Generalized Method of Moments

Econometrics

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Introduction

Generalized method of moments (GMM) provides a unifying framework for the analysis of many familiar estimators.

It includes OLS, IV, and GLS as special cases.

It offers a convenient method of estimation in certain models which were computationally very burdensome to estimate by traditional methods.

The estimation relies on population moments whose expectation is zero when evaluated at the true parameters.

GMM estimate is then obtained by minimizing a quadratic form in the analogous sample moments.

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Example

Instrumental variables can be obtained as a GMM estimator.

We obtain a moment condition per each instrument.

If the number of instruments is the same as the number of regressors we recover IV.

For more instruments than regressors, we obtain GIV.

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GMM

Suppose we have r moment conditions given by

$$E[f(x_t, \beta_0)] = 0,$$

where x is a stationary ergodic random vector, f is a $r \times 1$ vector of (possibly nonlinear) continuous functions of β a $q \times 1$ parameter vector, and β_0 is the true value of β.

We further assume that a law of large numbers can be applied to f so that the sample mean converges a.s. to the population mean,

$$\lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} f(x_t, \beta) = E[f(x_t, \beta)].$$

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The core principle of GMM is to estimate the parameters by using a quadratic form on the sample moments.

Let $g_n(\beta) = n^{-1} \sum_{t=1}^{T} f(x_t, \beta)$, and define the quadratic form

$$Q_n(\beta) = g_n'(\beta) W_n g_n(\beta),$$

where W_n is a positive definite symmetric matrix which converges in probability to a positive definite symmetric matrix W.

The GMM estimator for β is the value that minimizes $Q_n(\beta)$.

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The first order conditions are given by

$$G_n'(\hat{\beta})W_n g_n(\hat{\beta}) = 0,$$

where $G_n(\hat{\beta})$ is the $r \times q$ matrix given by $[G_n(\hat{\beta})]_{ij} = \partial g_{ni}(\beta)/\partial\beta_j$.

If the system is just identified, $r = q$, the matrices $G_n(\hat{\beta})$ and W_n are nonsingular, so that the solution does not depend on W_n.

Unlike the linear case, the f.o.c. does not imply an analytical solution, so they must be solved numerically.

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Although we cannot obtain an explicit solution, it can be shows that GMM is a consistent estimator.

From the population moment condition and the LLN, we have that $g_n(\beta_0) \to 0$.

W_n positive definite implies that $Q_n(\beta) \geq 0 \; \forall \beta$. Moreover, we assume that the model is asymptotically identified; that is, $E[f(x_t, \beta)] \neq 0 \; \forall \beta \neq \beta_0$.

Thus, if $\hat{\beta}$ minimizes $Q_n()$, then $\hat{\beta} \to \beta_0$.

Note that the estimator is consistent regardless of the W_n matrix used.

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GMM

Asymptotic Normality

To show that GMM is asymptotically normal, we use a first-order Taylor expansion approximation of $g_n(\beta)$ around β_0,

$$g_n(\hat{\beta}) = g_n(\beta_0) + G_n(\bar{\beta})(\hat{\beta} - \beta_0) + o_p(n^{-1}),$$

where $||\hat{\beta} - \bar{\beta}|| \leq ||\hat{\beta} - \beta_0||$.

Premultiplying both sides by $G_n'(\hat{\beta})W_n$, the left-hand side is zero by construction, so asymptotically,

$$n^{1/2}(\hat{\beta} - \beta_0) = -[G_n'(\beta_0)W_nG_n(\beta_0)]^{-1}G_n'(\beta_0)W_nn^{1/2}g_n(\beta_0).$$

Hence, under regularity conditions that guarantee that the CLT for $g_n(\beta_0)$ holds, $n^{1/2}(\hat{\beta} - \beta_0)$ follows a normal distribution.

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From the previous derivation, note that the variance of $n^{1/2}(\hat{\beta} - \beta_0)$ is given by

$$[G_n'(\beta_0)W_nG_n(\beta_0)]^{-1}G_n'(\beta_0)W_nS_WW_nG_n(\beta_0)[G_n'(\beta_0)W_nG_n(\beta_0)]^{-1},$$

where $S_W = E[f(x_t,\beta_0)f'(x_t,\beta_0)]$ is the variance-covariance matrix for $g_n(\beta_0)$.

Note that it has a sandwich form, which suggests that it is an inefficient estimator.

We obtain the efficient estimator by taking $W_n = S_w^{-1}$.

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We do not know S_w in general.

Hence, we can use a multi-step estimation methodology:

- We use an initial (most likely inefficient) W_0 matrix to estimate $\hat{\beta}$ and construct an estimate for $\hat{S}_w$. A typical implementation will make $W_0 = I$.
- We use our estimate for $\hat{S}_w$ in a second iteration of GMM.
- Keep iterating if necessary.

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Test for Overidentifaction

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Similar to the GIV case, we can/should test the model specification using the overidentifiying restrictions.

The test-statistic is given by

$$\tau_n = n g_n'(\hat{\beta}) \hat{S}_w^{-1} g_n(\hat{\beta}),$$

which follows a chi-square distribution with $r - q$ degrees of freedom.

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Summing Up

- We have studied a method to obtain consistent estimates based on population moments.
- The method relies on minimizing a quadratic form in the sample moments.
- There must be at least as many moment conditions as parameters to estimate.
- GMM can estimate nonlinear models.
- Weak instruments can have perverse finite sample effects on the estimation.
- The test for overidentifaction allows us to check the validity of the moment conditions.

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