Feilanalyse trapez

$f: [a,b] \to \mathbb{R}$ kont., C^2

variabelskift $f \to e$

$x = a + t(b-a)$, $t \in [0,1]$

$F(t) = f(a + t(b-a))$,

$\int_a^b f(x)\,dx = (b-a) \int_0^1 F(t)\,dt$

$E = \int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b))$

$= (b-a)\underbrace{\left[\int_0^1 F(t)\,dt - \frac{1}{2}(F(0)+F(1))\right]}_{I}$

$G(t) = \int_0^t F(t)\,dt - \frac{t}{2}(F(0)+F(t))$

$I = G(1)$

Reduceret til beregning af $G(1)$

$H(t) = G(t) - t^3 G(1)$

$H(0) = 0$, $H(1) = 0$, Rolles sætning giver

der findes et $\xi_1 \in (0,1)$, så $H'(\xi_1) = 0$

$H'(t) = F(t) - \frac{1}{2}(F(0) + F(t)) - \frac{t}{2}F'(t) - 3t^2 G(1)$

$H'(0) = F(0) - \frac{1}{2}(F(0) + F(0)) = 0$

Rolles sætning giver at der findes et $\xi_2 \in (0, \xi_1)$, så at $H''(\xi_2) = 0$

$$\underbrace{H''(t) = F'(t) - \frac{1}{2}F'(t) - \frac{1}{2}F'(t)}_{0} - \frac{t}{2}F''(t) - 6tG(1)$$

$$0 = H''(\xi_2) = -\frac{\xi_2}{2}F''(\xi_2) - 6\xi_2 G(1)$$

Heraf

$$G(1) = -\frac{1}{12}F''(\xi_2)$$

$$F''(t) = (b-a)^2 f^{(2)}(a + t(b-a))$$

altså

$$G(1) = -\frac{1}{12}(b-a)^2 f^{(2)}\underbrace{(a + \xi_2(b-a))}_{\xi}$$

Vi har vist, at der findes $\xi \in (a,b)$ så at

$$\int_a^b f(x)\,dx - \frac{b-a}{2}(f(a)+f(b))$$
$$= -\frac{1}{12}(b-a)^3 f^{(2)}(\xi)$$

M: $\int_a^b f(x)\,dx - (b-a) f\left(\frac{b+a}{2}\right) = \frac{(b-a)^3}{24} f^{(2)}(\xi_M)$

T: $\int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b)) = -\frac{(b-a)^2}{12} f^{(2)}(\xi_T)$

S: $\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right] = -\frac{(b-a)^3}{2880} f^{(3)}(\xi_S)$

Sammensat trapez

Del $[a,b]$ i N delintervaller, længde $h = \frac{b-a}{N}$

$a = x_0 < x_1 \dots < x_N = b$

$x_j = a + jh \quad , j = 0, 1, \dots, N$

$$\int_a^b f(x)\,dx = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x)\,dx = \sum_{j=0}^{N-1} \frac{h}{2}\left(f(x_j) + f(x_{j+1})\right)$$

$$+ \sum_{j=0}^{N-1} \frac{h^3}{12} f^{(2)}(\xi_j)$$

Omskriv fejlleddet

$$-\frac{h^2}{12}(b-a) \cdot \frac{1}{N} \sum_{j=0}^{N-1} f^{(2)}(\xi_j)$$

$$\min_j f^{(2)}(\xi_j) \leq \frac{1}{N} \sum_{j=0}^{N-1} f^{(2)}(\xi_j) \leq \max_j f^{(2)}(\xi_j)$$

mellem-liggende værdi sætn. giver

$$\frac{1}{N} \sum_{j=0}^{N-1} f^{(2)}(\xi_j) = f^{(2)}(\xi_T)$$

sammensat trapez fejlled er

$$-\frac{1}{12} h^2 (b-a) f^{(2)}(\xi_T)$$