Tutoría 4. Integración compleja

Ejercicio 1

Esboce gráficamente las siguientes trayectorias, indicando su sentido, y además exprese la trayectoria con una ecuación no paramétrica (que no depende de t)

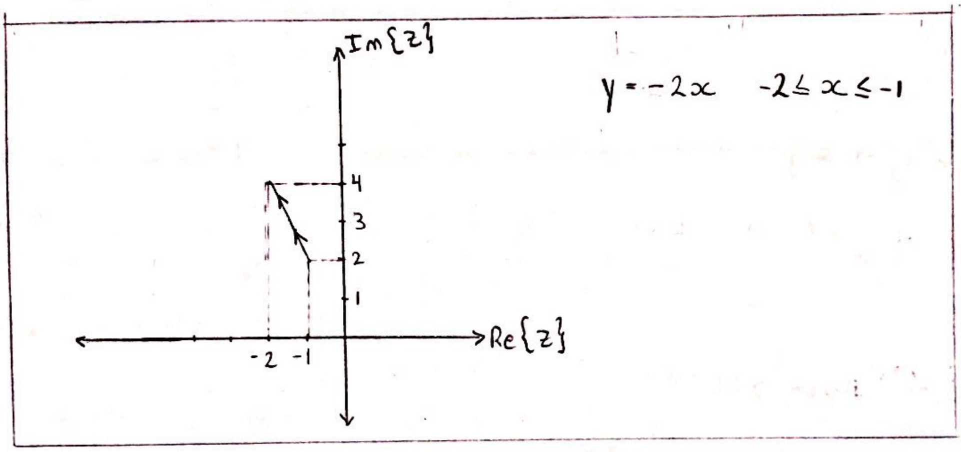
a) $z(t) = (-1+2j)t$ para $1 \leq t \leq 2$

b) $z(t) = 2-jt$ para $-3 \leq t \leq 1$

c) $z(t) = 1+j+e^{-j\pi t}$ para $0 \leq t \leq 1$

d) $z(t) = \min(t+1;2) + j\max(t-2;-1)$ para $0 \leq t \leq 4$

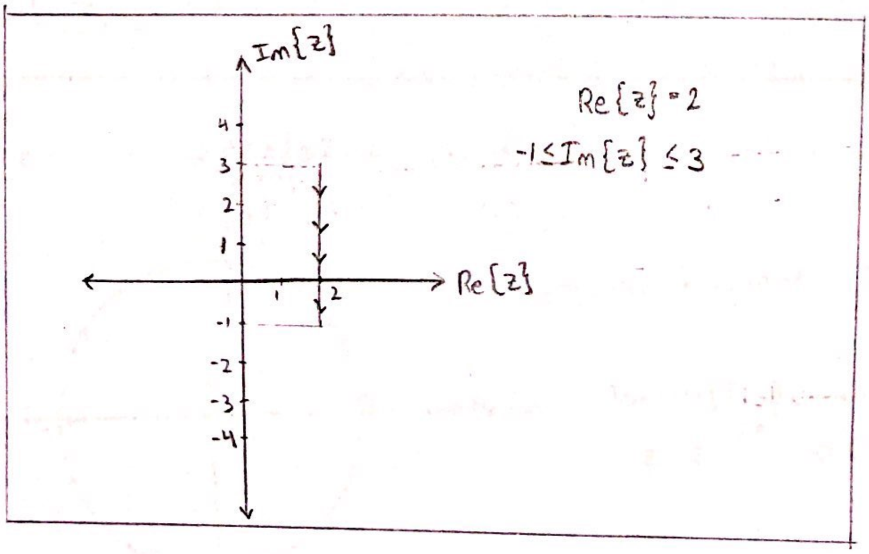
a) $z(t) = -t+j2t$ para $1 \leq t \leq 2$



$m = \frac{4-2}{-2--1} = \frac{2}{-1} = -2$

$b = 4 - -2 \cdot -2 = 0$

b) $z(t) = 2-jt$ para $-3 \leq t \leq 1$



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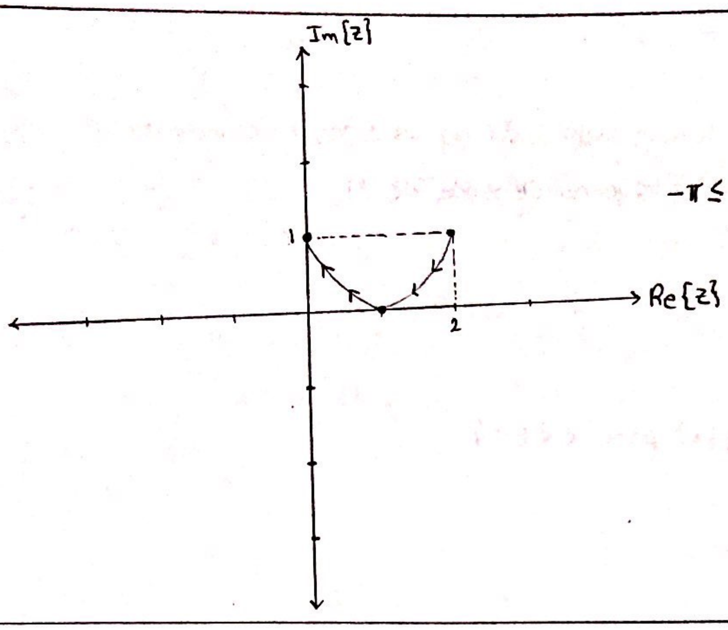
c) $z(t) = 1+j+e^{-j\pi t}$ para $0 \leq t \leq 1$



$|z-(1+j)| = 1$

$-\pi \leq \angle[z-(1+j)] \leq 0$

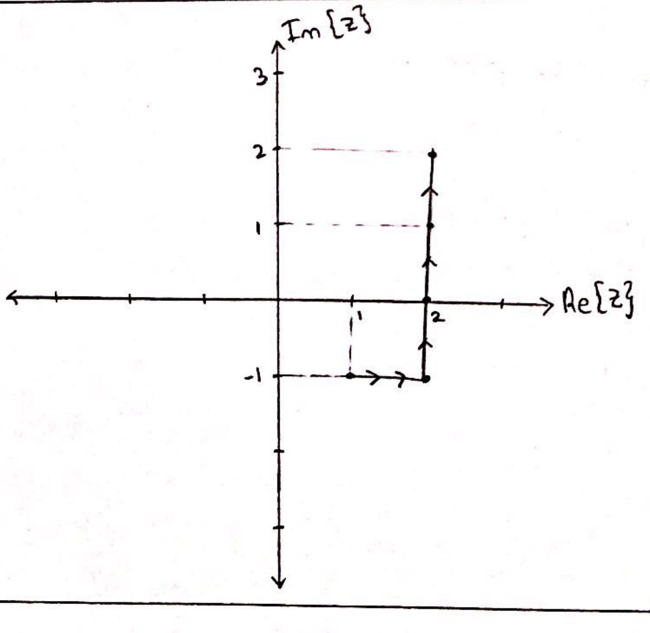
$t=0: \ e^{-j\pi(0)} = e^{0} = 1 \Rightarrow z(t) = 1+j+1 = 2+j$

$t=1: \ e^{-j\pi(1)} = e^{-j\pi} = -1 \Rightarrow z(t) = 1+j-1 = j$

$t=\frac{1}{2}: \ e^{-j\pi(\frac{1}{2})} = e^{-j\frac{\pi}{2}} = -j \Rightarrow z(t) = 1+j-j = 1$

d) $z(t) = \min(t+1;2) + j\max(t-2;-1)$ para $0 \leq t \leq 4$

- Con $t=0$ $\quad z(t) = 1-j$
- Con $t=1$ $\quad z(t) = 2-j$
- Con $t=2$ $\quad z(t) = 2$
- Con $t=3$ $\quad z(t) = 2+j$
- Con $t=4$ $\quad z(t) = 2+j2$



$Im\{z\} = -1 \ ; \ 1 \leq Re\{z\} \leq 2$

$Re\{z\} = 2 \ ; \ -1 \leq Im\{z\} \leq 2$

Ejercicio 2

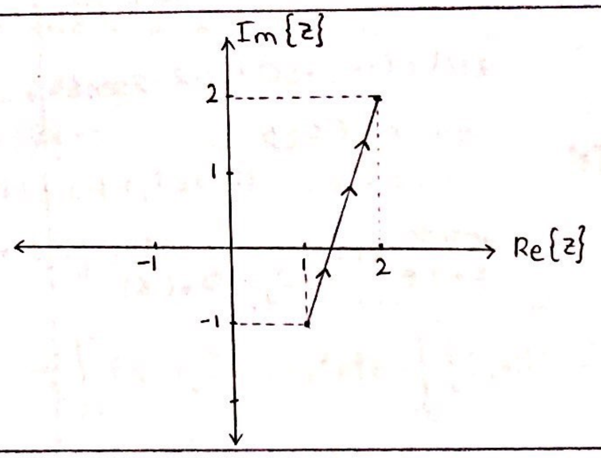
Esboce gráficamente y represente de forma paramétrica las siguientes trayectorias con $0 \leq t \leq 1$:

a) Segmento de recta entre $1-j$ a $2+2j$

b) Círculo unitario en sentido horario

c) $|z-1+2j| = 2$ en sentido antihorario

a)



$z(t) = t+1 + j(3t-1)$
con $0 \leq t \leq 1$

$m = \frac{2--1}{2-1} = \frac{3}{1} = 3$

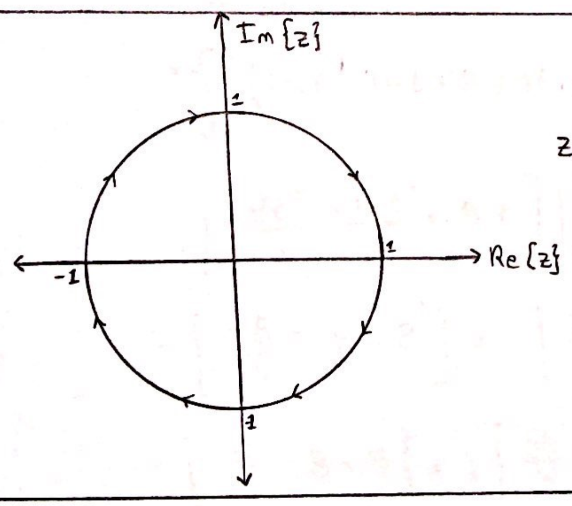
$b = y - mx = -1 - 3\cdot 1 = -4$

$y = 3x - 4$

$x = t+1$
$y = 3(t+1) - 4 = 3t + 3 - 4 = 3t - 1$

$z(t) = x + jy = t+1 + j(3t-1)$

b)



$z(t) = \cos t + j \operatorname{sen} t \quad -2\pi \leq t \leq 0$

También:
$z = e^{-j2\pi t}$; $0 \leq t \leq 1$

La ecuación del círculo es

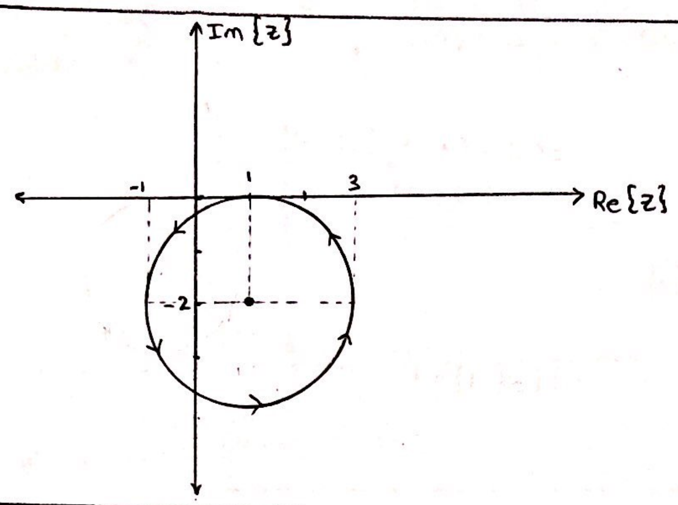
$$x^2+y^2=1$$

La forma paramétrica es

$$x=\cos t$$
$$y=\operatorname{sen} t$$

$$z(t)=\cos t+j\operatorname{sen} t \quad -2\pi \le t \le 0$$

c)



$$z(t)=(1+2\cos t)+j(-2+2\operatorname{sen} t)$$

con $0\le t\le 2\pi$

También:

$$z=2e^{j2\pi t}+1-2j;\quad 0\le t\le 1$$

La ecuación del círculo es

$$(x-1)^2+(y+2)^2=2^2$$

La forma paramétrica es:

$$x=1+2\cos t$$
$$y=-2+2\operatorname{sen} t$$

$$z(t)=(1+2\cos t)+j(-2+2\operatorname{sen} t) \quad 0\le t\le 2\pi$$

Ejercicio 3

Encuentre el valor de las integrales:

$$\int_c (x^2 + j2xy + y^2) dz$$

$$\int_c z^2 dz$$

Para las trayectorias de integración de los puntos 1.d y 2.a

• Para la trayectoria 2.a

$x = t+1$
$y = 3t-1$
$z(t) = t+1 + j(3t-1)$ con $0 \leq t \leq 1$
$dz = 1 + j3\,dt$

$$\rightarrow \int_c (x^2 + j2xy + y^2) dz = \int_0^1 \left[(t+1)^2 + j2(t+1)(3t-1) + (3t-1)^2\right](1+3j)\,dt$$

$$= \int_0^1 \left[t^2+2t+1 + (j2t+j2)(3t-1) + 9t^2 - 6t + 1\right](1+3j)\,dt$$

$$= \int_0^1 \left[10t^2 - 4t + 2 + j6t^2 - j2t + j6t - j2\right](1+3j)\,dt$$

$$= \int_0^1 \left[10t^2 - 4t + 2 + j6t^2 + j4t - j2\right](1+3j)\,dt$$

$$= \int_0^1 (10t^2 - 4t + 2 + j6t^2 + j4t - j2 + j30t^2 - j12t + j6 - 18t^2 - 12t + 6)\,dt$$

$$= \int_0^1 (-8t^2 - 16t + 8 + j36t^2 - j8t + j4)\,dt$$

$$= \int_0^1 (-8t^2 - 16t + 8)\,dt + j\int (36t^2 - 8t + 4)\,dt$$

$$= \left[\frac{-8t^3}{3} - \frac{16t^2}{2} + 8t\right]\Big|_0^1 + j\left[\frac{36t^3}{3} - \frac{8t^2}{2} + 4t\right]\Big|_0^1$$

$$= \left[-\frac{8}{3} - \frac{16}{2} + 8\right] + j\left[\frac{36}{3} - \frac{8}{2} + 4\right]$$

$$= \left[\frac{-8}{3} - 8 + 8\right] + j\left[\frac{36}{3} - 4 + 4\right]$$

$$= \frac{-8}{3} + j\frac{36}{3} = \frac{-8}{3} + j12$$

$$\boxed{\int_c (x^2 + j2xy + y^2)dz = -\frac{8}{3} + 12j \quad \text{para} \quad \begin{array}{c} z(t) = t+1+j(3t-1) \\ 0 \leq t \leq 1 \end{array}}$$

$$\rightarrow \int_c z^2 dz = \int_0^1 (x+jy)^2 dz = \int_0^1 (x^2 + j2xy - y^2) dz \quad \text{con} \quad x = t+1 \wedge y = 3t-1$$

$$= \int_0^1 \left[(t+1)^2 + j2(t+1)(3t-1) - (3t-1)^2 \right] (1+3j) dt$$

$$= \int_0^1 \left[t^2 + 2t + 1 + (j2t + j2)(3t-1) - (9t^2 - 6t + 1) \right] (1+3j) dt$$

$$= \int_0^1 \left[t^2 + 2t + 1 + j6t^2 - j2t + j6t - j2 - 9t^2 + 6t - 1 \right] (1+3j) dt$$

$$= \int_0^1 \left[-8t^2 + 8t + j6t^2 + j4t - 2j \right] (1+3j) dt$$

$$= \int_0^1 \left[-8t^2 + 8t + j6t^2 + j4t - 2j + -j24t^2 + j24t - 18t^2 - 12t + 6 \right] dt$$

$$= \int_0^1 \left[-26t^2 - 4t + 6 \right] dt + j\int_0^1 \left[-18t^2 + 28t - 2 \right] dt$$

$$= \left[-\frac{26t^3}{3} - \frac{4t^2}{2} + 6t \right] \Big|_0^1 + j\left[-\frac{18t^3}{3} + \frac{28t^2}{2} - 2t \right] \Big|_0^1$$

$$= \left(\frac{-26}{3} - 2 + 6 \right) + j\left(-\frac{18}{3} + 14 - 2 \right)$$

$$= \left(-\frac{26}{3} + 4 \right) + j\left(-6 + 12 \right)$$

$$\boxed{\int_c z^2 dz = -\frac{14}{3} + 6j \quad \text{para } z(t) = t+1+j(3t-1) \text{ con } 0 \leq t \leq 1}$$

Para la trayectoria 1.d

Primera parte:

$x = t \quad 1 \leq t \leq 2$

$y = -1$

$z(t) = t - j$

$dz = 1\,dt$

Segunda parte:

$x = 2$

$y = t \quad -1 \leq t \leq 2$

$z(t) = 2 + jt$

$dz = j\,dt$

$\rightarrow \int_c z^2 dz$

$z = x + jy$

$$f(z) = z^2 = (x+jy)^2 = x^2 + j2xy - y^2 = \underbrace{x^2 - y^2}_{u} + j\underbrace{2xy}_{v}$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial y} = 2x \quad \frac{\partial v}{\partial x} = 2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \text{Es analítica.}$$

Como la funcion es analitica, no depende de la trayectoria, solo del punto inicial y final:

$$\int_c z^2 dz = \int_{1-j}^{2+2j} z^2 dz = \left.\frac{z^3}{3}\right|_{1-j}^{2+2j} = \frac{1}{3}\left[(2+2j)^3 - (1-j)^3\right]$$

$$= \frac{1}{3}\left[2^3 + 3(2)^2 \cdot 2j + 3 \cdot 2 \cdot (2j)^2 + (2j)^3 - \left(1^3 - 3 \cdot 1^2 \cdot j + 3 \cdot 1 \cdot j^2 - j^3\right)\right]$$

$$= \frac{1}{3}\left[8 + 24j - 24 - 8j - (1 - 3j - 3 + j)\right]$$

$$= \frac{1}{3}\left[-16 + 16j + 2 + 2j\right]$$

$$= \frac{1}{3}(-14 + 18j)$$

$$\boxed{\int_c z^2 dz = \frac{-14}{3} + 6j}$$

$$\rightarrow \int_c (x^2 + j2xy + y^2)dz = \int_1^2 \left[t^2 - j2t + 1\right]dt + \int_{-1}^2 \left[4 + j4t + t^2\right] \cdot j\,dt$$

$$= \left[\int_1^2 (t^2+1)dt - j\int_1^2 2t\,dt\right] + \int_{-1}^2 \left[4j - 4t + jt^2\right]dt$$

$$= \left[\left(\frac{t^3}{3} + t \right) \Big|_1^2 - 2j \left(\frac{t^2}{2} \right) \Big|_1^2 \right] + \left[\int_{-1}^{2} -4t\,dt + j \int_{-1}^{2} (4 + t^2)\,dt \right]$$

$$= \left[\left(\frac{2^3}{3} + 2 - \frac{1^3}{3} + 1 \right) - 2j \left(\frac{2^2}{2} - \frac{1^2}{2} \right) \right] + \left[-4 \left(\frac{t^2}{2} \right)_{-1}^{2} + j \left(4t + \frac{t^3}{3} \right) \Big|_{-1}^{2} \right]$$

$$= \left[\frac{8}{3} + 2 - \frac{1}{3} - 1 - 2j \left(2 - \frac{1}{2} \right) \right] + \left[-4 \left(2 - \frac{1}{2} \right) + j \left(8 + \frac{8}{3} - \left(-4 + \frac{-1}{3} \right) \right) \right]$$

$$= \left[1 + \frac{7}{3} - 2j \left(\frac{3}{2} \right) \right] + \left[-4 \cdot \frac{3}{2} + j \left(12 + \frac{9}{3} \right) \right]$$

$$= \left[\frac{10}{3} - \frac{6}{2} j \right] + \left[-\frac{12}{2} + j \frac{45}{3} \right]$$

$$= \frac{10}{3} - 3j + -6 + j15$$

$$\boxed{\int_c (x^2 + j2xy + y^2)\,dz = -\frac{8}{3} + 12j}$$

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Ejercicio 4

Evalúe las integrales

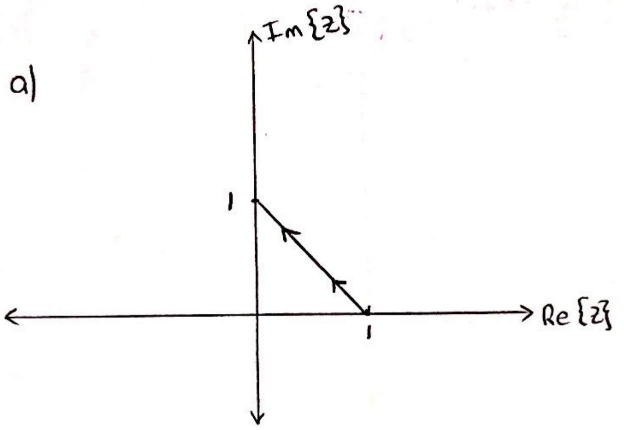
$$\int_c z^2 dz \qquad \int_c (x^2+y^2)dz$$

Para los contornos:

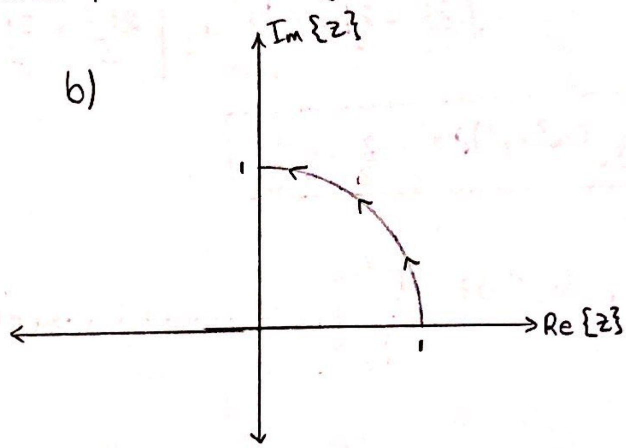
a) Segmento de recta 1 a j

b) Segmento de círculo $|z|=1$ que va en sentido positivo de 1 a j

a)



b)



$y = t$

$x = 1-t$

$z(t) = (1-t)+jt$ con $0 \leq t \leq 1$

$dz = (-1+j)dt$

$z = e^{jt}$ con $0 \leq t \leq \frac{\pi}{2}$

$z = \cos t + j\operatorname{sen} t$

$x = \cos t$

$y = \operatorname{sen} t$

$dz = (-\operatorname{sen} t + j\cos t)dt$

→ Para el contorno $\int_c z^2 dz$

$z = x+jy$

$$f(z) = z^2 = (x+jy)^2 = x^2 + j2xy - y^2 = \underbrace{x^2-y^2}_{u} + j\underbrace{2xy}_{v}$$

$$\frac{du}{dx} = 2x \qquad \frac{du}{dy} = -2y$$

$$\frac{dv}{dy} = 2x \qquad \frac{dv}{dx} = 2y$$

$$\Rightarrow \frac{du}{dx} = \frac{dv}{dy} \qquad \frac{du}{dy} = -\frac{dv}{dx} \Rightarrow$$ Es analítica.

Como la función es analítica, no depende del contorno, solo del punto de inicio y final, que son los mismos para ambos contornos:

$$\int_c z^2 dz = \int_1^j z^2 dz = \left.\frac{z^3}{3}\right|_1^j = \frac{-j}{3} - \frac{1}{3}$$

$$\boxed{\int_c z^2 dz = \frac{-1-j}{3}}$$

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→Para el contorno $\int_c (x^2+y^2)dz$

a) $\int_c (x^2+y^2)dz = \int_0^1 [(t-1)^2+t^2](1+j)dt = \int_0^1 (t^2-2t+1+t^2)(1+j)dt = \int_0^1 (2t^2-2t+1)(-1+j)dt$

$= \int_0^1 (-2t^2+2t-1+j2t^2-j2t+j)dt = -\int_0^1 (2t^2-2t+1)dt + j\int_0^1 (2t^2-2t+1)dt$

$= -\left[\frac{2t^3}{3} - \frac{2t^2}{2} + t\right]\Big|_0^1 + j\left[\frac{2t^3}{3} - \frac{2t^2}{2} + t\right]\Big|_0^1 = -\left(\frac{2}{3}-1+1\right) + j\left(\frac{2}{3}-1+1\right)$

$$\boxed{\int_c (x^2+y^2)dz = -\frac{2}{3} + j\frac{2}{3}}$$

b) $\int_c (x^2+y^2)dz = \int_0^{\pi/2} \underbrace{[\cos^2 t + \operatorname{sen}^2 t]}_{1} (-\operatorname{sen} t + j\cos t)dt$

$= \int_0^{\pi/2} -\operatorname{sen} t\,dt + j\int_0^{\pi/2} \cos t\, dt$

$= (\cos t)_0^{\pi/2} + j(\operatorname{sen} t)_0^{\pi/2}$

$= -1 + j$

$$\boxed{\int_c (x^2+y^2)dz = -1+j}$$

Escaneado con CamScanner

Ejercicio 5

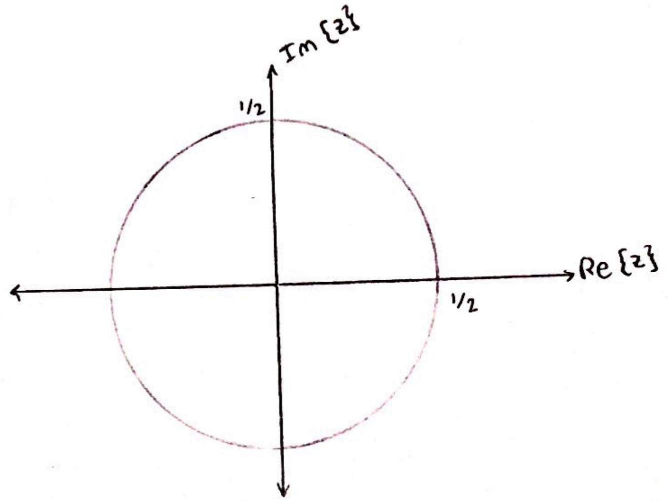
Evalúe la integral: $\oint_C \frac{1}{z^2(1+z^2)^2} dz$

Donde la trayectoria de integración es:

a) El círculo $|z| = \frac{1}{2}$

b) El círculo $|z| = 2$

a)



Como el radio es de $1/2$, solo incluye el polo $z=0$:

Por la fórmula de la integral de Cauchy:

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$

$$f(z) = \frac{1}{(1+z^2)^2} \Rightarrow f'(z) = \frac{-2 \cdot 2z}{(1+z^2)^3} = \frac{-4z}{(1+z^2)^3}$$

$$\oint_C \frac{1}{z^2(1+z^2)^2} dz = \frac{2\pi j}{1!} \cdot \left(\frac{-4z}{(1+z^2)^3} \right) \Bigg|_{z=0} = 0$$

$$\boxed{\oint_C \frac{1}{z^2(1+z^2)^2} dz = 0 \quad \text{para } |z| = \frac{1}{2}}$$

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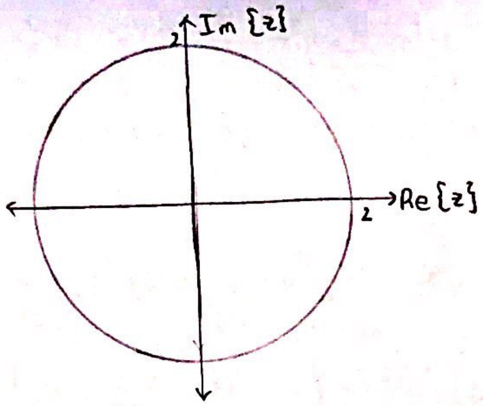
b)



Por el teorema del residuo:

$$\oint_c f(z)dz = 2\pi j \sum_{i=1}^{n} a_{-1}^{(i)}$$

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}$$

$$\oint_c \frac{1}{z^2(1+z^2)^2} dz = \frac{dz}{z^2(z+j)^2(z-j)^2}$$

$$a_{-1}^{(1)} = \frac{1}{(2-1)!} \lim_{z\to 0} \left\{ \frac{d}{dz} \left(\cancel{z^2} \cdot \frac{1}{\cancel{z^2}(1+z^2)^2} \right) \right\} = \lim_{z \to 0} \frac{-4z}{(1+z^2)^3} = 0$$

$$a_{-1}^{(2)} = \lim_{z \to j} \left\{ \frac{d}{dz} \left(\cancel{(z-j)^2} \cdot \frac{1}{z^2(z+j)^2\cancel{(z-j)^2}} \right) \right\} = \lim_{z \to 0} \left(\frac{-2}{z^3(z+j)^2} + \frac{-2}{z^2(z+j)^3} \right)$$

$$= \frac{-2}{j^3(2j)^2} + \frac{-2}{j^2(2j)^3} = \frac{-2}{-j(4)} - \frac{2}{-1(-8j)} = \frac{-2}{4j} - \frac{2}{8j} = -\frac{1}{2j} - \frac{1}{4j}$$

$$= \frac{j}{2} + \frac{j}{4} = \frac{4j+2j}{8} = \frac{6j}{8} = \frac{3}{4}j$$

$$a_{-1}^{(2)} = \lim_{z \to -j} \left\{ \frac{d}{dz} \left(\cancel{(z+j)^2} \cdot \frac{1}{z^2\cancel{(z+j)^2}(z-j)^2} \right) \right\} = \lim_{z \to -j} \left(\frac{-2}{z^3(z-j)^2} + \frac{-2}{z^2(z-j)^3} \right)$$

$$= \frac{-2}{(-j)^3(-2j)^2} - \frac{2}{(-j)^2(-2j)^3} = \frac{-2}{j(-4)} - \frac{2}{-1\cdot(8j)} = \frac{1}{2j} + \frac{1}{4j}$$

$$= \frac{-j}{2} - \frac{j}{4} = \frac{-4j-2j}{8} = \frac{-6j}{8} = -\frac{3}{4}j$$

$$\oint_c \frac{1}{z^2(1+z^2)^2} = 2\pi j \left(0 + \frac{3}{4}j - \frac{3}{4}j \right)$$

$$\boxed{\oint_c \frac{1}{z^2(1+z^2)^2} = 0}$$