6. Polynomial interpolation. General results. Comparison of polynomial interpolation and piecewise polynomial interpolation. Use `interpgui` to give examples.

Numerical Analysis E2021

Institute of Mathematics
Aalborg University

AALBORG UNIVERSITY
DENMARK

Lagrange Interpolation

For a given set of data points (x_i, y_i), with no x_i's being equal, we wish to interpolate a function that describes this data.

Lagrange Interpolation Theorem

Let $\mathbb{F}$ be a field, and let $m \in \mathbb{N}$. If $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ are all distinct, then the evaluation map $\mathcal{E} \colon \mathbb{F}[X]_{\leq m-1} \to \mathbb{F}^m$, such that $f(X) \mapsto (f(\alpha_1), \ldots, f(\alpha_m))$, is an isomorphism.

Proof.

$\mathcal{E}$ is a linear map, and the two vector spaces has the same dimension. Thus, it suffices to show that it is injective. This fact immediately follows by the fundamental theorem of algebra. $\square$

In other words: if we have m ordered pairs $(\alpha_1, y_1), \ldots, (\alpha_m, y_m)$ all in $\mathbb{F}^2$, such that $\alpha_1, \ldots, \alpha_m$ are all distinct. Then there exists a unique polynomial $f \in \mathbb{F}[X]_{\leq m-1}$, which satisfies $f(\alpha_i) = y_i$ for $i = 1, \ldots, m$.

Constructing an inverse

Let $\mathbb{F}$ be a field, and let $m \in \mathbb{N}$ such that $m \geq 1$. Furthermore, let $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ be pairwise distinct. Then

$$\delta_i(X) := \prod_{\substack{k=1 \\ k \neq i}}^{m} \frac{X - \alpha_k}{\alpha_i - \alpha_k} \tag{1}$$

for $i = 1, \ldots, m$. These are then clearly well-defined, and satisfy

1. $\deg(\delta_i) = m - 1$
2. $\delta_i(\alpha_i) = 1$
3. $\delta_i(\alpha_j) = 0$ if $1 \leq j \leq m$ and $j \neq i$

Proposition.
The inverse of $\mathcal{E}$ is the map $\mathcal{E}^{-1} : \mathbb{F}^m \to \mathbb{F}[X]_{\leq m-1}$, such that $(y_1, \ldots, y_m) \mapsto \sum_{i=1}^{m} y_i \delta_i(X)$.

Assume that $f : [a, b] \to \mathbb{R}$ such that $\alpha_j \in [a, b]$, where $j = 0, 1, \ldots, N$, are all distinct. Then $y_i := f(\alpha_i)$, and we say that the interpolating polynomium interpolates f. Then

Theorem

Assume that f is $N + 1$ times differentiable on $[a, b]$. Let p be the interpolating polynomial corresponding to the dataset $(\alpha_i, (f(\alpha_i))$. For every $x \in (a, b)$ there exists $\xi \in (a, b)$ such that

$$f(x) - p(x) = \frac{\prod_{i=0}^{N}(x - \alpha_i)}{(N + 1)!} f^{(N+1)}(\xi) \tag{2}$$

High-degree interpolation yields oscillatory polynomials, which can lead to poor results. See MATLAB demo of Runge. A way around this is to use piecewise interpolation.

Piecewise polynomial interpolation

We have several methods for polynomial interpolation:

- Piecewise linear interpolation
  - Hardly any smoothness
  - Preserves local monotonicity
- Lagrange interpolation
  - Infinitely differentiable
  - Often fails to preserve shape, particularly near ends, as seen with Runge
- pchip
  - Guaranteed to preserve shape
  - Discontinuous curvature due to non-continuous second-order derivaties
- spline
  - Has continuous second-order derivatives
  - Not guaranteed to preserve shape

MATLAB demo of `interpgui`