Dynamics of an extended Green Solow model

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Abstract

The Green Solow model provides an extension to the classical Solow model by adding emissions/pollution as a result of production. We further extend the Green Solow model in two ways: Firstly, by extending it to multiple heterogeneous countries affected by a common emissions pool created by production; Secondly, by including environmental capital as an explicit factor of production, affected by the production of all countries. Our more advanced model, through its variety of parameters and flexible formulation, can be used to simulate a wide variety of scenarios. The extension to multiple countries is particularly enriching and a significant departure from the classical Solow model, enabling us to model the long-term effects of heterogeneity between countries or in the policies they choose to enact to combat issues such as climate change. Although its calibration is imprecise in its current form, the modelling framework presented here provides an important step forward on the path to model the link between economic growth and the oft-forgotten environment permitting it.

1. Introduction and Literature Review

The relationship between economic growth and environmental degradation has been a major point of attention in environmental economics over the past decades (Tibaa & Omrib, 2017). The aim of this project is to modify and extend the Green Solow Model developed by Brock & Taylor (2010) to multi-regional/ multi-country calibration. The Green Solow model is itself an extension of the neoclassical growth model by Solow & Swan (1956), exploring interactions between income per capita and output levels. This section will be broken down into four subparts. In §A, we discuss the Solow (1956) model of economic growth and how it differs from the Ramsey (1928) growth model. In §B, we discuss how these growth models have been extended to factor in environmental degradation and in §C, how they have been modified to dynamic environments such as multiple regions or countries. In §D, we argue that our extension Green Solow model can provide innovative and comprehensive insights to the literature on sustainable economic growth and policy making.

A.The Solow neoclassical model of economic growth

In 1956, Solow redesigned Harrod & Domar' s growth model, dismissing the restrictive assumption that capital is the only factor of growth. He also rejected the standard Keynesian assumption that there is a fixed ratio between production factors. For Solow (1956), technological progress is a key component of economic growth in the long run. And so, in contrast to previous models, productivity is now factored in. This is because of exogenous improvements in technology over time; new capital becomes more attractive than old capital (Moroianu & Moroianu, 2012). Concretely, the Solow model is one of long-run economic growth, factoring in capital accumulation, labour (or population) growth, and increases in productivity through technological progress (Solow, 1956). It relies on an aggregate Cobb-Douglas (1926) production function, modelling the evolution of stock of capital per person. In Solow' s model, labour and technology grow exogenously. At equilibrium, the capital/labour ratio depends only on saving, growth, and capital depreciation rates. Capital accumulation is governed by two factors: capital investment and depreciation (Weil, 2014). Savings and capital investment decisions of output are exogenous, i.e. the savings rate is not determined within the model but given to the model. In contrast, the Ramsey (1928) model assumes an endogenous savings rate, meaning this choice arises as a result from initial parameters in the model, specifically utility-maximization of consumption over the long run. This latter is a representative agent model, with a finite number of agents and an infinite time horizon, building a macroeconomic model from microeconomic factors. Hence, in the Ramsey model, households make consumption and savings decisions at the microeconomic level, generating an endogenous savings rate (Chu, 2020). The savings rate determines the willingness of households to consume less now and invest in capital, and so to consume more in the future. We now explore how these models have integrated a dimension of environmental degradation.

B.Incorporating Sustainability in growth models

In recent decades, policy makers have started to thoroughly investigate policies that could enable us to achieve sustainable economic growth, that is, increasing income per capita without added pollution (UN, 2022). Consequently, economists have dedicated much attention to integrating economic damages from pollution into the above mentioned growth models. For instance, the Ramsey (1928) model of growth has been most used as a basis for aggregate cost-benefit integrated assessment models such as the Dynamic Integrated Climate - Economy (DICE) model (Nordhaus, 1992). By adding an "additional form of unnatural capital that has negative effect on economic output through its influence on the global average surface temperature" (the concentration of CO_2 in the atmosphere), the DICE model optimizes savings and emissions reductions over a multi century planning horizon (Newbold, 2010). The overarching objective of the DICE model, trough optimising these two economic quantities, is to maximize the discounted sum of all yearly utilities from consumption, which are themselves the product of the number of people alive and the discounted average income (Newbold, 2010).Other Integrated assessment models include process-based models, which are widely used by the Intergovernmental Panel on Climate Change (2022) to quantify environmental scenarios and to assess how mitigation policies can change these scenarios. Examples of these models include the IMAGE, AIM/GCE and GCAM models. While these models have proved useful in exploring pathways to stay on climate policy targets such as the 1.5 °C set during the Paris Agreement (Rogelj, et al., 2018), they have faced criticisms such as their inability to capture certain factors accurately such as climate sensitivity, inertia in climate response, carbon cycle feedbacks and so on (van Vuuren et al., 2011). Similarly, the Solow (1956) model of growth has been extended into the Green Solow model by Brock & Taylor (2010). The Green Solow model establishes an intimate relationship between Environmental Kuznets Curves and the Solow model, and simply aims to link growth rates, income levels and environmental quality. In contrast to integrated assessment models such as the DICE, savings rate and abatement choices are exogenously decided in the Green Solow model, and typically, pollution abatement amounts are an increasing and strictly concave function of the total economic activity. Therefore, the Green Solow model should generate an income per capita and environmental quality path that draws out an Environmental Kuznets Curve (Brock & Taylor, 2010). We now discuss how these models have been extended from a single-economy model to a multiple-economy one.

C. Extending sustainable models to multiple regions

Practically, the world economy is a complex one, and little environmental change can happen if not unilaterally. Policy makers need to factor in interactions between players and thus extending these models to multiple countries or regions has proved useful. Therefore, on one hand, integrated assessment models have been extended to integrate multiple economies to understand how domestic regulations can impact other regions or countries at economic and environmental levels. Examples of regional integrated assessment models include a continuation of the DICE model, the RICE model, and the FUND model (Tol & Anthoff, 2022).Nordhaus' (1992) RICE model displays how policy implementation could be done in one specific region, instead of providing a global picture. The FUND model on its part serves as a testing model for policies on climate change in a dynamic context. In a similar fashion to the RICE model, the FUND model breaks down the world into 16 regions and investigates the dynamics of population, technology, economics, emissions, atmospheric chemistry, climate, sea level, and other parameters as new policies are implemented in one region or another (Tol & Anthoff, 2022). Interesting applications of the RICE and FUND models included the evaluation of the social cost of carbon, that is usually simply considered as a negative externality in conventional markets. Indeed, policy makers such as the European Commission (2022) can then use this social cost of carbon to implement carbon taxes for example. On the other hand, less advanced but similarly interdisciplinary models, extending growth dynamics to multiple countries or regions are used in policy making by the OECD. One example of a such model is the ENV Linkages Model Version 3 (Château, et al., 2014). It is a computable General Equilibrium Model for several regions and sectors, built on database of national economies and underpinned by an input output table from national statistical agencies. The ENV Linkages Model Version 3 links economic activity to environmental pressure through GHG emissions, and nothing else; it assumes perfect markets, CRS technologies, and CES, no forward-looking behaviour, nor endogenous savings rate (Château, et al., 2014). One of the major discussions of this model, however, lies in its lack of integration of natural capital.We now move on to discuss why an extension of the Green Solow Model is relevant in the context of the current literature and the benefits and challenges that it can bring about.

D. Why extend the Green Solow Model?

Beyond extending the Green Solow model to numerous countries, we also intend to integrate natural capital in the model. Naturally, nature has a marginal product, and so our neoclassical economic framework considers it within factors of production, although not independently (Marshall, 1947). Common natural elements such as the air we breathe, international oceans, forests or even the ozone layer indeed are factors of production, even if some of these are not tradable and do not carry a market price. In short, natural capital comprises all natural endowments that better our standards of living, under the assumption that natural capital stock can be summed over in monetary units. Summing over this stock nonetheless becomes tricky when discussing ideas such as ecosystem services and the value of biodiversity, although under Arrow et al' s (2012) shadow pricing mechanism those could be derived as well. Extending the Green Solow model in this way will make it more comprehensive and integrated, providing interesting and original insights on the efficiency and effects of policies and how these affect parameters. However, specifically, this paper will not specifically consider natural resources such as oil, gas and various minerals as direct natural capital inputs in the production function (Brandt, et al., 2013), but rather as a state of the environment that determines our capacity to fully utilize the other inputs in producing output. Our project will follow a four-part structure. Following the introduction and the literature review in Section 1, we will outline the methodology in Section 2. This section will include a detailed description of the mathematical groundwork of the basic Solow model that will be used in Section 3. The latter will contain the bulk of our modelling work.In Section 4, we will discuss our findings and conclude on them.

2. Methodology

2.1 Presentation of the models

2.1.1 The Solow Model and the Green Solow Model

To take pollution into account in the economic growth process, we employ a simplified “Green Solow” structural model developed by Brock and Taylor (2004). Despite being criticized on many occasions for its strong assumption of constant emissions intensity growth (W.-J. Chen, 2015) and its functional form (Felipe and McCombie, 2014) its underlying neoclassical Solow growth model (Solow, 1957) has shown high explanatory power, is widely used in research, and most importantly for us is a very tractable modelling tool.

2.1.1.1 The model and its governing equations

The model is pinned down by a Cobb-Douglas production function:

A1 $$F[K[t], L[t]] = A * K[t]^{a} * L[t]^{1-a} \tag{1}$$

where labour (L) and capital (K) are inputs.

Without any emissions-reducing technological investment, under these parameters, we can formulate the growth rates of capital and labour at t in continuous time (i.e. where the length of period over which growth happens Δt goes to 0):

$$\begin{aligned}
&\text{A2} && E[t] = \mu * F[K[t], L[t]] = \mu * A * K[t]^{a} L[t]^{1-a} \\
&\text{A3} && \partial_t K[t] = s\,c * A * K[t]^{a} L[t]^{1-a} - \gamma * K[t] \\
&\text{A4} && \partial_t L[t] = (\beta - \eta\, E[t])\, L[t] \\
&\text{A5} && K[0] = K0 \\
&\text{A6} && L[0] = L0
\end{aligned} \tag{2}$$

By solving the above differential equations (after specifying initial conditions for capital and labour stocks at t=0), we could arrive at the growth paths for this country (e.g. of per capita stock of capital) plotted over time, which converge to the steady-state over time as neoclassical models predict. We can see that the rate of change of capital is determined by the amount of production that is "saved", sc, or equivalently not consumed. This allows the economy to accumulate capital over time which they can reinvest in themselves, increasing output further in the future.

To start, we assume that producing an amount X of output also produces an amount μ*X of pollution, hence μ is the emission intensity of production. Finally, we assume that the rate of natural population increase is β (the fertility rate), minus a term proportional to total pollution at time t. Note that to derive per capita capital over time, we would simply divide the expression for capital over time generated by solving the model by the expression for labour over time.

2.1.1.2 What is A?

Total Factor Productivity (TFP), in our model, A, and also referred to as the Solow residual, is the amount of output not accounted for by the amount of inputs used in production (Comin, 2006). Aggregate production functions estimate the quantity of output that can be traced back to different amounts of input factors used in production, such as capital and labor. The residual, unexplained, output generated by technological advance and intensity, and efficiency, and hence determine TFP (Burkett, 2006).

2.1.1.3 What is a?

The “a” parameter in the Cobb-Douglas production function is the output elasticity of capital, and the “1–a” of labor. These parameters are determined by technological availabilities, and measure the responsiveness of output to changes in levels of one of the input factors (Cobb & Douglas, 1928). Barro (1991) empirically investigated the original Solow growth model and verified that under this specification per capita income converged from the initial to steady-state level.

2.1.1.4 Why does capital depreciate?

One of the key characteristics of capital is that it wears out, triggering a process called depreciation. It happens when use or passage of time reduce the quantities of capital available. Depreciation is part of economic life, hence investment in new capital compensates for the loss of old capital (Weil, 2013). In our model, in line with the standard one-sector Solow model, we assume that capital depreciates at a constant rate.

2.1.1.5 Does pollution affect fertility?

In recent years, a similar trend has been observed for female fertility and air pollution: both rates have increased. There has thus been growing concern regarding the correlation between the two rates. Anthropogenic activities, such as traffic, industrial facilities and fossil fuel combustion, particularly in large urban areas, have contributed in the worsening of air quality, and are responsible for health-related pollutants. Through ingestion or inhalation, these pollutants can disrupt good endocrine activity. They could interfere with the thyroid axis and the metabolism, triggering endocrine disorders that could lead to infertility (Conforti, et al., 2018). In developed countries, with most industrial activity, one in seven couples are infertile. Hence, there is strong evidence that air pollution contributes to low fertility (Carrington, 2022).
The *effect* of pollution in this baseline model is introduced as the fraction of the total emissions reducing the population growth rate (i.e. fertility effect), as Nieuwenhuijsen, Mark J., et al. (2014) find. The total effect on the labour growth rate is thus β-η times pollution, an approach inspired by Witte (2022).

2.1.2 The Green Solow Model with emissions-reducing technological progress

2.1.2.1 Why distinguish between green investment and capital investment?

Witte (2022) simplified the Green Solow model to just include mitigation in the form of investment in “green” technology, on top of traditional investment sc. This “green” investment reduces the proportion of emissions generated by production that enters the biosphere, and will run between 0 and 1, with 0 being optimal. The previous equations for capital, emissions and labour now become:

$$
\begin{array}{ll}
\text{B1} & F[K[t], L[t]] = A * K[t]^{a} * L[t]^{1-a} \\
\text{B2} & E[t] = \mu * \Omega[t] * F[K[t], L[t]] = \mu * \Omega[t] * A * K[t]^{a} L[t]^{1-a} \\
\text{B3} & \partial_t K[t] = s c * A * K[t]^{a} L[t]^{1-a} - \gamma * K[t] \\
\text{B4} & \partial_t L[t] = (\beta - \eta\, E[t])\, L[t] \\
\text{B5} & \partial_t \Omega[t] = -k * sg * \Omega[t] \\
\text{B6} & K[0] = K0
\end{array} \tag{3}
$$

B7 $L[0] = L0$

B8 $\Omega[0] = \Omega 0$

Notice the rate of change of our brown tech factor is *naturally* diminishing (-k), but this rate can be increased by more investment into green technologies. Green investment is different from capital investment in the sense that it is restricted to a specific class of assets; those aligned with environmental preservation, regeneration or environmentally friendly business practises (Eyraud, et al., 2013). Major destinations for green investment include renewable technologies such as wind, solar, and hydropower. More recently, hydrogen, carbon dioxide absorption, and battery technology have been attractive investment areas. Eyraud, et al. (2013) have presented the macroeconomic trends and determinants of green investment over the past couple of decades and found that green investment had become a key driver of the energy sector, especially in China. To find the growth paths for Ω[*t*], K[t], and L[t], once again need to specify initial values at t=0. Ω[0] = 1 is the standard assumption we previously made, since all emissions are “uncontrolled” before investment in green technology is made.

2.1.3 The multiple-country Green Solow model with emissions-reducing technological progress

2.1.3.1 The equations

To extend the Green Solow model for growth, pollution, and mitigation to a world with multiple countries or regions that make their own decisions about investments in capital and green technology, but suffer from the same global pollution scenario, we have first come up with adapted versions of the equations from Witte’s (2022) Week 7 practical. These equations describe the periodic changes and the initial values of capital, labour, and green technology in each region, as well as the production and emissions function:

$$
\begin{array}{ll}
\text{C1} & F_i[K_i[t],\ L_i[t]] = A_i * K_i[t]^{a_i} * L_i[t]^{1-a_i} \\
\text{C2} & E[t] = \sum_{j=1}^{n} \mu_i\, \Omega_i[t] * F_i[K_i[t],\ L_i[t]] \\
\text{C3} & \partial_t K_i[t] = sc_i * F_i[K_i[t],\ L_i[t]] - \gamma_i * K_i[t] \\
\text{C4} & \partial_t L_i[t] = \left(\beta_i - \eta_i * \sum_{j=1}^{n} E[t]\right) * L_i[t] \\
\text{C5} & \partial_t \Omega_i[t] = -k_i * sg_i * \Omega_i[t] \\
\text{C6} & K_i[0] = K0_i \\
\text{C7} & L_i[0] = L0_i \\
\text{C8} & \Omega_i[0] = \Omega 0_i
\end{array}
\tag{4}
$$

We have also allowed for there to be regional differences in technological development Ω_i, natural fertility rates β_i, as well as in the initial values of the three key variables and other parameters. If we were to model 3 different countries for example, our complete system of differential equations to solve would become:

```
In[ ]:= SimEquationsGT[3];
```

$$\{KK_1'[t] == -\gamma_1 KK_1[t] + A_1 sc_1 KK_1[t]^{a_1} L_1[t]^{1-a_1},$$
$$KK_2'[t] == -\gamma_2 KK_2[t] + A_2 sc_2 KK_2[t]^{a_2} L_2[t]^{1-a_2},$$
$$KK_3'[t] == -\gamma_3 KK_3[t] + A_3 sc_3 KK_3[t]^{a_3} L_3[t]^{1-a_3},$$
$$KK_1[0] == K0_1,\ KK_2[0] == K0_2,\ KK_3[0] == K0_3,$$

$$L_1'[t] == L_1[t] \left(\beta_1 - \eta_1 \left(A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t] + A_3 \mu_3 KK_3[t]^{a_3} L_3[t]^{1-a_3} \Omega_3[t]\right)\right),$$
$$L_2'[t] == L_2[t] \left(\beta_2 - \eta_2 \left(A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t] + A_3 \mu_3 KK_3[t]^{a_3} L_3[t]^{1-a_3} \Omega_3[t]\right)\right),$$
$$L_3'[t] == L_3[t] \left(\beta_3 - \eta_3 \left(A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t] + A_3 \mu_3 KK_3[t]^{a_3} L_3[t]^{1-a_3} \Omega_3[t]\right)\right),$$
$$L_1[0] == L0_1,\ L_2[0] == L0_2,\ L_3[0] == L0_3,$$

$$\Omega_1'[t] == -k_1 sg_1 \Omega_1[t],$$
$$\Omega_2'[t] == -k_2 sg_2 \Omega_2[t],$$
$$\Omega_3'[t] == -k_3 sg_3 \Omega_3[t],$$
$$\Omega_1[0] == \Omega0_1,\ \Omega_2[0] == \Omega0_2,\ \Omega_3[0] == \Omega0_3\}$$

Which Mathematica can easily solve for us. We can also now define regional/country and aggregate world consumption in period t as:

$$\begin{array}{ll} \text{C9} & C_i[t] = (1 - \mathrm{sc}_i - \mathrm{sg}_i) * F[\mathrm{K}_i[t], \mathrm{L}_i[t]] \\ \text{C10} & C[t] = \sum_{i=1}^{n} C_i[t] \\ \text{C11} & U\left(\frac{C_i[t]}{L_i[t]}\right) = \ln\left(\frac{C_i[t]}{L_i[t]}\right) \\ \text{C12} & \mathrm{PV}_i = \int_0^T \left[L_i[t] * U\left(\frac{C_i[t]}{L_i[t]}\right)\right] e^{-\delta * t}\, dt \end{array} \tag{5}$$

C11 describes the utility of a representative agent as the natural logarithm of per-capita consumption while C12 shows how the present value of population utility from time 0 to T is calculated.

2.1.3.2 Why model fertility-impacting pollution globally and not locally?

Pollution is the result of damaging human economic activity on the environment. While some problems, such as biochemical contamination or waste deriving from human economic activities, such as mining, can be treated at the local level, a very significant proportion of hazardous pollution cannot be restricted to individual countries. This includes but is not limited to air pollution from the combustion of fossil fuels or other emissions: as all parts of the world are connected through fast atmospheric transport (Ramanathan & Feng, 2009). Hence, air pollution, to take an example, in one country can spillover to neighbouring ones. Similarly, although water pollution through the dumping of hazardous waste into rivers and oceans.Therefore, we model pollution at the global level.

It is important to note that each i now represents a parameter or function specific to a certain region or country. Hence $\mu_i * \Omega_i[t] * F[\mathrm{K}_i[t],\ \mathrm{L}_i[t]]$ is now the emission fraction of production in region i (fraction of emissions that is pollution). Of key assumption is that each country pollutes individually, but the effect of pollution on a country's fertility rate depends on the **global** stock of pollution, which will be out of a specific country's control. This introduces well known public-good dynamics

(Uitto, 2016) into our simulation. However, we do allow each country's fertility to be distinctly sensitive to worldwide emissions, this being represented by η_i.

2.1.4 The multiple-country Green Solow model with emissions-reducing technological progress and environmental capital

2.1.4.1 Why introduce environmental capital?

An obvious weakness of our previous model and of this approach is that the effect of emissions is limited to the negative impact of air pollution and other resulting environmental damages **only** on population/ labour growth rates. However, emissions are known to also cause damages that reduce output directly, through channels other than the impact on human health. For instance, CO2 build up in the atmosphere causes surface temperatures to rise, which hastens the melting of the ice caps and causes sea levels to rise (Church & Clark, 2013). This leads to the loss of coastal areas, which should cause a direct reduction in output. On the other hand, CO2 absorbed by the ocean leads to ocean acidification, which negatively impacts shellfish harvests or crucial ecosystem services provided by coral reefs. To incorporate these direct damages on output in our analysis, we have decided to alter the model by making production a function of three inputs instead of two: country-specific physical capital, country-specific labour, and **global** natural/ environmental capital (no subscript i).

2.1.4.2 Extending Dasgupta's (2021) production function

This approach is largely inspired by Chapter 4* of Dasguptas (2021) "The economics of biodiversity" report, but we extend his ideas by treating this in the context of our multiple-country Green Solow Model, while he applied this in a one country context. However, while he proposes decomposing environmental capital into a "flow of extracted provisioning service (R); and (ii) as a stock supplying regulating and maintenance services (S) in the form of a global public good" (Dasgupta, 2021), we decide, for simplicity purposes to only deal with the latter. *S*[*t*] isn't necessarily supposed to be viewed as a direct input to production but rather as a state of the environment which determines our capacity to fully utilize the other inputs in producing output. It can be represented of as regulation and maintenance services, such as disease control, soil regeneration, or ecosystems critical to our agricultural activities, that are required to exist for production to run smoothly.

Our final equations are thus:

D1

$$F_i[K_i[t],\ L_i[t],\ S[t]] = A_i * K_i[t]^{a_i} * L_i[t]^{1-a_i} * S[t]^{\theta_i}[t]$$

D2
$$E[t] = \sum_{j=1}^{n} \mu_i * \Omega_i[t] * F_i[K_i[t],\ L_i[t], S[t]]$$

D3
$$\partial_t K_i[t] = sc_i * F[K_i[t], L_i[t], S[t]] - \gamma_i * K_i[t]$$

D4
$$\partial_t L_i[t] = \left(\beta_i - \eta_i * \sum_{j=1}^{n} E[t]\right) * L_i[t]$$

D5
$$\partial_t \Omega_i[t] = -k_i * sg_i * \Omega_i[t]$$

D6 $$\partial_t S[t] = r * S[t] \left(1 - \frac{S[t]}{SU}\right) \left(\frac{S[t]-SL}{SU}\right) - \chi * \log \sum_{j=1}^{n} \left(\mu_i * \Omega_i[t] * F_i[K_i[t], L_i[t], S[t]]\right)$$

D7 $$K_i[0] = K0_i$$

D8 $$L_i[0] = L0_i$$

D9 $$\Omega_i[0] = \Omega 0_i$$

D10 $$S[0] = S0$$

D11 $$C_i[t] = (1 - sc_i - sg_i) * F[K_i[t], L_i[t], S[t]]$$

D12 $$C[t] = \sum_{i=1}^{n} C_i[t]$$

D13 $$PV_i = \int_0^T \left[L_i[t] * U\left(\frac{C_i[t]}{L_i[t]}\right)\right] e^{-\delta * t} dt$$

D14 $$U\left(\frac{C_i[t]}{L_i[t]}\right) = \ln\left(\frac{C_i[t]}{L_i[t]}\right)$$

Equation D6 represents the change in environmental capital, once again adapted from Dasgupta (2021). We assume environmental capital presents a certain natural capital regeneration rate, modelled here as r. However, this regeneration is bounded by an elaborated logistic function, where SU is the upper bound on natural capital (pre-industrial level), SL can be understood as the *Safe Minimum Threshold* for natural capital (hence SL<SU), and χ is global susceptibility of global natural capital to emissions.

2.1.4.3 What is environmental regeneration?

Environmental regeneration can be defined as the natural ability of our planet and the organisms and resources on it to renew, regenerate, restore, or recover from damage (McCaughey & Tomback, 2001). Simply put, our ecosystem can replenish what has been depleted, polluted or destroyed by humans up to a certain extent. Ecosystems can regenerate in various ways. One of the major ways that an ecosystem can regenerate is through photosynthesis, for example, turning CO2 into oxygen and plant biomass. This process cleans our air and allows a dynamic equilibrium that we can live by on a global level (Calama, R., et al., 2013). Another example is the regeneration of biomass such as forests. As trees are cut in correct circumstances and the right parameters such as soil fertility, availability of nutrients, animal migration paths, air quality remain present, they will regenerate and grow back (McCaughey & Tomback, 2001).

2.1.4.4 Why assume an upper bound and lower bound tipping points for natural capital?

It seems only logical that there is an upper bound on natural capital in the sense that regeneration of natural capital can only continue to a certain extent while allowing humans to develop, such as planting more and more trees, would help regenerate forests and air quality but in turn reduce our availability of space to build homes. Naturally, ecosystems have tipping points, and the resilience of an ecosystem to minor disturbances is a sign of the health of an ecosystem (GGKP, 2020). On a small scale, ecosystems have tipping points. For example, a forest fire can destroy and entire forest which will not regenerate, because the soil will not be fertile enough for a long period of time and so a return to the previous dynamic equilibrium will not be possible. This is also the case for living species during viral outbreaks such as pandemics for example. On a larger scale, our entire global ecosystem also has a tipping point, which will be our lower bound. An example of this is climate change. If a certain amount of pollution is reached for example, temperatures can rise enough to destabilize sea levels, air concentrations etc. which can have ripple on effects on other parameters and cause the entire stock of natural capital to collapse. In our simulations, we will choose an arbitrary lower bound: no empirical estimates are associated with this figure yet as we are the first to introduce this in such an extended model.

2.4.4.5 Some dynamics of the extended logistic function

Mathematically speaking, as long as $S[t] \times \epsilon\,[SL,\ SU]$, since SL and SU are >0, the regeneration of natural capital is positive because $S[t]$,$1-\frac{S[t]}{SU}$, and $\frac{S[t]-SL}{SU}$ are all positive.

If $S[t] > SU$, $1-\frac{S[t]}{SU}$<0, and the negative rate of change term will guide S[t] back to $S[t] \times \epsilon\,[SL,\ SU]$. If $S[t]\,\epsilon\,[0,\ SL]$, $S[t]$ starts spiralling downwards, in equivalent fashion to crossing a tipping point threshold, because $\frac{S[t]-SL}{SU}$ becomes negative while the other terms remain positive. However, since we don't implement the constraint that S[t]>0, if $S[t] < 0$, our whole system becomes nonsensical. Regarding the initial value of natural capital, we will assume $S[0] = S0 \cong SU$ because the environment wasn't originally degraded by emissions, so pre-industrial natural capital should be at the upper bound. This can change if we choose a post-industrial starting point.

2.2 Application of the model for various simulations

The above models, particularly the one developed in section 2.1.4, present an extremely varied range of simulations that can now be run. Principally through Mathematica's powerful DSolve Command, all the above systems can now be solved for a very large number of countries, and for a strong variety of initial conditions both with fixed policy regime **(2.2.1)** and when policy regime is changed **(2.2.2)** and exploring the implications.

2.2.1 Multiple-Country Scenarios **(Simulations 3.1 and 3.2)**

2.2.1.1 Extending the model to n countries: "Proof of concept"

After briefly presenting the baseline model we discussed in class, we present its application to n countries. Given the issues with calibrating the parameters it's only a schematic representation with randomly generated parametric set-up.

2.2.1.2 North-South models of economic development without environmental capital

We use our extended model to tackle the idea of there being a North-South divide between the countries of the world. The idea will be to run two scenarios with unequal starting conditions where we distinguish between low damage of emissions on output **(3.1.2.1.1)** and high damage **(3.1.2.1.2).** We then observe the effect on output, emissions, and capital per capita over time without there being any investments possible to mitigate this pollution.

Then, we implement the constant green investment rate but with varying scenarios between North and South. We explore model's predictions when only North invests in green technology **(3.1.2.2.1)** and when both countries invest buy to varying extent **(3.1.2.2.2).** This only handful of all the possible simulation scenarios one could run with the North-South framework but also with different distinction e.g. prediction of potential consequences of East-West divide over climate action that seems to prevail between EU countries.

2.2.1.3 North-South models of economic development with environmental capital

We conduct same simulation as above but the model is enriched with the environmental capital. We look at the predictions for North-South block when none of the regions invests in green technology, when only North invests and when both countries do. We pay special attention to examining the sensitivity of tipping points to initial conditions.

2.2.2 Policy Questions **(Simulations 3.3)**

On top of simulating a range of different scenarios with fixed policy, we now consider different policy scenarios. Among those that will be explored will be:

2.2.2.1 How effective is policy change at a given time t?

We use the parameters from North-South framework explored above, to simulate how does change in the environmental policy introduced at a given time t shapes the predictions. First we look at both countries investing in green technology at t=350 .

Next we motivate the global policy response by Principle 7 of the 1992 Rio United Nations Conference on Environment and Development : "In view of the different contributions to global environmental degradation, States have common but differentiated responsibilities". We translate this into our model by revising our assumptions and assume the investment in green technology is now a function of capital per capita. This is to reflect higher mobilisation of investments in green technology in countries with higher functional share of per capita income going to capital instead of labour. Third simulation also employs the new assumption but for the case with 5 countries where each makes an investment in green technology at different time. The new differential equations for simulations 3.2.2.2 and 3.2.2.3 are thus:

$$S'[t] == -\chi \, \mathrm{Log}\left[1 + S[t]^{\theta_1} A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + S[t]^{\theta_2} A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t]\right] + \frac{r\,(SU - S[t])\, S[t]\, (-SL + S[t])}{SU^2}$$

$$KK_1'[t] == -\gamma_1 KK_1[t] + S[t]^{\theta_1} A_1 sc_1 KK_1[t]^{a_1} L_1[t]^{1-a_1}$$

$$L_1'[t] == L_1[t] \left(\beta_1 - \eta_1 \left(S[t]^{\theta_1} A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + S[t]^{\theta_2} A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t]\right)\right)$$

$$L_2'[t] == L_2[t] \left(\beta_2 - \eta_2 \left(S[t]^{\theta_1} A_1 \mu_1 KK_1[t]^{a_1} L_1[t]^{1-a_1} \Omega_1[t] + S[t]^{\theta_2} A_2 \mu_2 KK_2[t]^{a_2} L_2[t]^{1-a_2} \Omega_2[t]\right)\right)$$

$$\Omega_1'[t] == -k_1 sg_1 \Omega_1[t] * KK_1[t] / L_1[t]$$

$$\Omega_2'[t] == -k_2 sg_2 \Omega_2[t] * \frac{t-500}{(2000)} * \mathrm{UnitStep}[t-500] * KK_2[t] / L_2[t]$$

$$\Omega_1[0] == \Omega 0_1, \; KK_2'[t] == -\gamma_2 KK_2[t] + S[t]^{\theta_2} A_2 sc_2 KK_2[t]^{a_2} L_2[t]^{1-a_2}$$

$$KK_1[0] == K0_1$$

$$KK_2[0] == K0_2$$

$$L_1[0] == L0_1$$

$$L_2[0] == L0_2$$

2.2.2.2 Is there an optimal global target for investment in green technology?

The model can also be used to find an approximation of the optimal international target for the s_g parameter which, if implemented by all countries, would maximize the sum of all regional/country present values of total utility, i.e. $\sum_{i=1}^{n} PV_i = \sum_{i=1}^{n} \int_0^T \left[L_i[t] * U\left(\frac{C_i[t]}{L_i[t]}\right)\right] e^{-\delta * t}\, d\,t$. This approximation can be found by running multiple iterations of the model with different international targets for s_g while keeping all other parameters constant. The model that will give the highest sum of regional present values of total utility will correspond to the optimal international s_g target.

2.2.2.3 Do regions have an incentive to deviate from the global target for investment in green technology?

The model can now tell us whether one of the regions would, individually, have an incentive to deviate from the global target. In other words, we want to see whether a region can obtain a higher present value of total utility by implement an s_g target that's different from internationally agreed one, given that all other regions have respected the mandated target. Using the model in this way can inform us about the likelihood of success of an international cooperation scenario, such as the one discussed above, and can give an indication of the penalties that would need to be imposed by the other regions to induce the defying region to implement the global target. The model can thus be a useful tool in the field of climate policy game theory.

3. Simulations and Discussion

3.1 Green Solow model with emissions-reducing technological progress without environmental capital

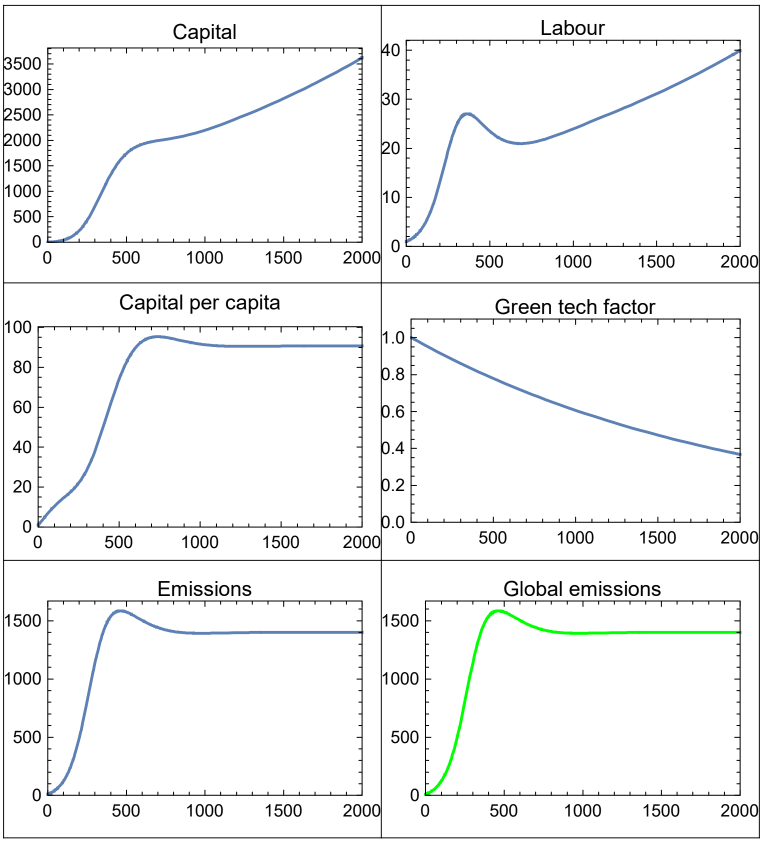
3.1.1 Introductory simulations: Extending the basic model to N countries

3.1.1.1 Model seen in class

In[]:=
```
SimOPM = {{10}, {0.01}, {0.005}, {0.5},
    {0.01}, {0.0145}, {0.00001}, {1}, {1}, {1}, {1}, {0.1}};
SimOEqs = SimEquationsGT[1];

Sim0 = SimulateGT[1, 2000, Sim0Eqs, Sim0PM, 1];
{Sim0[[5]], ParTableGT[Sim0PM]}
```

Out[]= {



,

	Country 1
A	10
s_c	0.01
s_g	0.005
a	0.5
γ	0.01
β	0.0145
η	0.00001
μ	1
K0	1
L0	1
Ω0	1
k	0.1

}

We first run our model as just one country. With the initial starting conditions inputed above, we can see that a traditional steady-state for capital per capita is reached as predicts most Solow models. Interestingly, we also see that total emissions peak around T=500 and then stabilize at a lower level. However, we seem to observe an “overshooting” process for both emissions and capital per capita presenting an interesting stabilization process at play. We observe labour grow at a much faster rate than capital at first, leading to an increase in output (as labour is in an input in our production function), which subsequently increases emissions which ends up backfiring on the growth of labour, meaning the original growth in labour was unsustainable.

This could be viewed as an extension of Malthus’ (1798) ideas on the relationship between capital growth and labour growth but via the stabilisation of incomes coming through the environment and not a change in technological progress. Neo-Malthusian theory, centered around similar ideas, argues that uncontrolled labour growth will contribute to ecological collapse (Hardin, 1968). Our results, nonetheless, show more concordance with the seminal “The Limits to growth” report (1972) that predicted a decline in population once emissions reached unsustainable levels.

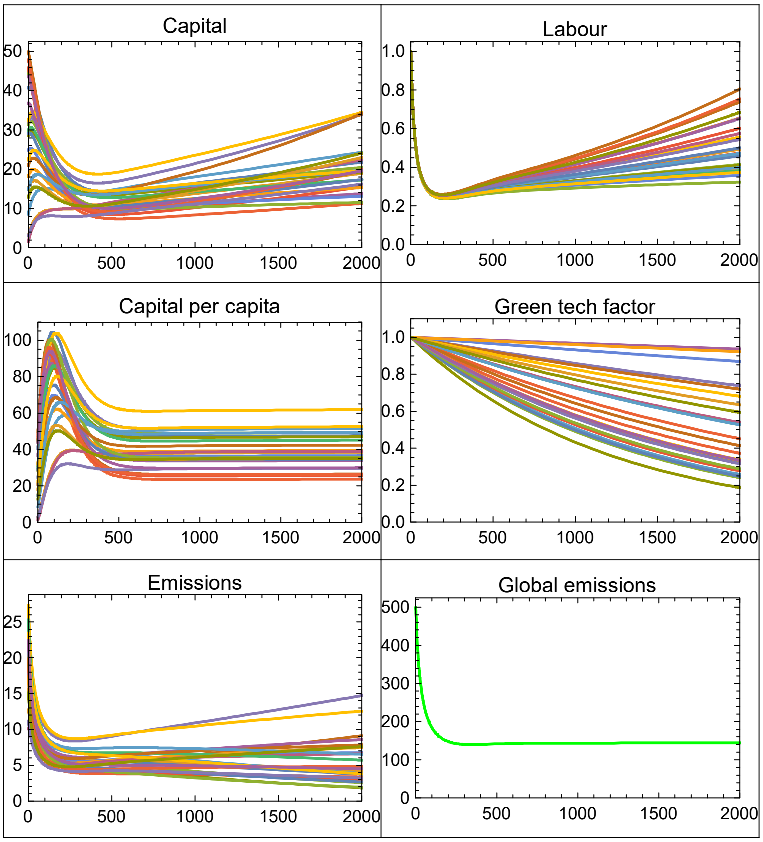
Labour thus peaks locally at a around T=300, at which point it starts falling. This ends up reducing emissions allowing a gradual re-improvement, but at a lower pace, in the growth rate of labour. This adjustment mechanism finalises around T=650 at which point we see total emissions and per capita capital become steady

3.1.1.2 Extending to n countries: Proof of concept

Most of the following simulations are run with only 2 countries. Our model can be run with any number of countries, but this makes it almost impossible to calibrate the parameters to obtain sensible output. We make the choice to run limited simulations for clarity of simulation and show-casing the possibilities of our model, and only include a few simulations with more countries. One such is seen here, with 25 countries initialized with random parameters.

In[ ]:=
```
IntrorandomMatrix =
  ParameterMatrixRandom[25, {{10, 8}, {0.02, 0.015}, {0, 0.01}, {0.3, 0.2}, {0.01, 0.01},
    {0.0145, 0.015}, {0.0001, 0.0001}, {1, 1}, {50, 1}, {1, 1}, {1, 1}, {0.1, 0.1}}];

SimulateGT[25, 2000, SimEquationsGT[25], IntrorandomMatrix, 1][[5]]
```

Out[ ]=


3.1.2 2-block “North-South” model

We can start by studying a simple 2-block, “North-South”/Developed-Developing model.
Block 1 starts with higher TFP (A), Capital intensity (a), savings (sc), capital (K0)
NB: In all plots with 2 blocks, block 1 is represented by a blue line, block 2 with a yellow line. Labels are omitted for cleaner graphs.

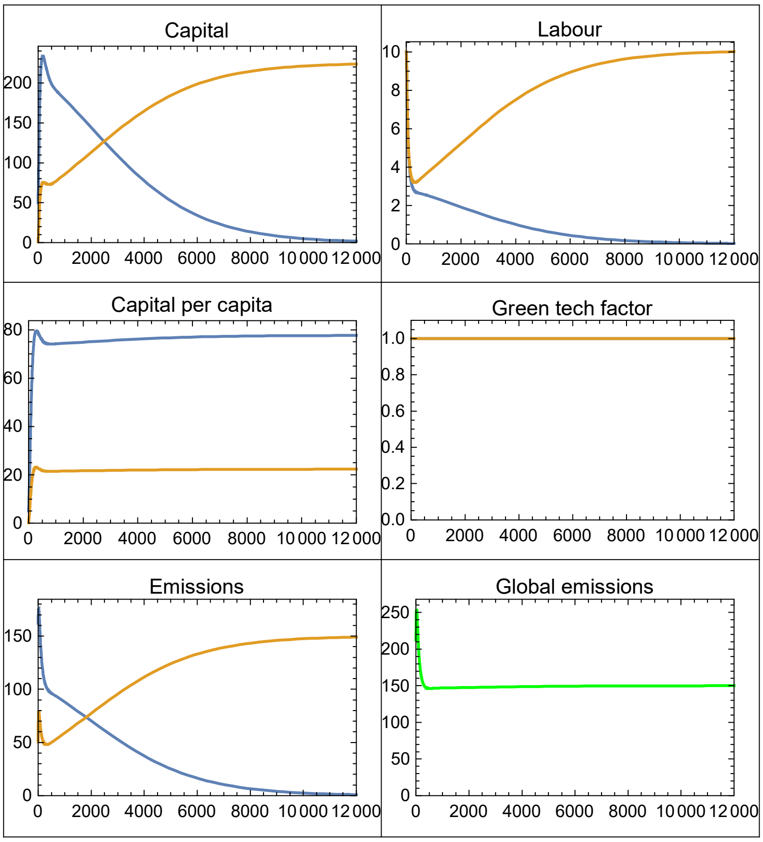
3.1.2.1 No green investment

We firstly examine the case with no savings dedicated to green investment: sc = 0.

3.1.2.1.1 Low damages, η

```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0, 0}, {0.3, 0.2}, {0.01, 0.01}, {0.0145, 0.015},
   {0.0001, 0.0001}, {1, 1}, {50, 1}, {10, 10}, {1, 1}, {0.1, 0.1}};
Sim1Eqs = SimEquationsGT[2];
Sim1 = SimulateGT[2, 12000, Sim1Eqs, Sim1PM, 1];
{Sim1[[5]], ParTableGT[Sim1PM]}
```

Out[=]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0	0
a	0.3	0.2
γ	0.01	0.01
β	0.0145	0.015
η	0.0001	0.0001
μ	1	1
K0	50	1
L0	10	10
Ω0	1	1
k	0.1	0.1

}

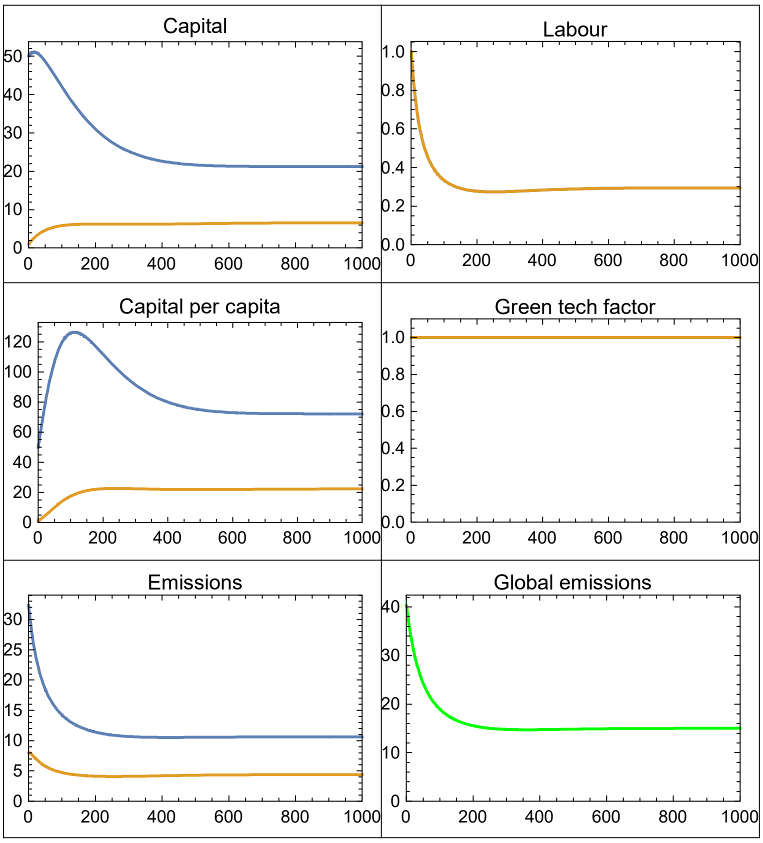
Here we run a model where we start one country with no possibility to invest in emissions reducing technology, and start the one country (North) with higher level of capita, TFP, savings rate, and slightly lower population growth rate, ceteris paribus,. We observe North sharply accelerating their overall capital investment, even though we see emissions, which in this case also represent output since there is no green tech factor, is crashing, which seems paradoxical. Isn't capital simply a fraction of output, which in this case is just pollution too? Then how could capital investment increase while output is falling? The distinction lies in appreciating the fact that the change in capital will be positive as long as saving, which is a fraction of output (and thus emissions here) is greater than depreciation. Even though output may be falling, if this is falling from a high value, it will allow capital to be increasing at first. It can only do so nonetheless, until the fall in output reaches such a point that depreciation outpaces capital investment.

We see that since North's labour diminishes, capital increases at first (as highlighted as above), such that per capita capital increases. However, after a while both growth rates start becoming negative, culminating in quite a strange process where as n tends to infinity, so do labour and capital, but in a ratio that **gives a higher capital per capita amount** than South! So while only glancing at the behaviour of capital per capita, we could conclude North still ends up benefiting more than the South. However, this example shows the limitation of our model in the sense we have not implemented a constraint on labour such that, similar to how we later model environmental capital and general models of animal populations, once a critical depensation threshold is reached the species collapses and cannot reach infinitely small amounts. Nonetheless, we do observe that although South starts with much lower parameters overall, it ends up becoming the more dominant country in terms of country's emissions as a fraction of total emissions. This is very likely due to extreme sensitivity of our model to population growth rates.

3.1.2.1.2 High damages (η)

```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0, 0}, {0.3, 0.2}, {0.01, 0.01},
   {0.015, 0.015}, {0.001, 0.001}, {1, 1}, {50, 1}, {1, 1}, {1, 1}, {0.1, 0.1}};
Sim1Eqs = SimEquationsGT[2];
Sim1 = SimulateGT[2, 1000, Sim1Eqs, Sim1PM, 1];
{Sim1[[5]], ParTableGT[Sim1PM]}
```

Out[ ]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0	0
a	0.3	0.2
γ	0.01	0.01
β	0.015	0.015
η	0.001	0.001
μ	1	1
K0	50	1
L0	1	1
Ω0	1	1
k	0.1	0.1

}

We observe that under the same scenario, although this time making sure to not let the model's be skewed by fertility rates, that with higher effects of pollution, we get very different results. Both countries and the whole world adjusts to a steady state of everything relatively quickly. Emissions in both countries fall to a level such that all growth paths balance correctly. These results also concord with our discussion in section 3.1.1.1, with our growth being fundamentally constrained by our production of emissions.

3.1.2.2 Constant green investment

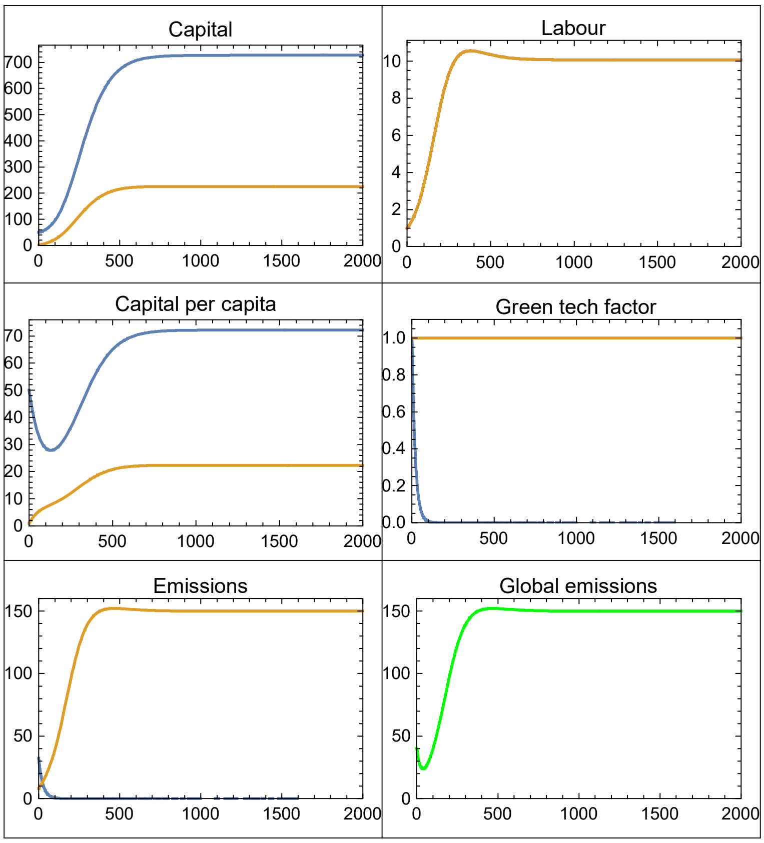
We can implement a constant green investment rate by choosing a constant value for s_g.

3.1.2.2.1 North invests 50% of output into green technology

In[ ]:=
```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0.5, 0}, {0.3, 0.2}, {0.01, 0.01},
   {0.015, 0.015}, {0.0001, 0.0001}, {1, 1}, {50, 1}, {1, 1}, {1, 1}, {0.1, 0.1}};
Sim1Eqs = SimEquationsGT[2];

Sim1 = SimulateGT[2, 2000, Sim1Eqs, Sim1PM, 1];
{Sim1[[5]], ParTableGT[Sim1PM]}
```

Out[ ]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.5	0
a	0.3	0.2
γ	0.01	0.01
β	0.015	0.015
η	0.0001	0.0001
μ	1	1
K0	50	1
L0	1	1
Ω0	1	1
k	0.1	0.1

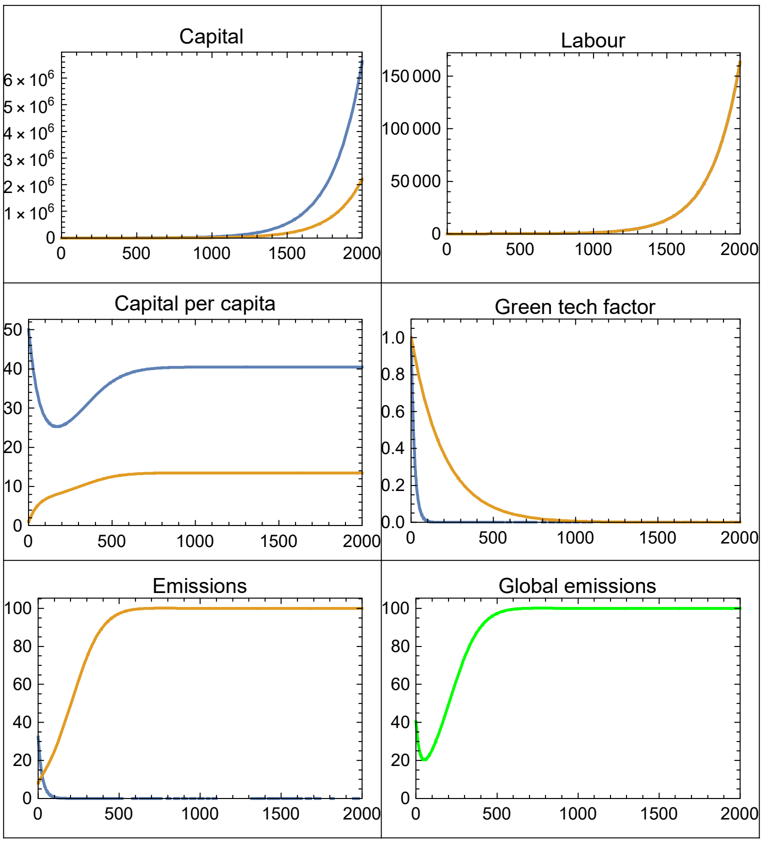
}

We here make only North invest into green technology. We observe that both countries converge to a steady state with North having a higher capital per capita, but no emissions produced, whilst South converges towards a lower steady state in per capita capital, even though North stops pollution entirely. We observe once again this process of autocorrection at play seen in the introductory simulation, with overshooting present in total emissions and labour.

3.1.2.2.2 North invests a lot, South a little, in green technology

```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0.5, 0.05}, {0.3, 0.2}, {0.01, 0.01},
   {0.015, 0.015}, {0.0001, 0.0001}, {1, 1}, {50, 1}, {1, 1}, {1, 1}, {0.1, 0.1}};
Sim1Eqs = SimEquationsGT[2];
Sim1 = SimulateGT[2, 2000, Sim1Eqs, Sim1PM, 1];
{Sim1[[5]], ParTableGT[Sim1PM]}
```

Out[ ]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.5	0.05
a	0.3	0.2
γ	0.01	0.01
β	0.015	0.015
η	0.0001	0.0001
μ	1	1
K0	50	1
L0	1	1
Ω0	1	1
k	0.1	0.1

}

However, if South invests even a tiny fraction too, such that both countries save, we observe an interesting finding : this diminishes level of steady sate capital per capita as compared to before! This could be because the incremental decrease in pollution (150 to 100) allows fertility rates to go through the roof, rising faster than capital. This leads us to think that the role of emissions in this system is once again as a “check” on labour growth rate and find that a certain level of emissions thus raises per capita capital, since this reduces labour.

This echoes some arguments (Bjorn, 1998) concerning the most efficient way to allow long run standards of living to rise is not through immediate investment into green technology or less emissions, but more pressing problems, such as global health, which could in theory also limit labour growth. Similarly, countries at first developing with emissions letting them accumulate a higher stock of output could raise the amount of capital to be invested in green technology in the future, and could also change the fertility rates, a finding corroborated by Chatterjee and Vog (2018).

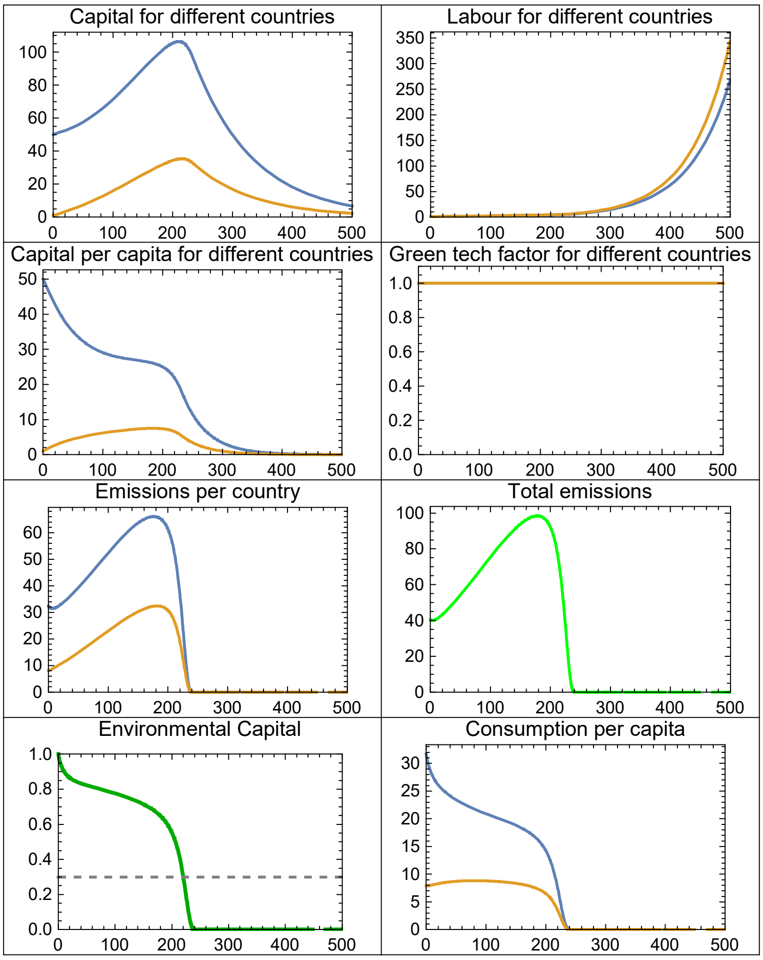
3.2 Green Solow model with emissions-reducing technological progress and environmental capital

3.2.1 2-block “North-South” model

3.2.1.1 No green investment

In[]:=
```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0, 0}, {0.3, 0.2}, {0.01, 0.01},
   {0.0145, 0.015}, {0.0001, 0.0001}, {1, 1}, {1, 1}, {50, 1}, {1, 1}, {1, 1},
   {0.25, 0.25}, {1, 1}, {0.3, 0.3}, {1, 1}, {0.005, 0.005}, {0.1, 0.1}};
SimulateEC[2, 500, SimEquationsEC[2], Sim1PM, 1][[5]];
ParTableEC[Sim1PM];
{%%, %}
```

Out[]=



	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0	0
a	0.3	0.2
γ	0.01	0.01
β	0.0145	0.015
η	0.0001	0.0001
μ	1	1
θ	1	1
K0	50	1
L0	1	1
Ω0	1	1
r	0.25	0.25
SU	1	1
SL	0.3	0.3
S0	1	1
χ	0.005	0.005
k	0.1	0.1

Here we run a similar model as in as before, with no green investment, 2.2.1, but introducing Environmental Capital as a necessary input to the production function, as well as plotting consumption per capita over time, which is simply output minus savings (green and normal). The clearest difference between this model and our previous is the disappearance of stabilization processes by both countries. Capital and emissions clearly increase up to a point which induces our environmental capital system to collapse, triggering unrecoverable damages to output and capital such that both capital per capita and consumption per capita completely crash for both countries. This could correspond greater than RCP 8.5 greenhouse gas trajectory modelled under the worst case scenario for climate change, which predicts widespread societal collapse due to the majority of planet becoming inhospitable by breakdown of our bodies cooling mechanisms' (Sherwood and Huber, 2010).

We also observe yet another limitation of the model: the way we've coded for labour growth does not depend on any level of consumption per capita, meaning once emissions cause production to crash, labour keeps growing infinitely. The same goes for the accumulation of capital, which as described before only depends on the previous level of capital, meaning this can accumulate even after emissions and output fully crash.

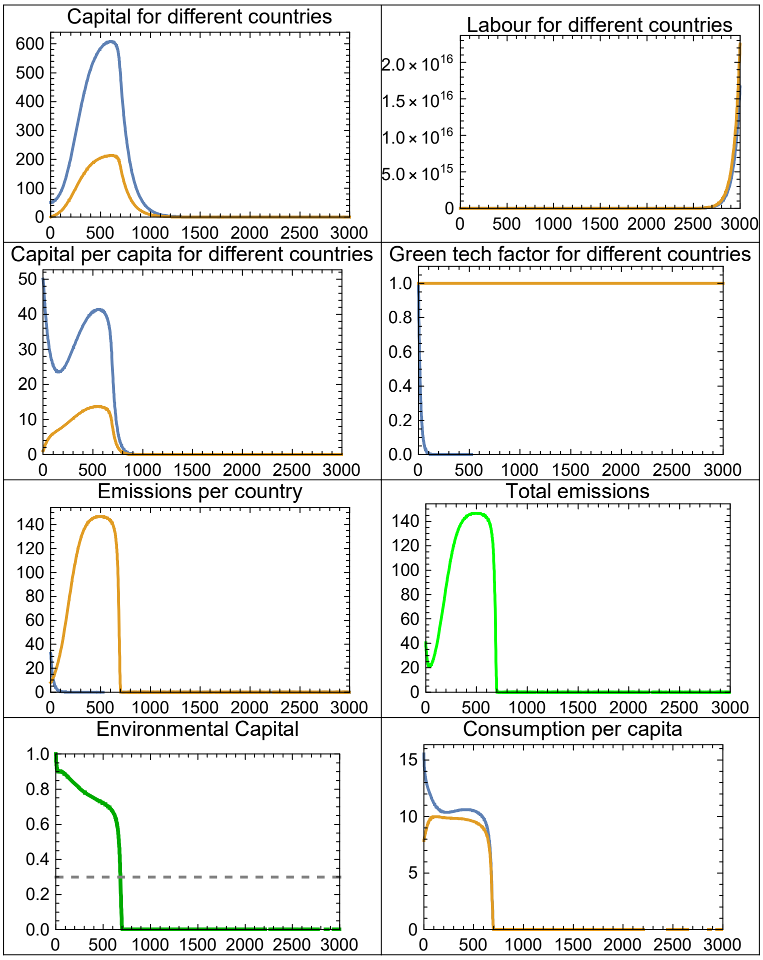
3.2.1.2 Constant green investment

3.2.1.2.1 North only invests in emissions reducing technology

In[22]:=

```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0.5, 0}, {0.3, 0.2}, {0.01, 0.01},
   {0.0149, 0.015}, {0.0001, 0.0001}, {1, 1}, {1, 1}, {50, 1}, {1, 1}, {1, 1},
   {0.25, 0.25}, {1, 1}, {0.3, 0.3}, {1, 1}, {0.0043, 0.0043}, {0.1, 0.1}};
SimulateEC[2, 3000, SimEquationsEC[2], Sim1PM, 1][[5]];
ParTableEC[Sim1PM];
{%%, %}
```

Out[25]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.5	0
a	0.3	0.2
γ	0.01	0.01
β	0.0149	0.015
η	0.0001	0.0001
μ	1	1
θ	1	1
K0	50	1
L0	1	1
Ω0	1	1
r	0.25	0.25
SU	1	1
SL	0.3	0.3
S0	1	1
χ	0.0043	0.0043
k	0.1	0.1

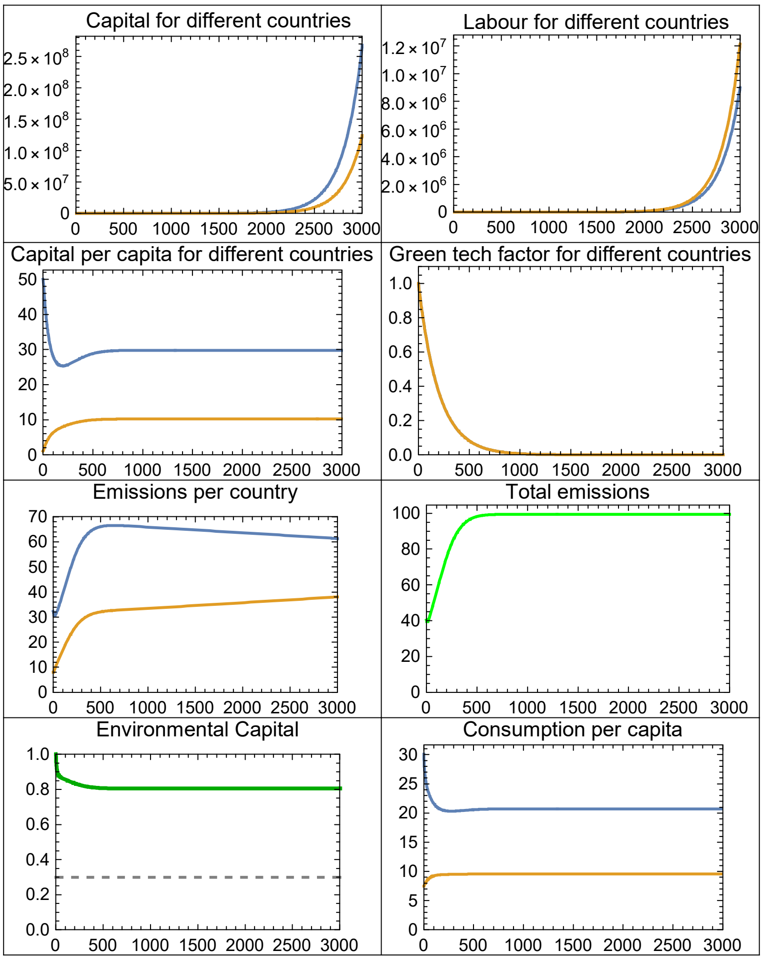
}

We run the same model as previously, except we make North invest **half their output** into green savings technology. We clearly see that this is not enough to prevent collapse, however we see that the extraordinary increase in green technology by North creates an immediate reduction in total emissions (from 40 to 20), but this is then clearly overtaken by the fact that South invests nothing into green technology and hence produces emissions, leading the system to fail.

3.2.1.2.2 Both invest in emissions reducing technology

In[26]:=
```
Sim1PM = {{10, 8}, {0.02, 0.015}, {0.05, 0.05}, {0.3, 0.2}, {0.01, 0.01},
    {0.0149, 0.015}, {0.0001, 0.0001}, {1, 1}, {1, 1}, {50, 1}, {1, 1}, {1, 1},
    {0.25, 0.25}, {1, 1}, {0.3, 0.3}, {1, 1}, {0.0043, 0.0043}, {0.1, 0.1}};
SimulateEC[2, 3000, SimEquationsEC[2], Sim1PM, 1][[5]];
ParTableEC[Sim1PM];
{%%, %}
```

Out[29]= {



,

	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.05	0.05
a	0.3	0.2
γ	0.01	0.01
β	0.0149	0.015
η	0.0001	0.0001
μ	1	1
θ	1	1
K0	50	1
L0	1	1
Ω0	1	1
r	0.25	0.25
SU	1	1
SL	0.3	0.3
S0	1	1
χ	0.0043	0.0043
k	0.1	0.1

}

However, if both countries only save **5%**, they avoid the crash. This is a very important policy finding as it suggests a tiny effort from both will go further than one country, even the richer one, investing everything it has. We observe an overshooting process followed by a correction in consumption per capita and capital per capita steady states for North.

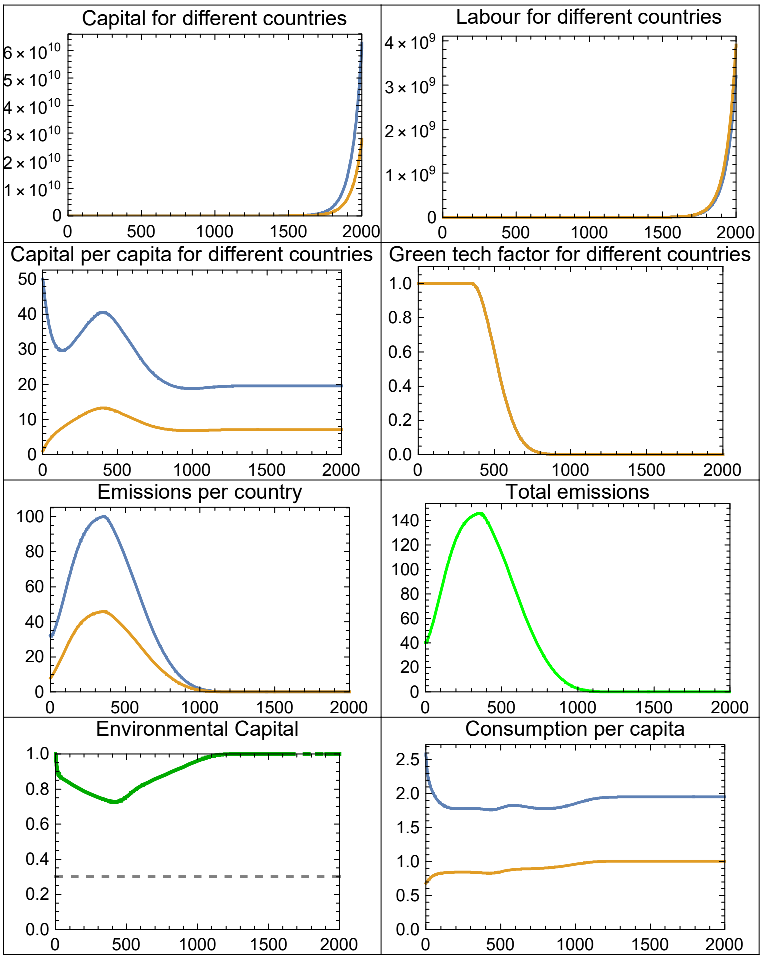
3.3 Policy Questions

3.3.1 How effective is policy change at a given time t?

3.3.1.1 Equal green investment

Occurring at t=350:

Out[∘]=



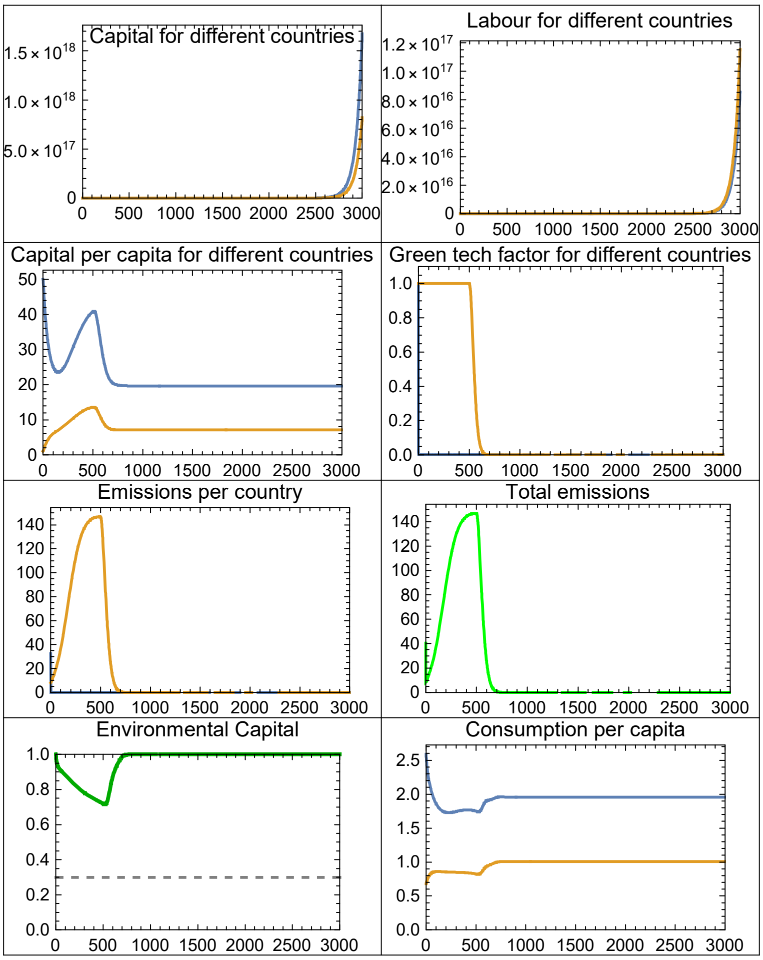
	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.9	0.9
a	0.3	0.2
γ	0.01	0.01
β	0.0149	0.015
η	0.0001	0.0001
μ	1	1
θ	1	1
K0	50	1
L0	1	1
Ω0	1	1
r	0.25	0.25
SU	1	1
SL	0.3	0.3
S0	1	1
χ	0.0043	0.0043
k	0.1	0.1

We now allow for there to be a change in green investment at t=350, with both countries investing 90% of their capital into green technology from an original level of 0. We observe environmental capital endure negative growth rates until the new policy implemented, which limits emissions to such an extent that emissions crash, and environmental capital fully recovers. Per capita capital does exhibit some strange oscillating behaviour around t=350. We observe it first diminishes, then increases, before falling to a lower steady state level than previously.

3.3.1.2 Capita-per-capita scaled green investment for 2 countries, simultaneous

Now introduce a model where green investment is scaled by capital per capita.
In the first case, only 2 countries, they all start investing after a certain period of time.

```
SimulateEC[2, 3000, eqs, Sim1PM, 1][[5]];
ParTableEC[Sim1PM];
{%%, %}
```

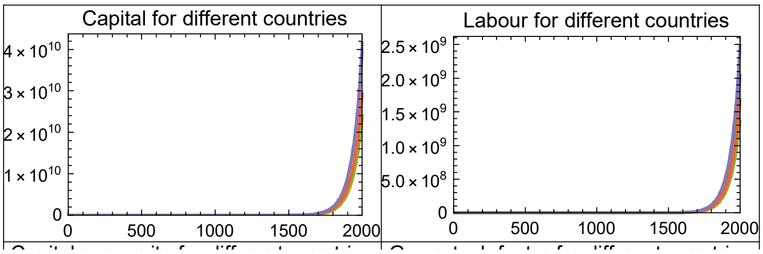


	Country 1	Country 2
A	10	8
s_c	0.02	0.015
s_g	0.9	0.9
a	0.3	0.2
γ	0.01	0.01
β	0.0149	0.015
η	0.0001	0.0001
μ	1	1
θ	1	1
K0	50	1
L0	1	1
Ω0	1	1
r	0.25	0.25
SU	1	1
SL	0.3	0.3
S0	1	1
χ	0.0043	0.0043
k	0.1	0.1

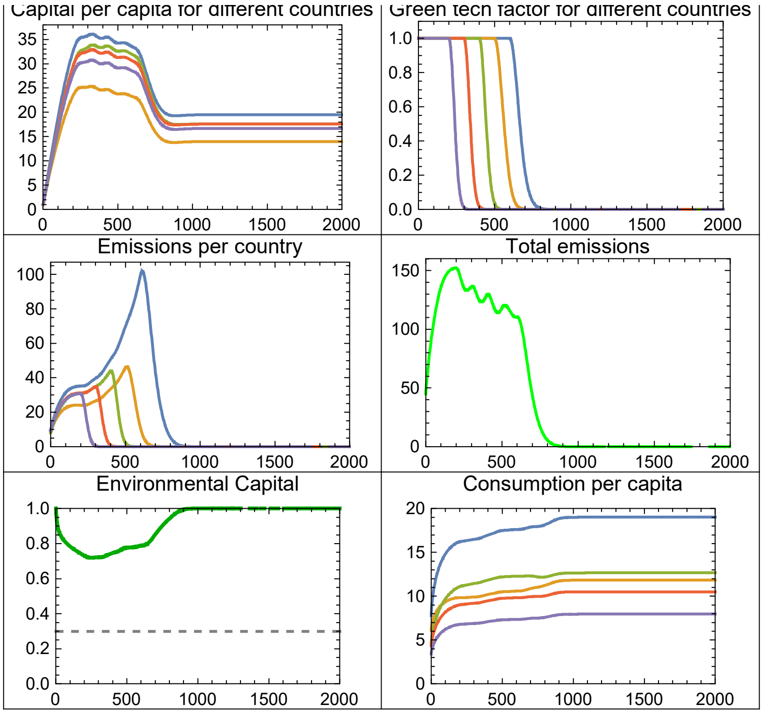
Note that the sg inputs in the above table are not correct, as these are then scaled per capital per capita. We observe that under this scenario, environmental capital recovers quicker and total emissions fall more dramatically. Note North immediately acquires pollution-reducing green technology, while South first waits a little bit. This is a reassuring in the global context, although obvious calibration is further needed.

3.3.1.3 Capita-per-capita scaled green investment for 5 countries, random initial conditions, lagged investment times

```
SimulateEC[5, 2000, PanicSim5, Sim5PM, 1][[5]];
ParTableEC[Sim5PM];
{%%, %}
```



Out[ ]= {



,

	Country 1	Country 2	Country 3	Country 4	Country 5
A	10	9	8	9	9
s_c	0.02	0.02	0.02	0.02	0.02
s_g	0.2	0.3	0.4	0.5	0.6
a	0.3	0.25	0.35	0.31	0.3
γ	0.01	0.01	0.01	0.01	0.01
β	0.015	0.0149	0.0148	0.0149	0.0151
η	0.0001	0.0001	0.0001	0.0001	0.0001
μ	1	1	1	1	1
θ	1	1	1	1	1
K0	1	1	1	1	1
L0	1	1	1	1	1
Ω0	1	1	1	1	1
r	0.25	0.25	0.25	0.25	0.25
SU	1	1	1	1	1
SL	0.3	0.3	0.3	0.3	0.3
S0	1	1	1	1	1
χ	0.0043	0.0043	0.0043	0.0043	0.0043
k	0.1	0.1	0.1	0.1	0.1

}

We can clearly observe the effect of lagged green investment: every time a country starts investing into technology emissions: we observe a dip in total emissions, which tend to pick up again until the next country reduces their emissions. We observe these investments all take place within enough time to avoid a tipping point collapse.

3.3.2 Is there an optimal global target for investment in green technology?

An interesting application that we could explore through this model is finding an international target for the s_g parameter which, if implemented by all countries, would maximize the sum of the present values of total utility of per-capita consumption in all regions of the world, i.e. $\sum_{i=1}^{n} PV_i = \int_0^T \left[L_i[t] * U\left(\frac{C_i[t]}{L_i[t]}\right)\right] e^{-\delta * t}\, dt$. To find an approximation for this international target, we would first need to fix the values for all the other parameters in the model, and then run the model for these parameters, which remain fixed, coupled with different targets for s_g. For all the s_g targets considered, the model can calculate the present values of total utility in all the regions and sum these up. The optimal international target for s_g will correspond to the largest sum. We can pick s_g targets at a regular interval, such as 0.005. For example, we could consider $s_g \in \{0, 0.005, 0.01, 0.015, ...\}$. Obviously, this method will only allow us to find a rough approximation for the optimal target.

In this application, we will consider a world that has three regions. The first step is to fix the other parameters for the three regions. We will randomly pick some parameters from specified ranges, while other parameters are constant for all regions:

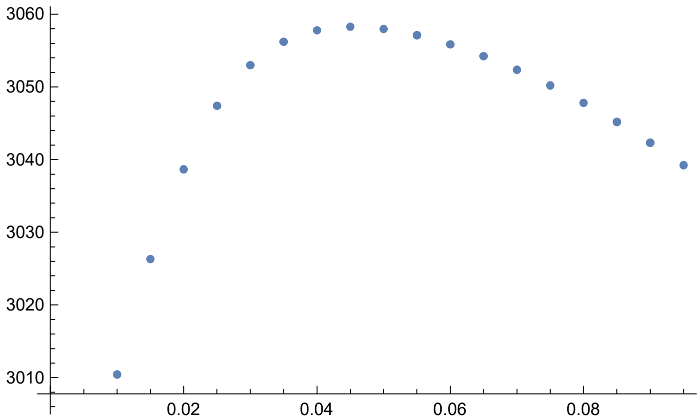
```
ThreeRegionParMatrixEC = ParameterMatrixGreenTech[3, {{9, 11}, {0.04, 0.06}, {0.3, 0.3}, {0.025, 0.035}, {0.009, 0.011}, {0.0000001, 0.0000001}, {1, 1}, {1, 1}, {1.8, 2.2}, {1.8, 2.2}, {1, 1}, {0.2, 0.2}, {1, 1}, {0.3, 0.3}, {1, 1}, {0.0016, 0.0016}, {0.2, 0.2}}];
```

```
Out[•]= {{9.5731, 10.219, 9.51794}, {0.0561592, 0.0585907, 0.0449503}, {0.3, 0.3, 0.3},
 {0.0346525, 0.0301739, 0.0311}, {0.0092477, 0.00970828, 0.0109773},
 {1. × 10^-7, 1. × 10^-7, 1. × 10^-7}, {1., 1., 1.}, {1., 1., 1.}, {1.90798, 2.19251, 1.90638},
 {2.16875, 1.87029, 1.97041}, {1., 1., 1.}, {0.2, 0.2, 0.2}, {1., 1., 1.},
 {0.3, 0.3, 0.3}, {1., 1., 1.}, {0.0016, 0.0016, 0.0016}, {0.2, 0.2, 0.2}}
```

The parameters above are, in order: A, s_c, a, γ, β, η, μ, θ, K0, L0, Ω_0, r, SU, SL, S0, x, k. We haven't included s_g yet, which comes between s_c and a.

In what follows, we run the model for the above parameters coupled with different s_g targets for 1000 time units. In the For loop below, we start at s_g=0.01, and then progressively increase s_g by 0.005 until it reaches 0.095. We start at s_g=0.01 because the economy and the environment collapse for smaller values before t=1000, and for some reason the sums of the present values evaluate to complex numbers if this happens.

```
ListPlot[lis]
```

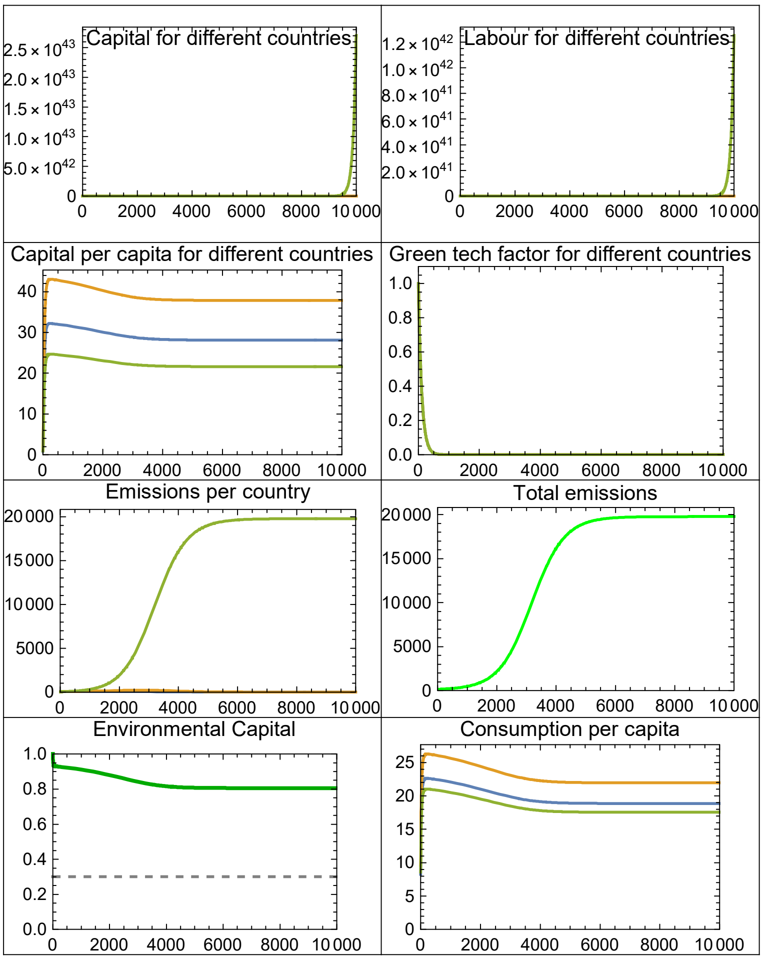


It appears that the optimal international target for s_g, given our chosen parameters and considering only the first 1000 time periods, is 0.045. We can check whether this target will lead to a collapse in the longer term:

```
OptimalThreeRegionParMatrixEC =
  Insert[ThreeRegionParMatrixEC, {0.045, 0.045, 0.045}, 3];
```

```
OptimalECSim =
  SimulateEC[3, 10000, SimEquationsEC[3], OptimalThreeRegionParMatrixEC, 0.016];
```

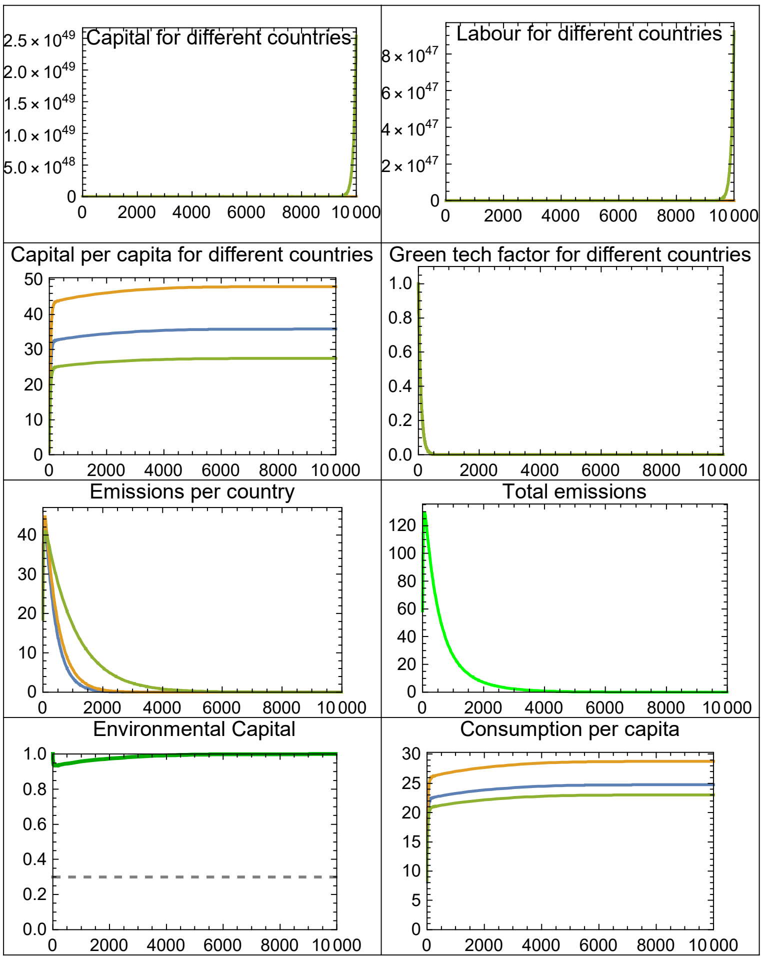
```
OptimalECSim[[5]]
```



It seems that environmental capital stabilizes around 0.8 in the longer term but it doesn't recover to the original value of 1 when s_g=0.045. If we were to pick a higher s_g, such as s_g=0.06, then environmental capital would have recovered to 1:

In[ ]:=
```
ThreeRegionParMatrixECwithsg = Insert[ThreeRegionParMatrixEC, {0.06, 0.06, 0.06}, 3];
ECSim = SimulateEC[3, 10000, SimEquationsEC[3], ThreeRegionParMatrixECwithsg, 0.016];
ECSim[[5]]
```

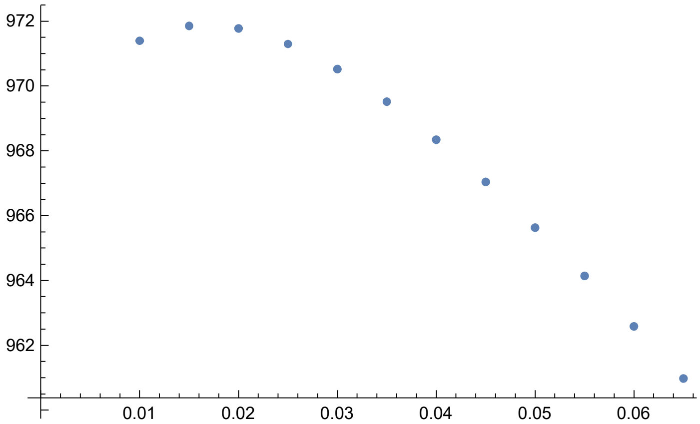
Out[ ]=



3.3.3 Do regions have an incentive to deviate from the global target for investment in green technology?

Even though a higher s_g might be better in the longer run, let's assume that the s_g target of 0.045 is implemented. An interesting thing to see now would be whether one of the three regions has an incentive to disregard the international target and instead implement the s_g which maximizes its own present value of total utility, given that the two other regions have implemented the international target. In the For loop below, we fix s_g at 0.045 for regions 2 and 3, but we let s_g start at 0.01 and increase progressively by 0.005 up to 0.065 for region 1, to see whether region 1 can obtain a higher present value of total utility with another s_g target:

In[ ]:=
```
ListPlot[lisR1]
```

Out[ ]=


Interestingly, the optimal s_g for region 1 would be 0.015 instead of the international target of 0.045. We therefore find that region 1 would have an incentive to free ride on the mitigation of the other 2 regions. However, lowering s_g by too much would not be optimal. Below we look at the present values of total utilities for the three regions when s_g=0.045 is implemented by all three regions and when s_g=0.045 is only implemented by regions 2 and 3 while region 1 adopts s_g=0.015:

In[ ]:=
```
OptimalThreeRegionParMatrixEC =
  Insert[ThreeRegionParMatrixEC, {0.045, 0.045, 0.045}, 3];
OptimalECSim = SimulateEC[3, 1000, SimEquationsEC[3],
   OptimalThreeRegionParMatrixEC, 0.016];
PVUOptimalECSim = Table[NIntegrate[OptimalECSim[[4, i]], {t, 0, 1000}], {i, 1, 3}]
Total[PVUOptimalECSim]
```

Out[ ]= {967.04, 937.573, 1153.66}

Out[ ]= 3058.28

In[ ]:=
```
OptimalR1ThreeRegionParMatrixEC =
  Insert[ThreeRegionParMatrixEC, {0.025, 0.045, 0.045}, 3];
OptimalR1ECSim = SimulateEC[3, 1000, SimEquationsEC[3],
   OptimalR1ThreeRegionParMatrixEC, 0.016];
PVUOptimalECSim = Table[NIntegrate[OptimalR1ECSim[[4, i]], {t, 0, 1000}], {i, 1, 3}]
Total[PVUOptimalECSim]
```

Out[ ]= {971.296, 934.641, 1148.3}

Out[ ]= 3054.23

The sum of the present values of total utility for all regions falls from 3058 to 3054, but the present value of region 1 increases from 967 to 971. The other two regions could use the results from this model to perhaps find an appropriate penalty to impose on region 1 if it doesn't adopt the international target.

This application shows that this model can be used to evaluate different mitigation policies for a particular region by calculating the present value of the total utility of per-capita consumption in that region that results from implementing different targets for s_g, given that other regions have already chosen their s_g targets. Thus, after finding the optimal global s_g target, we find find that regions will benefit individually from disregarding the international target and implementing lower investments in green technology, given that the other regions respect the target. This indicates that an equilibrium of international climate cooperation, where all regions implement the optimal global s_g, is unlikely to be reached, at least in the absence of other mechanisms such as international sanctions. Our findings are in line with what the literature predicts on this topic. Nordhaus (2015) analyzes whether a cooperative equilibrium can be reached in climate agreements. Countries are assumed to adopt climate policies that maximize their national economic welfare. One of the major results in the study is that a regime without trade sanctions will dissipate to the low-abatement (low s_g in our case), non-cooperative equilibrium, and that the global non-cooperative carbon price and control rate will be in the order of one-tenth of the efficient cooperative levels. The fact that our analysis shows that region 1 can capture a higher present value of population utility (economic welfare) for itself by lowering green investment (abatement) shows that regions implementing the efficient global target isn't an equilibrium. However, Nordhaus finds that a regime with small trade penalties on non-participants (countries that don't implement the optimal global target for abatement), a Climate Club, can induce a large stable coalition with high levels of abatement. Our model can provide an indication of the magnitude of the sanctions that would need to be imposed to induce a cooperative equilibrium. If regions 2 and 3 imposed sanctions that lowered region 1's present value of utility by 5 if it disregarded the global target, then region 1 would have been induced to cooperate. Similar values could have been computed for the other two regions.

4.Limitations and extensions

Our extended model and conducted simulations are based on number of assumptions that allowed us to arrive at sensible results but at the same time are limiting its predictive and interpretive power.

From a purely practical perspective, it is clear from the simulations that our model needs further calibration. We've observed extreme sensitivity to multiple parameters, particularly fertility rate, which limits our ability to engage in any type of realistic policy discussion. One limitation to our approach concerns the structure of of labour growth rate equation, which does not depend on a minimal level of per capita consumption. As a consequence, we observe that under total system crashes, labour keeps growing, which is obviously unrealistic. Furthermore, we observed under the first simulations that sometimes both capital and labour tended to 0 with infinity, but that this ratio kept a certain steady level of capital per capita, a nonsensical result. Further work could include a constraint on this type of behaviour.

A more "structural" limitation of our work lies at the base of our model. The use of the Solow model as groundwork for analysing how economic growth can be achieved in tandem with environmental protection, and the spillover growth effects between countries, is problematic. The Solow model assumes that the savings rate - the key factor for determining the level of capital intensity (Sharipov, 2015), leading to economic growth - is exogenous. This means that the savings rate is determined outside of the model, and not by its agents. It does not include microeconomic factors, and is not forward-looking. Hence, it fails to accurately model economic growth behaviour (McCallum, 1996), as the rate of growth is determined exogenously, and is independent of preferences and policy implementations. The government can thereby have no impact on long-term growth, as the only way it can act is through policy instruments affecting the savings rate. These failings have been stressed by Romer & Lucas, and the neoclassical growth models have been rejected to introduce endogenous growth models (e.g. the Ramsey-Cass-Koopmans model).

Construction of the model also leaves a room for a debate. We relax neoclassical assumption of economy's independence by introducing the interaction of two or more countries or regions through the global emissions level. We assume this is the only channel of interaction and otherwise the model is spatially homogeneous i.e. it does not take into account labour or capital mobility between countries or regions. This hence does not allow for the possibility of countries to invest into each other in the same way global economies do, which could be an interesting extension, perhaps by extending this model via network theory.

Although it still produces suggestive results in simulations, the biophysic, economic and social system is much more nuanced. For instance, we motivate the introduction of environmental capital S[t] by existence of environmental damage via multiple channels e.g., rising sea levels eroding coastal areas. Such risk will surely bring harm to country's i output but in reality, it would also affect L[t] and consequently Y[t] of many neighboring countries or regions due to large-scale coastal migration over the long term (Lincke and Hinkel, 2021). Migration of highly skilled workers would also affect L[t] of neighbouring country and likely yield productivity gains for destination country through A_i, as has been the case with the circular migration between "North" and "South" over last two decades (Samet, 2013). Hence, there are other relevant spatial channels of interactions that, if accounted for, could predict different results. In fact, Camacho and Zhou (2004) extended the Solow-Swan model to the open economy by including factor mobility and found that such countries not always converge to a steady-state.

Similarly, the underlying neoclassical assumption of substitutability between capital and labour is limiting in the context of our project. In simulations in which the environmental damage depletes S[t], moves economy beyond the tipping point and consequently makes output converge to zero (what mitigates capital accumulation $sc_i \times F[K_i[t], L_i[t], S[t]]$ and from that point forward capital also depreciates to zero) our model predicts infinite increase in the the level of labour. Although the results are still interpretable, the infinite and non-linear growth of `Li[t]` is not a sensible prediction here since as the capital depletes, declining "capital deepening effect" would cause the decline in the productivity of labor. This calls for more elaborate construction of the differential equation for `Li[t]` but more broadly, is also one of many arguments for endogenizing the productivity parameter. Models in which this is the case would, depending on parametric set-up, either predict environmental disaster consistently across all factors or would allow agents to try to avoid it by optimizing savings rate accordingly (e.g. Ramsey-Cass-Koopman).

We also specify that the "green tech factor" $\Omega_i[t]$ is strictly decreasing i.e. the fraction of output that generates harmful emissions approaches zero with each subsequent time period at a constant rate. This doesn't necessarily have to be the case for all countries and regions. Depending on the baseline level of regulation at t=0 it may stay constant for several periods when little is done in a given country to mitigate harmful emissions. In the open economy this may be even more subtle. Tight environmental regulation (high sg_i requirements) in country i may encourage domestic firms to shift their profitable polluting activities to some other country j with weaker level of regulation - a phenomenon called Pollution Haven Hypothesis (PHH) (Bogmans and Withagen, 2010). What follows, backward countries or regions may adopt lenient environmental regulation to attract foreign companies. Therefore, even though the effect of domestic regulation on global level of pollution (both in our model and according to studies (Ben-David et al. 2021)) is clearly positive, the country/region-level effect of sg_i on output is not unambiguous if firms and regulators follow PHH.

Another signification limitation comes from formulation of model's parameters. For convenience, we make strong assumption of time-invariance of parameters whereas there is some contrary evidence in studies using Solow-Swan model. For instance, Park and Ryu (2006) reported higher capital and labor elasticities (a_i and 1-a_i) in the early stage of development of East Asian economies suggesting increasing returns to scale. Similarly, Dhawan and Gerdes (1997) found declining productivity in the U.S. between 1970 to 1989. This also speaks to our previous point - the assumption of exogenous which is the major shortcoming of this class of models. Romer (1990) along with Aghion and Howitt (1992) shifted the endogenous growth paradigm by postulating that the technological change is in fact determined from within the model by the process of innovation with increasing returns to scale driven by entrepreneurs and researchers. Under this view long-term growth in output per capita is attributed to technological change rather than capital accumulation (Aghion and Howitt, 2007) what accordingly changes optimal policy decisions. For the above reasons, the multiple countries model could be misspecified if e.g. growth of certain developing countries is much better characterized by time-invariant and not independent parameters over large time period T. Extending our simulation to include time-varying parameters would be very difficult given that we would need to simulate both the stocks of L[t], K[t], S[t] and time-varying parameters at once, although it has been done using EKF - based estimation (Munguia et al. 2019).

The Ramsey growth model, although much more computationally intensive, lends itself to many interesting extensions that would fit the of narrative of our project. The most relevant one would compare (on top of global emissions influencing fertility rate/labour growth) output growth paths with utility functions which would interact perceived emission mitigation efforts of multiple countries, allowing for simulation of optimal collaborative emission reduction schemes, similar to RICE model (Nordhaus, 2013) but with a different interpretation of natural capital. A Ramsay model extension similar to this one here could also incorporate utilities being interdependent: thus allowing endogeneous notions of altruism to determine the savings rate.

In addition, to mitigate the difficulties with calibrating the model against the estimates from existing literature and to attempt to make it more predictive, another possible extension would be (after solving the model for the given path equation) estimating chosen parameters by OLS or MLE from a publicly available (e.g. OECD) data for output, labour and capital stock for sample of countries over different sectors similar to the study of Boyko et al. (2020). Obtaining data on environmental capital could be more problematic given no records of its accounts in which case one could use the accounting value estimates of environmental capital of Managi and Kumar (2018) as suggested in the

Dasgupta Review. For the missing data points of natural capital, one could use imputation methods or fitted values (what would also remove country specific variance from cyclicality of technological progress, fertility rates or capital depreciation if one estimated stochastic parameter over large T). Although not ideal, this method with large-enough sample would improve its predictive power, but more importantly bring us closer to having well calibrated model and allow for more interesting simulation ideas.

5. Conclusion

Throughout this report, we have argued extending a simple Green Solow model to multiple countries by incorporating the notion of environmental capital and extending it to multiple countries interacting with a shared pool of emissions is a useful step in the literature on growth models, particularly because neoclassical production functions, even in the case of climate change and other environmental issues, never incorporate the idea of the environment being an input in our production. This has provided us with a range of different simulations to be run, providing us with interesting glimpses at what this model could become once properly calibrated. A key finding in the first simulations without environmental capital is the presence of Malthusian corrections: without any green investment, the dynamic pathway of our emissions will play a key role in our per capita capital levels. In line, we have on numerous occasions observed “overshooting” processes of emissions, highlighting the autocorrecting mechanisms at play.

However, once we introduced environmental capital, our model became more unstable, principally via the role of a tipping point. This now constrained the notion that we could always pollute and engage in environmental mitigation when we felt it necessary, and in theory would introduce forward-looking behaviour, through the exogenous role of government in choosing the amount invested into green technology. We’ve observed that small efforts by countries often supersede one large investment by one country, opening up the model to game theoretical ideas. We further investigated this by proving that within the context of our model, a region would benefit from deviating from a pre-calculated optimal international target for emissions reduction: the well known prisoner’s dilemma.

We stand by this report not as a useful policy tool, but as a stepping stone in a new type of green growth models, which through appropriate calibration and adjusting of equations as outlined might provide new and insightful alternatives to the current models used in the literature.

6. Literature and Code appendix

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Green Tech Simulation Code

In[1]:=
```
ParameterMatrixRandom[N_, ChoiceRanges_] :=
 Table[RandomReal[ChoiceRanges[[p]], N], {p, Length@ChoiceRanges}]
```

This command generates the basic set of equations underlying the model, with no policy intervention.

In[2]:=
```
SimEquationsGT[N_] := Table[{
    ∂t KKi[t] == sci * Ai * (KKi[t])^ai (Li[t])^(1-ai) - γi * KKi[t],
    KKi[0] == K0i,
    ∂t Li[t] ==
      (βi - ηi * Sum[ μj * Ωj[t] * Aj * KKj[t]^aj Lj[t]^(1-aj), {j, 1, N}]) * Li[t],
    Li[0] == L0i,
    ∂t Ωi[t] == - ki * sgi * Ωi[t],
    Ωi[0] == Ω0i},
   {i, 1, N}];
```

In[3]:=
```
ParTableGT[M_] := Block[
  {ParListGT = {"", "A", "sc", "sg", "a", "γ", "β", "η", "μ", "K0", "L0", "Ω0", "k"}},
  Inter = Prepend[M, Table[StringForm["Country ``", i], {i, 1, Length@M[[1]]}]];
  Final = MapThread[Prepend, {Inter, ParListGT}];
  grid = Grid[Final, Frame -> All,
    Background -> {{LightGray, None}, {LightGray, None}}, Spacings -> {1, 1}];
  Return[grid]
 ]
```

This command (numerically) solves the specified system with the given parameters.

In[4]:=
```
NSolGT[ParametersMatrix_, T_, eqns_, vars_] :=
 Block[
  {eqs = Flatten[eqns /. Flatten[Table[{
         A_i -> ParametersMatrix[[1, i]],
         sc_i -> ParametersMatrix[[2, i]],
         sg_i -> ParametersMatrix[[3, i]],
         a_i -> ParametersMatrix[[4, i]],
         γ_i -> ParametersMatrix[[5, i]],
         β_i -> ParametersMatrix[[6, i]],
         η_i -> ParametersMatrix[[7, i]],
         μ_i -> ParametersMatrix[[8, i]],
         K0_i -> ParametersMatrix[[9, i]],
         L0_i -> ParametersMatrix[[10, i]],
         Ω0_i -> ParametersMatrix[[11, i]],
         k_i -> ParametersMatrix[[12, i]]
        },
        {i, 1, Length@ParametersMatrix[[1]]}]]]},
  Return[NDSolve[eqs, vars, {t, 0, T}][[1]]]
 ]
```

These commands generate the emissions and consumption functions, based on the NDSolve output.

In[5]:=
```
EmissionsFunctionGT[NDSols_, ParameterList_] :=
 Block[{A = ParameterList[[1]], a = ParameterList[[4]],
   μ = ParameterList[[8]], N = Length@NDSols / 3},
  Return[Table[μ[[i]] * NDSols[[i + 2 * N]][t] * A[[i]] *
     (NDSols[[i]][t])^a[[i]] * (NDSols[[i + N]][t])^(1-a[[i]]), {i, 1, N}]]]

ConsumptionFunctionGT[NDSols_, ParameterList_] :=
 Block[{A = ParameterList[[1]], sc = ParameterList[[2]],
   sg = ParameterList[[3]], a = ParameterList[[4]], N = Length@NDSols / 3},
  Return[Table[(1 - sc[[i]] - sg[[i]]) A[[i]] * (NDSols[[i]][t])^a[[i]] *
     (NDSols[[i + N]][t])^(1-a[[i]]), {i, 1, N}]]]

UtilityFunctionGT[NDSols_, ParameterList_, δ_] :=
 Block[{A = ParameterList[[1]], sc = ParameterList[[2]],
   sg = ParameterList[[3]], a = ParameterList[[4]], N = (Length@NDSols) / 3},
  Return[Table[Log[((1 - sc[[i]] - sg[[i]]) A[[i]] *
        (NDSols[[i]][t])^a[[i]] * (NDSols[[i + N]][t])^(1-a[[i]])) /
       NDSols[[i + N]][t]] * NDSols[[i + N]][t] * E^(-δ*t), {i, 1, N}]]]
```

This command generates a grid of graphs with all the

In[8]:=
```
PlotEvolutionGT[IFList_, EMFuncs_, CFuncs_, T_, ParametersMatrix_] :=
 GraphicsGrid[{
   {Plot[Evaluate@Table[IFList[[i]][t], {i, 1, Length@IFList / 3}], {t, 0, T},
     PlotLabel → "Capital", PlotRange -> {{0, T}, {0, All}}, Frame → True],
    Plot[Evaluate@Table[IFList[[i]][t],
       {i, Length@IFList / 3 + 1, 2 * Length@IFList / 3}], {t, 0, T},
     PlotLabel → "Labour", PlotRange -> {{0, T}, {0, All}}, Frame → True]},

   {Plot[Evaluate@
      (Table[IFList[[i]][t], {i, 1, Length@IFList / 3}] / Table[IFList[[
           i + Length@IFList / 3]][t], {i, 1, Length@IFList / 3}]), {t, 0, T},
     PlotLabel → "Capital per capita", PlotRange -> {{0, T}, {0, All}},
     Frame → True],
    Plot[Evaluate@Table[IFList[[i]][t], {i, 2 * Length@IFList / 3 + 1,
        Length@IFList}], {t, 0, T}, PlotLabel → "Green tech factor",
     PlotRange -> {{0, T}, {0, 1.1}}, Frame → True]},

   {Plot[Evaluate@EMFuncs, {t, 0, T}, PlotLabel → "Emissions",
     PlotRange -> {{0, T}, {0, All}}, Frame → True],
    Plot[Total[EMFuncs], {t, 0, T}, PlotStyle → Green, PlotLabel →
      "Global emissions", PlotRange -> {{0, T}, {0, All}}, Frame → True]}},

  ImageSize → {400, 500}, Frame -> All,
  Spacings → {0, 0}, Alignment → Center, AspectRatio → 1.1]
```

All simulation

In[9]:=
```
SimulateGT[N_, T_, SimEqs_, ParameterMatrix_, δ_] :=
  Block[{SimVars =
     Union[Table[KKᵢ, {i, 1, N}], Table[Lᵢ, {i, 1, N}], Table[Ωᵢ, {i, 1, N}]]},
   SimOutput = SimVars /. NSolGT[ParameterMatrix, T, SimEqs, SimVars];
   SimEmissions = EmissionsFunctionGT[SimOutput, ParameterMatrix];
   SimConsumption = ConsumptionFunctionGT[SimOutput, ParameterMatrix];
   SimUtility = UtilityFunctionGT[SimOutput, ParameterMatrix, δ];
   SimPlot = PlotEvolutionGT[SimOutput,
     SimEmissions, SimConsumption, T, ParameterMatrix];

   Return[{SimOutput, SimEmissions, SimConsumption, SimUtility, SimPlot}]];
```

Environmental Capital Simulation Code

In[10]:=

```
SimEquationsEC[N_] := Flatten@Union[Table[{
      ∂t KKi[t] == sci * Ai * (KKi[t])^ai (Li[t])^(1-ai) S[t]^θi - γi * KKi[t],
      KKi[0] == K0i,
      ∂t Li[t] ==
       (βi - ηi * Sum[μj * Ωj[t] * Aj * KKj[t]^aj Lj[t]^(1-aj) S[t]^θj, {j, 1, N}]) * Li[t],
      Li[0] == L0i,
      ∂t Ωi[t] == - ki * sgi * Ωi[t],
      Ωi[0] == Ω0i},
    {i, 1, N}], {
    ∂t S[t] == r * S[t] * ((SU - S[t])/SU) ((S[t] - SL)/SU) -
      χ * Log[1 + Sum[μj * Ωj[t] * Aj * KKj[t]^aj Lj[t]^(1-aj) S[t]^θj, {j, 1, N}]],
    S[0] == S0}];
```

In[11]:=

```
ParTableEC[M_] :=
 Block[{ParListEC = {"", "A", "sc", "sg", "a", "γ", "β", "η", "μ", "θ", "K0",
     "L0", "Ω0", "r", "SU", "SL", "S0", "χ", "k"}},
  Inter = Prepend[M, Table[StringForm["Country ``", i], {i, 1, Length@M[[1]]}]];
  Final = MapThread[Prepend, {Inter, ParListEC}];
  grid = Grid[Final, Frame -> All,
    Background -> {{LightGray, None}, {LightGray, None}}, Spacings -> {1, 1}];
  Return[grid]
 ]
```

In[12]:=
```
NSolEC[ParametersMatrix_, T_, eqns_, vars_] :=
 Block[
  {eqs = Flatten[eqns /. Flatten[Table[{
         Ai -> ParametersMatrix[[1, i]],
         sci -> ParametersMatrix[[2, i]],
         sgi -> ParametersMatrix[[3, i]],
         ai -> ParametersMatrix[[4, i]],
         γi -> ParametersMatrix[[5, i]],
         βi -> ParametersMatrix[[6, i]],
         ηi -> ParametersMatrix[[7, i]],
         μi -> ParametersMatrix[[8, i]],
         θi -> ParametersMatrix[[9, i]],
         K0i -> ParametersMatrix[[10, i]],
         L0i -> ParametersMatrix[[11, i]],
         Ω0i -> ParametersMatrix[[12, i]],
         r -> ParametersMatrix[[13, i]],
         SU -> ParametersMatrix[[14, i]],
         SL -> ParametersMatrix[[15, i]],
         S0 -> ParametersMatrix[[16, i]],
         χ -> ParametersMatrix[[17, i]],
         ki -> ParametersMatrix[[18, i]]
        }, {i, 1, Length@ParametersMatrix[[1]]}]]]},
  Return[NDSolve[eqs, vars, {t, 0, T}][[1]]]
 ]
```

In[13]:=
```
EmissionsFunctionEC[NDSols_, ParameterList_] :=
 Block[{A = ParameterList[[1]], a = ParameterList[[4]],
   θ = ParameterList[[9]], μ = ParameterList[[8]], N = (Length@NDSols - 1) / 3},
  Return[Table[μ[[i]] * NDSols[[1 + i + 2 * N]][t] * A[[i]] * (NDSols[[1 + i]][t])^a[[i]] *
     (NDSols[[1 + i + N]][t])^(1-a[[i]]) * (NDSols[[1]][t])^θ[[i]], {i, 1, N}]]]

ConsumptionFunctionEC[NDSols_, ParameterList_] :=
 Block[{A = ParameterList[[1]], sc = ParameterList[[2]], sg = ParameterList[[3]],
   a = ParameterList[[4]], θ = ParameterList[[9]], N = (Length@NDSols - 1) / 3},
  Return[Table[(1 - sc[[i]] - sg[[i]]) A[[i]] * (NDSols[[1 + i]][t])^a[[i]] *
     (NDSols[[1 + i + N]][t])^(1-a[[i]]) * (NDSols[[1]][t])^θ[[i]], {i, 1, N}]]]

UtilityFunctionEC[NDSols_, ParameterList_, δ_] :=
 Block[{A = ParameterList[[1]], sc = ParameterList[[2]], sg = ParameterList[[3]],
   a = ParameterList[[4]], θ = ParameterList[[9]], N = (Length@NDSols - 1) / 3},
  Return[Table[Log[((1 - sc[[i]] - sg[[i]]) A[[i]] * (NDSols[[1 + i]][t])^a[[i]] *
         (NDSols[[1 + i + N]][t])^(1-a[[i]]) * (NDSols[[1]][t])^θ[[i]]) /
       NDSols[[1 + i + N]][t]] * NDSols[[1 + i + N]][t] * E^(-δ*t), {i, 1, N}]]]
```

In[16]:=
```
PlotEvolutionEC[IFList_, EMFuncs_, CFuncs_, T_, ParametersMatrix_, N_] :=
 GraphicsGrid[{
   {Plot[Evaluate@Table[IFList[[i]][t], {i, 2, 1 + N}], {t, 0, T},
     PlotLabel → "Capital for different countries",
     PlotRange -> {{0, T}, {0, All}}, Frame → True],
    Plot[Evaluate@Table[IFList[[i]][t], {i, 2 + N, 1 + 2 * N}], {t, 0, T},
     PlotLabel → "Labour for different countries",
     PlotRange -> {{0, T}, {0, All}}, Frame → True]},

   {Plot[Evaluate@(Table[IFList[[i]][t], {i, 2, 1 + N}] /
        Table[IFList[[i]][t], {i, 2 + N, 1 + 2 * N}]), {t, 0, T},
     PlotLabel → "Capital per capita for different countries",
     PlotRange -> {{0, T}, {0, All}}, Frame → True],
    Plot[Evaluate@Table[IFList[[i]][t], {i, 2 + 2 * N, 1 + 3 * N}],
     {t, 0, T}, PlotLabel → "Green tech factor for different countries",
     PlotRange -> {{0, T}, {0, 1.1}}, Frame → True]},

   {Plot[Evaluate@EMFuncs, {t, 0, T}, PlotLabel → "Emissions per country",
     PlotRange -> {{0, T}, {0, All}}, Frame → True],
    Plot[Total[EMFuncs], {t, 0, T}, PlotStyle → Green, PlotLabel → "Total emissions",
     PlotRange -> {{0, T}, {0, All}}, Frame → True]},
   {Plot[{IFList[[1]][t], 0.3}, {t, 0, T}, PlotLabel → "Environmental Capital",
     PlotRange → {{0, T}, {0, 1}},
     PlotStyle → {{Thick, Darker[Green]}, {Dashed, Gray}}, Frame → True],
    Plot[Evaluate@(Table[CFuncs[[i]], {i, 1, N}] /
        Table[IFList[[i]][t], {i, 2 + N, 1 + 2 * N}]), {t, 0, T}, PlotLabel →
      "Consumption per capita", PlotRange -> {{0, T}, {0, All}}, Frame → True]}
  },

  ImageSize → {400, 500}, Frame -> All,
  Spacings → {0, 0}, Alignment → Center, AspectRatio → 1.25]
```

All simulation

In[17]:=
```
SimulateEC[N_, T_, SimEqs_, ParameterMatrix_, δ_] :=
  Block[{SimVars =
     Union[Table[KKi, {i, 1, N}], Table[Li, {i, 1, N}], Table[Ωi, {i, 1, N}], {S}]},
   SimOutput = SimVars /. NSolEC[ParameterMatrix, T, SimEqs, SimVars];
   SimEmissions = EmissionsFunctionEC[SimOutput, ParameterMatrix];
   SimConsumption = ConsumptionFunctionEC[SimOutput, ParameterMatrix];
   SimUtility = UtilityFunctionEC[SimOutput, ParameterMatrix, δ];
   SimPlot = PlotEvolutionEC[SimOutput,
     SimEmissions, SimConsumption, T, ParameterMatrix, N];
   Return[{SimOutput, SimEmissions, SimConsumption, SimUtility, SimPlot}]];
```

3.3.1.1 Code

```
Tmax = 2000;
Tpanic = 350;
eqs = {S[0] == S0,
   S'[t] ==
    -χ Log[1 + S[t]^θ1 A1 μ1 KK1[t]^a1 L1[t]^(1-a1) Ω1[t] + S[t]^θ2 A2 μ2 KK2[t]^a2 L2[t]^(1-a2) Ω2[t]] +
     (r (SU - S[t]) S[t] (-SL + S[t]))/SU^2,
   KK1'[t] == -γ1 KK1[t] + S[t]^θ1 A1 sc1 KK1[t]^a1 L1[t]^(1-a1),
   KK1[0] == K01,
   L1'[t] == L1[t]
     (β1 - η1 (S[t]^θ1 A1 μ1 KK1[t]^a1 L1[t]^(1-a1) Ω1[t] + S[t]^θ2 A2 μ2 KK2[t]^a2 L2[t]^(1-a2) Ω2[t])),
   L1[0] == L01,
   Ω1'[t] == -k1 sg1 Ω1[t] * (t - Tpanic)/(2000) * UnitStep[t - Tpanic],
   Ω1[0] == Ω01,
   KK2'[t] == -γ2 KK2[t] + S[t]^θ2 A2 sc2 KK2[t]^a2 L2[t]^(1-a2),
   KK2[0] == K02,
   L2'[t] == L2[t]
     (β2 - η2 (S[t]^θ1 A1 μ1 KK1[t]^a1 L1[t]^(1-a1) Ω1[t] + S[t]^θ2 A2 μ2 KK2[t]^a2 L2[t]^(1-a2) Ω2[t])),
   L2[0] == L02,
   Ω2'[t] == -k2 sg2 Ω2[t] * (t - Tpanic)/(2000) * UnitStep[t - Tpanic],
   Ω2[0] == Ω02};
Sim1PM = {{10, 8}, {0.02, 0.015}, {0.9, 0.9}, {0.3, 0.2}, {0.01, 0.01},
   {0.0149, 0.015}, {0.0001, 0.0001}, {1, 1}, {1, 1}, {50, 1}, {1, 1}, {1, 1},
   {0.25, 0.25}, {1, 1}, {0.3, 0.3}, {1, 1}, {0.0043, 0.0043}, {0.1, 0.1}};
SimulateEC[2, 2000, eqs, Sim1PM, 1][[5]];
ParTableEC[Sim1PM];
{%%, %}
```

3.3.1.2 Code

$$
\begin{aligned}
&\text{eqs} = \Big\{S[0] == S0,\\
&\quad S'[t] == -\chi\,\text{Log}\Big[1 + S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t]\Big] + \frac{r\,(SU - S[t])\,S[t]\,(-SL + S[t])}{SU^2},\\
&\quad KK_1'[t] == -\gamma_1\,KK_1[t] + S[t]^{\theta_1} A_1\,sc_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1},\\
&\quad KK_1[0] == K0_1,\\
&\quad L_1'[t] == L_1[t]\,\Big(\beta_1 - \eta_1\,\Big(S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t]\Big)\Big),\\
&\quad L_1[0] == L0_1,\\
&\quad \Omega_1'[t] == -k_1\,sg_1\,\Omega_1[t] * KK_1[t] \,/\, L_1[t],\\
&\quad \Omega_1[0] == \Omega0_1,\; KK_2'[t] == -\gamma_2\,KK_2[t] + S[t]^{\theta_2} A_2\,sc_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2},\\
&\quad KK_2[0] == K0_2,\\
&\quad L_2'[t] == L_2[t]\,\Big(\beta_2 - \eta_2\,\Big(S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t]\Big)\Big),\\
&\quad L_2[0] == L0_2,\\
&\quad \Omega_2'[t] == -k_2\,sg_2\,\Omega_2[t] * \frac{t - 500}{(2000)} * \text{UnitStep}[t - 500] * KK_2[t] \,/\, L_2[t],\\
&\quad \Omega_2[0] == \Omega0_2\Big\};
\end{aligned}
$$

3.3.1.3 Code

$$
\begin{aligned}
&\text{Tpanic} = 600;\\
&\text{PanicSim5} = \Big\{S[0] == S0,\\
&\quad S'[t] == -\chi\,\text{Log}\Big[1 + S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t] +\\
&\qquad S[t]^{\theta_3} A_3\,\mu_3\,KK_3[t]^{a_3} L_3[t]^{1-a_3}\,\Omega_3[t] + S[t]^{\theta_4} A_4\,\mu_4\,KK_4[t]^{a_4} L_4[t]^{1-a_4}\,\Omega_4[t] +\\
&\qquad S[t]^{\theta_5} A_5\,\mu_5\,KK_5[t]^{a_5} L_5[t]^{1-a_5}\,\Omega_5[t]\Big] + \frac{r\,(SU - S[t])\,S[t]\,(-SL + S[t])}{SU^2},\\
&\quad KK_1'[t] == -\gamma_1\,KK_1[t] + S[t]^{\theta_1} A_1\,sc_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1},\\
&\quad KK_1[0] == K0_1,\\
&\quad L_1'[t] == L_1[t]\,\Big(\beta_1 - \eta_1\,\Big(S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t] +\\
&\qquad S[t]^{\theta_3} A_3\,\mu_3\,KK_3[t]^{a_3} L_3[t]^{1-a_3}\,\Omega_3[t] + S[t]^{\theta_4} A_4\,\mu_4\,KK_4[t]^{a_4} L_4[t]^{1-a_4}\,\Omega_4[t] +\\
&\qquad S[t]^{\theta_5} A_5\,\mu_5\,KK_5[t]^{a_5} L_5[t]^{1-a_5}\,\Omega_5[t]\Big)\Big),\\
&\quad L_1[0] == L0_1,\\
&\quad \Omega_1'[t] == -k_1\,sg_1\,\Omega_1[t] * KK_1[t] \,/\, L_1[t] * \frac{t - \text{Tpanic}}{(2000)} * \text{UnitStep}[t - \text{Tpanic}],\\
&\quad \Omega_1[0] == \Omega0_1,\\
&\\
&\quad KK_2'[t] == -\gamma_2\,KK_2[t] + S[t]^{\theta_2} A_2\,sc_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2},\\
&\quad KK_2[0] == K0_2,\\
&\quad L_2'[t] == L_2[t]\,\Big(\beta_2 - \eta_2\,\Big(S[t]^{\theta_1} A_1\,\mu_1\,KK_1[t]^{a_1} L_1[t]^{1-a_1}\,\Omega_1[t] + S[t]^{\theta_2} A_2\,\mu_2\,KK_2[t]^{a_2} L_2[t]^{1-a_2}\,\Omega_2[t] +\\
&\qquad S[t]^{\theta_3} A_3\,\mu_3\,KK_3[t]^{a_3} L_3[t]^{1-a_3}\,\Omega_3[t] + S[t]^{\theta_4} A_4\,\mu_4\,KK_4[t]^{a_4} L_4[t]^{1-a_4}\,\Omega_4[t] +\\
&\qquad S[t]^{\theta_5} A_5\,\mu_5\,KK_5[t]^{a_5} L_5[t]^{1-a_5}\,\Omega_5[t]\Big)\Big),\\
&\quad L_2[0] == L0_2,
\end{aligned}
$$

```
Ω₂′[t] == -k₂ sg₂ Ω₂[t] * KK₂[t] / L₂[t] * (t - Tpanic + 100)/(2000) * UnitStep[t - Tpanic + 100],
  Ω₂[0] == Ω0₂,

  KK₃′[t] == -γ₃ KK₃[t] + S[t]^θ₃ A₃ sc₃ KK₃[t]^a₃ L₃[t]^(1-a₃),
  KK₃[0] == K0₃,
  L₃′[t] ==
   L₃[t] (β₃ - η₃ (S[t]^θ₁ A₁ μ₁ KK₁[t]^a₁ L₁[t]^(1-a₁) Ω₁[t] + S[t]^θ₂ A₂ μ₂ KK₂[t]^a₂ L₂[t]^(1-a₂) Ω₂[t] +
        S[t]^θ₃ A₃ μ₃ KK₃[t]^a₃ L₃[t]^(1-a₃) Ω₃[t] + S[t]^θ₄ A₄ μ₄ KK₄[t]^a₄ L₄[t]^(1-a₄) Ω₄[t] +
        S[t]^θ₅ A₅ μ₅ KK₅[t]^a₅ L₅[t]^(1-a₅) Ω₅[t])),
  L₃[0] == L0₃,
  Ω₃′[t] == -k₃ sg₃ Ω₃[t] * KK₃[t] / L₃[t] * (t - Tpanic + 200)/(2000) * UnitStep[t - Tpanic + 200],
  Ω₃[0] == Ω0₃,

  KK₄′[t] == -γ₄ KK₄[t] + S[t]^θ₄ A₄ sc₄ KK₄[t]^a₄ L₄[t]^(1-a₄),
  KK₄[0] == K0₄,
  L₄′[t] ==
   L₄[t] (β₄ - η₄ (S[t]^θ₁ A₁ μ₁ KK₁[t]^a₁ L₁[t]^(1-a₁) Ω₁[t] + S[t]^θ₂ A₂ μ₂ KK₂[t]^a₂ L₂[t]^(1-a₂) Ω₂[t] +
        S[t]^θ₃ A₃ μ₃ KK₃[t]^a₃ L₃[t]^(1-a₃) Ω₃[t] + S[t]^θ₄ A₄ μ₄ KK₄[t]^a₄ L₄[t]^(1-a₄) Ω₄[t] +
        S[t]^θ₅ A₅ μ₅ KK₅[t]^a₅ L₅[t]^(1-a₅) Ω₅[t])),
  L₄[0] == L0₄,
  Ω₄′[t] == -k₄ sg₄ Ω₄[t] * KK₄[t] / L₄[t] * (t - Tpanic + 300)/(2000) * UnitStep[t - Tpanic + 300],
  Ω₄[0] == Ω0₄,

  KK₅′[t] == -γ₅ KK₅[t] + S[t]^θ₅ A₅ sc₅ KK₅[t]^a₅ L₅[t]^(1-a₅),
  KK₅[0] == K0₅,
  L₅′[t] ==
   L₅[t] (β₅ - η₅ (S[t]^θ₁ A₁ μ₁ KK₁[t]^a₁ L₁[t]^(1-a₁) Ω₁[t] + S[t]^θ₂ A₂ μ₂ KK₂[t]^a₂ L₂[t]^(1-a₂) Ω₂[t] +
        S[t]^θ₃ A₃ μ₃ KK₃[t]^a₃ L₃[t]^(1-a₃) Ω₃[t] + S[t]^θ₄ A₄ μ₄ KK₄[t]^a₄ L₄[t]^(1-a₄) Ω₄[t] +
        S[t]^θ₅ A₅ μ₅ KK₅[t]^a₅ L₅[t]^(1-a₅) Ω₅[t])),
  L₅[0] == L0₅,
  Ω₅′[t] == -k₅ sg₅ Ω₅[t] * KK₅[t] / L₅[t] * (t - Tpanic + 400)/(2000) * UnitStep[t - Tpanic + 400],
  Ω₅[0] == Ω0₅};
Sim5PM = {
  {10, 9, 8, 9, 9},
  {0.02, 0.02, 0.02, 0.02, 0.02},
  {0.2, 0.3, 0.4, 0.5, 0.6},
  {0.3, 0.25, 0.35, 0.31, 0.3},
  {0.01, 0.01, 0.01, 0.01, 0.01},
  {0.015, 0.0149, 0.0148, 0.0149, 0.0151},
  {0.0001, 0.0001, 0.0001, 0.0001, 0.0001},
  {1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1},
  {1, 1, 1, 1, 1},
```

```
{1, 1, 1, 1, 1},
 {1, 1, 1, 1, 1},
{0.25, 0.25, 0.25, 0.25, 0.25},
{1, 1, 1, 1, 1},
{0.3, 0.3, 0.3, 0.3, 0.3},
{1, 1, 1, 1, 1},
{0.0043, 0.0043, 0.0043, 0.0043, 0.0043},
{0.1, 0.1, 0.1, 0.1, 0.1}};
```

Code 3.3.2.1

```
ThreeRegionParMatrixEC =
  {{9.573096775404341`, 10.218953699359364`, 9.517939387554007`},
   {0.056159183136605666`, 0.0585906591782796`, 0.044950281362396996`}, {0.3`, 0.3`,
    0.3`}, {0.03465253082693404`, 0.030173908929235993`, 0.031100044133080432`},
   {0.009247701509468305`, 0.009708276533175953`, 0.01097732656415741`},
   {1.`*^-7, 1.`*^-7, 1.`*^-7}, {1.`, 1.`, 1.`}, {1.`, 1.`, 1.`},
   {1.9079801729003916`, 2.1925091109642647`, 1.9063805398075317`},
   {2.1687486837542655`, 1.8702891497387295`, 1.9704144912049009`},
   {1.`, 1.`, 1.`}, {0.2`, 0.2`, 0.2`}, {1.`, 1.`, 1.`}, {0.3`, 0.3`, 0.3`},
   {1.`, 1.`, 1.`}, {0.0016`, 0.0016`, 0.0016`}, {0.2`, 0.2`, 0.2`}};
```

```
In[ ]:= For[n = 2;
  lis = {}, n < 20, n++, ThreeRegionParMatrixECwithsg = Insert[ThreeRegionParMatrixEC,
    {0.005 * n, 0.005 * n, 0.005 * n}, 3]; ECSim = SimulateEC[3, 1000, SimEquationsEC[3],
    ThreeRegionParMatrixECwithsg, 0.016]; PVUECSim = Table[NIntegrate[ECSim[[4, i]],
      {t, 0, 1000}], {i, 1, 3}]; lis = Append[lis, {0.005 * n, Total[PVUECSim]}]; Print[
   StringJoin["sg ", ToString[0.005 * n], " Total PV ", ToString[Total[PVUECSim]]]]]
```

```
sg 0.01 Total PV 3010.43
sg 0.015 Total PV 3026.32
sg 0.02 Total PV 3038.66
sg 0.025 Total PV 3047.4
sg 0.03 Total PV 3052.99
sg 0.035 Total PV 3056.22
sg 0.04 Total PV 3057.8
sg 0.045 Total PV 3058.28
sg 0.05 Total PV 3057.98
sg 0.055 Total PV 3057.12
sg 0.06 Total PV 3055.85
sg 0.065 Total PV 3054.24
sg 0.07 Total PV 3052.35
sg 0.075 Total PV 3050.2
sg 0.08 Total PV 3047.81
sg 0.085 Total PV 3045.18
sg 0.09 Total PV 3042.32
sg 0.095 Total PV 3039.22
```

Code 3.3.3

```
In[ ]:= For[n = 2; lisR1 = {}, n < 14, n++, ThreeRegionParMatrixECwithsgR1 =
    Insert[ThreeRegionParMatrixEC, {0.005 * n, 0.045, 0.045}, 3]; ECSimR1 =
    SimulateEC[3, 1000, SimEquationsEC[3], ThreeRegionParMatrixECwithsgR1, 0.016];
  PVUECSimR1 = Table[NIntegrate[ECSimR1[[4, i]], {t, 0, 1000}], {i, 1, 3}];
  lisR1 = Append[lisR1, {0.005 * n, PVUECSimR1[[1]]}];
  Print[StringJoin["sg ", ToString[0.005 * n],
     " PVRegion1 ", ToString[PVUECSimR1[[1]]]]]]
```

```
sg 0.01 PVRegion1 971.394
sg 0.015 PVRegion1 971.853
sg 0.02 PVRegion1 971.776
sg 0.025 PVRegion1 971.296
sg 0.03 PVRegion1 970.519
sg 0.035 PVRegion1 969.519
sg 0.04 PVRegion1 968.346
sg 0.045 PVRegion1 967.04
sg 0.05 PVRegion1 965.629
sg 0.055 PVRegion1 964.138
sg 0.06 PVRegion1 962.583
sg 0.065 PVRegion1 960.977
```