4.7.13

We have the following linear system

$$\begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

We carry out two iterations using the Jacobi method with our initial guess $x^{(0)} = (0, 0, 0)$.
This means to find $x^{(1)}$ we need to solve

$$\begin{bmatrix} 2x_1^{(1)} & 0 & 0 \\ 0 & -10x_2^{(1)} & 0 \\ 0 & 0 & 4x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Where 0 has replaced the coordinates that are multiplied with coordinates from $x^{(0)}$. We find that $x^{(1)} = (\frac{1}{2}, \frac{6}{5}, \frac{1}{2})$. With our new approximation of (x_1, x_2, x_3) we solve the following

$$\begin{bmatrix} 2x_1^{(2)} & 0x_2^{(1)} & -1x_3^{(1)} \\ -2x_1^{(1)} & -10x_2^{(2)} & 0x_3^{(1)} \\ -1x_1^{(1)} & -1x_2^{(1)} & 4x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Rewritten to

$$\begin{bmatrix} 2x_1^{(2)} & 0 & -\frac{1}{2} \\ -1 & -10x_2^{(2)} & 0 \\ -\frac{1}{2} & -\frac{6}{5} & 4x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Using Maple to calculate these simple equations we find $x^{(2)} = (\frac{3}{4}, \frac{11}{10}, \frac{37}{40})$.
We can see the convergence towards the actual solution $(1, 1, 1)$
Now we do two iteration of the Gauss Seidel method with the same initial guess. This means we need to solve

$$\begin{bmatrix} 2x_1^{(2)} & 0 & 0 \\ -2x_1^{(2)} & -10x_2^{(2)} & 0 \\ -1x_1^{(2)} & -1x_2^{(2)} & 4x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Where 0 has replaced the coordinates that are multiplied with coordinates from $x^{(0)}$. We find that $x^{(1)} = (\frac{1}{2}, \frac{11}{10}, \frac{9}{10})$. For the second iteration we need to solve the following

$$\begin{bmatrix} 2x_1^{(2)} & 0x_2^{(1)} & -1x_3^{(1)} \\ -2x_1^{(2)} & -10x_2^{(2)} & 0x_3^{(1)} \\ -1x_1^{(2)} & -1x_2^{(2)} & 4x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Rewritten to

$$\begin{bmatrix} 2x_1^{(2)} & 0 & -\frac{9}{10} \\ -2x_1^{(2)} & -10x_2^{(2)} & 0 \\ -x_1^{(2)} & -1x_2^{(2)} & 4x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

Using Maple, we find that $x^{(2)} = (\frac{19}{20}, \frac{101}{100}, \frac{99}{100})$.
This is close to the actual solution $(1, 1, 1)$
Finally, we want to do two iterations with the Conjugate gradient method. Our initial guess is the same. We start off by calculating the residual vector r_0 associated with x_0.

$$r_0 = b - Ax_0 = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$$

This is the first iteration, so we set our search direction $p_0 = r_0$. Now we compute the scalar α_0

$$\alpha_0 = \frac{r_0^T r_0}{p_0^T A p_0} = \frac{\begin{bmatrix} 1 & -12 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & -12 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}} = -\frac{149}{1378}$$

This means we can compute x_1 as

$$x_1 = x_0 + \alpha_0 p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \frac{149}{1378} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{149}{1378} \\ \frac{894}{689} \\ -\frac{149}{689} \end{bmatrix} \approx \begin{bmatrix} -0.11 \\ 1.30 \\ -0.22 \end{bmatrix}$$

Now we need to do the second iteration. We start by computing our new residual vector r_1 by the following formula

$$r_1 = r_0 - \alpha_0 A p_0 = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} + \frac{149}{1378} \begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{523}{689} \\ \frac{5587}{1378} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.76 \\ 4.05 \end{bmatrix}$$

Now we need to compute the next scalar β_0 so we can later find p_1. This is done by the following formula

$$\beta_0 = \frac{r_1^T r_1}{r_0^T r_0} = \frac{\begin{bmatrix} 1 & \frac{523}{689} & \frac{5587}{1378} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{523}{689} \\ \frac{5587}{1378} \end{bmatrix}}{\begin{bmatrix} 1 & -12 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}} = \frac{\frac{34207569}{1898884}}{149} \approx 0.12$$

This means we can now compute p_1 as

$$p_1 = r_1 + \beta_0 p_0 = \begin{bmatrix} 1 \\ \frac{523}{689} \\ \frac{5587}{1378} \end{bmatrix} + \frac{\frac{34207569}{1898884}}{149} \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2128465}{1898884} \\ -\frac{328396}{474721} \\ \frac{2039512}{474721} \end{bmatrix} \approx \begin{bmatrix} 1.12 \\ -0.69 \\ 4.30 \end{bmatrix}$$

Now that we found p_1 we can compute the new scalar α_1 and then finally find x_2.

$$\alpha_1 = \frac{r_1^T r_1}{p_1^T A p_1} = \frac{\begin{bmatrix} 1 & \frac{523}{689} & \frac{5587}{1378} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{523}{689} \\ \frac{5587}{1378} \end{bmatrix}}{\begin{bmatrix} \frac{2128465}{1898884} & -\frac{328396}{474721} & \frac{2039512}{474721} \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} \frac{2128465}{1898884} \\ -\frac{328396}{474721} \\ \frac{2039512}{474721} \end{bmatrix}}$$
$$= \frac{\frac{34207569}{1898884}}{\frac{173875590081}{2616662152}} \approx 0.27$$

And finally, we can find x_2 in the samme manner as x_1

$$x_2 = x_1 + \alpha_1 p_1 = \begin{bmatrix} -\frac{149}{1378} \\ \frac{894}{689} \\ -\frac{149}{689} \end{bmatrix} + \frac{\frac{34207569}{1898884}}{\frac{173875590081}{2616662152}} \begin{bmatrix} \frac{2128465}{1898884} \\ -\frac{328396}{474721} \\ \frac{2039512}{474721} \end{bmatrix} = \begin{bmatrix} \frac{117367462}{599574001} \\ \frac{665523174}{599574001} \\ \frac{568673243}{599574001} \end{bmatrix} \approx \begin{bmatrix} 0.20 \\ 1.11 \\ 0.95 \end{bmatrix}$$

The fractions may be a little heavy to look at, but we need to work with exact arithmetics or the values may vary greatly.
We see that x_2 converges closer to 1 in two of the coordinates but not the first one.