15. Error analysis for ODE IVP solvers.

Numerical Analysis E2021

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Errors in ODE problems

When solving ODEs numerically, we get errors from two sources:

- *Discretisation error:* a property of the differential equation and the applied method.
  - *Local:* the error that would be made in one step if the previous values were exact and if there were no roundoff error.
  - *Global:* the difference between the computed solution, still ignoring roundoff, and the true solution.
- *Roundoff error:* a property of the computer hardware and the program used.

Error in ODE problems

Discretisation error

Let u be a function given by

$$u_n' = f(t, y_n),$$
$$u_n(t_n) = y_n.$$

Then the local resp. global discretisation errors are given by

$$d_n = y_{n+1} - u_n(t_{n+1}), \quad e_n = y_n - y(t_n).$$

We say that an ODE solver is of order p if $d_n = \mathcal{O}(h_n^{p+1})$.

Order of Euler's method

Recall Euler's method

$$y_{n+1} = y_n + h_n f(t_n, y_n).$$

Assume the local solution $u_n(t)$ has a continuous second derivative. Then, using Taylor series near the point t_n,

$$u_n(t) = u_n(t_n) + (t - t_n)u_n'(t_n) + \mathcal{O}((t - t_n)^2).$$

Now, by using the differential equation and the initial condition defining $u_n(t)$, we obtain

$$u_n(t_{n+1}) = y_n + h_n f(t_n, y_n) + \mathcal{O}(h_n^2).$$

Consequently, we have

$$d_n = y_{n+1} - u_n(t_{n+1}) = \mathcal{O}(h_n^2),$$

thus Euler's method is of order 1.

Order and global error

Consider the global discretisation error at a fixed point t_f. As accuracy requirements increase, the step size h_n decreases, thus the total number of steps N required to reach t_f increases. Rough estimate:

$$N \approx (t_f - t_0)/h,$$

where h is the average step size. We roughly have that if the local error is $\mathcal{O}(h^{p+1})$, then the global error is $N \cdot \mathcal{O}(h^{p+1}) = \mathcal{O}(h^p)$.

This explains the use of $p+1$ as the exponent in the definition of order.

MATLAB demo of exercise 7.7.