Fundamental Data Structure Problem

The fundamental data structuring problem is the problem of storing $S_1, \ldots, S_n$ items such that we can efficiently support different kinds of queries and operations. These operations/queries are Make-Set, Insert, Delete, Find, Join, Paste and Split.

Random Treaps

Random Treaps is a datastructure that has a set of keys like a normal binary search tree, but each node also has a priority associated with it. This priority is given at random. The treap satisfies that it is a binary search tree with respect to keys and also a heap with respect to the priorities. Theorem 8.1 states there is a unique treap for each set of key-priority values.
The operations Make-Set and Find are implemented normally. yadda yadda for the others.

Proof of Lemma 8.6: We will show that $E[depth(x_k)] = H_k + H_{n-k+1} + 1$. We define a variable X_{ik} if x_i is an ancestor of x_k. If we look at the case when $i < k$, then:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i<k} X_{ik}\right] &= \sum_{i<k} p_i && \text{(where } p_i = \max\{p_i, \ldots, p_k\}\text{)} \\
&= \sum_{i<k} \frac{1}{k-i+1} \\
& && \text{(Only ancestor if it was the largest key at its insertion)} \\
&= \sum_{i=2}^{k} \frac{1}{i} \\
&= H_k - 1
\end{aligned}$$

The same analysis follows for $i > k$, yielding an expected depth of $(H_k - 1) + (H_{n-k+1} - 1) + 1$ (+1 from itself) which gives us $\mathcal{O}(\lg n)$ depth.

Proof of Lemma 8.7: