14. Present some single step methods for solving an IVP for a system of ODE. Compare the methods presented.

Numerical Analysis E2021

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ODE initial value problems

The initial value problem for an ordinary differential equation involves finding a function $y(t)$ that satisfies

$$\frac{dy(t)}{dt} = f(t, y(t))$$

together with the initial condition $y(t_0) = y_0$. A numerical solution to this problem generates a sequence of values for the independent variable, $t_0, t_1, \ldots$, and a corresponding sequence of values for the dependent variable, $y_0, y_1, \ldots$, so that each y_n approximates the solution at t_n:

$$y_n \approx y(t_n), n = 0, 1, \ldots$$

Euler's method Uses fixed step size h and approximates solution

$$
\begin{aligned}
y_{n+1} &= y_n + hf(t_n, y_n) \\
t_{n+1} &= t_n + h.
\end{aligned}
$$

Problem: this method does not provide an error estimate. There is no automatic way to determine what step size is needed to achieve a specified accuracy.

Two natural improvements, both needing an additional evaluation of f.

Single-step methods

Midpoint and trapezoid analogues

Midpoint analogue: uses Euler to step halfway across the interval, evaluates the function at this intermediate point, then uses that slope to take the actual step:

$$\begin{aligned}
s_1 &= f(t_n, y_n), \\
s_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_1\right), \\
y_{n+1} &= y_n + h s_2, \\
t_{n+1} &= t_n + h.
\end{aligned}$$

Trapezoidal analogue: uses Euler to take a tentative step across the interval, evaluates the function at this exploratory point, then averages the two slopes to take the actual step:

$$\begin{aligned}
s_1 &= f(t_n, y_n), \\
s_2 &= f(t_n + h, y_n + h s_1), \\
y_{n+1} &= y_n + h \frac{s_1 + s_2}{2}, \\
t_{n+1} &= t_n + h.
\end{aligned}$$

Single-step methods

Runge-Kutta 4

Acts as a **Simpson's analogue**, however uses four evaluations of f per step:

$$\begin{aligned}
s_1 &= f(t_n, y_n),\\
s_2 &= f(t_n + h/2, y_n + hs_1/2),\\
s_3 &= f(t_n + h/2, y_n + hs_2/2),\\
s_4 &= f(t_n + h, y_n + hs_3),\\
y_{n+1} &= y_n + h/6(s_1 + 2s_2 + 2s_3 + s_4),\\
t_{n+1} &= t_n + h.
\end{aligned}$$

MATLAB demo of comparision of methods.