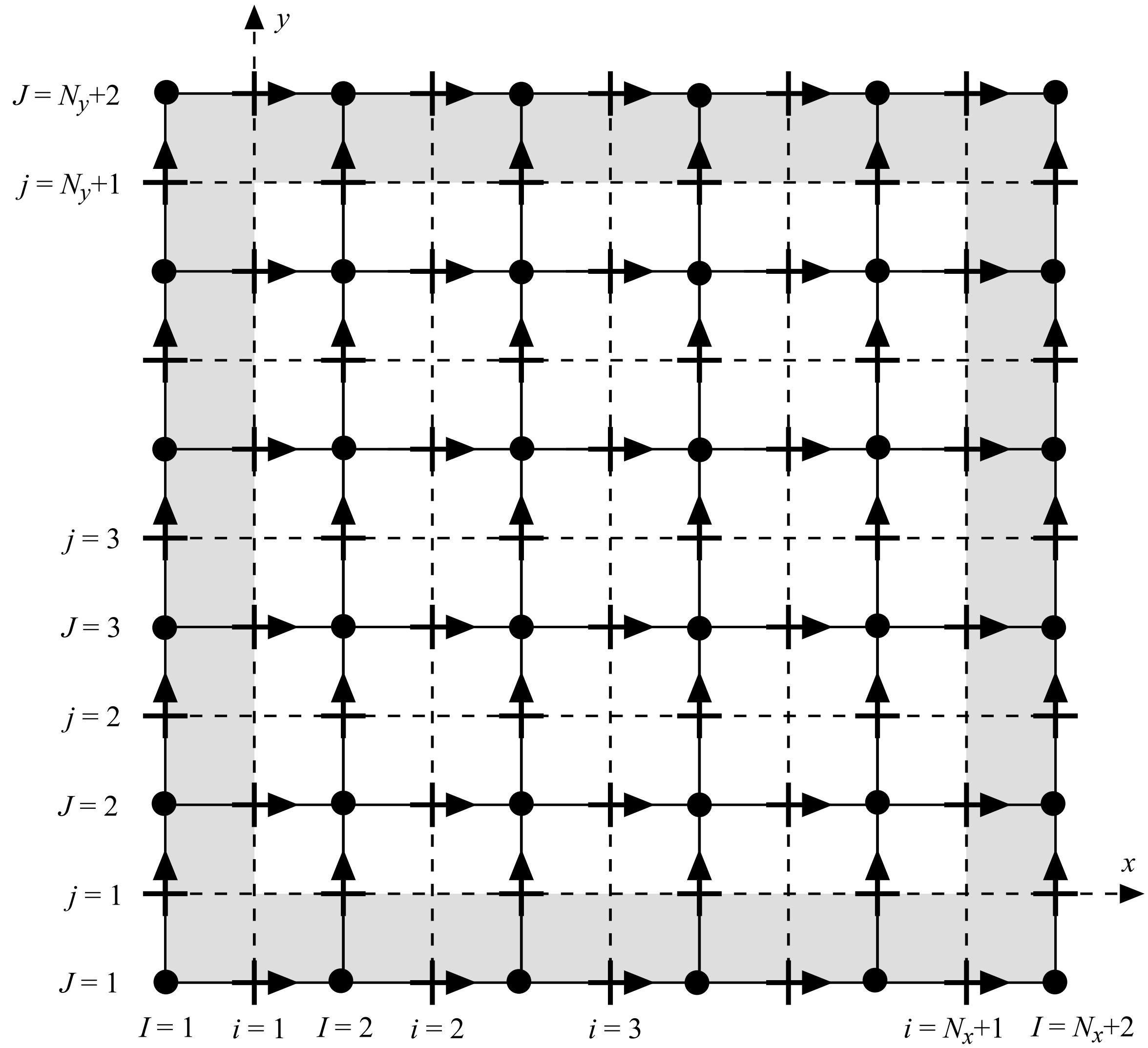
Staggered Grid for SIMPLE with Main Nodes Surrounding Domain:



Nomenclature for Grid Geometry:

I, J main grid points for scalar properties $\phi(I,J)$, $I = 1,2,\ldots N_x + 2$ and $J = 1,2,\ldots N_y + 2$

i, j staggered grid points for $u(i,J)$ and $v(I,j)$, $i = 1,2,\ldots N_x + 1$ and $j = 1,2,\ldots N_y + 1$

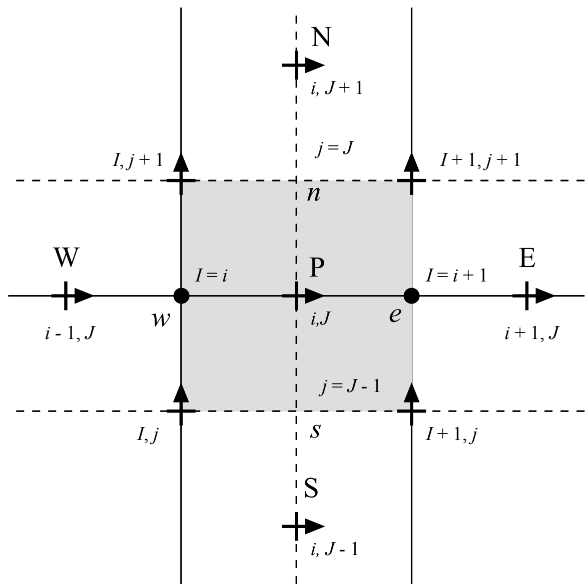
L_x	domain length in x-direction	N_x	number of main grid points in x-direction
L_y	domain length in y-direction	N_y	number of main grid points in y-direction

$0 \le x \le L_x$, $dx = L_x / N_x$, $x = (i-1)dx = (I-3/2)dx$

$0 \le y \le L_y$, $dy = L_y / N_y$, $y = (j-1)dy = (J-3/2)dy$

Derivation of *x*-Momentum Equation for Interior Node:

u **- velocity cell**



Recall steady *x*-momentum equation for finite volume formulation:

$$\sum_{f=1}^{N_f} u_f \left(\rho \vec{V} \cdot \vec{n}\, A\right)_f = \sum_{f=1}^{N_f} \left(\mu \nabla u \cdot \vec{n}\, A\right)_f - \left(\frac{\partial p}{\partial x}\right)_f \cancel{V}$$

For two-dimensional domain for interior elements aligned with the *x* and *y* axis

$$u_e\left(\rho u A\right)_e - u_w\left(\rho u A\right)_w + u_n\left(\rho u A\right)_n - u_s\left(\rho u A\right)_s =$$
$$\left(\mu \frac{\partial u}{\partial x} A\right)_e - \left(\mu \frac{\partial u}{\partial x} A\right)_w + \left(\mu \frac{\partial u}{\partial y} A\right)_n - \left(\mu \frac{\partial u}{\partial y} A\right)_s + \left(p_w - p_e\right) dy\, dz$$

Define the following advection (or *flow*) *coefficient*, *F*, and *diffusion coefficient*, *D*, at the faces where the gradient is evaluated using 2-point central finite difference approximation:

$F_w = \left(\rho u A\right)_w = \rho\, dy\, dz\left[u\left(i-1,J\right) + u\left(i,J\right)\right]/2$	$D_w = \left(\mu/dx\right)_w A_w = \left(\mu/dx\right)_w dy\, dz$
$F_e = \left(\rho u A\right)_e = \rho\, dy\, dz\left[u\left(i,J\right) + u\left(i+1,J\right)\right]/2$	$D_e = \left(\mu/dx\right)_e A_e = \left(\mu/dx\right)_e dy\, dz$
$F_s = \left(\rho v A\right)_s = \rho\, dx\, dz\left[v\left(I,j\right) + v\left(I+1,j\right)\right]/2$	$D_s = \left(\mu / dy\right)_s A_s = \left(\mu / dy\right)_s dx\, dz$
$F_n = \left(\rho v A\right)_n = \rho\, dx\, dz\left[v\left(I,j+1\right) + v\left(I+1,j+1\right)\right]/2$	$D_n = \left(\mu/dy\right)_n A_n = \left(\mu/dy\right)_n dx\, dz$

NOTE: For constant properties: $D_x = D_w = D_e = (\mu/dx)\,dy\,dz$, $D_y = D_s = D_n = (\mu/dy)\,dx\,dz$

$$F_e u_e - F_w u_w + F_n u_n - F_s u_s = D_e(u_E - u_P) - D_w(u_P - u_W) + D_n(u_N - u_P) - D_s(u_P - u_S) + (p_w - p_e)\,dy\,dz$$

Evaluate properties at the faces using the *hybrid differencing scheme*.

$Pe_f > 2$	$Pe_f < -2$	$-2 \le Pe_f \le 2$
$u_w = u_W$, $D_w = 0$	$u_w = u_P$, $D_w = 0$	$u_w = (u_W + u_P)/2$
$u_e = u_P$, $D_e = 0$	$u_e = u_E$, $D_e = 0$	$u_e = (u_E + u_P)/2$
$u_s = u_S$, $D_s = 0$	$u_s = u_P$, $D_s = 0$	$u_s = (u_S + u_P)/2$
$u_n = u_P$, $D_n = 0$	$u_n = u_N$, $D_n = 0$	$u_n = (u_N + u_P)/2$

where *Pe* is the Peclet number (ratio of advection to diffusion rate). At high *Pe* number this is 1st order upwinding with no diffusion. For moderate *Pe* this is central differencing.

$$Pe_f = \frac{(\rho u)_f}{(\mu / dx)_f} = \frac{F_f}{D_f}$$

For $2 \le Pe \le -2$ for all faces we get

$$\frac{F_e}{2}(u_E + u_P) - \frac{F_w}{2}(u_W + u_P) + \frac{F_n}{2}(u_N + u_P) - \frac{F_s}{2}(u_S + u_P) = D_e(u_E - u_P) - D_w(u_P - u_W) + D_n(u_N - u_P) - D_s(u_P - u_S) + (p_w - p_e)\,dy\,dz$$

$$\left(\frac{F_e}{2} + D_e - \frac{F_w}{2} + D_w + \frac{F_n}{2} + D_n - \frac{F_s}{2} + D_s\right)u_P = \left(D_e - \frac{F_e}{2}\right)u_E + \left(D_w + \frac{F_w}{2}\right)u_W + \left(D_n - \frac{F_n}{2}\right)u_N + \left(D_s + \frac{F_s}{2}\right)u_S + (p_w - p_e)\,dy\,dz$$

Rewrite as

$$a_P u_P = a_W u_W + a_E u_E + a_S u_S + a_N u_N + b$$

$$a_E = D_e - F_e/2\ ,\quad a_W = D_w + F_w/2\ ,\quad a_N = D_n - F_n/2\ ,\quad a_S = D_s + F_s/2$$

$$a_P = a_W + a_E + a_S + a_N + \Delta F\ ,\quad \Delta F = F_e - F_w + F_n - F_s\ ,\quad b = (p_w - p_e)\,dy\,dz$$

For $Pe > 2$ for all faces we get

$$F_e u_P - F_w u_W + F_n u_P - F_s u_S = \left(p_w - p_e\right) dy\, dz$$

$$\left(F_e + F_n\right)\phi_P = F_w \phi_W + F_s \phi_S + \left(p_w - p_e\right) dy\, dz$$

$$a_E = 0 \ , \quad a_W = F_w \ , \quad a_N = 0 \ , \quad a_S = F_s$$

For $Pe < -2$ for all faces we get

$$F_e u_E - F_w u_P + F_n u_N - F_s u_P = 0$$

$$-\left(F_w + F_s\right) u_P = -F_e u_W - F_n u_S$$

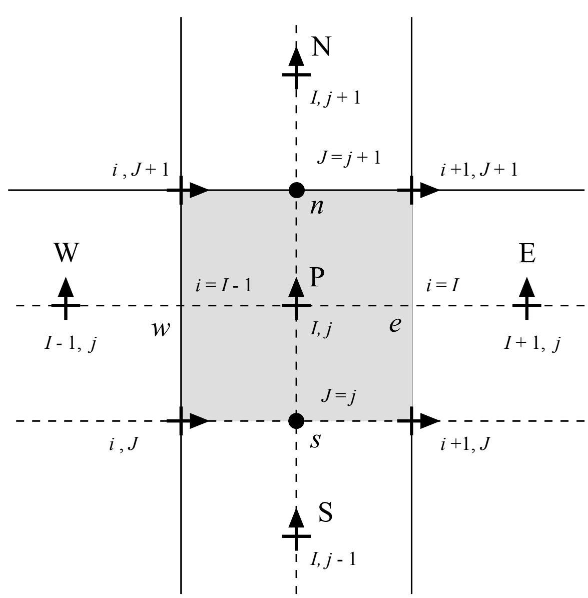
$$a_E = -F_e \ , \quad a_W = 0 \ , \quad a_N = -F_n \ , \quad a_S = 0$$

These three possibilities can be programed using the following:

$a_W = \max\left(0, F_w, D_w + F_w/2\right)$	$a_S = \max\left(0, F_s, D_s + F_s/2\right)$
$a_E = \max\left(0, -F_e, D_e - F_e/2\right)$	$a_N = \max\left(0, -F_n, D_n - F_n/2\right)$

Derivation of *y*-Momentum Equation for Interior Node:

v **- velocity cell**



Recall steady *y*-momentum equation for finite volume formulation:

$$F_w = \left(\rho u A\right)_w = \rho\, dy\, dz \left[u\left(i, J\right) + u\left(i, J+1\right) \right] / 2$$

For two-dimensional domain for interior elements aligned with the *x* and *y* axis

$$v_e\left(\rho v A\right)_e - v_w\left(\rho v A\right)_w + v_n\left(\rho v A\right)_n - v_s\left(\rho v A\right)_s =$$
$$\left(\mu\frac{\partial v}{\partial x}A\right)_e - \left(\mu\frac{\partial v}{\partial x}A\right)_w + \left(\mu\frac{\partial v}{\partial y}A\right)_n - \left(\mu\frac{\partial v}{\partial y}A\right)_s + \left(p_s - p_n\right)dx\,dz$$

Define the following advection (or *flow*) *coefficient*, F, and *diffusion coefficient*, D, at the faces where the gradient is evaluated using 2-point central finite difference approximation:

$F_w = \left(\rho u A\right)_w = \rho\,dy\,dz\left[u\left(i,J\right)+u\left(i,J+1\right)\right]/2$	$D_w = \left(\mu/dx\right)_w A_w = \left(\mu/dx\right)_w dy\,dz$
$F_e = \left(\rho u A\right)_e = \rho\,dy\,dz\left[u\left(i+1,J\right)+u\left(i+1,J+1\right)\right]/2$	$D_e = \left(\mu/dx\right)_e A_e = \left(\mu/dx\right)_e dy\,dz$
$F_s = \left(\rho v A\right)_s = \rho\,dx\,dz\left[v\left(I,j-1\right)+v\left(I,j\right)\right]/2$	$D_s = \left(\mu/dy\right)_s A_s = \left(\mu/dy\right)_s dx\,dz$
$F_n = \left(\rho v A\right)_n = \rho\,dx\,dz\left[v\left(I,j\right)+v\left(I,j+1\right)\right]/2$	$D_n = \left(\mu/dy\right)_n A_n = \left(\mu/dy\right)_n dx\,dz$

NOTE: For constant properties: $D_x = D_w = D_e = \left(\mu/dx\right)dy\,dz$, $D_y = D_s = D_n = \left(\mu/dy\right)dx\,dz$

$$F_e v_e - F_w v_w + F_n v_n - F_s v_s =$$
$$D_e\left(v_E - v_P\right) - D_w\left(v_P - v_W\right) + D_n\left(v_N - v_P\right) - D_s\left(v_P - v_S\right) + \left(p_s - p_n\right)dx\,dz$$

Again, use hybrid differencing scheme (with same coefficients as above) and rewrite as:

$$a_P v_P = a_W v_W + a_E v_E + a_S v_S + a_N v_N + b \;,\quad b = \left(p_s - p_n\right)dx\,dz$$

Derivation of Advection-Diffusion Equation for Property at Main Node:

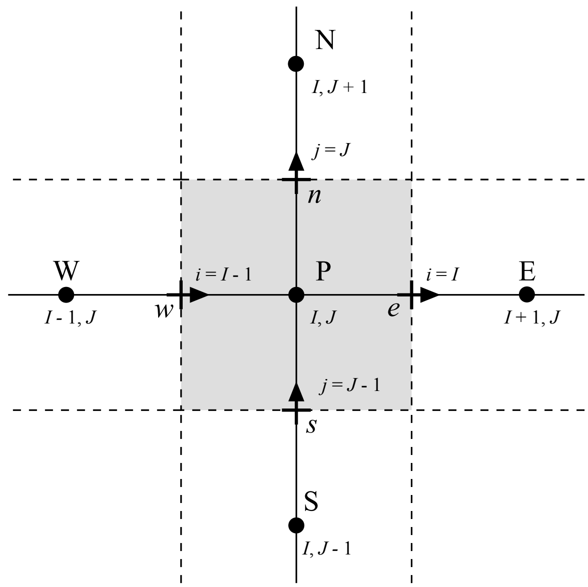
Recall general steady, advection-diffusion equation for finite volume formulation:

$$\sum_{f=1}^{N_f}\phi_f\left(\rho\vec{V}\cdot\vec{n}\,A\right)_f = \sum_{f=1}^{N_f}\left(\Gamma\nabla\phi\cdot\vec{n}\,A\right)_f + S_\phi \cancel{V}$$

For two-dimensional domain for interior elements aligned with the x and y axis

$$\phi_e\left(\rho u A\right)_e - \phi_w\left(\rho u A\right)_w + \phi_n\left(\rho u A\right)_n - \phi_s\left(\rho u A\right)_s =$$
$$\left(\Gamma\frac{\partial\phi}{\partial x}A\right)_e - \left(\Gamma\frac{\partial\phi}{\partial x}A\right)_w + \left(\Gamma\frac{\partial\phi}{\partial y}A\right)_n - \left(\Gamma\frac{\partial\phi}{\partial y}A\right)_s + S_\phi \cancel{V}$$

scalar property cell



Define the following advection (or *flow*) *coefficient*, *F*, and *diffusion* coefficient, *D*, at the faces where the gradient is evaluated using 2-point central finite difference approximation:

$F_w = (\rho u A)_w = \rho u(i-1,J)\,dy\,dz$	$D_w = (\Gamma/dx)_w A_w = (\Gamma/dx)_w\,dy\,dz$
$F_e = (\rho u A)_e = \rho u(i,J)\,dy\,dz$	$D_e = (\Gamma/dx)_e A_e = (\Gamma/dx)_e\,dy\,dz$
$F_s = (\rho v A)_s = \rho v(I,j-1)\,dx\,dz$	$D_s = (\Gamma / dy)_s A_s = (\Gamma / dy)_s\,dx\,dz$
$F_n = (\rho v A)_n = \rho v(I,j)\,dx\,dz$	$D_n = (\Gamma/dy)_n A_n = (\Gamma/dy)_n\,dx\,dz$

NOTE: For constant properties: $D_x = D_w = D_e = (\Gamma/dx)\,dy\,dz$, $D_y = D_s = D_n = (\Gamma/dy)\,dx\,dz$

$$F_e\phi_e - F_w\phi_w + F_n\phi_n - F_s\phi_s = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) + D_n(\phi_N - \phi_P) - D_s(\phi_P - \phi_S) + S_\phi \cancel{V}$$

Again, use hybrid differencing scheme (with same coefficients as above) and rewrite as:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + b \ , \quad b = S_\phi \cancel{V}$$

Velocity Correction Equations:

Define actual, guessed (or starred) and corrected (or primed) values as follows:

$u = u^* + u'$, $v = v^* + v'$, and $p = p^* + p'$

Calculate an updated velocity field, u^* and v^*, using *x* and *y*-momentum equations with guessed pressure field, p^*, and coefficients estimated using guessed velocity fields:

$$a_P u_P^* = a_W u_W^* + a_E u_E^* + a_S u_S^* + a_N u_N^* + \left(p_W^* - p_E^*\right) dy\,dz$$

$$a_P v_P^* = a_W v_W^* + a_E v_E^* + a_S v_S^* + a_N v_N^* + \left(p_S^* - p_N^*\right) dx\,dz$$

Subtract these equations from original momentum equation and substitute in corrections to get:

$$a_P u_P' = a_W u_W' + a_E u_E' + a_S u_S' + a_N u_N' + \left(p_W' - p_E'\right) dy\,dz$$

$$a_P v_P' = a_W v_W' + a_E v_E' + a_S v_S' + a_N v_N' + \left(p_S' - p_N'\right) dx\,dz$$

Simplify by neglecting neighboring correction values and rewrite as:

$$u_P' = du_P\left(p_W' - p_E'\right) \;,\quad du_P = \begin{cases} dy\,dz/a_P & \text{SIMPLE} \\ dy\,dz\Big/\left(a_P - \sum a_{nb}\right) & \text{SIMPLE-C} \end{cases}$$

$$v_P' = dv_P\left(p_S' - p_N'\right) \;,\quad du_P = \begin{cases} dy\,dz/a_P & \text{SIMPLE} \\ dy\,dz\Big/\left(a_P - \sum a_{nb}\right) & \text{SIMPLE-C} \end{cases}$$

Finally, substituting back into definitions for

$$u_P = u_P^* + du_P\left(p_W' - p_E'\right) \quad \text{or} \quad u_{i,J} = u_{i,J}^* + du_{i,J}\left(p_{I,J}' - p_{I+1,J}'\right)$$

$$v_P = v_P^* + dv_P\left(p_S' - p_N'\right) \quad \text{or} \quad v_{I,j} = v_{I,j}^* + dv_{I,j}\left(p_{I,J}' - p_{I,J+1}'\right)$$

Pressure Correction Equation from Conservation of Mass for Main Node

Recall conservation of mass for finite volume formulation:

$$\sum_{f=1}^{N_f} \left(\rho \vec{V} \cdot \vec{n} \, A \right)_f = 0$$

For two-dimensional domain for interior elements aligned with the x and y axis

$$\left(\rho u \, A \right)_e - \left(\rho u \, A \right)_w + \left(\rho v \, A \right)_n - \left(\rho v \, A \right)_s = 0$$

Substitute in corrected velocities to get

$$\rho \, dy \, dz \left[u_e^* + du_e \left(p'_P - p'_E \right) - u_w^* - du_w \left(p'_W - p'_P \right) \right] + \\ \rho \, dx \, dz \left[v_n^* + dv_n \left(p'_P - p'_N \right) - v_s^* - dv_s \left(p'_S - p'_P \right) \right] = 0$$

$$\left[\rho \, dy \, dz \left(du_e + du_w \right) + \rho \, dx \, dz \left(dv_n + dv_s \right) \right] p'_P = \left(\rho \, dy \, dz \, du_e \right) p'_E + \left(\rho \, dy \, dz \, du_w \right) p'_W + \\ \left(\rho \, dx \, dz \, dv_n \right) p'_N + \left(\rho \, dx \, dz \, du_s \right) p'_S + \left[\rho \, dy \, dz \left(u_w^* - u_e^* \right) + \rho \, dx \, dz \left(v_s^* - v_n^* \right) \right]$$

Rewrite as

$$a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b'$$

$$a_P = a_W + a_E + a_S + a_N \;, \quad b' = \rho \, dy \, dz \left(u_w^* - u_e^* \right) + \rho \, dx \, dz \left(v_s^* - v_n^* \right)$$

$a_W = \rho \, dy \, dz \, du_w$	$a_S = \rho \, dx \, dz \, dv_s$
$a_E = \rho \, dy \, dz \, du_e$	$a_N = \rho \, dx \, dz \, dv_n$