Tutoría 5. Integración Compleja y funciones de variable compleja

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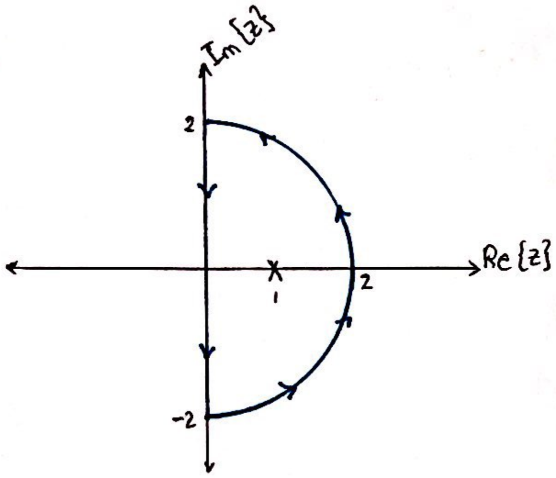
Ejercicio 1

Si C es un cemicírculo en sentido positivo conformado por los puntos de frontera $z \in \mathbb{C}$, de la región $|z| \leq 2$, $\mathrm{Re}\{z\} > 0$, entonces determine el resultado de la siguiente integral compleja:

$$\int_C \frac{\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{1-z^3}\,dz$$



$$1^3 - z^3 = (1-z)(1+z+z^2)$$

$$z = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$$

Utilizando el criterio de la integral de Cauchy

Por formulario: $\displaystyle \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)\,\frac{2\pi j}{n!}$

$$\Rightarrow \oint_C \frac{\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{(1-z)(z^2+z+1)}\,dz = -\oint_C \frac{\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{(z-1)(z^2+z+1)}\,dz$$

Utilizando la fórmula de la integral de Cauchy

$$\oint_C \frac{\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{(1-z)(z^2+z+1)}\,dz = \left.\frac{-\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{z^2+z+1}\right|_{z_0} \cdot 2\pi j$$

$$\boxed{\int_C \frac{\operatorname{sen}\left(z\,\frac{\pi}{2}\right)}{1-z^3}\,dz = -j\,\frac{2\pi}{3}}$$

Escaneado con CamScanner

Ejercicio 2

Evalúe las siguientes integrales reales utilizando métodos de integración compleja

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4x+5)^2}\,dx$$

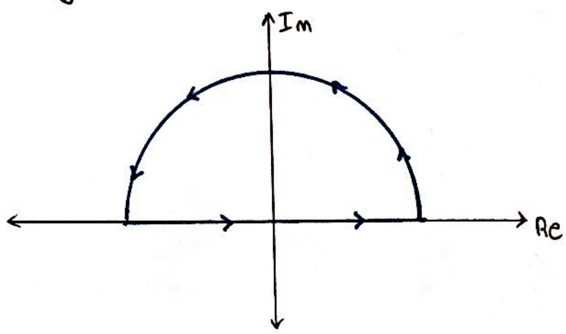
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\,dx$$

① $$\int_{-\infty}^{\infty} \frac{1}{(x^2+4x+5)^2}\,dx$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-4\cdot 1\cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j2}{2} = -2 \pm j$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x+2-j)^2(x+2+j)^2}$$

Como la integral es de $-\infty$ a ∞ el contorno es:



Como solo es el semicírculo de arriba, solo afecta el polo $(x+2-j)^2$

Usando el teorema del residuo:

$$\int_{-\infty}^{\infty} \frac{dz}{(z+2-j)^2(z+2+j)^2} = 2\pi j \cdot \text{residuo de } f(z) \text{ en el polo } (z+2-j)^2$$

$$= 2\pi j \cdot \lim_{z \to -2+j} \left(\frac{1}{(z+2+j)^2}\right)'$$

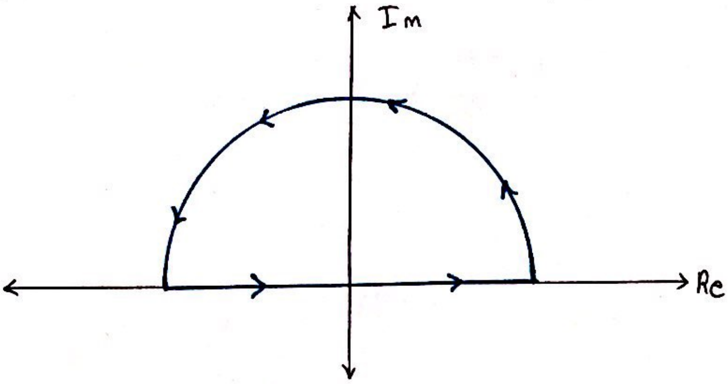
$$= 2\pi j \cdot \lim_{z \to -2+j} \left(\frac{-2}{(z+2+j)^3}\right)$$

$$= 2\pi j \cdot \frac{-2}{(2j)^3} = \frac{-4\pi j}{-8j} = \frac{\pi}{2}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dz}{(z+2-j)^2(z+2+j)^2} = \frac{\pi}{2}}$$

② $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\,dx$

Como la integral es de $-\infty$ a $+\infty$:



$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{0 \pm \sqrt{0-4(1)(4)}}{2} = \frac{\pm\sqrt{-16}}{2} = \pm j\,\frac{4}{2} = \pm 2j$$

$$\int_{-\infty}^{\infty} \frac{1}{(x+2j)^3(x-2j)^3}\,dx$$

La región del semicírculo solo contiene el polo $(x-2j)^3$

Usando el teorema del residuo

$$\int_{-\infty}^{\infty} \frac{dz}{(z+2j)^3(z-2j)^3} = 2\pi j \cdot \text{residuo de } f(z) \text{ en el polo } (z-2j)^3$$

$$= 2\pi j \cdot \frac{1}{(3-1)!} \lim_{z\to 2j} \left\{ \frac{d^2}{dz^2}\left[(z-2j)^3 \cdot \frac{1}{(z+2j)^3(z-2j)^3} \right] \right\}$$

$$= \frac{2\pi j}{2!} \lim_{z\to 2j} \left(\frac{1}{(z+2j)^3} \right)'' = \pi j \cdot \lim_{z\to 2j} \left(\frac{-3}{(z+2j)^4} \right)'$$

$$= \pi j \cdot \lim_{z\to 2j} \left(\frac{12}{(z+2j)^5} \right) = \pi j \cdot \left(\frac{12}{(2j+2j)^5} \right) = \pi j \left(\frac{12}{1024 j} \right)$$

$$= \frac{6\pi}{512} = \frac{3\pi}{256}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dz}{(z+2j)^3(z-2j)^3} = \frac{3\pi}{256}}$$

Escaneado con CamScanner

Ejercicio 3

Resuelva las siguientes integrales trigonométricas por medio de integración compleja:

$$\int_0^{2\pi} \frac{\cos\theta}{3+2\cos\theta}\,d\theta$$

$$\int_0^{\pi} \frac{d\theta}{2+\operatorname{sen}^2\theta}$$

① $$\int_0^{2\pi} \frac{\cos\theta}{3+2\cos\theta}\,d\theta$$

$$\int_0^{2\pi} G(\operatorname{sen}\theta, \cos\theta)\,d\theta = \oint_C f(z)\,dz \quad \text{con} \quad z = e^{j\theta} \qquad |z| = 1$$

$$\Downarrow$$

$$\frac{dz}{jz} = d\theta \qquad \operatorname{sen}\theta = \frac{1}{2j}\left(z - \frac{1}{z}\right)$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$I_1 = \int_0^{2\pi} \frac{\cos\theta}{3+2\cos\theta}\,d\theta = \oint_C \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{3+\frac{2}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{jz} = -\frac{j}{2}\oint_C \frac{z^2+1}{z\left[z^2+1+3z\right]}\,dz$$

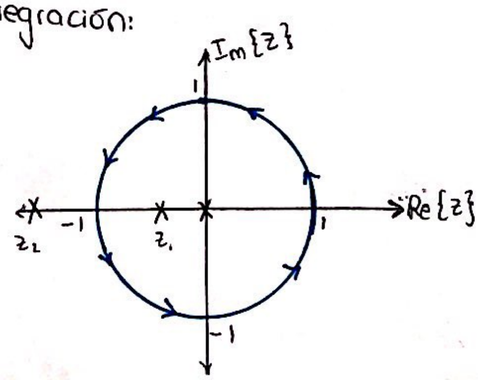
<u>Polos:</u> $z = 0$

$$z^2+3z+1 = 0 \Rightarrow z = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$\rightarrow \quad z_1 = \frac{-3+\sqrt{5}}{2} \quad \rightarrow \approx -0{,}38$$

$$z_2 = \frac{-3-\sqrt{5}}{2} \quad \rightarrow \approx -2{,}62 \quad \leftarrow$$ <u>No</u> está dentro del círculo unitario

Trayectoria de integración:



$$I_1 = -\frac{j}{2} \oint_c \frac{z^2+1}{z\left[z-\left(\frac{\sqrt{5}-3}{2}\right)\right]\left[z+\frac{3+\sqrt{5}}{2}\right]} dz$$

$$I_1 = 2\pi j \sum_{i=1}^{N} a_{-1}^{i}$$

$\rightarrow$ Para $z=0$

$$a_{-1}^{(1)} = 1 \cdot \lim_{z\to 0} \frac{z^2+1}{\left(z+\frac{3-\sqrt{5}}{2}\right)\left(z+\frac{3+\sqrt{5}}{2}\right)}$$

$$a_{-1}^{(1)} = 1$$

$\rightarrow$ Para $z = \frac{-3+\sqrt{5}}{2}$

$$a_{-1}^{(2)} = 1 \cdot \lim_{z\to \frac{-3+\sqrt{5}}{2}} \left[\frac{z^2+1}{z\left(z+\frac{3+\sqrt{5}}{2}\right)}\right]$$

$$a_{-1}^{(2)} = \frac{-3\sqrt{5}}{5}$$

$$I_1 = 2\pi j \left[a_{-1}^{(1)} + a_{-1}^{(2)}\right]\left(\frac{-j}{2}\right) = 2\pi j \left[1 + \frac{-3\sqrt{5}}{5}\right] \cdot \left(\frac{-j}{2}\right)$$

$$I_1 = \pi\left[1 + \frac{-3\sqrt{5}}{5}\right]$$

$$\boxed{\int_0^{2\pi} \frac{\cos\theta}{3+2\cos\theta} d\theta = \pi\left[1 - \frac{3\sqrt{5}}{5}\right]}$$

② $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\operatorname{sen}^2\theta}$

$$\alpha = 2\theta$$

Si $\theta = 0 \rightarrow \alpha = 0$

$\theta = \pi \rightarrow \alpha = 2\pi$

$$d\alpha = 2\, d\theta$$

Reescribiendo la integral:

$$I_2 = \int_0^{2\pi} \frac{1}{2+\operatorname{sen}^2\left(\frac{\alpha}{2}\right)} \cdot \frac{d\alpha}{2}$$

$$I_2 = \frac{1}{2}\int_0^{2\pi} \frac{1}{2 + \frac{1}{2} - \frac{\cos(\alpha)}{2}}\, d\alpha = \int_0^{2\pi} \frac{1}{5 - \cos\alpha}\, d\alpha$$

$$\text{Sea } z = e^{j\alpha} \Rightarrow \frac{dz}{jz} = d\alpha$$

$$\cos\alpha = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$$\Rightarrow I_2 = \frac{1}{2}\oint_c \frac{1}{5 - \frac{1}{2}\left[z + \frac{1}{z}\right]} \cdot \frac{dz}{jz}$$

$$I_2 = \oint_c \frac{2\, dz}{j(10z - z^2 - 1)} = 2j \oint_c \frac{dz}{z^2 - 10z + 1}$$

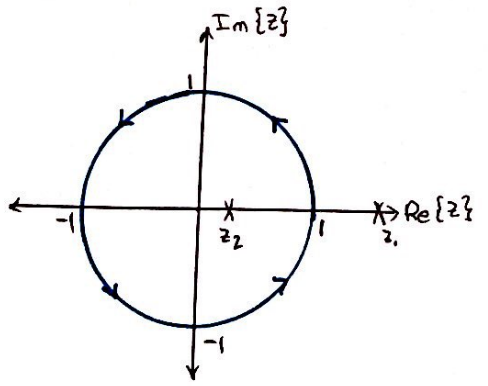
Polos:

$$z^2 - 10z + 1 = 0$$

$$\Rightarrow z = \frac{10 \pm \sqrt{100 - 4}}{2} = 5 \pm 2\sqrt{6} \leftarrow$$ Solo el polo $5 - 2\sqrt{6}$ está dentro del círculo unitario

$$I_2 = 2j \oint_c \frac{dz}{(z - 5 - 2\sqrt{6})(z - 5 + 2\sqrt{6})}$$

Trayectoria de integración:



$$I_2 = 2\pi j \cdot 2j\, a_{-1}$$

$$a_{-1} = \frac{1}{(1-1)!} \cdot \lim_{z \to 5 - 2\sqrt{6}} \left(\frac{1}{z - 5 - 2\sqrt{6}}\right) = -\frac{\sqrt{6}}{24}$$

$$I_2 = -4\pi\left(-\frac{\sqrt{6}}{24}\right) = \frac{\pi\sqrt{6}}{6}$$

$$\boxed{\int_0^{\pi} \frac{d\theta}{2+\operatorname{sen}^2\theta} = \frac{\pi\sqrt{6}}{6}}$$

Ejercicio 4

Determine α de manera que la funcion dada sea armónica y determine el conjugado

$$u(x,y) = \operatorname{sen}(x)\cosh(\alpha y)$$

Para que la función sea armónica, se tiene que cumplir que:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

$$\frac{du}{dx} = \cosh(\alpha y)\cdot\cos x$$

$$\frac{du^2}{dx^2} = \cosh(\alpha y)\cdot -\operatorname{sen} x = -\operatorname{sen} x\cosh(\alpha y)$$

Por formulario:

$$\cosh(\alpha y) = \frac{e^{\alpha y}+e^{-\alpha y}}{2}$$

$$\Rightarrow u(x,y) = \operatorname{sen} x \cdot \frac{e^{\alpha y}+e^{-\alpha y}}{2}$$

$$\frac{du}{dy} = \operatorname{sen} x\cdot\left(\frac{e^{\alpha y}\cdot\alpha + e^{-\alpha y}\cdot -\alpha}{2}\right)$$

$$\frac{du^2}{dy^2} = \operatorname{sen} x\left(\frac{e^{\alpha y}\cdot\alpha^2 + e^{-\alpha y}\cdot\alpha^2}{2}\right)$$

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

$$-\operatorname{sen} x\cdot\left(\frac{e^{\alpha y}+e^{-\alpha y}}{2}\right) + \operatorname{sen} x\left(\frac{\alpha^2 e^{\alpha y}+\alpha^2 e^{-\alpha y}}{2}\right) = 0$$

Para que esto se cumpla, se tiene que cumplir que:

$$\frac{e^{\alpha y}+e^{-\alpha y}}{2} = \frac{\alpha^2 e^{\alpha y}+\alpha^2 e^{-\alpha y}}{2}$$

$$\alpha^2 = 1$$
$$\alpha = \pm 1$$

$\rightarrow$ Como $\cosh(\alpha y) = \cosh(-\alpha y)$

$$\boxed{\alpha = 1}$$

Para determinar el conjugado:

$$\frac{du}{dx} = \cosh(\alpha y)\cdot\cos x \Rightarrow \int \cosh(\alpha y)\cdot \cos x\, dy = \int \frac{dv}{dy} = \cos x \int \cosh(a y)$$

$$= \cos x \cdot \frac{1}{\alpha}\,\operatorname{senh}(a y) + F(x)$$

Como $\alpha = 1$

$$\Rightarrow \int \frac{dv}{dy} = v = \cos x \cdot \operatorname{senh}(y) + F(x)$$

$$\frac{du}{dy} = \operatorname{sen} x \cdot \left(\frac{e^{y}-e^{-y}}{2}\right) = \operatorname{sen} x \cdot \operatorname{senh}(y)$$

$$\Rightarrow -\frac{dv}{dx} = -\int \operatorname{sen} x \cdot \operatorname{senh}(y)\, dx = v$$

$$= -\left(\operatorname{senh}(y)\cdot \int \operatorname{sen} x\, dx\right) = -\left(\operatorname{senh}(y)\cdot -\cos x + F(y)\right)$$

$$= \operatorname{senh}(y)\cdot \cos x + F(y)$$

$$\boxed{\Rightarrow V = \cos(x)\cdot \operatorname{senh}(y) + C \quad \text{con } C = \text{cte.}}$$