Max Flows

Flow networks is a directed graph $G = (V, E)$ with a capacity function c. We have a source s and and a sink t. A *flow* is a function $f : V \times V \to \mathbb{R}$ that satisfies.

Capacity constraint: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$.
Flow conservation: For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

Maximum-flow problems are flow problems where we wish to find a flow of maximum value.
A **Residual network**, dentoted G_f, is a graphh (V, E_f) where edges represent how we can change the flow in G.
An **Augmentation** of a flow is a function $f \uparrow f'$ with value $|f \uparrow f'| = |f| + |f'|$. This value is actually stricly larger than $|f|$ (Corollary 26.3)

Ford-Fulkerson is an algorithm that simply keeps finding augmenting paths until there is no more. Since an augmenting path always increases the flow by at least one unit and we can find an augmenting path in $O(E)$ time (breadth first with right data structure), the running time is $\mathcal{O}(E|f^*|)$.

A cut (S, T) is a removal of edges so that the graph is split into two subgraphs. The net flow is the flow from S to T minus the flow from T to S. The capacity of a cut is the total capacity from S to T. A **Minimum cut** is a cut that minimizes the capacity of the cut.

Max-flow min-cut theorem

These are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

$(1 \Rightarrow 2)$: Suppose there is an augmenting path in G_f and f is a maximum flow, Corollary 26.3 says that the resulting flow value is stricly greater than f, which is a contradiction.
$(2 \Rightarrow 3)$: Suppose there are no augmenting paths in G_f. If we make a cut in the graph so nodes that can be reached from s in G_f is in S and let $T = V - S$. We now consider a pair $(u \in S, v \in T)$. If $(u, v) \in E$, then $f(u, v) = c(u, v)$ as otherwise the edge would exist in E_f and v would be a part of S. If $(v, u) \in E$, then $f(v, u) = 0$ since otherwise it would be positive and (u, v) would exist in E_f which places v in S, if neither is in EE then $f(u, v) = f(v, u) = 0$, thus

$$\begin{aligned} f(S, T) &= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{v \in T} \sum_{u \in S} f(v, u) \\ &= \sum_{u \in S} \sum_{v \in T} c(u, v) \\ &= c(S, T) \end{aligned}$$

By Lemma 26.4, we get $|f| = f(S, T) = c(S, T)$.
$(3 \Rightarrow 1)$: We have from Corollary 26.5 that any flow f is bounded above by any cut (S, T), thus when $|f| = c(S, T)$ means that f must be a maximum flow.

Edmonds-Karp is when we pick the augmented path as the shortest path from s to t. This runs in $\mathcal{O}(VE^2)$, since the number of augmententations is $\mathcal{O}(VE)$ and we can find one in $\mathcal{O}(E)$.