Concepts: Probabilistic method, probability amplification, Locasz local lemma, conditional probabilities.
Probabiltic Method:

1. Any ran. var. x assume ≥ 1 value $\geq E[x]$ and assume ≥ 1 value $\leq E[x]$.
2. If o chosen at random from U satifies property p with non-zero probability, the there $\exists o \in U$ with property p.

Application: Max Cut
Thrm: For any grah $G(V,E)$, $|V| = n$, $|E| = m$, there exist partition of V to A, B st.

$$|\{(u,v)|u \in A \wedge v \in B\}| \geq m/2$$

Consider assign $v \in V$ at random to A or B with 1/2 prob. $\Rightarrow$ Prob $(u,v) = u \in A \wedge v \in B = 1/2$. By linarity of expectation we expect $m/2$ edges satisfying $p = (u,v) = u \in A \wedge v \in B$. $\Rightarrow \exists p$ satisfying the thrm by Probabilistic Method 2.
Not necessarily an efficient ran. alg., if prob. is miniscule.

Max-Sat:m clauses in cnf over n vars.
Problem: assign vars values st. max num. of cluase are satisfied.
Thrm 1: For any set of m clauses, there is a thruth assign. st var satisfy $\geq m/2$ clauses.
Proof: Ran assign var to 0,1 indp. and equiprob. For $1 \leq i \leq$,, $Z_i = 1$ iff i'th clause satisfied. With k literals, $\mathbf{Pr}[\bar{Z}_i] = 2^{-k}$, since all literals must zero. $\Rightarrow \mathbf{Pr}(Z_i) = 1 - 2^{-k} \geq 1/2 \Rightarrow \forall i.E[Z_i] \geq 1/2$. By linarity of expectation

$$E[\sum_{i=1}^{m} Z_i \geq m/2$$

By probabilistic method 1. $\exists$ assign. st $\sum_{i=1}^{m} Z_i \geq m/2$.
3/4-approx Max Sat.
α-ran-approx alg: for instance I $m.(I)$ max num of satif. clauses $E[M_A(I)]$ expected num of satisf. clause by $\mathcal{A}$

$$inf_{I \in \mathcal{I}} \frac{E[m_A(I)]}{m.(I)} = \alpha$$

Thrm 1 is $1 - 2^{-k}$-approx alg if $\forall i |C_i| \geq k \Rightarrow k \geq 2$ Thrm 1 $\geq 3/4$-approx alg.
Issue $k = 1$ (1/2 approx)
Solution: New randomized rounding alg. Run both, return better.
Formulate problem as Linear programming relaxation and use randomize rounding.
Let $z_j \in \{0,1\}$ be ind. sat. C_j. For each var x_i let y_i be indp. st. $y_i = 1$ iff $x_i = TRUE$. C_j^+ be set pf indeces in C_j of vars in uncomplemented form and let C_j^- be indeces of complemented vars.
Linear Program

$$\max \sum_{j=1}^{m} z_j \qquad\qquad y_i, z_j \in \{0,1\}$$

subject to

$$\sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j$$

Relaxation: $y_i, z_j \in [0, 1$. Let $\hat{y}_i$ be relaxed y_i when solved. $\hat{z}_j$ be val obt. for $z_j \Rightarrow \max \sum_j \hat{z}_j \geq \max \sum_j z_j$.
Show $E[Z] \geq (1 - 1/e) \sum_j \hat{z}_j$, then show best of two algs atleast $3/4 \sum_j \hat{z}_j$.
Let $\mathbf{Pr}[y_i = 1] = \hat{y}_i$ and

$$\beta_k = 1 - (1 - k^{-1})^k$$

Note $\beta_k \geq (1 - 1/e)$ for all $k \in \mathcal{Z}^+$.
Lemma 1: Let c_j be a clause with k literals. The prob that it is satis. by rand rounding is $\geq \beta_k \hat{z}_j$

Proof: Focus on c_j and assume $c_j^+ = c_j$, assume $x_1 \vee .. \vee x_k$. By constraint $\hat{y}_1 + .. + \hat{y}_k \geq \hat{z}_j$. c_j remains unsatis if all y_i are rounded to zero.
$\mathbf{Pr}[c_j = FALSE] = \prod_{i=1}^{k}(1-\hat{y}_i)$ since ran rounding is indep.
Remain to show

$$1 - \prod_{i=1}^{k}(1-\hat{y}_i) \geq \beta_k \hat{z}_j$$

LHS min when $\hat{y}_i = \hat{z}_j/k \Rightarrow$

$$1-(1-x/k)^k \geq \beta_k x$$

for $x \in [0,1]$. Note $g(x) = 1-(1-x/k)^k$ is concave, so it remains to show $f(0) \geq g(0)$ and $f(1) \geq g(1)$, for $f(x) = \beta_k x$. Simple calc gives $g(0) = f(0) = 0$ and $f(1) = g(1) = 1-(1-1/k)^k$.
Thrm 3: Given instance I of Max-Sat. $E[satisclauses]$ by a Linear Prog with randomized rounding is (1-1/e) times the max num of number clauses satifyable on I.
Proof: Lemma 1 over all j with linearity of expectation. Thrm 4: Let n_1 be the expected of alg from thrm 1. Let n_2 be expected of alg from Thrm 2 then

$$\max(n_1, n_2) \geq 3/4 \sum_j \hat{z}_j$$

Proof: Suffice to show $(n_1+n_2)/2 \geq 3/4 \sum \hat{z}_j$.
Denote by S^k the set of clause c where $|c| = k$

$$n_1 = \sum_k \sum_{c_j \in S^k} (1-2^{-k} \geq \sum_k \sum_{c_j \in S^k} (1-2^{-k})\hat{z}_j$$

By lemma 1

$$n_2 \geq \sum_k \sum_{c_j \in S^k} \beta_k \hat{z}_j \Rightarrow$$

$$\frac{n_1+n_2}{2} \geq \sum_k \sum_{c_j \in S^k} \frac{(1-2^{-k})+\beta_k}{2}\hat{z}_j$$

Since $\forall k.(1-2^{-k}) + \beta_k \geq 3/2 \Rightarrow$

$$\frac{n_1+n_2}{2} \geq \frac{3}{4}\sum_k \sum_{c_j \in S^k} \hat{z}_j = \frac{3}{4}\sum_j \hat{z}_j$$

Expanding graph: number of any S is large than some constant times $|S|$, ie $|\tau(S)| \geq c|S|$. $\tau(S) = \{w \in V | \exists v \in S, (v,w) \in E\}$.
An $(n, d, \alpha, c$ OR-concentrator is a bipartite multigraph $G(L, R, E)$ with indp. set $|L| = |R| = n$ st.

1. $\forall v \in L.deg(v) \leq d$
2. $\forall S \subseteq L$ st. $|S| \leq \alpha n$, there are $|\tau(S) \leq R| \geq c|S|$

Thrm 5: $\exists n_0$ st. $\forall n > n_0$ there is an $(n, 18, 1/3, 2)$ OR-concentrator.
Proof: Consider bipartite rand graph on vert. in L and R st $v \in L$ choose $\tau(v)$ by sampling d vertices indep and unif from R. $\Rightarrow \tau(v) \leq d$ since multi edges removed.
Let $\mathcal{E}_s$ denote event that $\tau(S) < c|S|$ neighbors in R.
Bound $\mathbf{Pr}[\mathcal{E}_s]$ then $\sum_{s \in S:|S| \leq \alpha n} \mathbf{Pr}[\mathcal{E}_s]$ to upperbound prob that rand. graph fails to be OR-conctrator (n, d, α, c) as specified.
Fix $S \subseteq L$ st $|S| = s$ and $T \subseteq R$ st $|T| = cs$. there are $\binom{n}{s}$ ways of choosing S, and $\binom{n}{cs}$ ways of choosing T.
$\mathbf{Pr}[\tau(S) \subseteq T] = (cs/n)^{ds}$, since $\forall v \in S.\tau(v) \leq d \Rightarrow \mathbf{Pr}[\mathcal{E}_S] \leq \binom{n}{s}\binom{n}{cs}(cs/n)^{ds}$
Use $\binom{n}{k} \leq (ne/k)^k$

$$\mathbf{Pr}[\mathcal{E}_s] \leq (ne/s)^s (ne/cs)^{cs} (cs/n)^{ds} = \left[(\frac{s}{n})^{d-c-1} e^{1+c} c^{d-c}\right]^s$$

$\alpha = 1/3$ and $s \leq \alpha n$

$$\begin{aligned}
\mathbf{Pr}[\mathcal{E}_s] &\leq \left[\left(\frac{1}{3}\right)^{d-c-1} e^{1+d} c^{d-c}\right]^s \\
&\leq \left[\left(\frac{1}{3}\right)^{d} (3e)^{1+d}\right]^s
\end{aligned}$$

Using $c = 2 \wedge d = 18$

$$\mathbf{Pr}[\mathcal{E}] \leq \left[\left(\frac{2}{3}\right)^{18} (3e)^3\right]^s$$

let $r = (2/3)^{18}(3e)^3$ and note $r < 1/2$ $Rightarrow$

$$\sum \mathbf{Pr}[\mathcal{E}_s] \leq \sum_{s \geq 1} r^s = \frac{r}{1-r} < 1$$