Randomized Algorithms
Assignment 2 - Resubmission

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Problem 3.1

When want to count the number of empty bins when we throw n balls into n bins, consider a single bin i. The probability that the bin is empty is equal to throwing the ball into another bin each of the n times:

$$\begin{aligned}\mathbb{P}[\text{bin } i \text{ is empty}] &= \left(\frac{n-1}{n}\right)^n \\ &= \left(1 - \frac{1}{n}\right)^n \\ &\approx 1/e && \text{(Approaches for } n \to \infty\text{)}\end{aligned}$$

Let Z_i be 1 if bin i is empty and 0 otherwise. Then we have exactly that:

$$\mathbb{E}[Z_i] = \mathbb{P}[\text{bin } i \text{ is empty}] \approx 1/e$$

Since every bin has the same probability of being empty, we can find the expected number of empty bins:

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] &= \sum_{i=1}^{n} \mathbb{E}[Z_i] && \text{(By linearity of expectation)} \\ &\approx \sum_{i=1}^{n} 1/e \\ &= n/e\end{aligned}$$

Which is what we wanted to show.
When we throw m balls into n bins, we can calculate the expected number of empty bins in the same manner. That is, look at the probability of bin i being empty (that we hit another bin m times):

$$\begin{aligned}\mathbb{P}[\text{bin } i \text{ is empty}] &= \left(\frac{n-1}{n}\right)^m \\ &= \left(1 - \frac{1}{n}\right)^m\end{aligned}$$

If we want to show an approximation as before, we can approximate the probability above to $\frac{1}{e^{m/n}}$. Then again, we let Z_i be 1 if bin i is empty and 0 otherwise and as before we can calculate the expected number of empty bins:

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] &= \sum_{i=1}^{n} \mathbb{E}[Z_i] && \text{(By linearity of expectation)} \\ &\approx \sum_{i=1}^{n} \frac{m1}{e^{m/n}} \\ &= ne^{-m/n}\end{aligned}$$

This probability hold generally. If we let $r = m/n$, where m are the balls we throw into the n bins. Then the number of empty bins will approximately be ne^{-r}.