Graph Coloring
Assignment 1

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March 2, 2016

1 Problem 1

The first step of a `FunnyGraph`, $F(n)$, is to remove a cycle from a complete graph G. Say the graph has n vertices, so the starting graph has $\chi(G) = n$. Removing the cycle results in the graph G' with $\chi(G') = \lceil \frac{n}{2} \rceil$. To see this is correct, consider the coloring where two vertices that were adjacent in the removed cycle are colored with the same color. Thus, if n is even, then it can be done with $n/2$ colors, and if n is odd, then we have a vertex left over, so we must use $\lceil \frac{n}{2} \rceil$ colors. To prove this is an optimal coloring, we use the formula

$$\frac{|V(G)|}{\alpha(G)} \leq \chi(G)$$

Since the maximum independent set can easily be seen to be 2, that means it is optimal to color sets of 2 as the left hand side is exactly equal to $\chi(G')$ when n is even. It is also optimal when n is odd as we can not have an independent set of 3.
The second step is the Mycielskistep, which increases the chromatic number of a graph by 1. So now we have a graph G'' with $\chi(G'') = \lceil \frac{n}{2} \rceil + 1$.
The last step is to join it with the wheel graph W_{n+1}. The wheel graph has chromatic number 3 when $n+1$ is odd, and chromatic number 4 when $n+1$ is even. Since joining two graphs makes the graphs completely connected, they cannot have any of the same colors. Now we know that

$$\chi(G_1 + \cdots + G_n) = \sum_{i=1}^{n} \chi(G_i)$$

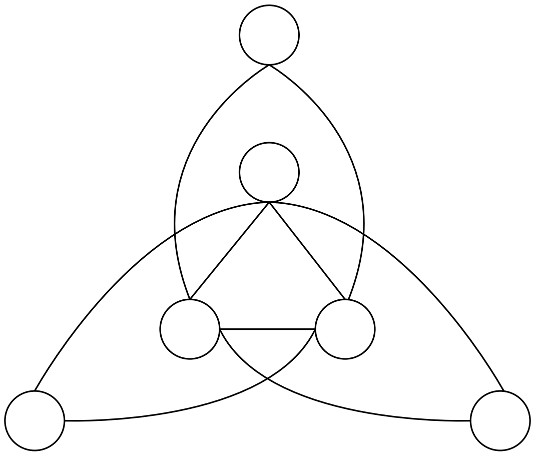
So the resulting graph, G''', has chromatic number $\chi(G''') = \chi(G'') + \chi(W_{n+1})$, so the chromatic number of the `FunnyGraph` of size n can be expressed as:

$$\chi(F(n)) = \begin{cases} \lceil \frac{n}{2} \rceil + 5 & \text{if } n \text{ is odd} \\ \frac{n}{2} + 4 & \text{if } n \text{ is even} \end{cases}$$

As W_{n+1} will have an odd cycle when n is odd and the other way around. We can also remove the ceilings as we know $n/2$ will give us an integer for even n. So for $G = F(99)$, the chromatic number is $\lceil \frac{n}{2} \rceil + 5 = 55$. Note that the expression for $\chi(F(n))$ only works for $n \geq 4$.

2 Problem 2

To prove that this greedy method does not provide optimal results, I will show a counter example which can be extended to make infinitely many graphs where it is not optimal. Consider the following graph G:



It is constructed by K_3 in the middle, and then adding three new vertices, which connect to two of the vertices from K_3. The proposed greedy method would pick the three outer vertices as it is the largest independent set. Then only K_3 would be left, which requires 3 colors, so the coloring would use a total of 4 colors. However, it is optimal to pick independent sets of size 2, the sets that pick one vertex for the inner and outer "triangle". This results in a coloring using only 3 colors.
This can be extended by adding infinity many of these "triangles" on the outside that connects to K_3 in the same way. The largest independent set will always have size $|V(G)| - 3$, where the last 3 vertices are in K_3, so the greedy method will always use 4 colors, while it is optimal to pick 3 equally large sets, one vertex from each "triangle", as it uses only 3 colors. This can actually be extended for any K_n as long as you add n vertices that connects to $n-1$ vertices from K_n for each "iteration" of this.

3 Problem 3

Say we have an optimal coloring of G using $\chi(G)$ colors. Then we have a partition of G into n $(= \chi(G))$ independent sets. Let these sets have size $\alpha_1(G), .., \alpha_n(G)$. We know that at least one of the independent sets has size larger than or equal to $\frac{|V(G)|}{\chi(G)}$, since otherwise all vertices are not colored. Therefore we can say that:

$$\begin{aligned} \frac{|V(G)|}{\chi(G)} &\leq \max(\alpha_i(G)) && \text{(For } i = 1..n\text{)} \\ &= \max(\omega_i(\overline{G})) && \text{(As } \omega(\overline{G}) = \alpha(G)\text{)} \\ &\leq \chi(\overline{G}) && \text{(Since } \omega(\overline{G}) \leq \chi(G)\text{)} \end{aligned}$$

If we now multiply both sides by $\chi(G)$, we get:

$$|V(G)| \leq \chi(\overline{G})\chi(G)$$

We now easily see that either $\chi(\overline{G})$ or $\chi(G)$ must be larger than or equal to $\sqrt{|V(G)|}$, as the above inequality would not hold otherwise, therefore we have:

$$\max(\chi(G), \chi(\overline{G})) \geq \sqrt{|V(G)|}$$

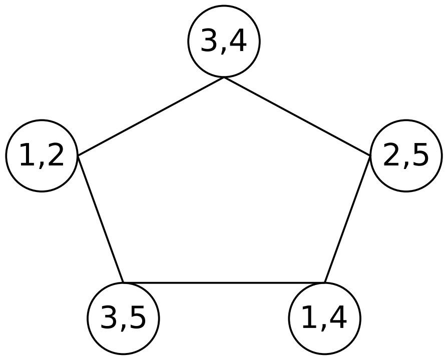
Which is what we wanted to show.

4 Problem 4

We start by proving that $\omega(G)\chi(H) \leq \chi(G[H])$. We know that for any graph G', then $\omega(G') \leq \chi(G')$. Since the graph H has chromatic number $\chi(H)$, the substituted graph $G[H]$ must use $\omega(G) \cdot \chi(H)$ colors as the clique consisting of $\omega(G)$ components of H is completely connected, since when they are completely connected, it means that any two components cannot use the same color. Thus, it holds that $\omega(G)\chi(H) \leq \chi(G[H])$.

Now we need to prove that $\chi(G[H]) \leq \chi(G)\chi(H)$. Let us denote V_i the vertex i in G and let H_i be the graph H replacing the vertex V_i. Let their (optimal) colorings be denoted $c(V_i)$ and $c(H_i)$ (remember that each H_i can be colored differently). If we had an optimal coloring of G, then using the same coloring where each component in $G[H]$ has optimal coloring $c(H_i)$ whenever G had coloring $c(V_i)$ must produce a coloring of $G[H]$ using at most $\chi(G)\chi(H)$ colors. Note that it is only an upper bound since colorings of H_i and H_j might use *some* of the same colors when they are not adjacent, but can still be different colorings even though both uses $\chi(H)$ colors. If we use entirely different set of colors (as we are forced to do if the components are adjacent) for all components H_i having a different coloring, then we have exactly that $\chi(G[H]) = \chi(G)\chi(H)$.

To demonstrate that it is in fact an upper bound and not an exact bound, consider $G[H]$, where $G = C_5$ and $H = K_2$. Since each component after the substitution is completely connected if there was an edge in G, we cannot use any of the same colors for two adjacent components. Each component H_i in the substituted graph requires 2 colors. We can perform the following coloring of $G[H]$:



Since $\chi(G) = 3$ as it is an odd cycle and $\chi(H) = 2$, the upper bound is 6. However, this coloring only uses 5 colors, which means it is possible to have $\chi(G[H]) < \chi(G)\chi(H)$.

5 Problem 5

The chromatic number of the constructed graph G of size n as described in the assignment text is equal to $\pi(n)$, where π is the function that counts primes under n plus one. The plus one comes from the number 1. Thus the chromatic number, $\chi(G)$, for $n = 20162016$ is $\pi(20162016) = 1280233$.

`Proof:` Imagine we introduce vertex 1 first and iteratively adds vertices in order so that the last vertex that is introduced is V_n. The proof consists of two parts. Firstly, we want to prove that every time we introduce a prime-numbered vertex, we need an additional color. And secondly, when we introduce a composite-numbered vertex, we do not need any additional colors.
The first part is easy to see. Say we introduce a prime-numbered vertex V_p. Then for any prime q, where $q < p$, then $\gcd(q, p) = 1$ or p would not be a prime. This means there is an edge to all primes less than p. Thus all primes (and including 1) must form a clique in G.
For the second part, when we introduce a composite-numbered vertex V_k, then k has a prime factorization. We know that k is divisible by any of these prime factors, thus there cannot be an edge between two of these vertices. So if we color V_k the same as any of the prime factors, we have a valid coloring of G. To see this is true, we prove it by contradiction. Suppose we have a vertex V_l where there is an edge to V_k, so $\gcd(l, k) = 1$. If the vertices were colored the same, then it means that l and k have a common prime factor p. But if this is true, then $p \setminus l$ and $p \setminus k$, which implies $\gcd(l, k) \neq 1$ as $p > 1$. This is a contradiction, so this coloring must be a valid coloring of G. $\blacksquare$