# Two Wheel (name TBD)

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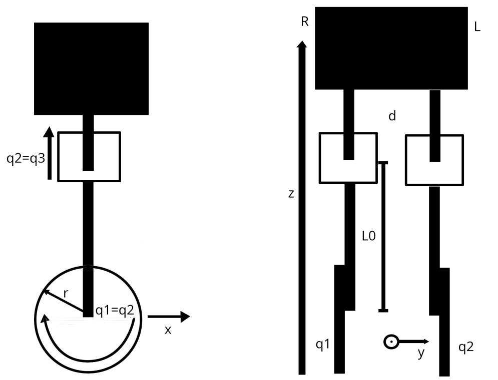
# 1 Structure




# 2 Kinematic

## 2.1 Variables

Coordinates:

$$p = \begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_z \end{bmatrix}$$

Joint Variables:

$$q = \begin{bmatrix} \theta_R \\ \theta_L \\ d_L \\ d_R \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Simplified system(q3=q4=const)
Generalized Coordinates:

$$u = \begin{bmatrix} d \\ \theta_x \\ \theta_z \end{bmatrix}$$

Joint Variables:

$$q = \begin{bmatrix} \theta_R \\ \theta_L \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

## 2.2 Constraint Definition

Diff drive constrain:

$$\frac{dy}{dx} = tan(\theta_z)$$

$$\begin{bmatrix} sin(\theta_z) \\ cos(\theta_z) \\ 0 \\ 0 \\ 0 \end{bmatrix} * p = 0$$

We obtain the following G matrix

$$G = \begin{bmatrix} c_{\theta_z} & 0 & 0 & 0 \\ s_{\theta_z} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{p} = G\dot{u} = G \begin{bmatrix} v \\ v_z \\ \omega_x \\ \omega_z \end{bmatrix}$$

We can find a correlation between the u vector and q as

$$\dot{u} = T\dot{q}$$

$$T = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{d} & -\frac{1}{d} \\ -\frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & \frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & 0 & 0 \end{bmatrix}$$

$$G_q = G * T = \begin{bmatrix} r\frac{c_{\theta_z}}{2} & r\frac{c_{\theta_z}}{2} & 0 & 0 \\ r\frac{s_{\theta_z}}{2} & r\frac{s_{\theta_z}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{d} & -\frac{1}{d} \\ -\frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & \frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & 0 & 0 \end{bmatrix}$$

$$\dot{p} = G_q\dot{q}$$

For the simplified system we already have the minimum number of coordinates so we only need to convert the generalized coordinates into physically meaning coordinate (joint variable)

$$\dot{q} = T * \dot{u}$$

$$\omega_R = \frac{v + \omega_z * \frac{d}{2}}{r} - \omega_y$$

$$\omega_L = \frac{v - \omega_z * \frac{d}{2}}{r} - \omega_y$$

$$T = \begin{bmatrix} \frac{1}{r} & -1 & \frac{d}{2r} \\ \frac{1}{r} & -1 & -\frac{d}{2r} \end{bmatrix}$$

# 3 Dynamic

For the dynamic we need to consider a third variable $\theta_y$

$$p = \begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}$$

And since no joint directly controls the orientation we can expand $G_q$ as follows:

$$G_q = \begin{bmatrix} r\frac{c_{\theta_z}}{2} & r\frac{c_{\theta_z}}{2} & 0 & 0 \\ r\frac{s_{\theta_z}}{2} & r\frac{s_{\theta_z}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{d} & -\frac{1}{d} \\ 0 & 0 & 0 & 0 \\ -\frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & \frac{r}{\sqrt{d^2+(d_R-d_L)^2}} & 0 & 0 \end{bmatrix}$$

## 3.1 Kinetic energy

Let's take into consideratio one body at a time ignoring for the moment the motor's contribution:

1. Right Wheel
2. Left Wheel
3. Right Leg
4. Left Leg
5. Body

# Change everything as function of u

Express everything within respect to $\dot{u} = \begin{bmatrix} v \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$$K_1 = \begin{cases} K_1 = \frac{1}{2}m_W v_{W_R}^2 + \frac{1}{2}\omega_{W_R}^T \Gamma_W \omega_{W_R} \\ v_{W_R} = v + \omega_z \frac{\sqrt{d^2+(d\,tan(\theta_x))^2}}{2} \\ \omega_{W_R} = \begin{bmatrix} \dot{\theta}_x \\ \dot{q}1 + \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \frac{v_{W_R}}{r} + \omega_y \\ \omega_z \end{bmatrix} \\ Gamma_{W_R} = ... \end{cases}$$

$$K_2 = \begin{cases} K_2 = \frac{1}{2} m_W v_{W_L}^2 + \frac{1}{2} \omega_{W_L}^T \Gamma_W \omega_{W_L} \\ v_{W_L} = v - \omega_z \frac{\sqrt{d^2 + (d\,tan(\theta_x))^2}}{2} \\ \omega_{W_L} = \begin{bmatrix} \omega_x \\ \dot{q}2 + \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \frac{v_{W_L}}{r} + \omega_y \\ \omega_z \end{bmatrix} \end{cases}$$

$$K_3 = \begin{cases} K_3 = \frac{1}{2} m_L v_{L_R}^2 + \frac{1}{2} \omega_{L_R}^T \Gamma_L \omega_{L_R} \\ v_{L_R} = v + \omega_z \frac{\sqrt{d^2 + (d\,tan(\theta_x))^2}}{2} \\ \omega_{L_R} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \end{cases}$$

$$K_4 = \begin{cases} K_4 = \frac{1}{2} m_L v_{L_L}^2 + \frac{1}{2} \omega_{L_L}^T \Gamma_L \omega_{L_L} \\ v_{L_L} = v - \omega_z \frac{\sqrt{d^2 + (d\,tan(\theta_x))^2}}{2} \\ \omega_{L_L} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \end{cases}$$

$$K_5 = \begin{cases} K_4 = \frac{1}{2} m_B v_B^2 + \frac{1}{2} * \omega_B^T \Gamma_B \omega_B \\ v_B = v + p_B X \omega_B \\ \omega_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\ p_B = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \end{cases}$$

### 3.2 Potential energy

With the same assumptio made before we can notice that we have only gravitational contribution. We can consider the 0 plane at the wheel height and

$$U = \begin{cases} U_1 = 0 \\ U_2 = 0 \\ U_3 = \frac{L_0 sin(\theta_x) sin(\theta_y)}{2} m_L\, g \\ U_4 = \frac{L_0 sin(\theta_x) sin(\theta_y)}{2} m_L\, g \\ U_5 = z sin(\theta_x) sin(\theta_y)\, m_B\, g \end{cases} = g(u)$$

### 3.3 External forces

As first approximation we can neglect the dissipative forces as frictions. The forces taken into account are the reaction force and the gravity force of the body mass.

$$f_r : \frac{\delta W_{f_r}}{\delta u} = 0$$

$$f_g : \begin{cases} \frac{\delta W_{f_g}}{\delta d} = 0 \\ \frac{\delta W_{f_g}}{\delta z} = \end{cases}$$

### 3.4 Lagrange Equations

Knowing both kinetic and potential energy we can start computing the left part of the lagrange equations

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{u}}\right) - \left(\frac{\delta L}{\delta u}\right)$$

$$\frac{\delta L}{\delta \dot{u}} = \begin{bmatrix} 2(m_L + m_W + \frac{\Gamma_W}{r^2})v + \frac{m_B(2v-2\omega_y z)}{4\sqrt{\omega_x^2 z^2+\omega_y^2 z^2-2\omega_y v\, z+v^2+v_z^2}} + \frac{2\Gamma_W \omega_y}{r} \\ \frac{m_B}{2\sqrt{\omega_x^2 z^2+\omega_y^2 z^2-2\omega_y v\, z+v^2+v_z^2}} v_z \\ 2(\frac{\Gamma_B}{2} + \Gamma_L + \Gamma_W)\omega_x + \frac{m_B}{2\sqrt{\omega_x^2 z^2+\omega_y^2 z^2-2\omega_y v\, z+v^2+v_z^2}}\omega_x z^2 \\ 2(\frac{\Gamma_B}{2} + \Gamma_L + \Gamma_W)\omega_y + \frac{2\Gamma_W}{r} v - \frac{m_B z(v-\omega_y z)}{2\sqrt{\omega_x^2 z^2+\omega_y^2 z^2-2\omega_y v\, z+v^2+v_z^2}} \\ 2\omega_z(\frac{\Gamma_B}{2} + \Gamma_L + \Gamma_W + \frac{d\, m_L(tan(\theta_x)^2+d)}{4} + \frac{d\, m_W(tan(\theta_x)^2+d)}{4} + \frac{\Gamma_W |d(tan(\theta_x)^2+d)|}{4r^2}) \end{bmatrix}$$

We can make some assumpion, first the

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{u}}\right) = ] \, [$$

# 4 Simplified Dynamic

### 4.1 Kinetic Energy

$$K1 = \begin{cases} K1 = \frac{1}{2} m_W v_1^2 + 1/2 \omega_1^T \Gamma_W \omega_1 \\ v_1 = v + \frac{d}{2}\omega_z \\ \omega_1 = \begin{bmatrix} 0 \\ \frac{v1}{r} \\ \omega_z \end{bmatrix} \end{cases}$$

$$K1 = \begin{cases} K2 = \frac{1}{2} m_W v_2^2 + 1/2 \omega_2^T \Gamma_W \omega_2 \\ v_2 = v - \frac{d}{2}\omega_z \\ \omega_2 = \begin{bmatrix} 0 \\ \frac{v2}{r} \\ \omega_z \end{bmatrix} \end{cases}$$

$$K3 = \begin{cases} K3 = \frac{1}{2} m_B v_3^2 + 1/2 \omega_3^T \Gamma_B \omega_3 \\ v_3 = v + l\omega_y \\ \omega_3 = \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix} \end{cases}$$

## 4.2 Potential Energy

$$U = l\,cos(\theta_y)\,g$$

## 4.3 Lagrange equation

$$L = \frac{\Gamma_{b,\mathrm{yy}} \left|\frac{\partial}{\partial t}\theta_y\,(t)\right|^2}{2} + \frac{\Gamma_{b,\mathrm{zz}} \left|\frac{\partial}{\partial t}\theta_z\,(t)\right|^2}{2} + \Gamma_{w,\mathrm{zz}} \left|\frac{\partial}{\partial t}\theta_z\,(t)\right|^2 + \frac{m_w \left|-\frac{d\,\frac{\partial}{\partial t}\theta_z(t)}{2} + \frac{\partial}{\partial t}p\,(t)\right|^2}{2} + \frac{m_w \left|\frac{d\,\frac{\partial}{\partial t}\theta_z(t)}{2} + \frac{\partial}{\partial t}p\,(t)\right|^2}{2} + \frac{m_w \left|l\,\frac{\partial}{\partial t}\theta_y\,(t) + \frac{\partial}{\partial t}p\,(t)\right|^2}{2} - g\,l\,\cos\left(\theta_y\,(t)\right) + \frac{\Gamma_{w,\mathrm{yy}} \left|-\frac{d\,\frac{\partial}{\partial t}\theta_z(t)}{2} + \frac{\partial}{\partial t}p\,(t)\right|^2}{r^2\,2} + \Gamma_{w,\mathrm{yy}} \Big| d\,\frac{\partial}{\partial t}$$

$$B\ddot{u} + g(u) = \gamma$$

$$B = \begin{bmatrix} \Gamma_{w,\mathrm{yy}} + 3\,m_w & l\,m_w & 0 \\ l\,m_w & m_w\,l^2 + \Gamma_{b,\mathrm{yy}} & 0 \\ 0 & 0 & \Gamma_{b,\mathrm{zz}} + 2\,\Gamma_{w,\mathrm{zz}} + \frac{d^2\,m_w}{2} + \frac{\Gamma_{w,\mathrm{yy}}\,d^2}{2\,r^2} \end{bmatrix}$$

$$g(u) = \begin{bmatrix} 0 \\ -g\,l\,\sin\theta_y \\ 0 \end{bmatrix}$$

$$\gamma = \ forces\,on\,generalized\,coordinates = T^T\,\tau$$

$$\tau = (T^{+})^T\,\gamma$$