Hausmann:

$$Cov(\hat\beta^{ols}, \hat\beta^{ols} - \hat\beta^{IV}) = 0 \quad Var(\hat\beta^{ols}) - Cov(\hat\beta^{ols}, \hat\beta^{IV})$$

$$Var(\hat\beta^{IV} - \hat\beta^{ols}) =$$

$$Var(\hat\beta^{IV}) + Var(\hat\beta^{ols}) - Cov(\hat\beta^{IV}, \hat\beta^{ols}) - Cov(\hat\beta^{IV}, \tilde\beta^{ols})$$

$$= Var(\hat\beta^{IV}) - Var(\hat\beta^{ols})$$

$Cov(\hat\beta^{E}, \hat\beta^{E} - \hat\beta^{I})$, $\left.\begin{array}{l}\hat\beta^{E} \to \text{efficient} \\ \hat\beta^{I} \to \text{inefficient}\end{array}\right\}$ both consistent

$q = \hat\beta^{E} - \hat\beta^{I}$ to show that $Cov(\hat\beta^{E}, q) = 0$

$$\text{plim}\, q = \text{plim}\,(\hat\beta^{E} - \hat\beta^{I}) = \beta - \beta = 0$$

Define $\hat\beta^{0} = \hat\beta^{E} + rAq$, $r \in \mathbb{R}$, A inv matrix

$\hat\beta^{0}$ is consistent

$$Var(\hat\beta^{0}) = Var(\hat\beta^{E} + rAq) = Var(\hat\beta^{E}) + r^2 A Var(q) A^T$$
$$+ rA\,Cov(\hat\beta^{E}, q) + r\,Cov^T(\hat\beta^{E}, q) A^T$$

$$F(r) = Var(\hat\beta^{0}) - Var(\hat\beta^{E})$$

$F(0) = 0 \qquad F(r) \geq 0$ pos. semi. def.

$$F'(r) = 2rA\,Var(q)A^T + A\,Cov(\hat\beta^{E}, q) + Cov^T(\hat\beta^{E}, q) A^T$$

Let $A = -\mathrm{cov}^T(\hat{\beta}^E, q)$ and compute $F'(0)$

$$F'(0) = -\mathrm{cov}(\hat{\beta}^E, q)\mathrm{cov}(\hat{\beta}^E, q)$$
$$- \mathrm{cov}(\hat{\beta}^E, q)\mathrm{cov}(\hat{\beta}^E, q)$$
$$= -2\,\mathrm{cov}^T(\hat{\beta}^E, q)\mathrm{cov}(\hat{\beta}^E, q) \leq 0$$

unless $\mathrm{cov}(\hat{\beta}^E, q) = 0$

thus $F'(0) \leq 0$, $F(r) \geq 0$, $F(0) = 0$