NP-Completeness

Ill start with explaining the question of $P = NP$. The problems in P are problems that can be solved in polynomial time. The problems in NP are problems that, given a certificate, can verify a solution in polynomial time, so obviously $P \subseteq NP$. Instead of studying optimization problems, we will focus on decision problems, that is, a problem for which the output is yes/no.
We say a language L is accepts in polynomial time, if an algorithm can correctly determine if some $x \in L$ for some input x. We can show that if a language is accepted in polynomial time, it also decides it in polynomial time.
Reductions (with example linear equations) and we say $L_1 \leq_p L_2$.
To show that $NP \subseteq P$, a key concept is reducebility of problems. The idea is we want to reduce problems to other problems, so called NP-complete problems, in polynomial time. If these problems can be solved in polynomial time, then $P = NP$.

To give a concrete example, lets consider the NP-complete problem $3 - CNF - SAT$. We want to show the clique problem (decided version) is NP-complete by reducing to it.
We first prove it is in NP. It clearly is, as given a set $V' \subseteq V$, we just check if for each vertex, that there is an edge to all other vertices. Now we want to prove the reduction to clique.
For each clause in ϕ (up to k), we create three vertices in a graph and contruct an edge, (u, v), between vertices if they are in different triples and u is not the negation of v. This graph can be built in polynomial time. Now if ϕ has a satisfying assignment, that means there is a clique of size k since the construction of the graph ensures there is no edge between negation of literals.

Another NP-complete problem is the vertex cover. We will show that CLIQUE $\leq_p$ VERTEX-COVER. Obviously VERTEX-COVER $\in NP$ as we can just, for all edges, check if either u og v is in the cover set. Now, given a graph G, we also have the complement of it, $\overline{G} = (V, \overline{E})$ (computed in polynomial time). This is a reduction. G has a clique of size k only if $\overline{G}$ has a vertex cover of size $|V| - k$. If there is an instance of clique $< G, k >$, then we claim that the instance $< \overline{G}, |V| - k >$ of vertex cover is a reduction. If we have an edge $(u, v) \in \overline{E}$, then one of the vertices is not in the clique vertice set, V'. This means the edge is covered by a vertex in $V - V'$. This applies to all edges in $\overline{E}$, so all edges in $\overline{E}$ are covered by $V - V'$.
Conversively, if $\overline{G}$ has a vertex cover V' of size $|V| - k$, then for all edges $(u, v) \in \overline{E}$, at least one of u and v is in V', and the opposite, if neither is, then the edge must be in E. Therefore $V - V'$ is a clique of size k.