10. Basic quadrature rules. Composite quadrature rule. Order of a quadrature rule.

Numerical Analysis E2021

Institute of Mathematics
Aalborg University

AALBORG UNIVERSITY
DENMARK

Motivation

Let $f : [a, b] \to \mathbb{R}$. We seek to compute the value

$$\int_a^b f(x)dx.$$

- The integrand $f(x)$ may be known only at certain points, such as obtained by sampling.
- A formula for the integrand may be known, but it may be difficult or impossible to find an antiderivative that is an elementary function.
  - *Example:* $\exp(-x^2)$.
- It may be easier to compute a numerical approximation than to compute the antiderivative.

Basic quadrature rules

Two basic rules:

- The **midpoint rule** $M = hf\left(\frac{a+b}{2}\right)$,
- The **trapezoid rule** $T = h\frac{f(a)+f(b)}{2}$,

both of order 2.

Example:

$$\int_0^1 x^2 dx = \frac{1}{3}, \quad M = 1\left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad T = 1\left(\frac{0+1}{2}\right) = \frac{1}{2},$$

thus the error of M is $1/12$ and the error of T is $-1/6$, i.e. the error of T is -2 times the error of M.

Simpson's rule

Suppose the error in T is exactly -2 times the error in M. Solve

$$S - T = -2(S - M)$$

for S to obtain

$$S = \frac{2}{3}M + \frac{1}{3}T = \frac{h}{6}(f(a) + 4f(c) + f(b)),\ c = \frac{a+b}{2},$$

which is of order 4.

Composite Simpson's rule

Let $d = (a+c)/2$ and $e = (c+b)/2$. Apply Simpson's rule to each subinterval to obtain a quadrature rule over $[a,b]$:

$$S_2 = \frac{h}{12}(f(a) + 4f(d) + 2f(c) + 4f(e) + f(b)).$$

Both S and S_2 are of order 4, but the S_2 step size is half the S step size, so S_2 is roughly 2^4 times as accurate. Thus, a combination Q is obtained by solving

$$Q - S = 16(Q - S_2),$$

which yields

$$Q = S_2 + (S_2 - S)/15.$$

MATLAB demo of exercise 6.6.