Unit 2

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Solution of the non homogeneous or complete equation

Theorem: If $y_c(k)$ is the complementary solution of homo eq $(a_0E^n + a_1E^{n-1} + \dots + a_n)y_k = 0$ and $y_p(k)$ is any particular solution of the complete equation $(a_0E^n + a_1E^{n-1} + \dots + a_n)y_k = R(k)$ then the general solution $y_k = y_c(k) + y_p(k)$ and all solutions are special cases of it.

Methods for finding particular solutions

Method of undetermined coefficients

Terms in $R(k)$	Trial solution
β^k	$A\beta^k$
$\sin ak$ or $\cos ak$	$A\cos ak + B\sin ak$
poly $P(k)$ of degree m	$A_0k^m + A_1k^{m-1} + \dots + A_m$
$\beta^k \sin ak$ or $\beta^k \cos ak$	$\beta^k(A\cos ak + B\sin ak)$

No term of the trial solution can appear in the complementary function. If any term of the trial sol does happen to be in the complementary function then the entire trial ~~function~~ sol corresponding to this term must be multiplied by a positive integer power of k which is just large enough so that no term of the new trial sol will appear in complementary function

5.20 Solve $y_{k+2} - 6y_{k+1} + 8y_k = 3k^2+2-5\cdot3^k$

$(E^2-6E+8)\,y_k = 3k^2+2-5\cdot3^k$

Aux eq: $r^2-6r+8=0$

$(r-2)(r-4)=0$

$r=2,4$

CF $y_c(k) = c_1 2^k + c_2 4^k$

Now trial sol

$y_p(k) = Ak^2+Bk+C+D\cdot3^k$

$$\begin{aligned}
y_{k+2}-6y_{k+1}+8y_k &= A(k+2)^2+B(k+2)+C+D\cdot3^{k+2} \\
&\quad -6\left[A(k+1)^2+B(k+1)+C+D\cdot3^{k+1}\right] \\
&\quad +8\left[Ak^2+Bk+C+D\cdot3^k\right]
\end{aligned}$$

$$\begin{aligned}
&= A\left[k^2+4k+4\right]+Bk+2B+C+9D\cdot3^k \\
&\quad -6A(k^2+2k+1)-6Bk-6B-6C-18D\,3^k \\
&\quad +8Ak^2+8Bk+8C+8D\,3^k
\end{aligned}$$

$= 3Ak^2+(-8A+3B)k-2A-4B+3C-D\,3^k$

Now $3k^2+2-5\cdot3^k = 3Ak^2+(-8A+3B)k-2A-4B+3C-D\cdot3^k$

$\therefore\ 3=3A \Rightarrow A=1$

$-8A+3B=0 \Rightarrow 3B=8 \Rightarrow B=8/3$

$-2A-4B+3C=2 \Rightarrow -2-\frac{32}{3}+3C=2$

$\Rightarrow 3C = 2+\frac{38}{3} = \frac{44}{3}$

$\Rightarrow C=\frac{44}{9}$

$-5\cdot3^k = -D\cdot3^k$

$\Rightarrow D=5$

$\therefore$ particular sol is $k^2+\frac{8}{3}k+\frac{44}{9}+5\cdot3^k$

Gen sol is $c_1 2^k + c_2 4^k + k^2+\frac{8}{3}k+\frac{44}{9}+5\cdot3^k$

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5.21 Solve $y_{k+2}-4y_{k+1}+4y_k = 3.2^k+5.4^k$

$(E^2-4E+4)y_k = 3.2^k+5.4^k$

Aux eq: $r^2-4r+4=0 \Rightarrow (r-2)^2=0 \Rightarrow r=2,2$

$y_c(k) = (c_1+c_2k)2^k$

Trial sol: $Ak^2 2^k + B.4^k$

$y_{k+2}-4y_{k+1}+4y_k = A(k+2)^2 2^{k+2} + B.4^{k+2}$
$= 3.2^k+5.4^k$
$- 4(A(k+1)^2 2^{k+1} + B.4^{k+1})$
$+ 4(Ak^2 2^k + B.4^k)$

$=4A(k^2+4k+4)2^k + 16B4^k$
$- 8(A(k^2+2k+1)2^k) - 16B4^k$
$+ 4Ak^2 2^k + 4B.4^k$

$=(\cancel{4Ak^2}+\cancel{16Ak}+16A)2^k - (\cancel{8Ak^2}+\cancel{16Ak}+8A)2^k + (\cancel{4Ak^2})2^k + 4B.4^k$

$= (\cancel{32Ak} + \overset{8A}{\cancel{24A}})2^k + 4B.4^k$

$\cancel{3 = 32Ak + 24A}$

$5 = 4B \Rightarrow B = 5/4$

$3 = 8A \Rightarrow A = 3/8$

PS $y_p(k) = \frac{3}{8}k^2 2^k + \frac{5}{4}.4^k$

GS $y_k = (c_1+c_2k)2^k + \frac{3}{8}k^2 2^k + \frac{5}{4}.4^k$

$= (c_1+c_2k)2^k + 3k^2 2^{k-3} + 5.4^{k-1}$

5.22 Solve $y_{k+3}-3y_{k+2}+3y_{k+1}-y_k = 24(k+2)$

$(E^3-3E^2+3E-1)y_k = 24(k+2)$

Aux eq: $r^3-3r^2+3r-1=0$

$(r-1)^3=0 \Rightarrow r=1,1,1$

$$y_c(k) = (c_1 + c_2 k + c_3 k^2) 1^k$$
$$= c_1 + c_2 k + c_3 k^2.$$

Trial sol: $(A_1 k + A_2) k^3 = A_1 k^4 + A_2 k^3.$

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = A_1(k+3)^4 + A_2(k+3)^3$$
$$= 24k + 48$$
$$-3(A_1(k+2)^4 + A_2(k+2)^3)$$
$$+3(A_1(k+1)^4 + A_2(k+1)^3)$$
$$-(A_1 k^4 + A_2 k^3)$$

$$= A_1(\cancel{k^4} + \cancel{12k^3} + \cancel{54k^2} + 108k + 81) + A_2(\cancel{k^3} + \cancel{9k^2} + \cancel{27k} \ldots$$
$$-3A_1(\cancel{k^4} + \cancel{8k^3} + \cancel{24k^2} + 32k + 16) - 3A_2(\cancel{k^3} + \cancel{6k^2} + \cancel{12k} \ldots$$
$$+3A_1(\cancel{k^4} + \cancel{4k^3} + 6k^2 + 4k + 1) + 3A_2(\cancel{k^3} + \cancel{3k^2} + \cancel{3k} + \ldots$$
$$-A_1 \cancel{k^4} - A_2 \cancel{k^3}$$

$$= 24A_1 k + 36A_1 + 6A_2.$$

$$24A_1 = 24 \Rightarrow A_1 = 1$$
$$36A_1 + 6A_2 = 48$$
$$6A_2 = 48 - 36 = 12$$
$$A_2 = 2$$

$$y_k = c_1 + c_2 k + c_3 k^2 + 2k^3 + k^4$$

12 - 24 + 12
54
18
72
120
96
24
84
-48
36
27
9

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Special operators Method

Any linear difference equation can be written as $\phi(E)\, y_k = R(k)$ — (1)

Define the operator $\frac{1}{\phi(E)}$ called the inverse of $\phi(E)$ by the relationship $\frac{1}{\phi(E)} R(k) = U$

where $\phi(E)\, U = R(k)$

We assume U doesnot have any arbitrary constants and so is the particular solution of (1)

1. $\frac{1}{\phi(E)} \beta^k = \frac{\beta^k}{\phi(\beta)}$, $\phi(\beta) \neq 0$, otherwise use method 4.

2. $\frac{1}{\phi(E)} \sin ak$ or $\frac{1}{\phi(E)} \cos ak$

$\sin ak = \frac{e^{iak} - e^{-iak}}{2i}$, $\cos ak = \frac{e^{iak} + e^{-iak}}{2}$

and use 1

3) $\frac{1}{\phi(E)} P(k) = \frac{1}{\phi(1+\Delta)} P(k) = (b_0 + b_1 \Delta + \dots + b_m \Delta^m + \dots) P(k)$

where the expansion needs to be carried out only as far as Δ^m since $\Delta^{m+1} P(k) = 0$

($P(k)$ is a usual poly of degree m)

4) $\frac{1}{\phi(E)} \beta^k P(k) = \beta^k \frac{1}{\phi(\beta E)} P(k)$ then use 3.

Result also holds for any function $F(k)$ is place of $P(k)$

Q 5.25 Solve $y_{k+2} - 2y_{k+1} + 5y_k = 2\cdot 3^k - 4\cdot 7^k$

$(E^2 - 2E + 5)\, y_k = 2\cdot 3^k - 4\cdot 7^k$

Aux eq: $r^2 - 2r + 5 = 0$

$r = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\rho = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\theta = \tan^{-1}\dfrac{2}{1}$

$y_c(k) = (\sqrt{5})^k (c_1 \cos k\theta + c_2 \sin k\theta)$

$y_p(k) = \dfrac{1}{E^2 - 2E + 5}\; 2\cdot 3^k - 4\cdot 7^k$

$= 2\cdot \dfrac{1}{3^2 - 2\cdot 3 + 5}\, 3^k - 4\, \dfrac{1}{7^2 - 2\cdot 7 + 5}\, 7^k$

$= \dfrac{2}{8}\cdot 3^k - \dfrac{4}{40}\cdot 7^k$

$= \dfrac{3^k}{4} - \dfrac{7^k}{10}$

sol $y_k = 5^{k/2}(c_1 \cos k\theta + c_2 \sin k\theta) + \dfrac{1}{4}\cdot 3^k - \dfrac{1}{10}\cdot 7^k$

Q 5.28 Find $\dfrac{1}{E^2 - 4E + 4}\; 2^k$

Here $\phi(E) = E^2 - 4E + 4 = (E-2)^2$

$\phi(2) = 0$

So, case 1 fails.

So, using case 4

$\dfrac{1}{\phi(E)}\, 2^k = 2^k \dfrac{1}{\phi(2E)}\cdot 1 = 2^k \dfrac{1}{(2E)^2 - 4(2E) + 4}\cdot 1$

$= 2^k \dfrac{1}{4E^2 - 8E + 4}\cdot 1$

$= 2^k \dfrac{1}{4(1+\Delta)^2 - 8(1+\Delta) + 4}\cdot 1$

$= 2^k \dfrac{1}{4(1 + \Delta^2 + 2\Delta) - 8 - 8\Delta + 4}\cdot 1$

$= 2^k \dfrac{1}{4\Delta^2}\cdot 1$

$= 2^{k-2}\, \Delta^{-1}\Delta^{-1}(1)$

$= 2^{k-2}\, \Delta^{-1} k^{(1)} = 2^{k-2} \dfrac{k^{(2)}}{2} = \dfrac{2^k k(k-1)}{8}$

by Dr. Varsha H Patil

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Q5.29. Find $\frac{1}{E^2-6E+8}(3k^2+2)$

$= \frac{1}{(\Delta+1)^2-6(\Delta+1)+8}(3k^2+2)$

$= \frac{1}{\Delta^2+2\Delta+1-6\Delta-6+8}(3k^2+2)$

$= \frac{1}{\Delta^2-4\Delta+3}(3k^2+2)$

$= \frac{1}{3\left(1-\left(\frac{4\Delta}{3}-\frac{\Delta^2}{3}\right)\right)}(3k^2+2)$

$= \frac{1}{3}\left[1+\left(\frac{4\Delta}{3}-\frac{\Delta^2}{3}\right)+\left(\frac{4\Delta}{3}-\frac{\Delta^2}{3}\right)^2\right](3k^2+2)$

$= \frac{1}{3}\left[1+\frac{4}{3}\Delta-\frac{\Delta^2}{3}+\frac{16}{9}\Delta^2\right](3k^2+2)$

$\cancel{= \frac{1}{3}[3k^2+2+4}$

$= \frac{1}{3}\left[1+\frac{4}{3}\Delta+\frac{13}{9}\Delta^2\right](3k^{(2)}+3k^{(1)}+2)$

$= \frac{1}{3}(3k^2+2)+\frac{4}{9}\Delta(3k^{(2)}+3k^{(1)}+2)$
$+\frac{13}{27}\Delta^2(3k^{(2)}+3k^{(1)}+2)$

$= k^2+\frac{2}{3}+\frac{4}{9}\left[6k^{(1)}+3\right]+\frac{13}{27}[6]$

$= k^2+\frac{2}{3}+\frac{8}{3}k+\frac{4}{3}+\frac{26}{9}$

$2+\frac{26}{9}$ $\quad 44$

$= k^2+\frac{8k}{3}+\frac{44}{9}$

Rough Sheet

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5.30 Solve $(E^2-4E+4)y_k = 3.2^k + 5.4^k$

$r^2-4r+4=0$

$(r-2)^2=0, \quad r=2,2$

$y_c(k) = (c_1+c_2k)\,2^k$

$$y_p(k) = \frac{1}{(E-2)^2}\cdot 3.2^k + \frac{1}{(E-2)^2}\,5.4^k$$

$$= 3.2^k \frac{1}{(2E-2)^2}\,1 + 5\cdot\frac{1}{(4-2)^2}\,4^k$$

$$= \frac{3.2^k}{4}\frac{1}{(E-1)^2}\,1 + \frac{5}{4}\cdot 4^k$$

$$= \frac{3.2^k}{4}\cdot\frac{k^{(2)}}{2} + \frac{5}{4}\cdot 4^k$$

$$= \frac{3}{8}k(k-1)\,2^k + \frac{5}{4}\cdot 4^k$$

$$= \frac{3}{8}k^2 2^k - \frac{3}{8}k\,2^k + \frac{5}{4}\cdot 4^k$$

G.S $\quad y_k = (c_1+c_2k)\,2^k + \frac{3}{8}k^2 2^k - \frac{3}{8}k\,2^k + \frac{5}{4}\cdot 4^k$

$y_k = c_1 2^k + c\,k\,2^k + \frac{3}{8}k^2 2^k + \frac{5}{4}\cdot 4^k$

where $c = c_2 - \frac{3}{8}$

Q5.85 Solve each of the following by:
Method of undetermined coefficients

a) $y_{k+2} - 3y_{k+1} + 2y_k = 4^k$

$r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$
$\Rightarrow r = 1, 2$

$y_c(k) = c_1 + c_2(2)^k$

$y_p = A \cdot 4^k$

$A\,4^{k+2} - 3A\,4^{k+1} + 2A\,4^k = 4^k$

$16A\,4^k - 12A\,4^k + 2A\,4^k = 4^k$

$6A \cdot 4^k = 4^k$

$6A = 1$

$\Rightarrow A = 1/6$

$y_p = \frac{4^k}{6}$

$y_k = y_c + y_p = c_1 + c_2(2)^k + \frac{1}{6}4^k$

b) $y_{k+2} - 4y_{k+1} + 4y_k = k^2$

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2, 2$

$y_c = (c_1 + c_2 k)2^k$

$y_p = Ak^2 + Bk + C$

$A(k+2)^2 + B(k+2) + C - 4[A(k+1)^2 + B(k+1) + C]$
$+ 4[Ak^2 + Bk + C] = k^2$

$A(k^2+4k+4) + Bk + 2B + C - 4[A(k^2+2k+1) + Bk + B + C]$
$+ 4Ak^2 + 4Bk + 4C = k^2$

$Ak^2 + (4A + B - 8A - \cancel{4B} + \cancel{4B})k + \cancel{4A} + C + 2B$
$\cancel{-4A} - 4B - \cancel{4C} + \cancel{4C} = k^2$

$A = 1$ $\quad -4A + B = 0$ $\quad \cancel{4A} - 2B + C = 0$

$B = 4A = 4$ $\quad \cancel{4} - 8 + C = 0$

$C = \cancel{4}\ 8$

$y_p = k^2 + 4k + \cancel{4}\ 8$

$y_k = (c_1 + c_2 k)2^k + k^2 + 4k + \cancel{4}\ 8.$

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c) $4y_{k+2} + y_k = 2k^2 - 5k + 3$

$4r^2 + 1 = 0$

$r^2 = -1/4 \Rightarrow r = \pm \frac{1}{2} i$

$y_c(k) = \left(\frac{1}{2}\right)^k \left(C_1 \cos \frac{k\pi}{2} + C_2 \sin \frac{k\pi}{2}\right)$

$p = \sqrt{0^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1/2}{0}\right) = \pi/2$

$y_p(k) = Ak^2 + Bk + C$

$4A(k+2)^2 + B(k+2) + C + Ak^2 + Bk + C = 2k^2 - 5k + 3$

$4A(k^2 + 4k + 4) + Bk + 2B + C + Ak^2 + Bk + C = 2k^2 - 5k + 3$

$5Ak^2 + (16A + 2B)k + 16A + 2B + 2C = 2k^2 - 5k + 3$

$5A = 2$

$A = \frac{2}{5}$

$16A + 2B = -5$

$16 \times \frac{2}{5} + 2B = -5$

$2B = -5 - \frac{32}{5} = \frac{-57}{5}$

$B = -\frac{57}{10}$

$2B + 2C = 3$

$-\frac{57}{5} + 2C = 3$

$2C = 3 + \frac{57}{5} = \frac{72}{5}$

$C = \frac{36}{5}$

$y_p(k) = \frac{2}{5}k^2 - \frac{57}{10}k + \frac{36}{5}$

$y_k = \left(\frac{1}{2}\right)^k \left(C_1 \cos k\pi/2 + C_2 \sin k\frac{\pi}{2}\right) + \frac{2}{5}k^2 - \frac{57}{10}k + \frac{36}{5}$

d) $y_{k+1} - y_k = k, \quad y_0 = 1$

$r - 1 = 0 \Rightarrow r = 1$

$y_c(k) = C_1$

$y_p(k) = Ak^2 + Bk$

$A(k+1)^2 + B(k+1) - Ak^2 - Bk = k$

$A(\cancel{k^2} + 2k + 1) + B - \cancel{Ak^2} = k$

$2Ak + A + B = k$

$2A = 1$

$A = 1/2$

$A + B = 0$

$B = -A = -1/2$

$y_p(k) = -\frac{k^2}{2} + \frac{k}{2}$

$$y_k = C_1 - \frac{k^2}{2} + \frac{k}{2}$$

$$y_0 = 1$$

$$1 = y_0 = C_1$$

$$y_k = 1 - \frac{k^2}{2} + \frac{k}{2}$$

e) $y_{k+2} + 2y_{k+1} + y_k = k + 2^k$

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1$$

$$y_c(k) = (C_1 + C_2 k)(-1)^k$$

$$y_p(k) = Ak + B + C2^k.$$

$$A(k+2) + B + C2^{k+2} + 2(A(k+1) + B + C \cdot 2^{k+1}) + Ak + B + C2^k = k + 2^k$$

$$Ak + 2A + B + 4C2^k + 2Ak + 2A + 2B + 4C2^k + Ak + B + C2^k = k + 2^k$$

$$4Ak + 4A + 4B + 9C2^k = k + 2^k$$

$4A = 1$, $A = \frac{1}{4}$ $\quad$ $4A + 4B = 0$, $A = -B$, $B = -\frac{1}{4}$ $\quad$ $9C = 1$, $C = \frac{1}{9}$

$$y_p(k) = \frac{k}{4} - \frac{1}{4} + \frac{1}{9}2^k$$

$$y_k = (C_1 + C_2 k)(-1)^k + \frac{k}{4} - \frac{1}{4} + \frac{1}{9}2^k$$

f) $y_{k+2} - 8y_{k+1} + 16y_k = 3 \cdot 4^k$, $y_0 = 0$, $y_1 = 0$

$$r^2 - 8r + 16 = 0$$

$$(r-4)^2 = 0 \Rightarrow r = 4, 4$$

$$y_c(k) = (C_1 + C_2 k)4^k$$

$$y_p(k) = Ak^2 4^k$$

$$A(k+2)^2 4^{k+2} - 8A(k+1)^2 4^{k+1} + 16Ak^2 4^k = 3 \cdot 4^k$$

$$16A(\cancel{k^2} + \cancel{4k} + 4)4^k - 32A(\cancel{k^2} + \cancel{2k} + 1)4^k + 16A\cancel{k^2}4^k = 3 \cdot 4^k$$

$$(64A - 32A)4^k = 3 \cdot 4^k.$$

$$32A = 3 \Rightarrow A = 3/32$$

$$y_p(k) = \frac{3}{32}k^2 4^k$$

$$y_k = (C_1 + C_2 k)4^k + \frac{3}{32}k^2 4^k$$

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$0 = y_0 = c_1$

$0 = y_1 = 4C_2 + \frac{3}{32_8} \cdot 4$

$4C_2 = -\frac{3}{8} \Rightarrow C_2 = -\frac{3}{32}$

$y_k = \cancel{c}\ \frac{-3}{32} k\, 4^k + \frac{3}{32} k^2\, 4^k$

$= \frac{3}{32} k\, 4^k (k-1)$

g) $2y_{k+2} - 3y_{k+1} + y_k = k^2 - 4k + 3$

$2r^2 - 3r + 1 = 0$

$2r^2 - 2r - r + 1 = 0$

$2r(r-1) - (r-1) = 0 \Rightarrow r = 1, 1/2$.

$y_c(k) = c_1 + c_2\left(\frac{1}{2}\right)^k$

$y_p(k) = Ak^3 + Bk^2 + Ck$

$2\left(A(k+2)^3 + B(k+2)^2 + C(k+2)\right) - 3\left(A(k+1)^3 + B(k+1)^2 + C(k+$

$+ Ak^3 + Bk^2 + Ck = k^2 - 4k + 3$.

$2A(\cancel{k^3} + 6k^2 + 12k + 8) + 2B(\cancel{k^2} + 4k + 4) + 2C(k+2)$

$-3A(\cancel{k^3} + 3k^2 + 3k + 1) - 3B(\cancel{k^2} + 2k + 1) - 3C(k+1)$

$+ A\cancel{k^3} + B\cancel{k^2} + Ck = k^2 - 4k + 3$

$3A = 1$

$A = 1/3$

$15A + 2B - C = -4$
$+C$

$15A + 2B = -4$

$5 + 4 = -2B$

$B = -9/2$.

$13A + 5B + C = 3$

$\frac{13}{3} - \frac{45}{2} - 3 = -C$

$\frac{26 - 135 - 18}{6} = -C$

$+\frac{127}{6} = C$

$y_p(k) = \frac{k^3}{3} - \frac{9k^2}{2} + \frac{127}{6}k$

$y_k = c_1 + c_2\left(\frac{1}{2}\right)^k + \frac{k^3}{3} - \frac{9k^2}{2} + \frac{127}{6}k$

h) $y_{k+2} + 4y_k = 5(-3)^k + 10k$

$r^2 + 4 = 0 \Rightarrow r^2 = \pm 2i$

$y_c(k) = 2^k \left(C_1 \cos \frac{k\pi}{2} + C_2 \sin \frac{k\pi}{2}\right)$

$\rho = \sqrt{0^2 + 2^2} = 2$

$\theta = \tan^{-1}\left(\frac{2}{0}\right) = \pi/2.$

$y_p(k) = A(-3)^k + Bk + C$

$A(-3)^{k+2} + B(k+2) + C + 4A(-3)^k + 4Bk + 4C = 5(-3)^k + 10k$

$9A(-3)^k + 4A(-3)^k = 5(-3)^k$

$13A = 5$

$A = 5/13$

$Bk + 4Bk = 10k$

$5B = 10$

$B = 2$

$2B + C + 4C = 0$

$2B = -5C$

$-\frac{4}{5} = C$

$y_p = \frac{5}{13}(-3)^k + 2k - \frac{4}{5}$

$y_k = 2^k\left(C_1 \cos \frac{k\pi}{2} + C_2 \sin \frac{k\pi}{2}\right) + \frac{5}{13}(-3)^k + 2k - \frac{4}{5}$

i) $y_{k+2} - y_k = \frac{1}{3^k}$

$(E^2 - 1)y_k = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = 1, -1$

$y_c(k) = C_1 + C_2(-1)^k$

$y_p(k) = A\left(\frac{1}{3}\right)^k$

$A\left(\frac{1}{3}\right)^{k+2} - A\left(\frac{1}{3}\right)^k = \frac{1}{3^k}$

$\frac{A}{9}\left(\frac{1}{3}\right)^k - A\left(\frac{1}{3}\right)^k = \frac{1}{3^k}$

$-\frac{8A}{9} = 1 \Rightarrow A = -\frac{9}{8}$

$y_p(k) = -\frac{9}{8}\left(\frac{1}{3}\right)^k$

$y_k = C_1 + C_2(-1)^k - \frac{9}{8}\,\frac{1}{3^k}$

5.88 Solve $y_{k+3} - 6y_{k+2} + 11y_{k+1} - 6y_k = 4k + 3.2^k - 5^k$

$r^3 - 6r^2 + 11r - 6 = 0 \Rightarrow r = 1, 2, 3$

$y_c(k) = c_1 + c_2 2^k + c_3 3^k$

$y_p(k) = Ak^2 + Bk + Ck2^k + D5^k$

$A(k+3)^2 + B(\cancel{k}+3) + C(k+3)2^{k+3} + D5^{k+3}$
$-6[A(k+2)^2 + B(\cancel{k}+2) + C(k+2)2^{k+2} + D5^{k+2}]$
$+11[A(k+1)^2 + B(\cancel{k}+1) + C(k+1)2^{k+1} + D5^{k+1}]$
$-6[Ak^2 + B\cancel{k} + Ck2^k + D5^k] = 4k + 3.2^k - 5^k$

$\Rightarrow A(\cancel{k^2}+6k+9) + 3B + 8C(k+3)2^k + 125D5^k - 6A(\cancel{k^2}+4k+4)$
$-12B - 24C(k+2)2^k - 150D5^k + 11A(\cancel{k^2}+2k+1) + 11B$
$+22C(k+1)2^k + 55D5^k - 6A\cancel{k^2} - 6Ck2^k - 6D5^k$
$= 4k + 3.2^k - 5^k$

$\Rightarrow (6A - 24A + 22A)k + \underbrace{(\cancel{8C} - 24C + 22C)}_{-6C}k2^k$
$+9A + 3B - 24A - 12B + 11B + 11A + (24C - 48C + 22C)2^k$
$+24D5^k = 4k + 3.2^k - 5^k$

$4A = 4$
$A = 1$

$-4A + 2B = 0$
$4A = 2B$
$2A = B$
$B = 2$

$-2C = 3$
$C = -\frac{3}{2}$

$24D = -1$
$D = -\frac{1}{24}$

$y_p(k) = k^2 + 2k - \frac{3}{2}k2^k - \frac{1}{24}5^k$

Q 5.89 Solve $y_{k+4} - 16y_k = k^2 - 5k + 2 - 4.3^k$

$r^4 - 16 = 0 \Rightarrow (r^2+4)(r^2-4) = 0$
$\Rightarrow (r^2+4)(r+2)(r-2) = 0$
$r = 2, -2, 2i, -2i$

$y_c(k) = c_1 2^k + c_2(-2)^k + 2^k\left(c_3 \cos\frac{\pi}{2}k + c_4 \sin\frac{\pi k}{2}\right)$

$y_p(k) = Ak^2 + Bk + C + D3^k$

$A(k+4)^2 + B(k+4) + C + D3^{k+4} - 16Ak^2 - 16Bk - 16C$
$- 16D3^k = k^2 - 5k + 2 - 4.3^k$

$\Rightarrow A(k^2 + 8k + 16) + Bk + 4B + C + 81D3^k - 16Ak^2 - 16Bk - 16C$
$- 16D3^k$
$= k^2 - 5k + 2 - 4.3^k$

$-15A = 1$
$A = -\frac{1}{15}$

$8A + B - 16B = -5$
$8A - 15B = -5$
$5 - \frac{8}{15} = 15B$
$\frac{67}{225} = B$

$16A + 4B + C - 16C = 2$
$-\frac{16}{15} + \frac{268}{225} - 2 = 15C$
$\frac{-240 + 268 - 450}{225 \times 15} = C$
$C = \frac{-422}{3375}$

$65D = -4$
$D = -4/65$

$$y_p = -\frac{k^2}{15} + \frac{67}{225}k - \frac{422}{3375} - \frac{4}{65} \cdot 3^k$$

Q 5.90 Solve $y_{k+3} + y_k = 2^k \cos 3k$

$r^3 + 1 = 0 \Rightarrow (r+1)(r^2 - r + 1) = 0$
$r = -1, \frac{1 \pm \sqrt{1-4}}{2}$
$= -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$y_c(k) = c_1(-1)^k + c_1 \cos \frac{\pi}{3} k + c_2 \sin \frac{\pi k}{3}$

$\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$
$= \sqrt{\frac{1+3}{4}} = 1$
$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$
$= \pi/3 = 60°$

$y_p = 2^k(A \cos 3k + B \sin 3k)$

$8.2^k(A \cos 3(k+3) + B \sin 3(k+3))$
$+ 2^k(A \cos 3k + B \sin 3k) = 2^k \cos 3k$

$8.2^k(A \cos 3k \cos 9 - A \sin 3k \sin 9 + B \sin 3k \cos 9$
$+ B \cos 3k \sin 9) + 2^k(A \cos 3k + B \sin 3k)$
$= 2^k \cos 3k$

$8A \cos 9 + 8B \sin 9 + A = 1$
$(8 \cos 9 + 1)A + (8 \sin 9)B = 1$

$-8A \sin 9 + 8B \cos 9 + B = 0$
$(-8 \sin 9)A + (8 \cos 9 + 1)B = 0$

$$\begin{vmatrix} 8\cos 9 + 1 & 8 \sin 9 \\ -8 \sin 9 & 8 \cos 9 + 1 \end{vmatrix} = 64\cos^2 9 + 1 + 16 \cos 9 + 64 \sin^2 9$$
$$= 65 + 16 \cos 9$$

$$\begin{vmatrix} 1 & 8 \sin 9 \\ 0 & 8 \cos 9 + 1 \end{vmatrix} = 8 \cos 9 + 1 \qquad \begin{vmatrix} 8\cos 9 + 1 & 1 \\ -8 \sin 9 & 0 \end{vmatrix} = 8 \sin 9$$

$A = \frac{8\cos 9 + 1}{65 + 16\cos 9}$, $B = \frac{8\sin 9}{65+16\cos 9}$

$y_p(k) = 2^k \left(\frac{8\cos 9 + 1}{65+16\cos 9} \cos 3k + \frac{8\sin 9}{65+16\cos 9} \sin 3k \right)$

G.S $y_k = y_c(k) + y_p(k)$

Que 5.91 Solve $3y_{k+4} - 4y_{k+2} + y_k = 3^k(k-1)$

$3r^4 - 4r^2 + 1 = 0$

$3r^4 - 3r^2 - r^2 + 1 = 0$

$3r^2(r^2-1) - (r^2-1) = 0$

$(3r^2-1)(r^2-1) = 0$

$r = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 1, -1$

$y_c(k) = C_1 + C_2(-1)^k + C_3\left(\frac{1}{\sqrt{3}}\right)^k + C_4\left(\frac{-1}{\sqrt{3}}\right)^k$

$y_p(k) = 3^k(Ak + B)$

$3.3^{k+4}(A(k+4)+B) - 4.3^{k+2}(A(k+2)+B) + 3^k(Ak+B) = 3^k(k-1)$

$243.3^k(Ak+4A+B) - 36.3^k(Ak+2A+B) + 3^k(Ak+B) = 3^k(k-1)$.

$243A - 36A + A = 1$

$208A = 1$

$A = \frac{1}{208}$

$243(4A+B) - 36(2A+B) + B = -1$

$243\left(\frac{1}{52} + B\right) - 36\left(\frac{1}{104} + B\right) + B = -1$

$\frac{486 - 36}{104} + 208B = -1$

$208B = -1 - \frac{450}{104}$

$B = \frac{-554}{208} = -\frac{277}{104}$

$y_p(k) = 3^k\left(\frac{k}{208} - \frac{277}{104}\right)$

$y_k = C_1 + C_2(-1)^k + C_3\left(\frac{1}{\sqrt{3}}\right)^k + C_4\left(\frac{-1}{\sqrt{3}}\right)^k + 3^k\left(\frac{k}{208} - \frac{277}{104}\right)$

5.92 Solve by operator method

a) $y_{k+2} - 3y_{k+1} + 2y_k = 4\cdot 3^k - 2\cdot 5^k$

$r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$

$y_c(k) = C_1 + C_2\, 2^k$

$y_p(k) = \frac{1}{E^2 - 3E + 2}(4\cdot 3^k - 2\cdot 5^k)$

$= 4\cdot \frac{1}{3^2 - 3\cdot 3 + 2} 3^k - 2\frac{1}{5^2 - 3\cdot 5 + 2} 5^k$

$= \frac{4}{2} 3^k - \frac{2}{12} 5^k = 2\cdot 3^k - \frac{1}{6}\cdot 5^k$

$y_k = C_1 + C_2\cdot 2^k + 2\cdot 3^k - \frac{1}{6}\cdot 5^k$

b) $y_{k+2} + 4y_{k+1} + 4y_k = k^2 - 3k + 5$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2, -2$

$y_c(k) = (C_1 + C_2 k)(-2)^k$

$y_p(k) = \frac{1}{(E+2)^2}\cdot k^2 - 3k + 5$

$k(k-1) = k^{(2)}$

$= \frac{1}{(\Delta+3)^2}\cdot k^{(2)} - 2k^{(1)} + 5$

$= \frac{1}{9\left(1+\frac{\Delta}{3}\right)^2} k^{(2)} - 2k^{(1)} + 5$

$= \frac{1}{9}\left(1 - \frac{2}{3}\Delta + \frac{3}{9}\Delta^2\right)(k^{(2)} - 2k^{(1)} + 5)$

$= \frac{1}{9}\left(k^{(2)} - 2k^{(1)} + 5 - \frac{2}{3}(2k^{(1)} - 2) + \frac{3}{9}(2)\right)$

$= \frac{1}{9}\left[k^2 - k - 2k + 5 - \frac{4}{3}k + \frac{4}{3} + \frac{2}{3}\right]$

$= \frac{1}{9}\left[k^2 - \frac{13}{3}k + 7\right]$

$-3 - \frac{4}{3}$

$y_k = (C_1 + C_2 k)(-2)^k + \frac{k^2}{9} - \frac{13}{27}k + \frac{7}{9}$

c) $3y_{k+2} - 8y_{k+1} - 3y_k = 3^k - 2k + 1$

$3r^2 - 8r - 3 = 0$

$3r^2 - 9r + r - 3 = 0$

$3r(r-3) + r - 3 = 0 \Rightarrow (3r+1)(r-3) = 0$

$\Rightarrow r = -\frac{1}{3}, 3$

$y_c(k) = C_1\, 3^k + C_2\left(-\frac{1}{3}\right)^k$

$$y_p(k) = \frac{1}{3E^2-8E-3}\, 3^k + \frac{1}{3E^2-8E-3}\,(1-2k)$$

$$= 3^k \frac{1}{3(3E)^2-8(3E)-3}\, 1 + \frac{1}{3(1+\Delta)^2-8(1+\Delta)-3}\,(1-2k)$$

$$= 3^k \frac{1}{27E^2-24E-3}\, 1 + \frac{1}{3(\Delta^2+1+2\Delta)-8-8\Delta-3}\,(1-2k)$$

$$= 3^k \frac{1}{27(1+\Delta^2+2\Delta)-24-24\Delta-3}\, 1 + \frac{1}{3\Delta^2-2\Delta-8}\,(1-2k)$$

$$= 3^k \frac{1}{27\Delta^2+30\Delta}\, 1 - \frac{1}{8}\left(1-\left(\frac{3}{8}\Delta^2 - \frac{\Delta}{4}\right)\right)^{-1}\left(1-2k^{(1)}\right)$$

$$= \frac{3^k}{3} \frac{1}{\Delta(9\Delta+10)}\, 1 - \frac{1}{8}\left(1+\frac{3}{8}\Delta^2 - \frac{\Delta}{4}\right)\left(1-2k^{(1)}\right)$$

$$= \frac{3^k}{3\times 10} \frac{1}{\Delta\left(1+\frac{9}{10}\Delta\right)}\, 1 - \frac{1}{8}\left(1-2k+\frac{1}{2}\right)$$

$$= \frac{3^k}{30} \frac{1}{\Delta}\left(1-\frac{9}{10}\Delta\right) 1 - \frac{1}{8}\left(\frac{3}{2}-2k\right)$$

$$= \frac{3^k}{30} \frac{1}{\Delta}\, 1 - \frac{3}{16} + \frac{k}{4}$$

$$= \frac{k\,3^k}{30} + \frac{k}{4} - \frac{3}{16}.$$

$$\frac{1}{10}\,\frac{1}{E-3}\,3^k$$

$$\frac{1}{10}\,3^k\,\frac{1}{3E-3}\,1$$

$$\frac{3^k}{3\times 10}\,\frac{1}{E-1}\,1$$

$$\frac{3^k}{30}\,k$$

Variation of parameter

5.100

2) $y_{k+2} - 5y_{k+1} + 6y_k = k^2$ — (I)

$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2, 3$

$y_c(k) = c_1 2^k + c_2 3^k$

Now

$y_k = y_p(k) = K_1 2^k + K_2 3^k$ — (II)

where K_1 and K_2 are functions of k.

$\Delta y_k = \Delta(K_1 2^k) + \Delta(K_2 3^k)$

$= K_1 \Delta 2^k + (\Delta K_1) 2^k + (\Delta K_1)(\Delta 2^k)$

$+ K_2 \Delta 3^k + (\Delta K_2) 3^k + (\Delta K_2)(\Delta 3^k)$

$= K_1 (2^{k+1} - 2^k) + 2^k \Delta K_1 + (2^{k+1} - 2^k)(\Delta K_1)$

$+ K_2 (3^{k+1} - 3^k) + 3^k \Delta K_2 + (3^{k+1} - 3^k)(\Delta K_2)$

$= 2^k K_1 + 2K_2 3^k + 2^k \Delta K_1 + 3^k \Delta K_2 + 2^k \Delta K_1 + 2 \cdot 3^k \Delta K_2$

$= 2^k K_1 + 2 \cdot 3^k K_2 + \underline{2^{k+1} \Delta K_1 + 3^{k+1} \Delta K_2}$.

Now we need to find values of K_1 & K_2 for which we need two condition, of which one is that (II) satisfies (I)

Now we assume the second cond. to be

$2^{k+1} \Delta K_1 + 3^{k+1} \Delta K_2 = 0$ — (III)

So $\Delta y_k = 2^k K_1 + 2 \cdot 3^k K_2$.

Now $\Delta^2 y_k = K_1 2^k + 2^k \Delta K_1 + 2^k \Delta K_1$

$+ 2[2K_2 3^k + 3^k \Delta K_2 + 2 \cdot 3^k \Delta K_2]$

$= K_1 2^k + 4K_2 3^k + 2^{k+1} \Delta K_1 + 2 \cdot 3^{k+1} \Delta K_2$

(I) $\Rightarrow (E^2 - 5E + 6) y_k = k^2$

$E = 1 + \Delta$

$(1 + \Delta^2 + 2\Delta - 5 - 5\Delta + 6) y_k = k^2$

$\Rightarrow (\Delta^2 - 3\Delta + 2) y_k = k^2$

$\Rightarrow \cancel{K_1 2^k} + \cancel{4K_2 3^k} + 2^{k+1} \Delta K_1 + 2 \cdot 3^{k+1} \Delta K_2$

$- 3[\cancel{2^k K_1} + \cancel{2 \cdot 3^k K_2}] + 2[\cancel{K_1 2^k} + \cancel{K_2 3^k}] = k^2$

$2^{k+1} \Delta K_1 + 2 \cdot 3^{k+1} \Delta K_2 = k^2$. — (IV)

So $2^{k+1}\Delta K_1 + 3^{k+1}\Delta K_2 = 0$ — III

$2^{k+1}\Delta K_1 + 2.3^{k+1}\Delta K_2 = k^2$ — IV

$$\Delta K_1 = \frac{\begin{vmatrix} 0 & 3^{k+1} \\ k^2 & 2.3^{k+1} \end{vmatrix}}{\begin{vmatrix} 2^{k+1} & 3^{k+1} \\ 2^{k+1} & 2.3^{k+1} \end{vmatrix}} \qquad \Delta K_2 = \frac{\begin{vmatrix} 2^{k+1} & 0 \\ 2^{k+1} & k^2 \end{vmatrix}}{\begin{vmatrix} 2^{k+1} & 3^{k+1} \\ 2^{k+1} & 2.3^{k+1} \end{vmatrix}}$$

$$= \frac{3^{k+1}\begin{vmatrix} 0 & 1 \\ k^2 & 2 \end{vmatrix}}{2^{k+1}3^{k+1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} \qquad = \frac{2^{k+1}\begin{vmatrix} 1 & 0 \\ 1 & k^2 \end{vmatrix}}{2^{k+1}3^{k+1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$= \frac{-k^2}{2^{k+1}} \qquad = \frac{k^2}{3^{k+1}}$$

OR IV − III $\Rightarrow 3^{k+1}\Delta K_2 = k^2$

$$\Delta K_2 = \frac{k^2}{3^{k+1}}$$

$2^{k+1}\Delta K_1 + 3^{k+1}\Delta K_2 = 0$

$2^{k+1}\Delta K_1 + k^2 = 0$

$$\Delta K_1 = -\frac{k^2}{2^{k+1}}$$

Now finding K_1

$$K_1 = \frac{1}{\Delta}\left(-\frac{k^2}{2^{k+1}}\right) = -\frac{1}{\Delta}\left(\left(\frac{1}{2}\right)^k \frac{k^2}{2}\right)$$

$$= -\frac{1}{E-1}\left(\left(\frac{1}{2}\right)^k \frac{k^2}{2}\right)$$

$$= -\left(\frac{1}{2}\right)^k \frac{1}{\frac{E}{2}-1}\left(\frac{k^2}{2}\right)$$

$$= -\left(\frac{1}{2}\right)^k \frac{1}{E-2}(k^2)$$

$$= -\left(\frac{1}{2}\right)^k \frac{1}{\Delta+1-2} k^2$$

$$= -\left(\frac{1}{2}\right)^k \frac{1}{\Delta-1}\left(k^{(2)} + k^{(1)}\right)$$

$$= \left(\frac{1}{2}\right)^k (1-\Delta)^{-1}\left(k^{(2)} + k^{(1)}\right)$$

$$= \left(\frac{1}{2}\right)^k (1+\Delta+\Delta^2)\left(k^{(2)} + k^{(1)}\right)$$

$$= \left(\frac{1}{2}\right)^k \left(k^{(2)} + k^{(1)} + 2k^{(1)} + 1 + 2\right)$$

$$= \left(\frac{1}{2}\right)^k \left(k(k-1) + 3k + 3\right)$$

$$= \left(\frac{1}{2}\right)^k \left(k^2 + 2k + 3\right)$$

$$\Delta K_2 = \frac{k^2}{3^{k+1}}.$$

$$K_2 = \frac{1}{\Delta}\left(\frac{k^2}{3^{k+1}}\right) = \frac{1}{\Delta}\left(\frac{1}{3}\right)^k\left(\frac{k^2}{3}\right)$$

$$= \left(\frac{1}{3}\right)^k \frac{1}{\frac{E}{3}-1}\left(\frac{k^2}{3}\right)$$

$$= \frac{1}{3^k}\,\frac{1}{E-3}\left(k^2\right)$$

$$= \frac{1}{3^k}\,\frac{1}{\Delta+1-3}\,k^2$$

$$= \frac{1}{3^k}\,\frac{1}{\Delta-2}\,k^2$$

$$= \frac{-1}{2\cdot 3^k}\left(1 - \frac{\Delta}{2}\right)^{-1}\left(k^{(2)} + k^{(1)}\right)$$

$$= \frac{-1}{2\cdot 3^k}\left(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4}\right)\left(k^{(2)} + k^{(1)}\right)$$

$$= \frac{-1}{2\cdot 3^k}\left(k^{(2)} + k^{(1)} + \frac{2k^{(1)}+1}{2} + \frac{2}{4}\right)$$

$$= \frac{-1}{2\cdot 3^k}\left(k^{(2)} + 2k^{(1)} + 1\right)$$

$$= \frac{-1}{2\cdot 3^k}\left(k^2 - k + 2k + 1\right)$$

$$= \frac{-1}{2\cdot 3^k}\left(k^2 + k + 1\right)$$

Now $y_p(k) = K_1 u_1 + K_2 u_2$

$$= \frac{1}{2^k}\left(k^2 + 2k + 3\right)2^k - \frac{1}{2\cdot 3^k}\left(k^2 + k + 1\right)3^k$$

$$= k^2 + 2k + 3 - \frac{1}{2}\left(k^2 + k + 1\right)$$

$$= \frac{1}{2}k^2 + \frac{3}{2}k + \frac{5}{2}.$$

* solve $y_{k+2} - 3y_{k+1} + 2y_k = 4^k$ by variation of parameters

$r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$

$y_c(k) = C_1 + C_2 2^k$

$y_p(k) = K_1 + K_2 2^k$

$K_1 = -\frac{1}{\Delta} \frac{R(k)}{\alpha^{k+1}}$

$= -\frac{1}{\Delta} \frac{4^k}{1^{k+1}}$

$= -\frac{1}{E-1} 4^k$

$= -\frac{4^k}{3}$

$K_2 = \frac{1}{\Delta} \frac{R(k)}{\beta^{k+1}}$

$= \frac{1}{\Delta} \frac{4^k}{2^{k+1}}$

$= \frac{1}{2} \frac{1}{\Delta} 2^k$

$= \frac{1}{2} \frac{1}{E-1} 2^k$

$= \frac{1}{2} 2^k$

$y_p(k) = -\frac{4^k}{3} + \frac{1}{2} 2^k 2^k$

$= -\frac{4^k}{3} + \frac{1}{2} 4^k = \left(\frac{1}{2} - \frac{1}{3}\right) 4^k = \frac{4^k}{6}$

$y_k = y_c(k) + y_p(k)$

$= C_1 + C_2 2^k + \frac{1}{6} 4^k$

* $y_{k+1} - y_k = k \quad , y_0 = 1$

$r - 1 = 0 \Rightarrow r = 1$

$y_c(k) = C_1$

$y_p(k) = K_1$

$K_1 = -\frac{1}{\Delta} k = -\frac{1}{2} k^{(2)} = -\frac{1}{2}(k^2 - k)$

$y_p(k) = -\frac{k^2}{2} + \frac{k}{2}$

$y_k = C_1 - \frac{k^2}{2} + \frac{k}{2}$

$1 = y_0 = C_1$

$y_k = 1 - \frac{k^2}{2} + \frac{k}{2}$

Q Solve $y_{k+2} - 5y_{k+1} + 6y_k = k^2$ by reduction of order

$$(E^2 - 5E + 6)\, y_k = k^2 \Rightarrow (E-3)(E-2)\, y_k = k^2$$

Let $(E-2)\, y_k = z_k$

then $(E-3)\, z_k = k^2$

$$r - 3 = 0 \Rightarrow r = 3$$

$$z_c(k) = c_1 3^k$$

$$z_p(k) = \frac{1}{E-3} k^2$$

$$= \frac{1}{\Delta - 2} k^{(2)} + k^{(1)}$$

$$= -\frac{1}{2}\left(1 - \frac{\Delta}{2}\right)^{-1} \left(k^{(2)} + k^{(1)}\right)$$

$$= -\frac{1}{2}\left(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4}\right)\left(k^{(2)} + k^{(1)}\right)$$

$$= -\frac{1}{2}\left(k^2 + k^{(1)} + \frac{1}{2} + \frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(k^2 + k + 1\right)$$

$$z_k = c_1 3^k - \frac{1}{2}\left(k^2 + k + 1\right)$$

Now $(E-2)\, y_k = c_1 3^k - \frac{1}{2}(k^2 + k + 1)$

$$r - 2 = 0 \Rightarrow r = 2$$

$$y_c(k) = c_2 2^k$$

$$y_p(k) = \frac{1}{E-2}\left(c_1 3^k\right) - \frac{1}{2}\frac{1}{E-2} k^2 + k + 1$$

$$= c_1 3^k - \frac{1}{2}\frac{1}{\Delta - 1} k^{(2)} + 2k^{(1)} + 1$$

$$= c_1 3^k + \frac{1}{2}\left(1 + \Delta + \frac{\Delta^2}{2}\right)\left(k^{(2)} + 2k^{(1)} + 1\right)$$

$$= c_1 3^k + \frac{1}{2}\left(k^2 + k + 1 + 2k + 2 + 2\right)$$

$$= c_1 3^k + \frac{1}{2}\left(k^2 + 3k + 5\right)$$

$$y_k = c_2 2^k + c_1 3^k + \frac{1}{2}\left(k^2 + 3k + 5\right)$$

If an n^{th} order equation can be written as

$$(E-r_1)(E-r_2)(E-r_3)\ldots(E-r_n)\,y_k = R^k$$

then on letting $z_k = (E-r_2)\ldots(E-r_n)\,y_k$

we are led to first order equation

$$(E-r_1)\,z_k = R_k$$

which has solution

$$z_k = r_1^k\,\Delta^{-1}\left(\frac{R_k}{r_1^{k+1}}\right) = r_1^k \sum_{p=1}^{k-1} \frac{R_p}{r_1^{p+1}} + c_1 r_1^k$$

$$\Delta^{-1}\left(\frac{R_k}{r_1^{k+1}}\right) = \left(\sum_{p=1}^{k-1} \frac{R_p}{r_1^{p+1}} + c_1\right)$$

$$\Delta\left(\sum_{p=1}^{k-1} \frac{R_p}{r_1^{p+1}} + c_1\right) = \sum_{p=1}^{k} \frac{R_p}{r_1^{p+1}} + c_1 - \sum_{p=1}^{k-1} \frac{R_p}{r_1^{p+1}} - c_1$$

$$= \frac{R_k}{r_1^{k+1}}$$

$$\Rightarrow \Delta^{-1}\left(\frac{R_k}{r_1^{k+1}}\right) = \sum_{p=1}^{k-1} \frac{R_p}{r_1^{p+1}} + c_1$$

* solve $y_{k+2} - 4y_{k+1} + 4y_k = 3\cdot 2^k \cancel{+5\cdot 4^k}$

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2, 2$

$y(k) = (c_1 + c_2 k)\,2^k$

$(E-2)^2 y_k = 3\cdot 2^k$

$(E-2)\,y_k = z_k$

$(E-2)\,z_k = 3\cdot 2^k$

$r - 2 = 0, \quad r = 2$

$z_k = c_1 2^k + \frac{1}{E-2}\,3\cdot 2^k$

$= c_1 2^k + 2^k \frac{1}{2E-2}\,3$

$= c_1 2^k + \frac{2^k}{2}\,\frac{1}{\Delta}\,3 = c_1 2^k + 2^k\left(\frac{3k}{2}\right)$

$z_k = 2^k \frac{1}{\Delta}\left(\frac{3\cdot 2^k}{2^{k+1}}\right) + c_1 2^k$

$= 2^k \frac{1}{\Delta}\,\frac{3}{2} + c_1 2^k$

$= 2^k\,\frac{3k}{2} + c_1 2^k$

$$(E-2)y_k = c_1 2^k + 2^k\left(\frac{3k}{2}\right)$$

$$y_k = c_2 2^k + \frac{1}{E-2} c_1 2^k + \frac{1}{E-2}\left(\frac{3}{2} k 2^k\right)$$

$$= c_2 2^k + \frac{2^k}{2} \frac{1}{\Delta} c_1 + \frac{3}{2} \frac{2^k}{2} \frac{1}{\Delta} k$$

$$= c_2 2^k + c_1 \frac{k}{2} 2^k + \frac{3}{4} 2^k \frac{k(k-1)}{2}$$

$$= c_2 2^k + \frac{c_1}{2} k 2^k + \frac{3}{8}(k^2 - k) 2^k$$

$$= c_2 2^k + \left(\frac{c_1}{2} - \frac{3}{8}\right) k 2^k + \frac{3}{8} k^2 2^k$$

$$= c_2 2^k + c_3 k 2^k + \frac{3}{8} k^2 2^k$$

$$c_2 2^k + 2^k \frac{1}{\Delta} \frac{c_1 2^k + 2^k \frac{3k}{2}}{2^{k+1}}$$

$$= c_2 2^k + \frac{2^k}{2} \frac{1}{\Delta}\left(c_1 + \frac{3k}{2}\right)$$

$$= c_2 2^k + \frac{2^k}{2}\left[c_1 k + \frac{3}{4} k(k-1)\right]$$

P

2. $y_{k+2} - 4y_{k+1} + 4y_k = 3\cdot 2^k + 5\cdot 4^k$

$$(E-2)^2 y_k = 3\cdot 2^k + 5\cdot 4^k$$

$$(E-2) y_k = z_k$$

$$(E-2) z_k = 3\cdot 2^k + 5\cdot 4^k$$

$$z_k = c_1 2^k + 2^k \frac{1}{\Delta} \frac{3\cdot 2^k + 5\cdot 4^k}{2^{k+1}}$$

P

$$= c_1 2^k + 2^k \frac{1}{\Delta}\left(\frac{3}{2} + \frac{5}{2} 2^k\right)$$

$$= c_1 2^k + 2^k\left(\frac{3}{2}k + \frac{5}{2} 2^k\right)$$

$$(E-2)y_k = c_1 2^k + \frac{3}{2} k 2^k + \frac{5}{2} 4^k$$

$$y_k = c_2 2^k + 2^k \frac{1}{\Delta}\left(\frac{c_1 2^k + \frac{3}{2} k 2^k + \frac{5}{2} 4^k}{2^{k+1}}\right)$$

$$= c_2 2^k + \frac{2^k}{2} \frac{1}{\Delta}\left(c_1 + \frac{3}{2} k + \frac{5}{2} 2^k\right)$$

$$= c_2 2^k + \frac{2^k}{2}\left(c_1 k + \frac{3}{4} k(k-1) + \frac{5}{2} 2^k\right)$$

$$= c_2 2^k + \frac{c_1}{2} k 2^k + \frac{3}{8} k(k-1) 2^k + \frac{5}{4} 4^k$$

$$= A 2^k + B k 2^k + \frac{3}{8} k^2 2^k + \frac{5}{4} 4^k$$

Q $2y_{k+1} - y_k = 1$, $y_0 = 0$

$(2E - 1) y_k = 1$

$(E - \frac{1}{2}) y_k = \frac{1}{2}$

$y_k = c_1 \frac{1}{2^k} + (\frac{1}{2})^k \frac{1}{\Delta} \frac{1/2}{(1/2)^{k+1}}$

$= c_1 \frac{1}{2^k} + \frac{1}{2^{k+1}} \frac{1}{\Delta} 2^{k+1}$

$= c_1 \frac{1}{2^k} + \frac{1}{2^k} \frac{1}{\Delta} 2^k$

$= c_1 \frac{1}{2^k} + \frac{1}{2^k} \cdot 2^k$

$= 1 + c_1 \frac{1}{2^k}$

$0 = y_0 = 1 + c_1$

$c_1 = -1$

$y_k = 1 - \frac{1}{2^k}$

Q $y_{k+1} - 2y_k = 3.2^k$

$(E - 2) y_k = 3.2^k$

$y_k = c_1 2^k + 2^k \frac{1}{\Delta} \left(\frac{3.2^k}{2^{k+1}} \right)$

$= c_1 2^k + 2^k \frac{1}{\Delta} \left(\frac{3}{2} \right)$

$= c_1 2^k + 2^k \frac{3k}{2}$

$= \left(c_1 + \frac{3k}{2} \right) 2^k$

$\frac{1}{E-2} 3.2^k$

$\frac{2^k}{2} \frac{1}{E-1} 3$

$\frac{2^k}{2} . 3k$

Method of generating functions

Q. Solve $y_{k+2} - 3y_{k+1} + 2y_k = 0$

if $y_0 = 2,\ y_1 = 3$

Let $G(t) = \sum_{k=0}^{\infty} y_k t^k$

Multiplying the diff. eq by t^k and summing from $k=0$ to ∞, we have

$$\sum_{k=0}^{\infty} y_{k+2} t^k - 3\sum_{k=0}^{\infty} y_{k+1} t^k + 2\sum_{k=0}^{\infty} y_k t^k = 0$$

$$\Rightarrow (y_2 + y_3 t + y_4 t^2 + \ldots) - 3(y_1 + y_2 t + y_3 t^2 + \ldots)$$
$$+ 2(y_0 + y_1 t + y_2 t^2 + \ldots) = 0$$

Now $y_2 + y_3 t + y_4 t^2 + \ldots$ $= \dfrac{y_2 t^2 + y_3 t^3 + y_4 t^4}{t^2}$

$$= \frac{y_0 + y_1 t + y_2 t^2 + \ldots - y_0 - y_1 t}{t^2}$$

$$= \frac{G(t) - y_0 - y_1 t}{t^2}$$

$y_1 + y_2 t + y_3 t^2 + \ldots$ $= \dfrac{y_1 t + y_2 t^2 + y_3 t^3 + \ldots}{t}$

$$= \frac{y_0 + y_1 t + y_2 t^2 + \ldots - y_0}{t}$$

$$= \frac{G(t) - y_0}{t}$$

$$\Rightarrow \frac{G(t) - y_0 - y_1 t}{t^2} - 3\left(\frac{G(t) - y_0}{t}\right) + 2G(t) = 0$$

$$\Rightarrow \frac{G(t) - 2 - 3t}{t^2} - 3\left(\frac{G(t) - 2}{t}\right) + 2G(t) = 0$$

$$\Rightarrow G(t) - 2 - 3t - 3t\,G(t) + 6t + 2t^2 G(t) = 0$$

$$\Rightarrow G(t) = \frac{2 - 3t}{2t^2 - 3t + 1} = \frac{2 - 3t}{2t^2 - 2t - t + 1} = \frac{2-3t}{(1-t)(1-2t)}$$

$$\frac{2-3t}{(1-t)(1-2t)} = \frac{A}{1-t} + \frac{B}{1-2t}$$

$$2-3t = A(1-2t) + B(1-t)$$

Put $t = 1$

$$-1 = A(-1)$$

$$A = 1$$

Put $t = 1/2$

$$2 - \frac{3}{2} = \frac{B}{2}$$

$$B = 1$$

$$\therefore G(t) = \frac{1}{1-t} + \frac{1}{1-2t}$$

$$= \sum_{k=0}^{\infty} t^k + \sum_{k=0}^{\infty} (2t)^k$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k$$

$$\sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} (1 + 2^k) t^k$$

$$\Rightarrow y_k = 1 + 2^k$$

which is required solution.

5.106 a) $y_{k+2} - 6y_{k+1} + 8y_k = 0$, $y_0 = 0$, $y_1 = 2$

$$G(t) = \sum_{k=0}^{\infty} y_k t^k$$

$$\sum_{k=0}^{\infty} y_{k+2} t^k - 6 \sum_{k=0}^{\infty} y_{k+1} t^k + 8 \sum_{k=0}^{\infty} y_k t^k = 0$$

$$\sum_{k=0}^{\infty} y_{k+2} t^k = y_2 t \; y_3 t + y_4 t^2 + \dots$$

$$= \frac{y_2 t^2 + y_3 t^3 + y_4 t^4 + \dots}{t^2}$$

$$= \frac{y_0 + y_1 t + y_2 t^2 + \dots - y_0 - y_1 t}{t^2} = \frac{G(t) - y_0 - y_1 t}{t^2}$$

$$\sum_{k=0}^{\infty} y_{k+1} t^k = y_1 + y_2 t + y_3 t^2 + \dots$$

$$= \frac{y_0 + y_1 t + y_2 t^2 + \dots - y_0}{t}$$

$$= \frac{G(t) - y_0}{t}$$

$$\frac{G(t) - y_0 - y_1 t}{t^2} - 6\left[\frac{G(t) - y_0}{t}\right] + 8G(t) = 0$$

$$\frac{G(t) - 2t}{t^2} - 6\frac{G(t)}{t} + 8G(t) = 0$$

$$G(t) - 2t - 6t\,G(t) + 8t^2 G(t) = 0$$

$$(1-6t+8t^2)\,G(t) = 2t$$

$$G(t) = \frac{2t}{1-6t+8t^2} = \frac{2t}{1-2t-4t+8t^2}$$

$$= \frac{2t}{(1-2t)-4t(1-2t)} = \frac{2t}{(1-2t)(1-4t)}$$

$$\frac{2t}{(1-2t)(1-4t)} = \frac{A}{1-2t} + \frac{B}{1-4t}$$

$$2t = A(1-4t) + B(1-2t)$$

If $t = \frac{1}{4}$

$$\frac{1}{2} = B\left(1-\frac{1}{2}\right) \Rightarrow B = 1$$

If $t = \frac{1}{2}$

$$1 = A(1-2) \Rightarrow A = -1$$

$$\therefore G(t) = \frac{-1}{1-2t} + \frac{1}{1-4t}$$

$$= -\sum_{k=0}^{\infty}(2t)^k + \sum_{k=0}^{\infty}(4t)^k$$

$$\sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty}\left(-2^k+4^k\right)t^k$$

$$\therefore y_k = -2^k + 4^k$$

$\cancel{r = 2, 4}$

$\cancel{y_k = c_1 2^k + c_2 4^k}$

$\cancel{0 = y_0 = c_1 + c_2}$

$\cancel{2 = y_1 = 2c_1 + 4c_2}$

b) $2y_{k+1} - y_k = 1$, $y_0 = 0$

$$G(t) = \sum_{k=0}^{\infty} t^k y_k$$

$$2\sum_{k=0}^{\infty} y_{k+1}\, t^k - \sum_{k=0}^{\infty} t^k y_k = \sum_{k=0}^{\infty} t^k$$

$2r = 1$

$r = 1/2$

$$\sum_{k=0}^{\infty} y_{k+1} t^k = y_1 + y_2 t + y_3 t^2 + \dots$$

$$= \frac{y_0 + y_1 t + y_2 t^2 + \dots - y_0}{t}$$

$$= \frac{G(t) - y_0}{t}$$

$$2\,\frac{G(t)-0}{t} - G(t) = (1-t)^{-1}$$

$$2G(t) - t\,G(t) = \frac{t}{1-t}$$

$$G(t) = \frac{t}{(2-t)(1-t)} = \cancel{\frac{t}{2}\,\frac{1}{1-t/2}} = \cancel{\frac{t}{2}\sum_{k=0}^{\infty}(t/2)^k} = \cancel{\sum_{k=0}^{\infty}\left(\frac{t}{2}\right)^{k+1}}$$

$$\frac{t}{(2-t)(1-t)} = G(t) = \frac{A}{2-t} + \frac{B}{1-t}$$

$$t = A(1-t) + B(2-t)$$

$$t = 1$$
$$1 = B$$
$$t = 2$$
$$2 = -A$$

$$G(t) = \frac{-2}{2-t} + \frac{1}{1-t}$$
$$= \frac{-1}{1-t/2} + \frac{1}{1-t} = -\sum (t/2)^k + \sum t^k$$

$$\sum_{k=0}^{\infty} y_k t^k = \sum_{k=0}^{\infty} \left(-\left(\frac{1}{2}\right)^k + 1\right) t^k$$

$$y_k = -\left(\frac{1}{2}\right)^k + 1$$

$1 + 2t + 3t^2$
$+ t + 2t^2 + 3t^3$
$1 + 2t$

g) $y_{k+2} + 2y_{k+1} + y_k = 1 + k \;,\; y_0 = 0 \,,\; y_1 = 1$

$\ldots t^k + \sum y_k t^k = \sum_{k=0}^{\infty} (1+k)\, t^k$