7.3.3

Since it is a 4-point Gaussian quadratue rule, we write our $(2n-1)$-th polynomium as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

So we want to find the left hand side which can be written as

$$\begin{aligned}
\int_{-1}^{1} f(x)dx &= [a_0x + a_1(\frac{x^2}{2}) + a_2(\frac{x^3}{3}) + a_3(\frac{x^4}{4}) + a_4(\frac{x^5}{5}) \\
&\quad + a_5(\frac{x^6}{6}) + a_6(\frac{x^7}{7}) + a_7(\frac{x^8}{8})]_{-1}^{1} \\
&= a_0(1-(-1)) + a_1(\frac{1^2-(-1)^2}{2}) + a_2(\frac{1^3-(-1)^3}{3}) + a_3(\frac{1^4-(-1)^4}{4}) \\
&\quad + a_4(\frac{1^5-(-1)^5}{5}) + a_5(\frac{1^6-(-1)^6}{6}) + a_6(\frac{1^7-(-1)^7}{7}) + a_7(\frac{1^8-(-1)^8}{8}) \\
&= a_0(2) + a_1(\frac{0}{2}) + a_2(\frac{2}{3}) + a_3(\frac{0}{4}) + a_4(\frac{2}{5}) + a_5(\frac{0}{6}) + a_6(\frac{2}{7}) + a_7(\frac{0}{8}) \\
&= 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 \qquad (1)
\end{aligned}$$

Our formula from the exercise text can be written as

$$\begin{aligned}
\int_{-1}^{1} f(x)dx &\approx A_0(a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + a_4x_0^4 + a_5x_0^5 + a_6x_0^6 + a_7x_0^7) \\
&\quad + A_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4 + a_5x_1^5 + a_6x_1^6 + a_7x_1^7) \\
&\quad + A_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + a_4x_2^4 + a_5x_2^5 + a_6x_2^6 + a_7x_2^7) \\
&\quad + A_3(a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^3 + a_4x_3^4 + a_5x_3^5 + a_6x_3^6 + a_7x_3^7) \\
&= a_0(A_0 + A_1 + A_2 + A_3) + a_1(A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3) \\
&\quad + a_2(A_0x_0^2 + A_1x_1^2 + A_2x_2^2 + A_3x_3^2) + a_3(A_0x_0^3 + A_1x_1^3 + A_2x_2^3 + A_3x_3^3) \\
&\quad + a_4(A_0x_0^4 + A_1x_1^4 + A_2x_2^4 + A_3x_3^4) + a_5(A_0x_0^5 + A_1x_1^5 + A_2x_2^5 + A_3x_3^5) \\
&\quad + a_6(A_0x_0^6 + A_1x_1^6 + A_2x_2^6 + A_3x_3^6) + a_7(A_0x_0^7 + A_1x_1^7 + A_2x_2^7 + A_3x_3^7)
\end{aligned}$$

Now we want equation (1) to be the same as what we found above. Obviously we only need to look at a_0, a_2, a_4 and a_6, meaning we can set up the following equations

$$2 = A_1 + A_2 + A_3 + A_4$$
$$\frac{2}{3} = A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2$$
$$\frac{2}{5} = A_0 x_0^4 + A_1 x_1^4 + A_2 x_2^4 + A_3 x_3^4$$
$$\frac{2}{7} = A_0 x_0^6 + A_1 x_1^6 + A_2 x_2^6 + A_3 x_3^6$$

We know that the x_i nodes are the roots of the n-th polynomium, which for $n = 4$ is those given in the book. To find the weights, you can solve the linear system with 4 equations and 4 unknowns. Thus you find A_0, A_1, A_2 and A_3. Since we have now estimated our integral with these found values, we can say

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)$$