estimating the autoregressive component.

The ADF-GLS test estimates the deterministic components by the regression

$$Y_t - \bar{\rho} Y_{t-1} = (X_t - \bar{\rho} X_{t-1})\gamma + U_t,$$

where X_t contains the deterministic components and $\bar{\rho} = 1 + \bar{c}/n$, with $\bar{c} = -7$ when X_t is just a constant, and $\bar{c} = -13.5$ when it contains a constant and a trend.

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Unit Root Tests

After the first step, construct $\tilde{Y}_t$ given by

$$\tilde{Y}_t = Y_t - X_t\hat{\gamma},$$

and run the test regression

$$\Delta\tilde{Y}_t = \beta\tilde{Y}_{t-1} + \sum_{i=1}^{p} \delta_i \Delta\tilde{Y}_{t-i} + V_t,$$

which has a distribution with more power against the alternative.

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Cointegration

Economic theory often suggests that two or more economic variables should be linked more or less closely, ex: interest rates of different maturities, prices of similar commodities in different countries, the money supply and the price level, etc.

Relationships among these variables are usually assumed to hold only in the long run. Economic forces are expected to act in the direction of eliminating short-run deviations.

In this sense, variables that are all individually $I(1)$, and hence divergent, can in a certain sense diverge together.

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Cointegration Single-Equation

Let $Y_{1,t}$ and $Y_{2,t}$ be unit root processes, we say that the series cointegrate if there exists a linear combination between them that is stationary.

In other words we look for α and β such that

$$V_t = Y_{1,t} - \alpha - \beta Y_{2,t},$$

is a stationary process.

Intuitively, cointegration means that the distance between the series is a stationary process.

The concept of cointegration is more general, allowing $I(d)$ series to have a $I(d-b)$ linear combination with $b > 0$.

Cointegration Single-Equation

The steps to test for cointegration according to the **Engle-Granger method** are

- Estimate by OLS the equation

$$Y_{1,t} = \alpha + \beta Y_{2,t} + V_t.$$

- Obtain the residuals from the regression

$$\hat{V}_t := Y_{1,t} - \hat{\alpha} - \hat{\beta} Y_{2,t},$$

  and test them for a unit root using an ADF test.

- If we reject the null of a unit root, the series cointegrate.

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Cointegration Multivariate Model

An alternative to the single-equation method is to test for cointegration using a vector autoregression, the **Johansen method**.
Consider the VAR (see p. 623)

$$\mathbb{Y}_t = \mathbb{X}_t B + \sum_{i=1}^{p+1} \mathbb{Y}_{t-i} \Phi_i + U_t,$$

where $\mathbb{Y}_t$ is a $1 \times g$ vector of $I(1)$ variables, and $\mathbb{X}_t$ is a row vector of deterministic components.
By a reparametrization similar to the one used in the DF-test, we can write the system as

$$\Delta \mathbb{Y}_t = \mathbb{X}_t B + \mathbb{Y}_{t-1} \Pi + \sum_{i=1}^{p} \Delta \mathbb{Y}_{t-i} \Gamma_i + U_t.$$

Cointegration Multivariate Model

$$\Delta \mathbb{Y}_t = \mathbb{X}_t B + \mathbb{Y}_{t-1}\Pi + \sum_{i=1}^{p} \Delta \mathbb{Y}_{t-i}\Gamma_i + U_t .$$

- If Π has rank zero, there is no cointegration.
- If Π has full rank, then all linear combinations from the vector $\mathbb{Y}$ are $I(0)$, meaning that the vector itself is $I(0)$.
- If Π has rank r with $0 < r < g$, then there are r cointegrating relations.

Cointegration Multivariate Model

Suppose the matrix Π is has rank r, then we can find matrices η and α, both of size $g \times r$, such that

$$\Pi = \eta\alpha^T.$$

Replacing this expression,

$$\Delta\mathbb{Y}_t = \mathbb{X}_t B + \mathbb{Y}_{t-1}\eta\alpha^T + \sum_{i=1}^{p} \Delta\mathbb{Y}_{t-i}\Gamma_i + U_t,$$

which shows that $\mathbb{Y}_{t-1}\eta$ are the cointegrating relations and η the cointegrating vector.

Note that we have written the system as an Error-Correction Model. In this sense, α captures the short-term adjustment parameter to deviations in the long-run equilibrium.

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Cointegration Multivariate Model

The Johansen method for cointegration estimates the model by ML and test for r cointegrating relations by looking at the eigenvalues.
Let λ_i be the eigenvalues in descending order.

- The **trace test**: for $H_0 : r = r_1$ versus $H_1 : r = r_2$ for $r_1 < r_2 \leq g$ we compute the statistic

$$-n \sum_{i=r_1+1}^{r_2} \log(1 - \lambda_i).$$

- The **maximum eigenvalue test**:for $H_0 : r = r_0$ versus $H_1 : r = r_0 + 1$ we compute the statistic

$$-n \log(1 - \lambda_{r+1}) = -n \log(1 - \lambda_{max}),$$

where λ_{max} is the largest eigenvalue not incorporated in the cointegrating relations.

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Summing Up

- Some time series show deterministic and/or stochastic trends that do not correspond to stationary processes.
- Unit root processes violate the standard assumptions made in OLS.
- One notable consequence of the violation of the assumptions is the uncovering of spurious regressions: finding statistical significance between independent series.
- It is thus imperative that we test if our series are stationary.
- A notable exception occurs when there is a stationary linear relation between nonstationary series: cointegration.
- Cointegration implies that the processes may differ in the short-run but will return to a long-run equilibrium.