4.3.1

a

We have our linear system

$$\begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

First we solve it using Gaussian elimination. We make the following matrix

$$\begin{bmatrix} -1 & 1 & -4 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 3 & 2 & \frac{1}{2} \end{bmatrix}$$

We then apply the row operations to the matrix and keep track of our multipliers to create a lower triangular matrix later.

$$\begin{bmatrix} -1 & 1 & -4 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 3 & 2 & \frac{1}{2} \end{bmatrix} \qquad R_2 - (-2)R_1$$

$$\begin{bmatrix} -1 & 1 & -4 & 0 \\ 0 & 4 & -8 & 1 \\ 0 & 6 & -10 & \frac{1}{2} \end{bmatrix} \qquad R_3 - (-3)R_1$$

$$\begin{bmatrix} -1 & 1 & -4 & 0 \\ 0 & 4 & -8 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \qquad R_3 - \frac{3}{2}R_2$$

The multipliers can be used to define a lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & \frac{3}{2} & 1 \end{bmatrix}$$

And we can define an upper triangular matrix by the results from the Gaussian elimination

$$U = \begin{bmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

And thus the factorization $A = LU$ is

$$\begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

We find the solution by back substitution $Ux = \hat{b}$ to be

$$\begin{aligned} x_3 &= -\frac{1}{2} && \text{solving} : 2x = -1 \\ x_2 &= -\frac{3}{4} && \text{solving} : -8 \cdot -\frac{1}{2} + 4 \cdot x = 1 \\ x_1 &= \ \ \frac{5}{4} && \text{solving} : -4 \cdot -\frac{1}{2} - \frac{3}{4} - x = 0 \end{aligned}$$

Now we solve it again, using Gaussian Elimination with scaled row pivoting.

$$\begin{bmatrix} -1 & 1 & -4 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 3 & 2 & \frac{1}{2} \end{bmatrix}$$

Initially, we have that $S = (4, 2, 3)$ and $P = (1, 2, 3)$. If we look at the ratios $\{1/4, 2/2, 3/3$. Row 2 and 3 have the same ratio, so we pick row 2 to be the first pivot row, so $p = (2, 1, 3)$. Now we use multiples of row 2 to subtract from the other 2 rows to get zeroes in the first column.

$$\begin{bmatrix} 2 & 2 & 0 & 1 \\ -1 & 1 & -4 & 0 \\ 3 & 3 & 2 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & -4 & \frac{1}{2} \\ 3 & 3 & 2 & \frac{1}{2} \end{bmatrix} \qquad R2 - (-\frac{1}{2})R1$$

$$\begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & -4 & -\frac{1}{2} \\ 0 & 0 & 2 & -1 \end{bmatrix} \qquad R3 - \frac{3}{2}R1$$

And we do not need to pick another pivot row as we are done.
The factorization $PA = LU$ will be

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 3 & 2 \end{bmatrix}$$

Where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

As row 1 and 2 were switched.
By back substitution we get (solving $Ux = \hat{b}$)

$$\begin{aligned} x_3 &= -\frac{1}{2} && \text{solving : } 2x = -1 \\ x_2 &= -\frac{3}{4} && \text{solving : } -4 \cdot -\frac{1}{2} + 2 \cdot x = -\frac{1}{2} \\ x_1 &= \frac{5}{4} && \text{solving : } 2 \cdot -\frac{3}{4} + 2 \cdot x = 1 \end{aligned}$$

In the following exercise (b-e), we have written two different procedures. One that is with pivoting and one without. The procedure has been tweaked to also produce the permutation matrix P if pivoting is desired. We use a Maple function to calculate the values of $x_1, x_2, .., x_n$, however it is not displayed twice as it is the same result.

b

In the appendix in **??** there is Maple code implementing Gaussian Elimination with and without pivoting. From section 4.3.1b, we see that using Gaussian elimination on b gives

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 2m & -11 & 0 & 1 \\ 0 & -\frac{2}{11}m & 1 & \frac{15}{22} \end{bmatrix}$$

where m's represent the multipliers used. This allow us to create a lower triangle matrix L and an upper U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives us the factorization $A = LU$

$$\begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can use U with $\hat{b}$ to solve the system $Ux = \hat{b}$ and find x_1, x_2, x_3 with `backSub` that we made in (cp4.2.2).

$$x_1 = \frac{3}{11}$$
$$x_2 = \frac{5}{11}$$
$$x_3 = \frac{1}{11}$$

Using the implemented Gaussian elimination with scaled pivoting from Maple we see that we produce a matrix

$$\begin{bmatrix} \frac{1}{2}m & \frac{11}{4} & -\frac{11}{4} & -\frac{1}{4} \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

This means we have L and U to be

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{11}{4} & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{4} \end{bmatrix}$$

And from maple we also have our produced permutation matrix P, so we get the following factorization $PA = LU$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{11}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{4} \end{bmatrix}$$

Furthermore, we see from Maple, that solving the system $Ux = \hat{b}$ gives us the same x_1, x_2 and x_3 as before.

c

Using the same procedure as in (b), we see that using Gaussian elimination gives us

$$\begin{bmatrix} -1 & 1 & 0 & -3 & 4 \\ -m & 1 & 3 & -2 & 4 \\ 0 & m & -4 & 1 & -1 \\ -3m & 3m & 2m & -3 & 3 \end{bmatrix}$$

We can then create the matrices L and U

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{bmatrix} U = \begin{bmatrix} -1 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Which gives us the factorization $A = LU$

$$\begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The values x_1, x_2, x_3, x_4 are computed in the same way as (b) to be (solving $Ux = \hat{b}$)

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 0 \\ x_4 &= -1 \end{aligned}$$

Using the implemented Gaussian elimination with scaled pivoting from Maple we see that we produce a matrix

$$\begin{bmatrix} -\frac{1}{3}m & m & \frac{4}{3} & -\frac{4}{3} & \frac{4}{3} \\ \frac{1}{3}m & 0 & 2m & 3 & -3 \\ 0 & 1 & -1 & -1 & 3 \\ 3 & 0 & 1 & 2 & 1 \end{bmatrix}$$

Now we got the matrices L and U

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 1 & 0 \\ \frac{1}{3} & 0 & 2 & 1 \end{bmatrix} U = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

And since we have our permutation matrix P, we can put the factorization $PA = LU$ as

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 1 & 0 \\ \frac{1}{3} & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Furthermore, we see from Maple, that solving the system $Ux = \hat{b}$ gives us the same x_1, x_2 and x_3 as before.

d

From the appendix, we see that Gaussian elimination yields us

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 0 \\ 2m & -4 & 0 & 2 & -10 \\ \frac{1}{2}m & 3m & 2 & -5 & -9 \\ -m & -\frac{1}{2}m & 2m & -3 & -3 \end{bmatrix}$$

The factorization $A = LU$ becomes

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 4 & 10 \\ 3 & -13 & 3 & 3 \\ -6 & 4 & 2 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

And the values x_1, x_2, x_3, x_4 are computed to be (solving $Ux = \hat{b}$)

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 3 \\ x_3 &= -2 \\ x_4 &= 1 \end{aligned}$$

Using the implemented Gaussian elimination with scaled pivoting from Maple we see that we produce a matrix

$$\begin{bmatrix} 6 & -2 & 2 & 4 & 0 \\ 2m & -2m & -\frac{4}{13}m & -\frac{6}{13} & -\frac{6}{13} \\ \frac{1}{2}m & -6m & 26 & -83 & -135 \\ -m & 2 & 4 & -14 & -16 \end{bmatrix}$$

We can the define our matrices L and U

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & -6 & 1 & 0 \\ 2 & -2 & \frac{4}{13} & 1 \end{bmatrix} U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 2 & 4 & -14 \\ 0 & 0 & 26 & -83 \\ 0 & 0 & 0 & -\frac{6}{13} \end{bmatrix}$$

And given P, we have the factorization $PA = LU$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 4 & 10 \\ 3 & -13 & 3 & 3 \\ -6 & 4 & 2 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & -6 & 1 & 0 \\ 2 & -2 & \frac{4}{13} & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 2 & 4 & -14 \\ 0 & 0 & 26 & -83 \\ 0 & 0 & 0 & -\frac{6}{13} \end{bmatrix}$$

Furthermore, we see from Maple, that solving the system $Ux = \hat{b}$ gives us the same x_1, x_2 and x_3 as before.

e

Gaussian elimination gives us

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 4m & -9 & -6 & -3 & 6 \\ 8m & -\frac{16}{9}m & -\frac{62}{3} & -\frac{25}{3} & -\frac{25}{3} \\ 2m & -\frac{1}{3}m & \frac{6}{31}m & -\frac{12}{31} & -\frac{12}{31} \end{bmatrix}$$

We find the factorization $A = LU$ to

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & -\frac{16}{9} & 1 & 0 \\ 2 & -\frac{1}{3} & \frac{6}{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & -\frac{62}{3} & -\frac{25}{3} \\ 0 & 0 & 0 & -\frac{12}{31} \end{bmatrix}$$

The values x_1, x_2, x_3, x_4 are computed to be (solving $Ux = \hat{b}$)

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 0 \\ x_4 &= 1 \end{aligned}$$

Using the implemented Gaussian elimination with scaled pivoting from Maple we see that we produce a matrix

$$\begin{bmatrix} \frac{1}{2}m & -\frac{3}{2} & 1 & \frac{1}{2} & 2 \\ 2m & 10m & -12 & -6 & -6 \\ 4m & -\frac{8}{3}m & -\frac{1}{18}m & 2 & 2 \\ 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

We can then define our matrices L and U

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 \\ 4 & -\frac{8}{3} & -\frac{1}{18} & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

With our permutation matrix P we can do the factorization $PA = LU$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 \\ 4 & -\frac{8}{3} & -\frac{1}{18} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Furthermore, we see from Maple, that solving the system $Ux = \hat{b}$ gives us the same x_1, x_2 and x_3 as before.