Completeness exercise

We want to prove completeness of

$$(p \to q) \to q \vdash (q \to p) \to p$$

We follow the steps described in the assignment text.
We start with

$$(p \to q) \to q \models (q \to p) \to p$$

Now we move the premises to the right hand side, so we get

$$\models ((p \to q) \to q) \to ((q \to p) \to p)$$

We proceed by creating a truth table

p	q	$p \to q$	$(p \to q) \to q$	$q \to p$	$(q \to p) \to p$	$((p \to q) \to q) \to ((q \to p) \to p)$
T	T	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	F	T	T
F	F	T	F	T	F	T

This means we have the following constructed sequents

$$\begin{aligned} p, q &\vdash ((p \to q) \to q) \to ((q \to p) \to p) && : \alpha_1 \\ p, \neg q &\vdash ((p \to q) \to q) \to ((q \to p) \to p) && : \alpha_2 \\ \neg p, q &\vdash ((p \to q) \to q) \to ((q \to p) \to p) && : \alpha_3 \\ \neg p, \neg q &\vdash ((p \to q) \to q) \to ((q \to p) \to p) && : \alpha_4 \end{aligned}$$

As explained in the assignment text, we start by proving α_1, meaning we prove

$$\begin{aligned} p &\vdash \gamma_l[p] && (a) \\ q &\vdash \gamma_l[q] && (b) \\ p, q &\vdash \gamma_l[p \to q] && (c) \\ p, q &\vdash \gamma_l[(p \to q) \to q] && (d) \\ p, q &\vdash \gamma_l[q \to p] && (e) \\ p, q &\vdash \gamma_l[(q \to p) \to p] && (f) \\ p, q &\vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] && (g) \end{aligned}$$

In (a), we prove $p \vdash \gamma_l[p]$ where $\gamma_l[p] = p$ meaning it is trivially true.
In (b), we prove $q \vdash \gamma_l[q]$ where $\gamma_l[q] = q$ meaning it is trivially true.

In (c), we prove $p, q \vdash \gamma_l[p \to q]$ where $\gamma_l[p \to q] = p \to q$. A box proof is provided below

1	p	premise
2	q	premise
3	p	assumption
4	q	(2)
5	$p \to q$	$\to I(3-4)$

In (d), we prove $p, q \vdash \gamma_l[(p \to q) \to q]$ where $\gamma_l[(p \to q) \to q] = (p \to q) \to q$. A box proof is provided below

1	p	premise
2	q	premise
3	$p \to q$	assumption
4	q	$\to E(1,3)$
5	$(p \to q) \to q$	$\to I(3-4)$

In (e), we prove $p, q \vdash \gamma_l[q \to p]$ where $\gamma_l[q \to p] = q \to p$. This is the exact same proof as in (c), so we reuse this proof with the atoms p and q switched.
In (f), we prove $p, q \vdash \gamma_l[(q \to p) \to p]$ where $\gamma_l[(q \to p) \to p] = (q \to p) \to p$. This is the exact same proof as in (d), so we reuse this proof with the atoms p and q switched.
In (g), we prove $p, q \vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)]$ where $\gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] = ((p \to q) \to q) \to ((q \to p) \to p)$. A boxproof is provided below

1	p	premise
2	q	premise
3	$(p \to q) \to q$	proved in (d)
4	$(q \to p) \to p$	proved in (f)
5	$(p \to q) \to q$	assumption
6	$(q \to p) \to p$	(4)
7	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to I(5-6)$

We now prove α_2 in the same manner.

$$
\begin{aligned}
p &\vdash \gamma_l[p] && (a)\\
\neg q &\vdash \gamma_l[q] && (b)\\
p, \neg q &\vdash \gamma_l[p \to q] && (c)\\
p, \neg q &\vdash \gamma_l[(p \to q) \to q] && (d)\\
p, \neg q &\vdash \gamma_l[q \to p] && (e)\\
p, \neg q &\vdash \gamma_l[(q \to p) \to p] && (f)\\
p, \neg q &\vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] && (g)
\end{aligned}
$$

In (a), we prove $p \vdash \gamma_l[p]$ where $\gamma_l[p] = p$ meaning it is trivially true.
In (b), we prove $\neg q \vdash \gamma_l[q]$ where $\gamma_l[q] = \neg q$ meaning it is trivially true.
In (c), we prove $p, \neg q \vdash \gamma_l[p \to q]$ where $\gamma_l[p \to q] = \neg(p \to q)$. A box proof is provided below

1	p	premise
2	$\neg q$	premise
3	$(p \to q)$	assumption
4	q	$\to E(1,3)$
5	$\perp$	$\neg E(4,2)$
6	$\neg(p \to q)$	$\neg I(3-5)$

In (d), we prove $p, \neg q \vdash \gamma_l[(p \to q) \to q]$ where $\gamma_l[(p \to q) \to q] = (p \to q) \to q$. A box proof is provided below

1	p	premise
2	$\neg q$	premise
3	$p \to q$	assumption
4	q	$\to E(1,3)$
5	$(p \to q) \to q$	$\to I(3-4)$

In (e), we prove $p, \neg q \vdash \gamma_l[q \to p]$ where $\gamma_l[q \to p] = q \to p$. A box proof is provided below

1	p	premise
2	$\neg q$	premise
3	q	assumption
4	p	$\neg E(3,2)$
5	$q \to p$	$\to I(3-5)$

In (f), we prove $p, q \vdash \gamma_l[(q \to p) \to p]$ where $\gamma_l[(q \to p) \to p] = (q \to p) \to p$. A box proof is provided below

1	p	premise
2	$\neg q$	premise
3	$(q \to p)$	assumption
4	p	(1)
5	$(q \to p) \to p$	$\to I(3-4)$

In (g), we prove $p, \neg q \vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)]$ where $\gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] = ((p \to q) \to q) \to ((q \to p) \to p)$. A boxproof is provided below

1	p	premise
2	$\neg q$	premise
3	$(p \to q) \to q$	proved in (d)
4	$(q \to p) \to p$	proved in (f)
5	$(p \to q) \to q$	assumption
6	$(q \to p) \to p$	(4)
7	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to I(5-6)$

We now prove α_3 in the same manner.

$$
\begin{aligned}
\neg p &\vdash \gamma_l[p] && (a)\\
q &\vdash \gamma_l[q] && (b)\\
\neg p, q &\vdash \gamma_l[p \to q] && (c)\\
\neg p, q &\vdash \gamma_l[(p \to q) \to q] && (d)\\
\neg p, q &\vdash \gamma_l[q \to p] && (e)\\
\neg p, q &\vdash \gamma_l[(q \to p) \to p] && (f)\\
\neg p, q &\vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] && (g)
\end{aligned}
$$

In (a), we prove $\neg p \vdash \gamma_l[p]$ where $\gamma_l[p] = \neg p$ meaning it is trivially true.
In (b), we prove $q \vdash \gamma_l[q]$ where $\gamma_l[q] = q$ meaning it is trivially true.

In (c), we prove $\neg p, q \vdash \gamma_l[p \to q]$ where $\gamma_l[p \to q] = p \to q$. A box proof is provided below

1	$\neg p$	premise
2	q	premise
3	p	assumption
4	$\bot$	$\neg E(3,1)$
5	q	$\bot E(4)$
6	$p \to q$	$\to I(3-5)$

In (d), we prove $\neg p, q \vdash \gamma_l[(p \to q) \to q]$ where $\gamma_l[(p \to q) \to q] = (p \to q) \to q$. A box proof is provided below

1	$\neg p$	premise
2	q	premise
3	$p \to q$	assumption
4	q	(2)
5	$(p \to q) \to q$	$\to I(3-4)$

In (e), we prove $\neg p, q \vdash \gamma_l[q \to p]$ where $\gamma_l[q \to p] = \neg(q \to p)$. A box proof is provided below

1	$\neg p$	premise
2	q	premise
3	$q \to p$	assumption
4	p	$\to E(2,3)$
5	$\bot$	$\neg I(4,1)$
6	$\neg(q \to p)$	$\neg E(3-5)$

In (f), we prove $\neg p, q \vdash \gamma_l[(q \to p) \to p]$ where $\gamma_l[(q \to p) \to p] = (q \to p) \to p$. A box proof is provided below

1	$\neg p$	premise
2	q	premise
3	$q \to p$	assumption
4	p	$\to E(2,3)$
5	$(q \to p) \to p$	$\to I(3-4)$

In (g), we prove $\neg p, q \vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)]$ where $\gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] = ((p \to q) \to q) \to ((q \to p) \to p)$. A boxproof is provided below

1	$\neg p$	premise
2	q	premise
3	$(p \to q) \to q$	proved in (d)
4	$(q \to p) \to p$	proved in (f)
5	$(p \to q) \to q$	assumption
6	$(q \to p) \to p$	(4)
7	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to I(5-6)$

We now prove α_4 in the same manner.

$$\begin{aligned} \neg p &\vdash \gamma_l[p] && (a) \\ \neg q &\vdash \gamma_l[q] && (b) \\ \neg p, \neg q &\vdash \gamma_l[p \to q] && (c) \\ \neg p, \neg q &\vdash \gamma_l[(p \to q) \to q] && (d) \\ \neg p, \neg q &\vdash \gamma_l[q \to p] && (e) \\ \neg p, \neg q &\vdash \gamma_l[(q \to p) \to p] && (f) \\ \neg p, \neg q &\vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] && (g) \end{aligned}$$

In (a), we prove $\neg p \vdash \gamma_l[p]$ where $\gamma_l[p] = \neg p$ meaning it is trivially true.
In (b), we prove $\neg q \vdash \gamma_l[q]$ where $\gamma_l[q] = \neg q$ meaning it is trivially true.
In (c), we prove $\neg p, \neg q \vdash \gamma_l[p \to q]$ where $\gamma_l[p \to q] = p \to q$. A box proof is provided below

1	$\neg p$	premise
2	$\neg q$	premise
3	p	assumption
4	$\bot$	$\to E(3,1)$
5	q	$\bot E(4)$
6	$p \to q$	$\to I(3-5)$

In (d), we prove $\neg p, \neg q \vdash \gamma_l[(p \to q) \to q]$ where $\gamma_l[(p \to q) \to q] = \neg((p \to q) \to q)$. A box proof is provided below

1	$\neg p$	premise
2	$\neg q$	premise
3	$p \to q$	proved in (c)
4	$(p \to q) \to q$	assumption
5	q	$\to E(3,4)$
6	$\perp$	$\neg E(5,2)$
7	$\neg((p \to q) \to q)$	$\neg I(4-6)$

In (e), we prove $\neg p, \neg q \vdash \gamma_l[q \to p]$ where $\gamma_l[q \to p] = q \to p$. A box proof is provided below

1	$\neg p$	premise
2	$\neg q$	premise
3	q	assumption
4	$\perp$	$\neg E(3,2)$
5	p	$\perp E(4)$
6	$q \to p$	$\to I(3-5)$

In (f), we prove $\neg p, \neg q \vdash \gamma_l[(q \to p) \to p]$ where $\gamma_l[(q \to p) \to p] = \neg((q \to p) \to p)$. A box proof is provided below

1	$\neg p$	premise
2	$\neg q$	premise
3	$(q \to p)$	proved in (e)
4	$(q \to p) \to p$	assumption
5	p	$\to E(3,4)$
6	$\perp$	$\neg E(5,1)$
7	$\neg((q \to p) \to p)$	$\neg I(4-6)$

In (g), we prove $\neg p, \neg q \vdash \gamma_l[((p \to q) \to q) \to ((q \to p) \to p)]$ where $\gamma_l[((p \to q) \to q) \to ((q \to p) \to p)] = ((p \to q) \to q) \to ((q \to p) \to p)$. A boxproof is provided below

1	p	premise
2	q	premise
3	$\neg((p \to q) \to q)$	proved in (d)
4	$\neg((q \to p) \to p)$	proved in (f)
5	$(p \to q) \to q$	assumption
6	$\bot$	$\neg E(5,3)$
7	$(q \to p) \to p$	$\bot E(6)$
8	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to I(5-7)$

These proofs $\alpha_1, \alpha_2, \alpha_3$ and α_4 together can be used to prove that

$$\vdash ((p \to q) \to q) \to ((q \to p) \to p)$$

as all combination of truth values are used.
We extend this to to the proof

$$(p \to q) \to q \vdash (q \to p) \to p$$

As so we have proved completeness of the formula.