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Note

The diameter of directed graphs

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Abstract

Let D be a strongly connected oriented graph, i.e., a digraph with no cycles of length 2, of order n and minimum out-degree δ. Let D be eulerian, i.e., the in-degree and out-degree of each vertex are equal. Knyazev (Mat. Z. 41(6) 1987 829) proved that the diameter of D is at most $\frac{5}{2\delta+2}n$ and, for given n and δ, constructed strongly connected oriented graphs of order n which are δ-regular and have diameter greater than $\frac{4}{2\delta+1}n-4$. We show that Knyazev's upper bound can be improved to $\mathrm{diam}(D)\leqslant \frac{4}{2\delta+1}n+2$, and this bound is sharp apart from an additive constant.

Keywords: Diameter; Directed graph; Distance; Eulerian; Minimum degree; Oriented graph

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The problem of determining a sharp upper bound on the diameter of an undirected graph G in terms of its order and minimum degree was solved by several authors (e.g. [2,4]), who proved that

$$\mathrm{diam}(G)\leqslant \frac{3}{\delta+1}n+c, \tag{1}$$

where n is the order of G, δ is the minimum degree of G and c is a constant.

We consider the corresponding problem for *directed* graphs. If the minimum degree is replaced by the minimum out-degree, then (1) does not generalize to directed graphs: Soares [5] constructed strongly connected digraphs of order n and minimum out-degree δ of diameter $n-2\delta+1$ and showed that this is the maximum diameter of strong digraphs

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of order n and minimum degree δ. However, Knyazev [3] and, independently, Soares [5] demonstrated that for eulerian digraphs, i.e., each vertex has equal in-degree and out-degree, (1) holds and that the factor $3/(\delta+1)$ is best possible. Essentially the same bound was given by Knyazev [3]. For oriented graphs, i.e., digraphs with no 2-cycle, Knyazev [3] improved the factor $3/(\delta+1)$ to $5/(2\delta+2)$.

Theorem 1 (*Knyazev [3]*). *Let D be a strong, eulerian digraph of order n and minimum out-degree δ, $2 \leqslant \delta \leqslant n/2$. If D contains no 2-cycle then*

$$\operatorname{diam}(D) \leqslant \frac{5}{2\delta+2}\, n.$$

For given n, δ with $2 \leqslant \delta \leqslant n/2$ there exists a δ-regular eulerian oriented graph D of order n and

$$\operatorname{diam}(D) \geqslant \frac{4}{2\delta+1}\, n - 4 + \frac{1}{2\delta+1}.$$

In this note we prove that the factor $\frac{5}{2\delta+2}$ can be improved to $\frac{4}{2\delta+1}$. In conjunction with the second part of Theorem 1, it follows that for fixed $\delta \geqslant 2$ and large n, the maximum diameter of an eulerian oriented graph of order n and minimum degree δ is $\frac{4}{2\delta+1}\, n + O(1)$.

The digraphs considered in this note have no multiple arcs or cycles of length 2 and are strongly connected. The out-degree of a vertex v is denoted by $\deg^+(v)$. We denote the minimum out-degree by δ. For subsets A, B of the vertex set of a digraph, we denote the number of arcs with tail in A and head in B by $q(A, B)$. For $q(A, A)$ we write $q(A)$. The diameter $\operatorname{diam}(D)$ is the maximum distance between any two vertices of D.

Theorem 2. *Let D be a strong, eulerian digraph of order n and minimum out-degree δ, $2 \leqslant \delta \leqslant n/2$. If D contains no 2-cycle then*

$$\operatorname{diam}(D) \leqslant \frac{4}{2\delta+1}\, n + 2.$$

Proof. Let v be an arbitrary vertex and for an integer i let $V_{\leqslant i}$, V_i, $V_{\geqslant i}$ be the set of vertices at distance at most i, exactly i and at least i, respectively. Let $n_i = |V_i|$ and let $\operatorname{ex}(v)$ be the largest i with $n_i > 0$. We prove that for $i = 1, 2, \ldots, \operatorname{ex}(v) - 2$,

$$n_{i-1} + n_i + n_{i+1} + n_{i+2} \geqslant 2\delta + 1. \tag{2}$$

Since D is eulerian, we have $q(V_{\geqslant i}, V_{\leqslant i-1}) = q(V_{\leqslant i-1}, V_{\geqslant i})$ and thus

$$\begin{aligned} q(V_i, V_{\leqslant i-1}) + q(V_{i+1}, V_{\leqslant i-1}) &\leqslant q(V_{\geqslant i}, V_{\leqslant i-1}) \\ &= q(V_{\leqslant i-1}, V_{\geqslant i}) \\ &= q(V_{i-1}, V_i) \\ &\leqslant n_{i-1} n_i. \end{aligned} \tag{3}$$

Since D is an oriented graph we have

$$q(V_i, V_{i+1}) + q(V_{i+1}, V_i) \leqslant n_i n_{i+1}. \tag{4}$$

Adding all out-degrees and applying (4) and (3), we obtain

$$\begin{aligned}
0 &\leqslant \sum_{x\in V_i} \deg^+(x) + \sum_{x\in V_{i+1}} \deg^+(x) - \delta(n_i + n_{i+1}) \\
&= \Big(q(V_i, V_{\leqslant i-1}) + q(V_i) + q(V_i, V_{i+1})\Big) \\
&\quad +\Big(q(V_{i+1}, V_{\leqslant i-1}) + q(V_{i+1}, V_i) + q(V_{i+1}) \\
&\quad +q(V_{i+1}, V_{i+2})\Big) - \delta(n_i + n_{i+1}) \\
&\leqslant n_{i-1}n_i + \frac{1}{2}\, n_i(n_i - 1) + n_i n_{i+1} + \frac{1}{2}\, n_{i+1}(n_{i+1} - 1) \\
&\quad +n_{i+1}n_{i+2} - \delta(n_i + n_{i+1}) \\
&= \frac{1}{2}\,(n_i + n_{i+1})(n_i + n_{i+1} - 2\delta - 1) + n_{i-1}n_i + n_{i+1}n_{i+2}.
\end{aligned} \tag{5}$$

Define the function $g(n_{i-1}, n_i, n_{i+1}, n_{i+2})$ to be the right-hand side of (5). In order to prove (2) we minimize the function $f(n_{i-1}, n_i, n_{i+1}, n_{i+2}) = n_{i-1} + n_i + n_{i+1} + n_{i+2}$ subject to the constraints $n_{i-1}, n_i, n_{i+1}, n_{i+2} \geqslant 1$ and, by (5), $g(n_{i-1}, n_i, n_{i+1}, n_{i+2}) \geqslant 0$. If $\max\{n_{i-1}, n_i, n_{i+1}, n_{i+2}\} > 2\delta$ then (2) holds, hence we can assume that $n_{i-1}, n_i, n_{i+1}, n_{i+2}$ are in the closed interval $[1, 2\delta]$. Since the set of all solutions of $g \geqslant 0$ with $n_{i-1}, n_i, n_{i+1}, n_{i+2} \in [1, 2\delta]$ is compact, f attains its minimum on it. Let $n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2}$ be chosen such that $f(n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2})$ is minimum subject to the above conditions. We first show that

$$g(n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2}) = 0. \tag{6}$$

Suppose to the contrary that $g > 0$. Then at least one of $n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2}$ is strictly greater than 1. Reducing it by a sufficiently small value will leave $g > 0$ but reduce f, a contradiction. Hence (6) holds.

We have to minimize f subject to $g = 0$ and $n_{i-1}, n_i, n_{i+1}, n_{i+2} \geqslant 1$.

Case 1: $\min\{n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2}\} > 1$.

Then $(n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2})$ is a local minimum of f subject to $g = 0$. By the Lagrange multiplier theorem (see for example [1, p.164f]) there exists a real λ such that at $(n^*_{i-1}, n^*_i, n^*_{i+1}, n^*_{i+2})$ we have

$$\frac{\partial f}{\partial n_j} + \lambda\, \frac{\partial g}{\partial n_j} = 0$$

for $j = i-1, i, i+1, i+2$. Hence

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} n^*_i \\ n^*_{i-1} + n^*_i + n^*_{i+1} - \delta - \frac{1}{2} \\ n^*_i + n^*_{i+1} + n^*_{i+2} - \delta - \frac{1}{2} \\ n^*_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies $\lambda = -1/n_i^*$, $n_i^* = n_{i+1}^*$ and $n_{i-1}^* = n_{i+2}^*$. Hence $1 - (n_{i-1}^* + 2n_i^* - \delta - \frac{1}{2})/n_i^* = 0$, which, after simplification, yields

$$n_{i-1}^* + n_i^* = n_{i+1}^* + n_{i+2}^* = \delta + \frac{1}{2}.$$

Hence $f \geqslant 2\delta + 1$ and (2) holds.

Case 2: ($n_{i-1}^* = 1$ and $n_i^*, n_{i+1}^*, n_{i+2}^* > 1$) or ($n_{i+2}^* = 1$ and $n_{i-1}^*, n_i^*, n_{i+1}^* > 1$).

We assume that $n_{i-1}^* = 1$; the case $n_{i+2}^* = 1$ is analogous. Then $(n_i^*, n_{i+1}^*, n_{i+2}^*)$ is a local minimum of f subject to $g = 0$ and $n_{i-1} = 1$. Again by the Lagrange multiplier method, there exist reals λ_1, λ_2 with

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} n_i^* \\ n_{i-1}^* + n_i^* + n_{i+1}^* - \delta - \frac{1}{2} \\ n_i^* + n_{i+1}^* + n_{i+2}^* - \delta - \frac{1}{2} \\ n_{i+1}^* \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Equality of the second and third row above implies $n_{i-1}^* = n_{i+2}^* = 1$, a contradiction.

Case 3: $n_{i-1}^* = n_{i+2}^* = 1$ and $n_i^*, n_{i+1}^* > 1$.

In this case we have $g = g(1, n_i^*, n_{i+1}^*, 1) = \frac{1}{2}(n_i^* + n_{i+1}^*)(n_i^* + n_{i+1}^* - 2\delta) \geqslant 0$, which implies $n_i^* + n_{i+1}^* - 2\delta \geqslant 1$ and thus $f \geqslant 2\delta + 3$, so (2) holds.

Case 4: ($n_i^* = 1$ and $n_{i-1}^*, n_{i+1}^*, n_{i+2}^* > 1$) or ($n_{i+1}^* = 1$ and $n_{i-1}^*, n_i^*, n_{i+2}^* > 1$).

This case is analogous to Case 2; we obtain the contradiction $n_i^* = n_{i+1}^* = 1$.

Case 5: $n_i^* = n_{i+1}^* = 1$.

In this case we have $g = g(n_{i-1}^*, 1, 1, n_{i+2}^*) = n_{i-1}^* + n_{i+2}^* + 1 - 2\delta \geqslant 0$, which implies $n_{i-1}^* + n_{i+2}^* \geqslant 2\delta - 1$ and thus $f \geqslant 2\delta + 1$.

In each case we have $f(n_{i-1}^*, n_i^*, n_{i+1}^*, n_{i+2}^*) \geqslant 2\delta + 1$, which implies (2).

Now let v be a vertex such that the eccentricity of v is maximum and thus equals $\mathrm{diam}(D) =: d$. Let $a = \lfloor \frac{d+1}{4} \rfloor$. Then

$$n = \sum_{i=0}^{d} n_i \geqslant \sum_{i=0}^{a-1} (n_{4i} + n_{4i+1} + n_{4i+2} + n_{4i+3}) \geqslant a(2\delta + 1).$$

Hence, by $\frac{d-2}{4} \leqslant a$, we obtain $d \leqslant \frac{4}{2\delta+1} n + 2$, as desired.

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