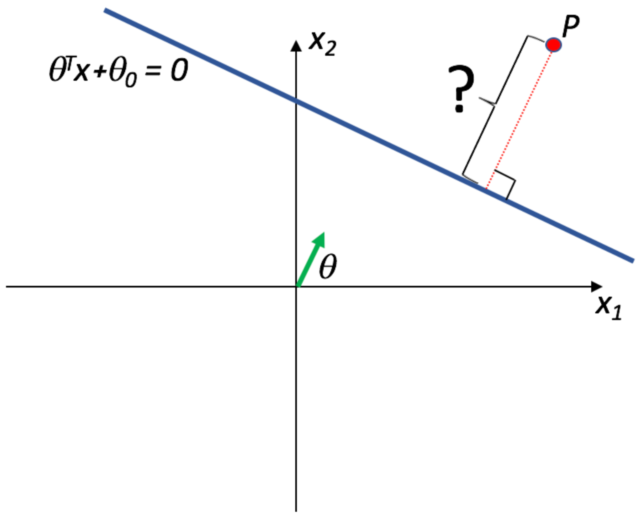
1) Numpy procedures for hyperplanes and separators

Relevant material on linear classifiers in the notes
Helpful numpy explanations at the bottom of the page.

1.1) General hyperplane, distance to point

Let p be an arbitrary point in R^d. Give a formula for the **signed** perpendicular distance from the hyperplane specified by θ, θ_0 to this point p.



Enter your answer as a Python expression. Use `theta` for θ, `theta_0` for θ_0, `p` for the point p, `transpose(x)` for transpose of an array, `norm(x)` for the length (L2-norm) of a vector, and `x@y` to indicate a matrix product of two arrays.

Formula for signed distance: ((transpose(theta)@p)+theta_0)/norm(theta)

Ask for Help **100.00%**

You have infinitely many submissions remaining.

Multiple Possible Solutions:

Solution 1: `(transpose(theta)@p + theta_0) / norm(theta)`

$$\frac{(\theta)^T \mathbf{p} + \theta_0}{\|\theta\|}$$

Solution 2: `(transpose(p)@theta + theta_0) / norm(theta)`

$$\frac{(\mathbf{p})^T \theta + \theta_0}{\|\theta\|}$$

Explanation:

Consider the proof for the equation of signed distance from origin to hyperplane in exercise 1. Instead of representing the projected vector as $\langle 0, 0 \rangle$ - x, where x was our random point on the line, the projected vector's may be represented by $p - x$, where p is the point in query.

The length of the projection is then:

$$\frac{\theta^T(p-x)}{\|\theta\|}$$

Distributing the dot product, we arrive at:

$$\frac{\theta^T p - \theta^T x}{\|\theta\|}$$

Performing a similar substitution to 1.2, $-\theta^T x$ is equal to θ_0 by the equation for the hyperplane $\theta^T p + \theta_0 = 0$. Thus, the equation for the distance of a point to a hyperplane is the following:

$$\frac{\theta^T p + \theta_0}{\|\theta\|}$$

1.2) Code for signed distance!

Write a Python function using numpy operations (no loops!) that takes column vectors (d by 1) `x` and `th` (of the same dimension) and scalar `th0` and returns the signed perpendicular distance (as a 1 by 1 array) from the hyperplane encoded by `(th, th0)` to `x`. Note that you are allowed to use the "length" function defined in previous coding questions (includig week 1 exercises).

```
import numpy as np
def signed_dist(x, th, th0):
    return (th.T.dot(x)+th0)/length(th)
```

Ask for Help **100.00%**

You have infinitely many submissions remaining.

Here is the solution we wrote:

```
import numpy as np
# x is dimension d by 1
# th is dimension d by 1
# th0 is a scalar
# return 1 by 1 matrix of signed distance
def signed_dist(x, th, th0):
   return ((th.T@x) + th0) / length(th)
```

Explanation:

First, recall from Problem 1.3 the formula for the signed perpendicular distance of a general hyperplane defined by θ, θ_0 to a point x:

$$\frac{\theta^T x + \theta_0}{||\theta||}$$

In order to code this, we can think of it as a 3-step process

1. Matrix multiply the transpose of th and x (this can be done as np.dot(np.transpose(th), x), np.matmul(np.transpose(th), x), or np.transpose(th)@x but NOT as np.transpose*x, which is element-wise multiplication
2. Add th0 to this product (+ th0)
3. Divide the entire thing by the norm of th (length(th) as defined in 2.6). Make sure you divide the *entire* sum by length(th), not just th0!

Putting these all together gives us our desired solution: (np.dot(np.transpose(th), x) + th0)/length(th))

1.3) Code for side of hyperplane

Write a Python function that takes as input

- a column vector `x`
- a column vector `th` that is of the same dimension as `x`
- a scalar `th0`

and returns

- `+1` if `x` is on the positive side of the hyperplane encoded by `(th, th0)`
- `0` if on the hyperplane
- `-1` otherwise.

The answer should be a 2D array (a 1 by 1). Look at the *sign* function. Note that you are allowed to use any functions defined in week 1's exercises.

```python
import numpy as np
def positive(x, th, th0):
    return np.sign((th.T.dot(x)+th0)/length(th))

```

Ask for Help **100.00%**

You have infinitely many submissions remaining.

Here is the solution we wrote:

```python
import numpy as np
# x is dimension d by 1
# th is dimension d by 1
# th0 is dimension 1 by 1
# return 1 by 1 matrix of +1, 0, -1
def positive(x, th, th0):
   return np.sign(np.dot(np.transpose(th), x) + th0)
```

Explanation:

First, recall the formula for how we determine which side of the hyperplane defined by θ, θ_0 a point x lies on:

$$\text{sign}\left(\theta^T x + \theta_0\right)$$

The expression inside the sign() function can be coded the same way we did in the previous problem, leading to our desired solution: np.sign(np.dot(np.transpose(θ), x) + θ_0).

Another clever way to solve this problem uses the signed_distance function from the previous problem. Note that the expression inside the sign() function above is equal to $||\theta||$ times the signed perpendicular distance from the previous problem. Thus, we could write our solution as np.sign(signed_dist(x, θ, θ_0)*length(θ)). However, length(θ) is a positive scalar, so it doesn't actually affect which side of the hyperplane x lies on, so we can remove that term entirely, leading to our solution np.sign(signed_dist(x, θ, θ_0)).

Now, given a hyperplane and a set of data points, we can think about which points are on which side of the hyperplane. This is something we do in many machine-learning algorithms, as we will explore soon. It is also a chance to begin using numpy on larger chunks of data.

1.4) Expressions operating on data

We define `data` to be a 2 by 5 array (two rows, five columns) of scalars. It represents 5 data points in two dimensions. We also define `labels` to be a 1 by 5 array (1 row, five columns) of 1 and -1 values.

```
data = np.transpose(np.array([[1, 2], [1, 3], [2, 1], [1, -1], [2, -1]]))
labels = rv([-1, -1, +1, +1, +1])
```

For each subproblem, provide a Python expression that sets `A` to the quantity specified. Note that `A` should always be a 2D numpy array. Only one relatively short expression is needed for each one. No loops!

You can use (our version) of the `length` and `positive` functions; they are already defined, don't paste in your definitions. Those functions if written purely as matrix operations should work with a 2D data array, not just a single column vector as the first argument, with no change.

1. `A` should be a 1 by 5 array of values, either +1, 0 or -1, indicating, for each point in `data`, whether it is on the positive side of the hyperplane defined by `th, th0`. **Use `data, th, th0` as variables in your submission.**

```
1 import numpy as np
2 A = positive(data, th, th0)
```

Run Code | Submit | View Answer | Ask for Help **100.00%**

You have infinitely many submissions remaining.

2. `A` should be a 1 by 5 array of boolean values, either `True` or `False`, indicating for each point in `data` and corresponding label in `labels` whether it is correctly classified by hyperplane `th = [1, 1], th0 = -2` . That is, return `True` when the side of the hyperplane (specified by θ, θ_0) that the point is on agrees with the specified label.

```
import numpy as np
A = positive(data, th, th0) == labels
```

Ask for Help 100.00%

You have infinitely many submissions remaining.

Here is the solution we wrote:

```
import numpy as np
A = (labels == positive(data, cv([1, 1]), -2))
```

Explanation:

We want to compare the values of the labels in $labels$ to their corresponding values calculated in our previous problem. We can do this element-wise comparison in numpy in two ways:

1. (labels == positive(data, cv([1,1]), -2))
2. Using the *equal* operator in numpy: np.equal(labels, positive(data, cv([1,1]), -2))

1.5) Score

Write a procedure that takes as input

- `data`: a `d` by `n` array of floats (representing `n` data points in `d` dimensions)
- `labels`: a `1` by `n` array of elements in `(+1, -1)`, representing target labels
- `th`: a `d` by `1` array of floats that together with
- `th0`: a single scalar or 1 by 1 array, represents a hyperplane

and returns the number of points for which the label is equal to the output of the `positive` function on the point.

Since `numpy` treats `False` as 0 and `True` as 1, you can take the sum of a collection of Boolean values directly.

```
import numpy as np
def score(data, labels, th, th0):
    return np.sum(labels == positive(data,th,th0))

```

Run Code Submit View Answer Ask for Help 100.00%

You have infinitely many submissions remaining.

1.6) Best separator

Now assume that we have some "candidate" classifiers that we want to pick the best one out of. Assume you have `ths`, a d by m array of m candidate θ's (each θ has dimension d by 1), and `th0s`, a 1 by m array of the corresponding m candidate θ_0's. Each of the θ, θ_0 pair represents a hyperplane that characterizes a binary classifier.

Write a procedure that takes as input

- `data`: a `d` by `n` array of floats (representing `n` data points in `d` dimensions)
- `labels`: a `1` by `n` array of elements in `(+1, -1)`, representing target labels
- `ths`: a `d` by `m` array of floats representing m candidate θ's (each θ has dimension d by 1)
- `th0s`: a `1` by `m` array of the corresponding m candidate θ_0's.

and finds the hyperplane with the highest score on the data and labels. In case of a tie, return the first hyperplane with the highest score, in the form of

- a tuple of a `d` by 1 array and an offset in the form of 1 by 1 array.

The function `score` that you wrote above was for a single hyperplane separator. Think about how to generalize it to multiple hyperplanes and include this modified (if necessary) definition of `score` in the answer box.

Note: Look below the answer box for useful numpy functions!

```
import numpy as np

def best_separator(data, labels, ths, th0s):
    best_index = np.argmax(np.sum((np.sign(ths.T.dot(data)+th0s.T)==labels), axi
    return (np.array([ths.T[best_index]]).T, np.array([th0s.T[best_index]]))
```

Ask for Help **100.00%**

You have infinitely many submissions remaining.

Here is the solution we wrote:

```
import numpy as np
# data is dimension d by n
# labels is dimension 1 by n
# ths is dimension d by m
# th0s is dimension 1 by m
# return matrix of integers indicating number of data points correct for
# each separator:  dimension m x 1
def score_mat(data, labels, ths, th0s):
   pos = np.sign(np.dot(np.transpose(ths), data) + np.transpose(th0s))
   return np.sum(pos == labels, axis = 1, keepdims = True)
def best_separator(data, labels, ths, th0s):
   best_index = np.argmax(score_mat(data, labels, ths, th0s))
   return cv(ths[:,best_index]), th0s[:,best_index:best_index+1]
```

Explanation:

First, let's break up the best classifier problem into three subproblems:

1. Extend the score() function to a second dimension, allowing us to generate scores for multiple hyperplanes.
2. Apply this new score() function to the data and the array of hyperplanes and select the best score (across the second dimension).
3. Once we've found the best score (or the index of the best score), use that to return the correspondingly best (θ, θ_0) hyperplane parameters.

Let's tackle each subproblem in order:

To extend the score() function to generate scores for multiple hyperplanes, we can start by using the same expression in 3.3.1 to generate an $m \times n$ array of 1, 0, or -1 values corresponding to how each hyperplane classifies each point:

pos = positive(data, ths, th0s.T)

Be careful of dimension matching: np.dot(np.transpose(ths), data) has dimensions $m \times n$ and th0s has dimensions $1 \times m$

Now that we've generated an array of classification values, we can compare them to the label values the same way we did in problem 3.3.2, using (pos == labels) or np.equal(pos, labels). Since the second dimension of the two arrays are both n, there's no danger of dimension mismatch (although, if you like, you can create m copies of $labels$ and tile them along the first dimension using the $np.tile$ function) - this will do an element-wise comparison over the first dimension.

Finally, we want to sum these over the second dimension to create a $m \times 1$ array of scores corresponding to each hyperplane. We can achieve this with $np.sum()$ in the following way:

score_mat = np.sum((pos == labels), axis=1, keep_dims=True)

Two important things to keep note of here: first, we need to sum over $only$ the second dimension, so we need to use the axis parameter so that we only reduce the second dimension. Second, np.sum() will remove all dimensions we sum over, so just writing np.sum((pos == labels), axis=1) will return a 1-D array. We still want a 2-D array, so we set the keep_dims flag to True so the axis we sum over is left as a dimension of size 1.

Now that we have a matrix corresponding to each hyperplane's score on classifying $data$, we want to then find the highest score and its corresponding hyperplane. Note that we don't actually care about the value of the highest score, just the index so we can select the corresponding values in ths and th0s. To do this, we can use the $np.argmax()$ function as such:

best_index = np.argmax(score_mat)

Finally, we can select the corresponding θ and θ_0 from ths and th0s, remembering to select along the second dimension and convert the final θ into a column vector as we did in 3.3.1:

return cv(ths[:, best_index]), th0s[:, best_index]

Reference Material: Handy Numpy Functions and Their Usage

In order to avoid using for loops, you will need to use the following numpy functions. (So that you replace for loops with matrix operations)

A. `np.sum` with axis

`np.sum` can take an optional argument `axis`. Axis 0 is row and 1 is column in a 2D numpy array. **The way to understand the "axis" of numpy sum is that it sums(collapses) the given matrix in the direction of the specified axis. So when it collapses the axis 0 (row), it becomes just one row and column-wise sum.** Let's look at examples.

```
>>> np.sum(np.array([[1,1,1],[2,2,2]]), axis=1)
array([3, 6])
>>> np.sum(np.array([[1,1,1],[2,2,2]]), axis=0)
array([3, 3, 3])
```

Note that `axis=1` (column) will "squash" (or collapse) sum `np.array([[1,1,1],[2,2,2]])` in the column direction. On the other hand, `axis=0` (row) will collapse-sum `np.array([[1,1,1],[2,2,2]])` in the row direction.

B. Comparing matrices of different dimensions / advanced np.sum

Note that two matrices `A, B` below have same number of columns but different row dimensions.

```
>>> A = np.array([[1,1,1],[2,2,2],[3,3,3]])
>>> B = np.array([[1,2,3]])
>>> A==B
array([[ True, False, False],
       [False,  True, False],
       [False, False,  True]])
```

The operation `A==B` copies `B` three times row-wise so that it matches the dimension of `A` and then element-wise compaires `A` and `B`.

We can apply `A==B` to np.sum like below.

```
>>> A = np.array([[1,1,1],[2,2,2],[3,3,3]])
>>> B = np.array([[1,0,0],[2,2,0],[3,3,3]])
>>> np.sum(A==B, axis=1)
array([1, 2, 3])
```

C. np.sign

`np.sign`, given a numpy array as input, outputs a numpy array of the same dimension such that its element is the sign of each element of the input. Let's look at an example.

```
>>> np.sign(np.array([-3,0,5]))
array([-1,  0,  1])
```

D. np.argmax

`np.argmax`, given a numpy array as input, outputs the index of the maximum element of the input. Let's look at an example.

```
>>> np.argmax(np.array([[1,2,3],[4,5,6]]))
5
```

Note that the argmax index is given assuming the input array is flattened. So in our case, with 6 being the maximum element, 5 was returned instead of something like (1,2).

E. np.reshape

For a np array `A`, you can call `A.reshape((dim1_size,dim2_size,...))` in order to change the shape of the array.

```
>>> A = np.array([[1,2,3],[4,5,6]])
>>> A.reshape((3,2))
array([[1, 2],
       [3, 4],
       [5, 6]])
```

Note, the new shape has to have the same number of elements as the original.