Seemingly Unrelated Regressions

Econometrics

J. Eduardo Vera-Valdés

eduardo@math.aau.dk

Department of Mathematical Sciences
Aalborg University
Denmark

AALBORG UNIVERSITY
DENMARK

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J. Eduardo Vera-Valdés
eduardo@math.aau.

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Aalborg University
Denmark

Introduction

Up to this point, almost all models we have discussed have involved just one equation with one dependent variable.

Today, we will discuss models which jointly determine the values of two or more dependent variables using two or more equations.

Such models are called multivariate because they attempt to explain multiple dependent variables.

Examples of multivariate models are: capital asset pricing models, investment and production decisions by firms, inflation rates for different countries, etc.

J. Eduardo Vera-Valdés
eduardo@math.aau.

Department of Mathematical Sciences
Aalborg University
Denmark

Multivariate Models

We suppose that there are g dependent variables indexed by i.

Let Y_i denote the dependent variable, X_i denote the $n \times k_i$ matrix of regressors for the i-th equation, β_i denote parameters, and U_i the error terms.

Then the i-th equation of a multivariate linear regression model may be written as

$$Y_i = X_i\beta_i + Ui, \quad E(U_i Ui^T) = \sigma_{ii} I_n.$$

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Multivariate Models

Equation by equation, the expression

$$Y_i = X_i \beta_i + Ui, \quad E(U_i Ui^T) = \sigma_{ii} I_n,$$

is a linear regression that satisfies the OLS assumptions. We can estimate it by OLS if we assume that X_i are either exogenous or predetermined.

However, we ignore the possibility that the error terms may be correlated across the equations of the system. For example, a financial shock may affect the price of more than one stock.

We look to extract more information from the data by modelling this correlation. The method to do so is called **Seemingly Unrelated Regressions (SUR)**.

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SUR

The SUR model assumes that

$$E[U_{ti}U_{tj}] = \sigma_{ij} \quad \forall t,$$

and

$$E[U_{ti}U_{sj}] = 0 \quad \forall t \neq s,$$

where σ_{ij} is the (i, j) element of the $g \times g$ positive definite matrix Σ.

The assumption allows the errors to be contemporaneously correlated, but they are homoskedastic and independent across time.

The matrix Σ is called the contemporaneous covariance matrix. The method to estimate the parameters of the system depend on the assumptions we are willing to make on Σ.

J. Eduardo Vera-Valdés
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To estimate the model, we write the system as

$$Y_\bullet = X_\bullet \beta_\bullet + U_\bullet,$$

where we have stacked the observations vertically like $Y_\bullet = [Y_1^T, \cdots, Y_g^T]^T$.

Note that the errors' covariance matrix is given by

$$E[U_\bullet U_\bullet^T] = \Sigma \otimes I_n$$

Thus, OLS estimates of $\beta_\bullet$ will be consistent but inefficient.

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Assume to begin that we know Σ completely, we can use that information to estimate the system by GLS.

The GLS estimator is given by

$$\hat{\beta}_{\bullet}^{GLS} = (X_{\bullet}^{T}\Sigma_{\bullet}^{-1}X_{\bullet})^{-1}(X_{\bullet}^{T}\Sigma_{\bullet}^{-1}Y_{\bullet}),$$

where $\Sigma_{\bullet} = \Sigma \otimes I_n$.

Its covariance matrix is

$$Var(\hat{\beta}_{\bullet}^{GLS}) = (X_{\bullet}^{T}\Sigma_{\bullet}^{-1}X_{\bullet})^{-1}.$$

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Alternatively, assume that we know Δ where $\Sigma = \sigma^2\Delta$; that is, we know the covariance matrix up to a constant.

The GLS estimator is the same as before, and we estimate its covariance matrix by

$$Var(\hat{\beta}_{\bullet}^{GLS}) = \hat{\sigma}^2(X_{\bullet}^T(\Delta^{-1} \otimes I_n)X_{\bullet})^{-1},$$

where

$$\hat{\sigma}^2 = \frac{1}{gn}\hat{U}_{\bullet}^T(\Delta^{-1} \otimes I_n)\hat{U}_{\bullet}.$$

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In practice, Σ is not known (even up to a constant) so we have to estimate it.

Analogous to our general discussion on FGLS, we can use the residuals from OLS to construct estimates of the covariance matrix.

Thus, let

$$\hat{\Sigma} = \frac{1}{n} \hat{U}_{OLS}^T \hat{U}_{OLS},$$

where $\hat{U}_{OLS}$ is a $n \times g$ matrix with the OLS residuals.

The FGLS estimator is thus a similar expression as before replacing Σ with $\hat{\Sigma}$.

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There are two special cases where estimating a model using the SUR is not more efficient as just estimating each equation by OLS:

- If the equations are truly unrelated; that is, the matrix Σ is diagonal.
- When the regressors in all equations are the same.

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Since the diagonality of Σ is at the heart of using SUR estimation methods, it is important to test if whether it is diagonal.

Breusch and Pagan derived a simple and easy to use statistic based on the sample correlation coefficients of the OLS residual.

Let

$$LM = T \sum_{i=2}^{g} \sum_{j=1}^{i-1} r_{ij}^2,$$

where

$$r_{ij} = \frac{\hat{s}_{ij}}{(\hat{s}_{ii}\hat{s}_{jj})^{1/2}}, \quad \text{and} \quad \hat{s}_{ij} = \frac{1}{T} \sum \hat{U}_{it}^{OLS} \hat{U}_{jt}^{OLS}$$

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Under the null,

$$LM = T \sum_{i=2}^{g} \sum_{j=1}^{i-1} r_{ij}^2$$

follows a chi-square with $g(g-1)/2$ degrees of freedom.

For example, for the two equations case, $LM = Tr_{21}^2 \sim \chi_1^2$. For the three equations case, $LM = T(r_{21}^2 + r_{31}^2 + r_{32}^2) \sim \chi_3^2$.

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Another test of interest is whether some parameters between equations are the same.

We can use a Wald-type test. Define test statistic by

$$W = (R\hat{\beta}_{\bullet} - r)^T [RVar(\hat{\beta}_{\bullet})R^T]^{-1}(R\hat{\beta}_{\bullet} - r),$$

which follows a chi-square distribution with degrees of freedom equal to the number of restrictions. Where R and r are matrices for the linear restrictions.

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Summing Up

- We have studied models for multiple dependent variables.
- We show that if the error terms are contemporaneously related, which may be the case if there is a simultaneous shock to all equations, OLS is not efficient.
- We can recover efficiency if we estimate the model by GLS or FGLS.
- GLS is numerically the same as OLS when the equations are truly unrelated or when they have the same regressors.
- We can make tests regarding the covariance matrix of the regressors to check if the equations are related.
- We can make hypothesis tests on whether some parameters should be the same across equations.

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