9. Zero finding for real functions of one variable. Present Newton's method and discuss its advantages and disadvantages. Present your solution to Exercise 4.3 in Moler.

Numerical Analysis E2021

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Motivation

For function of one variable we are often interested in solving for x in the following equation

$$F(x) = y \tag{1}$$

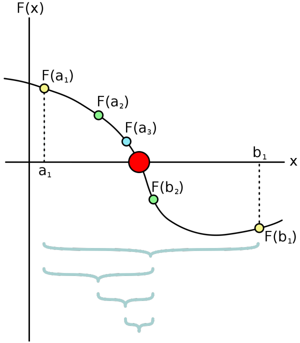
which can be transformed into a root-finding problem by defining $f(x) := F(x) - y$, resulting in

$$f(x) = 0 \Leftrightarrow F(x) = y \tag{2}$$

Thus, we wish to have methods for determining zeroes.

Bisection

If we have an interval $[a, b]$ where the function changes sign, then bisection is a slow but sure method, where we in every iteration bisect the search interval.



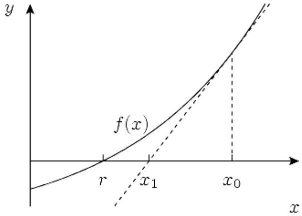
In IEEE double-precision it is guaranteed to find two successive floating-point numbers in 52 steps.

Newton's Method

Newton's Method is based on the following iterative process

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

the idea is we draw the tangent to the graph, and determine this tangents intersection with the x-axis to get our next point.



However, f must be smooth, f' should be nice to compute, and starting guess must be close to root.

However, we do have nice local convergence properties, **if** f' and f'' exist, and are continuous, and our initial point is close to our computed solution. Furthermore,

$$e_{n+1} = \mathcal{O}(e_n^2) \tag{4}$$

Secant Method

The secant method is Newton's method where we replace the derivative with an approximation using the secant. That is

$$x_{n+1} := x_n - \frac{f(x_n)}{s_n}, \quad \text{where } s_n := \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{5}$$

The advantage over Newton's is of course then that we do not have to compute the derivative at every step. It also has similar convergence properties, and assuming continuity of f' and f'', one can prove that

$$e_{n+1} = \mathcal{O}(e_n^{\phi}) \tag{6}$$

Inverse Quadratic Interpolation

IQI uses the three previous points in order to determine the next one. For the point

$$(a, f(a)), \quad (b, f(b)), \quad (c, f(c)) \tag{7}$$

we interpolate the three points into a quadratic function of y, $P(y)$, such that

$$a = P(f(a)), \quad b = P(f(b)), \quad c = P(f(c)) \tag{8}$$

our next iterative step will then be $x = P(0)$. However, we require that our three points $f(a), f(b), f(c)$ are all distinct, which we cant guarantee. Furthermore, it can be shown that

$$e_{n+1} = \mathcal{O}(e_n^{1.84}) \tag{9}$$

Summary

We have several methods for polynomial interpolation:

- Bisection
  - Always has a solution, but slow
  - Linear convergence
- Newton's Method
  - Quick convergence guaranteed under assumptions
  - Quadratic convergence
- Secant Method
  - No convergence-guarantee.
  - Doesn't require derivative, but may have issues if root is a critical point
  - Superlinear convergence
- IQI
  - Converges quickly when close to a root
  - Can fail entirely
  - Poor if initial point is far from solution

MATLAB demo of exercise 4.3.