6.4.19

We want to find a natural cubic spline function whose knots are $-1, 0, 1$ and that takes these values

x	-1	0	1
y	5	7	9

Let us expand the table to include all coefficients.

i	0	1	2
x	-1	0	1
y	5	7	9
h = $x_{i+1} - x_i$	1	1	
u = $2(h_i + h_{i-1})$		4	
b = $\frac{6}{h_i}(y_{i+1} - y_i)$	12	12	
v = $b_i - b_{i-1}$		0	

By **KC, page 351** equation **(6)** and inserting known values we are looking for solutions to

$$S_i(x) = \begin{cases} \frac{z_0}{6}(-x)^3 + \frac{z_1}{6}(x+1)^3 + C(x+1) - Dx & x \in [x_0, x_1] \\ \frac{z_1}{6}(1-x)^3 + \frac{z_2}{6}(x)^3 + Cx + D(1-x) & x \in [x_1, x_2] \end{cases}$$

First off we want to determine values of z_i. This is done by solving the following linear system using [**KC, page 352**] with $z_0 = z_2 = 0$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Where obviously $z_1 = 0$, thus we get

$$S(x) = \begin{cases} C(x+1) - Dx & x \in [x_0, x_1] \\ Cx + D(1-x) & x \in [x_1, x_2] \end{cases}$$

Where we can find C and D so we get the following

$$S(x) = \begin{cases} (\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6})(x+1) - (\frac{y_i}{h_i} - \frac{z_i h_i}{6})x & x \in [x_0, x_1] \\ (\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6})x + (\frac{y_i}{h_i} - \frac{z_i h_i}{6})(1-x) & x \in [x_1, x_2] \end{cases}$$

Where all values are now known, and S is

$$S(x) = \begin{cases} (\frac{7}{1} - \frac{0}{6})(x+1) - (\frac{5}{1} - \frac{0}{6})x & x \in [x_0, x_1] \\ (\frac{9}{1} - \frac{0}{6})x + (\frac{7}{1} - \frac{0}{6})(1-x) & x \in [x_1, x_2] \end{cases}$$

$$S(x) = \begin{cases} 2x + 7 & x \in [x_0, x_1] \\ 2x + 7 & x \in [x_1, x_2] \end{cases}$$

Which makes sense with a linear function, since the second derivative is 0.