4. Error analysis for solutions to systems of linear equations. The condition number and its role in the analysis.

Numerical Analysis E2021

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Motivation

We wish to analyse errors for when we compute solutions to linear equations. Thus, we introduce the condition number

$$\kappa(A) = \|A^{-1}\| \cdot \|A\| \tag{1}$$

satisfying

$$\begin{aligned} \kappa(cA) &= \kappa(A), \quad c \neq 0 \\ \kappa(A^{-1}) &= \kappa(A) \\ \kappa(AB) &\leq \kappa(A)\kappa(B) \\ \kappa(Q) &= 1 \\ \kappa(QA) &= \kappa(AQ) = 1 \end{aligned} \tag{2}$$

for orthogonal Q.

κ is a measure of “how” singular a matrix is.

Non-pertubed error analysis

For invertible A and the system $Ax = b$ we consider

$$A(x_t + \underbrace{\delta x}_{\text{error}}) = b + \underbrace{\delta b}_{\text{res.}} \tag{3}$$

and one can then show that

$$\frac{1}{\kappa(A)} \frac{||\delta b||}{||b||} \leq \frac{||\delta x||}{||x_t||} \leq \kappa(A) \frac{||\delta b||}{||b||} \tag{4}$$

so κ is a relative error magnification factor.

Pertubed error analysis

We consider a case where our computed solution satisfies a pertubed form of the original equation

$$(A + \delta A)(x_t + \delta x) = (A + E)x_c = b \tag{5}$$

and under the assumption $||A^{-1}|| \cdot ||\delta A|| < 1$ then $A + \delta A$ is invertible. From this we can make estimates for the relative error, most importantly

$$\frac{||x_t - x_c||}{||x_t||} \leq \frac{\kappa(A)||E||/||A||}{1 - \kappa(A)||E||/||A||} \tag{6}$$

MATLAB

MATLAB demo for poorly conditioned system.