Multivariate Time Series, part 1

Stationary Dynamic Models: ECM and VAR

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Introduction

One reason why it may be more interesting to consider several series simultaneously is that it may improve forecasts. The history of a variable, X_t, may help forecasting future values of Y_t.

Today we consider multivariate time series models: explaining one variable from its own past including current or lagged values of other variables.

We typically require stationarity to apply standard estimation or testing procedures in a dynamic time series model.

The use of nonstationary variables may result in invalid estimators and spurious regression, (see part 2).

An important exception arises when two or more I(1) variables are **cointegrated**; that is, there exists a linear combination which is stationary. In such cases a long-run relationship between these variables exists, see part 2.

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Autoregressive Distributed Lag

Consider the $ADL(1,1)$ given by

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \gamma_0 X_t + \gamma_1 X_{t-1} + u_t \quad u_t \sim IID(0, \sigma^2).$$

- If $\beta_1 = \gamma_1 = 0$, the model collapses to a standard static regression.
- When $\gamma_0 = \gamma_1 = 0$, we have a univariate $AR(1)$ model.
- When $\gamma_1 = 0$, we have a partial adjustment model.
- If $\gamma_1 = -\beta_1\gamma_0$, the model collapses to a static regression model with $AR(1)$ errors.
- When $\beta_1 = 1$ and $\gamma_1 = -\gamma_0$, we have a model in first differences.

We can tests for these special cases using t or F tests.

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Error-Correction Models

Moreover, under the condition that $|\beta_1| < 1$, the $ADL(1,1)$ model can be written as an Error-Correction Model (EDM) given by

$$\Delta Y_t = \beta_0 + \gamma_0 \Delta X_t + (\beta_1 - 1)(Y_{t-1} - \lambda X_{t-1}) + u_t,$$

where λ is defined as $\lambda = \frac{\gamma_0 + \gamma_1}{1 - \beta_1}$.

The term $(\beta_1 - 1)(Y_{t-1} - \lambda X_{t-1})$ is the error-correction term.

The difference between Y_{t-1} and λX_{t-1} measures the extent to which the long-run equilibrium relationship is not satisfied.

In this sense, the parameter $\beta_1 - 1$ can be interpreted as the short-term adjustment parameter from deviations from the long-run equilibrium.

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Vector Autoregressions

In single-equation models we express one variable in terms of the others.

Nonetheless, we often want to model the dynamic relationships among several time series variables

A simple way to do so without making many assumptions is to use a vector autoregression (VAR) model, the multivariate analog of an autoregressive model for a single time series.

Vector Autoregressions

Let $\mathbb{Y}_t$ be a $1 \times g$ (row) vector denoting the t-th observation of a set of g variables, a vector autoregressive model of order p, $VAR(p)$, can be written as

$$\mathbb{Y}_t = \alpha + \sum_{j=1}^{p} \mathbb{Y}_{t-j}\Phi_j + U_t, \quad U_t \sim IID(0, \Sigma),$$

where U_t is a $1 \times g$ vector of error terms. Note that Φ_j is $(g \times g)$.

If Y_{ti} denotes the i-th element of $\mathbb{Y}_t$ and $\phi_{j,ki}$ denotes the ki-th element of Φ_j, then

$$Y_{ti} = \alpha_i + \sum_{j=1}^{p} \sum_{k=1}^{g} Y_{t-j,k}\phi_{j,ki} + u_{ti},$$

which shows that it is a linear regression on a constant and p lags of all g variables. Thus the VAR has the form of a SUR model (see sect. 12.2.

Vector Autoregressions

Estimation can be done by OLS followed by maximizing the concentrated likelihood function.
The maximised loglikelihood function is

$$-\frac{gn}{2}(\log 2\pi + 1) - \frac{n}{2}\log|\hat{\Sigma}|,$$

where

$$\hat{\Sigma} = \frac{1}{n}\sum_{t=1}^{n}\hat{U}_t^T\hat{U}_t.$$

We can use the loglikelihood to test the number of lags in the system. To test the null of p lags against $p+1$, we compute the LR statistic

$$n(\log|\hat{\Sigma}(p)| - \log|\hat{\Sigma}(p+1)|),$$

which approximately follows a χ^2 with g^2 degrees of freedom.
Additionally, AIC and BIC can be used.

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Vector Autoregressions

The long-run equilibrium of $\mathbb{Y}_t$ can be determined if we assume stationarity.

Note that

$$E[\mathbb{Y}_t] = \alpha + \sum_{j=1}^{p} E[\mathbb{Y}_{t-j}]\Phi_j,$$

thus,

$$\mu := E[\mathbb{Y}_t] = \Phi(1)^{-1}\alpha,$$

where we have assumed that $\Phi(1) := I - \sum_{j=1}^{p} \Phi_j$ is invertible.

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Impulse-Response Function

(Also see `https://www.r-econometrics.com/timeseries/irf/`) If the *VAR* has a long-run equilibrium; that is, if $\Phi(1)$ is invertible, we can write the model as a Vector Moving Average (*VMA*) model by premultiplying with $\Phi(L)^{-1}$,

$$\mathbb{Y}_t = \mu + \Phi(L)^{-1} U_t.$$

This describes $\mathbb{Y}_t$ as a weighted sum of all current and past shocks in the system.

Write $\Phi(L)^{-1} = I + A_1 L + A_2 L^2 + \cdots$, then

$$A_s = \frac{\partial \mathbb{Y}_{t+s}}{U_t},$$

measures the effect of a one-unit increase in U_t.

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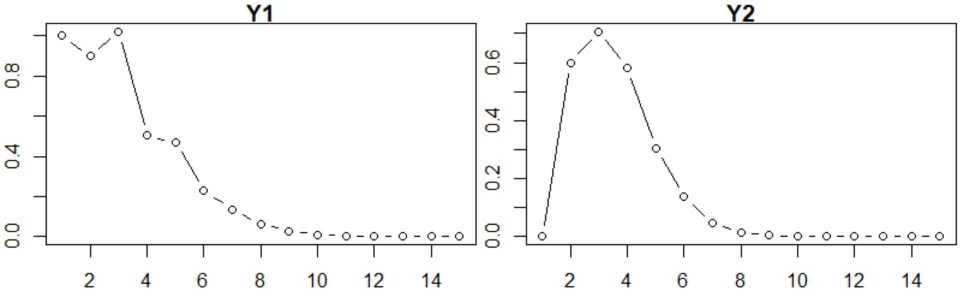
Impulse-Response Function

Impulse-Response Function in a bivariate $VAR(1)$ given by

$$\mathbb{Y}_t = \begin{bmatrix} 0.9 & -0.7 \\ 0.6 & 0.8 \end{bmatrix} \mathbb{Y}_{t-1} + U_t$$

to a shock in the first variable.



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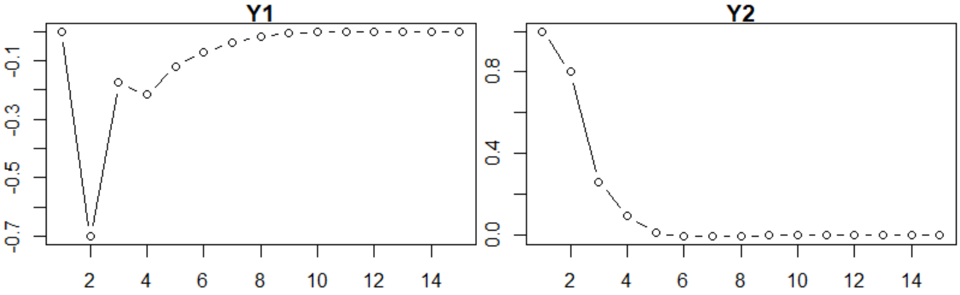

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Granger Causality

One common use of vector autoregressions is to test if one or more of the variables do not "Granger cause" the others.

Suppose we divide the variables in a *VAR* into two groups, Y_{t1} and Y_{t2}. We say that Y_{t2} does not Granger cause Y_{t1} if the distribution of Y_{t1}, conditional on past values of both Y_{t1} and Y_{t2}, is the same as the distribution of Y_{t1} conditional only on its own past values.

In practice, it is very difficult to analyse the entire distribution, and we almost always content ourselves with asking whether the conditional mean of Y_{t1} depends on past values of Y_{t2}.

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Granger Causality

We can rewrite the *VAR* as

$$[Y_{t1}\;\; Y_{t2}] = [\alpha_1\;\; \alpha_2] + \sum_{j=1}^{p} [Y_{t-j,1}\;\; Y_{t-j,2}] \begin{bmatrix} \Phi_{j,11} & \Phi_{j,12} \\ \Phi_{j,21} & \Phi_{j,22} \end{bmatrix} + [U_{t1}\;\; U_{t2}].$$

If Y_{t2} does not cause Y_{t1}, then all $\Phi_{j,21}$ must be zero matrices.

Similarly, if Y_{t1} does not cause Y_{t2}, then all $\Phi_{j,12}$ must be zero matrices.

We can test either hypothesis by estimating the restricted and unrestricted models by OLS and compute the LR statistic as before.

In practice, we are very commonly interested in testing Granger causality for a single dependent variable. In that case, the specification is a univariate regression and we can then perform an asymptotic F test.

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Summing Up

- Some time series show a relationship such that a change to one of the variables may affect current or future values of another.
- We may use of this dynamics to strengthen the analysis and produce better forecasts.
- If the effect is unidirectional, one variable is affected by the other but without feedback effects, then we can model it using single-equation models.
- If such a feedback effect exists, we can model all of the variables simultaneously using a multivariate model.
- Under stationarity, dynamic models allows us to obtain long-run equilibriums.
- Short-run and long-run effects can be tested using standard tests.

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