Logic in Computer Science - Assignment 3

Structural Induction

a

We want to show, by structural induction, the following hypothesis.
Inductive hypothesis: We claim that any tree with height h has at most 2^h propositional atoms (leaves).
Base case: We see that if $\phi = \perp$ or $\phi = p$ then our height is 0 and contains 1 propositional atom. This holds to be less or equal to 2^h, as $2^0 = 1$ and $1 \leq 2^0$.
Inductive step: We let a T be a tree of height $k+1$.
Since any propositional logic of form $\phi \oplus \psi$ has two children, we have two subtrees of height k. By out inductive hypothesis, we know both of these trees have at most 2^k leaves, and as such, the amount of leaves in T is equal to the number of leaves in each subtree. This amount is guaranteed to be less than or equal to $2^k + 2^k = 2^{k+1}$.
For the case where $\phi = \neg\phi$, we have only one subtree of height k. The maximum amount of leaves for this subtree is 2^k.
As this is strictly less than 2^{k+1}, we can conclude that the sum of leaves of both subtrees is at most 2^{k+1}, thus it holds for $k+1$ and the hypothesis is proved.

b

We want to show the following using structural induction.
Induction hypothesis: Any propositional logic of height h contains strictly less than 2^h propositional atoms if the logic contains a $\neg$ anywhere.
Base case: Our base is that $\neg p$ or $\neg\perp$ has a height of 1 and p adds 0 to the height. It has 1 propositional atom, which is strictly less than $2^1 = 2$.
Inductive step: Lets assume we have a subtree T with height $h+1$. We have two subtrees of height h and we assume one of these subtrees have a negation in it.
Let the subtree containing a negation be T_2 with height h. As the negation has only one child with height $h-1$, we know the subtree T_2 has less than or equal to 2^{h-1} (using proof from (a)).
For the other subtree, T_3 of also at most height h, we can also use (a) to show that it has at most 2^h.
Now let $l(T)$ be the number of leaves in T, we want to show

$$l(T) > l(T_2) + l(T_3)$$

We know from (a) that $l(T) \leq 2^{h+1}$, $l(T_3) \leq 2^h$ and $l(T_2) \leq 2^{h-1}$, so we can rewrite the above to a worst case

$$2^{h+1} > 2^{h-1} + 2^h$$

or

$$2^h > 2^{h-1}$$

By using $2^{h+1} = 2^h + 2^h$ and eliminating 2^h on both sides. This is seen to be true. Had our T_3 had a negation we would have

$$2^{h+1} > 2^{h-1} + 2^{h-1}$$

Which is obviously true.
Thus, we can prove our induction hypothesis.