Logic in Computer Science - Assignment 4

Exercise 3.2.2

a

We have a formula $\phi = G\, a$.

i

We find the following path that satisfies ϕ

$$q_3 \rightarrow q_4 \rightarrow q_3 \rightarrow q_4 \rightarrow \ldots$$

ii

It does ***not*** hold for all paths as

$$q_3 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

does not contain a in every node.

b

We have a formula $\phi = a\, U\, b$.

i

The following path satisfies ϕ

$$q_3 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

ii

It does ***not*** as we can pick a path

$$q_3 \rightarrow q_1 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

where a does not hold in q_1 and b does not hold either.

c

We have a formula $\phi = a\, U\, X\, (a \wedge \neg b)$.

i

The following path satisfies ϕ

$$q_3 \rightarrow q_4 \rightarrow q_3 \rightarrow q_4 \rightarrow \ldots$$

ii

It does ***not*** hold as we can go to either q_1 or q_2 and the path will never contain $X(a \wedge \neg b)$.

d

We have a formula $\phi = X \neg b \wedge G(\neg a \vee \neg b)$.

i

The following path satisfies ϕ

$$q_3 \rightarrow q_1 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

ii

It does ***not*** hold as we can pick the path

$$q_3 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

which will never contain $X \neg b$.

e

We have a formula $\phi = X(a \wedge b) \wedge F(\neg a \wedge \neg b)$.

i

The following path satisfies ϕ

$$q_3 \rightarrow q_4 \rightarrow q_3 \rightarrow q_1 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

ii

It does ***not*** hold as we can pick a path

$$q_3 \rightarrow q_2 \rightarrow q_2 \rightarrow \ldots$$

And it will neither contain $X(a \wedge b)$ or $F(\neg a \wedge \neg b)$ and both are required.

Exercise 2.3.7

To prove the equivalence $\phi\, W\, \psi \wedge F\, \psi \equiv \phi\, U\, \psi$ we suppose there exists a path that satisfies $\phi\, W\, \psi \wedge F\, \psi$. Then, from clause 5, we know that

$$\pi \models F\, \psi \tag{1}$$
$$\pi \models \phi\, W\, \psi \tag{2}$$

Using (1) with clause 10, we have that *for some* $i \geq 1$ *that* $\pi^i \models \psi$.
Since the above applies, this means using (2) with clause 12 we have that *for some* $i \geq 1$ *we have* $\pi^i \models \psi$ *and for* $j = 1, .., i-1$ *we have* $\pi^j \models \phi$.
Combining what we have found above, we can use clause 11 to show $\pi \models \phi\, U\, \psi$ as the two conditions needed is what we have found above.

PCP Exercise

Exercise 1

ProofWeb has been used for this task. The code can be seen below and it is also uploaded separately in a `.txt` file for raw text.

```
Require Import ProofWeb.

Variable P : D * D -> Prop.
Variable f0 f1 : D -> D.
Variable e : D.

Theorem PCP_ex_simp_b :
(P(f1 e,f1(f0(f1 e))) /\
 P(f0(f1 e), f0(f0 e)) /\
 P(f1(f1(f0 e)),f1(f1 e)))
->
((all v, all w, (P(v,w) -> P(f1 v,f1(f0(f1 w))))) /\
 (all v, all w, (P(v,w) -> P(f0(f1 v), f0(f0 w)))) /\
 (all v, all w, (P(v,w) -> P(f1(f1(f0 v)),f1(f1 w)))))
->
exi z, P(z,z).

Proof.
imp_i H1.
imp_i H2.
insert C1 ((all v, all w, (P(v,w) -> P(f0(f1 v), f0(f0 w)))) /\
            (all v, all w, (P(v,w) -> P(f1(f1(f0 v)),f1(f1 w))))).
f_con_e2 H2.
exi_i (f1(f1(f0(f0(f1(f1(f1(f0(f1 e))))))))).
imp_e (P(f0(f1(f1(f1(f0(f1 e))))), (f0(f0(f1(f1(f1(f0(f1 e))))))))).
all_e (all w, (P(f0(f1(f1(f1(f0(f1 e))))), w) ->
                P(f1(f1(f0(f0(f1(f1(f1(f0(f1 e)))))))), f1(f1 w)))).
all_e (all v, all w, (P(v,w) -> P(f1(f1(f0 v)),f1(f1 w)))).
f_con_e2 C1.
imp_e (P((f1(f1(f0(f1 e)))), (f1(f1(f1(f0(f1 e))))))).
all_e (all w, (P((f1(f1(f0(f1 e)))), w) -> P(f0(f1(f1(f1(f0(f1 e))))), f0(f0 w)))).
all_e (all v, all w, (P(v,w) -> P(f0(f1 v), f0(f0 w)))).
f_con_e1 C1.
imp_e (P((f1 e), (f1(f0(f1 e))))).
all_e (all w, (P((f1 e), w) -> P (f1(f1(f0((f1 e)))), f1(f1 w)))).
all_e (all v, all w, (P(v,w) -> P(f1(f1(f0 v)),f1(f1 w)))).
f_con_e2 C1.
f_con_e1 H1.

Qed.
```

Exercise 2

We want to find the predicate logic for the PCP instance

$$((001, 0), (01, 011), (01, 101), (10, 001))$$

This means we write,from [H&R, page 134], $\psi = \phi_1 \wedge \phi_2 \rightarrow \phi_3$ where we find ϕ_1 to

$$\begin{aligned}
\phi_1 &\stackrel{def}{=} \bigwedge_{i=1}^{k} P(f_{s_i}(e), f_{t_i}(e)) \\
&= (P(f_{001}(e), f_0(e)) \wedge P(f_{01}(e), f_{011}(e)) \wedge P(f_{01}(e), f_{101}(e)) \wedge P(f_{10}(e), f_{001}(e)))
\end{aligned}$$

And ϕ_2 is found to be

$$\begin{aligned}
\phi_2 &\stackrel{def}{=} \forall v \forall w (P(v, w) \rightarrow \bigwedge_{i=1}^{k} P(f_{s_i}(v), f_{t_i}(w))) \\
&= \forall v \forall w (P(v, w) \rightarrow (P(f_{001}(v), f_0(w)) \wedge P(f_{01}(v), f_{011}(w)) \wedge P(f_{01}(v), f_{101}(w)) \\
&\qquad \wedge P(f_{10}(v), f_{001}(w)))
\end{aligned}$$

And lastly ϕ_3 is defined to be

$$\phi_3 \stackrel{def}{=} \exists z P(z, z)$$

Which defines our predicate logic formula ψ.