Solution to the Heat Equation in Polar Coordinates for a Circular Disk

1. Introduction

The primary goal of this work is to deduce the final form of the temperature distribution as a series solution involving Bessel functions, which are well-suited to problems in cylindrical or spherical coordinates.

In many situation we often find that objects are symmetrical in nature, consider a circular pan, say you want to find the temperature distribution and variation of it over time and space, this requires solving heat equation, but... **Wait!** Consider heat equation, and now at boundaries it takes lot of effort to solve for this equation with **MANY CONDITIONS!!**

Now let's consider solving it in the Polar Co-ordinates Space

2. The Heat Equation in Polar Coordinates

The heat equation in polar coordinates (assuming no angular dependence due to radial symmetry) is:

$$\frac{\partial u(r,t)}{\partial t} = \alpha \left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} \right)$$

where:

- $u(r,t)$ is the temperature at radial position r and time t,
- α is the thermal diffusivity of the material,
- r is the radial distance from the center of the disk.

This form of the heat equation accounts for heat flow in the radial direction, assuming the temperature is uniform along any circle centered at $r = 0$.

3. Boundary and Initial Conditions

To fully define the problem, we specify:

1. **Boundary condition**: At the boundary of the disk (radius R), the temperature is held constant at T_0. Thus,

   $$u(R,t) = T_0, \quad \text{for all } t \geq 0.$$

   In many practical situations, T_0 is taken to be 0 (e.g., a cooling disk with a constant temperature maintained at the boundary).

2. **Initial condition**: The temperature distribution at $t = 0$ is prescribed by a function $f(r)$, giving the initial temperature throughout the disk:

   $$u(r,0) = f(r), \quad 0 \leq r \leq R.$$

4. Method of Solution: Separation of Variables

To solve this PDE, we use the method of separation of variables. We assume the solution can be written as the product of two functions:

$$u(r,t) = R(r)T(t).$$

Substituting this into the heat equation gives:

$$R(r)\frac{dT(t)}{dt} = \alpha T(t) \left(\frac{d^2 R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} \right).$$

Dividing through by $\alpha R(r)T(t)$, we obtain:

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{\alpha}{R(r)}\left(\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr}\right) = -\lambda,$$

where λ is a separation constant. This gives two ordinary differential equations (ODEs):

- Temporal equation: $\frac{dT(t)}{dt} + \lambda\alpha T(t) = 0$,
- Spatial equation: $r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + \lambda r^2 R(r) = 0$.

5. Solution to the Temporal Equation

The temporal ODE is straightforward to solve:

$$\frac{dT(t)}{dt} + \lambda\alpha T(t) = 0.$$

This is a first-order linear differential equation, and its solution is:

$$T(t) = Ce^{-\lambda\alpha t},$$

where C is a constant to be determined later based on conditions we have.

6. Solution to the Spatial Equation: Bessel's Equation

The spatial ODE is a form of Bessel's equation:

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + (\lambda r^2)R(r) = 0.$$

The general solution to this equation is given by:

$$R(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r),$$

where J_0 and Y_0 are Bessel functions of the first and second kind, respectively. Since $Y_0(\sqrt{\lambda}r)$ is singular at $r = 0$, we discard it for a physically meaningful solution, because the temperature can't tend to infinite at point of time. Therefore, the solution reduces to:

$$R(r) = AJ_0(\sqrt{\lambda}r).$$

7. Applying the Boundary Conditions

We now apply the boundary condition $u(R, t) = T_0$. This gives:

$$R(R) = AJ_0(\sqrt{\lambda}R) = T_0.$$

For nontrivial solutions (i.e., $T_0 = 0$), the argument $\sqrt{\lambda}R$ must be a zero of J_0, denoted by β_n, the n-th zero of the Bessel function J_0. Therefore,

$$\lambda_n = \left(\frac{\beta_n}{R}\right)^2.$$

Thus, the solution is indexed by these values λ_n.

*(Else in genral case where T_0 not necessarily be zero, we have zero of Bessel function of some Non-Zero order)

8. General Solution: Bessel-Fourier Series

The general solution to the heat equation is then a sum over all possible modes:

$$u(r, t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\left(\frac{\beta_n}{R}\right)^2 t} J_0\left(\frac{\beta_n r}{R}\right),$$

where A_n are the Fourier-Bessel coefficients, determined by the initial condition.

A Small Review Into Strun-Liouville Theorem

The coefficients A_n in the Bessel-Fourier series can be determined using the orthogonality of the Bessel functions, which is a consequence of the Sturm-Liouville theorem. This theorem states that for any second-order linear differential equation of the form:

$$\frac{d}{dr}\left(p(r)\frac{dR}{dr}\right) + \left(\lambda w(r) - q(r)\right) R = 0,$$

the eigenfunctions corresponding to distinct eigenvalues λ_n are orthogonal with respect to the weight function $w(r)$. In our case, the Bessel equation can be written in Sturm-Liouville form, where the weight function is $w(r) = r$. Thus, the Bessel functions $J_0\left(\frac{\beta_n r}{R}\right)$ are orthogonal with respect to the inner product:

$$\int_0^R r J_0\left(\frac{\beta_n r}{R}\right) J_0\left(\frac{\beta_m r}{R}\right) dr = 0 \quad \text{for} \quad n \neq m.$$

Using this orthogonality property, the coefficients A_n can be found by projecting the initial temperature distribution $f(r)$ onto the eigenfunctions $J_0\left(\frac{\beta_n r}{R}\right)$ as follows:

$$A_n = \frac{2}{R^2 J_1^2(\beta_n)} \int_0^R r f(r) J_0\left(\frac{\beta_n r}{R}\right) dr,$$

where J_1 is the first derivative of the Bessel function J_0 and β_n are the zeros of J_0. This integral determines how much of each Bessel mode contributes to the initial condition, leading to the full solution as a sum of these modes weighted by the coefficients A_n.

9. Determining the Coefficients A_n

Now The coefficients A_n are found by projecting the initial condition $f(r)$ onto the basis functions $J_0\left(\frac{\beta_n r}{R}\right)$:

$$A_n = \frac{2}{R^2 J_1^2(\beta_n)} \int_0^R r f(r) J_0\left(\frac{\beta_n r}{R}\right) dr,$$

where $J_1(\beta_n)$ is the derivative of J_0 evaluated at β_n.

10. Conclusion

This example demonstrates the Simplicity of solving Heat diffusion differential equation using Polar co-ordinates which otherwise in cartesian co-ordinates makes us to look at many conditions at boundary!

At the same time it turn out that the solutions are summations of Bessel-Fourier Basis and also it very Simple to calculate coefficients of those basis due to orthogonality of the Bessel Functions of same order.

But This is just one of the applications of Bessel-Fourier Series, there are many intresting application of this Series. Invoking the properties of Completeness and Orthogonality of these Series, It is easier to represent functions which have Radial or Cylindrical Symmetries! in this Basis.

One of the interesting applications where these are used is for reconstruction of MRI Images (because human body has Radial Symmetry!!) But that is another Story to be explored!

References

[1] **"Bessel Functions"** Wikepedia

[2] **"Bessel Functions and 2D Problems"** LibreTexts Mathematics

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