13. Error estimates for quadrature rules.

Numerical Analysis E2021

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Quadrature rules

Basic rules:

- **Midpoint rule** $M = hf\left(\frac{a+b}{2}\right)$,
- **Trapezoid rule** $T = h\frac{f(a)+f(b)}{2}$,
- **Simpson's rule** $S = \frac{h}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$.

Error estimates

Proposition. Let $f : [a,b] \to \mathbb{R}$ be n times continuously differentiable. Then for each $x \in [a,b]$ there exists a $\psi \in (a,b)$ such that

$$f(x) - p(x) = \frac{\prod_{j=1}^{n}(x - x_j)}{n!} f^{(n)}(\psi),$$

where $p(x)$ is the polynomial that interpolates $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

Error estimates

Apply proposition to quadrature rules to obtain

$$\int_a^b f(x)dx - hf((a/b)/2) = \frac{h^3}{24} f^{(2)}(\psi_M)$$

$$\int_a^b f(x)dx - \frac{h}{2}\Big(f(a) + f(b)\Big) = \frac{h^3}{12} f^{(2)}(\psi_T)$$

$$\int_a^b f(x)dx - \frac{h}{6}(f(a) + 4f((a/b)/2) + f(b)) = \frac{h^5}{180} f^{(4)}(\psi_S),$$

respectively, where ψ_M, ψ_T, ψ_S are points in the interval.

Error estimate of composite trapezoid rule

Given $f:[a,b]\to\mathbb{R}$ and $N\geq 2$ we divide $[a,b]$; let $h=(b-a)/N$ and $x_j=a+(j-1)h$, $j=1,\ldots,N+1$. Then $x_1=a$ and $x_{N+1}=a+Nh=b$.
Assume we have a bound $|f^{(2)}(\psi)|\geq K_2$ for all $\psi\in[a,b]$. Apply the single-step estimate to each subinterval $[x_j,x_{j+1}]$

$$T^j=\frac{h}{2}\Big(f(x_j)+f(x_{j+1})\Big),$$

thus we obtain

$$\int_{x_j}^{x_{j+1}} f(x)dx-T^j=-\frac{h^3}{12}f^{(2)}(\psi).$$

Error estimate of composite trapezoid rule

Composite trapezoid rule:

$$T_N = \frac{h}{2} f(x_1) + h \sum_{j=2}^{N-1} f(x_j) + \frac{h}{2} f(x_N).$$

Error estimate

$$\left| \int_a^b f(x) dx - T_N \right| \leq \sum_{j=1}^{N} \frac{h^3}{12} |f^{(2)}(\psi_j)| \leq \frac{K_2}{12} N h^3 = \frac{K_2}{12} (b-a) h^2.$$

Other error estimates

Composite midpoint rule and error estimate

$$M_N = h \sum_{j=1}^{N} f((x_j + x_{j+1})/2),$$

$$\left| \int_a^b f(x)dx - M_N \right| \leq \frac{K_2}{24}(b-a)h^2.$$

Composite Simpson's rule and error estimate

$$S_N = 2/3 M_N + 1/3 T_N,$$

$$\left| \int_a^b f(x)dx - S_N \right| \leq \frac{K_4}{180}(b-a)h^4,$$

Assuming $|f^{(4)}(\psi)| \leq K_4$ for all $\psi \in [a,b]$.