Tutoría II. Transformada Z

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Ejercicio #1

$$x(n) = \frac{u(n-2)}{4^n}$$

Por el formulario:

$$a^n u(n) \circ\!\!-\!\!\bullet \frac{1}{1-az^{-1}} \quad |z|>|a|$$

$$x(n) = \frac{1}{4^n} \cdot u(n-2) = \left(\frac{1}{4}\right)^n u(n-2)$$

Por la propiedad de desplazamiento en n

$$x(n-k) \circ\!\!-\!\!\bullet z^{-k} X(z) \quad R\setminus\{0\} \text{ si } k>0 \text{ y } R\setminus\{\infty\} \text{ si } k<0$$

$$x(n) = \frac{1}{4^n} u(n-2) = \frac{1}{4^2} \cdot \frac{1}{4^{n-2}} u(n-2) = \frac{1}{16} \cdot \left(\frac{1}{4}\right)^{n-2} u(n-2)$$

$$u(n-2) = \frac{1}{16}\left(\frac{1}{4}\right)^n u(n)$$

Aplicando la transformada:

$$Z\{u(n-2)\} = \frac{1}{16} z^{-2} \cdot \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$X(n-2) = \frac{1}{16} \cdot \frac{z^{-2}}{1-\frac{1}{4}z^{-1}} \quad \text{ROC: } |z| > \left|\frac{1}{4}\right|$$

$$\boxed{\frac{u(n-2)}{4^n} \circ\!\!-\!\!\bullet \frac{1}{16}\left(\frac{z^{-2}}{1-\frac{1}{4}z^{-1}}\right) \quad \text{ROC: } |z| > \left|\frac{1}{4}\right|}$$

Ejercicio 2

$X(z) = \cos(z)$

Utilizando la transformada por definición

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

$$x(n) = \frac{1}{2\pi j} \oint_c \cos(z)\, z^{n-1}\, dz$$

→ Si $n>0$:

$$x(n) = \frac{1}{2\pi j} \oint_c \underbrace{\cos(z)\, z^{n-1}}_{\text{Analítico}}\, dz \Rightarrow \text{Integral va a ser cero}$$

$$x(n) = 0 \quad \forall n > 0$$

→ Si $n<0$:

$$x(n) = \frac{1}{2\pi j} \oint_c \cos z\, z^{-(n-1)}\, dz = \frac{1}{2\pi j} \oint_c \frac{\cos z}{z^{1-n}}\, dz$$

Fórmula de la integral de Cauchy

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz = f^{(n)}(z_0)\, \frac{2\pi j}{n!}$$

$$\Rightarrow x(n) = \frac{1}{2\pi j}\left(\frac{2\pi j\, [\cos(z)]^{-n}}{(-n)!} \right) = \frac{(\cos(z))^{-n}}{(-n)!} \quad \text{con } n<0$$

Con n impar:

$$x(-1) = \frac{(\cos(z))'}{1!} = \frac{-\operatorname{sen}(0)}{1} = 0$$

$$x(-3) = \frac{(\cos(z))^{(3)}}{3!} = \frac{\operatorname{sen}(0)}{3!} = 0$$

$$x(-5) = \frac{(\cos(z))^{(5)}}{5!} = \frac{-\operatorname{sen}(0)}{5!} = 0$$

$x(n) = 0 \quad \forall n < 0$ y con n impar con n par

$$x(-2) = \frac{(\cos(z))''}{2!} = \frac{-\cos(0)}{2!} = \frac{-1}{2!}$$

$$x(-4) = \frac{(\cos(z))^{(4)}}{4!} = \frac{\cos(0)}{4!} = \frac{1}{4!}$$

$$x(-6) = \frac{(\cos(z))^{(6)}}{6!} = \frac{-\cos(0)}{6!} = \frac{-1}{6!}$$

$$x(n) = \frac{(-1)^{n/2}}{(-n)!} \quad \forall n<0 \quad y \; n \text{ par}$$

$\rightarrow$ Si $n=0$:

$$x(n) = \frac{1}{2\pi j} \oint_c \cos(z) z^{-1} dz = \frac{1}{2\pi j}(2\pi j \cdot 1) = 1$$

$$\Rightarrow \boxed{\cos(z) \bullet\!\!-\!\!\circ \begin{cases} \frac{(-1)^{n/2}}{(-n)!} & \text{con } n \leq 0 \text{ y } n \text{ par} \\ 0 & \text{para el resto.} \end{cases}}$$

Ejercicio 3

$$X(z) = \frac{1+z^{-1}}{1+\frac{1}{3}z^{-1}} \qquad \text{ROC: } |z| > \frac{1}{3} \text{ y } |z| < \frac{1}{3}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-1}$$

① ROC $\cdot$ $|z > \frac{1}{3}|$

$$\begin{array}{l|l} 1 + z^{-1} & 1+\frac{1}{3}z^{-1} \\ \hline -1+\frac{1}{3}z^{-1} & 1+\frac{2}{3}z^{-1} - \frac{2}{3^2}z^{-2} + \frac{2}{3^3}z^{-3} - \ldots \\ \hline \frac{2}{3}z^{-1} & \\ -(\frac{2}{3}z^{-1} + \frac{2}{3^2}z^{-2}) & \\ \hline \frac{-2}{3^3}z^{-2} & \\ -\left(-\frac{2}{3^3}z^{-2} - \frac{2}{3^3}z^{-3}\right) & \\ \hline \end{array}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$x(n) = 0 \quad \forall\, n < 0$

$x(0) = 1$ para $n = 0$

para $n > 0 \Rightarrow x(n) = \dfrac{2(-1)^{n+1}}{3^n}$

$$\boxed{X(z) = \frac{1+z^{-1}}{1+\frac{1}{3}z^{-1}} \;\bullet\!\!-\!\!\!-\!\!\circ\; \begin{cases} 0 & n<0 \\ 1 & n=0 \\ \dfrac{2(-1)^{n+1}}{3^n} & n>0 \end{cases} \qquad \text{ROC: } |z| > \frac{1}{3}}$$

② ROC: $|z| < \frac{1}{3}$

$$X(z) = \frac{z^{-1}+1}{\frac{1}{3}z^{-1}+1}$$

$$\begin{array}{l|l} z^{-1} + 1 & \frac{1}{3}z^{-1} + 1 \\ \hline -(z^{-1}+3) & 3 - 2\cdot 3z + 2\cdot 3^2 z^2 - 2\cdot 3^3 z^3 \\ \quad -2 & \\ -(-2 - 2\cdot 3z) & \\ \quad 2\cdot 3z & \\ -(2\cdot 3z + 2\cdot 3^2 z^2) & \\ \quad -2\cdot 3^2 z^2 & \end{array}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Para $n<0$: $x(n) = 2(-1)^n \cdot 3^n$

Para $n=0$: $x(0) = 3$

Para $n>0$ $\quad x(n) = 0$

$$\boxed{X(z) = \frac{1+z^{-1}}{1+\frac{1}{3}z^{-1}} \;\bullet\!\!-\!\!\!-\!\!\circ\; \begin{cases} 2(-1)^n \cdot 3^n & n<0 \\ 3 & n=0 \\ 0 & n>0 \end{cases} \qquad \text{ROC: } |z| < \frac{1}{3}}$$

Ejercicio 4

$$X = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$$

$1-\frac{1}{z}=0 \Rightarrow 1=\frac{1}{z} \Rightarrow z=1$ | $1+\frac{2}{z}=0 \Rightarrow \frac{2}{z}=-1 \Rightarrow -2=z$

Por fracciones parciales

$$\frac{1-\frac{z^{-1}}{3}}{(1-z^{-1})(1+2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1+2z^{-1}}$$

$$A = \lim_{z\to 1} \cancel{(1-z^{-1})}\cdot \frac{1-\frac{z^{-1}}{3}}{\cancel{(1-z^{-1})}(1+2z^{-1})} = \frac{1-\frac{1}{3}}{1+2} = \frac{\frac{3-1}{3}}{3} = \frac{\frac{2}{3}}{3} = \frac{2}{9}$$

$$B = \lim_{z\to -2} (1+2z^{-1})\cdot \frac{1-\frac{z^{-1}}{3}}{(1-z^{-1})(1+2z^{-1})} = \frac{1-\frac{1}{3\cdot -2}}{1-\frac{1}{-2}} = \frac{1+\frac{1}{6}}{1+\frac{1}{2}} = \frac{\frac{6+1}{6}}{\frac{2+1}{2}} = \frac{7\cdot 2}{6\cdot 3} = \frac{14}{18} = \frac{7}{9}$$

$$x(z) = \frac{\frac{2}{9}}{(1-z^{-1})} + \frac{\frac{7}{9}}{(1+2z^{-1})}$$

Por el formulario:

$u(n) \circ\!\!-\!\!\bullet \frac{1}{1-z^{-1}} \qquad |z|>1$

$a^n u(n) \circ\!\!-\!\!\bullet \frac{1}{1-az^{-1}} \qquad |z|>a$

$$\Rightarrow x(t) = \frac{2}{9}u(n) + \frac{7}{9}(-2)^n u(n)$$

$$\boxed{x(t) = \frac{1}{9}\left(2+7(-2)^n\right)u(n)}$$

Ejercicio 5

$$X(z) = \frac{1}{256}\left[\frac{256 - z^{-8}}{1-\frac{1}{2}z^{-1}}\right] \quad ROC \quad |z|>0$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{256}\frac{z^{-8}}{1-\frac{1}{2}z^{-1}}$$

$$x(t) = Z^{-1}\{X(z)\} = Z^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{256}\cdot\frac{z^{-8}}{1-\frac{1}{2}z^{-1}}\right\}$$

$$x(n) = Z^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\} - \frac{1}{256}\left\{\frac{z^{-8}}{1-\frac{1}{2}z^{-1}}\right\}$$

Por el formulario

$$a^n u(n) \multimap \frac{1}{1-az^{-1}} \quad |z|>|a|$$

$$x(n) = \left(\frac{1}{2}\right)^n\cdot u(n) - \frac{1}{256}\left\{\frac{z^{-8}}{1-\frac{1}{2}z^{-1}}\right\}$$

Por la propiedad de desplazamiento en el tiempo:

$$x(n-k) \multimap z^{-k}X(z)$$

$$\Rightarrow x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{256}\, Z^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\}\Bigg|_{n=n-8}$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{256}\left[\left(\frac{1}{2}\right)^n u(n)\right]\Bigg|_{n=n-8}$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{256}\left(\frac{1}{2}\right)^{n-8} u(n-8)$$

$$x(n) = \left(\frac{1}{2}\right)^n\left[u(n) - \frac{2^8}{256}u(n-8)\right]$$

$$\boxed{x(n) = \left(\frac{1}{2}\right)^n\left(u(n) - u(n-8)\right)}$$