Logic in Computer Science - Assignment 3

Exercise 2.2

3a

In the following, m is a constant, f is a function with one argument, S and B are predicate symbols with two arguments.

i

$S(m, x)$ is a valid formula.

ii

$B(m, f(m))$ is a valid formula.

iii

$f(m)$ is a valid formula.

iv

$B(B(m, x), y)$ is **not** a valid formula as B can only take terms as arguments.

v

$S(B(m), z)$ is **not** a valid formula as S can not take terms as arguments and the inner B only takes one argument.

vi

$(B(x, y) \rightarrow (\exists z S(z, y)))$ is a valid formula.

vii

$(S(x, y) \rightarrow S(y, f(f(x))))$ is a valid formula.

viii

$(B(x) \rightarrow B(B(x)))$ is **not** a valid formula as all B's only take one argument and one of them takes another predicate as argument.

4c

We have ϕ is $\exists x(P(y,z) \land (\forall y(\neg Q(y,x) \lor P(y,z))))$ with P and Q as predicate symbols with two arguments.

Yes, we see that the variable y is both free and bound in ϕ. It is free on the left side of the "or" in the inner expression and bound on the right side.

4d

i

We compute $\phi[w/x]$

$$\exists x(P(y,z) \land (\forall y(\neg Q(y,x) \lor P(y,z))))$$

We see this is the same. This is because there are no free variable x in ϕ that can be replaced.
We now compute $\phi[w/y]$.

$$\exists x(P(\mathbf{w},z) \land (\forall y(\neg Q(y,x) \lor P(y,z))))$$

Where one free y has been replaced by w (marked in bold).
We then compute $\phi[f(x)/y$.

$$\exists x(P(\boldsymbol{f(x)},z) \land (\forall y(\neg Q(y,x) \lor P(y,z))))$$

Again, the same y has been replaced.
Lastly we compute $\phi[g(y,z)/z]$

$$\exists x(P(y,\boldsymbol{g(y,z)}) \land (\forall y(\neg Q(y,x) \lor P(y,\boldsymbol{g(y,z)}))))$$

Where two replacement has been made.

ii

Since there are no free x in ϕ, all three of them are free for x in ϕ (using definition 2.8).

iii

There is one free y.
We see that w is free for y in ϕ as a replacement will not cause a w to be under a quantifier involving w.
f(x) is not free for y as the variable x will become bound due to the quantifier "$\exists x$".
g(y,z) is free for y as both y and z will not be bound by any quantifiers.

Exercise 2.3

1a

We want to prove the validity of the sequent $(y = 0) \wedge (y = x) \vdash 0 = x$.

1	$(y = 0) \wedge (y = x)$	premise
2	$y = 0$	$\wedge E_1(1)$
3	$y = x$	$\wedge E_2(1)$
4	$0 = x$	$= E(2, 3)$

7c

We want to prove the validity of $\exists x \forall y P(x, y) \vdash \forall y \exists x P(x, y)$.

1	$\exists x \forall y P(x, y)$	premise
2	y_0	
3	x_0	
4	$\forall y P(x_0, y)$	assumption
5	$P(x_0, y)$	$\forall E(4)$
6	$\exists x P(x, y_0)$	$\exists I(5)$
7	$\exists x P(x, y_0)$	$\exists E(1, 3-6)$
8	$\forall y \exists x P(x, y)$	$\forall I(2-7)$

9a

We want to prove the validity of $\exists x(S \to Q(x)) \vdash S \to \exists x Q(x)$ where Q has arity 1 and S has arity 0.

1	$\exists x(S \to Q(x))$	premise
2	S	assumption
3	x_0	
4	$S \to Q(x_0)$	$Somethingsomething$
5	$Q(x_0)$	$\to E(2, 4)$
6	$\exists x Q(x)$	$\exists I(5)$
7	$\exists x Q(x)$	$\exists E(1, 3-6)$
8	$S \to \exists x Q(x)$	$\to I(2-7)$

9l

We want to prove the validity of $\forall x P(x) \vee \forall x Q(x) \vdash \forall x(P(x) \vee Q(x))$ where P and Q have arity 1.

1	$\forall x P(x) \vee \forall x Q(x)$	premise
2	x_0	
3	$\forall x P(x)$	assumption
4	$P(x_0)$	$\forall I(3)$
5	$P(x_0) \vee Q(x_0)$	$\vee I_1(4)$
6	$\forall x Q(x)$	assumption
7	$Q(x_0)$	$\forall I(6)$
8	$P(x_0) \vee Q(x_0)$	$\vee I_2(7)$
9	$P(x_0) \vee Q(x_0)$	$\vee E(1, 3-5, 6-8)$
10	$\forall x(P(x) \vee Q(x))$	$\forall I(2-9)$

Exercise 2.4

5

ϕ is the sentence $\forall x \forall y \exists z (R(x, y) \to R(y, z))$ and R has arity 2.

a

We let $A \stackrel{def}{=} \{a, b, c, d\}$ and $R^{\mathcal{M}} \stackrel{def}{=} \{(b, c), (b, b), (b, a)\}$. We want to determine if we have $\mathcal{M} \models \phi$.

ϕ is not valid for $\mathcal{M}$ as we have $(b, c) \in R^{\mathcal{M}}$, but there exists no element in $R^{\mathcal{M}}$ for which there is (c, z) where z an arbitrary element in A.

b

We let $A \stackrel{def}{=} \{a, b, c\}$ and $R^{\mathcal{M}} \stackrel{def}{=} \{(b, c), (a, b), (c, b)\}$. We want to determine if we have $\mathcal{M} \models \phi$.

ϕ is true in this model $\mathcal{M}$ as for any element in $R^{\mathcal{M}}$, (x, y), there also exists an element in the form (y, z) so the implication holds true.

8

We want to show the semantic entailment $\forall x P(x) \vee \forall x Q(x) \models \forall x(P(x) \vee Q(x))$.

We let a model, $\mathcal{M}$, satisfy $\forall x P(x) \vee \forall x Q(x)$. We then want to show that $\mathcal{M}$ satisfies $\forall x(P(x) \vee Q(x))$. This means that either the model satisfies $\forall x P(x)$ or $\forall x Q(x)$.
We can conclude that all elements are in either $P^{\mathcal{M}}$ or they are in $Q^{\mathcal{M}}$. This means that all elements must also be in the union set, $P^{\mathcal{M}} \cup Q^{\mathcal{M}}$.
This union can also be rewritten as $\forall x(P(x) \vee Q(x))$, thus $\mathcal{M} \models \forall x(P(x) \vee Q(x))$.

10

We want to determine if $\forall x(P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$ is a semantic entailment or not.

We suppose we have a model, $\mathcal{M}'$, that satisfy $\forall x(P(x) \vee Q(x))$. To show that it is ***not*** a semantic entailment, we construct a counter-example model. Assume that either $P(x)$ or $Q(x)$ is always true (as in **xor**) and both $P(x)$ and $Q(x)$ assume F at one x, then $\forall(P(x) \vee Q(x))$ will always be true. However, since $P(x)$ and $Q(x)$ both can assume F, then it will not hold for $\forall x P(x) \vee \forall x Q(x)$ as there exists x where $P(x)$ is false and there exists x where $Q(x)$ is false.

Structural Induction

a

We want to show, by structural induction, the following hypothesis.
Inductive hypothesis: We claim that any tree with height h has at most 2^h propositional atoms (leaves).
Base case: We see that if $\phi = \perp$ or $\phi = p$ then our height is 0 and contains 1 propositional atom. This holds to be less or equal to 2^h, as $2^0 = 1$ and $1 \leq 2^0$.
Inductive step: We let a T be a tree of height $k+1$. Since any propositional logic has at most two children, we have in worst case two subtrees of height k. By out inductive hypothesis, we know both of these trees have at most 2^k leaves, and as such, the amount of leaves in T is equal to the number of leaves in each subtree. This amount is guaranteed to be less than or equal to $2^k + 2^k = 2^{k+1}$.
This means that it holds for $k+1$, and the hypothesis is proved.

b

We want to show the following using structural induction.
Induction hypothesis: Any propositional logic of height h contains strictly less than 2^h propositional atoms if the logic contains a $\neg$ anywhere.
Base case: Our base is that $\neg p$ has a height of 1 and p adds 0 to the height. It has 1

propositional atom, which is strictly less than $2^1 = 2$.
Inductive step: Lets assume we have a subtree T_2 with height h_2 that is a full binary tree. We know from our proof in (a) that the subtree has at most 2^h atoms. If now we add a $\phi = \neg\psi$ at the top, which we can do as it only has one child (the subtree T_2), we have a new subtree T_3 with height $h_2 + 1$. However, as there is only one subtree of height h_2 which is T_2 the amount of leaves will be the same.
Let $l(\phi)$ be the amount of leaves of subtree ϕ. We want to show

$$\begin{aligned} l(T_3) &< 2^{h_2+1} \\ &= 2^h + 2^h \end{aligned}$$

Since $l(T_3) = l(T_2) = 2^h$ and $2^h < 2^h + 2^h$ we can conclude that any subtree ϕ with a $\neg$ has

$$l(\phi) \leq 2^{height(\phi)-1} \Rightarrow l(\phi) < 2^{height(\phi)}$$

If we add to a subtree of this kind, we can use the proof from (a) to show that the number of leaves will still be strictly less than 2^h. Thus, we can prove our induction hypothesis.